ODE FOR THE DERIVATIVES OF THE GREEN FUNCTION IN THE INFINITE DEPTH CASE (FREQUENCY DOMAIN)

Notations are similar as in References R2 and R3.

G_r derivative of G with respect to r

G_Z derivative of G with respect to Z

MAIN RESULTS

$$(*)\quad \frac{\omega^2}{4} G_r^{(2)} - \omega\left(\omega^2 Z + \frac{9}{4}\right) G_R^{(1)} + \left(\omega^4\left(r^2+Z^2\right)+3\omega^2 Z+3\right) G_R = \frac{-6r\left(1+Z\omega^2\right)}{\left(r^2+Z^2\right)^{3/2}}$$

with initial conditions

$$\begin{cases} G_R(r,Z,0) = \dfrac{-2r}{\left(r^2+Z^2\right)^{3/2}} - i(0) \\ \lim\limits_{\omega\to 0}\left\{\dfrac{d}{d\omega} G_r(r,Z,i\omega)\right\} = 0 + i(0) \end{cases}$$

(*) $\frac{\omega^2}{4} G_Z^{(2)} - \omega\left(\omega^2 Z + \frac{9}{4}\right) G_Z^{(1)} + \left(\omega^4 (r^2+Z^2) + 3\omega^2 Z + 4\right) G_Z$

$$= \frac{-8Z - \omega^2 (4Z^2 - 2r^2)}{(r^2+Z^2)^{3/2}}$$

with initial conditions

$$\begin{cases} G_Z(r, Z, 0) = \dfrac{-2Z}{(r^2+Z^2)^{3/2}} - i(0) \\ \lim\limits_{\omega\to 0} \left\{ \dfrac{\partial}{\partial \omega} G_Z(r, Z, i\omega) \right\} = 0 + i(0) \end{cases}$$

EXPRESSION OF $\frac{\partial G}{\partial n_{\Pi'}}$ using G_Z and G_R

To compute influences coefficient, we need $\frac{\partial G}{\partial n_{\Pi'}}$ but we only know $\frac{\partial G}{\partial Z} = G_Z$ and $\frac{\partial G}{\partial r} = G_R$

According to R4 (page 72 on 4) and to reference R5 (page 388) we have

$$\frac{\partial G}{\partial n_{\Pi'}} = p' \frac{\partial G}{\partial x'} + q' \frac{\partial G}{\partial y'} + r' \frac{\partial G}{\partial z'}$$

with (p', q', r') the cosine direction in Π'

we know that $r = \sqrt{(x-x')^2 + (y-y')^2}$

and $Z = z + z'$

So $\frac{\partial r}{\partial x'} = -\frac{x-x'}{r}$ $\quad \frac{\partial r}{\partial y'} = -\frac{y-y'}{r}$

$$\frac{\partial Z}{\partial z'} = 1$$

Thus,

$$\frac{\partial G}{\partial n_{P'}} = -p' \frac{x - x'}{r} \frac{\partial G}{\partial r} - q' \frac{y - y'}{r} \frac{\partial G}{\partial r} + r' \frac{\partial G}{\partial Z}$$

$$= -p' \frac{x - x'}{r} G_R - q' \frac{y - y'}{r} G_R + r' G_Z$$

The information available up to now (here) is enough to completely compute the influence coefficients if we take into account ODE of equation 11 in Reference R2.

DETAILED DERIVATION OF THE ODE FOR G_R

Here are some known identities (verify them using a calculus book, reference R2 and/or reference R3)

$\mathcal{F}$ is fourier transform ; $f^{(n)}(t)$ is n^{th} derivative of f

$\hat{f}(w)$ is fourier transform of $f(t)$

1) $\mathcal{F}\{f^{(n)}(t)\} = (iw)^n \hat{f}(iw)$

2) $\mathcal{F}\{t^m f(t)\} = i^m \dfrac{d^m}{dw^m} \hat{f}(iw)$

(3) $(fg)' = f'g + g'f$

(4) $(fg)'' = f''g + 2f'g' + fg''$

S_r or S_R is same thing so as G_R G_r

From equation 6.5 of R 3 we have

$$\left(r^2+z^2\right) F_r^{(4)} - z t F_r^{(3)} + \left(\frac{1}{4} t^2 - 6z\right) F_r^{(2)} + \frac{11 t}{4} F_r^{(1)}$$

$$+ \frac{21}{4} F_r = 0$$

F_r is Green Function derivative with respect to r in time domain

Similarly as in section 2 of R2 (1st ligne)

