NONLINEAR MARKOVIAN STOCHASTIC APPROXIMATION

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1 Preliminaries

Notations The Euclidean norm is denoted by $\|.\|$. The lowercase letter c and its derivatives c', c_0, etc. denote universal numerical constants, whose value may change from line to line. As we are primarily interested in dependence of α and k, we adopt the following big-O notation: $\|f\| = \mathcal{O}(h(\alpha, k))$ if it holds that $\|f\| \le s \cdot \|h(\alpha, k)\|$ for some constant $s > 0$.

We use of the following iteration scheme:

$$\theta_{t+1} = \theta_t + \alpha\left(g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t)\right)$$

where $g : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^d$ is a deterministic function, $\{\xi_k\}_{k\ge1}$ are $i.i.d$ zero-mean random fields, and $\alpha > 0$ is a constant stepsize. We shall omit the superscript $^{(\alpha)}$ in θ_k when the dependence on α is clear from the context.

In our settings, $\{X_n, n \in \mathbb{N}\}$ is a *state-dependent* (or controlled) Markov chain, *i.e.,* for any bounded measurable function $g : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^d$,

$$\mathbb{E}\left[g(\theta_n, X_{n+1}) \,|\mathcal{F}_n\right] = P_{\theta_n} g(\theta_n, X_n) = \int g(\theta_n, x) P_\theta(X_n, dx),$$

where $P_\theta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is a Markov kernel such that, for each $\theta \in \Theta$, P_θ has a unique stationary distribution π_θ. Also, $\mathcal{F}_n$ denotes the filtration generated by the random variables $(\theta_0, \{\xi_{m+1}\}_{m\le n-1}, \{X_m\}_{m\le n})$.

1.1 Assumptions

Assumption 1 *For each $X \in \mathcal{X}$, the function $g(\theta, X)$ is three times continuously differentiable in θ with uniformly bounded first to third derivatives, i.e., $\sup_{\theta\in\mathbb{R}^d} \|g^{(i)}(\theta, X)\| < \infty$ for $i = 1,2,3, X \in \mathcal{X}$. Moreover, there exists a constant $L_1 > 0$ such that (1) $\|g^{(i)}(\theta, X) - g^{(i)}(\theta', X)\| \le L_1$, for all $\theta, \theta' \in \mathbb{R}^d, i = 0,1,2$ and $X \in \mathcal{X}$, and (2) $\|g^{(i)}(0, X)\| \le L_1$ for all $X \in \mathcal{X}$ and $i = 0,1,2$.*

Assumption 1 implies that $g(\theta, X)$ is L_1-Lipschitz w.r.t θ uniformly in X. The above assumption immediately implies that the growth of $\|g\|$ and $\|\bar{g}\|$ will be at most linear in θ, i.e., $\|g(\theta, X)\| \le L_1(\|\theta - \theta^*\| + 1)$ and $\|\bar{g}(\theta)\| \le L_1(\|\theta - \theta^*\| + 1)$. Similar implications also hold for the first and second derivatives in the same manner.

Assumption 2 *There exists $\mu > 0$ such that $\langle \theta - \theta', \bar{g}(\theta) - \bar{g}(\theta')\rangle \le -\mu\|\theta - \theta'\|^2, \forall \theta, \theta' \in \mathbb{R}^d$.* Consequently, the target equation $\bar{g}(\theta) = 0$ has a unique solution θ^*.

Denote by $\mathcal{F}_k$ the filtration generated by $\{X_{t+1}, \theta_t, \xi_{t+1}\}_{t=0}^{k-1} \cup \{X_{k+1}, \theta_k\}$.

Assumption 3 *Let $p \in \mathbb{Z}_+$ be given. The noise sequence $(\xi_k)_{k\ge1}$ is a collection of i.i.d random fields satisfying the following conditions with $L_{2,p} > 0$:*

$$\mathbb{E}[\xi_{k+1}(\theta)|\mathcal{F}_k] = 0 \quad \textit{and} \quad \mathbb{E}^{1/(2p)}\left[\|\xi_1(\theta)\|^{2p}\right] \le L_{2,p}\left(\|\theta - \theta^*\| + 1\right), \quad \forall \theta \in \mathbb{R}^d.$$

Define $C(\theta) = \mathbb{E}\left[\xi_1(\theta)^{\otimes 2}\right]$ and assume that $C(\theta)$ is at least twice differentiable. There also exists $M_\epsilon, k_\epsilon \ge 0$ such that for $\theta \in \mathbb{R}^d$, we have $\max_{i=1,2} \|C^{(i)}(\theta)\| \le M_\epsilon\{1 + \|\theta - \theta^*\|^{k_\epsilon}\}$. In the sequel, we set $L := L_1 + L_2$, and without loss of generality, we assume $L \ge 2\mu$ for some technical reasons.

Assumption 4 *There exists a Borel measurable function* $\hat{g}: \mathbb{R}^d \times \mathcal{X} \rightarrow \mathbb{R}^d$ *where for each* $\theta \in \mathbb{R}^d, X \in \mathcal{X}$,

$$\hat{g}(\theta, X) - P_\theta \hat{g}(\theta, X) = g(\theta, X) - \bar{g}(\theta).$$

Assumption 5 *There exists* $L_{PH}^{(0)} < \infty$ *and* $L_{PH}^{(1)} < \infty$ *such that, for all* $\theta \in \mathbb{R}^d$ *and* $X \in \mathcal{X}$, *one has* $\|\hat{g}(\theta, X)\| \leq L_{PH}^{(0)}$, $\|P_\theta \hat{g}(\theta, X)\| \leq L_{PH}^{(0)}$. *Moreover, for* $(\theta, \theta') \in \mathcal{H}^2$,

$$\sup_{X \in \mathcal{X}} \|P_\theta \hat{g}(\theta, X) - P_{\theta'} \hat{g}(\theta', X)\| \leq L_{PH}^{(1)} \|\theta - \theta'\|.$$

Assumption 6 *For any* $\theta, \theta' \in \mathbb{R}^d$, *we have* $\sup_{X \in \mathcal{X}} \|P_\theta(X, .) - P_{\theta'}(X, .)\|_{TV} \leq L_P \|\theta - \theta'\|$.
Assumption 7 *For any* $\theta, \theta' \in \mathbb{R}^d$, *we have* $\sup_{X \in \mathcal{X}} \|g(\theta, X) - g(\theta', X)\| \leq L_H \|\theta - \theta'\|$.
Assumption 8 *There exists* $\rho < 1$, $K_P < \infty$ *such that*

$$\sup_{\theta \in \mathbb{R}^d, X \in \mathcal{X}} \|P_\theta^n(X, .) - \pi_\theta(.)\|_{TV} \leq \rho^n K_P,$$

Assumption 9 *For any* $\theta, \theta' \in \mathbb{R}^d$, *we have* $\sup_{X \in \mathcal{X}} \|\pi_\theta(X) - \pi_{\theta'}(X)\| \leq L_S \|\theta - \theta'\|$.
Lemma 1 *Assume that assumptions 6-8 hold. Then, for any* $\theta \in \mathbb{R}^d$ *and* $X \in \mathcal{X}$,

$$\|\hat{g}(\theta, X)\| \leq \frac{\sigma K_P}{1 - \rho},$$

$$\|P_\theta \hat{g}(\theta, X)\| \leq \frac{\sigma \rho K_P}{1 - \rho}.$$

Moreover, for any $\theta, \theta' \in \mathbb{R}^d$ *and* $X \in \mathcal{X}$,

$$\|P_\theta \hat{g}(\theta, X) - P_{\theta'} \hat{g}(\theta', X)\| \leq \|\theta - \theta'\|,$$

where

$$L_{PH}^{(1)} = \frac{K_P^2 \sigma L_P}{(1 - \rho)^2}(2 + K_P) + \frac{K_P}{1 - \rho} L_H.$$

Proof of this lemma can be found in [2], Lemma 7.

2 Error Bound

2.1 Base Case

We first prove the following lemma because we are going to use that calculation in many different parts of the proof:

Lemma2. *Using Assumptions 1, 3, 5, and for sufficiently small* α *and* $t \geq 1$, *we have*

$$\mathbb{E}[\|\theta_t - \theta_{t-1}\|] \leq \alpha L \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|\right] + 1\right).$$

Proof. We have

$$\begin{aligned}\mathbb{E}[\|\theta_t-\theta_{t-1}\|] &\le \alpha\mathbb{E}\left[\|g(\theta_{t-1},X_t)+\xi_t(\theta_{t-1})\|\right]\\ &\le \alpha L_1\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)+\alpha L_2\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)\\ &\le \alpha L\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)\end{aligned}$$

where the first line follows from ??, second line from the Lipschitzness condition and the assumption of

$$\mathbb{E}^{1/2}\left[\|\xi_t(\theta_{t-1})\|^2|\mathscr{F}_{t-1}\right]\le L_2\left(\mathbb{E}[\|\theta_{t-1}\|]+1\right),$$

and the third line from ??.

