Calculus III (Math 241)

In what follows, [Ste21] refers to our adopted textbook J. Stewart, D. Clegg, S. Watson, *Calculus: Early Transcendentals*, 9th edition, metric version, Cengage Learning, 2021

H1 Solve [Ste21], Section 12.1, Exercise 48.

H2 Solve [Ste21], Section 12.2, Exercises 32, 34, 36 and 38; cp. Example 7 on p. 842.

H3 Solve [Ste21], Section 12.3, Exercises 40, 42, 46, 48. For the definition of $\operatorname{orth}_{\mathbf{a}} \mathbf{b}$ see Exercise 45.

H4 Show that the non-terminating decimal expansion of a real number $a \in (0,1]$ is unique.

Hint: Assume $0.a_1a_2a_3\ldots = 0.b_1b_2b_3\ldots$ and n is the smallest index with $a_n \neq b_n$. Derive a contradiction by subtracting the two corresponding series.

H5 *Optional Exercise*

For each of the following real numbers a, find a nonzero polynomial with integer coefficients having a as a zero: $\sqrt{2}+\sqrt[3]{5}$, $\sqrt{2}-\sqrt[3]{5}$, $\sqrt{2}\cdot\sqrt[3]{5}$, and $\sqrt{2}/\sqrt[3]{5}$.

Hint: In each case there is a polynomial of degree 6 having this property. You are required to produce the coefficients of the polynomial (and not merely a representation as a product of certain polynomials, say).

H6 *Optional Exercise*

The goal of this exercise is to construct a bijection (i.e., a one-to-one correspondence) between $\mathbb{R}$ and $\mathbb{R}^2$. For this you may use without proof the following fact already used in the lecture (in CANTOR's diagonal argument): Every real number $a \in (0,1]$ has a unique "non-terminating" decimal expansion $a = 0.a_1a_2a_3\ldots$ with $a_i \in \{0,1,\ldots,9\}$ and $a_i \neq 0$ for infinitely many i.

a) Find a bijection from $\mathbb{R}$ to $(0,1]$.

   Hint: Use an elementary one-variable function to map $\mathbb{R}$ bijectively onto $(0,1)$, and then adjust.

b) Find a bijection from $(0,1]\times(0,1] = \{(a,b) \in \mathbb{R}^2; 0 < a \leq 1, 0 < b \leq 1\}$ to $(0,1]$.

   Hint: This is tricky. Interleaving $(0.a_1a_2\ldots, 0.b_1b_2\ldots) \to (0.a_1b_1a_2b_2\ldots)$ of non-terminating decimal fractions can't be used, since this wouldn't produce, e.g., $0.101010\ldots$ Work with subwords instead of single digits.

c) Use a) and b) to find the desired bijection from $\mathbb{R}$ to $\mathbb{R}^2$.

H7 *Optional Exercise*

Show that the set $\mathbb{A}$ of all real algebraic numbers is countable.

Hint: Every $\alpha \in \mathbb{A}$ is a zero of some polynomial $a(X) = a_0 + a_1 X + \cdots + a_d X^d$ with coefficients $a_i \in \mathbb{Z}$ and degree $d \geq 1$ (i.e, $a_d \neq 0$). Define the *height* of $a(X)$ as $d + |a_0| + |a_1| + \cdots + |a_d|$, and show that for every integer $h \in \mathbb{Z}^+$ there exist only finitely many polynomials of height h.

Instructions For your homework it is best to maintain 2 notebooks, which are handed in on alternate Wednesdays. You may also use A4 sheets, provided they are firmly stapled together.

Don't forget to write your name (English and Chinese) and your student ID on the first page.

Homework is handed in on Wednesdays before the discussion session starts (late homework won't be accepted!) and will be returned on the next Wednesday.

Answers to exercises must be justified; it is not sufficient to state only the final result of a computation.

Answers must be written in English.

For a full homework score it is sufficient to solve ca. 80 % of the mandatory homework exercises. Optional exercises contribute to the homework score, but they are usually more difficult and you should work on them only if you have sufficient spare time.

Due on Wed Sep 27, 6 pm

The optional exercises can be handed in until Wed Oct 11, 6 pm.

Solutions

1

Ex. 48: The condition is $|\mathbf{x}-(-1,5,3)| = 2\,|\mathbf{x}-(6,2,-2)|$. Squaring and inserting $\mathbf{x} = (x_1, x_2, x_3)$ gives

$$
\begin{aligned}
& (x_1+1)^2 + (x_2-5)^2 + (x_3-3)^2 = 4(x_1-6)^2 + 4(x_2-2)^2 + 4(x_3+2)^2 \\
\iff \quad & 3x_1^2 - 50x_1 + 3x_2^2 - 6x_2 + 3x_3^2 + 22x_3 + 141 = 0 \\
\iff \quad & x_1^2 - \tfrac{50}{3}x_1 + x_2^2 - 2x_2 + x_3^2 + \tfrac{22}{3}x_3 = -\tfrac{141}{3}.
\end{aligned}
$$

Completing the three squares further gives

$$\left(x_1 - \tfrac{25}{3}\right)^2 + (x_2-1)^2 + \left(x_1 + \tfrac{11}{3}\right)^2 = \tfrac{625}{9} + 1 + \tfrac{121}{9} - \tfrac{141}{3} = \tfrac{332}{9}.$$

This is the equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3} = \frac{2}{3}\sqrt{83}$.

2

Ex. 32: The given force vectors, measured in Newton, are $(20\cdot\cos 45°, 20\cdot\sin 45°) = (10\sqrt{2}, 10\sqrt{2})$ and $\big(16\cdot\cos(-30°), 16\cdot\sin(-30°)\big) = (8\sqrt{3}, -8)$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = (10\sqrt{2}+8\sqrt{3}, 10\sqrt{2}-8) \approx (28.00, 6.142)$ [N],
Then the magnitude of $\mathbf{F}$ is $|\mathbf{F}| = \sqrt{(10\sqrt{2}+8\sqrt{3})^2 + (10\sqrt{2}-8)^2} \approx 28.66$ [N]. and the angle $\mathbf{F}$ makes with the positive x-axis is $\arctan(\frac{10\sqrt{2}-8}{10\sqrt{2}+8\sqrt{3}}) \approx 0.2160 \approx 12.37°$.

Ex. 34: By symmetry, the two tensions $\mathbf{T}_1, \mathbf{T}_2$ have the same length $T = |\mathbf{T}_1| = |\mathbf{T}_2|$ and are of the form $T\,(\pm\cos(60°), \sin(60°)) = T\left(\pm\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$. The equilibrium condition in the position of the hook gives $T\left(0, \sqrt{3}\right) = \mathbf{T}_1 + \mathbf{T}_2 = 500\cdot 9.81(0,1)$ [N], so that $T = \frac{4905}{\sqrt{3}} \approx 2832$ [N]

Ex. 36: $\mathbf{G} = 2\cdot 25\cdot\sin 37° \approx 30.09$ [N].

Ex. 38: We can set up plane coordinates in such a way that the initial position of the rower is $(0,0)$, one bank of the channel is the y-axis, and the desired landing point has coordinates $(400, 250)$. If the rower steers her kayak in the direction $(\cos\phi, \sin\phi)$, the resulting velocity of the kayak will be $2(\cos\phi, \sin\phi) + (0, -0.5) = \left(2\cos\phi, 2\sin\phi - \frac{1}{2}\right)$ [ms^{-1}], which must be a multiple of $(400, 250)$, or of $(8,5)$. This gives the equation $10\cos\phi - 16\sin\phi + 4 = 0$, i.e., $5\cos\phi - 8\sin\phi = -2$. The solution, using the addition theorems for cos, sin (or the equivalent complex numbers point of view $\mathrm{Re}\left[(5+8\mathrm{i})\mathrm{e}^{\mathrm{i}\phi}\right] = -2$) is $\phi = \arccos\frac{-2}{\sqrt{89}} - \arctan\frac{8}{5} \approx 0.772$, or about $44.2°$. The time for the journey, measured in seconds, is the reciprocal multiple, i.e., $\frac{400}{2\cos\phi} \approx 279$.

3

Ex. 40: $|\mathbf{a}| = \sqrt{1^2+4^2} = \sqrt{17}$, $\mathbf{a}\cdot\mathbf{b} = 1\cdot 2 + 4\cdot 3 = 14$
$\Longrightarrow$ The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\mathrm{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{14}{\sqrt{17}}$, and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\mathrm{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a} = \frac{14}{17}(1,4) = \left(\frac{14}{17}, \frac{56}{17}\right)$.

Ex. 42: $|\mathbf{a}| = \sqrt{1+16+64} = \sqrt{81} = 9$, $\mathbf{a}\cdot\mathbf{b} = -12+4+16 = 8$, $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \frac{8}{9}$, $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{8}{81}(-1,4,8) = \left(-\frac{8}{81}, \frac{32}{81}, \frac{64}{81}\right)$.

Ex. 46: Using the formula $\operatorname{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \operatorname{proj}_{\mathbf{a}}\mathbf{b}$ from Exercise 45 and the result of Exercise 40, we obtain $\operatorname{orth}_{\mathbf{a}}\mathbf{b} = (2,3) - \left(\frac{14}{17}, \frac{56}{17}\right) = \left(\frac{20}{17}, -\frac{5}{17}\right)$

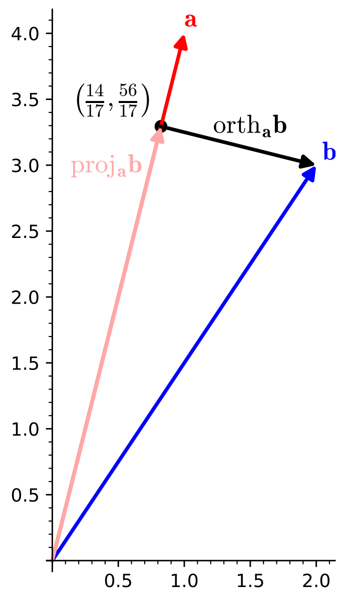


Figure 1: Illustration for Ex. 46. Actually, the vector $\mathbf{a}$ points from the origin to the point $(1,4)$ but is hidden by $\operatorname{proj}_{\mathbf{a}}\mathbf{b}$.

Ex. 48: (a) $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \operatorname{comp}_{\mathbf{b}}\mathbf{a} \iff \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|} \iff \frac{1}{|\mathbf{a}|} = \frac{1}{|\mathbf{b}|}$ or $\mathbf{a}\cdot\mathbf{b} = 0$
$\iff |\mathbf{a}| = |\mathbf{b}|$ or $\mathbf{a}\cdot\mathbf{b} = 0$
That is, $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \operatorname{comp}_{\mathbf{b}}\mathbf{a}$ iff $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or have the same length.