Let's define $S_R(r, z, t)$

This will similarly leads to

$$\left(r^2+z^2\right) S_r^{(4)} - z t S_r^{(3)} + \left(\frac{1}{4} t^2 - 6z\right) S_R^{(2)}$$

$$+ \frac{11}{4} t S_r =$$

$$\delta \left(4\left(r^2+z^2\right) F_r^{(3)} - 3zt F_r^{(2)} + 2\left(\frac{1}{4} t^2 - 6z\right) F_r^{(1)} + \frac{11}{4} t F_r\right)$$

$$+ \delta^{(1)} \left[6\left(r^2+z^2\right) F_r^{(2)} - 3 z t F_r^{(1)} + \left(\frac{1}{4} t^2 - 6z\right) F_r\right]$$

$$+ \delta^{(2)} \left[4\left(r^2+z^2\right) F_r^{(1)} - z t F_r\right]$$

$$+ \delta^3 \left(r^2 + z^2\right) F_r$$

The LHS of above equation can be rewritten as (using known identities (1) and (2))

$$L = (r^2+z^2)(i\omega)^4 \hat{S}_r - Zi^1 \frac{d}{d\omega} \mathcal{F} S_r^{(3)}$$
$$+ \frac{1}{4} i^2 \frac{d^2}{d\omega^2} \mathcal{F} S_r^{(2)} - 6z(i\omega)^2 \hat{S}_r +$$
$$\frac{11}{4} i \frac{d}{d\omega} \mathcal{F} S_r^{(1)} + \frac{21}{4} \hat{S}_r$$

$$= \hat{S}_r\left(\omega^4(r^2+z^2) + 6z\omega^2 + \frac{21}{4}\right)$$
$$- Zi \frac{d}{d\omega} (i\omega)^3 \hat{S}_r - \frac{1}{4} \frac{d^2}{d\omega^2} (i\omega)^2 \hat{S}$$
$$+ \frac{11}{4} i \frac{d}{d\omega} (i\omega) \hat{S}_r$$

[using identities (3) and (4)]

$$= \hat{S}_r\left(\omega^4(r^2+z^2) + 6z\omega^2 + \frac{21}{4}\right)$$
$$- Zi(3) i^3 \omega^2 \hat{S}_r - Zi(i\omega)^3 \hat{S}_r^{(1)}$$
$$+ \frac{11}{4} i(i) \hat{S}_r + \frac{11}{4} i (i\omega) \hat{S}_r^{(1)}$$
$$- \frac{1}{4} 2i^2 \hat{S} - \frac{1}{4} i^2 \omega^2 \hat{S}_r^{(2)} + \frac{1}{4} i^2 2\omega \hat{S}_r^{(1)}$$

$$L = \hat{S}_n\left(w^4(r^2+z^2) + 6zw^2 + \frac{21}{4} - 3zw^2 - \frac{11}{4} + \frac{1}{2}\right)$$

$$+ \hat{S}_n^{(1)}\left(-z w^3 - \frac{11}{4} w + \frac{1}{2} w\right)$$

$$+ \hat{S}_n^{(2)}\left(+\frac{1}{4} w^2\right)$$

$$= \hat{S}_n\left(w^4(r^2+z^2) + 3zw^2 + 3\right) - \hat{S}_n^{(1)} w\left(zw^2 + \frac{9}{4}\right)$$

$$+ \hat{S}_n^{(2)}\left(\frac{1}{4} w^2\right)$$

To derive the equation for the right hand size we need to use equation (9) of R2 and initial condition for F_n in equation (6.6) of reference R3

Without replacing by their values in 6.6 and by using known identities (1) to (4) It in easy to see that the right hand size is :

$$(r^2+z^2) F_n^{(3)}(0) + F_n^{(1)}(0)\left[-w^2(r^2+z^2) - \underline{\underline{4z}} + z \right]$$

(8) (8)

Replacing $F_r^{(3)}(0)$ and $F_r^{(1)}(0)$ by their value leads to

$$\frac{-6r(1+Z\omega^2)}{(r^2+Z^2)^{3/2}}$$