NONLINEAR MARKOVIAN STOCHASTIC APPROXIMATION

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1 Preliminaries

Notations The Euclidean norm is denoted by $\|.\|$. The lowercase letter c and its derivatives c', c_0, etc. denote universal numerical constants, whose value may change from line to line. As we are primarily interested in dependence of α and k, we adopt the following big-O notation: $\|f\| = \mathcal{O}(h(\alpha,k))$ if it holds that $\|f\| \le s \cdot \|h(\alpha,k)\|$ for some constant $s > 0$.

We use of the following iteration scheme:

$$\theta_{t+1} = \theta_t + \alpha\left(g(\theta_t, X_{t+1}) + \xi_{t+1}(\theta_t)\right)$$

1.1 Assumptions

Assumption 1 *For each $X \in \mathcal{X}$, the function $g(\theta, X)$ is three times continuously differentiable in θ with uniformly bounded first to third derivatives, i.e., $\sup_{\theta\in\mathbb{R}^d} \|g^{(i)}(\theta,X)\| < \infty$ for $i = 1,2,3, X \in \mathcal{X}$. Moreover, there exists a constant $L_1 > 0$ such that (1) $\|g^{(i)}(\theta,X) - g^{(i)}(\theta',X)\| \le L_1$, for all $\theta,\theta' \in \mathbb{R}^d, i = 0,1,2$ and $X \in \mathcal{X}$, and (2) $\|g(0,X)\| \le L_1$ for all $X \in \mathcal{X}$.*

Assumption 1 implies that $g(\theta,X)$ is L_1-Lipschitz w.r.t θ uniformly in X. The above assumption immediately implies that the growth of $\|g\|$ and $\|\bar{g}\|$ will be at most linear in θ, i.e., $\|g(\theta,X)\| \le L_1(\|\theta-\theta^*\|+1)$ and $\|\bar{g}(\theta)\| \le L_1(\|\theta-\theta^*\|+1)$.

Assumption 2 *There exists $\mu > 0$ such that $\langle \theta-\theta', \bar{g}(\theta)-\bar{g}(\theta')\rangle \le -\mu\|\theta-\theta'\|^2, \forall\theta,\theta' \in \mathbb{R}^d$. Consequently, the target equation $\bar{g}(\theta) = 0$ has a unique solution θ^*.*

Denote by $\mathcal{F}_k$ the filtration generated by $\{X_{t+1},\theta_t,\xi_{t+1}\}_{t=0}^{k-1} \cup \{X_{k+1},\theta_k\}$.

Assumption 3 *Let $p \in \mathbb{Z}_+$ be given. The noise sequence $(\xi_k)_{k\ge1}$ is a collection of i.i.d random fields satisfying the following conditions with $L_{2,p} > 0$:*

$$\mathbb{E}[\xi_{k+1}(\theta)|\mathcal{F}_k] = 0 \quad \textit{and} \quad \mathbb{E}^{1/(2p)}\left[\|\xi_1(\theta)\|^{2p}\right] \le L_{2,p}\left(\|\theta-\theta^*\|+1\right), \quad \forall\theta \in \mathbb{R}^d.$$

Define $C(\theta) = \mathbb{E}\left[\xi_1(\theta)^{\otimes 2}\right]$ and assume that $C(\theta)$ is at least twice differentiable. There also exists $M_\epsilon, k_\epsilon \ge 0$ such that for $\theta \in \mathbb{R}^d$, we have $\max_{i=1,2}\|C^{(i)}(\theta)\| \le M_\epsilon\{1+\|\theta-\theta^*\|^{k_\epsilon}\}$. In the sequel, we set $L := L_1 + L_2$, and without loss of generality, we assume $L \ge 1$.

