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# NONLINEAR MARKOVIAN STOCHASTIC APPROXIMATION

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# 1 Preliminaries

**Notations** The Euclidean norm is denoted by  $\|\cdot\|$ . The lowercase letter  $c$  and its derivatives  $c', c_0$ , etc. denote universal numerical constants, whose value may change from line to line. As we are primarily interested in dependence of  $\alpha$  and  $k$ , we adopt the following big- $O$  notation:  $\|f\| = \mathcal{O}(h(\alpha, k))$  if it holds that  $\|f\| \leq s \cdot \|h(\alpha, k)\|$  for some constant  $s > 0$ .

We use of the following iteration scheme:

$$\theta_{t+1} = \theta_t + \alpha (g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t))$$

## 1.1 Assumptions

**Assumption 1** For each  $X \in \mathcal{X}$ , the function  $g(\theta, X)$  is three times continuously differentiable in  $\theta$  with uniformly bounded first to third derivatives, i.e.,  $\sup_{\theta \in \mathbb{R}^d} \|g^{(i)}(\theta, X)\| < \infty$  for  $i = 1, 2, 3, X \in \mathcal{X}$ . Moreover, there exists a constant  $L_1 > 0$  such that (1)  $\|g^{(i)}(\theta, X) - g^{(i)}(\theta', X)\| \leq L_1$ , for all  $\theta, \theta' \in \mathbb{R}^d, i = 0, 1, 2$  and  $X \in \mathcal{X}$ , and (2)  $\|g(0, X)\| \leq L_1$  for all  $X \in \mathcal{X}$ .

Assumption 1 implies that  $g(\theta, X)$  is  $L_1$ -Lipschitz w.r.t  $\theta$  uniformly in  $X$ . The above assumption immediately implies that the growth of  $\|g\|$  and  $\|\bar{g}\|$  will be at most linear in  $\theta$ , i.e.,  $\|g(\theta, X)\| \leq L_1(\|\theta - \theta^*\| + 1)$  and  $\|\bar{g}(\theta)\| \leq L_1(\|\theta - \theta^*\| + 1)$ .

**Assumption 2** There exists  $\mu > 0$  such that  $\langle \theta - \theta', \bar{g}(\theta) - \bar{g}(\theta') \rangle \leq -\mu \|\theta - \theta'\|^2, \forall \theta, \theta' \in \mathbb{R}^d$ . Consequently, the target equation  $\bar{g}(\theta) = 0$  has a unique solution  $\theta^*$ .

Denote by  $\mathcal{F}_k$  the filtration generated by  $\{X_{t+1}, \theta_t, \xi_{t+1}\}_{t=0}^{k-1} \cup \{X_{k+1}, \theta_k\}$ .

**Assumption 3** Let  $p \in \mathbb{Z}_+$  be given. The noise sequence  $(\xi_k)_{k \geq 1}$  is a collection of i.i.d random fields satisfying the following conditions with  $L_{2,p} > 0$ :

$$\mathbb{E}[\xi_{k+1}(\theta) | \mathcal{F}_k] = 0 \quad \text{and} \quad \mathbb{E}^{1/(2p)}[\|\xi_1(\theta)\|^{2p}] \leq L_{2,p}(\|\theta - \theta^*\| + 1), \quad \forall \theta \in \mathbb{R}^d.$$

Define  $C(\theta) = \mathbb{E}[\xi_1(\theta)^{\otimes 2}]$  and assume that  $C(\theta)$  is at least twice differentiable. There also exists  $M_\epsilon, k_\epsilon \geq 0$  such that for  $\theta \in \mathbb{R}^d$ , we have  $\max_{i=1,2} \|C^{(i)}(\theta)\| \leq M_\epsilon \{1 + \|\theta - \theta^*\|^{k_\epsilon}\}$ . In the sequel, we set  $L := L_1 + L_2$ , and without loss of generality, we assume  $L \geq 1$ .

**Assumption 4** There exists a Borel measurable function  $\hat{g}: \mathbb{R}^d \times \mathcal{X} \rightarrow \mathbb{R}^d$  where for each  $\theta \in \mathbb{R}^d, X \in \mathcal{X}$ ,

$$\hat{g}(\theta, X) - P_\theta \hat{g}(\theta, X) = g(\theta, X) - \bar{g}(\theta).$$

**Assumption 5** There exists  $L_{PH}^{(0)} < \infty$  and  $L_{PH}^{(1)} < \infty$  such that, for all  $\theta \in \mathbb{R}^d$  and  $X \in \mathcal{X}$ , one has  $\|\hat{g}(\theta, X)\| \leq L_{PH}^{(0)}, \|P_\theta \hat{g}(\theta, X)\| \leq L_{PH}^{(0)}$ . Moreover, for  $(\theta, \theta') \in \mathcal{H}^2$ ,

$$\sup_{X \in \mathcal{X}} \|P_\theta \hat{g}(\theta, X) - P_{\theta'} \hat{g}(\theta', X)\| \leq L_{PH}^{(1)} \|\theta - \theta'\|.$$

**Assumption 6** For any  $\theta, \theta' \in \mathbb{R}^d$ , we have  $\sup_{X \in \mathcal{X}} \|P_\theta(X, \cdot) - P_{\theta'}(X, \cdot)\|_{TV} \leq L_P \|\theta - \theta'\|$ .