(b) $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \operatorname{proj}_{\mathbf{b}}\mathbf{a} \iff \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\mathbf{b} \iff \mathbf{a}\cdot\mathbf{b} = 0$ or $\frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{b}}{|\mathbf{b}|^2}$.
But $\frac{\mathbf{a}}{|\mathbf{a}|^2} = \frac{\mathbf{b}}{|\mathbf{b}|^2} \implies \frac{|\mathbf{a}|}{|\mathbf{a}|^2} = \frac{|\mathbf{b}|}{|\mathbf{b}|^2} \implies |\mathbf{a}| = |\mathbf{b}|$. Substituting this into the previous equation given $\mathbf{a} = \mathbf{b}$.
So $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \operatorname{proj}_{\mathbf{b}}\mathbf{a}$ iff $\mathbf{a}$ and $\mathbf{b}$ are either orthogonal or equal.

4 We have

$$0 = a - a = \sum_{k=0}^{\infty}(b_k - a_k)10^{-k} = \sum_{k=n}^{\infty}(b_k - a_k)10^{-k} \quad \text{or} \quad b_n - a_n = \sum_{k=n+1}^{\infty}(a_k - b_k)10^{n-k}.$$

Now $|b_n - a_n| \geq 1$ and $|a_k - b_k| \leq 9$ for $k \geq n+1$. This gives

$$1 \leq \left|\sum_{k=n+1}^{\infty}(a_k - b_k)10^{n-k}\right| \leq \sum_{k=n+1}^{\infty}|a_k - b_k|\,10^{n-k} \leq \sum_{k=n+1}^{\infty} 9\cdot 10^{n-k}$$
$$= 9\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots\right) = \frac{9}{10}\frac{1}{1-1/10} = 1.$$

(The last equality is just $0.999\cdots = 1$.) Hence equality should hold everywhere. In particular, $|a_k - b_k| = 9$ for $k \geq n+1$. This leaves the two possibilities $(a_k, b_k) = (0,9)$ or $(9,0)$. If the first alternative holds for all $k \geq n+1$, the expansion $0.a_1a_2\ldots$ would be terminating, contradiction. Similarly, the second alternative cannot hold for all $k \geq n+1$. Thus there exist $k, l \geq n+1$ such that $a_k - b_k > 0$ and $a_l - b_l < 0$. But then the triangle inequality (2nd inequality in the computation above) cannot hold with equality. This is the final contradiction, and the assertion is proved.

5 $\sqrt{2}$ is a root of $X^2 - 2$, which has roots $\pm\sqrt{2}$, and $\sqrt[3]{5}$ is a root of $X^3 - 5$, which has roots $\sqrt[3]{5}$, $\omega\sqrt[3]{5}$, $\omega^2\sqrt[3]{5}$, where $\omega = \frac{1}{2}\left(-1 + \mathrm{i}\sqrt{3}\right)$ (a primitive 3rd root of unity). There are six possible sums of a root of $X^2 - 2$ and a root of $\sqrt[3]{5}$, and the number $\sqrt{2} + \sqrt[3]{5}$ is one of them. It turns out that the polynomial

$$p(X) = \left(X - \sqrt{2} - \sqrt[3]{5}\right)\left(X - \sqrt{2} - \omega\sqrt[3]{5}\right)\left(X - \sqrt{2} - \omega^2\sqrt[3]{5}\right)\left(X + \sqrt{2} - \sqrt[3]{5}\right)\left(X + \sqrt{2} - \omega\sqrt[3]{5}\right)\left(X + \sqrt{2} - \omega^2\sqrt[3]{5}\right)$$

having all six sums as roots has integer coefficients:

$$p(X) = X^6 - 6X^4 - 10X^3 + 12X^2 - 60X + 17,$$

as a somewhat tedious hand computation shows (but I have used my computer algebra system). Another idea to find this polynomial is as follows: Writing $a = \sqrt{2} + \sqrt[3]{5}$, we have $(a - \sqrt{2})^3 = 5$, so that a is a root of

$$(X - \sqrt{2})^3 - 5 = X^3 - 3\sqrt{2}X^2 + 6X - 2\sqrt{2} - 5.n$$

Multiplying this polynomial by $(X + \sqrt{2})^3 - 5 = X^3 + 3\sqrt{2}X^2 + 6X + 2\sqrt{2} - 5$ gives a polynomial with integer coefficients having a as a root, which is in fact the same as above:

$$\begin{aligned}\left((X - \sqrt{2})^3 - 5\right)\left((X + \sqrt{2})^3 - 5\right) &= (X^2 - 2)^3 - 10X^3 - 60X + 25 \\ &= X^6 - 6X^4 - 10X^3 + 12X^2 - 60X + 17 = p(X)\end{aligned}$$

This also shows how to simplify the computation of $p(X)$, so that no calculator is needed.
For the other 3 numbers almost no further computations are necessary: Since $-\sqrt{2} + \sqrt[3]{5}$ is a root of $p(X)$, the number $b = \sqrt{2} - \sqrt[3]{5}$ is a root of $p(-X) = X^6 - 6X^4 + 10X^3 + 12X^2 + 60X + 17$, and $c = \sqrt{2}\sqrt[3]{5}$, $d = \sqrt{2}/\sqrt[3]{5}$ are roots of $X^6 - 8 \cdot 25$ and $X^6 - \frac{8}{25}$, respectively. Thus c is a root of $X^6 - 200$, and d is a root of $25X^6 - 8$.
Remarks: Algebraic numbers can also be defined as numbers in $\mathbb{R}$ (or $\mathbb{C}$) that are roots of nonzero polynomials with rational coefficients. (To see this, multiply such a polynomial by the least common multiple of the denominators of its coefficients to make it integral.) The monic polynomial in $\mathbb{Q}[X]$ of least degree having an algebraic number a as a root is uniquely determined and called *minimal polynomial* of a. The four polynomials we have determined above (provided we take the last one as $X^6 - \frac{8}{25}$) are all minimal polynomials.
If a, b are algebraic numbers then so are $a \pm b$, ab, and a/b if $b \neq 0$. This can be shown in very much the same way as for the numbers $a = \sqrt{2}$, $b = \sqrt[3]{5}$ above, arguing with the different roots of their minimum polynomials. As a consequence, the algebraic numbers form itself a field (a subfield of $\mathbb{C}$), and so do the real algebraic numbers (which form a subfield of $\mathbb{R}$). These fields are rather big (but still countable; cf. H7). Smaller extension fields of $\mathbb{Q}$ can be constructed by considering a fixed algebraic number a and determining the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and a. This field is commonly denoted by $\mathbb{Q}(a)$. As examples we have $\mathbb{Q}\left(\sqrt{2}\right) = \left\{r + s\sqrt{2}; r, s \in \mathbb{Q}\right\}$ and $\mathbb{Q}\left(\sqrt[3]{5}\right) = \left\{r + s\sqrt[3]{5} + t\sqrt[3]{5}^2; r, s, t \in \mathbb{Q}\right\}$.

6 a) $x \mapsto \arctan x$ maps $\mathbb{R}$ bijectively to $(-\pi/2, \pi/2)$. Hence $x \mapsto \frac{1}{2} + \frac{1}{\pi}\arctan x$ maps $\mathbb{R}$ bijectively to $(0,1)$. In order to include 1 in the range, we use the idea of Hilbert's Hotel. Take a countable subset S of $(0,1)$, e.g., $S = \{1/n; n = 2,3,\dots\}$, and map this bijectively to $S \cup \{1\}$, e.g., using $\frac{1}{n} \mapsto \frac{1}{n-1} = \frac{1/n}{1-1/n}$. Thus a bijection from $\mathbb{R}$ to $(0,1]$ is

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{\pi}\arctan x & \text{if } \frac{1}{2} + \frac{1}{\pi}\arctan x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}, \\ \frac{\frac{1}{2}+\frac{1}{\pi}\arctan x}{\frac{1}{2}-\frac{1}{\pi}\arctan x} & \text{if } \frac{1}{2} + \frac{1}{\pi}\arctan x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}. \end{cases}$$

The inverse map $f^{-1}\colon (0,1] \to \mathbb{R}$ has a slightly less complicated form:

$$f^{-1}(y) = \begin{cases} \tan\left(\pi\left(y - \frac{1}{2}\right)\right) & \text{if } y \notin \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}, \\ \tan\left(\pi\left(\frac{y}{1+y} - \frac{1}{2}\right)\right) & \text{if } y \in \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}. \end{cases}$$

b) Since the sequence $a_1, a_2, a_3, \dots$ has infinitely many nonzeros, it can be uniquely decomposed into words $s_1, s_2, s_3, \dots$ containing a single nonzero digit at the end and otherwise zeros. For example, for $0.120340056789\dots$ the corresponding decomposition is $s_1 = 1$, $s_2 = 2$, $s_3 = 03$, $s_4 = 4$, $s_5 = 005$, $s_6 = 6$, $\dots$

Similarly $b_1, b_2, b_3, \dots$ admits a unique decomposition into words t_1, t_2, t_3, $\dots$ of the indicated form. Now consider the map $g\colon (0,1] \times (0,1] \to (0,1]$ defined by

$$g(a,b) = g(0.a_1a_2\dots, 0.b_1b_2\dots) = 0.s_1t_1s_2t_2\dots$$

Since a non-terminating decimal fraction $c = 0.c_1c_2c_3\dots \in (0,1]$ can be written in exactly one way as $c = 0.u_1u_2u_3\dots$ with words $u_1, u_2, u_3, \dots$ of the indicated form, we have $c = g(0.u_1u_3u_5\dots, 0.u_2u_4u_6\dots)$, and this is the only way to obtain c as an image of some $(a,b) \in (0,1] \times (0,1]$. Thus g is bijective.

c) The desired bijection is obtain by composing the three bijections $f\colon \mathbb{R} \to (0,1]$ from a), $g^{-1}\colon (0,1] \to (0,1] \times (0,1]$ from b) and $(0,1] \times (0,1] \to \mathbb{R} \times \mathbb{R}$, $(a,b) \mapsto \left(f^{-1}(a), f^{-1}(b)\right)$ (which is clearly a bijection as well) in this order.