For the base case analysis, we can write:

$$\begin{aligned}&\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right]-\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]=\\ &2\alpha\mathbb{E}\left[\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})\rangle\right]+\alpha^2\mathbb{E}\left[\|g(\theta_k,X_{k+1})\|^2\right]+\alpha^2\mathbb{E}\left[\|\xi_{k+1}(\theta_k)\|^2\right]=\\ &2\alpha\mathbb{E}\left[\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})-\bar{g}(\theta_k)\rangle\right]+2\alpha\mathbb{E}\left[\langle\theta_k-\theta^*,\bar{g}(\theta_k)\rangle\right]+\alpha^2\mathbb{E}\left[\|g(\theta_k,X_{k+1})\|\right]+\alpha^2\mathbb{E}\left[\|\xi_{k+1}(\theta_k)\|^2\right].\end{aligned}$$

It is easy to see that under Strong Monotonicity assumption, we have

$$\langle\theta_k-\theta^*,\bar{g}(\theta_k)\rangle=\langle\theta_k-\theta^*,\bar{g}(\theta_k)-\bar{g}(\theta^*)\rangle\le-\mu\|\theta_k-\theta^*\|^2.$$

Additionally, under Assumption 1 and 3, we have the following upper bound

$$\begin{aligned}&\alpha^2\left(\mathbb{E}\left[\|g(\theta_k,X_{k+1})\|^2\right]+\mathbb{E}\left[\|\xi_{k+1}(\theta_k)\|^2\right]\right)\\ &\le\alpha^2\left(L_1^2\mathbb{E}\left[\left(\|\theta_k-\theta^*\|+1\right)^2\right]+L_2^2\mathbb{E}\left[\left(\|\theta_k-\theta^*\|+1\right)^2\right]\right)\\ &\le 2\alpha^2L^2\left(\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]+1\right).\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right]\le\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]+2\alpha^2L^2+2\alpha\mathbb{E}\left[\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})-\bar{g}(\theta_k)\rangle\right]$$

Solving this recursion gives us the following inequality:

$$\begin{aligned}\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right]&\le\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k+1}\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]\\ &+\sum_{t=0}^{k}\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t 2\alpha^2L^2\\ &+\sum_{t=0}^{k}2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k-t}\mathbb{E}\left[\langle\theta_t-\theta^*,g(\theta_t,X_{t+1})-\bar{g}(\theta_t)\rangle\right].\end{aligned}$$

For notational simplicity we define $\gamma_t:=2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k-t}$ for $0\le t\le k$.

The second term above is just a geometric series. According to Lemma 12 of [1], this equals to $\frac{\alpha L^2\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k+1}\right]}{-\alpha L^2+\mu}$.

Now, we can upper bound the third summand using below decomposition:

$$\mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*,g(\theta_t,X_{t+1})-\bar{g}(\theta_t)\right\rangle\right]=A_1+A_2+A_3+A_4+A_5$$

with

$$\begin{aligned}
A_1 :=& \mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*,\hat{g}(\theta_t,X_{t+1})-P_{\theta_t}\hat{g}(\theta_t,X_t)\right\rangle\right],\\
A_2 :=& \mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*,P_{\theta_t}\hat{g}(\theta_t,X_t)-P_{\theta_{t-1}}\hat{g}(\theta_{t-1},X_t)\right\rangle\right],\\
A_3 :=& \mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta_{t-1},P_{\theta_{t-1}}\hat{g}(\theta_{t-1},X_t)\right\rangle\right],\\
A_4 :=& \mathbb{E}\left[\sum_{t=1}^{k}\left(\gamma_t-\gamma_{t-1}\right)\left\langle\theta_{t-1}-\theta^*,P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}-\theta^*,X_t\right)\right\rangle\right],\\
A_5 :=& \mathbb{E}\left[\gamma_0\left\langle\theta_0-\theta^*,\hat{g}(\theta_0,X_0)\right\rangle\right]+\mathbb{E}\left[\gamma_k\left\langle\theta_k-\theta^*,P_{\theta_k}\hat{g}(\theta_k,X_{k+1})\right\rangle\right]
\end{aligned}$$

For A_1, we note that $\hat{g}(\theta_t,X_{t+1})-P_{\theta_t}\hat{g}(\theta_t,X_t)$ is a martingale difference sequence [cf. ?] and therefore we have $A_1=0$ by taking the total expectation.

For A_2, applying Cauchy-Schwarz inequality and **??**, we have

$$\begin{aligned}
A_2 &\leq \sum_{t=1}^{k}L_{PH}^{(1)}\gamma_t\mathbb{E}\left[\|\theta_t-\theta^*\|\,\|\theta_t-\theta_{t-1}\|\right]\\
&=\sum_{t=1}^{k}\alpha L_{PH}^{(1)}\gamma_t\mathbb{E}\left[\|\theta_t-\theta^*\|\,\|g(\theta_{t-1},X_t)+\xi_t(\theta_{t-1})\|\right]\\
&\leq\sum_{t=1}^{k}\alpha L_{PH}^{(1)}\gamma_t\mathbb{E}\left[\left(\|\theta_t-\theta_{t-1}\|+\|\theta_{t-1}-\theta^*\|\right)\left(\|g(\theta_{t-1},X_t)\|+\|\xi_t(\theta_{t-1})\|\right)\right]\\
&\leq\sum_{t=1}^{k}\alpha L_{PH}^{(1)}\gamma_t\Big(L_1\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^2\right]+\mathbb{E}\left[\|\theta_t-\theta_{t-1}\|\,\|\theta_{t-1}-\theta^*\|\right]+\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+\mathbb{E}[\|\theta_t-\theta_{t-1}\|]\right)\\
&+\mathbb{E}[\|\theta_t-\theta_{t-1}\|\,\|\xi_t(\theta_{t-1})\|]+\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\,\|\xi_t(\theta_{t-1})\|\right]\Big)
\end{aligned}$$

where the second line follows from ?? and the third line follows from the triangle inequality. Now we upper the compound terms in the last line's parentheses:

$$\begin{aligned}
\mathbb{E}\left[\|\theta_t-\theta_{t-1}\|\,\|\theta_{t-1}-\theta^*\|\right]&=\mathbb{E}\left[\mathbb{E}\left[\|\theta_t-\theta_{t-1}\|\,\|\theta_{t-1}-\theta^*\||\mathscr{F}_{t-1}\right]\right]\\
&\leq\mathbb{E}\left[\alpha L\left(\|\theta_{t-1}-\theta^*\|+1\right)\|\theta_{t-1}-\theta^*\|\right]\\
&\leq\frac{\alpha L\left(3\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^2\right]+1\right)}{2}
\end{aligned}$$

where in the second line we used Lemma 2 and in the last line we used $u\leq\frac{u^2+1}{2}$.

$$\begin{aligned}
\mathbb{E}[\|\theta_t-\theta_{t-1}\|\,\|\xi_t(\theta_{t-1})\|]&\leq\mathbb{E}\left[\alpha\left(\|g(\theta_{t-1},X_t)\|+\|\xi_t(\theta_{t-1})\|\right)\|\xi_t(\theta_{t-1})\|\right]\\
&\leq\mathbb{E}\left[\alpha\|\xi_t(\theta_{t-1})\|^2+\alpha L_1\left(\|\theta_{t-1}-\theta^*\|+1\right)\|\xi_t(\theta_{t-1})\|\right]\\
&\leq L\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)
\end{aligned}$$

where the first inequality follows from ??, second line from Lemma 2 and in the last line we used boundedness property of the noise and sufficiently small α.

$$\begin{aligned}\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\,\|\xi_t(\theta_{t-1})\|\right] &= \mathbb{E}\left[\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\,\|\xi_t(\theta_{t-1})\|\,|\mathscr{F}_{t-1}\right]\right]\\ &\le \mathbb{E}\left[L_2\|\theta_{t-1}-\theta^*\|\left(\|\theta_{t-1}-\theta^*\|+1\right)\right]\\ &\le \frac{L_2\left(3\|\theta_{t-1}-\theta^*\|^2+1\right)}{2}\end{aligned}$$

where the second line follows from ?? and in the last line we used $u \le \frac{u^2+1}{2}$.

Summing up all these bounds, we can write for A_2:

$$\begin{aligned}A_2 &\le \sum_{t=1}^{k} \alpha L_{PH}^{(1)}\gamma_t\left(2L\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)+2L\left(3\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^2\right]+1\right)+\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^2\right]\right)\\ &\le \sum_{t=1}^{k} \alpha L L_{PH}^{(1)}\gamma_t\left(8\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^2\right]+5\right)\end{aligned}$$

which in the last line we again used $u \le \frac{u^2+1}{2}$ property.

For A_3, we obtain

$$\begin{aligned}A_3 &\le \sum_{t=1}^{k} \gamma_t\mathbb{E}\left[\|\theta_t-\theta_{t-1}\|\,\|P_{\theta_{t-1}}\hat{g}(\theta_{t-1},X_t)\|\right]\\ &\le \sum_{t=1}^{k} L_{PH}^{(0)}\gamma_t\mathbb{E}\left[\|g(\theta_{t-1},X_t)+\xi_t(\theta_{t-1})\|\right]\\ &\le \sum_{t=1}^{k} L_{PH}^{(0)}\gamma_t\left(L_1\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)+L_2\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)\right)\\ &\le \sum_{t=1}^{k} \alpha L L_{PH}^{(0)}\gamma_t\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)\end{aligned}$$

where second line follows from **??** and third line follows from ?? .

For A_4, we have

$$\begin{aligned}A_4 &\le \sum_{t=1}^{k} |\gamma_t-\gamma_{t-1}|\,\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\,\|P_{\theta_{t-1}}\hat{g}(\theta_{t-1},X_t)\|\right]\\ &\le \sum_{t=1}^{k} L_{PH}^{(0)}|\gamma_t-\gamma_{t-1}|\,\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]\end{aligned}$$

Finally, for A_5, we obtain

$$A_5 \le L_{PH}^{(0)}\left(\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]\right)$$

which follows from Cacuhy-Scwarz inequality and **??**.