Assumption 4 *There exists a Borel measurable function $\hat{g}: \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^d$ where for each $\theta \in \mathbb{R}^d, X \in \mathcal{X}$,*

$$\hat{g}(\theta,X) - P_\theta\hat{g}(\theta,X) = g(\theta,X) - \bar{g}(\theta).$$

Assumption 5 *There exists $L_{PH}^{(0)} < \infty$ and $L_{PH}^{(1)} < \infty$ such that, for all $\theta \in \mathbb{R}^d$ and $X \in \mathcal{X}$, one has $\|\hat{g}(\theta,X)\| \le L_{PH}^{(0)}$, $\|P_\theta\hat{g}(\theta,X)\| \le L_{PH}^{(0)}$. Moreover, for $(\theta,\theta') \in \mathcal{H}^2$,*

$$\sup_{X\in\mathcal{X}} \|P_\theta\hat{g}(\theta,X) - P_{\theta'}\hat{g}(\theta',X)\| \le L_{PH}^{(1)}\|\theta-\theta'\|.$$

Assumption 6 *For any $\theta,\theta' \in \mathbb{R}^d$, we have $\sup_{X\in\mathcal{X}}\|P_\theta(X,.) - P_{\theta'}(X,.)\|_{TV} \le L_P\|\theta-\theta'\|$.*

Assumption 7 *For any $\theta,\theta' \in \mathbb{R}^d$, we have $\sup_{X\in\mathcal{X}}\|g(\theta,X) - g(\theta',X)\| \le L_H\|\theta-\theta'\|$.*

Assumption 8 *There exists $\rho < 1$, $K_P < \infty$ such that*

$$\sup_{\theta \in \mathbb{R}^d, X \in \mathcal{X}} \|P_\theta^n(X,.) - \pi_\theta(.)\|_{TV} \leq \rho^n K_P,$$

Lemma 1 *Assume that assumptions 6-8 hold. Then, for any $\theta \in \mathbb{R}^d$ and $X \in \mathcal{X}$,*

$$\|\hat{g}(\theta, X)\| \leq \frac{\sigma K_P}{1-\rho},$$

$$\|P_\theta \hat{g}(\theta, X)\| \leq \frac{\sigma \rho K_P}{1-\rho}.$$

Moreover, for any $\theta, \theta' \in \mathbb{R}^d$ and $X \in \mathcal{X}$,

$$\|P_\theta \hat{g}(\theta, X) - P_{\theta'} \hat{g}\left(\theta', X\right)\| \leq \|\theta - \theta'\|,$$

where

$$L_{PH}^{(1)} = \frac{K_P^2 \sigma L_P}{(1-\rho)^2}(2 + K_P) + \frac{K_P}{1-\rho} L_H.$$

Proof of this lemma can be found in [1], Lemma 7.

2 Error Bound

2.1 Base Case

For the base case analysis, we can write:

$$\begin{aligned}
&\mathbb{E}\left[\|\theta_{k+1} - \theta^*\|^2\right] - \mathbb{E}\left[\|\theta_k - \theta^*\|^2\right] = \\
&2\alpha \mathbb{E}\left[\langle \theta_k - \theta^*, g(\theta_k, X_{k+1})\rangle\right] + \alpha^2 \mathbb{E}\left[\|g(\theta_k, X_{k+1})\|^2\right] + \alpha^2 \mathbb{E}\left[\|\xi_{k+1}(\theta_k)\|^2\right] = \\
&2\alpha \mathbb{E}\left[\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k)\rangle\right] + 2\alpha \mathbb{E}\left[\langle \theta_k - \theta^*, \bar{g}(\theta_k)\rangle\right] + \alpha^2 \mathbb{E}\left[\|g(\theta_k, X_{k+1})\|\right] + \alpha^2 \mathbb{E}\left[\|\xi_{k+1}(\theta_k)\|^2\right].
\end{aligned}$$

It is easy to see that under Strong Monotonicity assumption, we have

$$\langle \theta_k - \theta^*, \bar{g}(\theta_k)\rangle = \langle \theta_k - \theta^*, \bar{g}(\theta_k) - \bar{g}\left(\theta^*\right)\rangle \leq -\mu \|\theta_k - \theta^*\|^2.$$

Additionally, under Assumption 1 and 3, we have the following upper bound

$$\begin{aligned}
&\alpha^2\left(\mathbb{E}\left[\|g(\theta_k, X_{k+1})\|^2\right] + \mathbb{E}\left[\|\xi_{k+1}(\theta_k)\|^2\right]\right) \\
&\leq \alpha^2\left(L_1^2 \mathbb{E}\left[\left(\|\theta_k - \theta^*\| + 1\right)^2\right] + L_2^2 \mathbb{E}\left[\left(\|\theta_k - \theta^*\| + 1\right)^2\right]\right) \\
&\leq 2\alpha^2 L^2\left(\mathbb{E}\left[\|\theta_k - \theta^*\|^2\right] + 1\right).
\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \le \left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]+2\alpha^2L^2+2\alpha\mathbb{E}\left[\left\langle\theta_k-\theta^*, g\left(\theta_k, X_{k+1}\right)-\bar{g}\left(\theta_k\right)\right\rangle\right]$$