7 It suffices to prove the assertion in the hint, since then we can enumerate $\mathbb{A}$ in the following way. Since a polynomial has only finitely many zeros (no more than its degree), the set $\mathbb{A}_h$ of real numbers that are roots of some integral polynomial of fixed height h is finite for every h as well. Now we list all elements in $\mathbb{A}_1$ (in fact $\mathbb{A}_1 = \emptyset$), then all elements in $\mathbb{A}_2$ that have not been listed yet, then all elements in $\mathbb{A}_3$ that have not been listed yet, and so on.
For the proof of the assertion note that an integer polynomial $a(X)$ of height h must satisfy $\deg a(X) \leq h-1$ and $|a_i| \leq h-1$ for all i. Hence there are no more than $(2h-1)^h$ possibilities for $a(X)$, which is a finite upper bound.
For example, the integer polynomials of height 2 are $\pm X$, so that $\mathbb{A}_2 = \{0\}$; those of height 3 are $\pm X^2$, $\pm 2X$, $\pm 1 \pm X$, so that $\mathbf{A}_3 = \{0, \pm 1\}$; those of height 4 are $\pm X^3$, $\pm 2X^2$, $\pm 1 \pm X^2$, $\pm X \pm X^2$, $\pm 3X$, $\pm 1 \pm 2X$, $\pm 2 \pm X$, so that $\mathbb{A}_4 = \left\{0, \pm 1, \pm\frac{1}{2}, \pm 2\right\}$. The first irrationals appear at height 5, which is the height of (among others) $X^2 - 2$, $2X^2 - 1$, $X^2 \pm X - 1$.

Solutions where prepared by Dai Yue, Li Yulin, Wang Tengyue, Zheng Han, Shi Fangping, and Thomas Honold

Calculus III (Math 241)

W1 a) Find an equational representation $a_1x_1+a_2x_2+a_3x_3=b$ of the plane with parametric representation

$$\mathbf{x}=\begin{pmatrix}1\\0\\-2\end{pmatrix}+c_1\begin{pmatrix}0\\2\\-2\end{pmatrix}+c_2\begin{pmatrix}3\\3\\1\end{pmatrix},\quad c_1,c_2\in\mathbb{R}.$$

What is the geometric meaning of the vector $\mathbf{a}=(a_1,a_2,a_3)$?

b) Find a parametric representation of the plane $x_1+x_2+x_3=1$.

c) Explain how to make the equational representation of planes in $\mathbb{R}^3$ *canonical* (i.e., every plane should have a unique associated linear equation of the given form).

W2 a) Compute a parametric representation for the intersection of the two planes in $\mathbb{R}^3$ with equations $x_1+x_2-2x_3=4$ and $-2x_1-x_2+5x_3=0$, thereby showing that this intersection is a line.

b) Represent the line in $\mathbb{R}^3$ through the two points $(1,2,1)$ and $(3,0,-1)$ as solution set of a system of 2 linear equations.

W3 A plane H in $\mathbb{R}^3$ with equation $a_1x_1+a_2x_2+a_3x_3=0$ partitions the whole space into 3 sets:

$$\begin{aligned}H^+&=\{\mathbf{x}\in\mathbb{R}^3;a_1x_1+a_2x_2+a_3x_3>0\},\\ H&=\{\mathbf{x}\in\mathbb{R}^3;a_1x_1+a_2x_2+a_3x_3=0\},\\ H^-&=\{\mathbf{x}\in\mathbb{R}^3;a_1x_1+a_2x_2+a_3x_3<0\},\end{aligned}$$

and similarly for lines in $\mathbb{R}^2$. Can you distinguish the “halfspaces” H^+ and H^- geometrically by a property satisfied by $\mathbf{a}=(a_1,a_2,a_3)$ and the points in H^+, H^- ?

W4 DAVID HILBERT’s *Hotel Infinitude* contains infinitely many rooms numbered by $1,2,3,\dots$. The hotel is fully booked during the Midautumn Festival and

a) a new guest arrives;

b) a new guest arrives and insists on being accommodated in Room no. 88;

c) countably many new guests arrive;

d) countably many new tourist groups, each consisting of countably many tourists, arrive;

e) a continuum of new guests arrives.

Explain what you as the hotel manager can do in each case.

Solutions

1 This and the next exercise is related to Section 12.5 of the textbook [Ste21] and the material on Analytic Geometry in the lecture. The terminology used in the textbook is to some extent different from that of the lecture. We do not follow [Ste21] in calling the parametric representation (with parameters c_1, c_2) in a) "vector equation" or "parametric equation" (since, strictly speaking, these are not equations). The "scalar equation" of a plane in [Ste21] is what a) asks for. We use the term "equational representation" for this, because it applies also to lines, which are represented by 2 linear equations. Finally, the "symmetric equation" for a line in [Ste21] won't be used at all in the course (since it is not needed and holds only for some lines).

a) There are several ways to solve this.

1st solution The desired equation has the form $a_1x_1 + a_2x_2 + a_3x_3 = b$ with $a_1, a_2, a_3, b \in \mathbb{R}$ and not all a_i equal to zero. Substituting the coordinates x_i gives

$$a_1(1+3c_2) + a_2(2c_1+3c_2) + a_3(-2-2c_1+c_2) = b \quad \text{for all } c_1, c_2 \in \mathbb{R}.$$

This is equivalent to

$$a_1 - 2a_3 + (2a_2 - 2a_3)c_1 + (3a_1 + 3a_2 + a_3)c_2 = b \quad \text{for all } c_1, c_2 \in \mathbb{R},$$

and in turn to

$$a_1 - 2a_3 = b \wedge 2a_2 - 2a_3 = 0 \wedge 3a_1 + 3a_2 + a_3 = 0.$$

Setting $a_3 = 1$ gives $a_2 = 1$, $a_1 = -\frac{4}{3}$, $b = -\frac{10}{3}$. Hence an equation for the plane is

$$-4x_1 + 3x_2 + 3x_3 = -10,$$

obtained by scaling the original equation by 3 (so that the coefficients become integers).

2nd solution Denoting the given plane by H, we have $\mathbf{p} = (1,0,-2) \in H$ (put $c_1 = c_2 = 0$), and hence a_1, a_2, a_3, b must satisfy $a_1p_1 + a_2p_2 + a_3p_3 = a_1 - 2a_3 = b$. This allows us to rewrite the equation as $a_1x_1 + a_2x_2 + a_3x_3 = a_1p_1 + a_2p_2 + a_3p_3$, or $a(x_1 - p_1) + a_2(x_2 - p_2) + a_3(x_3 - p_3) = 0$. Using the dot product, this can be concisely written as $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$ and says that $\mathbf{a}$ must be orthogonal to any difference of two points in H ("direction vectors" of H). Particular such vectors are $(0,2,-2)$ and $(3,3,1)$, giving $2a_2 - 2a_3 = 0 \wedge 3a_1 + 3a_2 + a_3 = 0$. Thus we obtain the same equations for a_1, a_2, a_3, b as in the 1st solution.

The vector $\mathbf{a}$ may also be taken as the cross product of $(0,2,-2)$, and $(3,3,1)$, but this is somewhat harder to compute than the above solution.

3rd solution We take 3 non-collinear points on the plane, e.g., $\mathbf{p}_1 = (1,0,-2)$ and the two points $\mathbf{p}_2 = (1,2,-4)$ (parameters $c_1 = 1$, $c_2 = 0$), $\mathbf{p}_3 = (4,3,-1)$ ($c_1 = 0$, $c_2 = 1$). These points determine the plane and hence its equation up to a scalar multiple. Substituting the points into the equation gives the system

$$\begin{array}{rcrcrcl} a_1 & & & - & 2a_3 & = & b \\ a_1 & + & 2a_2 & - & 4a_4 & = & b \\ 4a_1 & + & 3a_2 & - & a_3 & = & b \end{array}$$

This system is equivalent to that in the previous solutions (subtract the first equation from the other two equations).

The geometric meaning of $\mathbf{a} = (a_1, a_2, a_3)$ is that $\mathbf{a}$ must be orthogonal to any direction vector of H (i.e., a so-called *normal vector* of H); cp. the 2nd solution.

b) From any 3 non-collinear points $\mathbf{p_1}$, $\mathbf{p_2}$, $\mathbf{p_3}$ on the plane a parametric representation is obtained as

$$H = \{\mathbf{p}_1 + c_1(\mathbf{p}_2 - \mathbf{p}_1) + c_2(\mathbf{p}_3 - \mathbf{p}_1); c_1, c_2 \in \mathbb{R}\},$$

for which we also use the shorthand $H = \mathbf{p}_1 + \mathbb{R}(\mathbf{p}_2 - \mathbf{p}_1) + \mathbb{R}(\mathbf{p}_3 - \mathbf{p}_1)$. Here we can take $\mathbf{p}_1 = (1,0,0)$, $\mathbf{p}_2 = (0,1,0)$, $\mathbf{p}_3 = (0,0,1)$ and obtain

$$H = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

c) The equation of a plane can be made unique by setting the first nonzero coefficient among a_1, a_2, a_3 equal to 1. For example, the canonical equation of the plane in a) is $x_1 - \frac{3}{4}x_2 - \frac{3}{4}x_3 = \frac{5}{2}$. Such canonical forms are important for representing planes (and affine subspaces of $\mathbb{R}^n$ in general) on a computer. Later we will discuss the reduced row echelon form of a matrix, which generalizes this.

2 a) Setting $x_1 = c$ we get $x_2 - 2x_3 = 4 - c \wedge -x_2 + 5x_3 = 2c$. Solving for x_2, x_3 gives $x_3 = 4/3 + c/3$, $x_2 = 2x_3 + 4 - c = 20/3 - c/3$ and

$$L = \left\{ \begin{pmatrix} c \\ 20/3 - c/3 \\ 4/3 + c/3 \end{pmatrix}; c \in \mathbb{R} \right\} = \begin{pmatrix} 0 \\ 20/3 \\ 4/3 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ -1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 20/3 \\ 4/3 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix},$$

a line with the particular point $\mathbf{p}_1 = (0, 20/3, 4/3)$ and direction vector $(3, -1, 1)$.

Provided we know already that the solution is a line L, we can also determine two distinct points $\mathbf{p}_1, \mathbf{p}_2 \in L$ and obtain L as $\mathbf{p}_1 + \mathbb{R}(\mathbf{p}_2 - \mathbf{p}_1)$.

b) The coefficients of such an equation must satisfy

$$\begin{array}{rcrcrcl} a_1 & + & 2a_2 & + & a_3 & = & b \\ 3a_1 & & & - & a_3 & = & b \end{array}$$

Two linearly independent solutions of this linear system of equations for a_1, a_2, a_3, b are obtained by setting $(a_1, a_2) = (1,0)$ respectively $(0,1)$ and solving for a_3, b. The corresponding solutions are $(a_1, a_2, a_3, b) = (1,0,1,2)$ and $(0,1,-1,1)$, so that a suitable system is

$$\begin{array}{rcrcrcl} x_1 & & & + & x_3 & = & 2 \\ & & x_2 & - & x_3 & = & 1 \end{array}$$

3 The property in question is related to the angle between the normal vector $\mathbf{a} = (a_1, a_2, a_3)$ and the point $\mathbf{x} = (x_1, x_2, x_3)$, which is given by

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{x}}{|\mathbf{a}|\,|\mathbf{x}|} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{x_1^2 + x_2^2 + x_3^2}},$$

the 3-dimensional analogue of the angle formula given in the lecture.

