Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable and satisfies $f(x,y) = 2024$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$. Then f has at least one critical point.

b) Suppose $p(x)$ and $q(y)$ are polynomials of degree 3 and 4, respectively, without multiple zeros. Then the function $f(x,y) = p(x)q(y)$ (with domain $\mathbb{R}^2$) has at most 12 critical points.

c) The surface in $\mathbb{R}^3$ with equation $\mathrm{e}^{xyz} = x + y + z$ is smooth.

d) The quadric surface in $\mathbb{R}^3$ with equation $(x-y)(y-z) = z-x$ is a hyperbolic paraboloid.

e) Suppose $q\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable and satisfies $q(tx,ty) = t^2 q(x,y)$ for all $(x,y) \in \mathbb{R}^2$ and all $t > 0$. Then $x\, q_x(x,y) + y\, q_y(x,y) = 2\, q(x,y)$ holds for all $(x,y) \in \mathbb{R}^2$.

f) For the differential 1-form $\omega = (2xy + z^2)\,\mathrm{d}x + (2yz + x^2)\,\mathrm{d}y + (2xz + y^2)\,\mathrm{d}z$ and the curve $\gamma(t) = (\cos t, \sin t - \cos t, -\sin t)$, $t \in [0, \pi/2]$ the line integral $\int_\gamma \omega$ is zero.

Question 2 (ca. 11 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^3 + y^3 - 3y^2 - 3xy.$$

a) Determine all critical points of f and their types using the 2nd-order partial derivatives test (Hesse matrix test).

b) Is the graph G_f symmetric with respect to some plane in $\mathbb{R}^3$?

Question 3 (ca. 8 marks)

Using the method of Lagrange multipliers, solve the optimization problem

$$\begin{array}{rl} \text{Maximize} & 2x + 3y + 6z \\ \text{subject to} & x^2 + y^2 + z^2 + xy + xz + yz = 6. \end{array}$$

Note: Required are (i) a proof that the optimization problem has a solution, (ii) the optimal objective value z^*, and (iii) all optimal solutions (x^*, y^*, z^*).

Question 4 (ca. 6 marks)

Evaluate the integral

$$\int_0^\pi \frac{\ln(1 + \cos x)}{\cos x}\,\mathrm{d}x\,.$$

Hint: Show that the function

$$F(b) = \int_0^\pi \frac{\ln(1 + b\cos x)}{\cos x}\,\mathrm{d}x, \quad b \in [-1, 1]$$

is well-defined, differentiable in $(-1,1)$, and continuous at $b=1$ (and $b=-1$ as well). You may use without proof that

$$\int_0^\pi \frac{\mathrm{d}x}{1+b\cos x}=\frac{\pi}{\sqrt{1-b^2}},\quad b\in(-1,1).$$

The proof of this identity, which can be derived from $\int_0^{2\pi}\frac{\mathrm{d}t}{A^2\cos^2 t+B^2\sin^2 t}=\frac{2\pi}{AB}$ (cf. the lecture), gives 2 bonus marks.

Question 5 (ca. 11 marks)

a) Determine the centroid of the region in $\mathbb{R}^3$ consisting of all points above the surface $z=9x^2+4y^2$ and below the plane $18x+16y+z=0$.

b) Determine the area of the surface in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$z^2=2xy,\quad x,y,z\geq 0,\quad x+y\leq 3.$$

Solutions

1 a) True. Since differentiable functions are continuous, f attains on the closed unit disk a minimum and a maximum. These can be equal to 2024, but then f must be constant on the whole disk, which implies that all points inside the disk are critical. Otherwise either the minimum or the maximum (or both) are attained at some point in the open unit disk, and this point(s) must consequently be critical. $\boxed{2}$

b) False. A counter example is $p(x) = x(x-1)(x-2)(x-3)$, $q(y) = y(y-1)(y-2)$. Since $\nabla f(x,y) = \big(p'(x)q(y), p(x)q'(y)\big)$, the 12 points (x,y) with $x \in \{0,1,2,3\}$, $y \in \{0,1,2\}$ are critical. But points (x,y) with $p'(x) = q'(y) = 0$ are critical as well, and there exist such points by Rolle's Theorem, e.g., there exists $x \in (0,1)$ such that $p'(x) = 0$ and $y \in (0,1)$ such that $q'(y) = 0$. (In fact the number of critical points of f is 18, since p' has 3 zeros and q' has 2 zeros; cf. the solution to a related question in Midterm 3.) $\boxed{2}$

c) True. The surface is the 0-level set of $g(x,y,z) = \mathrm{e}^{xyz} - x - y - z$, which has $\nabla g(x,y,z) = (yz\,\mathrm{e}^{xyz} - 1, xz\,\mathrm{e}^{xyz} - 1, xy\,\mathrm{e}^{xyz} - 1)$. A critical point of g must have $xyz \neq 0$. Then $g_x = g_y = 0$ implies $x = y$. By symmetry we obtain $x = y = z$, so that the only critical point is (x,x,x) where x solves $x^2\mathrm{e}^{x^3} = 1$. (Calculus I shows that the solution is unique.) The point (x,x,x) is not on the surface, however, since $g(x,x,x) = \mathrm{e}^{x^3} - 3x = 0$ implies $3x^3 = x^2(3x) = x^2\mathrm{e}^{x^3} = 1$, and further $1 = \left(x^2\mathrm{e}^{x^3}\right)^3 = x^6\mathrm{e}^{3x^3} = x^6\mathrm{e}$, which in turn gives the absurd $\mathrm{e} = x^{-6} = -1/9$. (One can also show that the solution of $x^2\mathrm{e}^{x^3} = 1$ satisfies $0.5 < x < 1$ while $\mathrm{e}^{x^3} = 3x$ has two solutions, which are < 0.5 and > 1, respectively.) $\boxed{2}$

d) False. The change of variables $x' = x - y$, $y' = y - z$, $z' = z$ ($z' = z - x = -x' - y'$ is not possible, because the associated matrix has rank 2) transforms this quadric surface into $x'y' + x' + y' = 0$, which is degenerate, since the equation doesn't involve z'. (In the 4×4 matrix that decides degeneracy the row/column corresponding to z' is zero.) $\boxed{2}$

e) True. Fix the point (x,y) and consider the function $t \mapsto q(tx,ty) = t^2q(x,y)$ on $(0,\infty)$. Differentiating with respect to t on both sides of the equation gives

$$q_x(tx,ty)x + q_y(tx,ty)y = 2t\,q(x,y).$$

(On the left side we have used the chain rule.) Setting $t = 1$ proves the stated identity. $\boxed{2}$

f) True. The form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ is exact in $\mathbb{R}^2$, since $P_y = 2x = Q_x$, $P_z = 2z = R_x$, $Q_z = 2y = R_y$ and $\mathbb{R}^2$ is simply connected. The usual method for finding an antiderivative f gives $f_x = 2xy + z^2 \implies f(x,y,z) = x^2y + z^2x + g(y,z) \implies 2yz + x^2 = f_y = x^2 + g_y \implies g_y = 2yz \implies g(y,z) = y^2z + h(z) \implies f(x,y,z) = x^2y + z^2x + y^2z + h(z) \implies 2xz + y^2 = f_z = 2xz + y^2 + h'(z) \implies h(z) = C$ is a constant. So one particular antiderivative is $f(x,y,z) + x^2y + y^2z + z^2x$. The Fundamental Theorem for Line Integrals then gives

$$\int_\gamma \omega = f\big(\gamma(\pi/2)\big) - f\big(\gamma(0)\big) = f(0,1,-1) - f(1,-1,0) = -1 - (-1) = 0.$$

