Stochastic Signal Processing

Lecture 3 - Two Random Variables

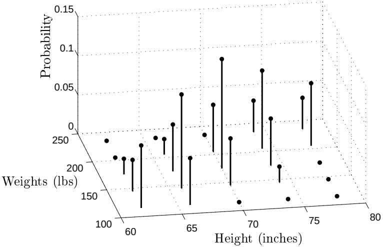
August, 2025

What will be presented today?

- The notion of two RVs
- Mathematical representation in terms of joint PMFs, CDFs and PDFs
- Independence of two RVs
- Moments for two RVs
- Joint Gaussian RVs

Introduction

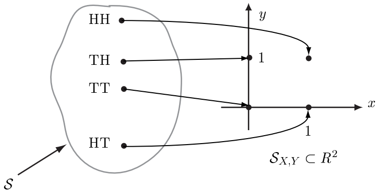
- Previous lectures dealt with a single RV (discrete or continuous) which measures 1 attribute of the outcome of a random experiment.
- In practice, we are often interested in measuring more than 1 attributes of the outcome of a random experiment
- Example: weight and height of random student from class.



Source: S. Kay (2006) Intuitive Probability and Random Processes using MATLAB. Kluwer Academic Publishers

Two Random Variables

- The concept of RV as a mapping can be extended to the case where 2 quantities are of interest.
- Consider a function which assigns a pair of real numbers, $\boldsymbol{X}(\varsigma) = (X(\varsigma), Y(\varsigma))$ to each outcome ς in the sample space S of a random experiment i.e., a vector function that maps S to $\mathcal{R}^2$.
- Examples: weight and height of random student from class; Number of users opting to see ads or have direct access of a web page; spin of a wheel.



Source: S. Kay (2006) Intuitive Probability and Random Processes using MATLAB. Kluwer Academic Publishers

Events and Probabilities I

- Events involving $\boldsymbol{X}$ are specified by conditions which can be represented by region in 2D plane.
- Examples: $A = \{X + Y \leq 10\}$; $B = \{min(X, Y) \leq 5\}$; and $C = X^2 + Y^2 \leq 100$

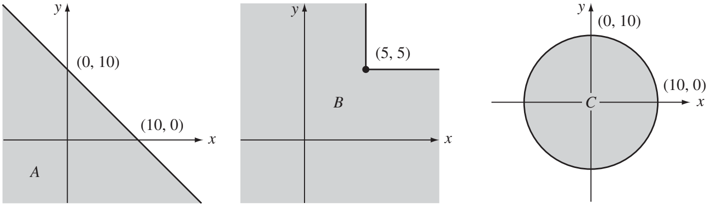


FIGURE 5.2

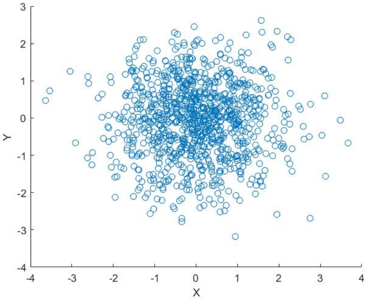
Source: A. Garcia (2008) Probability, Statistics and Random Processes for Electrical Engineering. New york:Pearson

Events and Probabilities II

- Equivalent Events: To determine the probability that $\boldsymbol{X}$ is in some region B in a plane, we can find *equivalent event* A in the underlying sample space S and find its probability:

$$P[\boldsymbol{X} \text{ in } B] = P[A] = P\{\varsigma : (X(\varsigma), Y(\varsigma)) \text{ in } B\}].$$

- Scattergram: provides graphical means to deduce the joint behaviour of two RVs.



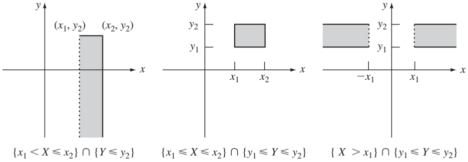
- *Joint Events*: Events of interest for a random vector $\boldsymbol{X}$ typically have the product form:

$$A = \{X \text{ in } A_1\} \cap \{Y \text{ in } A_2\}$$

where A_k is a one-dimensional event (subset of real line). Event A occurs when both events X in A_1 and Y in A_2 occur jointly.