Solving this recursion gives us the following inequality:

$$\begin{aligned}\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \le &\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k+1}\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]\\ &+\sum_{t=0}^{k}\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t 2\alpha^2L^2\\ &+\sum_{t=0}^{k}2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k-t}\mathbb{E}\left[\left\langle\theta_t-\theta^*, g\left(\theta_t, X_{t+1}\right)-\bar{g}\left(\theta_t\right)\right\rangle\right].\end{aligned}$$

For notational simplicity we define $\gamma_t := 2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{k-t}$ for $0 \le t \le k$.

The second term above is just a geometric series which is of $\mathcal{O}(\alpha)$.

Now, we can upper bound the third summand using below decomposition:

$$\mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*, g(\theta_t, X_{t+1})-\bar{g}(\theta_t)\right\rangle\right] = \mathbb{E}\left[A_1+A_2+A_3+A_4+A_5\right]$$

with

$$\begin{aligned}A_1 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, \hat{g}\left(\theta_t, X_{t+1}\right)-P_{\theta_t}\hat{g}\left(\theta_t, X_t\right)\right\rangle,\\ A_2 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, P_{\theta_t}\hat{g}\left(\theta_t, X_t\right)-P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}, X_t\right)\right\rangle,\\ A_3 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta_{t-1}, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}, X_t\right)\right\rangle,\\ A_4 &:= \sum_{t=1}^{k}\left(\gamma_t-\gamma_{t-1}\right)\left\langle\theta_{t-1}-\theta^*, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}-\theta^*, X_t\right)\right\rangle,\\ A_5 &:= \gamma_0\left\langle\theta_0-\theta^*, \hat{g}\left(\theta_0, X_0\right)\right\rangle+\gamma_k\left\langle\theta_k-\theta^*, P_{\theta_k}\hat{g}\left(\theta_k, X_{k+1}\right)\right\rangle\end{aligned}$$

For A_1, we note that $\hat{g}\left(\theta_t, X_{t+1}\right)-P_{\theta_t}\hat{g}\left(\theta_t, X_t\right)$ is a martingale difference sequence [cf. ?] and therefore we have $\mathbb{E}[A_1]=0$ by taking the total expectation.

For A_2, applying Cauchy-Schwarz inequality and **??**, we have

$$\begin{aligned}A_2 &\le \sum_{t=1}^{k}L_{PH}^{(1)}\gamma_t\|\theta_t-\theta^*\|\,\|\theta_t-\theta_{t-1}\|\\ &= \sum_{t=1}^{k}\alpha L_{PH}^{(1)}\gamma_t\|\theta_t-\theta^*\|\,\|g(\theta_t, X_{t+1})+\xi_{t+1}(\theta_t)\|\\ &\le \sum_{t=1}^{k}\alpha L_{PH}^{(1)}\gamma_t\|\theta_t-\theta^*\|\left(L_1\left(\|\theta_t-\theta^*\|+1\right)+L_2\left(\|\theta_t-\theta^*\|+1\right)\right)\\ &\le \sum_{t=1}^{k}\frac{\alpha L_{PH}^{(1)}\gamma_t}{2}\left(3\|\theta_t-\theta^*\|^2+1\right)\end{aligned}$$

where the third line follows from the Lipschitzness condition and the assumption of

$$\mathbb{E}^{1/2}\left[\|\xi_{t+1}(\theta_t)\|^2|\mathscr{F}_t\right] \leq L_2(\|\theta_t\|+1)$$

also, last line follows from the identity $u \leq \frac{1}{2}(1+u^2)$.

