NONLINEAR MARKOVIAN STOCHASTIC APPROXIMATION

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1 Preliminaries

Notations The Euclidean norm is denoted by $||.||$. The lowercase letter c and its derivatives c', c_0, etc. denote universal numerical constants, whose value may change from line to line. As we are primarily interested in dependence of α and k, we adopt the following big-O notation: $||f|| = \mathcal{O}(h(\alpha,k))$ if it holds that $||f|| \le s \cdot ||h(\alpha,k)||$ for some constant $s > 0$.

We use of the following iteration scheme:

$$\theta_{t+1} = \theta_t + \alpha\left(g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t)\right) \tag{1}$$

1.1 Assumptions

Assumption 1 *For each $X \in \mathcal{X}$, the function $g(\theta,X)$ is three times continuously differentiable in θ with uniformly bounded first to third derivatives, i.e., $\sup_{\theta\in\mathbb{R}^d} ||g^{(i)}(\theta,X)|| < \infty$ for $i = 1,2,3, X \in \mathcal{X}$. Moreover, there exists a constant $L_1 > 0$ such that (1) $||g^{(i)}(\theta,X) - g^{(i)}(\theta',X)|| \le L_1$, for all $\theta,\theta' \in \mathbb{R}^d, i = 0,1,2$ and $X \in \mathcal{X}$, and (2) $||g(0,X)|| \le L_1$ for all $X \in \mathcal{X}$.*

Assumption 1 implies that $g(\theta,X)$ is L_1-Lipschitz w.r.t θ uniformly in X. The above assumption immediately implies that the growth of $||g||$ and $||\bar{g}||$ will be at most linear in θ, i.e., $||g(\theta,X)|| \le L_1(||\theta-\theta^*||+1)$ and $||\bar{g}(\theta)|| \le L_1(||\theta-\theta^*||+1)$.

Assumption 2 *There exists $\mu > 0$ such that $\langle \theta-\theta', \bar{g}(\theta) - \bar{g}(\theta')\rangle \le -\mu||\theta-\theta'||^2, \forall \theta,\theta' \in \mathbb{R}^d$. Consequently, the target equation $\bar{g}(\theta) = 0$ has a unique solution θ^*.*

Denote by $\mathcal{F}_k$ the filtration generated by $\{X_{t+1},\theta_t,\xi_{t+1}\}_{t=0}^{k-1} \cup \{X_{k+1},\theta_k\}$.

Assumption 3 *Let $p \in \mathbb{Z}_+$ be given. The noise sequence $(\xi_k)_{k\ge1}$ is a collection of i.i.d random fields satisfying the following conditions with $L_{2,p} > 0$:*

$$\mathbb{E}[\xi_{k+1}(\theta)|\mathcal{F}_k] = 0 \quad \textit{and} \quad \mathbb{E}^{1/(2p)}\left[||\xi_1(\theta)^{2p}\right] \le L_{2,p}\left(||\theta-\theta^*||+1\right), \quad \forall \theta \in \mathbb{R}^d.$$

Define $C(\theta) = \mathbb{E}\left[\xi_1(\theta)^{\otimes 2}\right]$ and assume that $C(\theta)$ is at least twice differentiable. There also exists $M_\epsilon, k_\epsilon \ge 0$ such that for $\theta \in \mathbb{R}^d$, we have $\max_{i=1,2} ||C^{(i)}(\theta)|| \le M_\epsilon\{1 + ||\theta-\theta^*||^{k_\epsilon}\}$. In the sequel, we set $L := L_1 + L_2$, and without loss of generality, we assume $L \ge 1$.

Assumption 4 *There exists a Borel measurable function $\hat{g} : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^d$ where for each $\theta \in \mathbb{R}^d, X \in \mathcal{X}$,*

$$\hat{g}(\theta,X) - P_\theta \hat{g}(\theta,X) = g(\theta,X) - \bar{g}(\theta). \tag{2}$$

Assumption 5 *There exists $L_{PH}^{(0)} < \infty$ and $L_{PH}^{(1)} < \infty$ such that, for all $\theta \in \mathbb{R}^d$ and $X \in \mathcal{X}$, one has $||\hat{g}(\theta,X)|| \le L_{PH}^{(0)}$, $||P_\theta \hat{g}(\theta,X)|| \le L_{PH}^{(0)}$. Moreover, for $(\theta,\theta') \in \mathcal{H}^2$,*

$$\sup_{X\in\mathcal{X}} ||P_\theta \hat{g}(\theta,X) - P_{\theta'}\hat{g}(\theta',X)|| \le L_{PH}^{(1)}||\theta-\theta'||. \tag{3}$$

Assumption 6 *For any $\theta,\theta' \in \mathbb{R}^d$, we have* $\sup_{X\in\mathcal{X}} ||P_\theta(X,.) - P_{\theta'}(X,.)||_{TV} \le L_P||\theta-\theta'||$.