- The probability of the **product-form** events is:

$$P[A] = P[\{X \text{ in } A_1\} \cap \{Y \text{ in } A_2\}] = P[X \text{ in } A_1, Y \text{ in } A_2]$$

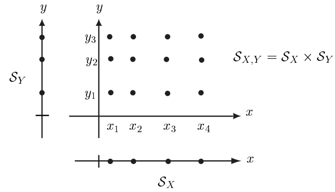


Pair of RVs I

- Let the vector RV $\mathbf{X} = (X, Y)$ assume values from some countable set $S = \{(x_j, y_k), j = 1, 2, \ldots; k = 1, 2, \ldots\}$. The joint probability mass function (PMF) of X specifies the probabilities of joint event $\{X = x_j\} \cap \{Y = y_k\}$:

$$\begin{aligned} p_{X,Y} &= P[\{X = x_j\} \cap \{Y = y_j\}] \\ &= P[X = x_j, Y = y_k] \text{ for } j = 1, 2, \ldots \;\; k = 1, 2, \ldots \end{aligned}$$

- Thus the joint PMF gives the probability of the occurrence of the pairs (x_j, y_k).



Pair of RVs II

- The probability of an event A:

$$P[\mathbf{X} \text{ in } A] = \sum \sum_{(x_j, y_k) \text{ in } A} p_{X,Y}(x_j, y_k)$$

- Probability of the sample space S is 1:

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k) = 1$$

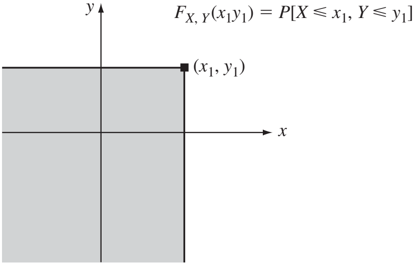
- Marginal PMF:

$$p_X(x_j) = P[X = x_j] = \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k)$$

Joint CDFs

The joint CDF of a pair of RVs X and Y is defined as the probability of product-form event $\{X \leq x_1\} \cup \{Y \leq y_1\}$

$$F_{X,Y}(x_1, y_1) = P[X \leq x_1, Y \leq y_1].$$

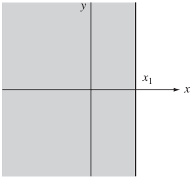


Properties of Joint CDFs I

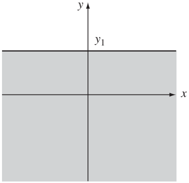
- Nondecreasing in the 'north-east' direction:

$$F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2) \text{ if } x_1 \leq x_2,\ \ y_1 \leq y_2,$$

- $F_{X,Y}(-\infty, y_1) = F_{X,Y}(x_1, -\infty) = 0$
- $F_{X,Y}(\infty, \infty) = 1$
- $F_X(x) = F_{X,Y}(x_1, \infty) = P[X \leq x_1, Y < \infty] = P[X \leq x_1]$



$F_X(x_1) = P[X \leq x_1, Y < \infty]$

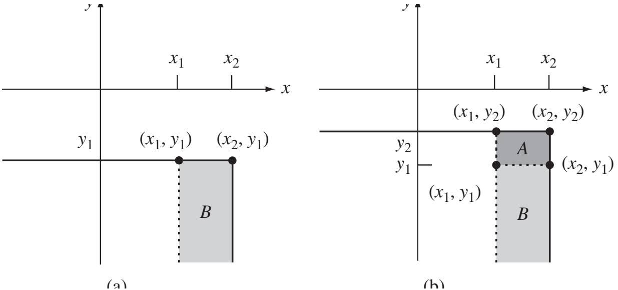


$F_Y(y_1) = P[X < \infty, Y \leq y_1]$

Finding probabilities of events using CDFs

The joint CDFs can be used to find the probabilities of events that can be expressed as a union or intersection of semi-infinite rectangles.

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) \tag{1}$$

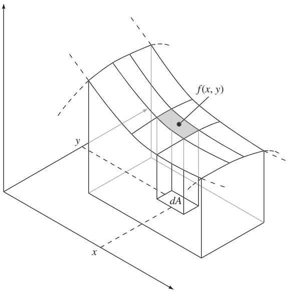


Source: A. Garcia (2008) Probability, Statistics and Random Processes for Electrical Engineering. New york:Pearson

Joint PDF of 2 RVs

- The RVs X and Y are jointly continuous if the probabilities of events involving those RVs can be expressed as an integral of a PDF.
- Let $f_{X,Y}(x, y)$ be a non-negative function, called the joint pdf, that is defined on the real plane such that for every event A, we have

$$P[\mathbf{X} \text{ in } A] = \int\int_A f_{X,Y}(x', y') dx' dy'.$$



A. Garcia (2008) Probability, Statistics and Random Processes for Electrical Engineering

Properties of joint PDF I

- Probability of Sample space S:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x',y')dx'dy' = 1$$