Computing $\int_\gamma \omega$ directly from the definition seems to be a terrible idea. I have computed the resulting integral with SageMath as $9\int_0^{\pi/2}\cos^2 t\sin t\,\mathrm{d}t - 3\int_0^{\pi/2}\cos t\sin^2 t\,\mathrm{d}t$ $-3\int_0^{\pi/2}\sin^3 t\,\mathrm{d}t = 9\,\frac{1}{3} - 3\,\frac{1}{3} - 3\,\frac{2}{3} = 0$. $\boxed{2}$

Remarks: No marks were assigned for wrong answers and answers without justification.

a) Only few students were able to justify the answer True correctly. Arguments restricting f to the unit circle cannot work, because computing $\nabla f(x_0, y_0)$ requires knowledge of $f(x, y)$ for all points (x, y) in some neighborhood of (x_0, y_0) (i.e., an open disk around (x_0, y_0)). For the same reason, arguments restricting f to line segments cut out of the unit disk can't work. (From $f(x_1, y_1) = f(x_2, y_2) = 2024$ at the end points of the line segment one can conclude that the corresponding directional derivative at some intermediate point must be zero, but "$\nabla f(x, y) = \mathbf{0}$" requires two independent directional derivatives at the *same* point to be zero.) Using polar coordinates also doesn't help, because this only transfers the problem to an equivalent one in the (r, θ) plane (that a differentiable function with constant value on ∂D must have a critical point in the interior D°).

Many students concluded that f, being continuous, has an extremum on the closed and bounded unit disk, but then claimed without justification that such a point must be critical. (For general differentiable functions the extrema usually fall onto the boundary of the unit disk and hence need not be critical.) In these cases I have assigned 0.5 marks. Students who went further and distinguished the two cases "f constant/non-constant on the disk" received 1.5 or 2 marks, depending on whether the argument, which relies on both the maximum and the minimum, was made clear.

b) During the exam we have announced that "without multiple zeros" ("with only simple zeros") means "with all zeros of multiplicity 1". Familiarity with this concept of polynomial algebra was necessary to answer this question correctly. If f is a nonzero polynomial with real coefficients and $a \in \mathbb{R}$, there exists a unique integer $m \geq 0$ such that $f(x) = (x-a)^m g(x)$ for a polynomial $g(x)$ which satisfies $g(a) \neq 0$. This integer m is called the *multiplicity* of a as a zero of f. Differentiating f shows that $m \geq 2$ ("a is a multiple zero") is equivalent to $f(a) = f'(a) = 0$. Students who didn't know this were likely to choose, e.g., $p(x) = x^3$ and $q(x)$ arbitrary, which gives infinitely many critical points of $f(x) = p(x)q(y)$ (viz., all points on the y-axis). Such "counterexamples", which are excluded by the assumption "no multiple zeros", couldn't be honored by any marks.

Many students answered correctly that the maximum number of critical points is 18, but I have assigned 2 marks only if they indicated in some way that the maximum number of zeros (equal to the degree) of a polynomial and its derivative can be realized simultaneously.

But likewise many students failed to determine the maximum number of critical points (with 6, 17, 36 the most frequent wrong answers), in which case 0 marks were assigned regardless of whether the resulting answer "True" or "False" was correct. Of those students who found the maximum to be < 12 several answered "False", which is a logic error in its own right, (e.g., if a number is ≤ 6 then it is also ≤ 12).

c) This question received the most correct answers. A few students got the partial derivatives of $g(x, y, z) = \mathrm{e}^{xyz} - x - y - z$ wrong, which invalidates everything. Some students claimed $\nabla g(x, y, z) = (0, 0, 0)$ has no solution, which is wrong. Also some students who determined correctly that $\nabla g(x, y, z) = (0, 0, 0)$ is solvable concluded that the surface isn't smooth. In all these cases I have assigned 0 marks. Deriving $x = y = z$ for a singular point and the equations $x^2 \mathrm{e}^{x^3} = 1$, $\mathrm{e}^{x^3} - 3x$ was honored by 1.5 marks, and any further substantial argument (such as deriving $x = \sqrt[3]{1/3}$) by 2 marks. A few students, who proved correctly that the surface has no singular point, afterwards answered inadvertently "False". In this case I have subtracted only 0.5 marks, deviating from the usual policy that a wrong answer gives automatically zero marks.

d) The question was answered correctly by only few students. Many students fell into the trap, set up with the corresponding Midterm 3 question in mind, and claimed that the surface is a hyerbolic paraboloid. Others didn't fall into the trap but used the representing matrix $\mathbf{A}$ of the associated quadratic form $q(x, y, z) = (x - y)(y - z)$ to arrive at the same wrong answer. Since $2\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, which has Sylvester canonical form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (alternatively, the eigenvalues $1, 0, -3$), they concluded that the quadric must be a hyerbolic paraboloid; cf. the proof of the classification theorem for space quadrics in the lecture. But, since the quadric is degenerate, the proof breaks down at the point where we made the change of variables $z' = 2b_1x + 2b_2y + 2b_3z + c$ (because of $b_3 = 0$).

Of course degeneracy can also be proved by the standard test using the 4×4 matrix $\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^\mathsf{T} & c \end{pmatrix}$. Scaling the equation by 2, the matrix becomes $\begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$, which has rank < 4. (The some of the first three rows/columns is zero.)

A few students noted that despite non-invertibility of $\mathbf{A}$ the equation $\mathbf{A}\mathbf{x} = -\mathbf{b}$ can be solved to obtain a "center", with respect to which the quadric is symmetric. But this doesn't help in any way with answering the question. The transformed equation $x'y' + x' + y' = 0$, which is equivalent to $(x' + 1)(y' + 1) = 1$, shows that the quadric is a hyperbolic cylinder whose "centers" form a line.

