Stochastic Signal Processing

Lecture 2 - Random Variable

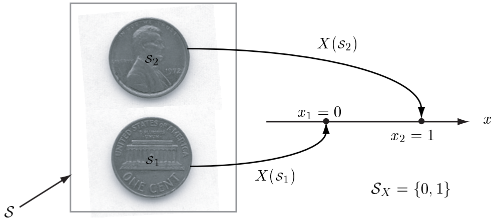
August, 2025

What will be presented today?

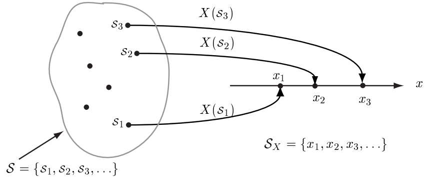
- The notion of discrete and continuous random variables (RV)
- Probability Mass Function (PMF) and Probability Distribution Function (PDF)
- Cumulative Distribution Function (CDF)
- Examples of commonly used PMFs and PDFs, e.g., Bernoulli, Binomial, Geometric and Uniform, Exponential and Gaussian
- Moments of RVs
- Conditional PDFs and PMFs

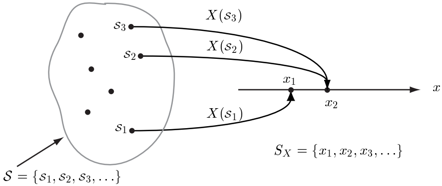
Random Variable

- A **Random variable** X is a function (or mapping) from sample space S of a random experiment to a set of real numbers.
- The sample space S is the domain while the new set S_X of all the values taken by X is the range of RV.
- The function X is fixed (deterministic) and the randomness in the observed values is due to the input argument ς.



- **Discrete RV**: maps the sample space S of a random experiment into a subset of the real line that consists of *finite or countably infinite set of points* e.g., number of heads in 3 consecutive coin tosses.
- One-to-one vs many-to-one mapping





- **Continuous RV**: is defined as a mapping from the experimental sample space S to a numerical sample space S_X which is a subset of a real line (consisting of *continuum of numbers).*

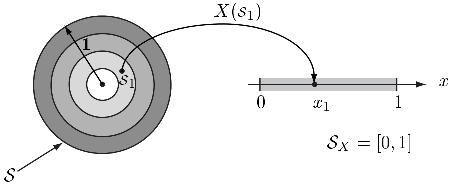


Figure 10.1: Mapping of the outcome of a thrown dart to the real line (example of continuous random variable).

Why do we need RVs?

- The events of interest in a random experiment inherently involve some measurement or numerical attribute of the outcomes e.g., number of heads in n coin tosses; in a randomly selected computer job, we may be interested in the execution time of the job.
- Mathematical convenience: easier to deal with and manipulate numbers rather than non-numeric outcomes.

Equivalent Events: A way to assign probabilities to RV

- Consider a random experiment and let ς and S denote an outcome and sample space of the experiment.
- Let X be a RV that maps S into a set of real numbers S_X. Let A be the set of outcomes ς in S that lead to values $X(\varsigma)$ in B as shown.

$$A = \{\varsigma : X(\varsigma) \text{ in } B\}$$

then the event B occurs whenever A occurs, implying that their probabilities must be equal.

$$P[B] = P[A] = P[\{\varsigma : X(\varsigma) \text{ in } B\}].$$

A and B are **equivalent events**.

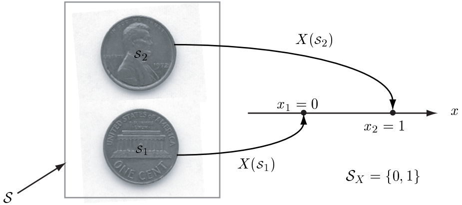
- Examples: One-to-one (coin toss) and many-to-one (even number after a dice throw) mapping.