**Assumption 7** For any  $\theta, \theta' \in \mathbb{R}^d$ , we have  $\sup_{X \in \mathcal{X}} \|g(\theta, X) - g(\theta', X)\| \leq L_H \|\theta - \theta'\|$ .

**Assumption 8** *There exists  $\rho < 1$ ,  $K_P < \infty$  such that*

$$\sup_{\theta \in \mathbb{R}^d, X \in \mathcal{X}} \|P_\theta^n(X, \cdot) - \pi_\theta(\cdot)\|_{TV} \leq \rho^n K_P,$$

**Lemma 1** *Assume that assumptions 6-8 hold. Then, for any  $\theta \in \mathbb{R}^d$  and  $X \in \mathcal{X}$ ,*

$$\|\hat{g}(\theta, X)\| \leq \frac{\sigma K_P}{1 - \rho},$$

$$\|P_\theta \hat{g}(\theta, X)\| \leq \frac{\sigma \rho K_P}{1 - \rho}.$$

Moreover, for any  $\theta, \theta' \in \mathbb{R}^d$  and  $X \in \mathcal{X}$ ,

$$\|P_\theta \hat{g}(\theta, X) - P_{\theta'} \hat{g}(\theta', X)\| \leq \|\theta - \theta'\|,$$

where

$$L_{PH}^{(1)} = \frac{K_P^2 \sigma L_P}{(1 - \rho)^2} (2 + K_P) + \frac{K_P}{1 - \rho} L_H.$$

Proof of this lemma can be found in [1], Lemma 7.

## 2 Error Bound

### 2.1 Base Case

For the base case analysis, we can write:

$$\begin{aligned} & \mathbb{E}[\|\theta_{k+1} - \theta^*\|^2] - \mathbb{E}[\|\theta_k - \theta^*\|^2] = \\ & 2\alpha \mathbb{E}[\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) \rangle] + \alpha^2 \mathbb{E}[\|g(\theta_k, X_{k+1})\|^2] + \alpha^2 \mathbb{E}[\|\xi_{k+1}(\theta_k)\|^2] = \\ & 2\alpha \mathbb{E}[\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) \rangle] + 2\alpha \mathbb{E}[\langle \theta_k - \theta^*, \bar{g}(\theta_k) \rangle] + \alpha^2 \mathbb{E}[\|g(\theta_k, X_{k+1})\|^2] + \alpha^2 \mathbb{E}[\|\xi_{k+1}(\theta_k)\|^2]. \end{aligned}$$

It is easy to see that under Strong Monotonicity assumption, we have

$$\langle \theta_k - \theta^*, \bar{g}(\theta_k) \rangle = \langle \theta_k - \theta^*, \bar{g}(\theta_k) + \bar{g}(\theta^*) \rangle \leq -\mu \|\theta_k - \theta^*\|^2.$$

Additionally, under Assumption 1 and 3, we have the following upper bound

$$\begin{aligned} & \alpha^2 (\mathbb{E}[\|g(\theta_k, X_{k+1})\|^2] + \mathbb{E}[\|\xi_{k+1}(\theta_k)\|^2]) \\ & \leq \alpha^2 \left( L_1^2 \mathbb{E}[(\|\theta_k - \theta^*\| + 1)^2] + L_2^2 \mathbb{E}[(\|\theta_k - \theta^*\| + 1)^2] \right) \\ & \leq 2\alpha^2 L^2 (\mathbb{E}[\|\theta_k - \theta^*\|^2] + 1). \end{aligned}$$

Therefore, we have

$$\mathbb{E} [\|\theta_{k+1} - \theta^*\|^2] \leq (1 - 2\alpha(\alpha L^2 + \mu)) \mathbb{E} [\|\theta_k - \theta^*\|^2] + 2\alpha^2 L^2 + 2\alpha \mathbb{E} [\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) \rangle]$$

Solving this recursion gives us the following inequality:

$$\begin{aligned} \mathbb{E} [\|\theta_{k+1} - \theta^*\|^2] &\leq (1 - 2\alpha(\alpha L^2 + \mu))^{k+1} \mathbb{E} [\|\theta_0 - \theta^*\|^2] \\ &\quad + \sum_{t=0}^k (1 - 2\alpha(\alpha L^2 + \mu))^t 2\alpha^2 L^2 \\ &\quad + \sum_{t=0}^k 2\alpha (1 - 2\alpha(\alpha L^2 + \mu))^{k-t} \mathbb{E} [\langle \theta_t - \theta^*, g(\theta_t, X_{t+1}) - \bar{g}(\theta_t) \rangle]. \end{aligned}$$

For notational simplicity we define  $\gamma_t := 2\alpha(1 - 2\alpha(\alpha L^2 + \mu))^{k-t}$  for  $0 \leq t \leq k$ .