For A_3, we obtain

$$\begin{aligned}
A_3 &\leq \sum_{t=1}^{k}\gamma_t\|\theta_t-\theta_{t-1}\|\,\|P_{\theta_{t-1}}\hat{g}(\theta_{t-1},X_t)\| \\
&\leq \sum_{t=1}^{k}\alpha L_{PH}^{(0)}\gamma_t\|g(\theta_t,X_{t+1})+\xi_{t+1}(\theta_t)\| \\
&\leq \sum_{t=1}^{k}\alpha L_{PH}^{(0)}\gamma_t\left(L_1\left(\|\theta_t-\theta^*\|+1\right)+L_2\left(\|\theta_t-\theta^*\|+1\right)\right) \\
&\leq \sum_{t=1}^{k}\alpha L L_{PH}^{(0)}\gamma_t\left(\|\theta_t-\theta^*\|+1\right)
\end{aligned}$$

where second line follows from **??** and third line is similarly done to the previous part, using Lipschitzness condition and noise assumption.

For A_4, we have

$$\begin{aligned}
A_4 &\leq \sum_{t=1}^{k}|\gamma_t-\gamma_{t-1}|\,\|\theta_{t-1}-\theta^*\|\,\|P_{\theta_{t-1}\hat{g}(\theta_{t-1},X_t)}\| \\
&\leq \sum_{t=1}^{k}L_{PH}^{(0)}|\gamma_t-\gamma_{t-1}|\,\|\theta_{t-1}-\theta^*\|
\end{aligned}$$

Finally, for A_5, we obtain

$$A_5 \leq L_{PH}^{(0)}\left(\gamma_0\|\theta_0-\theta^*\|+\gamma_k\|\theta_k-\theta^*\|\right)$$

which follows from Cacuhy-Scwarz inequality and **??**.

Combining the above terms and taking expectations, gives us:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*,g\left(\theta_t,X_{t+1}-\bar{g}(\theta_t)\right)\right\rangle\right] \leq{}& \sum_{t=1}^{k}\frac{\alpha L_{PH}^{(1)}\gamma_t}{2}\left(1+3\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\right)+\sum_{t=1}^{k}\alpha L L_{PH}^{(0)}\gamma_t\left(\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+1\right)+ \\
&\sum_{t=0}^{k-1}L_{PH}^{(0)}|\gamma_t-\gamma_{t+1}|\,\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+L_{PH}^{(0)}\left(\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]\right)
\end{aligned}$$

now it should be noticed that as long as the α satisfies $\alpha \leq \frac{\mu}{L^2}$, we have $\gamma_t \leq \gamma_{t+1}$. Thus, we can simplify the above upper bound and write it this way:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*,g\left(\theta_t,X_{t+1}-\bar{g}(\theta_t)\right)\right\rangle\right] \leq{}& \sum_{t=1}^{k}\frac{\alpha L_{PH}^{(1)}\gamma_t}{2}\left(1+3\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\right)+ \\
&\sum_{t=1}^{k-1}L_{PH}^{(0)}\left((\alpha L-1)\gamma_t+\gamma_{t+1}\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+ \\
&\sum_{t=1}^{k}\alpha L L_{PH}^{(0)}\gamma_t+L_{PH}^{(0)}\left(\gamma_1\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+(\alpha L+1)\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]\right)
\end{aligned}$$

Hence, using the derived upper bounds from the above terms, we have:

$$\begin{aligned}\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \leq& \sum_{t=1}^{k}\frac{\alpha L_{PH}^{(1)}\gamma_t}{2}\left(1+3\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]\right)+\sum_{t=1}^{k-1}L_{PH}^{(0)}\left((\alpha L-1)\gamma_t+\gamma_{t+1}\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+\\&\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_1\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+(\alpha L+1)L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+\\&\left(\frac{L}{L_{PH}^{(0)}}+1\right)\frac{\gamma_1\left(1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k\right)}{\left(1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\right)}+c_t\cdot\alpha\end{aligned}$$

for further notation simplicity we define $c_{1,t}:=\left(\frac{L}{L_{PH}^{(0)}}+1\right)\frac{\gamma_1\left(1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t\right)}{\left(1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\right)}+c_t\cdot\alpha$ for $0\leq t\leq k$. Now to write down this upper bound in a way in which it only depends on $\|\theta_0-\theta^*\|$related terms and constants, we can write:

$$\begin{aligned}\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \leq& \sum_{t=1}^{k}\left[\frac{3\alpha L_{PH}^{(1)}\gamma_t}{2}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\frac{\alpha L_{PH}^{(1)}\gamma_t}{2}\right]+\sum_{t=1}^{k-1}L_{PH}^{(0)}\left((\alpha L-1)\gamma_t+\gamma_{t+1}\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+\\&\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_1\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+(\alpha L+1)L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}\\=&\sum_{t=1}^{k}\frac{3\alpha L_{PH}^{(1)}\gamma_t}{2}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\frac{\alpha L_{PH}^{(1)}}{2}\sum_{t=1}^{k}\gamma_t+\sum_{t=1}^{k-1}L_{PH}^{(0)}\left((\alpha L-1)\gamma_t+\gamma_{t+1}\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+\\&\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_1\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+(\alpha L+1)L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}\\=&\sum_{t=1}^{k}\frac{3\alpha L_{PH}^{(1)}\gamma_t}{2}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\sum_{t=1}^{k-1}L_{PH}^{(0)}\left((\alpha L-1)\gamma_t+\gamma_{t+1}\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+\\&\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_1\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+(\alpha L+1)L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}+\\&\frac{L_{PH}^{(1)}\gamma_1\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k\right]}{4\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\right]}\end{aligned}$$

where the last equality follows from the definition of γ_ts. Similarly we define $c_{2,t}:=\frac{L_{PH}^{(1)}\gamma_1\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t\right]}{4\left[1-\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\right]}$ for $0\leq t\leq k$. So we can write it as

$$\begin{aligned}\mathbb{E}\left[\|\theta_{k+1}-\theta^*\|^2\right] \leq& \sum_{t=1}^{k}\frac{3\alpha L_{PH}^{(1)}\gamma_t}{2}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\sum_{t=1}^{k-1}L_{PH}^{(0)}\left((\alpha L-1)\gamma_t+\gamma_{t+1}\right)\mathbb{E}\left[\|\theta_t-\theta^*\|\right]+\\&\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)\gamma_0\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+L_{PH}^{(0)}\gamma_1\mathbb{E}\left[\|\theta_0-\theta^*\|\right]+(\alpha L+1)L_{PH}^{(0)}\gamma_k\mathbb{E}\left[\|\theta_k-\theta^*\|\right]+c_{1,k}+c_{2,k}\end{aligned}$$

Now for the second term on RHS, we note that

$$(\alpha L-1)\gamma_t+\gamma_{t+1}\leq\alpha L\gamma_{t+1},\quad \mathbb{E}\left[\|\theta_t-\theta^*\|\right]\leq\sqrt{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]},$$

and consequently

$$\begin{aligned}
&\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^k}\sum_{t=1}^{k-1} L_{PH}^{(0)}\big((\alpha L-1)\gamma_t+\gamma_{t+1}\big)\mathbb{E}\big[\|\theta_t-\theta^*\|\big]\\
&\le 2L_{PH}^{(0)}L\alpha^2\sum_{t=1}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\sqrt{\mathbb{E}\big[\|\theta_t-\theta^*\|^2\big]}\\
&\le 2L_{PH}^{(0)}L\alpha^2\Big(\sum_{t=1}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\Big)^{1/2}\Big(\sum_{t=1}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\mathbb{E}\big[\|\theta_t-\theta^*\|^2\big]\Big)^{1/2}\\
&\le 2L_{PH}^{(0)}L\alpha^2\cdot\sum_{t=1}^{k-1}\frac{1}{(1-2\alpha(-\alpha L^2+\mu))^{t+1}}\mathbb{E}\big[\|\theta_t-\theta^*\|^2\big]+\frac{1}{-\alpha L^2+\mu}\cdot\frac{2L_{PH}^{(0)}L\alpha}{(1-2\alpha(-\alpha L^2+\mu))^k}.
\end{aligned}$$