Discrete RVs and Probability Mass Functions I

- The PMF of a discrete RV X is defined as:

$$p_X(x) = P[X = x] = P[\{\varsigma : X(\varsigma) = x\}], \quad \text{for } x \text{ a real number}$$

- $p_X(x)$ is nonzero (defined?) only at discrete values $x_1, x_2, \ldots$
- Probabilities of x_k, i.e., the PMF of X, can be obtained from the probabilities of the events $A_k = \{\varsigma : X(\varsigma) = x_k\}$. That is $p_X(x_k) = P[X = x_k] = P[\varsigma : X(\varsigma) = x_k]$



Discrete RVs and Probability Mass Functions II

- **Properties of PMF**:
  1. $p_X(x) \geq 0 \quad \text{for all } x$
  2. $\sum_{x \in S_X} p_X(x) = \sum_{\text{all k}} p_X(x_k) = \sum_{\text{all k}} P[A_k] = 1$
- Examples:
  1. Let X be the number of heads in 3 independent tosses of a coin. Find the PMF of X.
  2. Let A be an event of interest in some random experiment e.g., the device is defective. A "success" occurs is A occurs when an experiment is performed. The RV I_A is equal to 1 if A occurs, and 0 otherwise. Find the PMF of I_A.
  3. Let a random number generator outputs an integer number X which is equally likely within the set $\{0, 1, \ldots, M-1\}$. What is the PMF of X.
  4. A binary communication channel introduces bit error in a transmission with probability p. Let X be the number of errors in n independent transmissions. Find the PMF of X.

Cumulative Distribution Function

- **Cumulative distribution function (CDF)** of a RV X evaluated at x is defined by the probability of the event $\{X \leq x\}$:

$$F_X(x) = P[X \leq x] \text{ for } -\infty \leq x \leq +\infty \tag{1}$$

that is, the probability that X takes on values in the range $[(-\infty, x)]$.

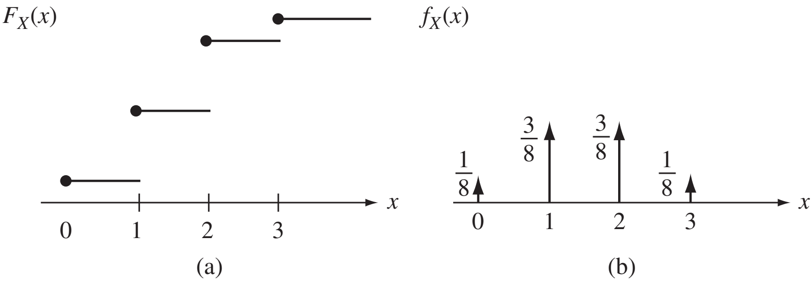
- The probabilities corresponding to all intervals on real line can be computed from $F_X(x)$.

Properties of CDF

1. $$0 \leq F_x(x) \leq 1$$
2. $$\lim_{x \to \infty} F_x(x) = 1$$
3. $$\lim_{x \to -\infty} F_x(x) = 0$$
4. $F_X(x)$ is a non-decreasing function of x i.e., if $a < b$, then $F_X(a) < F_X(b)$.
5. $$P[a < X \leq b] = F_X(b) - F_X(a)$$
6. $$P[X = b] = F_X(b) - F_X(b^-)$$
7. $$P[X > x] = 1 - F_X(x)$$

Example of CDF (Discrete RV)

A random experiment consists in counting the number of heads in 3 consecutive coin tosses. Let X denote the number of heads in 3 consecutive coin tosses. Find the CDF of X.



Probability Density Function (PDF)

- PDF of X is defined as the derivative of the CDF of X i.e.,

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

- Represents the 'density' of the probability of x in the following sense:

$$P[x < X \leq x + h] \quad = F_X(x+h) - F_X(x) \quad = \frac{F_X(x+h) - F_X(x)}{h} h.$$