The second term above is just a geometric series which is equal to  $2\alpha^2 L^2 (\alpha L^2 + \mu)^k$ .

Now, we can upper bound the third summand using below decomposition:

$$\mathbb{E} \left[ \sum_{t=0}^k \gamma_t \langle \theta_t - \theta^*, g(\theta_t, X_{t+1}) - \bar{g}(\theta_t) \rangle \right] = \mathbb{E} [A_1 + A_2 + A_3 + A_4 + A_5]$$

with

$$\begin{aligned} A_1 &:= \sum_{t=1}^k \gamma_t \langle \theta_t - \theta^*, \hat{g}(\theta_t, X_{t+1}) - P_{\theta_t} \hat{g}(\theta_t, X_t) \rangle, \\ A_2 &:= \sum_{t=1}^k \gamma_t \langle \theta_t - \theta^*, P_{\theta_t} \hat{g}(\theta_t, X_t) - P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t) \rangle, \\ A_3 &:= \sum_{t=1}^k \gamma_t \langle \theta_t - \theta_{t-1}, P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t) \rangle, \\ A_4 &:= \sum_{t=1}^k (\gamma_t - \gamma_{t-1}) \langle \theta_{t-1} - \theta^*, P_{\theta_{t-1}} \hat{g}(\theta_{t-1} - \theta^*, X_t) \rangle, \\ A_5 &:= \gamma_0 \langle \theta_0 - \theta^*, \hat{g}(\theta_0, X_0) \rangle + \gamma_k \langle \theta_k - \theta^*, P_{\theta_k} \hat{g}(\theta_k, X_{k+1}) \rangle \end{aligned}$$

For  $A_1$ , we note that  $\hat{g}(\theta_t, X_{t+1}) - P_{\theta_t} \hat{g}(\theta_t, X_t)$  is a martingale difference sequence [cf. ?] and therefore we have  $\mathbb{E}[A_1] = 0$  by taking the total expectation.

For  $A_2$ , applying Cauchy-Schwarz inequality and **??**, we have

$$\begin{aligned} A_2 &\leq \sum_{t=1}^k L_{PH}^{(1)} \gamma_t \|\theta_t - \theta^*\| \|\theta_t - \theta_{t-1}\| \\ &= \sum_{t=1}^k \alpha L_{PH}^{(1)} \gamma_t \|\theta_t - \theta^*\| \|g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t)\| \\ &\leq \sum_{t=1}^k \alpha L_{PH}^{(1)} \gamma_t \|\theta_t - \theta^*\| (L_1 (\|\theta_t - \theta^*\| + 1) + L_2 (\|\theta_t - \theta^*\| + 1)) \\ &\leq \sum_{t=1}^k \frac{\alpha L_{PH}^{(1)} \gamma_t}{2} (3\|\theta_t - \theta^*\|^2 + 1) \end{aligned}$$

where the third line follows from the Lipschitzness condition and the assumption of

$$\mathbb{E}^{1/2} [\|\xi_{t+1}(\theta_t)\|^2 | \mathcal{F}_t] \leq L_2 (\|\theta_t\| + 1)$$

also, last line follows from the identity  $u \leq \frac{1}{2}(1 + u^2)$ .

For  $A_3$ , we obtain

$$\begin{aligned} A_3 &\leq \sum_{t=1}^k \gamma_t \|\theta_t - \theta_{t-1}\| \|P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t)\| \\ &\leq \sum_{t=1}^k \alpha L_{PH}^{(0)} \gamma_t \|g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t)\| \\ &\leq \sum_{t=1}^k \alpha L_{PH}^{(0)} \gamma_t (L_1 (\|\theta_t - \theta^*\| + 1) + L_2 (\|\theta_t - \theta^*\| + 1)) \\ &\leq \sum_{t=1}^k \alpha L L_{PH}^{(0)} \gamma_t (\|\theta_t - \theta^*\| + 1) \end{aligned}$$

where second line follows from **??** and third line is similarly done to the previous part, using Lipschitzness condition and noise assumption.