We also note that

$$\frac{\gamma_k}{(1-2\alpha(-\alpha L^2+\mu))^k}\mathbb{E}\big[\|\theta_k-\theta^*\|\big]\le\alpha\frac{\mathbb{E}\big[\|\theta_k-\theta^*\|^2\big]}{(1-2\alpha(-\alpha L^2+\mu))^k}+\frac{\alpha}{(1-2\alpha(-\alpha L^2+\mu))^k}.$$

similarly

$$\frac{\gamma_1}{(1-2\alpha(-\alpha L^2+\mu))^k}\mathbb{E}\big[\|\theta_0-\theta^*\|\big]\le\alpha\frac{\mathbb{E}\big[\|\theta_0-\theta^*\|^2\big]}{(1-2\alpha(-\alpha L^2+\mu))^1}+\frac{\alpha}{(1-2\alpha(-\alpha L^2+\mu))^1}.$$

and we also define for $0\le t\le k$

$$c_{3,t}:=\frac{1}{\alpha L^2+\mu}\frac{2\alpha L_{PH}^{(0)}L}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}+\frac{\alpha\left(\alpha L+1\right)L_{PH}^{(0)}}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}+\frac{\alpha L_{PH}^{(0)}}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)}$$

to wrap up all the remainder terms.

Substituting back and rearranging with also defining $c'_{2,k}:=\frac{c_{2,k}}{(1-2\alpha(-\alpha L^2+\mu))^k}$ and $c'_{1,k}:=\frac{c_{1,k}}{(1-2\alpha(-\alpha L^2+\mu))^k}$, yields

$$\begin{aligned}
\frac{\mathbb{E}\big[\|\theta_{k+1}-\theta^*\|^2\big]}{(1-2\alpha(-\alpha L^2+\mu))^k}\le{}&\frac{\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+\frac{3}{2}L_{PH}^{(1)}\right)}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^k}\mathbb{E}\left[\|\theta_k-\theta^*\|^2\right]+\sum_{t=1}^{k-1}\frac{\alpha\left(\frac{3}{2}L_{PH}^{(1)}+2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{-1}LL_{PH}^{(0)}\right)}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]+\\
&\left(\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)+\alpha L_{PH}^{(1)}\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{-1}\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+c'_{1,k}+c'_{2,k}+c_{3,k}.
\end{aligned}$$

by choosing $\alpha\le min\left\{\frac{\sqrt{4\mu^2-4L^2}-4\mu}{4L^2},\frac{1}{2}\right\}$, we have $2\alpha\left(-\alpha L^2+\mu\right)\le\frac{1}{2}$ and thus we have

$$\alpha\left(\frac{3}{2}L_{PH}^{(1)}+2\alpha\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{-1}LL_{PH}^{(0)}\right)\le 4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)$$

and again in a similar fashion we have

$$\left(\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)+\alpha L_{PH}^{(1)}\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^{-1}\right)\le 2\alpha L_{PH}^{(1)}+1$$

using above simplifications we can rewrite our upper bound as

$$\frac{\mathbb{E}\big[\|\theta_{k+1}-\theta^*\|^2\big]}{(1-2\alpha(-\alpha L^2+\mu))^k}\le 4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\sum_{t=1}^{k}\frac{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}+\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+c_{1,k}+c_{2,k}+c_{3,k}.$$

For solving the above recursion, we first define $S_t := 4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\sum_{l=1}^{t}\frac{\mathbb{E}[\|\theta_l-\theta^*\|^2]}{(1-2\alpha(-\alpha L^2+\mu))^l}$ for $0\le t\le k$. Also we use $C_t := c'_{1,t}+c'_{2,t}+c_{3,t}$ and $C'_t=\sum_{l=1}^{t}C_{l-1}$ for $0\le t\le k$, defining constant terms. Now we can write

$$\frac{\mathbb{E}\left[\|\theta_{t+1}-\theta^*\|^2\right]}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}\le S_t+\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+C_{t-1}.$$

using this expansion, we can write for S_k

$$\begin{aligned}
S_k &= 4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\sum_{t=1}^{k}\frac{\mathbb{E}\left[\|\theta_t-\theta^*\|^2\right]}{\left(1-2\alpha\left(-\alpha L^2+\mu\right)\right)^t}\\
&= 4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\sum_{t=1}^{k}\left[S_{t-1}+\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+C_{t-1}\right]\\
&= 4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\sum_{t=1}^{k}S_{t-1}+k\left(2\alpha L_{PH}^{(1)}+1\right)\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+C'_k
\end{aligned}$$

to solve S_k, we define $C''_t := \sum_{l=1}^{t} l\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+C'_l$ for $1\le t\le k$. Now we can write S_k as