- Connection b/w CDF and PDF:

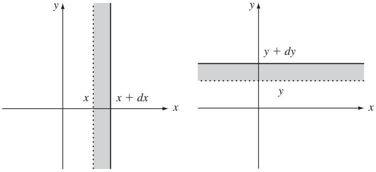
$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x',y')dx'dy'$$

Properties of joint PDF II

- Marginal PDF:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y')dy'$$

- What does $f_X(x)dx$ and $f_Y(y)dy$ represent?



$f_X(x)dx \cong P[x < X \leq x + dx, Y < \infty]$
(a)

$f_Y(y)dy \cong P[X < \infty, y < Y \leq y + dy]$
(b)

Properties of joint PDF III

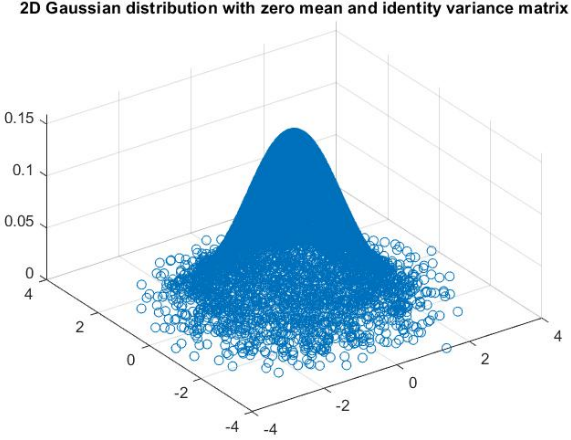
- Probability of a joint event corresponding to rectangular region is obtained by letting $A = \{(x, y) : a_1 < x \leq b_1, a_2 < y \leq b_2\}$:

$$P[a_1 < X \leq b_1, a_2 < Y \leq b_2] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{X,Y}(x', y') dy' dx'.$$

- What does $f_{X,Y}(x, y)dxdy$ represent?

Examples

- **Jointly Gaussian RVs**:



- **Jointly Uniform RVs**: Can you visualize what the density function would look like for 2D case?

Independence of 2 RVs

- The RVs X and Y are independent if any event A_1 defined in terms of X is independent of any event A_2 defined in terms of Y

$$P[X \text{ in } A_1, Y \text{ in } A_2] = P[X \text{ in } A_1]P[Y \text{ in } A_2].$$

- The two RVs are independent if and only if:
  1. For Discrete RVs:

$$\begin{aligned} p_{X,Y}(x_j, y_k) &= P[X = x_j, Y = y_k] \\ &= P[X = x_j]P[Y = y_k] \\ &= P[X = x_j]P[Y = y_k] \text{ for all } x_j \text{ and } y_k \end{aligned}$$

  2. In terms of CDF:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \text{ for all } x \text{ and } y$$

  3. For continuous RVs in terms of PDF:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \text{ for all } x \text{ and } y$$

Expected Values of Function of RVs

- The expected value of $Z = g(X, Y)$ can be found from the following expression:

$$E[Z] = \sum_i \sum_n g(x_i, y_n) p_{X,Y}(x_i, y_n), \ X, \ Y: \text{ Discrete}$$

$$E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy, \ X, \ Y: \text{ Continuous}$$

- Problem: Let $Z = X + Y$. Find $E[Z]$. If we extend the summation over n RVs: $X_1 + X_2 + \ldots + X_n$, what will be the mean?

Correlation and Covariance of two RVs I

- The jk-th joint moment of X and Y is:

$$E[X^j Y^k] = \sum_i \sum_n x_i^j y_n^k p_{X,Y}(x_i, y_n), \; X, \; Y\text{: Discrete}$$

$$E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x, y) dxdy, \; X, \; Y\text{: Continuous}$$

- If $j = 0$, we obtain the moments of Y, e.g. $E[Y^2]$ for $k = 2$ (and $j = 0$).
- If $k = 0$, we obtain the moments of X, , e.g. $E[X^3]$ for $j = 3$ (and $k = 0$).
- Correlation of X and Y, $E[XY]$: obtained by putting $j = k = 1$.
    - $E[XY] = 0$ implies that X and Y are orthogonal.