For  $A_4$ , we have

$$\begin{aligned} A_4 &\leq \sum_{t=1}^k |\gamma_t - \gamma_{t-1}| \|\theta_{t-1} - \theta^*\| \|P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t)\| \\ &\leq \sum_{t=1}^k L_{PH}^{(0)} |\gamma_t - \gamma_{t-1}| \|\theta_{t-1} - \theta^*\| \end{aligned}$$

Finally, for  $A_5$ , we obtain

$$A_5 \leq L_{PH}^{(0)} (\gamma_0 \|\theta_0 - \theta^*\| + \gamma_k \|\theta_k - \theta^*\|)$$

which follows from Cacuchy-Schwarz inequality and **??**.

Combining the above terms and taking expectations, gives us:

$$\begin{aligned} \mathbb{E} \left[ \sum_{t=0}^k \gamma_t \langle \theta_t - \theta^*, g(\theta_t, X_{t+1} - \bar{g}(\theta_t)) \rangle \right] &\leq \sum_{t=1}^k \frac{\alpha L_{PH}^{(1)} \gamma_t}{2} (1 + 3\mathbb{E} [\|\theta_t - \theta^*\|^2]) + \sum_{t=1}^k \alpha L L_{PH}^{(0)} \gamma_t (\mathbb{E} [\|\theta_t - \theta^*\|] + 1) + \\ &\quad \sum_{t=0}^{k-1} L_{PH}^{(0)} |\gamma_t - \gamma_{t+1}| \mathbb{E} [\|\theta_t - \theta^*\|] + L_{PH}^{(0)} (\gamma_0 \mathbb{E} [\|\theta_0 - \theta^*\|] + \gamma_k \mathbb{E} [\|\theta_k - \theta^*\|]) \end{aligned}$$

now it should be noticed that as long as the  $\alpha$  satisfies  $\alpha \leq \frac{\sqrt{4\mu^2 + 8L^2} - \mu}{4L^2}$ , we have  $\gamma_t \leq \gamma_{t+1}$ . Thus, we can simplify the above upper bound and write it this way:

$$\begin{aligned} \mathbb{E} \left[ \sum_{t=0}^k \gamma_t \langle \theta_t - \theta^*, g(\theta_t, X_{t+1} - \bar{g}(\theta_t)) \rangle \right] &\leq \sum_{t=1}^k \frac{\alpha L_{PH}^{(1)} \gamma_t}{2} (1 + 3\mathbb{E} [\|\theta_t - \theta^*\|^2]) + \\ &\quad \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1) \gamma_t + \gamma_{t+1}) \mathbb{E} [\|\theta_t - \theta^*\|] + \\ &\quad \sum_{t=1}^k \alpha L L_{PH}^{(0)} \gamma_t + L_{PH}^{(0)} (\gamma_1 \mathbb{E} [\|\theta_0 - \theta^*\|] + (\alpha L + 1) \gamma_k \mathbb{E} [\|\theta_k - \theta^*\|]) \end{aligned}$$

Hence, using the derived upper bounds from the above terms, we have:

$$\begin{aligned}\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2] &\leq \sum_{t=1}^k \frac{\alpha L_{PH}^{(1)} \gamma_t}{2} (1 + 3\mathbb{E}[\|\theta_t - \theta^*\|^2]) + \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1)\gamma_t + \gamma_{t+1}) \mathbb{E}[\|\theta_t - \theta^*\|] + \\ &\quad (1 - 2\alpha(\alpha L^2 + \mu))\gamma_0 \mathbb{E}[\|\theta_0 - \theta^*\|^2] + L_{PH}^{(0)} \gamma_1 \mathbb{E}[\|\theta_0 - \theta^*\|] + (\alpha L + 1) L_{PH}^{(0)} \gamma_k \mathbb{E}[\|\theta_k - \theta^*\|] + \\ &\quad \left( \frac{L}{L_{PH}^{(0)}} + 1 \right) \frac{\gamma_1 (1 - (1 - 2\alpha(\alpha L^2 + \mu))^k)}{(1 - (1 - 2\alpha(\alpha L^2 + \mu)))} + (1 - 2\alpha(\alpha L^2 + \mu))^k 2\alpha^2 L^2\end{aligned}$$

for further notation simplicity we define  $c_{1,t} := \left( \frac{L}{L_{PH}^{(0)}} + 1 \right) \frac{\gamma_1 (1 - (1 - 2\alpha(\alpha L^2 + \mu))^t)}{(1 - (1 - 2\alpha(\alpha L^2 + \mu)))} + (1 - 2\alpha(\alpha L^2 + \mu))^t 2\alpha^2 L^2$  for  $0 \leq t \leq k$ . Now to write down this upper bound in a way in which it only depends on  $\|\theta_0 - \theta^*\|$  related terms and constants, we can write:

$$\begin{aligned}\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2] &\leq \sum_{t=1}^k \left[ \frac{3\alpha L_{PH}^{(1)} \gamma_t}{2} \mathbb{E}[\|\theta_t - \theta^*\|^2] + \frac{\alpha L_{PH}^{(1)} \gamma_t}{2} \right] + \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1)\gamma_t + \gamma_{t+1}) \mathbb{E}[\|\theta_t - \theta^*\|] + \\ &\quad (1 - 2\alpha(\alpha L^2 + \mu))\gamma_0 \mathbb{E}[\|\theta_0 - \theta^*\|^2] + L_{PH}^{(0)} \gamma_1 \mathbb{E}[\|\theta_0 - \theta^*\|] + (\alpha L + 1) L_{PH}^{(0)} \gamma_k \mathbb{E}[\|\theta_k - \theta^*\|] + c_{1,k} \\ &= \sum_{t=1}^k \frac{3\alpha L_{PH}^{(1)} \gamma_t}{2} \mathbb{E}[\|\theta_t - \theta^*\|^2] + \frac{\alpha L_{PH}^{(1)}}{2} \sum_{t=1}^k \gamma_t + \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1)\gamma_t + \gamma_{t+1}) \mathbb{E}[\|\theta_t - \theta^*\|] + \\ &\quad (1 - 2\alpha(\alpha L^2 + \mu))\gamma_0 \mathbb{E}[\|\theta_0 - \theta^*\|^2] + L_{PH}^{(0)} \gamma_1 \mathbb{E}[\|\theta_0 - \theta^*\|] + (\alpha L + 1) L_{PH}^{(0)} \gamma_k \mathbb{E}[\|\theta_k - \theta^*\|] + c_{1,k} \\ &= \sum_{t=1}^k \frac{3\alpha L_{PH}^{(1)} \gamma_t}{2} \mathbb{E}[\|\theta_t - \theta^*\|^2] + \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1)\gamma_t + \gamma_{t+1}) \mathbb{E}[\|\theta_t - \theta^*\|] + \\ &\quad (1 - 2\alpha(\alpha L^2 + \mu))\gamma_0 \mathbb{E}[\|\theta_0 - \theta^*\|^2] + L_{PH}^{(0)} \gamma_1 \mathbb{E}[\|\theta_0 - \theta^*\|] + (\alpha L + 1) L_{PH}^{(0)} \gamma_k \mathbb{E}[\|\theta_k - \theta^*\|] + c_{1,k} + \\ &\quad \frac{L_{PH}^{(1)} \gamma_1 [1 - (1 - 2\alpha(\alpha L^2 + \mu))^k]}{4[1 - (1 - 2\alpha(\alpha L^2 + \mu))]} \end{aligned}$$

where the last equality follows from the definition of  $\gamma_t$ s. Similarly we define  $c_{2,t} := \frac{L_{PH}^{(1)} \gamma_1 [1 - (1 - 2\alpha(\alpha L^2 + \mu))^t]}{4[1 - (1 - 2\alpha(\alpha L^2 + \mu))]}$  for  $0 \leq t \leq k$ . So we can write it as

$$\begin{aligned}\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2] &\leq \sum_{t=1}^k \frac{3\alpha L_{PH}^{(1)} \gamma_t}{2} \mathbb{E}[\|\theta_t - \theta^*\|^2] + \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1)\gamma_t + \gamma_{t+1}) \mathbb{E}[\|\theta_t - \theta^*\|] + \\ &\quad (1 - 2\alpha(\alpha L^2 + \mu))\gamma_0 \mathbb{E}[\|\theta_0 - \theta^*\|^2] + L_{PH}^{(0)} \gamma_1 \mathbb{E}[\|\theta_0 - \theta^*\|] + (\alpha L + 1) L_{PH}^{(0)} \gamma_k \mathbb{E}[\|\theta_k - \theta^*\|] + c_{1,k} + c_{2,k}\end{aligned}$$

Now for the second term on RHS, we note that

$$(\alpha L - 1)\gamma_t + \gamma_{t+1} \leq \alpha L \gamma_{t+1}, \quad \mathbb{E}[\|\theta_t - \theta^*\|] \leq \sqrt{\mathbb{E}[\|\theta_t - \theta^*\|^2]},$$