$$S_k = C''_{k-1}+\sum_{t=1}^{k-1}\left(\frac{(t-1)\,t}{2}+1\right)\left(4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\right)^t C''_{k-t}$$

solving for C''_t, we have

$$C''_t=\frac{t\,(t+1)}{2}\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\sum_{l=1}^{t}C'_l\le 2t^2\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\mathcal{O}(t)\cdot\alpha$$

where the inequality follows from the fact that $C_t=\mathcal{O}(\alpha)$ for each $0\le t\le k$. Plugging in the above in S_k gives us

$$\begin{aligned}
S_k &\le \left(2(k-1)^2\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\mathcal{O}(k-1)\right)+\\
&\sum_{t=1}^{k-1}\left(\frac{(t-1)\,t}{2}+1\right)\left(4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\right)^t\left[2t^2\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\mathcal{O}(t)\cdot\alpha\right]\\
&\le \left(2(k-1)^2\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\mathcal{O}(k-1)\right)+\\
&\sum_{t=1}^{k-1}4t^2\left(4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\right)^t\left[\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+\mathcal{O}(t)\cdot\alpha\right]\\
&\le \mathcal{O}\left(k^2\right)\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]\left(1+\sum_{t=1}^{k-1}\left(4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\right)^t\right)+\mathcal{O}\left(k^3\right)\cdot\alpha\cdot\sum_{t=1}^{k-1}\left(4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\right)^t+\mathcal{O}(k)
\end{aligned}$$

defining $Q_k := \left(4\alpha\left((\alpha L+1)\,L_{PH}^{(0)}+L_{PH}^{(1)}\right)\right)\frac{\left(4\alpha((\alpha L+1))L_{PH}^{(0)}+L_{PH}^{(1)}\right)^{k-1}-1}{\left(4\alpha\left((\alpha L+1)L_{PH}^{(0)}+L_{PH}^{(1)}\right)-1\right)}$ and plugging in the above upper bound to our error bound, we know constants $c_{4,k}$ and $c_{5,k}$ exists that we can write

$$\frac{\mathbb{E}[\|\theta_{k+1}-\theta^*\|^2]}{(1-2\alpha(-\alpha L^2+\mu))^k}\le c_{4,k}\cdot Q_k\left(k^2\left(2\alpha L_{PH}^{(1)}+1\right)\mathbb{E}\left[\|\theta_0-\theta^*\|^2\right]+k^3\cdot\alpha\right)+c_{5,k}\cdot k$$

2.2 General Case

In this case, we assume that the moment bound in [??] has been proven for $k \leq n-1$, we now proceed to show that the desired moment convergence holds for n with $2 \leq n \leq p$.

We start with the following decomposition of $\|\theta_{k+1} - \theta^*\|^{2n}$

$$\begin{aligned}
\|\theta_{k+1} - \theta^*\|^{2n} &= \left(\|\theta_k - \theta^*\|^2 + 2\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle + \alpha^2 \|g(\theta_x, X_{k+1}) + \xi_{k+1}(\theta_k)\|^2\right)^n \\
&= \sum_{\substack{i,j,l \\ i+j+l=n}} \binom{n}{i,j,l} \|\theta_k - \theta^*\|^{2i} \left(2\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle\right)^j \left(\alpha \|g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k)\|\right)^{2l}
\end{aligned}$$

We note the following cases.

1. $i = n,\ j = l = 0$. In this case, the summand is simply $\|\theta_k - \theta^*\|^{2i}$.
2. When $i = n-1, j = 1$ and $l = 0$. In this case, the summand is of order α, i.e.,

   $$\alpha 2n \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle^j \|\theta_k - \theta^*\|^{2(n-1)}.$$

   We can further decompose it as

   $$\begin{aligned}
   &2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) + \xi_{k+1}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)} \\
   &= \underbrace{2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) + \xi_{k+1}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)}}_{T_1} + \underbrace{2n\alpha \langle \theta_k - \theta^*, \bar{g}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)}}_{T_2}.
   \end{aligned}$$

   Note that, when (X_k) is i.i.d or from a martingale noise, we have

   $$\mathbb{E}[T_1 | \theta_k] = 0$$

   However, when (X_k) is Markovian, the above inequality does not hold and T_1 requires careful analysis.
   Nonetheless, under the strong monotonicity assumption, we have

   $$T_2 \leq -2n\alpha\mu \|\theta_k - \theta^*\|^{2n}.$$

3. For the remaining terms, we see that they are of higher orders of α. Therefore, when α is selected sufficiently small, these terms do not raise concern.