Combining the above terms gives us:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, g\left(\theta_t,X_{t+1}-\bar{g}(\theta_t)\right)\right\rangle\right] \le& \sum_{t=0}^{k-1}\alpha L L_{PH}^{(1)}\gamma_{t+1}\left(5+8\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\right)+\sum_{t=0}^{k-1}\alpha L L_{PH}^{(0)}\gamma_{t+1}\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|\right]+1\right)+\\ &\sum_{t=0}^{k-1} L_{PH}^{(0)}|\gamma_t-\gamma_{t+1}|\,\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+L_{PH}^{(0)}\left(\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]\right)\end{aligned}$$

now it should be noticed that as long as the α satisfies $\alpha \le \frac{\mu}{L^2}$, we have $\gamma_t \le \gamma_{t+1}$. Thus, we can simplify the above upper bound and write it this way:

$$\mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*, g\left(\theta_t, X_{t+1}-\bar{g}\left(\theta_t\right)\right)\right\rangle\right] \leq \sum_{t=0}^{k-1}\alpha L L_{PH}^{(1)}\gamma_{t+1}\left(5+8\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\right)+$$
$$\sum_{t=0}^{k-1}L_{PH}^{(0)}\left((\alpha L+1)\gamma_{t+1}-\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+$$
$$\sum_{t=0}^{k-1}\alpha L L_{PH}^{(0)}\gamma_{t+1}+L_{PH}^{(0)}\left(\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]\right)$$

Hence, using the derived upper bounds from the above terms, we have:

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \leq \sum_{t=0}^{k-1}\alpha L L_{PH}^{(1)}\gamma_{t+1}\left(5+8\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\right)+\sum_{t=0}^{k-1}L_{PH}^{(0)}\left((\alpha L+1)\gamma_{t+1}-\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+$$
$$\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+$$
$$2\alpha^2 L L_{PH}^{(0)}\left[\frac{1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k}{1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)}\right]+\frac{\alpha L^2\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k+1}\right]}{-\alpha L^2+\mu}$$

for further notation simplicity we define $c_{1,t} := 2\alpha^2 L L_{PH}^{(0)}\left[\frac{1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}{1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)}\right]+\frac{\alpha L^2\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{t+1}\right]}{-\alpha L^2+\mu}$ for $0\leq t\leq k$. Now to write down this upper bound in a way in which it only depends on $\|\theta_0-\theta^*\|$ related terms and constants, we can write:

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \leq \sum_{t=0}^{k-1}\left[8\alpha L L_{PH}^{(1)}\gamma_{t+1}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+5\alpha L L_{PH}^{(1)}\gamma_{t+1}\right]+\sum_{t=0}^{k-1}L_{PH}^{(0)}\left((\alpha L+1)\gamma_{t+1}-\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+$$
$$\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}$$
$$=\sum_{t=0}^{k-1}8\alpha L L_{PH}^{(1)}\gamma_{t+1}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+5\alpha L L_{PH}^{(1)}\sum_{t=0}^{k-1}\gamma_{t+1}+\sum_{t=0}^{k-1}L_{PH}^{(0)}\left((\alpha L_1)\gamma_{t+1}-\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+$$
$$\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}$$
$$=\sum_{t=0}^{k-1}8\alpha L L_{PH}^{(1)}\gamma_{t+1}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\sum_{t=0}^{k-1}L_{PH}^{(0)}\left((\alpha L+1)\gamma_{t+1}-\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+$$
$$\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}+$$
$$\frac{10\alpha^2 L L_{PH}^{(1)}\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k\right]}{\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\right]}$$

where the last equality follows from the definition of γ_ts. Similarly we define $c_{2,t} := \frac{10\alpha^2 L L_{PH}^{(1)}\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t\right]}{\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\right]}$ for $0\leq t\leq k$. So we can write it as

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \leq \sum_{t=0}^{k-1}8\alpha L L_{PH}^{(1)}\gamma_{t+1}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\sum_{t=0}^{k-1}L_{PH}^{(0)}\left((\alpha L+1)\gamma_{t+1}-\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+$$
$$\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}+c_{2,k}$$

Now for the second term on RHS, we note that since $L \geq 2\mu$,

$$\left(\alpha L+1\right)\gamma_{t+1}-\gamma_t \leq 2\alpha L\gamma_{t+1}, \quad \mathbb{E}\left[\|\theta_t-\theta^*\|\right] \leq \sqrt{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]},$$

and consequently

$$\begin{aligned}
&\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^k}\sum_{t=0}^{k-1} L_{PH}^{(0)}\left((\alpha L+1)\gamma_{t+1}+\gamma_t\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right] \\
&\leq 4L_{PH}^{(0)}L\alpha^2\sum_{t=0}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\sqrt{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]} \\
&\leq 4L_{PH}^{(0)}L\alpha^2\Big(\sum_{t=0}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\Big)^{1/2}\Big(\sum_{t=1}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\Big)^{1/2} \\
&\leq 4L_{PH}^{(0)}L\alpha^2\cdot\sum_{t=0}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\frac{1}{-\alpha L^2+\mu}\cdot\frac{2\alpha LL_{PH}^{(0)}}{(1-2\alpha(-\alpha L^2+\mu))^k}.
\end{aligned}$$

We also note that

$$\frac{\gamma_k}{(1-2\alpha(-\alpha L^2+\mu))^k}\mathbb{E}\left[\|\theta_k-\theta^*\|\right] \leq \alpha\frac{\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]}{(1-2\alpha(-\alpha L^2+\mu))^k}+\frac{\alpha}{(1-2\alpha(-\alpha L^2+\mu))^k}.$$

similarly

$$\frac{\gamma_0}{(1-2\alpha(-\alpha L^2+\mu))^k}\mathbb{E}\left[\|\theta_0-\theta^*\|\right] \leq \alpha\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\alpha.$$

and we also define for $0 \leq t \leq k$

$$c_{3,t} := \frac{1}{-\alpha L^2+\mu}\frac{2\alpha LL_{PH}^{(0)}}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}+\frac{\alpha L_{PH}^{(0)}}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}+\alpha L_{PH}^{(0)}$$

to wrap up all the remainder terms.

Substituting back and rearranging with also defining $c'_{2,t} := \frac{c_{2,t}}{(1-2\alpha(-\alpha L^2+\mu))^t}$ and $c'_{1,t} := \frac{c_{1,t}}{(1-2\alpha(-\alpha L^2+\mu))^t}$, yields

$$\begin{aligned}
\frac{\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right]}{(1-2\alpha(-\alpha L^2+\mu))^k} \leq{}& \frac{\alpha}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k}\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]+\sum_{t=0}^{k-1}\frac{\alpha\left(8LL_{PH}^{(1)}+4\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{-1}LL_{PH}^{(0)}\right)}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+ \\
&\alpha\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+c'_{1,k}+c'_{2,k}+c_{3,k}.
\end{aligned}$$

for sufficiently small αs, we have

$$\alpha\left(\frac{3}{2}LL_{PH}^{(1)}+2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{-1}LL_{PH}^{(0)}\right) \leq 4\alpha\left((\alpha L+1)L_{PH}^{(0)}+2LL_{PH}^{(1)}\right)$$

using above simplification we can rewrite our upper bound as

$$\frac{\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right]}{(1-2\alpha(-\alpha L^2+\mu))^k} \leq 4\alpha\left((\alpha L+1)L_{PH}^{(0)}+2LL_{PH}^{(1)}\right)\sum_{t=0}^{k}\frac{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}+\alpha\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+c'_{1,k}+c'_{2,k}+c_{3,k}.$$

For solving the above recursion, we first define $S_t := 4\alpha\left((\alpha L+1)L_{PH}^{(0)}+2LL_{PH}^{(1)}\right)\sum_{l=0}^{t}\frac{\mathbb{E}\left[\|\theta_l-\theta^*\|^2\right]}{(1-2\alpha(-\alpha L^2+\mu))^l}$ for

$0 \le t \le k$. Also we use $C_t := c'_{1,t} + c'_{2,t} + c_{3,t}$ and for $0 \le t \le k$, defining constant terms. Now we can write

$$\frac{\mathbb{E}\left[\|\theta_{t+1}-\theta^*\|^2\right]}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t} \le S_t + \alpha \mathbb{E}\left[\|\theta_0-\theta^*\|^2\right] + C_t.$$

using this expansion, we should first notice that

$$\begin{aligned}
\frac{S_t}{S_{t-1}} &= \frac{S_{t-1} + 4\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\frac{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}}{S_{t-1}} \\
&= 1 + 4\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\frac{S_{t-1} + \alpha\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right] + C_{t-1}}{S_{t-1}} \\
&\le 1 + 8\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right).
\end{aligned}$$

Now, since we have $S_0 = 4\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]$, thus

$$S_t \le 4\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\left[1 + 8\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\right]^t \mathbb{E}\left[\|\theta_0-\theta^*\|^2\right].$$

Substituting this upper bound into previous equations we get

$$\begin{aligned}
&\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \le \\
&\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k \left[4\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\left(1 + 8\alpha\left((\alpha L+1)L_{PH}^{(0)} + 2LL_{PH}^{(1)}\right)\right)^k \mathbb{E}\left[\|\theta_0-\theta^*\|^2\right] + \alpha\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right] + C_k\right]
\end{aligned}$$

Choosing α sufficiently small, we can simplify the above inequality and write it as follows

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \le \tilde{c}_1 \cdot \left(1-2\alpha\mu\right)^{k+1}\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right] + \tilde{c}_2 \cdot 2\alpha L\left(L_{PH}^{(0)} + \alpha L_{PH}^{(1)}\right)$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are $\mathcal{O}(1)$ constants.