and consequently

$$\begin{aligned}
& \frac{1}{(1-2\alpha(\alpha L^2 + \mu))^k} \sum_{t=1}^{k-1} L_{PH}^{(0)} ((\alpha L - 1)\gamma_t + \gamma_{t+1}) \mathbb{E}[\|\theta_t - \theta^*\|] \\
& \leq 2L_{PH}^{(0)} L \alpha^2 \sum_{t=1}^{k-1} \frac{1}{(1-2\alpha(\alpha L^2 + \mu))^{t+1}} \sqrt{\mathbb{E}[\|\theta_t - \theta^*\|^2]} \\
& \leq 2L_{PH}^{(0)} L \alpha^2 \left( \sum_{t=1}^{k-1} \frac{1}{(1-2\alpha(\alpha L^2 + \mu))^{t+1}} \right)^{1/2} \left( \sum_{t=1}^{k-1} \frac{1}{(1-2\alpha(\alpha L^2 + \mu))^{t+1}} \mathbb{E}[\|\theta_t - \theta^*\|^2] \right)^{1/2} \\
& \leq 2L_{PH}^{(0)} L \alpha^2 \cdot \sum_{t=1}^{k-1} \frac{1}{(1-2\alpha(\alpha L^2 + \mu))^{t+1}} \mathbb{E}[\|\theta_t - \theta^*\|^2] + \frac{1}{\alpha L^2 + \mu} \cdot \frac{2L_{PH}^{(0)} L \alpha}{(1-2\alpha(\alpha L^2 + \mu))^k}.
\end{aligned}$$

We also note that

$$\frac{\gamma_k}{(1-2\alpha(\alpha L^2 + \mu))^k} \mathbb{E}[\|\theta_k - \theta^*\|] \leq \alpha \frac{\mathbb{E}[\|\theta_k - \theta^*\|^2]}{(1-2\alpha(\alpha L^2 + \mu))^k} + \frac{\alpha}{(1-2\alpha(\alpha L^2 + \mu))^k}.$$

similarly

$$\frac{\gamma_1}{(1-2\alpha(\alpha L^2 + \mu))^k} \mathbb{E}[\|\theta_0 - \theta^*\|] \leq \alpha \frac{\mathbb{E}[\|\theta_0 - \theta^*\|^2]}{(1-2\alpha(\alpha L^2 + \mu))^1} + \frac{\alpha}{(1-2\alpha(\alpha L^2 + \mu))^1}.$$

and we also define for  $0 \leq t \leq k$

$$c_{3,t} := \frac{1}{\alpha L^2 + \mu} \frac{2\alpha L_{PH}^{(0)} L}{(1-2\alpha(\alpha L^2 + \mu))^t} + \frac{\alpha(\alpha L + 1) L_{PH}^{(0)}}{(1-2\alpha(\alpha L^2 + \mu))^t} + \frac{\alpha L_{PH}^{(0)}}{(1-2\alpha(\alpha L^2 + \mu))}$$

to wrap up all the remainder terms.

Substituting back and rearranging yields

$$\begin{aligned}
\frac{\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2]}{(1-2\alpha(\alpha L^2 + \mu))^k} & \leq \frac{\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + \frac{3}{2} L_{PH}^{(1)} \right)}{(1-2\alpha(\alpha L^2 + \mu))^k} \mathbb{E}[\|\theta_k - \theta^*\|^2] + \sum_{t=1}^{k-1} \frac{\alpha \left( \frac{3}{2} L_{PH}^{(1)} + 2\alpha(1-2\alpha(\alpha L^2 + \mu))^{-1} L L_{PH}^{(0)} \right)}{(1-2\alpha(\alpha L^2 + \mu))^t} \mathbb{E}[\|\theta_t - \theta^*\|^2] + \\
& \quad \left( (1-2\alpha(\alpha L^2 + \mu)) + \alpha L_{PH}^{(1)} (1-2\alpha(\alpha L^2 + \mu))^{-1} \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + c_{1,k} + c_{2,k} + c_{3,k}.
\end{aligned}$$

by choosing  $\alpha \leq \min \left\{ \frac{\sqrt{16\mu^2 + 16L^2} - 4\mu}{8L^2}, \frac{1}{2} \right\}$ , we have  $2\alpha(\alpha L^2 + \mu) \leq \frac{1}{2}$  and thus we have

$$\alpha \left( \frac{3}{2} L_{PH}^{(1)} + 2\alpha(1-2\alpha(\alpha L^2 + \mu))^{-1} L L_{PH}^{(0)} \right) \leq 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right)$$

and again in a similar fashion we have

$$\left( (1-2\alpha(\alpha L^2 + \mu)) + \alpha L_{PH}^{(1)} (1-2\alpha(\alpha L^2 + \mu))^{-1} \right) \leq 2\alpha L_{PH}^{(1)} + 1$$

using above simplifications we can rewrite our upper bound as

$$\frac{\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2]}{(1-2\alpha(\alpha L^2 + \mu))^k} \leq 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \sum_{t=1}^k \frac{\mathbb{E}[\|\theta_t - \theta^*\|^2]}{(1-2\alpha(\alpha L^2 + \mu))^t} + \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + c_{1,k} + c_{2,k} + c_{3,k}.$$