Assumption 7 *For any $\theta,\theta' \in \mathbb{R}^d$, we have* $\sup_{X\in\mathcal{X}} ||g(\theta,X) - g(\theta',X)|| \le L_H||\theta-\theta'||$.

Assumption 8 *There exists* $\rho < 1$, $K_P < \infty$ *such that*

$$\sup_{\theta \in \mathbb{R}^d, X \in \mathscr{X}} ||P_\theta^n(X,.) - \pi_\theta(.)||_{TV} \leq \rho^n K_P, \tag{4}$$

Lemma 1 *Assume that assumptions 6-8 hold. Then, for any* $\theta \in \mathbb{R}^d$ *and* $X \in \mathscr{X}$,

$$||\hat{g}(\theta, X)|| \leq \frac{\sigma K_P}{1-\rho}, \tag{5}$$

$$||P_\theta \hat{g}(\theta, X)|| \leq \frac{\sigma \rho K_P}{1-\rho}. \tag{6}$$

Moreover, for any $\theta, \theta' \in \mathbb{R}^d$ *and* $X \in \mathscr{X}$,

$$||P_\theta \hat{g}(\theta, X) - P_{\theta'} \hat{g}\left(\theta', X\right)|| \leq L_{PH}^{(1)} ||\theta - \theta'||, \tag{7}$$

where

$$L_{PH}^{(1)} = \frac{K_P^2 \sigma L_P}{(1-\rho)^2}(2 + K_P) + \frac{K_P}{1-\rho} L_H. \tag{8}$$

Proof of this lemma can be found in [1], Lemma 7.

2 Error Bound

2.1 Base Case

For the base case analysis, we can write:

$$\begin{aligned}
&\mathbb{E}\left[||\theta_{k+1} - \theta^*||^2\right] - \mathbb{E}\left[||\theta_k - \theta^*||^2\right] = \\
&2\alpha \mathbb{E}\left[\left\langle \theta_k - \theta^*, g(\theta_k, X_{k+1})\right\rangle\right] + \alpha^2 \mathbb{E}\left[||g(\theta_k, X_{k+1})||^2\right] + \alpha^2 \mathbb{E}\left[||\xi_{k+1}(\theta_k)||^2\right] = \\
&2\alpha \mathbb{E}\left[\left\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k)\right\rangle\right] + 2\alpha \mathbb{E}\left[\left\langle \theta_k - \theta^*, \bar{g}(\theta_k)\right\rangle\right] + \alpha^2 \mathbb{E}\left[||g(\theta_k, X_{k+1})\right] + \alpha^2 \mathbb{E}\left[||\xi_{k+1}(\theta_k)||^2\right].
\end{aligned} \tag{9}$$

It is easy to see that under Strong Monotonicity assumption, we have

$$\left\langle \theta_k - \theta^*, \bar{g}(\theta_k)\right\rangle = \left\langle \theta_k - \theta^*, \bar{g}(\theta_k) + \bar{g}\left(\theta^*\right)\right\rangle \leq -\mu ||\theta_k - \theta^*||^2. \tag{10}$$

Additionally, under Assumption 1 and 3, we have the following upper bound

$$\begin{aligned}
&\alpha^2\left(\mathbb{E}\left[||g(\theta_k, X_{k+1})||^2\right] + \mathbb{E}\left[||\xi_{k+1}(\theta_k)||^2\right]\right) \\
&\leq \alpha^2\left(L_1^2 \mathbb{E}\left[\left(||\theta_k - \theta^*|| + 1\right)^2\right] + L_2^2 \mathbb{E}\left[\left(||\theta_k - \theta^*|| + 1\right)^2\right]\right) \\
&\leq 2\alpha^2 L^2\left(\mathbb{E}\left[||\theta_k - \theta^*||^2\right] + 1\right).
\end{aligned} \tag{11}$$

Therefore, we have

$$\mathbb{E}\left[||\theta_{k+1} - \theta^*||^2\right] \leq \left(1 - 2\alpha\left(\alpha L^2 + \mu\right)\right) \mathbb{E}\left[||\theta_k - \theta^*||^2\right] + 2\alpha^2 L^2 + 2\alpha \mathbb{E}\left[\left\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k)\right\rangle\right] \tag{12}$$