The points in H^+ have $\cos\theta > 0$ and hence form an acute angle ($\theta < 90^0$) with $\mathbf{a}$. The points in H^- have $\cos\theta < 0$ and hence form an obtuse angle ($\theta > 90^0$) with $\mathbf{a}$. In particular, the normal vector $\mathbf{a}$ points into H^+ (or, viewed as a point, is contained in H^+).

A similar interpretation holds for a general plane $a_1x_1 + a_2x_2 + a_3x_3 = b$, as can be seen from the form $\mathbf{a}\cdot(\mathbf{x}-\mathbf{p}) = 0$. If we define $H^\pm = \{\mathbf{x} \in \mathbb{R}^3; a_1x_1 + a_2x_2 + a_3x_3 \gtrless b\}$, then again H^+ is characterized as the halfspace into which the normal vector $\mathbf{a}$ points (or as the halfspace which contains $\mathbf{a}+\mathbf{p}$).

4 a) Ask the guest in Room i to move to Room $i+1$, for $i = 1,2,3,\dots$, and accommodate the new guest in the then vacant Room 1.

b) Ask the guest in Room i, $i = 88,89,90,\dots$, to move to Room $i+1$, and accommodate the new guest in Room 88.

c) Ask the guest in Room i to move to Room $2i$ and accommodate the new guests in Rooms $1,3,5,\dots$

d) First move the guest in Room i to Room 2^i, $i = 1,2,3\dots$ Then accommodate the tourists of the 1st group in Rooms $3,6,12,24,\dots$, those of the 2nd group in Rooms $5,10,20,40,\dots$, those of the 3rd group in Rooms $7,14,28,56,\dots$, and so on. Since every positive integer n has a unique representation $n = m2^k$ with m odd and $k \in \{0,1,2,\dots\}$, this produces a booking schedule with exactly one guest in each room.

e) Sell the hotel to a classmate who doesn't yet now about set theory. There is no way to accommodate all new guests, because there is no bijection (or injection) from $\mathbb{R}$ to $\mathbb{N}$.

Notes: The purpose of Part a) is to illustrate the fact that $\mathbb{N}$ (in fact every infinite set) can be mapped bijectively onto a proper subset. Similarly, there exist surjections from $\mathbb{N}$ to $\mathbb{N}$ which are not injective, for example $1 \mapsto 1$ and $i \mapsto i-1$ for $i \geq 2$.

For finite sets this is not true. Whenever we have an injection (or surjection) from a finite set to itself, it must automatically be a bijection.

Parts c) and d) illustrate the (strange at the first glance) laws for computing with cardinalities of infinite sets: $|\mathbb{N}| + |\mathbb{N}| = |\mathbb{N}|$, $|\mathbb{N}|\cdot|\mathbb{N}| = |\mathbb{N}|$, and similarly for other infinite sets. On the contrary, $|S| + |S| > |S|$, $|S|^2 > |S|$ for any finite set S of size $|S| \geq 2$.

The cardinality of $\mathbb{R}$, or of any interval in $\mathbb{R}$ of positive length (which is the same, cf. Exercise H6 a) in the 1st homework assignment) is also referred to as the cardinality of the *continuum*.

Calculus III (Math 241)

H8 Solve [Ste21], Section 12.4, Exercises 48–52.

H9 Solve [Ste21], Section 12.5, Exercises 72, 74, 76, 80.

H10 Do Exercises 26, 28, 30 in [Ste21], Ch. 13.1.

H11 Do Exercise 58 in [Ste21], Ch. 13.1.

H12 *Optional Exercise*

The dot product of two complex vectors $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{C}^n$ is defined as $\mathbf{z} \cdot \mathbf{w} = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n$ and the length of $\mathbf{z} \in \mathbb{C}^n$ as $|\mathbf{z}| = \sqrt{\mathbf{z} \cdot \mathbf{z}}$.

a) Derive properties of the complex dot product that are analogous to Properties (D1)–(D4) of the real dot product; cf. the lecture.
b) Show that the Cauchy-Schwarz Inequality generalizes to $\mathbb{C}^n$ (where of course linear dependence of $\mathbf{z}, \mathbf{w}$ over $\mathbb{C}$ is involved in the 2nd part).

H13 *Optional Exercise*

Prove the follwing quantitative version of the Cauchy-Schwarz Inequality:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2$$

for any choice of real (or complex) numbers $a_1, \dots, a_n, b_1, \dots, b_n$.

Due on Wed Oct 11, 6 pm

The optional exercises can be handed in until Wed Oct 18, 6 pm. Exercises H10 and H11, which are quite short, should be done after the first lecture on vector functions (Sat Oct 7).

Solutions

8 **Ex. 48** We have $\mathbf{c} = -\mathbf{a} - \mathbf{b} \Longrightarrow \mathbf{b} \times \mathbf{c} = \mathbf{b} \times (-\mathbf{a} - \mathbf{b}) = -\mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, since $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{b} = \mathbf{0}$. Cyclically permuting $\mathbf{a}, \mathbf{b}, \mathbf{c}$, which doesn't affect the condition $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, gives the second equality.
Ex. 49 $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = \underbrace{\mathbf{a} \times \mathbf{a}}_{\mathbf{0}} - \underbrace{\mathbf{b} \times \mathbf{a}}_{-\mathbf{a} \times \mathbf{b}} + \mathbf{a} \times \mathbf{b} - \underbrace{\mathbf{b} \times \mathbf{b}}_{\mathbf{0}} = 2(\mathbf{a} \times \mathbf{b})$
Ex. 50 The proof is a rather tedious calculation, which is simplified a bit by the symmetry properties of the cross product.

$$
\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} = \begin{pmatrix} a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1) \\ a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \end{pmatrix} \\
&= \begin{pmatrix} (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 \\ (a_3c_3 + a_1c_1)b_2 - (a_3b_3 + a_1b_1)c_2 \\ (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 \end{pmatrix} \\
&= \begin{pmatrix} (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1 + b_1 + a_2b_2 + a_3b_3)c_1 \\ (a_2c_2 + a_3c_3 + a_1c_1)b_2 - (a_2 + b_2 + a_3b_3 + a_1b_1)c_2 \\ (a_3c_3 + a_1c_1 + a_2c_2)b_3 - (a_3 + b_3 + a_1b_1 + a_2b_2)c_3 \end{pmatrix} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\end{aligned}
$$

More conceptual proofs are possible, but I don't know of any substantially simpler proof. For example, one may observe that both sides of the identity are zero if $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent (since then $\mathbf{b} \times \mathbf{c} = \mathbf{0}$ and similarly for the right-hand side). Otherwise $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$ form a basis of $\mathbb{R}^3$ and we can write $\mathbf{a} = \lambda_1 \mathbf{b} + \lambda_2 \mathbf{c} + \lambda_3(\mathbf{b} \times \mathbf{c})$ for some $\lambda_i \in \mathbb{R}$. Since both sides of the identity are linear in $\mathbf{a}$, it suffices to check the identity in the three cases $\mathbf{a} = \mathbf{b}$, $\mathbf{c}$, $\mathbf{b} \times \mathbf{c}$. The last case is trivial. In the first case it reduces to $\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{b} - |\mathbf{b}|^2 \mathbf{c}$. It is clear that $\mathbf{v} = \mathbf{b} \times (\mathbf{b} \times \mathbf{c}) = \mu_1 \mathbf{b} + \mu_2 \mathbf{c}$ for some $\mu_1, \mu_2 \in \mathbb{R}$, since $\mathbf{v}$ is orthogonal to $\mathbf{b} \times \mathbf{c}$ and hence must be in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$. The constants μ_1, μ_2 can be computed from $\mathbf{v} \cdot \mathbf{b} = 0$, which gives $\mu_1 |\mathbf{b}|^2 + \mu_2(\mathbf{b} \cdot \mathbf{c}) = 0$, and $\mathbf{v} \cdot \mathbf{c} = -|\mathbf{b} \times \mathbf{c}|^2$ (a special case of a triple product identity), which gives $\mu_1(\mathbf{b} \cdot \mathbf{c}) + \mu_2 |\mathbf{c}|^2 = -|\mathbf{b} \times \mathbf{c}|^2$. They turn out to have the required form. The second case $\mathbf{a} = \mathbf{c}$ is done similarly.
Ex. 51 The 2nd and 3rd summand in the identity arise from the 1st by the cyclic permutation $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, so when using Ex. 50 we just need to apply the same cyclic permutation to the right-hand side of the identity. This gives

$$
\begin{aligned}
&\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b} \\
&\quad = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \\
&\quad = \mathbf{0},
\end{aligned}
$$

since the 1st and 6th, the 2nd and 3rd, and the 4th and 5th summand cancel due to the symmetry of the dot product.
Ex. 52 This is a rather straightforward consequence of Ex. 50 and the fact that the triple product

is invariant under cyclic permutations of its three arguments:

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \mathbf{d}\cdot\big((\mathbf{a}\times\mathbf{b})\times\mathbf{c}\big) && \text{(cyclic invariance)}\\
&= -\mathbf{d}\cdot\big(\mathbf{c}\times(\mathbf{a}\times\mathbf{b})\big)\\
&= -\mathbf{d}\cdot\big((\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}\big) && \text{(Ex. 50)}\\
&= -\big((\mathbf{c}\cdot\mathbf{b})(\mathbf{d}\cdot\mathbf{a})-(\mathbf{c}\cdot\mathbf{a})(\mathbf{d}\cdot\mathbf{b})\big)\\
&= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}.
\end{aligned}$$

9 Ex. 72 A particular point on the plane, call it E, is $\mathbf{a}=(8,0,0)$. The distance from $\mathbf{b}=(-6,3,5)$ to E is equal to the orthogonal projection of $\mathbf{b}-\mathbf{a}=(-14,3,5)$ to the subspace spanned by the normal vector $\mathbf{n}=(1,-2,-4)$ of E, which is a line through the origin.

$$d=\left|\frac{(\mathbf{b}-\mathbf{a})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right| = \frac{|(\mathbf{b}-\mathbf{a})\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{|-14\cdot 1+3(-2)+5\cdot(-4)|}{\sqrt{1^2+(-2)^2+(-4)^2}} = \frac{40}{\sqrt{21}} \approx 8.73.$$