2.2 General Case

Similar to the previous case, we first prove a useful lemma:

Lemma3. *Using Assumptions 1, 3, 5, and $m, n \ge 1$, we have*

$$\mathbb{E}\left[\|\theta_t-\theta_{t-1}\|^m \|\theta_{t-1}-\theta^*\|^n\right] \le 2^{2m+n-2}\alpha^m L^m\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^{m+n}\right] + 1\right).$$

Also when $m = 0$, this upper bound can be written as

$$\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^n\right] \le 2^{n-1}\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^n\right] + 1\right). \tag{1}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\|\theta_t-\theta_{t-1}\|^m\|\theta_{t-1}-\theta^*\|^n\right] &\le \alpha^m\mathbb{E}\left[\|g(\theta_{t-1},X_t)+\xi_t(\theta_{t-1})\|^m\|\theta_{t-1}-\theta^*\|^n\right]\\
&\le 2^{m-1}\alpha^m\mathbb{E}\left[\left(\|g(\theta_{t-1},X_t)\|^m+\|\xi_t(\theta_{t-1})\|^m\right)\|\theta_{t-1}-\theta^*\|^n\right]\\
&= 2^{m-1}\alpha^m\mathbb{E}\left[\mathbb{E}\left[\left(\|g(\theta_{t-1},X_t)\|^m+\|\xi_t(\theta_{t-1})\|^m\right)|\mathscr{F}_{t-1}\right]\|\theta_{t-1}-\theta^*\|^n\right]\\
&\le 2^{m-1}\alpha^m\mathbb{E}\left[\left(L_1^m\mathbb{E}\left[\left(\|\theta_{t-1}-\theta^*\|+1\right)^m\right]+L_2^m\mathbb{E}\left[\left(\|\theta_{t-1}-\theta^*\|+1\right)^m\right]\right)\|\theta_{t-1}-\theta^*\|^n\right]\\
&\le 2^{m-1}\alpha^m L^m\mathbb{E}\left[\left(\|\theta_{t-1}-\theta^*\|+1\right)^{m+n}\right]\\
&\le 2^{2m+n-2}\alpha^m L^m\left(\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^{m+n}\right]+1\right)
\end{aligned}$$

where the second and the last line follows from the mclaurin's inequality. Also the fourth line follows from ??. Proof for the other case is a trivial consequence of these calculations.

Now to do the proof in this case, we assume that the moment bound in [??] has been proven for $k\le n-1$, we now proceed to show that the desired moment convergence holds for n with $2\le n\le p$.

We start with the following decomposition of $\|\theta_{k+1}-\theta^*\|^{2n}$

$$\begin{aligned}
\|\theta_{k+1}-\theta^*\|^{2n} &= \left(\|\theta_k-\theta^*\|^2+2\alpha\left\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})+\xi_{k+1}(\theta_k)\right\rangle+\alpha^2\|g(\theta_x,X_{k+1})+\xi_{k+1}(\theta_k)\|^2\right)^n\\
&= \sum_{\substack{i,j,l\\ i+j+l=n}}\binom{n}{i,j,l}\|\theta_k-\theta^*\|^{2i}\left(2\alpha\left\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})+\xi_{k+1}(\theta_k)\right\rangle\right)^j\left(\alpha\|g(\theta_k,X_{k+1})+\xi_{k+1}(\theta_k)\|\right)^{2l}
\end{aligned}$$

We note the following cases.

1. $i=n,\ j=l=0$. In this case, the summand is simply $\|\theta_k-\theta^*\|^{2i}$.
2. When $i=n-1, j=1$ and $l=0$. In this case, the summand is of order α, i.e.,

   $$\alpha\cdot 2n\left\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})+\xi_{k+1}(\theta_k)\right\rangle^j\|\theta_k-\theta^*\|^{2(n-1)}.$$

   We can further decompose it as

   $$\begin{aligned}
   &2n\alpha\left\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})+\xi_{k+1}(\theta_k)\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}\\
   &=\underbrace{2n\alpha\left\langle\theta_k-\theta^*,g(\theta_k,X_{k+1})-\bar{g}(\theta_k)+\xi_{k+1}(\theta_k)\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}}_{T_1}+\underbrace{2n\alpha\left\langle\theta_k-\theta^*,\bar{g}(\theta_k)\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}}_{T_2}.
   \end{aligned}$$

   Note that, when (X_k) is i.i.d or from a martingale noise, we have

   $$\mathbb{E}\left[T_1|\theta_k\right]=0$$

   However, when (X_k) is Markovian, the above inequality does not hold and T_1 requires careful analysis.
   Nonetheless, under the strong monotonicity assumption, we have

   $$T_2\le -2n\alpha\mu\|\theta_k-\theta^*\|^{2n}.$$

3. For the remaining terms, we see that they are of higher orders of α. Therefore, when α is selected sufficiently small, these terms do not raise concern.

Therefore, to prove the desired moment bound, we spend the remaining section analyzing T_1. Immediately, we note that

$$\begin{aligned}\mathbb{E}[T_1] &= \mathbb{E}\left[2n\alpha\left\langle\theta_k-\theta^*, g(\theta_k,X_{k+1})-\bar{g}(\theta_k)+\mathbb{E}[\xi_{k+1}(\theta_k)|\theta_k]\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}\right]\\ &= 2n\alpha\mathbb{E}\left[\underbrace{\left\langle\theta_k-\theta^*, g(\theta_k,X_{k+1})-\bar{g}(\theta_k)\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}}_{T_1'}\right].\end{aligned}$$

Subsequently, we focus on analyzing T_1'; but before that, we write the general recursion of the error bound. First, we define $T_{1,t}' \coloneqq \left\langle\theta_t-\theta^*, g(\theta_t,X_{t+1})-\bar{g}(\theta_t)\|\theta_t-\theta^*\|^{2(n-1)}\right\rangle$ to make T_1' dependent on the iteration index. Now, following the above decomposition and taking the expectations, we have:

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^{2n}\right] \le \mathbb{E}\left[\|\theta_k-\theta^*\|^{2n}\right]+2n\alpha\mathbb{E}\left[T_{1,k}'\right]-2n\alpha\mu\mathbb{E}\left[\|\theta_k-\theta^*\|^{2n}\right]+o(\alpha) = \left(1-2n\alpha\mu\right)\mathbb{E}\left[\|\theta_k-\theta^*\|^{2n}\right]+2n\alpha\mathbb{E}\left[T_{1,k}'\right]+o(\alpha)$$

similarly to the previous case we define $\gamma_t \coloneqq 2n\alpha\left(1-2n\alpha\mu\right)^{k-t}$ for $0\le t\le k$. Solving the above recursion will give us

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^{2n}\right] \le \sum_{t=0}^{k}\gamma_t\mathbb{E}\left[T_{1,t}'\right]+\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right]+o(\alpha)$$

We have to upper bound the first term in the RHS above. For this purpose, we use a similar decomposition to our base case analysis:

$$\sum_{t=0}^{k}\gamma_t\mathbb{E}\left[T_{1,t}'\right] = \mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*, g(\theta_t,X_{t+1})-\bar{g}(\theta_t)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)}\right] = A_1+A_2+A_3+A_4+A_5$$

with

$$\begin{aligned}
A_1 &\coloneqq \mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, \hat{g}(\theta_t,X_{t+1})-P_{\theta_t}\hat{g}(\theta_t,X_t)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)}\right],\\
A_2 &\coloneqq \mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, P_{\theta_t}\hat{g}(\theta_t,X_t)-P_{\theta_{t-1}}\hat{g}(\theta_{t-1},X_t)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)}\right],\\
A_3 &\coloneqq \mathbb{E}\left[\sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta_{t-1}, P_{\theta_{t-1}}\hat{g}(\theta_{t-1}X_t)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)}\right],\\
A_4 &\coloneqq \mathbb{E}\left[\sum_{t=1}^{k}\left(\gamma_t-\gamma_{t-1}\right)\left\langle\theta_{t-1}-\theta^*, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}-\theta^*,X_t\right)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)}\right],\\
A_5 &\coloneqq \mathbb{E}\left[\sum_{t=1}^{k}\gamma_{t-1}\left\langle\theta_{t-1}-\theta^*, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}-\theta^*,X_t\right)\right\rangle\left(\|\theta_t-\theta^*\|^{2(n-1)}-\|\theta_{t-1}-\theta^*\|^{2(n-1)}\right)\right],\\
A_6 &\coloneqq \mathbb{E}\left[\gamma_0\left\langle\theta_0-\theta^*, \hat{g}(\theta_0,X_0)\right\rangle\|\theta_0-\theta^*\|^{2(n-1)}\right]+\mathbb{E}\left[\gamma_k\left\langle\theta_k-\theta^*, P_{\theta_k}\hat{g}(\theta_k,X_{k+1})\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}\right].
\end{aligned}$$

For A_1, we note that $\hat{g}(\theta_t,X_{t+1})-P_{\theta_t}\hat{g}(\theta_t,X_t)$ is a martingale difference sequence [cf. ?] and therefore we have $A_1=0$ by taking the total expectation.