There were also several students who, perhaps due to a computational error, classified the quadric wrongly as one of the other nondegenerate types. One or two students claimed that the quadric is a cone, which is false but closer to the truth as a nondegenerate type. A cone can be excluded, e.g., because the quadric has no singular point; cf. Exercise H64 of Homework 11: The critical points of $f(x, y, z) = (x - y)(y - z) + x - z$ are precisely the "centers", which form a line, viz., $L = (-1, 0, 1) + \mathbb{R}(1, 1, 1)$; since $\mathbf{b}^\mathsf{T}\mathbf{v} = (1, 0, -1)^\mathsf{T}(-1 + \lambda, \lambda, 1 + \lambda) = -2 \neq 0$ for $\mathbf{v} \in L$, none of them is on the quadric.

e) Only few students were able to obtain marks for this question.

Functions with the indicated property are said to be *homogeneous of degree* 2. Most students assumed that q must be of the form $q(x, y) = ax^2 + bxy + cy^2$, which is not true. For example, $q(x, y) = \frac{x^3+y^3}{x}$ also has the indicated property. (My apologies if the symbol 'q' we have used to denote quadratic forms in the lecture promoted this error!)

Some students derived from the indicated property that q_x and q_y are homogeneous of degree 1, viz., $q_x(tx, ty) = t\, q(x, y)$ and similarly for q_y. While this was honored by

0.5 marks, it doesn't help with answering the question, I think. Several students, who had the right idea to differentiate the equation with respect to t, applied the chain rule incorrectly to obtain $tx\, q_x(x, y) + ty\, q_y(x, y) = 2t\, q(x, y)$, from which the desired identity would follow without setting $t = 1$. In such cases I have assigned up to 1 mark. (Often, to the student's advantage, it wasn't clear whether the argument on the left-hand side is (x, y) or (tx, ty).)

f) Many students solved this question in the way done above. A handful of students observed that ω is exact in $\mathbb{R}^2$ without computing an antiderivative, and used that line integrals of ω are independent of path: Replacing γ by $\gamma_1\gamma_2\gamma_3$ with $\gamma_1(t) = (1-t, -1, 0)$, $t \in [0, 1]$ (going from $(1, -1, 0)$ to $(0, -1, 0)$), $\gamma_2(t) = (0, t, 0)$, $t \in [-1, 1]$ (going from $(0, -1, 0)$ to $(0, 1, 0)$), $\gamma_3(t) = (0, 1, -t)$, $t \in [0, 1]$ (going from $(0, 1, 0)$ to $(0, 1, -1)$), the line integrals $\int_{\gamma_i} \omega$ are easily evaluated ($\int_{\gamma_1} \omega = \int_0^1 [2(1-t)(-1) + 0^2]\,(-1)\,\mathrm{d}t = 1$, $\int_{\gamma_2} \omega = \int_{-1}^1 [2t\,0 + 0^2]\,1\,\mathrm{d}t = 0$, $\int_{\gamma_3} \omega = \int_0^1 [2\,0(-t) + 1^2]\,(-1)\,\mathrm{d}t = -1$), giving $\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega = 1 + 0 - 1 = 0$.

But there were also many wrong answers like concluding $\int_\gamma \omega = 0$ directly from the Fundamental Theorem for Line Integrals (which requires γ to be closed, but γ isn't closed) or Stokes' Theorem (the same error). In such cases I have assigned 0.5 marks for the observation that ω is exact. Some students noted that γ isn't closed and wrongly concluded from this $\int_\gamma \omega \neq 0$ (a frequent logic error, assuming the equivalence of $P \to Q$ and $\neg P \to \neg Q$). Partial credit (0.5 marks) was given for setting up $\int_\gamma \omega$ correctly as an ordinary integral $\int_0^{\pi/2} \dots \mathrm{d}t$ and then stopping or claiming without justification that this integral is zero.

$$\sum_1 = 12$$

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2 a) Using the shorthands f, f_x, f_y for $f(x, y)$, $f_x(x, y)$, $f_y(x, y)$, we compute

$$\begin{aligned} f &= x^3 + y^3 - 3y^2 - 3xy \\ f_x &= 3x^2 - 3y \\ f_y &= 3y^2 - 6y - 3x \end{aligned}$$

Hence $\nabla f(x, y) = (0, 0)$ iff $y = x^2 \wedge x = y^2 - 2y$. Substituting $y = x^2$ into the 2nd equation gives $x^4 - 2x^2 - x = 0$. This quartic equation factors as

$$x(x^3 - 2x - 1) = x(x+1)(x^2 - x - 1) = x(x+1)\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right) = 0.$$

Since $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$, $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$, wo obtain 4 critical points, viz.,

$$\mathbf{p}_1 = (0, 0), \quad \mathbf{p}_2 = (-1, 1), \quad \mathbf{p}_3 = \left(\tfrac{1+\sqrt{5}}{2}, \tfrac{3+\sqrt{5}}{2}\right), \quad \mathbf{p}_4 = \left(\tfrac{1-\sqrt{5}}{2}, \tfrac{3-\sqrt{5}}{2}\right). \qquad \boxed{4}$$

Further we compute

$$\mathbf{H}_f(x,y)=\begin{pmatrix}6x & -3\\ -3 & 6y-6\end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1)=\begin{pmatrix}0 & -3\\ -3 & -6\end{pmatrix},\qquad \mathbf{H}_f(\mathbf{p}_2)=\begin{pmatrix}-6 & -3\\ -3 & 0\end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_3)=\begin{pmatrix}3+3\sqrt{5} & -3\\ -3 & 3+3\sqrt{5}\end{pmatrix},\quad \mathbf{H}_f(\mathbf{p}_4)=\begin{pmatrix}3-3\sqrt{5} & -3\\ -3 & 3-3\sqrt{5}\end{pmatrix}.$$

Since $\det\mathbf{H}_f(\mathbf{p}_1)=\det\mathbf{H}_f(\mathbf{p}_2)=-9<0$, the points $\mathbf{p}_1$, $\mathbf{p}_2$ are saddle points. $\boxed{2}$

Since $\det\mathbf{H}_f(\mathbf{p}_3)=\left(3+3\sqrt{5}\right)^2-9=45+18\sqrt{5}>0$, $3+3\sqrt{5}>0$, the point $\mathbf{p}_3$ is a strict local minimum. $\boxed{1}$

Since $\det\mathbf{H}_f(\mathbf{p}_4)=45-18\sqrt{5}>0$, $3-3\sqrt{5}<0$, the point $\mathbf{p}_4$ is a strict local maximum. $\boxed{1}$

The corresponding values are $f(\mathbf{p}_1)=f(\mathbf{p}_2)=0$, $f(\mathbf{p}_3=\frac{-11-5\sqrt{5}}{2}\approx-11.09$, $f(\mathbf{p}_4=\frac{-11+5\sqrt{5}}{2}\approx-0.09$,

b) Such a plane must be vertical, i.e., spanned by $(0,0,1)$ and some line L in the (x,y)-plane.[1] An equivalent statement is that f is preserved by the planar reflection s_L at L.