Correlation and Covariance of two RVs II

- The jk-th joint and *central* moment of X and Y is $E[(X-E[X])^j(Y-E[Y])^k]$

$$E[(X-E[X])^j(Y-E[Y])^k] = \sum_i \sum_n (x_i - E[X])^j, (y_n - E[Y])^k p_{X,Y}(x_i, y_n),$$

$$E[(X-E[X])^j(Y-E[Y])^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-E[X])^i (y-E[Y])^k f_{X,Y}(x,y)dxdy,$$

- If $j=2$ and $k=0$, we get VAR[X]; If $j=0$ and $k=2$, we get VAR[Y].
- Covariance of X and Y, $E[(X-E[X])(Y-E[Y])]$: obtained by putting $j=k=1$.
    - $E[(X-E[X])(Y-E[Y])] = 0$ implies that X and Y are uncorrelated.
- What is the relation between correlation and covariance of X and Y?
- Independent vs Uncorrelated?

Correlation and Covariance of two RVs III

- Correlation Coefficient of X and Y:

$$\rho_{X,Y} = \frac{COV[X,Y]}{\sigma_x \sigma_y} = \frac{E[XY] - E[X]E[Y]}{\sigma_x \sigma_y}$$

  where $\sigma_x = \sqrt{VAR[X]}$ and $\sigma_y = \sqrt{VAR[Y]}$.

  1. $-1 \leq \rho_{X,Y} \leq 1$
  2. $\rho_{X,Y}$ quantifies linear relation between X and Y; the extreme values of ρ are achieved when $Y = aX + b$.
  3. If $\rho_{X,Y} = 0$, then X and Y are uncorrelated.

- Example (uncorrelated but dependent RVs): Let θ be uniformly distributed in $(0, 2\pi)$. Let $X = \cos(\theta)$ and $Y = \sin(\theta)$. Are X and Y independent? and/or uncorrelated?

Jointly Gaussian RVs: Two RV case

- The RVs X and Y are considered to be jointly Gaussian if their joint PDF has the form

$$f_{X,Y}(x,y) = \frac{exp\left\{\frac{-1}{2(1-\rho^2_{X,Y})}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 + \left(\frac{y-m_2}{\sigma_2}\right)^2 - 2\rho_{X,Y}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right)\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2_{X,Y}}}$$

  where $-\infty \leq (x, y) \leq \infty$
- The PDF is centered at (m_1, m_2) and has a bell shape that depends on the value of σ_1, σ_2 and ρ.

2D Gaussian PDF: Example

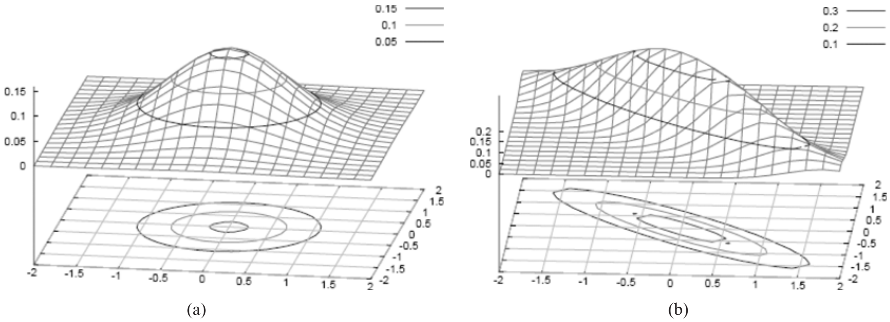


FIGURE 5.25
Jointly Gaussian pdf (a) $\rho = 0$ (b) $\rho = -0.9$.

- $m_1 = m_2 = 0$
- $\rho_{X,Y} = 0$ (left) and $\rho_{X,Y} = -0.9$ (right)
- Contours of constant PDF will be ellipse. How?
- $\rho_{X,Y} \neq 0$ implies the major axis of ellipse is oriented along an angle
- $\theta = \frac{1}{2} arctan\left(\frac{2\rho_{X,Y}\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right)$.

Marginal PDF from Joint Gaussian PDF

- One can integrate $f_{X,Y}(x,y)$ over all y to yield:

$$f_X(x) = \frac{1}{2\pi\sigma_1} e^{-(x-m_1)^2/2\sigma_1^2},$$

  which is Gaussian PDF with mean m_1 and variance σ_1.

- Show that marginal PDF of a 2D Gaussian with zero means and unit variances is another Gaussian with zero mean and unit variance.

$$f_{X,Y}(x,y) = \frac{e^{-(x^2+y^2-2\rho xy)/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}}.$$