For A_2, applying Cauchy-Schwarz inequality and ??, we have

$$
\begin{aligned}
A_2 &\le \sum_{t=1}^{k} L_{PH}^{(1)} \gamma_t \mathbb{E}\left[\|\theta_t - \theta_{t-1}\| \, \|\theta_t - \theta^*\|^{2n-1}\right] \\
&\le \sum_{t=1}^{k} L_{PH}^{(1)} \gamma_t \mathbb{E}\left[\|\theta_t - \theta_{t-1}\| \left(\|\theta_t - \theta_{t-1}\| + \|\theta_{t-1} - \theta^*\|\right)^{2n-1}\right] \\
&\le \sum_{t=1}^{k} 2^{2n-2} L_{PH}^{(1)} \gamma_t \mathbb{E}\left[\|\theta_t - \theta_{t-1}\|^{2n} + \|\theta_t - \theta_{t-1}\| \, \|\theta_{t-1} - \theta^*\|^{2n-1}\right] \\
&\le \sum_{t=1}^{k} 2^{2n-2} L_{PH}^{(1)} \gamma_t \left(2^{4n} \alpha^{2n} L^{2n} \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n}\right] + 1\right) + 2^{2n-1} \alpha L \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n}\right] + 1\right)\right) \\
&\le \sum_{t=1}^{k} 2^{4n-2} \alpha L L_{PH}^{(1)} \gamma_t \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n}\right] + 1\right)
\end{aligned}
$$

where the second line follows from triangle inequality, third line from mclaurin's inequality, fourth line from Lemma 3, and the last line for $\alpha < \frac{1}{4L}$.

For A_3, we obtain

$$
\begin{aligned}
A_3 &\le \sum_{t=1}^{k} \gamma_t \mathbb{E}\left[\|\theta_t - \theta_{t-1}\| \, \|P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t)\| \, \|\theta_t - \theta^*\|^{2(n-1)}\right] \\
&\le \sum_{t=1}^{k} L_{PH}^{(0)} \gamma_t \mathbb{E}\left[\|\theta_t - \theta_{t-1}\| \left(\|\theta_t - \theta_{t-1}\| + \|\theta_{t-1} - \theta^*\|\right)^{2(n-1)}\right] \\
&\le \sum_{t=1}^{k} 2^{2n-3} L_{PH}^{(0)} \gamma_t \mathbb{E}\left[\|\theta_t - \theta_{t-1}\|^{2n-1} + \|\theta_t - \theta_{t-1}\| \, \|\theta_{t-1} - \theta^*\|^{2(n-1)}\right] \\
&\le \sum_{t=1}^{k} 2^{2n-3} L_{PH}^{(0)} \gamma_t \left(2^{4n-1} \alpha^{2n-1} L^{2n-1} \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right) + 2^{2n-2} \alpha L \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right)\right) \\
&\le \sum_{t=1}^{k} 2^{4n-4} \alpha L L_{PH}^{(0)} \gamma_t \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right)
\end{aligned}
$$

where second line follows from triangle inequality, third line from mclaurin's inequality, fourth line from Lemma 3, and the last line for $\alpha < \frac{1}{4L}$.

For A_4, we have

$$
\begin{aligned}
A_4 &\le \sum_{t=1}^{k} |\gamma_t - \gamma_{t-1}| \, \mathbb{E}\left[\|\theta_{t-1} - \theta^*\| \|P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t)\| \, \|\theta_t - \theta^*\|^{2(n-1)}\right] \\
&\le \sum_{t=1}^{k} L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \, \mathbb{E}\left[\|\theta_{t-1} - \theta^*\| \, \|\theta_t - \theta^*\|^{2(n-1)}\right] \\
&\le \sum_{t=1}^{k} L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \mathbb{E}\left[\|\theta_{t-1} - \theta^*\| \left(\|\theta_t - \theta_{t-1}\| + \|\theta_{t-1} - \theta^*\|\right)^{2(n-1)}\right] \\
&\le \sum_{t=1}^{k} 2^{2n-3} L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \mathbb{E}\left[\|\theta_t - \theta_{t-1}\|^{2(n-1)} \, \|\theta_{t-1} - \theta^*\| \, + \|\theta_{t-1} - \theta^*\|^{2n-1}\right] \\
&\le \sum_{t=1}^{k} 2^{2n-3} L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \left(2^{4n-5} \alpha^{2n-2} L^{2n-2} \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right) + 2^{2n-2} \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right)\right) \\
&\le \sum_{t=1}^{k} 2^{4n-4} L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right)
\end{aligned}
$$

where the third line follows from triangle inequality, fourth line from mclaurin's inequality, fifth line from Lemma 3, and the last line for $\alpha < \frac{1}{4L}$.

Now for A_5, we have to first note that, using mean-value theorem and with $a \in [0,1]$, we'll get

$$
\begin{aligned}
\|\theta_t - \theta^*\|^{2(n-1)} - \|\theta_{t-1} - \theta^*\|^{2(n-1)} &= \|\theta_t - \theta_{t-1}\| \cdot 2(n-1) \|a\left(\theta_t - \theta^*\right) + (1-a)\left(\theta_{t-1} - \theta^*\right)\|^{2n-3} \\
&\leq \|\theta_t - \theta_{t-1}\| \cdot 2(n-1) \|a(\theta_t - \theta_{t-1}) + \theta_{t-1} - \theta^*\|^{2n-3} \\
&\leq 2^{2n-3}(n-1)\|\theta_t - \theta_{t-1}\| \left(\|\theta_t - \theta_{t-1}\|^{2n-3} + \|\theta_{t-1} - \theta^*\|^{2n-3}\right)
\end{aligned}
$$

where the last line follows using the mclaurin's inequality. Plugging in the above upper bound to A_5 gives us

$$
\begin{aligned}
A_5 &\leq \sum_{t=1}^{k} 2^{2n-3}(n-1)\gamma_{t-1}\mathbb{E}\left[\left\langle \theta_{t-1} - \theta^*, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1} - \theta^*, X_t\right)\right\rangle \|\theta_t - \theta_{t-1}\| \left(\|\theta_t - \theta_{t-1}\|^{2n-3} + \|\theta_{t-1} - \theta^*\|^{2n-3}\right)\right] \\
&\leq \sum_{t=1}^{k} 2^{2n-3}(n-1)\gamma_{t-1}\mathbb{E}\left[\|\theta_{t-1} - \theta^*\| \, \|P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}, X_t\right)\| \, \|\theta_t - \theta_{t-1}\| \left(\|\theta_t - \theta_{t-1}\|^{2n-3} + \|\theta_{t-1} - \theta^*\|^{2n-3}\right)\right] \\
&\leq \sum_{t=1}^{k} 2^{2n-3}(n-1)L_{PH}^{(0)}\gamma_{t-1}\mathbb{E}\left[\|\theta_{t-1} - \theta^*\| \, \|\theta_t - \theta_{t-1}\| \left(\|\theta_t - \theta_{t-1}\|^{2n-3} + \|\theta_{t-1} - \theta^*\|^{2n-3}\right)\right] \\
&\leq \sum_{t=1}^{k} 2^{2n-3}(n-1)L_{PH}^{(0)}\gamma_{t-1}\mathbb{E}\left[\|\theta_t - \theta_{t-1}\|^{2n-2} \|\theta_{t-1} - \theta^*\| + \|\theta_t - \theta_{t-1}\| \, \|\theta_{t-1} - \theta^*\|^{2n-2}\right] \\
&\leq \sum_{t=1}^{k} 2^{2n-3}(n-1)L_{PH}^{(0)}\gamma_{t-1}\left(2^{4n-5}\alpha^{2n-2}L^{2n-2}\left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right) + 2^{2n-2}\alpha L\left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right)\right) \\
&\leq \sum_{t=1}^{k} 2^{4n-4}(n-1)\alpha L L_{PH}^{(0)}\gamma_{t-1}\left(\mathbb{E}\left[\|\theta_{t-1} - \theta^*\|^{2n-1}\right] + 1\right)
\end{aligned}
$$

where the fifth line follows from Lemma 3 and the last line follows for $\alpha < \frac{1}{4L}$.

Finally, for A_6, we obtain

$$
A_6 \leq L_{PH}^{(0)}\left(\gamma_0\|\theta_0 - \theta^*\|^{2n-1} + \gamma_k\|\theta_k - \theta^*\|^{2n-1}\right)
$$

which follows from Cauchy-Schwarz inequality and ??.

Combining the above terms gives us:

$$
\begin{aligned}
\sum_{t=0}^{k}\gamma_t\mathbb{E}\left[T'_{1,t}\right] &= \mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle \theta_t - \theta^*, g\left(\theta_t, X_{t+1}\right) - \bar{g}\left(\theta_t\right)\right\rangle \|\theta_t - \theta^*\|^{2(n-1)}\right] \\
&\leq \sum_{t=0}^{k-1} 2^{4n-2}\alpha L L_{PH}^{(1)}\gamma_{t+1}\left(\mathbb{E}\left[\|\theta_t - \theta^*\|^{2n}\right] + 1\right) + \sum_{t=0}^{k-1} 2^{4n-4}\alpha L L_{PH}^{(0)}\gamma_{t+1}\left(\mathbb{E}\left[\|\theta_t - \theta^*\|^{2n-1}\right] + 1\right) \\
&+ \sum_{t=0}^{k-1} 2^{4n-4}L_{PH}^{(0)}|\gamma_{t+1} - \gamma_t|\left(\mathbb{E}\left[\|\theta_t - \theta^*\|^{2n-1}\right] + 1\right) + \sum_{t=0}^{k-1} 2^{4n-4}\left(n-1\right)\alpha L L_{PH}^{(0)}\gamma_{t+1}\left(\mathbb{E}\left[\|\theta_t - \theta^*\|^{2n-1}\right] + 1\right) \\
&+ L_{PH}^{(0)}\left(\gamma_0\|\theta_0 - \theta^*\|^{2n-1} + \gamma_k\|\theta_k - \theta^*\|^{2n-1}\right) \\
&= \sum_{t=0}^{k-1} 2^{4n-2}\alpha L L_{PH}^{(1)}\gamma_{t+1}\left(\mathbb{E}\left[\|\theta_t - \theta^*\|^{2n}\right] + 1\right) + \sum_{t=0}^{k-1} 2^{4n-4}L_{PH}^{(0)}\left(\left(n\alpha L + 1\right)\gamma_{t+1} - \gamma_t\right)\left(\mathbb{E}\left[\|\theta_t - \theta^*\|^{2n-1}\right] + 1\right) \\
&+ L_{PH}^{(0)}\left(\gamma_0\|\theta_0 - \theta^*\|^{2n-1} + \gamma_k\|\theta_k - \theta^*\|^{2n-1}\right)
\end{aligned}
$$

where in the last equality we used the fact that for $\alpha \leq \frac{\mu}{L^2}$, $\gamma_t \leq \gamma_{t+1}$.