Since f has only one local miminum and one local maximum, these must be located on L. (Local extrema off L must occur in pairs having the same type.) Since $\mathbf{p}_3$ and $\mathbf{p}_4$ have the form $(a,a+1)$ and $(b,b+1)$ with $a\neq b$, L must be the line $y=x+1$. $\boxed{1}$

The reflection at this line is $(x,y)\mapsto(y-1,x+1)$, as a picture shows. $\boxed{1}$

One can also find it using the observation that s_L must be the composition of the translation that maps $(0,1)\in L$ to $(0,0)$, the reflection at the line $y=x$, which is $(x,y)\mapsto(y,x)$, and the inverse translation that maps $(0,0)$ back to $(0,1)$: $(x,y)\mapsto(x,y-1)\mapsto(y-1,x)\mapsto(y-1,x+1)$.

That means we have to check the condition $f(y-1,x+1)=f(x,y)$:

$$\begin{aligned}f(y-1,x+1)&=(y-1)^3+(x+1)^3-3(x+1)^2-3(y-1)(x+1)\\&=y^3-3y^2+3y-1+x^3+3x^2+3x+1-3x^2-6x-3-3xy-3y+3x+3\\&=y^3-3y^2+x^3-3xy=f(x,y)\end{aligned}\qquad\boxed{1}$$

Thus G_f is indeed symmetric to the plane spanned by L and $(0,0,1)$, viz.,

$$\underbrace{(0,1,0)+\mathbb{R}(1,1,0)}_{L}+\mathbb{R}(0,0,1)=\bigl\{(x,x+1,z);x,z\in\mathbb{R}$$

Remarks: a) was solved completely (8 marks) by many students. One of the more frequent errors made was recognizing only extrema and designating saddle points as non-decidable type.

[1] The meaning of "spanned" is explained at the end of the solution.

For b) only few students received marks. Most students assumed that the putative symmetry must be of the form considered in the sample exams/midterms, i.e., with respect to the planes $x = 0$, $y = 0$, or $x = \pm y$. Showing that such symmetries don't exist couldn't be honored by any marks.

$$\sum_2 = 11$$

3 The continuous function $f(x, y, z) = 2x + 3y + 6z$ attains a maximum on the set $S = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 + xy + xz + yz = 6\}$, which is an ellipsoid and hence closed and bounded. (That the quadratic form $q(x, y, z) = x^2 + y^2 + z^2 + xy + xz + yz$ is positive definite was shown in H64 b) of Homework 11. One can also show this with repeatedly completing the square: $q(x, y, z) = \left(x + \frac{1}{2}\,y + \frac{1}{2}\,z\right)^2 + \frac{1}{2}\,yz + \frac{3}{4}\,y^2 + \frac{3}{4}\,z^2 = \left(x + \frac{1}{2}\,y + \frac{1}{2}\,z\right)^2 + \frac{3}{4}\left(y + \frac{1}{3}\,z\right)^2 + \frac{8}{12}\,z^2$.) $\boxed{1}$

Setting $g(x, y, z) = x^2 + y^2 + z^2 + xy + xz + yz$, the task is to minimize f on $\mathbb{R}^3$ under the constraint $g(x, y, z) = 6$.

Being an ellipsoid, S is nondegenerate and has no singular points. Hence the Lagrange Multiplier Theorem applies to all points $(x, y, z) \in S$. (This also follows from $\nabla g(x, y, z) = (2x + y + z, 2y + x + z, 2z + x + y)^\mathsf{T} = \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ and the invertibility of the 3×3 matrix, because $(0, 0, 0)$ is not on S.) $\boxed{1}$

Hence every optimal solution must satisfy $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for some $\lambda \in \mathbb{R}$, giving the system of equations

$$\begin{aligned} 2 &= \lambda(2x + y + z), \\ 3 &= \lambda(2y + x + z), \\ 6 &= \lambda(2z + x + y), \\ 6 &= x^2 + y^2 + z^2 + xy + xz + yz. \end{aligned} \qquad \boxed{2}$$

Clearly $\lambda = 0$ is impossible, so that the first three equations amount to the linear system

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2/\lambda \\ 1 & 2 & 1 & 3/\lambda \\ 1 & 1 & 2 & 6/\lambda \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2/\lambda \\ 0 & 3/2 & 1/2 & 2/\lambda \\ 1 & 1/2 & 3/2 & 5/\lambda \end{array}\right] \to \left[\begin{array}{ccc|r} 2 & 1 & 1 & 2/\lambda \\ 0 & 3/2 & 1/2 & 2/\lambda \\ 0 & 0 & 8/6 & 13/(3\lambda) \end{array}\right]$$
$$\to \left[\begin{array}{ccc|r} 2 & 1 & 1 & 2/\lambda \\ 0 & 3 & 1 & 4/\lambda \\ 0 & 0 & 4 & 13/\lambda \end{array}\right]$$

Backwards substitution yields $z = 13/(4\lambda)$, $y = 1/(4\lambda)$, $x = -3/(4\lambda)$. Substituting these into the 4th equation gives $96\,\lambda^2 = (-3)^2 + 1^2 + 13^2 + (-3)1 + (-3)13 + 1 \cdot 13 = 150$, i.e., $\lambda^2 = \frac{25}{16}$ and $\lambda = \pm\frac{5}{4}$. Thus there are two solutions, viz. $(x, y, z) = \left(\mp\frac{3}{5}, \pm\frac{1}{5}, \pm\frac{13}{5}\right)$. $\boxed{3}$

Since $f\left(-\frac{3}{5}, \frac{1}{5}, \frac{13}{5}\right) = 15$, $f\left(\frac{3}{5}, -\frac{1}{5}, -\frac{13}{5}\right) = -15$, the unique optimal solution is $(x^*, y^*, z^*) = \left(-\frac{3}{5}, \frac{1}{5}, \frac{13}{5}\right)$, and the corresponding maximum value is 15. $\boxed{1}$

Remarks: During the exam we have announced that in the statement of the question the first "z^*" should be deleted (because it conflicts with the subsequent use as z-coordinate" of an optimal solution). A better way to state the optimization problem is "Maximize $\zeta = 2x + 3y + 6z$ subject to ...". Then one can refer to the "optimal objective value ζ^*" afterwards; cf. the corresponding question in the 2nd sample exam.

My apologies, and thanks to the student who notified us of this conflict during the exam! I hope it hasn't caused too much headache.

Student performance on this question was probably the best among all 5 questions.