This is essentially the shortest solution.[1] Another way to solve the exercise has been discussed in the lecture: Compute the orthogonal projection $\mathbf{x}^*$ of $\mathbf{b}-\mathbf{a}$ to the direction space U of E (the plane $x-2y-4z=0$), and obtain d as $d=|\mathbf{b}-\mathbf{a}-\mathbf{x}^*|$. Two spanning vectors of U are $\mathbf{u}_1=(2,1,0)$, $\mathbf{u}_2=(4,0,1)$. Then $\mathbf{u}_1\cdot\mathbf{u}_1=5$, $\mathbf{u}_2\cdot\mathbf{u}_2=17$, $\mathbf{u}_1\cdot\mathbf{u}_2=\mathbf{u}_2\cdot\mathbf{u}_1=8$, $\mathbf{u}_1\cdot(\mathbf{b}-\mathbf{a})=-25$, $\mathbf{u}_2\cdot(\mathbf{b}-\mathbf{a})=-51$, so that the coefficients of $\mathbf{x}^*=x_1^*\mathbf{u}_1+x_2^*\mathbf{u}_2$ solve the system $5x_1+8x_2=-25$, $8x_1+17x_2=-51$. The solution is $x_1^*=-\frac{17}{21}$, $x_2^*=-\frac{55}{21}$, and hence

$$d=\left|\begin{pmatrix}-14\\3\\5\end{pmatrix}+\frac{17}{21}\begin{pmatrix}2\\1\\0\end{pmatrix}+\frac{55}{21}\begin{pmatrix}4\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}-40/21\\80/21\\160/21\end{pmatrix}\right| = \frac{40}{21}\left|\begin{pmatrix}-1\\2\\4\end{pmatrix}\right| = \frac{40}{\sqrt{21}}.$$

The vector $\frac{40}{21}(-1,2,4)$, which is the orthogonal projection of $\mathbf{b}-\mathbf{a}$ to the line $\mathbb{R}\mathbf{n}$ and also equal to $\mathbf{b}-\mathbf{a}-\mathbf{x}^*$, occurs in both solutions.

Ex. 74 Putting $x=y=0$ in the equation of the first plane, we get that the point $(0,0,0)$ is on the first plane. Since the planes are parallel the distance D between them is the distance from $(0,0,0)$ to the second plane $3x-6y+9z-1=0$. Writing this equation as $\mathbf{n}\cdot\mathbf{x}-\mathbf{n}\cdot\mathbf{a}=0$ with $\mathbf{n}=(3,6,9)$ and $\mathbf{a}$ any point on this plane, e.g., $\mathbf{a}=(\frac{1}{3},0,0)$, we obtain

$$D=\frac{|\mathbf{n}\cdot(\mathbf{0}-\mathbf{a})|}{|\mathbf{n}|} = \frac{|3(0)-6(0)+9(0)-1|}{\sqrt{3^2+(-6)^2+9^2}} = \frac{1}{\sqrt{126}} = \frac{1}{3\sqrt{14}} \approx 0.0891.$$

Here we are using the fact that the distance of a point $\mathbf{x}$ to the second plane is equal to the length of the orthogonal projection of $\mathbf{x}-\mathbf{a}$ onto the line $\mathbb{R}\mathbf{n}$. Alternatively, we could work with the orthogonal projection of $\mathbf{x}-\mathbf{a}$ onto $\mathbb{R}\mathbf{u}_1+\mathbb{R}\mathbf{u}_2$, where $\mathbf{u}_1,\mathbf{u}_2$ are linearly independent direction vectors of the second plane (the method used in the lecture), but this method requires a lot more effort.

[1] Th. 9 on p. 871 of [Ste21] gives a slightly simpler formula for d, which uses that an equation for E is $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{a}=8$. Thus $\mathbf{n}\cdot(\mathbf{b}-\mathbf{a})=\mathbf{n}\cdot\mathbf{b}-\mathbf{n}\cdot\mathbf{a}=1(-6)-2\cdot 3-4\cdot 5-8=-40$. One can drive Th. 9 even further and rewrite the plane equation as $n_1'x_1+n_2'x_2+n_3'x_3+d'=0$ with $\mathbf{n}'=(n_1',n_2',n_3')$ normalized to unit length. Now the distance is obtained simply by inserting $\mathbf{b}$ into the left-hand side of this equation and changing the sign to $+$ if necessary.

Ex. 76 The planes must have normal vectors parallel to the normal vector of the given plane. Since normal vectors are determined only up to scalar multiples, we can just take $\mathbf{n} = (1,2,-2)$, i.e., the planes parallel to the given plane have equations $x+2y-2z = k$, $k \in \mathbb{R}$. The distance between the two parallel planes is then

$$\frac{|1-k|}{\sqrt{1^2+2^2+(-2)^2}} = 2 \iff |1-k| = 6 \iff k = 7 \vee k = -5$$

Thus, the equations of the planes parallel to the given plane are $x+2y-2z = 7$ and $x+2y-2z = -5$.

Ex. 80 L_1: A direction vector for L_1 is $\mathbf{v}_1 = (1,2,2)$.
L_2: A normal vector for the plane π_1 is $\mathbf{n}_1 = (1,-1,2)$. Two linearly independent direction vectors of the plane π_2 are $(1,2,1)-(0,0,1) = (1,2,0)$ and $(3,2,-1)-(0,0,1) = (3,2,-2)$; thus, a normal vector for π_2 is $(1,2,0)\times(3,2,-2) = (-4,2,-4)$, or for simplicity, we can use $\mathbf{n}_2 = (2,-1,2)$. Since $(3,2,-1)$ lies on both planes, it also lies on L_2.
As L_1 and L_2 are neither parallel nor intersecting, they are skew. We can view the two lines as lying in two parallel planes. A common vector that is normal to both planes is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-2,-1,2)$. Notice that L_1 passes through the point $(1,2,6)$. An equational representation of the first plane is then $-2x-y+2z = 8$. Similarly, the point $(3,2,-1)$ lies on L_2; therefore, an equational representation of the second plane is $-2x-y+2z = -10$. The distance between two lines is then

$$D = \frac{|-10-8|}{\sqrt{2^2+1^2+2^2}} = \frac{18}{3} = 6.$$

Alternative solution (using the method discussed in the lecture): As in the first solution we obtain that the line L_1 has parametric representation

$$L_1 = \begin{pmatrix}1\\2\\6\end{pmatrix} + \mathbb{R}\begin{pmatrix}1\\2\\2\end{pmatrix},$$

and the planes π_1 and π_2 have equations $x-y+2z = -1$ and $2x-y+2z = 2$, respectively. A parametric representation of L_2 can be obtained by solving the linear system $x-y+2z = -1 \wedge 2x-y+2z = 2$. One step of Gaussian elimination yields

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 2 & -1 & 2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 4 \end{array}\right],$$

from which we infer that z is free, $y = 4+2z$, $x = -1+y-2z = 3$, and

$$L_2 = \left\{ \begin{pmatrix}3\\4+2z\\z\end{pmatrix} ; z \in \mathbb{R} \right\} = \begin{pmatrix}3\\4\\0\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\2\\1\end{pmatrix}.$$

From the lecture, D is equal to the distance between $(3,4,0)-(1,2,6) = (2,2,-6)$ and the orthogonal projection of $(2,2,-6)$ onto the subspace $\mathbb{R}(1,2,2)+\mathbb{R}(0,2,1)$. Computing the respective dot products gives the linear system

$$\begin{array}{rcrcr} 9x_1 & + & 6x_2 & = & -6, \\ 6x_1 & + & 5x_2 & = & -2. \end{array}$$

Solving the system, e.g. in the matrix way

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 6 & 5 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ -2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 & -6 \\ -6 & 9 \end{pmatrix} \begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix},$$

gives the projection point $\mathbf{x}^* = (-2)(1,2,2) + 2(0,2,1) = (-2,0,-2)$ and

$$D = \left| \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} \right| = \left| \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \right| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6.$$

10

Ex. 26: $x = \sin t$, $y = \cos t$ indicates that the projection onto the (x,y)-plane should be a circle centered at origin. Only I (it is very hard to discern that its projection on X-Y plane is a circle, but in fact it is), IV and VI have this property. In addition, $z = 1/(1+t^2)$ indicates that $t \in (0,1]$ and the graph should be much more dense when t approaches 0. Since IV extends to $+\infty$ and I ranges from -1 to $+1$, the correct one must be VI.

Ex. 28: As shown in Ex. 26, its projection onto the (x,y)-plane should be a circle centered at the origin. Besides, its z-range ranges from -1 to $+1$ and features a periodic function, which indicates that the graph should be a closed curve. Only I satisfies this.

Ex. 30: Since $\cos^2 t = \frac{1+\cos 2t}{2}$ and $\sin^2 t = \frac{1-\cos(2t)}{2}$, this curve can be written as

$$\mathbf{r}(t) = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \cos(2t) \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and looks like the graph of a sine/cosine function drawn in the plane $x+y=1$. Hence it matches III.

11 The paths (ranges of the curves) intersect in the two points

$$\begin{aligned} (1,1,1) &= \mathbf{r}_1(1) = \mathbf{r}_2(0), \\ (2,4,8) &= \mathbf{r}_1(2) = \mathbf{r}_2(1/2), \end{aligned}$$

the solutions of the equation $\mathbf{r}_1(s) = \mathbf{r}_2(t)$. The parametric curves itself do not intersect, since the equation $\mathbf{r}_1(t) = \mathbf{r}_2(t)$ has no solution.

12 a) For all $\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2, \mathbf{w} \in \mathbb{C}^n$ and $c \in \mathbb{C}$ the following are true:

(D1) $(\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{w} = \mathbf{z}_1 \cdot \mathbf{w} + \mathbf{z}_2 \cdot \mathbf{w}$;

(D2) $(c\mathbf{z}_1) \cdot \mathbf{w} = c(\mathbf{z}_1 \cdot \mathbf{w})$;

(D3) $\mathbf{w} \cdot \mathbf{z} = \overline{\mathbf{z} \cdot \mathbf{w}}$;

(D4) $\mathbf{z} \cdot \mathbf{z} \geq 0$ with equality iff $\mathbf{z} = \mathbf{0}$.