Now, if we make use of the inequality $2|x|^3 \leq x^2 + x^4$ and consolidating the terms, for sufficiently small

α, we have

$$
\begin{aligned}
\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^{2n}\right] \leq & \sum_{t=0}^{k-1} 2^{4n-5}\left[8\alpha L L_{PH}^{(1)}\gamma_{t+1} + L_{PH}^{(0)}\left((n\alpha L+1)\gamma_{t+1}-\gamma_t\right)\right]\left(\mathbb{E}\left[\|\theta_t-\theta^*\|^{2n}\right]+1\right) \\
& + \sum_{t=0}^{k-1} 2^{4n-5} L_{PH}^{(0)}\left((n\alpha L+1)\gamma_{t+1}-\gamma_t\right)\left(\mathbb{E}\left[\|\theta_t-\theta^*\|^{2(n-1)}\right]+1\right) + \frac{L_{PH}^{(0)}}{2}\left(\gamma_0\|\theta_0-\theta^*\|^{2(n-1)}+\gamma_k\|\theta_k-\theta^*\|^{2(n-1)}\right) \\
& + \left(\frac{L_{PH}^{(0)}}{2}+1\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right] + \frac{L_{PH}^{(0)}}{2}\gamma_k\|\theta_k-\theta^*\|^{2n} + o(\alpha) \\
\leq & \sum_{t=0}^{k-1} 2^{4n-4}\alpha L\gamma_{t+1}\left[4L_{PH}^{(1)} + nL_{PH}^{(0)}\right]\mathbb{E}\left[\|\theta_t-\theta^*\|^{2n}\right] + \sum_{t=0}^{k-1} 2^{4n-4} n\alpha L L_{PH}^{(0)}\mathbb{E}\left[\|\theta_t-\theta^*\|^{2(n-1)}\right] \\
& + \frac{L_{PH}^{(0)}}{2}\left(\gamma_0\|\theta_0-\theta^*\|^{2(n-1)}+\gamma_k\|\theta_k-\theta^*\|^{2(n-1)}\right) + \left(\frac{L_{PH}^{(0)}}{2}+1\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right] + \frac{L_{PH}^{(0)}}{2}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|^{2n}\right] + \tilde{c}_{1,k} + o(
\end{aligned}
$$

where the last inequality follows for $(n\alpha L+1)\gamma_{t+1}-\gamma_t \leq 2n\alpha L\gamma_{t+1}$ and also with

$$
\tilde{c}_{1,t} := \sum_{l=0}^{t-1} 2^{4n-4} n\alpha L\gamma_{t+1}\left[4L_{PH}^{(1)} + nL_{PH}^{(0)}\right] + \sum_{l=0}^{t-1} 2^{4n-4} n\alpha L L_{PH}^{(0)}
$$

being defined for $1 \leq t \leq k-1$. It can be seen that for sufficiently small α, $\tilde{c}_{1,t}$ would be of order $4^n n\alpha L\left(nL_{PH}^{(0)} + 4\alpha L_{PH}^{(1)}\right)$. Now from induction hypothesis we know that

$$
\mathbb{E}\left[\|\theta_t-\theta^*\|^{2(n-1)}\right] \leq \tilde{C}_{n-1,1}\cdot\left(1-2n\alpha\mu\right)^t\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right] + \tilde{C}_{n-1,2}\cdot 4^{n-1}(n-1)\alpha L\left((n-1)L_{PH}^{(0)} + 2\alpha L_{PH}^{(1)}\right).
$$

Thus, if we plug in this upper bound into the previous inequality, for $\alpha < \frac{1}{4^n}$ we will have

$$
\begin{aligned}
\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^{2n}\right] \leq & \sum_{t=0}^{k-1} 2^{4n-4}\alpha L\gamma_{t+1}\left[4L_{PH}^{(1)} + nL_{PH}^{(0)}\right]\mathbb{E}\left[\|\theta_t-\theta^*\|^{2n}\right] \\
& + \frac{L_{PH}^{(0)}}{2}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|^{2n}\right] + \tilde{c}_{2,k}\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right] + \tilde{c}_{1,k} + \tilde{c}_{3,k} + o(\alpha)
\end{aligned}
$$

in which $\tilde{c}_{3,k}$ would be of order $4^n n\alpha L\left(nL_{PH}^{(0)} + 4\alpha L_{PH}^{(1)}\right)$ and $\tilde{c}_{2,k}$ is an $\mathcal{O}(1)$ constant dependent to k.

To solve the above recursion, we define $\tilde{S}_t := \frac{L_{PH}^{(0)}}{2}\gamma_t\mathbb{E}\left[\|\theta_t-\theta^*\|^{2n}\right] + \sum_{l=0}^{t-1} 2^{4n-4}\alpha L\gamma_{l+1}\left[4L_{PH}^{(1)} + nL_{PH}^{(0)}\right]\mathbb{E}\left[\|\theta_l-\theta^*\|^{2n}\right]$ for $1 \leq t \leq k$. Using this we can write

$$
\mathbb{E}\left[\|\theta_{t+1}-\theta^*\|^{2n}\right] \leq \tilde{S}_t + \tilde{c}_{2,t}\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right] + \tilde{c}_{1,t} + \tilde{c}_{3,t} + o(\alpha).
$$

Now notice that we have(recall that $\gamma_t = 2n\alpha\left(1-2n\alpha\mu\right)^{k-t}$.)

$$
\begin{aligned}
\frac{\tilde{S}_t}{\tilde{S}_{t-1}} & \leq \frac{\tilde{S}_{t-1} + \left(2^{4n-4}\alpha L\gamma_t\left[4L_{PH}^{(1)} + nL_{PH}^{(0)}\right] - \frac{L_{PH}^{(0)}}{2}\gamma_{t-1}\right)\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^{2n}\right] + \frac{L_{PH}^{(0)}}{2}\gamma_t\mathbb{E}\left[\|\theta_t-\theta^*\|^{2n}\right]}{\tilde{S}_{t-1}} \\
& = \frac{\tilde{S}_{t-1} + \left(2^{4n-4}\alpha L\gamma_t\left[4L_{PH}^{(1)} + nL_{PH}^{(0)}\right] - \frac{L_{PH}^{(0)}}{2}\gamma_{t-1}\right)\mathbb{E}\left[\|\theta_{t-1}-\theta^*\|^{2n}\right] + \frac{L_{PH}^{(0)}}{2}\gamma_t\left[\tilde{S}_{t-1} + \tilde{c}_{2,t-1}\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right] + \tilde{c}_{1,t} + \tilde{c}_{3,t} + o(\alpha)\right]}{\tilde{S}_{t-1}}
\end{aligned}
$$

it is not very hard to see that the if we take α small enough above fraction could get as close as possible to 1. This means that we can write our final error bound as the following

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^{2n}\right] \le \tilde{C}_{n,1}\cdot\left(1-2n\alpha\mu\right)^{k+1}\mathbb{E}\left[\|\theta_0-\theta^*\|^{2n}\right]+\tilde{C}_{n,2}\cdot 4^n n\alpha L\left(nL_{PH}^{(0)}+4\alpha L_{PH}^{(1)}\right)$$

in which $\tilde{C}_{n,1}$ and $\tilde{C}_{n,2}$ are $\mathcal{O}(1)$ constants and also α has been taken small enough.

3 Bias Characterization

Before proceeding to the proof, we introduce the following shorthands and notations:

First, we recall that as a consequence of Assumption 2, the target equation $\bar{g}(\theta)=\mathbb{E}_{X\sim\pi_\theta}\left[g(\theta,X)\right]=0$, has a unique solution which we denote by θ^*. Moreover, we define X^* to be the samples being drawn from the stationary distribution of the Markov chain related to θ^*; *i.e.,* π_{θ^*}. Note that, from stationarity we know this is equal to the case that $\{X_n^*, n\in\mathbb{N}\}$ is a Markov chain initialized same as the other $\{X_n\}$ chain and evolving w.r.t. the P_{θ^*}. In this case we also take $\mathscr{F}_n=\{\theta_0,\{\xi_{m+1}\}_{m\le n-1},\{X_m\}_{m\le n},\{X_m^*\}_{m\le n}\}$. For notational simplicity, we omit indexes related to the *optimal Chain*(*i.e.,* $\{X_n^*\}$).