Several students noted the following elegant proof that S is an ellipsoid: Use $x^2 + y^2 + z^2 + xy + xz + yz = \frac{1}{2}(x+y)^2 + \frac{1}{2}(x+z)^2 + \frac{1}{2}(y+z)^2$, together with the fact that the change of variables $x' = x + y$, $y' = x + z$, $z' = y + z$ is bijective. Since that part of the question was worth only 1 mark, we have not insisted on proving bijectivity. But claiming without any justification that the constraint surface is bounded was penalized by 0.5 marks.

Showing that S is closed and bounded without any knowledge of quadrics is more demanding. Clearly S is closed as a level set of a continuous function, but how to show that S is bounded? The student's formula above gives for S the equation $(x+y)^2 + (x+z)^2 + (y+z)^2 = 12$, from which $|x+y|$, $|x+z|$, $|y+z|$ are $\leq \sqrt{12}$. Hence $x = \frac{1}{2}[(x+y) + (x+z) - (y+z)]$ is bounded in absolute value by $\frac{3}{2}\sqrt{12} = 3\sqrt{3}$. By symmetry, the same is true for y, z. A more direct proof is the following: Add $xy + xz + yz$ to the equation $x^2 + y^2 + z^2 + xy + xz + yz = 6$ to obtain $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 6 + xy + xz + yz$. Hence $xy + xz + yz \geq -6$ and $x^2 + y^2 + z^2 = 6 - xy - xz - yz \leq 12$. This is the sharpest possible bound, because $\left(\sqrt{6}, -\sqrt{6}, 0\right) \in S$ has length 12. (This isn't magic, but follows from the Principal Axis Theorem together with the fact that the smallest eigenvalue of the matrix $\left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right)$ representing the quadratic form $2(x^2 + y^2 + z^2 + xy + xz + yz)$ is equal to 1 and has $(1, -1, 0)$ as an associated eigenvector.)

As usual, many students forgot to check the smoothness condition $\nabla g(x, y, z) \neq (0, 0, 0)$ for $(x, y, z) \in S$. Some students wrote that the 2×3 matrix with rows ∇f, ∇g has full rank on S, which we haven't accepted without further justification ($\nabla f \neq \mathbf{0}$ isn't necessary at all for the Lagrange Multiplier Theorem).

$$\sum_3 = 8$$

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4 a) Writing $f(b, x) = \frac{\ln(1 + b\cos x)}{\cos x}$, we have

$$\int_0^\pi f_b(b, x)\,\mathrm{d}x = \int_0^\pi \frac{\cos x}{(1 + b\cos x)\cos x}\,\mathrm{d}x = \int_0^\pi \frac{\mathrm{d}x}{1 + b\cos x}$$

for $-1 < b < 1$. (For $b = \pm 1$ the integral is infinite.) $\boxed{1}$

The differentiation under the integral sign is justified, since $[0, \pi]$ is compact and $f_b(b, x)$ is a continuous 2-variable function on $(-1, 1) \times [0, \pi]$. $\boxed{1}$

Full justification: Given $b \in (-1, 1)$, choose $\delta > 0$ such that $-1 + \delta < b < 1 - \delta$. Then $f_b(b, x)$, being continuous and positive, attains a maximum $M > 0$ on the compact rectangle $[-1+\delta, 1-\delta] \times [0, \pi]$, and $\Phi(x) := M$ provides an integrable bound for $|f_b(b, x)| = f_b(b, x)$ that is independent of b. Hence we can apply Part 2 of the theorem on parameter integrals with $X := (-1+\delta, 1-\delta)$, $Y := [0, \pi]$ to obtain $F'(b) = \int_0^\pi f_b(b, x)\,\mathrm{d}x$. $\boxed{+1}$

It remains to show that F is continuous at $b = 1$. First we show that $F(1) = \int_0^\pi \frac{\ln(1+\cos x)}{\cos x}\,\mathrm{d}x$ is well-defined. At $x = \pi/2$ both numerator and denominator of the integrand have a simple zero, so that the integrand can be viewed as being continuous (and nonzero) there. At $x = \pi$ the integral is improper, but converges since $\int_0^1 \ln x \,\mathrm{d}x$ converges. (Use the estimate $1 + \cos x = 1 - \cos(x - \pi) = (x - \pi)^2/2 + \mathrm{o}\big((x - \pi)^2\big)$ for $x \to \pi$ together with an estimate $\ln y = \mathrm{o}(y^{-1/3})$ for $y \downarrow 0$.) $\boxed{1}$

For $\pi/2 < x < \pi$, $0 < b \leq 1$ we have $1 + b \cos x \geq 1 + \cos x$ and hence

$$0 < \frac{\ln(1 + b \cos x)}{\cos x} \leq \frac{\ln(1 + \cos x)}{\cos x} =: \Phi(x).$$

This bound, which is independent of b and integrable over $(\pi/2, \pi)$, implies the continuity of $b \mapsto \int_{\pi/2}^{\pi} \frac{\ln(1+b\cos x)}{\cos x} \,\mathrm{d}x$ in $b = 1$, from which F differs by a continuous function, viz. $b \mapsto \int_0^{\pi/2} \frac{\ln(1+b\cos x)}{\cos x} \,\mathrm{d}x$. This completes the proof. $\boxed{1}$

From the hint we have

$$\begin{aligned} F'(b) &= \frac{\pi}{\sqrt{1 - b^2}} \quad \text{for } -1 < b < 1. \\ &\implies F(b) = \pi \arcsin(b) + C \quad \text{for } -1 < b < 1 \end{aligned}$$

with a constant $C \in \mathbb{R}$. Since $F(0) = 0 = \pi \arcsin(0)$, the constant must be zero. $\boxed{1}$

Since both F and arcsin are continuous at $b = 1$, it finally follows that

$$\int_0^{\pi} \frac{\ln(1 + \cos x)}{\cos x} \,\mathrm{d}x = F(1) = \pi \arcsin(1) = \frac{\pi^2}{2}. \qquad \boxed{1}$$

In order to prove the auxiliary identity, observe that by substituting $1 = \cos^2(x/2) + \sin^2(x/2)$ and $\cos x = \cos^2(x/2) - \sin^2(x/2)$ into the integrand the integral becomes

$$\begin{aligned} \int_0^{\pi} \frac{\mathrm{d}x}{1 + b \cos x} &= \int_0^{\pi} \frac{\mathrm{d}x}{(1 + b)\cos^2(x/2) + (1 - b)\sin^2(x/2)} \\ &= 2 \int_0^{\pi/2} \frac{\mathrm{d}x}{(1 + b)\cos^2 t + (1 - b)\sin^2 t} \qquad \text{(Subst. } x = 2t,\ \mathrm{d}x = 2\,\mathrm{d}t) \\ &= \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}x}{(1 + b)\cos^2 t + (1 - b)\sin^2 t} \qquad \text{(See below)} \\ &= \frac{1}{2} \frac{2\pi}{\sqrt{1 + b}\sqrt{1 - b}} = \frac{\pi}{\sqrt{1 - b^2}}. \end{aligned}$$