The proofs are rather trivial and omitted. Thus the only property which changes is (D3). As a consequence of (D1), (D2), (D3), we have $\mathbf{z} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{z} \cdot \mathbf{w}_1 + \mathbf{z} \cdot \mathbf{w}_2$ and $\mathbf{z} \cdot (c\mathbf{w}) = \overline{c}(\mathbf{z} \cdot \mathbf{w})$ for $\mathbf{z}, \mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in \mathbb{C}^n$, $c \in \mathbb{C}$. One also says that the complex dot product is "half-linear" in the second argument and, in all, "$1\frac{1}{2}$-fold linear" (or "sesquilinear").

b) The correct generalization of the Cauchy-Schwarz Inequality to $\mathbb{C}^n$ is:

$$\left|\sum_{i=1}^{n} z_i\overline{w}_i\right| \leq \sqrt{\sum_{i=1}^{n}|z_i|^2}\cdot\sqrt{\sum_{i=1}^{n}|w_i|^2} \quad \text{for } z_1,\dots,z_n,w_1,\dots,w_n \in \mathbb{C},$$

equality holds iff $\mathbf{z}=(z_1,\dots,z_n)$ and $\mathbf{w}=(w_1,\dots,w_n)$ are linearly dependent in the $\mathbb{C}$-vectorspace $\mathbb{C}^n$, i.e., $\mathbf{z}=\mathbf{0}$ or there exists $c\in\mathbb{C}$ such that $\mathbf{w}=c\mathbf{z}$. Using the complex dot product, the inequality remains just "the same", viz. $|\mathbf{z}\cdot\mathbf{w}|\leq|\mathbf{z}|\,|\mathbf{w}|$.

For the proof we can use the identity quoted at the end of the solution to H13, whose proof is not difficult, or alternatively argue as follows: Consider the chain of inequalities

$$\left|\sum_{i=1}^{n} z_i\overline{w}_i\right| \leq \sum_{i=1}^{n}|z_i|\,|\overline{w}_i| = \sum_{i=1}^{n}|z_i|\,|w_i| \leq \sqrt{\sum_{i=1}^{n}|z_i|^2}\cdot\sqrt{\sum_{i=1}^{n}|w_i|^2}.$$

The first inequality follows from well-known properties of the absolute value of complex numbers, and the second inequality is a special case of the real Cauchy-Schwarx inequality. If equality holds in the complex Cauchy-Schwarx Inequality, equality must hold in both inequalities of the chain. This in turn implies $z_i\overline{w}_i=r_i\mathrm{e}^{\mathrm{i}\phi}$ with $r_i\geq 0$ and ϕ independent of i (from the first inequality) and $|w_i|=\lambda|z_i|$ with $\lambda\geq 0$ independent of i (from the second inequality, where w.l.o.g. we have assumed $\mathbf{z}\neq\mathbf{0}$). From this we get $r_i=|z_i\overline{w}_i|=\lambda|z_i|^2$ and $z_i\overline{w}_i=\lambda|z_i|^2\mathrm{e}^{\mathrm{i}\phi}$, which gives $\overline{w}_i=(\lambda\mathrm{e}^{\mathrm{i}\phi})\overline{z}_i$ for all $i\in\{1,\dots,n\}$. (Although we cannot cancel factors $z_i=0$, the identity $\overline{w}_i=(\lambda\mathrm{e}^{\mathrm{i}\phi})\overline{z}_i$ holds then nevertheless, since $z_i=0$ implies $w_i=0$.) This shows $\mathbf{w}=c\mathbf{z}$ with $c=\lambda\mathrm{e}^{-\mathrm{i}\phi}$, and completes the proof.

13 Expanding the left-hand side gives

$$\sum_{i=1}^{n} a_i^2b_i^2+\sum_{1\leq i<j\leq n} a_i^2b_j^2+a_j^2b_i^2-\left(\sum_{i=1}^{n}(a_ib_i)^2+\sum_{1\leq i<j\leq n}2a_ib_ia_jb_j\right)=$$
$$=\sum_{1\leq i<j\leq n}\left(a_i^2b_j^2+a_j^2b_i^2-2a_ib_ia_jb_j\right)=\sum_{1\leq i<j\leq n}(a_ib_j-a_jb_i)^2,$$

as asserted.
Remark: The formula also holds for complex numbers, but since in this case the involved quantities need not be real, we cannot conclude any inequality from it. There is, however, a related identity that provides a quantitative version of the complex Cauchy-Schwarz Inequality in H12:

$$\left(\sum_{i=1}^{n}|a_i|^2\right)\left(\sum_{i=1}^{n}|b_i|^2\right)-\left|\sum_{i=1}^{n}a_i\overline{b}_i\right|^2=\sum_{1\leq i<j\leq n}\left|a_ib_j-a_jb_i\right|^2$$

for complex numbers $a_1,\dots,a_n,b_1,\dots,b_n$. This identity is proved in a similar way.

Solutions where prepared by Chen Haonan and Thomas Honold

Calculus III (Math 241)

W5 a) Explain the following observation, which at the first glance seems somewhat paradoxical: When drawing the line connecting two distinct points $\mathbf{a}, \mathbf{b}$ on a whiteboard, we can add the midpoint $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ without knowing the position of the origin, but not the sum $\mathbf{a}+\mathbf{b}$.

b) Give an analytic geometry proof of the following well-known theorem from plane geometry: The three medians (lines connecting a vertex of a triangle to the midpoint of the opposite side) are concurrent in a point, which divides each median in the ratio $2:1$.

W6 Solve [Ste21], Section 12.5, Exercises 12, 70, 77.

W7 Determine the maximum value of the function $f(x_1, x_2, x_3) = x_1 + 2x_2 - x_3$ on the sphere $S = \{\mathbf{x} \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 3\}$.

Hint: Use the Cauchy-Schwarz Inequality.

W8 Determine the volume of the pyramid ("tetrahedron") with vertices $(1, -1, 0)$, $(2, 0, -1)$, $(2, 0, 2)$, $(0, 1, 7)$.

Solutions

5 a) Choosing a different origin corresponds to a translation $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \to \mathbf{x}+\mathbf{v}$. Since

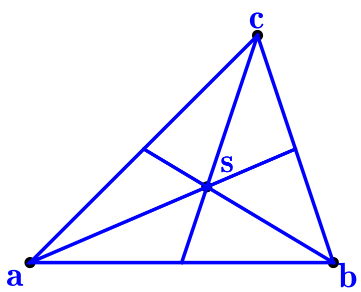
$$\frac{1}{2}(\mathbf{a}+\mathbf{v}+\mathbf{b}+\mathbf{v}) = \frac{1}{2}(\mathbf{a}+\mathbf{b})+\mathbf{v},$$

the midpoint is also translated by $\mathbf{v}$ and hence the whole picture is preserved. (This holds, more generally, for arbitrary affine combinations of $\mathbf{a}$ and $\mathbf{b}$.)

For the sum $\mathbf{a}+\mathbf{b}$ this is not true, since $(\mathbf{a}+\mathbf{v})+(\mathbf{b}+\mathbf{v}) = \mathbf{a}+\mathbf{b}+2\mathbf{v}$.

b) The easiest way to solve this exercise uses the observations that the point of concurrency is the centroid $\mathbf{s} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ of the triangle, and that the point dividing the line segment from $\mathbf{x}$ to $\mathbf{y}$ in the ratio $2:1$ is $\mathbf{x}+\frac{2}{3}(\mathbf{y}-\mathbf{x}) = \frac{\mathbf{x}+2\mathbf{y}}{3}$ (where $\frac{\mathbf{v}}{3}$ means $\frac{1}{3}\mathbf{v}$). The assertion then follows directly from

$$\mathbf{s} = \frac{\mathbf{a}}{3}+\frac{2}{3}\cdot\frac{\mathbf{b}+\mathbf{c}}{2} = \frac{\mathbf{b}}{3}+\frac{2}{3}\cdot\frac{\mathbf{a}+\mathbf{c}}{2} = \frac{\mathbf{c}}{3}+\frac{2}{3}\cdot\frac{\mathbf{a}+\mathbf{b}}{2}.$$



Another solution computes the intersection point of the lines $\mathbf{a}+\mathbb{R}\left(\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}\right)$ and $\mathbf{b}+\mathbb{R}\left(\frac{\mathbf{a}+\mathbf{c}}{2}-\mathbf{b}\right)$, which is given by

$$\mathbf{a}+\lambda\frac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{2} = \mathbf{b}+\mu\frac{\mathbf{a}+\mathbf{c}-2\mathbf{b}}{2} \iff \left(1-\lambda-\frac{\mu}{2}\right)\mathbf{a}+\left(\frac{\lambda}{2}-1+\mu\right)\mathbf{b}+\frac{\lambda-\mu}{2}\mathbf{c} = \mathbf{0}.$$

We can assume $\mathbf{a} = \mathbf{0}$ and must then have that $\mathbf{b}$, $\mathbf{c}$ are linearly independent. This implies

$$\frac{\lambda}{2}-1+\mu = \frac{\lambda-\mu}{2} = 0.$$

The solution is $\lambda = \mu = \frac{2}{3}$ and gives the centroid $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ as intersection point. Then one verifies that $\mathbf{s}$ is also on the third line $\mathbf{c}+\mathbb{R}\left(\frac{\mathbf{a}+\mathbf{b}}{2}-\mathbf{c}\right)$.

6 Ex. 12 Setting $x_1 = c$, we obtain the two equations $2y+3z = 1-c$, $-y+z = 1-c$ for y, z. Solving gives $z = \frac{1-c+2(1-c)}{5} = \frac{3}{5}-\frac{3}{5}c$, $y = z-1+c = -\frac{2}{5}+\frac{2}{5}c$, so that the line of intersection L has the parametric form

$$L = \left\{\begin{pmatrix} c \\ -\frac{2}{5}+\frac{2}{5}c \\ \frac{3}{5}-\frac{3}{5}c; c \in \mathbb{R}\end{pmatrix}\right\} = \begin{pmatrix} 0 \\ -\frac{2}{5} \\ \frac{3}{5}\end{pmatrix}+\mathbb{R}\begin{pmatrix} 1 \\ \frac{2}{5} \\ -\frac{3}{5}\end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{5} \\ \frac{3}{5}\end{pmatrix}+\mathbb{R}\begin{pmatrix} 5 \\ 2 \\ -3\end{pmatrix}.$$

Alternative solution: The line of intersection L (i.e., its direction vector) must be perpendicular to the normal vectors of both planes:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = (5, 2, -3)$$

Then we need to find a point on L. To do this, we can let $z = 0$, giving the equations

$$\begin{cases} x + 2y = 1 \\ x - y = 1 \end{cases}$$

Solving, we obtain the intersection point $(1, 0, 0)$. So the line L has the parametric form

$$\mathbf{x} = (1, 0, 0) + t\,(5, 2, -3)\,, \quad t \in \mathbb{R}.$$

or, in symmetric form, $\frac{x-1}{5} = \frac{y}{2} = \frac{z}{-3}$.

Setting $c = 1$ in the first solution gives $(1, 0, 0)^{\mathsf{T}} \in L$, showing that both solutions actually yield the same L (except for transposing the vectors), as it should be.