Additionally, For $X\in\mathscr{X}$,

$$\begin{aligned}
z_i(X) &:= \mathbb{E}\left[\left(\theta_\infty-\theta^*\right)^{\otimes i}|X^*=X\right],\\
\delta_i(X) &:= z_i(X)-\pi_{\theta^*}z_i.
\end{aligned}$$

We also denote

$$\begin{aligned}
&\bar{g}(\theta):=\mathbb{E}_{X\sim\pi_{\theta^*}}\left[g(\theta,X)\right],\quad \bar{g}_2(\theta):=\mathbb{E}_{X\sim\pi_{\theta^*}}\left[\left(g(\theta,X)\right)^{\otimes 2}\right]\\
&\bar{g}^{(1)}(\theta):=\mathbb{E}_{X\sim\pi_{\theta^*}}\left[g'(\theta,X)\right],\quad \bar{g}_2^{(1)}:=\mathbb{E}_{X\sim\pi_{\theta^*}}\left[\left(g'(\theta,X)\right)^{\otimes 2}\right],\quad \bar{g}^{(2)}(\theta):=\mathbb{E}_{X\sim\pi_{\theta^*}}\left[g''(\theta,X)\right].
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[g(\theta_\infty,X_{\infty+1})\right] &= \mathbb{E}\left[g\left(\theta^*,X_{\infty+1}\right)-g\left(\theta^*,X^*\right)\right]+\mathbb{E}\left[g\left(\theta^*,X^*\right)\right]\\
&+\mathbb{E}\left[\left(g'\left(\theta^*,X_{\infty+1}\right)-g'\left(\theta^*,X^*\right)\right)\left(\theta_\infty-\theta^*\right)\right]+\mathbb{E}\left[g'\left(\theta^*,X^*\right)\left(\theta_\infty-\theta^*\right)\right]\\
&+\frac{1}{2}\mathbb{E}\left[\left(g''\left(\theta^*,X_{\infty+1}\right)-g''\left(\theta^*,X^*\right)\right)\left(\theta_\infty-\theta^*\right)^{\otimes 2}\right]+\frac{1}{2}\mathbb{E}\left[g''\left(\theta^*,X^*\right)\left(\theta_\infty-\theta^*\right)^{\otimes 2}\right]\\
&+\mathbb{E}\left[R_3\left(\theta_\infty,X_{\infty+1}\right)\right]
\end{aligned}$$

Firstly, note that from the definition we have $\mathbb{E}\left[g(\theta^*,X^*)\right]=0$. Also,

$$\begin{aligned}
\mathbb{E}\left[g\left(\theta^*,X_{\infty+1}\right)-g\left(\theta^*,X^*\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[g\left(\theta^*,X_{\infty+1}\right)|\theta_\infty,X_\infty\right]-g\left(\theta^*,X^*\right)\right]\\
&= \mathbb{E}\left[P_{\theta_\infty}g\left(\theta^*,X_\infty\right)-g\left(\theta^*,X^*\right)\right]\\
&= \mathbb{E}\left[P_{\theta_\infty}g\left(\theta^*,X_\infty\right)-P_{\theta^*}g\left(\theta^*,X_\infty\right)+P_{\theta^*}g\left(\theta^*,X_\infty\right)-P_{\theta^*}g\left(\theta^*,X^*\right)\right]\\
&= \mathbb{E}\left[\left(P_{\theta_\infty}-P_{\theta^*}\right)g\left(\theta^*,X_\infty\right)+P_{\theta^*}\left(g\left(\theta^*,X_\infty\right)-g\left(\theta^*,X^*\right)\right)\right]
\end{aligned}$$

where the third equality follows from the fact that X^* comes from the stationary distribution of θ^*. Now notice that we have from Assumption 1 that $\mathbb{E}\left[g(\theta^*,X_\infty)\right]\le L_1$. Additionally,

$$\mathbb{E}\left[\|P_{\theta_\infty}-P_{\theta^*}\|\right]\le L_P\mathbb{E}\left[\|\theta_\infty-\theta^*\|\right]$$

from Assumption 7. For the remaining term we also have

$$
\begin{aligned}
\mathbb{E}\left[P_{\theta^*}\left(g\left(\theta^*, X_\infty\right) - g\left(\theta^*, X^*\right)\right)\right] &= \mathbb{E}\left[P_{\theta^*}\int_{\mathbb{R}} g\left(\theta^*, x\right)\pi_{\theta_\infty}(x) - g\left(\theta^*, x\right)\pi_{\theta^*}(x)dx\right] \\
&\leq \mathbb{E}\left[P_{\theta^*}L_S\|\theta_\infty - \theta^*\|\int_{\mathbb{R}} g\left(\theta^*, x\right)dx\right] \\
&\leq L_1 L_S \mathbb{E}\left[\|\theta_\infty - \theta^*\|\right]
\end{aligned}
$$

where the second line follows from Assumption 8. Summing up all these would give us

$$
\mathbb{E}\left[g\left(\theta^*, X_{\infty+1}\right) - g\left(\theta^*, X^*\right)\right] \leq L_1\left(L_P + L_S\right)\mathbb{E}\left[\|\theta_\infty - \theta^*\|\right],
$$

which is of $\mathcal{O}(\alpha^{1/2})$.

For the first and second derivative cases, conditioning on θ_∞ would give us a similar setting and thus we will have

$$
\begin{aligned}
\mathbb{E}\left[\left(g'\left(\theta^*, X_{\infty+1}\right) - g'\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\mathbb{E}\left[g'\left(\theta^*, X_{\infty+1}\right)|X_\infty, \theta_\infty\right] - g'\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)|\theta_\infty\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(P_{\theta_\infty}g'\left(\theta^*, X_\infty\right) - g'\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)|\theta_\infty\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(P_{\theta_\infty}g'\left(\theta^*, X_\infty\right) - P_{\theta^*}g'\left(\theta^*, X_\infty\right) + P_{\theta^*}g'\left(\theta^*, X_\infty\right) - P_{\theta^*}g'\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)|\theta_\infty\right.\right. \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\left(P_{\theta_\infty} - P_{\theta^*}\right)g'\left(\theta^*, X_\infty\right) + P_{\theta^*}\left(g'\left(\theta^*, X_\infty\right) - g'\left(\theta^*, X^*\right)\right)\right)\left(\theta_\infty - \theta^*\right)|\theta_\infty\right]\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[L_1\left(L_P + L_S\right)\|\theta_\infty - \theta^*\|\left(\theta_\infty - \theta^*\right)|\theta_\infty\right]\right] \\
&\leq L_1\left(L_P + L_S\right)\mathbb{E}\left[\|\theta_\infty - \theta^*\|\left(\theta_\infty - \theta^*\right)\right]
\end{aligned}
$$

which is of $\mathcal{O}(\alpha)$ in the end. In the same manner for the second derivative we can calculate this

$$
\begin{aligned}
\mathbb{E}\left[\left(g''\left(\theta^*, X_{\infty+1}\right) - g''\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)^{\otimes 2}\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\mathbb{E}\left[g''\left(\theta^*, X_{\infty+1}\right)|X_\infty, \theta_\infty\right] - g''\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)^{\otimes 2}|\theta_\infty\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(P_{\theta_\infty}g''\left(\theta^*, X_\infty\right) - g''\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta^*\right)^{\otimes 2}|\theta_\infty\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(P_{\theta_\infty}g''\left(\theta^*, X_\infty\right) - P_{\theta^*}g''\left(\theta^*, X_\infty\right) + P_{\theta^*}g''\left(\theta^*, X_\infty\right) - P_{\theta^*}g''\left(\theta^*, X^*\right)\right)\left(\theta_\infty - \theta\right.\right.\right. \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\left(P_{\theta_\infty} - P_{\theta^*}\right)g''\left(\theta^*, X_\infty\right) + P_{\theta^*}\left(g''\left(\theta^*, X_\infty\right) - g''\left(\theta^*, X^*\right)\right)\right)\left(\theta_\infty - \theta^*\right)^{\otimes 2}|\theta_\infty\right]\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[L_1\left(L_P + L_S\right)\|\theta_\infty - \theta^*\|\left(\theta_\infty - \theta^*\right)^{\otimes 2}|\theta_\infty\right]\right] \\
&\leq L_1\left(L_P + L_S\right)\mathbb{E}\left[\|\theta_\infty - \theta^*\|\left(\theta_\infty - \theta^*\right)^{\otimes 2}\right]
\end{aligned}
$$

in this case the difference will be of $\mathcal{O}\left(\alpha^{3/2}\right)$.

4 Experiments

Problem Setting

We consider a stochastic approximation (SA) scheme where the control parameter $\theta_t \in \mathbb{R}^d$ is updated via the rule:

$$
\theta_{t+1} = \theta_t + \alpha\left(g(\theta_t, X_{t+1}) + \xi_{t+1}\right),
$$

where:

- $\alpha > 0$ is a constant step size,

- $g: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz and strongly monotone function,
- ξ_t is a zero-mean noise sequence with bounded moments,
- $\{X_t\}$ is a state-dependent Markov chain with kernel parameterized by θ_t.

Markov Chain Structure

We define a family of Markov chains on $\mathbb{R}$, each corresponding to a parameter $\theta \in \mathbb{R}$, given by a discretized Langevin dynamics:

$$X_{t+1} = X_t - \frac{\epsilon^2}{2}(X_t - \theta_t) + \epsilon Z_t, \quad Z_t \sim \mathcal{N}(0,1),$$

which approximates the dynamics of an SDE with potential $U(x;\theta) = \frac{1}{2}(x-\theta)^2$. The stationary distribution of this Markov chain is approximately $\mathcal{N}(\theta,1)$.