To justify the second-to-last step, it suffices to show that $\int_0^{2\pi} \frac{\mathrm{d}t}{A^2 \cos^2 t + B^2 \sin^2 t} = 4 \int_0^{\pi/2} \frac{\mathrm{d}t}{A^2 \cos^2 t + B^2 \sin^2 t}$. Since the integrand $f(t) = \frac{1}{A^2 \cos^2 t + B^2 \sin^2 t}$ is π-periodic, we have $\int_0^{2\pi} f(t) \,\mathrm{d}t = 2 \int_0^{\pi} f(t) \,\mathrm{d}t$. Further, $\int_{\pi/2}^{\pi} f(t) \,\mathrm{d}t = -\int_{\pi/2}^{0} \frac{\mathrm{d}s}{A^2 \cos^2(\pi - s) + B^2 \sin^2(\pi - s)} = \int_0^{\pi/2} \frac{\mathrm{d}s}{A^2 \cos^2 s + B^2 \sin^2 s} = \int_0^{\pi/2} f(s) \,\mathrm{d}s$, where we have used the substitution $s = \pi - t$, $\mathrm{d}s = -\,\mathrm{d}t$ and the identities $\cos s = -\cos(\pi - s)$, $\sin s = \sin(\pi - s)$. This proves $\int_0^{2\pi} f(t) \,\mathrm{d}t = 4 \int_0^{\pi/2} f(t) \,\mathrm{d}t$. $\boxed{+2}$

Remarks: As in the past were lots of problems with this question, in particular with justifying the differentiation under the integral sign and the continuity of F at $b = 1$.

Some students misread the last sentence of the question and proved $\int_0^{2\pi} \frac{\mathrm{d}t}{A^2 \cos^2 t + B^2 \sin^2 t} = \frac{2\pi}{AB}$ by means of the standard substitution $s = \tan(t/2)$. This couldn't be honored by any marks, of course. Other students who attempted to obtain the bonus marks made the mistake, after expressing $\cos x$ in terms of $\cos(x/2)$, $\sin(x/2)$ and substituting $t = x/2$, that the interval of integration for t is $[0, 2\pi]$ (but it is $[0, \pi/2]$).

$$\sum_{4} = 6$$

5 a) Denoting the region by K, we have $(x, y, z) \in K$ iff $9x^2 + 4y^2 \leq z \leq -18x - 16y$. Hence the projection D of K onto the (x, y)-plane is described by $9x^2 + 4y^2 \leq -18x - 16y$, i.e., $9(x+1)^2 + 4(y+2)^2 \leq 25$, and thus a solid ellipse in normal form with center $(-1, -2)$ and semiaxes $a = 5/3$, $b = 5/2$.

$$\begin{aligned} \Longrightarrow \operatorname{vol}_3(K) &= \int_K 1 \, \mathrm{d}^3(x, y, z) = \int_D -18x - 16y - 9x^2 - 4y^2 \, \mathrm{d}^2(x, y) \qquad \boxed{1} \\ &= \int_D 25 - 9(x+1)^2 - 4(y+2)^2 \, \mathrm{d}^2(x, y). \end{aligned}$$

The change of variables $T(r, \theta) = \big(-1 + (r/3)\cos\theta, -2 + (r/2)\sin\theta\big)$, which has

$$\mathbf{J}_T(r, \theta) = \begin{pmatrix} \frac{1}{3}\cos\theta & -\frac{r}{3}\sin\theta \\ \frac{1}{2}\sin\theta & \frac{r}{2}\cos\theta \end{pmatrix}, \quad \det \mathbf{J}_T(r, \theta) = \frac{r}{6},$$

maps the rectangle $[0, 5] \times [0, 2\pi]$ onto D (modulo a null set) and turns the integral into

$$\begin{aligned} \Longrightarrow \operatorname{vol}_3(K) &= \int_{\substack{0 \leq r \leq 5 \\ 0 \leq \theta \leq 2\pi}} \big(25 - r^2\big)(r/6) \mathrm{d}^2(r, \theta) \qquad \boxed{1} \\ &= \frac{2\pi}{6} \int_0^5 25r - r^3 \, \mathrm{d}r \\ &= \pi/3 \left[\frac{25r^2}{2} - \frac{r^4}{4} \right]_0^5 = \frac{5^4 \pi}{12} = \frac{625\,\pi}{12}. \qquad \boxed{1} \end{aligned}$$

The remaining integrals involved in the centroid are computed in the same way:

$$\begin{aligned}
\int_K x\,\mathrm{d}^3(x,y,z) &= \int_D x\,\bigl(25-9(x+1)^2-4(y+2)^2\bigr)\,\mathrm{d}^2(x,y)\\
&= \int\limits_{\substack{0\le r\le 5\\ 0\le\theta\le 2\pi}} \bigl(-1+(r/3)\cos\theta\bigr)\bigl(25-r^2\bigr)(r/6)\mathrm{d}^2(r,\theta)\\
&= -\frac{625\,\pi}{12}, \qquad\qquad \left(\text{since } \int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = 0\right) \qquad \boxed{1}
\end{aligned}$$

$$\begin{aligned}
\int_K y\,\mathrm{d}^3(x,y,z) &= \int_D y\,\bigl(25-9(x+1)^2-4(y+2)^2\bigr)\,\mathrm{d}^2(x,y)\\
&= \int\limits_{\substack{0\le r\le 5\\ 0\le\theta\le 2\pi}} \bigl(-2+(r/2)\sin\theta\bigr)\bigl(25-r^2\bigr)(r/6)\mathrm{d}^2(r,\theta)\\
&= -\frac{625\,\pi}{6}, \qquad\qquad \left(\text{since } \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = 0\right) \qquad \boxed{\tfrac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\int_K z\,\mathrm{d}^3(x,y,z) &= \int_D \int_{9x^2+4y^2}^{-18x-16y} z\,\mathrm{d}z\,\mathrm{d}^2(x,y)\\
&= \frac{1}{2}\int_D (18x+16y)^2-(9x^2+4y^2)^2\mathrm{d}^2(x,y) \qquad \boxed{\tfrac{1}{2}}\\
&= \frac{1}{2}\int_D (-18x-16y-9x^2-4y^2)(-18x-16y+9x^2+4y^2)\mathrm{d}^2(x,y).
\end{aligned}$$