For solving the above recursion, we first define  $S_t := 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \sum_{l=1}^t \frac{\mathbb{E}[\|\theta_l - \theta^*\|^2]}{(1 - 2\alpha(\alpha L^2 + \mu))^l}$  for  $0 \leq t \leq k$ . Also we use  $C_t := c_{1,t} + c_{2,t} + c_{3,t}$  and  $C'_t = \sum_{l=1}^t C_{l-1}$  for  $0 \leq t \leq k$ , defining constant terms. Now we can write

$$\frac{\mathbb{E}[\|\theta_{t+1} - \theta^*\|^2]}{(1 - 2\alpha(\alpha L^2 + \mu))^t} \leq S_t + \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + C_{t-1}.$$

using this expansion, we can write for  $S_k$

$$\begin{aligned} S_k &= 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \sum_{t=1}^k \frac{\mathbb{E}[\|\theta_t - \theta^*\|^2]}{(1 - 2\alpha(\alpha L^2 + \mu))^t} \\ &= 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \sum_{t=1}^k \left[ S_{t-1} + \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + C_{t-1} \right] \\ &= 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \sum_{t=1}^k S_{t-1} + k \left( 2\alpha L_{PH}^{(1)} + 1 \right) \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + C'_k \end{aligned}$$

to solve  $S_k$ , we define  $C''_t := \sum_{l=1}^t l \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + C'_l$  for  $1 \leq t \leq k$ . Now we can write  $S_k$  as

$$S_k = C''_{k-1} + \sum_{t=1}^{k-1} \left( \frac{(t-1)t}{2} + 1 \right) \left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \right)^t C''_{k-t}$$

solving for  $C''_t$ , we have

$$C''_t = \frac{t(t+1)}{2} \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + \sum_{l=1}^t C'_l \leq 2t^2 \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + \mathcal{O}(t)$$

where the inequality follows from the fact that  $C_t = \mathcal{O}(1)$  for each  $0 \leq t \leq k$ . Plugging in the above in  $S_k$  gives us

$$\begin{aligned} S_k &\leq \left( 2(k-1)^2 \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + \mathcal{O}(k-1) \right) + \\ &\quad \sum_{t=1}^{k-1} \left( \frac{(t-1)t}{2} + 1 \right) \left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \right)^t \left[ 2t^2 \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + \mathcal{O}(t) \right] \\ &\leq \left( 2(k-1)^2 \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + \mathcal{O}(k-1) \right) + \\ &\quad \sum_{t=1}^{k-1} 4t^2 \left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \right)^t \left[ \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + \mathcal{O}(t) \right] \\ &\leq \mathcal{O}(k^2) \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] \left( 1 + \sum_{t=1}^{k-1} \left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \right)^t \right) \end{aligned}$$

defining  $Q_k := \left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \right) \frac{\left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) \right)^{k-1} - 1}{\left( 4\alpha \left( (\alpha L + 1) L_{PH}^{(0)} + L_{PH}^{(1)} \right) - 1 \right)}$  and plugging in the above upper bound to our error bound, we know constant  $c_{4,k}$  exists that we can write

$$\frac{\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2]}{(1 - 2\alpha(\alpha L^2 + \mu))^k} \leq c_{4,k} \cdot Q_k \cdot k^2 \left( 2\alpha L_{PH}^{(1)} + 1 \right) \mathbb{E}[\|\theta_0 - \theta^*\|^2] + C_k$$



## 2.2 General Case

In this case, we assume that the moment bound in [??] has been proven for  $k \leq n-1$ , we now proceed to show that the desired moment convergence holds for  $n$  with  $2 \leq n \leq p$ .

We start with the following decomposition of  $\|\theta_{k+1} - \theta^*\|^{2n}$

$$\begin{aligned} \|\theta_{k+1} - \theta^*\|^{2n} &= \left( \|\theta_k - \theta^*\|^2 + 2\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle + \alpha^2 \|g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k)\|^2 \right)^n \\ &= \sum_{\substack{i,j,l \\ i+j+l=n}} \binom{n}{i,j,l} \|\theta_k - \theta^*\|^{2i} \left( 2\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle \right)^j \left( \alpha \|g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k)\|^2 \right)^l \end{aligned}$$

We note the following cases.

1.  $i = n, j = l = 0$ . In this case, the summand is simply  $\|\theta_k - \theta^*\|^{2i}$ .
2. When  $i = n-1, j = 1$  and  $l = 0$ . In this case, the summand is of order  $\alpha$ , i.e.,

$$2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle^j \|\theta_k - \theta^*\|^{2(n-1)}.$$

We can further decompose it as

$$\begin{aligned} &2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)} \\ &= \underbrace{2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) + \xi_{k+1}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)}}_{T_1} + \underbrace{2n\alpha \langle \theta_k - \theta^*, \bar{g}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)}}_{T_2}. \end{aligned}$$

Note that, when  $(X_k)$  is i.i.d or from a martingale noise, we have

$$\mathbb{E}[T_1 | \theta_k] = 0$$

However, when  $(X_k)$  is Markovian, the above inequality does not hold and  $T_1$  requires careful analysis.