Choice of $g(\theta,x)$

We take

$$g(\theta,x) = \theta - x,$$

which is Lipschitz and strongly monotone. If $X \sim \mathcal{N}(\theta,1)$, then

$$\mathbb{E}[g(\theta,X)] = \theta - \mathbb{E}[X] = 0,$$

i.e., the expected value of g under the stationary distribution of the Markov chain with parameter θ is zero.

Poisson Equation

We require that for each θ, there exists a function $\hat{g}(\theta,x)$ satisfying the Poisson equation:

$$\hat{g}(\theta,x) - P_\theta \hat{g}(\theta,x) = g(\theta,x) - \bar{g}(\theta),$$

where P_θ is the transition kernel of the Markov chain with parameter θ, and $\bar{g}(\theta)$ is the expected value of g under the stationary distribution π_θ. In this setup, $\bar{g}(\theta^*) = 0$ where $\theta^* = 0$.

Bias Estimation

We estimate the instantaneous bias over time as:

$$\text{Bias}_t = \mathbb{E}[g(\theta_t, X_{t+1})] - \mathbb{E}[g(\theta^*, X^*)],$$

where $X^* \sim \mathcal{N}(0,1)$. We approximate the expectation over time using cumulative means.

Python Implementation

Listing 1: Stochastic Approximation with Bias Tracking

```
import numpy as np
import matplotlib.pyplot as plt
```

```
import seaborn as sns
from scipy.stats import norm

# ----------------------------------------
# PARAMETERS
# ----------------------------------------
np.random.seed(42)
T = 10000                 # number of SA iterations
alpha = 0.001             # SA step size (constant)
epsilon = 0.1             # discretized Langevin stepsize for X update
sigma_xi = 0.1            # std dev for noise in theta-update
d = 1                     # dimension (we work in R)

# ----------------------------------------
# FUNCTIONS
# ----------------------------------------

# Markov chain update for X using discretized Langevin dynamics.
# Here the potential U(x;theta) = 0.5*(x - theta)^2, so grad_U(x,theta)
def update_X(x, theta):
grad_U = x - theta  # gradient of U
noise = np.random.normal(0, 1)
x_next = x - 0.5 * epsilon**2 * grad_U + epsilon * noise
return x_next

# Function g: R x R -> R, here chosen as:
def g(theta, x):
return theta - x

# ----------------------------------------
# SIMULATION OF STOCHASTIC APPROXIMATION SCHEME
# ----------------------------------------
# Initialize theta and X
theta_vals = np.zeros(T+1)
X_vals = np.zeros(T+1)
g_vals = np.zeros(T)    # record g(theta_t, X_{t+1})

# For illustration, start with non-optimal theta and X
theta_vals[0] = 1.0     # initial theta (nonzero so that we see a drift t
X_vals[0] = 1.0         # initial state for X

# SA loop:
for t in range(T):
# Sample X_{t+1} from the Markov chain with kernel parameterized by thet
X_vals[t+1] = update_X(X_vals[t], theta_vals[t])
```

```
# Compute g(theta_t, X_{t+1})
g_val = g(theta_vals[t], X_vals[t+1])
g_vals[t] = g_val

# Sample the noise \xi_{t+1}
xi = np.random.normal(0, sigma_xi)

# SA update for theta
theta_vals[t+1] = theta_vals[t] + alpha * (g_val + xi)

# ----------------------------------------
# VISUALIZATIONS
# ----------------------------------------
plt.figure(figsize=(14, 6))

# Plot evolution of theta
plt.subplot(1,2,1)
plt.plot(theta_vals, label=r'$\theta_t$')
plt.axhline(0, color='black', linestyle='--', label=r'$\theta^*=0$')
plt.title('Evolution of control parameter $\\theta_t$')
plt.xlabel('Iteration')
plt.ylabel('$\\theta_t$')
plt.legend()
plt.grid(True)

# Plot evolution of g(theta, X) values
plt.subplot(1,2,2)
plt.plot(g_vals, label=r'$g(\theta_t, X_{t+1})$')
plt.axhline(0, color='black', linestyle='--')
plt.title(r'Evolution of $g(\theta_t, X_{t+1})$')
plt.xlabel('Iteration')
plt.ylabel('g value')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

# ----------------------------------------
# BIAS ESTIMATION
# ----------------------------------------
# After T iterations, we treat theta_T as our final parameter.
theta_final = theta_vals[-1]
print(f"Final theta: {theta_final:.4f}")
```

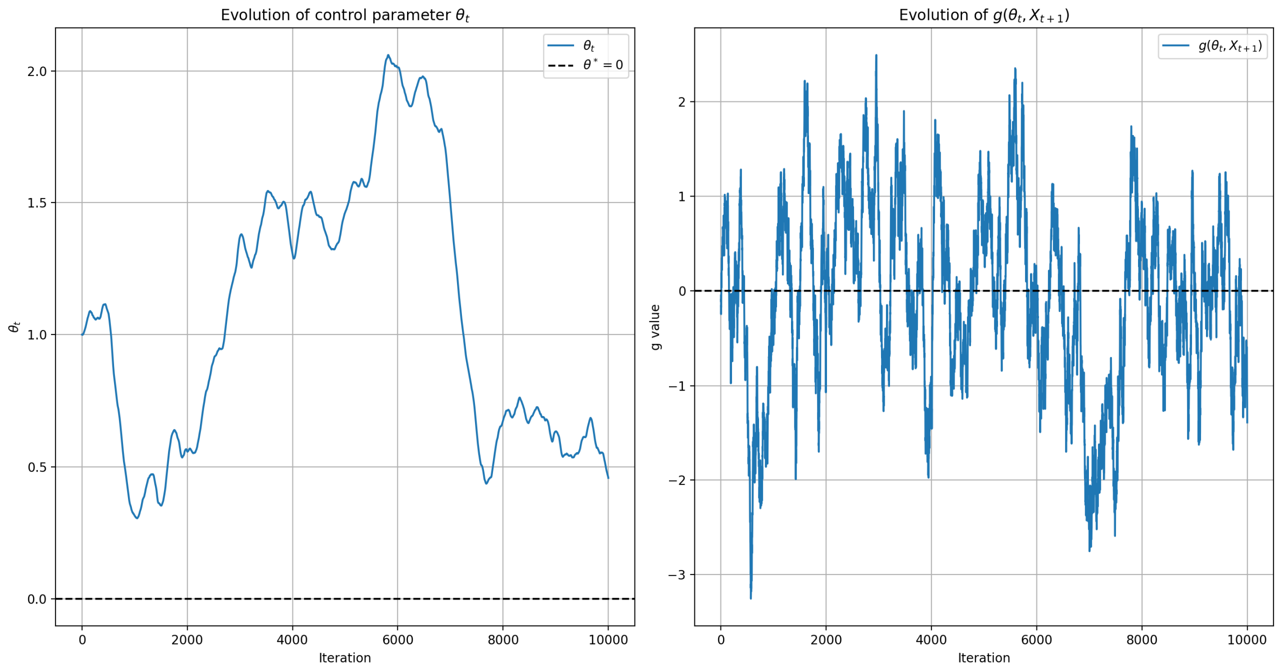
```
# (A) Estimate E[g(theta_final, X)] when X is drawn from the stationary
# For this, simulate a long chain with parameter theta_final.
T_chain = 5000
burn_in = 1000
X_chain = np.zeros(T_chain+1)
X_chain[0] = theta_final  # start at the mean
for t in range(T_chain):
X_chain[t+1] = update_X(X_chain[t], theta_final)
# Use the samples after burn_in:
g_final_samples = g(theta_final, X_chain[burn_in:])
E_g_final = np.mean(g_final_samples)

# (B) Estimate E[g(theta*, X)] when theta*=0 and X is drawn from the co
theta_star = 0.0
X_chain_star = np.zeros(T_chain+1)
X_chain_star[0] = theta_star
for t in range(T_chain):
X_chain_star[t+1] = update_X(X_chain_star[t], theta_star)
g_star_samples = g(theta_star, X_chain_star[burn_in:])
E_g_star = np.mean(g_star_samples)

bias = E_g_final - E_g_star

print(f"Estimated E[g(theta_final, X)] = {E_g_final:.4f}")
print(f"Estimated E[g(theta*, X)] = {E_g_star:.4f}")
print(f"Estimated Bias = {bias:.4f}")
```

Results




```
Final theta: 0.4580
Estimated E[g(theta_final, X)] = 0.2588
Estimated E[g(theta*, X)] = 0.2215
Estimated Bias = 0.0373
```

Conclusion

This simulation tracks the evolution of the control parameter θ_t, the evaluated update function $g(\theta_t, X_{t+1})$, and the bias over time. The algorithm converges toward the optimal value $\theta^* = 0$, and the empirical bias decreases as expected, supporting both theoretical intuition and the Poisson equation structure.

References

[1] M. Kaledin, E. Moulines, A. Naumov, V. Tadic, and H.-T. Wai. Finite time analysis of linear two-timescale stochastic approximation with markovian noise. In *Conference on Learning Theory*, pages 2144–2203. PMLR, 2020.

[2] B. Karimi, B. Miasojedow, E. Moulines, and H.-T. Wai. Non-asymptotic analysis of biased stochastic approximation scheme. In *Conference on Learning Theory*, pages 1944–1974. PMLR, 2019.