Ex. 70 $d = \min \left\{ \left| \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| ; t \in \mathbb{R} \right\} = d \left(\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}, \mathbb{R} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right)$, i.e., d is equal to the distance from the the point $(0, -5, 0)$ to the line $\mathbb{R}(2, -2, 1)$, which passes through the origin. The orthogonal projection of $(0, -5, 0)$ to the subspace $\mathbb{R}(2, -2, 1)$ is $\lambda^*(2, -2, 1)$ with $\lambda^* = \frac{(0,-5,0)\cdot(2,-2,1)}{(2,-2,1)\cdot(2,-2,1)} = \frac{10}{9}$. By Pythagoras' Theorem, $d^2 = |(0, -5, 0)|^2 - |\lambda^*(2, -2, 1)|^2 = 25 - \left(\frac{10}{9}\right)^2 \cdot 9 = \frac{125}{9}$ and $d = \frac{5}{3}\sqrt{5} \approx 3.73$.

Ex 77 The first line has parametric form $\mathbb{R}(1, 1, 1)$. The second line has parametric form $x = c$, $y = 2 + 2c$, $z = 3 + 3c$, i.e., $(0, 2, 3) + \mathbb{R}(1, 2, 3)$. According to the lecture, the distance d between the two lines is equal to the distance from the point $\mathbf{b} = (0, 2, 3)$ to the plane $U = \mathbb{R}\mathbf{u}_1 + \mathbb{R}\mathbf{u}_2 = \mathbb{R}(1, 1, 1) + \mathbb{R}(1, 2, 3)$.
$\mathbf{u}_1 \cdot \mathbf{u}_1 = 3$, $\mathbf{u}_1 \cdot \mathbf{u}_2 = 6$, $\mathbf{u}_2 \cdot \mathbf{u}_2 = 14$, $\mathbf{u}_1 \cdot \mathbf{b} = 5$, $\mathbf{u}_2 \cdot \mathbf{b} = 13$
$\Longrightarrow$ The orthogonal projection from $\mathbf{b}$ to $\mathbb{R}\mathbf{u}_1 + \mathbb{R}\mathbf{u}_2$ is $\lambda_1^*(1, 1, 1) + \lambda_2^*(1, 2, 3)$, where $(\lambda_1^*, \lambda_2^*)$ solves

$$\begin{array}{rcrcr} 3x_1 & + & 6x_2 & = & 5, \\ 6x_1 & + & 14x_2 & = & 13. \end{array}$$

The solution is $\lambda_1^* = -4/3$, $\lambda_2^* = 3/2$.

$$\Longrightarrow \mathbf{x}^* = -\frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 5/3 \\ 19/6 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 \\ 10 \\ 19 \end{pmatrix},$$

$$d = \left| \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ 10 \\ 19 \end{pmatrix} \right| = \frac{1}{6} \left| \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right| = \frac{1}{6}\sqrt{6},$$

in sync with the answer given in [Ste16], Appendix I.

Remark: A faster way to solve the exercise is to compute $\mathbf{b} - \mathbf{x}^*$ as the orthogonal projection of $\mathbf{b}$ onto the line through $\mathbf{0}$ spanned by a normal vector of U; see the afternote on Slide 49 of `lecture1-3_handout.pdf`. As normal vector we can take $\mathbf{n} = (1, -2, 1)$, so that

$$\mathbf{b} - \mathbf{x}^* = \frac{(0,2,3)\cdot(1,-2,1)}{|(1,-2,1)|^2}\begin{pmatrix}1\\-2\\1\end{pmatrix} = -\frac{1}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix},$$

$$d = |\mathbf{b} - \mathbf{x}^*| = \frac{|(0,2,3)\cdot(1,-2,1)|}{|(1,-2,1)|} = \frac{1}{\sqrt{6}}.$$

7 For $\mathbf{x} = (x_1, x_2, x_3) \in S$ the Cauchy-Schwarz inequality gives

$$\begin{aligned}|f(\mathbf{x})| &= \left|\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right| \leq \sqrt{1^2+2^2+(-1)^2}\cdot\sqrt{x_1^2+x_2^2+x_3^2}\\ &\leq \sqrt{6}\sqrt{3} = 3\sqrt{2} \approx 4.24\end{aligned}$$

with equality iff $\mathbf{x} = c(1,2,-1) = (c, 2c, -c)$ for some $c \in \mathbb{R}$. In order to have $\mathbf{x} \in S$, we must have $6c^2 = 3$, i.e., $c = \pm\frac{1}{2}\sqrt{2}$. Since $f(c, 2c, -c) = 6c$, the unique maximum of f on S is attained at the point corresponding to $c = \frac{1}{2}\sqrt{2}$, which is $\mathbf{x}^* = \left(\frac{1}{2}\sqrt{2}, \sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$.

Similarly, the unique minimum is attained at $-\mathbf{x}^* = \left(-\frac{1}{2}\sqrt{2}, -\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$.

8

First solution: The volume of the pyramid is 1/6 times the volume of the parallelepiped spanned by, say,

$$\begin{aligned}(2,0,-1) - (1,-1,0) &= (1,1,-1),\\ (2,0,2) - (1,-1,0) &= (1,1,2),\\ (0,1,7) - (1,-1,0) &= (-1,2,7).\end{aligned}$$

The latter is equal to the absolute value of the determinant

$$\begin{vmatrix}1 & 1 & -1\\ 1 & 1 & 2\\ -1 & 2 & 7\end{vmatrix} = \cdots = -9$$

(the determinant may change the sign if another vertex is used to translate), and the volume of the pyramid is 3/2.

Remark: This determinant is one of the few that are best evaluated by DODGSON's Formula; cf. Lecture 5. Since the top left 2×2 determinant vanishes, the "contracted" 2×2 determinant in the formula becomes $\left|\begin{smallmatrix}0 & 3\\ 3 & *\end{smallmatrix}\right| = -9$.

From this we also see that the bottom right entry of the 3×3 determinant has no effect on its computation, explaining that the results for Groups A and B are the same.

Second solution: This solution uses only elementary geometry and is considerably longer. You may view it as a double check that the solution using determinants works.

Denoting the vertices by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_2$, in order, we can view the triangle spanned by the first three vertices as the base and $\mathbf{v}_4$ as the the top of the pyramid. Using the formulas from elementary Geometry for the area of a triangle and volume of a pyramid, the volume V of the pyramid is $\frac{1}{6}|\mathbf{v}_2 - \mathbf{v}_1|\, h_1 h_2$, where h_1 is the distance from $\mathbf{v}_3$ to the line $\mathbf{v}_1 + \mathbb{R}(\mathbf{v}_2 - \mathbf{v}_1)$ ("height" of the triangle) and h_2 is the distance from $\mathbf{v}_4$ to the plane determined by the triangle. $\mathbf{v}_1 + \mathbb{R}(\mathbf{v}_2 - \mathbf{v}_1)$ ("height" of the pyramid). We obtain

$$\begin{aligned}
\mathbf{v}_2 - \mathbf{v}_1 &= (2, 0, -1) - (1, -1, 0) = (1, 1, -1), \\
|\mathbf{v}_2 - \mathbf{v}_1| &= \sqrt{3}3, \\
h_1 &= \left| \mathbf{v}_3 - \mathbf{v}_1 - \frac{(\mathbf{v}_3 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1)}{(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1)} (\mathbf{v}_2 - \mathbf{v}_1) \right| \\
&= \left| (1, 1, 2) - \frac{(1, 1, 2) \cdot (1, 1, -1)}{(1, 1, -1) \cdot (1, 1, -1)} (1, 1, -1) \right| = |(1, 1, 2)| = \sqrt{6}, \\
\mathbf{n} &= (1, -1, 0), \\
h_2 &= \left| \frac{(\mathbf{v}_4 - \mathbf{v}_1) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right| = \frac{|(\mathbf{v}_4 - \mathbf{v}_1) \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|(-1, 2, 7) \cdot (1, -1, 0)|}{|(1, -1, 0)|} = \frac{3}{\sqrt{2}},
\end{aligned}$$

so that $V = \frac{1}{6}\sqrt{3}\,\sqrt{6}\,\frac{3}{\sqrt{2}} = \frac{3}{2}$. (For the computation of h_2 the orthogonal projection to a normal vector $\mathbf{n}$ of the plane determined by the triangle, which has direction vectors $\mathbf{v}_2 - \mathbf{v}_1 = (1, 1, -1)$ and $\mathbf{v}_3 - \mathbf{v}_1 = (1, 1, 2)$, was used.)

Calculus III (Math 241)

H14 Do Exercises 34 and 58 in [Ste21], Ch. 13.2.

H15 Do Exercise 4 in [Ste21], Ch. 13.3.

H16 *Continuation of Exercise H11 of Homework 2*

Determine at which time t^* the particles are closest to each other, and the corresponding minimal distance d^*.

Hint: Work with the squared distance of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, which is easier to handle, and use Calculus I. The quantities t^*, d^* may be computed numerically.

H17 For a continuous curve $f\colon [a,b] \to \mathbb{R}^n$ prove that

$$\left| \int_a^b f(t)\,\mathrm{d}t \right| \leq \int_a^b |f(t)|\,\mathrm{d}t.$$

Hint: For $t \in [a,b]$ let $u(t) = \left| \int_a^t f(\tau)\,\mathrm{d}\tau \right|$ and $v(t) = \int_a^t |f(\tau)|\,\mathrm{d}\tau$. For the desired inequality $u(b) \leq v(b)$ it is enough to show that $u(a) = v(a)$ and $u'(t) \leq v'(t)$ for $t \in [a,b]$. Compute $u'(t)$, $v'(t)$ and apply the Cauchy-Schwarz Inequality for vectors in $\mathbb{R}^n$.

H18 (Continuation of Exercise W11 on Worksheet 3)

a) As shown in W11, $R(\phi)^\mathsf{T} R(\phi) = S(\phi)^\mathsf{T} S(\phi) = \mathbf{I}_2$. Show that, conversely, a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ satisfying $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}_2$ is necessarily a rotation or reflection matrix.

b) Express the matrix products $R(\phi_1)R(\phi_2)$, $S(\phi_1)S(\phi_2)$, $R(\phi_1)S(\phi_2)$, $S(\phi_1)R(\phi_2)$, which describe the compositions of the corresponding rotations/reflections, as either $R(\phi)$ or $S(\phi)$ for some angle ϕ.