Nonetheless, under the strong monotonicity assumption, we have

$$T_2 \leq -2n\alpha\mu \|\theta_k - \theta^*\|^{2n}.$$

3. For the remaining terms, we see that they are of higher orders of  $\alpha$ . Therefore, when  $\alpha$  is selected sufficiently small, these terms do not raise concern.

Therefore, to prove the desired moment bound, we spend the remaining section analyzing  $T_1$ . Immediately, we note that

$$\begin{aligned} \mathbb{E}[T_1] &= \mathbb{E} \left[ 2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) + \mathbb{E}[\xi_{k+1}(\theta_k) | \theta_k] \rangle \|\theta_k - \theta^*\|^{2(n-1)} \right] \\ &= 2n\alpha \mathbb{E} \left[ \underbrace{\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)}}_{T'_1} \right]. \end{aligned}$$

Subsequently, we focus on analyzing  $T'_1$ ; but before that, we write the general recursion of the error bound. First, we define  $T'_{1,t} := \langle \theta_t - \theta^*, g(\theta_t, X_{t+1}) - \bar{g}(\theta_t) \rangle \|\theta_t - \theta^*\|^{2(n-1)}$  to make  $T'_1$  dependent on the

iteration index. Now, following the above decomposition and taking the expectations, we have:

$$\mathbb{E}\|\theta_{k+1} - \theta^*\|^{2n} \leq \mathbb{E}\|\theta_k - \theta^*\|^{2n} + \mathbb{E}\left[T'_{1,k}\right] - 2n\alpha\mu\mathbb{E}\|\theta_k - \theta^*\|^{2n} + o(\alpha) = (1 - 2n\alpha\mu)\mathbb{E}\|\theta_k - \theta^*\|^{2n} + \mathbb{E}\left[T'_{1,k}\right] + o(\alpha)$$

similarly to the previous case we define  $\gamma_t := 2n\alpha(1 - 2n\alpha\mu)^{k-t}$  for  $0 \leq t \leq k$ . Solving the above recursion will give us

$$\mathbb{E}\|\theta_{k+1} - \theta^*\|^{2n} \leq \sum_{t=0}^k \gamma_t \mathbb{E}\left[T'_{1,t}\right] + \gamma_0 \mathbb{E}\|\theta_0 - \theta^*\|^{2n} + o(\alpha)$$

We have to upper bound the first term in the RHS above. For this purpose, we use a similar decomposition to our base case analysis:

$$\sum_{t=0}^k \gamma_t \mathbb{E}\left[T'_{1,t}\right] = \sum_{t=0}^k \gamma_t \langle \theta_t - \theta^*, g(\theta_t, X_{t+1}) - \bar{g}(\theta_t) \rangle \|\theta_t - \theta^*\|^{2(n-1)} = \mathbb{E}[A_1 + A_2 + A_3 + A_4 + A_5]$$

with

$$\begin{aligned} A_1 &:= \sum_{t=1}^k \gamma_t \langle \theta_t - \theta^*, \hat{g}(\theta_t, X_{t+1}) - P_{\theta_t} \hat{g}(\theta_t, X_t) \rangle \|\theta_t - \theta^*\|^{2(n-1)}, \\ A_2 &:= \sum_{t=1}^k \gamma_t \langle \theta_t - \theta^*, P_{\theta_t} \hat{g}(\theta_t, X_t) - P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t) \rangle \|\theta_t - \theta^*\|^{2(n-1)}, \\ A_3 &:= \sum_{t=1}^k \gamma_t \langle \theta_t - \theta_{t-1}, P_{\theta_{t-1}} \hat{g}(\theta_{t-1}, X_t) \rangle, \|\theta_t - \theta^*\|^{2(n-1)} \\ A_4 &:= \sum_{t=1}^k (\gamma_t - \gamma_{t-1}) \langle \theta_{t-1} - \theta^*, P_{\theta_{t-1}} \hat{g}(\theta_{t-1} - \theta^*, X_t) \rangle \|\theta_t - \theta^*\|^{2(n-1)}, \\ A_5 &:= \gamma_0 \langle \theta_0 - \theta^*, \hat{g}(\theta_0, X_0) \rangle \|\theta_0 - \theta^*\|^{2(n-1)} + \gamma_k \langle \theta_k - \theta^*, P_{\theta_k} \hat{g}(\theta_k, X_{k+1}) \rangle \|\theta_k - \theta^*\|^{2(n-1)} \end{aligned}$$

## References

- [1] B. Karimi, B. Miasojedow, E. Moulines, and H.-T. Wai. Non-asymptotic analysis of biased stochastic approximation scheme. In *Conference on Learning Theory*, pages 1944–1974. PMLR, 2019.