If CDF has the derivative at x, then as h becomes very small,

$$P[x < X \leq x + h] = f_X(x)h.$$

Probability Density Function

Illustration of PDF as a measure of the 'probability density' of X at point x

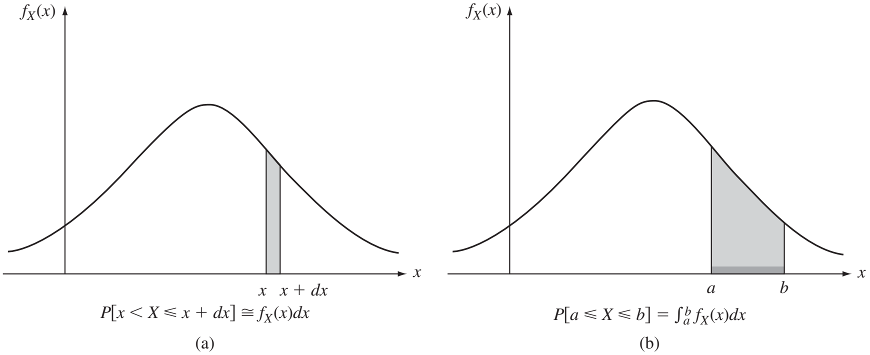


FIGURE 4.4
(a) The probability density function specifies the probability of intervals of infinitesimal width. (b) The probability of an interval $[a, b]$ is the area under the pdf in that interval.

Properties of PDF I

1. $$f_x(x) \geq 0$$

2. $$P[a \leq X \leq b] = \int_a^b f_X(x)dx$$

3. $$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

4. $$1 = \int_{-\infty}^{\infty} f_X(t)dt$$

Examples of PDFs

- PDF of a uniform RV in the range from $[a, b]$
- PDF of exponential RV: The transmission time X of messages in communication system obeys the exponential probability law with parameter λ, i.e., $P[X > x] = e^{-\lambda x}, \quad x > 0$. Find the CDF and PDF of X.

Expected value/Mean of a RV

The expected value/mean of a continuous RV X is denoted by $E[X]$ and is given by

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt. \tag{2}$$

For discrete RV X, the mean is defined by

$$m_X = E[X] = \sum_{x \in S_X} x p_X(x) = \sum_k x_k p_X(x_k).$$

The mean is defined only when the above sum/integral converges absolutely.

Expected value/Mean of $Y = g(X)$

The expected value/mean of a RV Y, in terms of the PDF of X, is given by

$$E[Y] = \int_{-\infty}^{\infty} g(t) f_X(t) dt. \tag{3}$$

For a function of discrete RV X, the mean is

$$E[Y] = \sum_k g(x_k) p_X(x_k).$$

1. What are the expected values of a constant c, cX and sum of functions of a RV, i.e., $\sum_k g_k(X)$?
2. Let $Y = a\cos(\omega t + \phi)$ where a, ω are constants and ϕ is a uniform RV in the interval $(0, 2\pi)$. What are the expected values of Y and Y^2?

Variance of RV

- The variance of an RV quantifies the variation of the RV about its mean value.
- Definition: $VAR[X] = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (t - E[X])^2 f_X(t)dt$.
- Standard Deviation: STD=$\sqrt{VAR[X]}$
- STD is a measure of the width or spread of a distribution.
- $VAR[X] = E[X^2] - (E[X])^2$.
- Variance of a discrete RV?

Prove the following:

1. $VAR[c] = 0$.
2. $VAR[X + c] = VAR[X]$.
3. $VAR[cX] = c^2 VAR[X]$.

nth moment of a RV

- $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$
- The characteristic function of a RV:

$$\begin{aligned} \Phi_X(\omega) &= E[e^{j\omega X}] \\ &= \int_{\infty}^{\infty} f_X(x) e^{j\omega x} dx. \end{aligned}$$

- Interpretation: i) Expected value of a function of X; ii) Fourier transform of the PDF of X (with change of ω sign).
- Usage: We can obtain moments of X conveniently by using the characteristic function.

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{dw^n} \Phi_X(\omega)|_{\omega=0}.$$

Prove the above result!

Examples of Continuous RV: Uniform RV I

- **Uniform RVs** arise in situations where all values in the interval of real line are equally likely to occur.
- Let a uniform RV X be distributed in the range $[a, b]$ on the real line, then
- Sample space: $S_X = \{a, b\}$
- Probability density function: $f_X(x) = \frac{1}{b-a}, \quad \text{for} \quad a \leq x \leq b$
- Expectation: $E[X] = \frac{a+b}{2}$ Prove
- Variance: $Var(X) = \frac{(b-a)^2}{12}$ Prove