Solving this recursion gives us the following inequality:

$$\begin{aligned}\mathbb{E}\left[||\theta_{k+1}-\theta^*||^2\right] \le &\left(1-2\alpha\left(\alpha L^2+\mu\right)\right)^{k+1}\mathbb{E}\left[||\theta_0-\theta^*||^2\right]\\ &+\sum_{t=0}^{k}\left(1-2\alpha\left(\alpha L^2+\mu\right)\right)^t 2\alpha^2L^2\\ &+\sum_{t=0}^{k}2\alpha\left(1-2\alpha\left(\alpha L^2+\mu\right)\right)^t\mathbb{E}\left[\left\langle\theta_t-\theta^*, g\left(\theta_t,X_{t+1}\right)-\bar{g}\left(\theta_t\right)\right\rangle\right].\end{aligned}\tag{13}$$

For notational simplicity we define $\gamma_t := 2\alpha\left(1-2\alpha\left(\alpha L^2+\mu\right)\right)^t$.

The second term above is just a geometric series which is equal to $2\alpha^2L^2\left(\alpha L^2+\mu\right)^k$.

Now, we can upper bound the third summand using below decomposition:

$$\mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*, g(\theta_t,X_{t+1})-\bar{g}(\theta_t)\right\rangle\right]=\mathbb{E}\left[A_1+A_2+A_3+A_4+A_5\right]\tag{14}$$

with

$$\begin{aligned}A_1 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, \hat{g}\left(\theta_t,X_{t+1}\right)-P_{\theta_t}\hat{g}\left(\theta_t,X_t\right)\right\rangle,\\ A_2 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, P_{\theta_t}\hat{g}\left(\theta_t,X_t\right)-P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1},X_t\right)\right\rangle,\\ A_3 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta_{t-1}, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1},X_t\right)\right\rangle,\\ A_4 &:= \sum_{t=1}^{k}\left(\gamma_t-\gamma_{t-1}\right)\left\langle\theta_{t-1}-\theta^*, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}-\theta^*,X_t\right)\right\rangle,\\ A_5 &:= \gamma_0\left\langle\theta_0-\theta^*, \hat{g}\left(\theta_0,X_0\right)\right\rangle+\gamma_k\left\langle\theta_t-\theta^*, P_{\theta_t}\hat{g}\left(\theta_t,X_{t+1}\right)\right\rangle\end{aligned}$$

For A_1, we note that $\hat{g}\left(\theta_t,X_{t+1}\right)-P_{\theta_t}\hat{g}\left(\theta_t,X_t\right)$ is a martingale difference sequence [cf. ?] and therefore we have $\mathbb{E}[A_1]=0$ by taking the total expectation.

For A_2, applying Cauchy-Schwarz inequality and **??**, we have

$$\begin{aligned}A_2 &\le \sum_{t=1}^{k}L_{PH}^{(1)}\gamma_t||\theta_t-\theta^*||\,||\theta_t-\theta_{t-1}||\\ &= \sum_{t=1}^{k}\alpha L_{PH}^{(1)}\gamma_t||\theta_t-\theta^*||\,||g(\theta_t,X_{t+1})+\xi_{t+1}(\theta_t)||\\ &\le \sum_{t=1}^{k}L_{PH}^{(1)}\gamma_t||\theta_t-\theta^*||\left(\alpha L_1\left(||\theta_t-\theta^*||+1\right)+\alpha L_2\left(||\theta_t-\theta^*||+1\right)\right)\\ &\le \sum_{t=1}^{k}\frac{L_{PH}^{(1)}\gamma_t}{2}(1+\alpha L)\left(1+3||\theta_t-\theta^*||^2\right)\end{aligned}\tag{15}$$

where the third line follows from the Lipschitzness condition and the assumption of

$$\mathbb{E}^{1/2}\left[||\xi_{t+1}\left(\theta_t\right)||^2|\mathscr{F}_t\right]\le L_2\left(||\theta_t||+1\right)$$

also, last line follows from the identity $u\le\frac{1}{2}(1+u^2)$.

For A_3, we obtain

$$\begin{aligned}
A_3 &\le \sum_{t=1}^{k} \gamma_t ||\theta_t - \theta_{t-1}|| \, ||P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t)|| \\
&\le \sum_{t=1}^{k} \alpha L_{PH}^{(0)} \gamma_t ||g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t)|| \\
&\le \sum_{t=1}^{k} L_{PH}^{(0)} \gamma_t \left(\alpha L_1 \left(||\theta_t - \theta^*|| + 1\right) + \alpha L_2 \left(||\theta_t - \theta^*|| + 1\right)\right) \\
&\le \sum_{t=1}^{k} L_{PH}^{(0)} \gamma_t (1 + \alpha L) ||\theta_t - \theta^*||
\end{aligned} \tag{16}$$

where second line follows from **??** and third line is similarly done to the previous part, using Lipschitzness condition and noise assumption.