Since $-18x-16y+9x^2+4y^2 = 9(x+1)^2+4(y+2)^2-36(x+1)-32(y+2)+75$, the same change of variables gives

$$\begin{aligned}
\int_K z\,\mathrm{d}^3(x,y,z) &= \frac{1}{2}\int\limits_{\substack{0\le r\le 5\\ 0\le\theta\le 2\pi}} (25-r^2)(r^2-12r\cos\theta-16r\sin\theta+75)(r/6)\mathrm{d}^2(r,\theta) \qquad \boxed{1}\\
&= \frac{2\pi}{12}\int_0^5 (25-r^2)(r^2+75)\,\mathrm{d}r \quad \left(\text{since } \textstyle\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = 0\right)\\
&= \frac{\pi}{6}\int_0^5 3\cdot 5^4 r - 50\,r^3 - r^5\,\mathrm{d}r\\
&= \frac{\pi}{6}\left(3\cdot 5^4\frac{5^2}{2} - 50\,\frac{5^4}{4} - \frac{5^6}{6}\right) = \frac{5^6\pi}{6}\left(\frac{3}{2}-\frac{1}{2}-\frac{1}{6}\right) = \frac{5^7\pi}{36} \qquad \boxed{1}\\
&= \frac{78125\,\pi}{36}.
\end{aligned}$$

$\Longrightarrow$ The centroid of K is

$$\frac{1}{\frac{625\,\pi}{12}}\left(-\frac{625\,\pi}{12}, -\frac{625\,\pi}{6}, \frac{5^7\pi}{36}\right) = \left(-1,-2,\frac{5^3}{3}\right) = \left(-1,-2,\tfrac{125}{3}\right). \qquad \boxed{1}$$

Alternative Solution: The defining inequalities for K can be stated as

$$0 \le -18x-16y-z \le -18x-16y-9x^2-4y^2 = 25-9(x+1)^2-4(y+2)^2.$$

$\Longrightarrow$ The change of variables $x' = x + 1$, $y' = y + 2$, $z' = -18x - 16y - z$, which is affine (linear+translation) and has (constant) Jacobi determinant -1, maps K onto the elliptic paraboloid $P = \{(x, y, z) \in \mathbb{R}^3; 0 \leq z \leq 25 - 9x^2 - 4y^2\}$. $\boxed{1}$

The centroid of P has the form $(0, 0, \overline{z})$ with $\overline{z} = \int_P z \, \mathrm{d}^3(x, y, z) / \operatorname{vol}_3(P)$. Since the z-section P_z of P is described by the equation $9x^2 + 4y^2 \leq 25 - z$, it is an ellipse with semiaxes $a = \frac{1}{3}\sqrt{25 - z}$ and $b = \frac{1}{2}\sqrt{25 - z}$ for $0 < z < 5$ and a 2-dimensional null set for $z \leq 0 \vee z \geq 5$.

$$\begin{aligned}
\Longrightarrow \quad \operatorname{vol}_3(P) &= \int_P 1 \, \mathrm{d}^3(x, y, z) = \int_{z=0}^{25} \operatorname{vol}_2(P_z) \, \mathrm{d}z \\
&= \frac{\pi}{6} \int_0^{25} (25 - z) \, \mathrm{d}z \qquad \text{(since } \operatorname{vol}_2(P_z) = \pi ab\text{)} \\
&= \frac{\pi}{6} \left[25z - \frac{z^2}{2} \right]_0^{25} = \frac{625\,\pi}{12}, \qquad \boxed{2} \\
\int_P z \, \mathrm{d}^3(x, y, z) &= \frac{\pi}{6} \int_0^{25} z(25 - z) \, \mathrm{d}z \\
&= \frac{\pi}{6} \left[25 \frac{z^2}{2} - \frac{z^3}{3} \right]_0^{25} = \frac{5^6 \pi}{6} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{5^6 \pi}{36}, \qquad \boxed{1} \\
\overline{z} &= \frac{5^6 \pi}{36} \Big/ \frac{5^4 \pi}{12} = \frac{25}{3}. \qquad \boxed{1}
\end{aligned}$$

Thus the centroid of P is $\left(0, 0, \frac{25}{3}\right)$.

Since bijective affine maps preserve centroids, the centroid of K is then obtained by setting $x' = y' = 0$, $z' = \frac{25}{3}$ in the said change of variables and solving for x, y, z. This gives $x = -1$, $y = -2$, $z = -18(-1) - 16(-2) - \frac{25}{3} = 50 - \frac{25}{3} = \frac{125}{3}$, so that the centroid of K is $\left(-1, -2, \frac{125}{3}\right)$. $\boxed{1}$

It remains to justify that bijective affine maps preserve centroids. This holds in any dimension n, but here we need and prove it for $n = 3$. Suppose $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ has the form $T(x, y, z) = (x', y', z')^\mathsf{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ with an invertible matrix $\mathbf{T} = (t_{ij}) \in \mathbb{R}^{3\times 3}$ and a vector $\mathbf{b} = (b_1, b_2, b_3)^\mathsf{T} \in \mathbb{R}^3$, and $K \subset \mathbb{R}^3$ is measurable and bounded with $\operatorname{vol}_3(K) > 0$. (These assumptions guarantee that K has a centroid $\mathbf{c}$.) The centroid of $T(K)$ has x-coordinate

$$\begin{aligned}
\frac{\int_{T(K)} x' \, \mathrm{d}^3(x', y', z')}{\int_{T(K)} 1 \, \mathrm{d}^3(x', y', z')} &= \frac{\int_K (t_{11}x + t_{12}y + t_{13}z + b_1) \, |\det \mathbf{J}_T(x, y, z)| \, \mathrm{d}^3(x, y, z)}{\int_K |\det \mathbf{J}_T(x, y, z)| \, \mathrm{d}^3(x, y, z)} \\
&= \frac{\int_K (t_{11}x + t_{12}y + t_{13}z + b_1) \, |\det \mathbf{T}| \, \mathrm{d}^3(x, y, z)}{\int_K |\det \mathbf{T}| \, \mathrm{d}^3(x, y, z)} \\
&\qquad \text{(since } T \text{ is linear)} \\
&= t_{11} \frac{\int_K x \, \mathrm{d}^3(x, y, z)}{\int_K 1 \, \mathrm{d}^3(x, y, z)} + t_{12} \frac{\int_K y \, \mathrm{d}^3(x, y, z)}{\int_K 1 \, \mathrm{d}^3(x, y, z)} + t_{13} \frac{\int_K z \, \mathrm{d}^3(x, y, z)}{\int_K 1 \, \mathrm{d}^3(x, y, z)} + b_1 \frac{\int_K 1 \, \mathrm{d}^3(x, y, z)}{\int_K 1 \, \mathrm{d}^3(x, y, z)} \\
&= t_{11}\overline{x} + t_{12}\overline{y} + t_{13}\overline{z} + b_1,
\end{aligned}$$