Examples of Continuous RV: Exponential RV I

- **Exponential RVs** arise in modelling the time between occurance of events and in modelling the lifetime of devices and systems.
- The exponential RV X with parameter $\lambda > 0$ has following properties:
- Sample space: $S_X = \{0, \infty\}$
- Probability density function: $f_X(x) = \lambda e^{-\lambda x} \quad x, \geq 0,\ \lambda > 0.$
- Expectation: $E[X] = \frac{1}{\lambda}$ Prove
- Variance: $Var(X) = \frac{1}{\lambda^2}$

Examples of Continuous RV: Exponential RV II

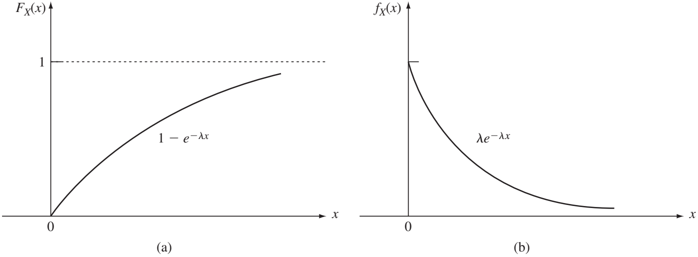


FIGURE 4.9
An example of a continuous random variable—the exponential random variable. Part (a) is the cdf and part (b) is the pdf.

Examples of Continuous RVs: Gaussian RV I

- The PDF of **Gaussian RV** is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \text{ for } -\infty \leq x \leq \infty,$$

  where m and $\sigma > 0$ are real numbers which are respectively the mean and the variance of X.

- The CDF of X is given by

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(x'-m)^2}{2\sigma^2}} dx'.$$

- With the change of variable, $t = \frac{(x'-m)}{\sigma}$, we get

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{\frac{-t^2}{2}} dt = \Phi(\frac{x-m}{\sigma}).$$

- $\Phi(x)$ is the CDF of Gaussian RV with $m = 0$ and $\sigma = 1$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt.$$

Any probability involving arbitrary Gaussian RV can be expressed in terms of $\Phi(x)$

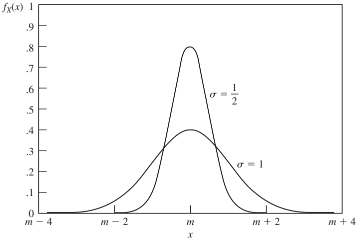


FIGURE 4.7
Probability density function of Gaussian random variable.

Examples of Continuous RVs: Gaussian RV III

In engineering, Q-function is defined by

$$
\begin{aligned}
Q(x) &= 1 - \Phi(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{\frac{-t^2}{2}} dt.
\end{aligned}
$$

$Q(x)$ is the probability of the tail of the function. No closed-form solution exists so its values are given in the form of look-up tables (See Table 4.2 in the text).

Examples of Discrete RV: Bernoulli RV

- A **Bernoulli random variable** X can take on two values, 1 and 0. It takes on a 1 if an experiment resulted in success, with probability p, and a 0 otherwise. Examples include coin flip, whether a disk drive crashed.
- Sample space: $\{0,1\}$
- Probability mass function: $P(X = 1) = p$; $P(X = 0) = (1 - p)$
- Expectation: $E[X] = p$
- Variance: $Var(X) = p(1 - p)$
- Bernoulli random variables and **indicator variables** are the same. As a review, a random variable I is called an indicator variable for an event A if $I = 1$ when A occurs and $I = 0$ if A does not occur.

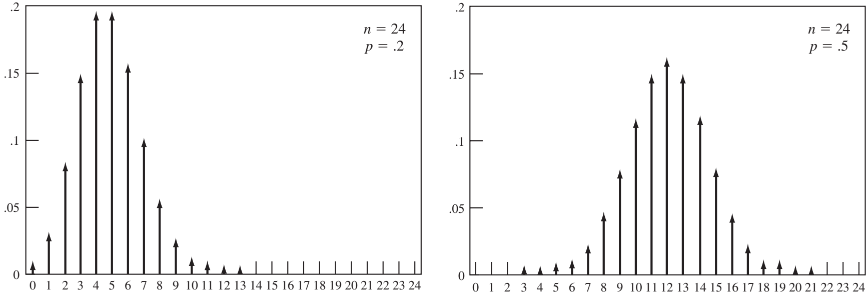
Examples of Discrete RV: Binomial RV I

- A **binomial random variable** is random variable that represents the number of successes in n successive independent trials of a Bernoulli experiment. Example include the number of heads in n coin flips, the number of disk drives that crashed in a cluster of 1000 computers.
- Sample space: $\{0,1,\ldots,n\}$
- If X is a Binomial random variable, where p is the probability of success in a given trial, then the pmf of X is given by:

$$P[X = k] = \binom{n}{k} p^k \times (1-p)^{n-k} \text{ for } k = 1, \ldots, n$$

- Expectation: $E[X] = np$ Prove
- Variance: $Var(X) = np(1-p)$ Prove

Examples of Discrete RV: Binomial RV II



(a) (b)

Examples of Discrete RV: Uniform RV I

- Discrete uniform RV Y take values from a set of consecutive integers i.e., $S_Y = j+1, \ldots, j+L$ with equal probability:

$$P_Y(k) = \frac{1}{L} \quad k \in \{j+1, \ldots, j+L\}.$$

- Sample space: $S_Y = \{j+1, \ldots, j+L\}$
- Expectation: $E[Y] = j + \frac{L+1}{2}$ Prove
- Variance: $Var(X) = \frac{L^2-1}{12}$

Conditional Probability Mass Function I

- Let X be a discrete RV and let C be an event such that $P[C] > 0$. The conditional PMF of X is defined by

$$p_X(x|C) = P[X = x|C] \quad \text{for } x \text{ a real number}$$
$$= \frac{P[\{X = x\} \cap C]}{P[C]}.$$

- The conditional probability of event $\{X = x_k\}$ is given by the probabilities of outcomes ς for which both $X(\varsigma) = x_k$ and ς are in C, normalized by $P[C]$.

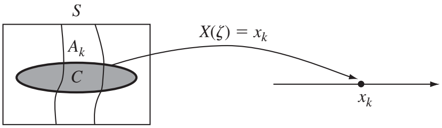


FIGURE 3.7

Conditional Probability Mass Function II

- **Theorem on total probability**: Let $B_1, \ldots, B_n$ forms the partition of sample space S. Let $p_X(x|B_i)$ be the conditional PMF of X given event B_i. Then

$$p_X(x) = \sum_{i=1}^{n} p_X(x|B_i) P[B_i].$$

- Example: A production line yields two types of devices.Type 1 devices occur with probability α and work for a relatively short time that is geometrically distributed with parameter r. Type 2 devices work much longer, occur with probability $1 - \alpha$ and have a lifetime that is geometrically distributed with parameter s. Let X be the lifetime of an arbitrary device. Find the PMF of X.

Conditional Mean and Variance I

- Let X be a discrete RV and suppose that we know that event B has occurred. The **conditional expected** value of X given B is

$$m_{X|B} = E[X|B] = \sum_k x_k p_X(x_k|B).$$

- The **conditional variance** of X given B is

$$VAR[X|B] = E[(X - m_{X|B})^2|B] = \sum_k (x_k - m_{X|B})^2 p_X(x_k|B).$$

Conditional Mean and Variance II

- Let $B_1, \ldots, B_n$ forms the partition of sample space S and let $p_X(x|B_i)$ be the conditional PMF of X given event B_i. Then

$$E[X] = \sum_{i=1}^{n} E[X|B_i]P[B_i]$$

Similarly,

$$E[g(X)] = \sum_{i=1}^{n} E[g(X)|B_i]P[B_i]$$

- Device Lifetime Example

Conditional PDF and CDF

- Conditional CDF:

$$F_X[x|A] = P[X \leq x|A]$$
$$= \frac{P[\{X \leq x\} \cap A]}{P[A]}, \text{ for } P[A] > 0$$

- Conditional PDF:

$$f_X(x|A) = \frac{d}{dx} F_X(x|A)$$

- **Example**: The lifetime of a machine has a continuous CDF $F_X(x)$. Find the conditional CDF and PDF of X given the event $C = \{X > t\}$ (i.e., machine is still working at time t).
- Using the theorem on total probability we get,
$F_X(x) = P[X \leq x] = \sum_{i=1}^{n} F_X(x|B_i)P[B_i]$.