For A_4, we have

$$\begin{aligned}
A_4 &\le \sum_{t=1}^{k} |\gamma_t - \gamma_{t-1}| \, ||\theta_{t-1} - \theta^*|| \, ||P_{\theta_{t-1} \hat{g}(\theta_{t-1}, X_t)}|| \\
&\le \sum_{t=1}^{k} L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \, ||\theta_{t-1} - \theta^*||
\end{aligned} \tag{17}$$

Finally, for A_5, we obtain

$$A_5 \le L_{PH}^{(0)} \left(\gamma_0 ||\theta_0 - \theta^*|| + \gamma_k ||\theta_k - \theta^*||\right) \tag{18}$$

which follows from Cacuhy-Scwarz inequality and **??**.

Combining the above terms and taking expectations, gives us:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{k} \gamma_t \langle \theta_t - \theta^*, g\left(\theta_t, X_{t+1} - \bar{g}(\theta_t)\right)\rangle\right] \le{}& \sum_{t=1}^{k} \frac{L_{PH}^{(1)} \gamma_t}{2} (1 + \alpha L) \left(1 + 3\mathbb{E}\left[||\theta_t - \theta^*||^2\right]\right) + \sum_{t=1}^{k} L_{PH}^{(0)} \gamma_t (1 + \alpha L) \mathbb{E}\left[||\theta_t - \theta^*||\right] + \\
& \sum_{t=0}^{k-1} L_{PH}^{(0)} |\gamma_{t+1} - \gamma_t| \, \mathbb{E}\left[||\theta_t - \theta^*||\right] + L_{PH}^{(0)} \left(\gamma_0 \mathbb{E}\left[||\theta_0 - \theta^*||\right] + \gamma_k \mathbb{E}\left[||\theta_k - \theta^*||\right]\right)
\end{aligned} \tag{19}$$

now it should be noticed that as long as we have $\alpha \le \frac{\sqrt{2\mu^2 + 4L^2} - \mu}{2L^2}$, we have $\gamma_{t+1} \le \gamma_t$. Thus, we can simplify the above upper bound and write it this way:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{k} \gamma_t \langle \theta_t - \theta^*, g\left(\theta_t, X_{t+1} - \bar{g}(\theta_t)\right)\rangle\right] \le{}& \sum_{t=1}^{k} \frac{L_{PH}^{(1)} \gamma_t}{2} (1 + \alpha L) \left(1 + 3\mathbb{E}\left[||\theta_t - \theta^*||^2\right]\right) + \\
& \sum_{t=1}^{k-1} L_{PH}^{(0)} \left((2 + \alpha L)\gamma_t - \gamma_{t+1}\right) \mathbb{E}\left[||\theta_t - \theta^*||\right] + \\
& L_{PH}^{(0)} \left((2\gamma_0 - \gamma_1) \, \mathbb{E}\left[||\theta_0 - \theta^*||\right] + \gamma_k \mathbb{E}\left[||\theta_k - \theta^*||\right]\right)
\end{aligned} \tag{20}$$

Hence, using the derived upper bounds from the above terms, we have:

$$\begin{aligned}
\mathbb{E}\left[||\theta_{k+1} - \theta^*||^2\right] \le{}& \sum_{t=1}^{k} \frac{L_{PH}^{(1)} \gamma_t}{2} (1 + \alpha L) \left(1 + 3\mathbb{E}\left[||\theta_t - \theta^*||^2\right]\right) + \sum_{t=1}^{k-1} L_{PH}^{(0)} \left((2 + \alpha L)\gamma_t - \gamma_{t+1}\right) \mathbb{E}\left[||\theta_t - \theta^*||\right] + \\
& \left(L_{PH}^{(0)} (2\gamma_0 - \gamma_1) + \frac{\gamma_{k+1}}{2\alpha}\right) \mathbb{E}\left[||\theta_0 - \theta^*||\right] + L_{PH}^{(0)} \gamma_k \mathbb{E}\left[||\theta_k - \theta^*||\right] + 2\alpha^2 L^2 \left(\alpha L^2 + \mu\right)^k
\end{aligned} \tag{21}$$

2.2 General Case

References

[1] B. Karimi, B. Miasojedow, E. Moulines, and H.-T. Wai. Non-asymptotic analysis of biased stochastic approximation scheme. In *Conference on Learning Theory*, pages 1944–1974. PMLR, 2019.