Therefore, to prove the desired moment bound, we spend the remaining section analyzing T_1. Immediately, we note that

$$\begin{aligned}
\mathbb{E}[T_1] &= \mathbb{E}\left[2n\alpha \langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) + \mathbb{E}[\xi_{k+1}(\theta_k) | \theta_k] \rangle \|\theta_k - \theta^*\|^{2(n-1)}\right] \\
&= 2n\alpha \mathbb{E}\left[\underbrace{\langle \theta_k - \theta^*, g(\theta_k, X_{k+1}) - \bar{g}(\theta_k) \rangle \|\theta_k - \theta^*\|^{2(n-1)}}_{T_1'}\right].
\end{aligned}$$

Subsequently, we focus on analyzing T_1'; but before that, we write the general recursion of the error bound. First, we define $T_{1,t}' := \langle \theta_t - \theta^*, g(\theta_t, X_{t+1}) - \bar{g}(\theta_t) \|\theta_t - \theta^*\|^{2(n-1)} \rangle$ to make T_1' dependent on the iteration index. Now, following the above decomposition and taking the expectations, we have:

$$\mathbb{E}\|\theta_{k+1}-\theta^*\|^{2n} \leq \mathbb{E}\|\theta_k-\theta^*\|^{2n} + \mathbb{E}\left[T'_{1,k}\right] - 2n\alpha\mu\mathbb{E}\|\theta_k-\theta^*\|^{2n} + o(\alpha) = \left(1-2n\alpha\mu\right)\mathbb{E}\|\theta_k-\theta^*\|^{2n} + \mathbb{E}\left[T'_{1,k}\right] + o(\alpha)$$

similarly to the previous case we define $\gamma_t := 2n\alpha\left(1-2n\alpha\mu\right)^{k-t}$ for $0 \leq t \leq k$. Solving the above recursion will give us

$$\mathbb{E}\|\theta_{k+1}-\theta^*\|^{2n} \leq \sum_{t=0}^{k}\gamma_t\mathbb{E}\left[T'_{1,t}\right] + \gamma_0\mathbb{E}\|\theta_0-\theta^*\|^{2n} + o(\alpha)$$

We have to upper bound the first term in the RHS above. For this purpose, we use a similar decomposition to our base case analysis:

$$\sum_{t=0}^{k}\gamma_t\mathbb{E}\left[T'_{1,t}\right] = \sum_{t=0}^{k}\gamma_t\left\langle\theta_t-\theta^*, g\left(\theta_t, X_{t+1}\right)-\bar{g}\left(\theta_t\right)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)} = \mathbb{E}\left[A_1+A_2+A_3+A_4+A_5\right]$$

with

$$\begin{aligned}
A_1 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, \hat{g}\left(\theta_t, X_{t+1}\right)-P_{\theta_t}\hat{g}\left(\theta_t, X_t\right)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)},\\
A_2 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta^*, P_{\theta_t}\hat{g}\left(\theta_t, X_t\right)-P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}, X_t\right)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)},\\
A_3 &:= \sum_{t=1}^{k}\gamma_t\left\langle\theta_t-\theta_{t-1}, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}, X_t\right)\right\rangle, \|\theta_t-\theta^*\|^{2(n-1)}\\
A_4 &:= \sum_{t=1}^{k}\left(\gamma_t-\gamma_{t-1}\right)\left\langle\theta_{t-1}-\theta^*, P_{\theta_{t-1}}\hat{g}\left(\theta_{t-1}-\theta^*, X_t\right)\right\rangle\|\theta_t-\theta^*\|^{2(n-1)},\\
A_5 &:= \gamma_0\left\langle\theta_0-\theta^*, \hat{g}\left(\theta_0, X_0\right)\right\rangle\|\theta_0-\theta^*\|^{2(n-1)} + \gamma_k\left\langle\theta_k-\theta^*, P_{\theta_k}\hat{g}\left(\theta_k, X_{k+1}\right)\right\rangle\|\theta_k-\theta^*\|^{2(n-1)}
\end{aligned}$$

References

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