where $\overline{x}, \overline{y}, \overline{z}$ denote the corresponding coordinates of $\mathbf{c}$. Repeating the computation for the 2nd and 3rd coordinate of the centroid of $T(K)$ gives

$$\begin{pmatrix}\overline{x}'\\ \overline{y}'\\ \overline{z}'\end{pmatrix}=\begin{pmatrix}\frac{\int_{T(K)}x'\,\mathrm{d}^3(x',y',z')}{\int_{T(K)}1\,\mathrm{d}^3(x',y',z')}\\ \frac{\int_{T(K)}y'\,\mathrm{d}^3(x',y',z')}{\int_{T(K)}1\,\mathrm{d}^3(x',y',z')}\\ \frac{\int_{T(K)}z'\,\mathrm{d}^3(x',y',z')}{\int_{T(K)}1\,\mathrm{d}^3(x',y',z')}\end{pmatrix}=\begin{pmatrix}t_{11}\overline{x}+t_{12}\overline{y}+t_{13}\overline{z}+b_1\\ t_{21}\overline{x}+t_{22}\overline{y}+t_{23}\overline{z}+b_2\\ t_{31}\overline{x}+t_{32}\overline{y}+t_{33}\overline{z}+b_3\end{pmatrix}=\begin{pmatrix}t_{11}&t_{12}&t_{13}\\ t_{21}&t_{22}&t_{23}\\ t_{31}&t_{32}&t_{33}\end{pmatrix}\begin{pmatrix}\overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix}+\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix},$$

as claimed. $\boxed{2}$

Remark: Nonlinear maps don't preserve centroids in general, as the following 1-dimensional example shows: The centroid of $[0,1]\subset\mathbb{R}$ is clearly $1/2$. The map $T(x)=x^2$ induces a bijection from $[0,1]$ to itself but, since $T(1/2)=1/4$, doesn't map the centroid of $[0,1]$ to itself.

b) The surface, let's name it S, is the graph of the function $f(x,y)=\sqrt{2xy}$ with domain $\Delta=\bigl\{(x,y,z)\in\mathbb{R}^3;x\geq 0,y\geq 0,x+y\leq 3\bigr\}$. Using the formula for surface volumes of graphs from the lecture (which was stated for functions with open domains, but holds here as well[2]), we obtain

$$\begin{aligned}
\Longrightarrow \mathrm{vol}_2(S)&=\int_\Delta\sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y)\\
&=\int_\Delta\sqrt{1+\left|\left(\sqrt{\frac{y}{2x}},\sqrt{\frac{x}{2y}}\right)\right|^2}\,\mathrm{d}^2(x,y)\\
&=\int_\Delta\sqrt{1+\frac{y}{2x}+\frac{x}{2y}}\,\mathrm{d}^2(x,y) && \boxed{1}\\
&=\int_\Delta\sqrt{\frac{2xy+y^2+x^2}{2xy}}\,\mathrm{d}^2(x,y)\\
&=\frac{1}{\sqrt{2}}\int_\Delta\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\,\mathrm{d}^2(x,y)\\
&=\sqrt{2}\int_\Delta\sqrt{\frac{x}{y}}\,\mathrm{d}^2(x,y) && \text{(by symmetry)}\\
&=\sqrt{2}\int_{x=0}^3\int_{y=0}^{3-x}\sqrt{\frac{x}{y}}\,\mathrm{d}y\,\mathrm{d}x\\
&=\sqrt{2}\int_{x=0}^3\bigl[2\sqrt{xy}\bigr]_0^{3-x}\,\mathrm{d}x\\
&=2\sqrt{2}\int_{x=0}^3\sqrt{x(3-x)}\,\mathrm{d}x && \boxed{1}\\
&=2\sqrt{2}\int_{x=0}^3\sqrt{\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}\,\mathrm{d}x\\
&=2\sqrt{2}\,\frac{9}{4}\,\frac{\pi}{2}=\frac{9\pi}{2\sqrt{2}}=\frac{9}{4}\sqrt{2}\,\pi, && \boxed{1}
\end{aligned}$$

[2] $f(\partial\Delta)$ consists of two line segments on the x-axis and y-axis and the smooth curve $\gamma(x)=\Bigl(x,3-x,\sqrt{2x(3-x)}\Bigr)$, $x\in(0,3)$, which all have area zero.

since the graph of the last integrand is a half-circle of radius $\frac{3}{2}$. (That integrands are not defined for some points on $\partial\Delta$ doesn't matter, since $\partial\Delta$ is a null set.)

Remarks: As announced on BB before the exam, a) carries substantially more marks than originally intended (8 vs. 5) and printed on the exam paper. This is due to the rather costly computation of the 3rd coordinate $\overline{z}$ of the centroid of K, which I realized only after the exam paper was printed. The threshold for a full exam score wasn't increased, i.e., the 3 additional marks are considered as bonus marks.

Very few students were able to solve a) and b) completely. In a) this was mostly due to not identifying the projection of K onto the x-y plane as the ellipse $9(x+1)^2+4(y+2)^2 \leq 25$. In b) students got stuck after not observing the symmetry of Δ and the integrand with respect to $x = y$, because this left a further rather complicated 1-dimensional integral to be solved.

In a) it was accepted if students stated intermediate results in factored form, i.e., writing 5^4, 5^7 in place of 625, 78125.

In b) it was accepted (but not honored by additional marks) if students computed the surface area of the solid bounded by $z^2 = 2xy$ and the 4 indicated planes.

In b) I found it surprising that the centroid of K lies vertically above the center of D. The z-sections of K, which have no particular symmetry (for $z > 0$ the plane region K_z is the intersection of the solid ellipse $9x^2 + 4y^2 \leq z$ and the half plane $18x + 16y \leq -z$), don't explain this. But an explanation is provided by the change of variables $T(x, y, z) = (x + 1, y + 2, -18x - 16y - z)$ used in the alternative solution, which maps the vertical line through $(-1, -2, 0)$ onto the z-axis. (This follows from $T(-1, -2, z) = (0, 0, 50 - z)$.) Since the centroid of P is on the z-axis, that of K must be on the vertical line through $(-1, -2, 0)$.

In the alternative solution to b), the two integrals involved in the computation of $\overline{z}$ can also be evaluated using polar-like coordinates (similar to those in the first solution), but the computation is more costly.

$$\sum_{5} = 11$$

$$\sum_{\text{Final Exam}} = 12 + 11 + 8 + 6 + 11 = 48 = 35 + 13$$

Owing to the overall higher difficulty of this year's final exam (pertaining to Questions 4, 5 and part of Question 2), the threshold for a full final exam score was set to 35 marks (compared with 40 in most previous years/semesters).