   Hint: $S(\phi)$ can be expressed in terms of $R(\phi)$ and $S(0)$, simplifying the computations.

c) Find all rotation and reflection matrices that leave the quadrangle in $\mathbb{R}^2$ with vertices $(\pm 1, 0)^\mathsf{T}$, $(0, \pm 1)^\mathsf{T}$ invariant.

d) Repeat c) for the regular hexagon (6-gon) in $\mathbb{R}^2$ with center $(0,0)$ and one vertex at $(1,0)$.

H19 Show that

$$h(t) = \begin{pmatrix} \sqrt{2}\cos t + \frac{\sin t}{\sqrt{2}} \\ \frac{\sqrt{3}\sin t}{\sqrt{2}} \end{pmatrix}, \quad t \in [0, 2\pi]$$

parametrizes an ellipse with principal axes of lengths $a = \sqrt{3}$, $b = 1$, which arises from the standard ellipse with equation $x^2/3 + y^2 = 1$ by a $30°$-degree rotation; cf. Lecture 11.

Hint: Compute $R(\pi/6)^{-1} h(t + \pi/4)$, where $R(\phi)$ denotes the rotation matrix with angle ϕ.

H20 *Optional Exercise*

This exercise provides an alternative route to the Bolzano-Weierstrass Theorem: *Every bounded sequence of real numbers has a convergent subsequence.*

a) Show that any sequence $a_1, a_2, a_3, \ldots$ of real numbers has either an increasing subsequence or a decreasing subsequence (or both). Here "increasing/decreasing" allow for successive sequence terms to be equal (also referred to as "weakly" increasing/decreasing).

   Hint: Let P be the set all indexes $n \in \mathbb{N}$ satisfying $a_n \geq a_m$ for all $m > n$. Distinguish the cases "$|P|$ is finite" and "$|P|$ is infinite".

b) Explain how to derive the Bolzano-Weierstrass Theorem from a).

Due on Wed Oct 18, 6 pm

Exercise H18 d) is also optional. The optional exercises can be handed in until Wed Oct 25, 7 pm.

Solutions (prepared by TA's and TH)

14

Ex. 34: $\mathbf{r}'(t) = (\pi\cos\pi t, 2\pi\cos\pi t, -\pi\sin\pi t)$, so the desired direction vectors are $\mathbf{r}'(0) = (\pi, 2\pi, 0)$ and $\mathbf{r}'(0.5) = (0, 0, -\pi)$. Also, we have $\mathbf{r}(0) = (0, 0, 1)$ and $\mathbf{r}(0.5) = (1, 2, 0)$. The tangent line at $t = 0$ is

$$\{(\pi t_1, 2\pi t_1, 1); t_1 \in \mathbb{R}\} = (0, 0, 1) + \mathbb{R}(1, 2, 0).$$

The tangent line at $t = 0.5$ is

$$\{(1, 2, -\pi t_2); t_2 \in \mathbb{R}\} = (1, 2, 0) + \mathbb{R}(0, 0, 1).$$

When $t_1 = 1/\pi$ and $t_2 = -1/\pi$, the two tangent lines intersect at the point $(1, 2, 1)$. A plot of the curve along with the tangent lines is shown in Fig. 1.

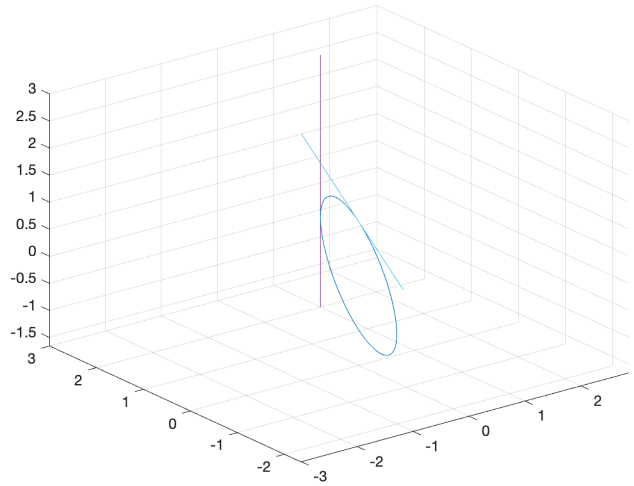


Figure 1: Plot for Ex. 32

Ex. 58: As shown in the lecture, we have

$$\frac{d}{dt}|\mathbf{r}(t)|^2 = \frac{d}{dt}\big(\mathbf{r}(t)\cdot\mathbf{r}(t)\big) = \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\,\mathbf{r}(t)\cdot\mathbf{r}'(t).$$

If $\mathbf{r}(t)$ is always perpendicular to the tangent vector $\mathbf{r}'(t)$ then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, so $\frac{d}{dt}|\mathbf{r}(t)|^2 = 0$, which yields $|\mathbf{r}(t)| = \text{constant}$. Denoting the constant by c, the curve must be contained in the sphere of radius c centered at the origin. (In the case $c = 0$ the curve is contained in the degenerate sphere $\mathrm{S}_0(\mathbf{0}) = \{\mathbf{0}\}$ of radius 0, i.e., the position is always the origin.)

Remarks: For domains I that are not intervals the assertion is false, as the example

$$\mathbf{r}(t) = \begin{cases} (\cos t, \sin t) & \text{if } 0 \le t < 1, \\ 2(\cos t, \sin t) & \text{if } 1 < t \le 2, \end{cases}$$

with domain $[0, 2] \setminus \{1\}$ shows. This curve is smooth in its domain, because the "jump discontinuity" at $t = 1$ has been removed.

15

$$
\begin{aligned}
\mathbf{r}(t) &= \left(2t, t^2, \tfrac{1}{3}t^3\right), \quad t \in [0,1] \\
\mathbf{r}'(t) &= (2, 2t, t^2), \\
\left|\mathbf{r}'(t)\right|^2 &= 4 + 4t^2 + t^4 = (t^2+2)^2, \\
L &= \int_0^1 \left|\mathbf{r}'(t)\right| \mathrm{d}t = \int_0^1 t^2 + 2\,\mathrm{d}t = \left[t^3/3 + 2t\right]_0^1 = \frac{7}{3}
\end{aligned}
$$

16 The squared distance between the particles is

$$d(t)^2 = |\mathbf{r}_1(t) - \mathbf{r}_2(t)|^2 = (t+1)^2 + (t^2-6t-1)^2 + (t^3-14t-1)^2 = t^6 - 27t^4 - 14t^3 + 231t^2 + 42t + 3,$$

which is a polynomial of degree 6. A plot of this function reveals that $d(t)^2$, and hence $d(t)$, is minimized (uniquely) at $t^* \approx -0.1$ with corresponding value $d(t^*) \approx 1.05$. The closest distance between the two particles is therefore ≈ 1.05.
A rigorous derivation of the minimum proceeds as follows: Consider the polynomial function

$$p(t) = d(t)^2 = p_1(t)^2 + p_2(t)^2 + p_3(t)^2$$

with $p_1(t) = t+1$, $p_2(t) = t^2 - 6t - 1$, $p_3(t) = t^3 - 14t - 1$. We have $p(0) = 3$, and we will show next that $p(t) > 3$ for $t \notin [-1,0]$. This implies that the minimum of $p(t)$ in $[-1,0]$ (which exists, because $p(t)$ is continuous) is a global minimum.
The roots of $p_2(t)$ are $t_1 = 3 - \sqrt{10} \in [-1,0]$ and $t_2 = 3 + \sqrt{10} \in [6,7]$. It follows that $p_2(t)^2$ is decreasing in $(-\infty,-1]$ and satisfies $p_2(t)^2 \geq p_2(-1)^2 = 36$ for $t < -1$. Since $p_1(t)^2 \geq 0$ and $p_3(t)^2 \geq 0$, this shows that $p(t) \geq 36$ for $t < -1$ as well. For $t > 1$ we have $p(t) \geq p_1(t)^2 > 4$. For $0 < t \leq 1$ we use that the three functions $p_1(t)^2$, $p_2(t)^2$, $p_3(t)^2$ are strictly increasing in $[0,1]$ (as is easily seen) and hence the same is true of $p(t)$. This gives $p(t) > p(0) = 3$ for $0 < t \leq 1$.
The global minimum $t^* \in [-1,0]$ is unique, since

$$p''(t) = 30t^4 - 324t^2 - 84t + 462 > 0 \quad \text{for } t \in [-1,0],$$

and hence $p'(t)$ is strictly increasing in $[-1,0]$. (Thus $p'(t)$ cannot have more than one zero in $[-1,0]$.)
Finally one can use Newton's method to compute t^* and $d(t^*)$ to any desired accuracy: t^* is a root of $p'(t)$, giving the iteration

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{p'(t_n)}{p''(t_n)} \quad \text{for } n = 0, 1, 2, \ldots$$

Running this iteration in SageMath produces $t^* \approx -0.0903394352931691$, $p(t^*) \approx 1.0995081616924252$, and $d(t^*) = \sqrt{p(t^*)} \approx 1.04857434724126$.

17 The auxiliary functions, written out in full, are

$$
\begin{aligned}
u(t) &= \sqrt{\left(\int_a^t f_1(\tau)\mathrm{d}\tau\right)^2 + \cdots + \left(\int_a^t f_n(\tau)\mathrm{d}\tau\right)^2}, \\
v(t) &= \int_a^t \sqrt{f_1(\tau)^2 + \cdots + f_n(\tau)^2}\,\mathrm{d}t.
\end{aligned}
$$

Clearly, $u(a)=v(a)=0$. The Fundamental Theorem of Calculus gives

$$\begin{aligned}
v'(t) &= \sqrt{f_1(t)^2+\cdots+f_n(t)^2},\\
u'(t) &= \frac{2f_1(t)\int_a^t f_1(\tau)\mathrm{d}\tau+\cdots+2f_n(t)\int_a^t f_n(\tau)\mathrm{d}\tau}{2u(t)}
\end{aligned}$$

The inequality $u'(t)\leq v'(t)$ is therefore equivalent to

$$\begin{pmatrix} f_1(t)\\ \vdots\\ f_n(t)\end{pmatrix}\cdot\begin{pmatrix}\int_a^t f_1(\tau)\mathrm{d}\tau\\ \vdots\\ \int_a^t f_n(\tau)\mathrm{d}\tau\end{pmatrix}\leq\left|\begin{pmatrix} f_1(t)\\ \vdots\\ f_n(t)\end{pmatrix}\right|\cdot\left|\begin{pmatrix}\int_a^t f_1(\tau)\mathrm{d}\tau\\ \vdots\\ \int_a^t f_n(\tau)\mathrm{d}\tau\end{pmatrix}\right|,$$

which is an instance of the Cauchy-Schwarz Inequality. Hence the functions $u(t)$, $v(t)$ satisfy the stated conditions, which clearly imply $u(b)\leq v(b)$ (draw a picture or use Calculus I to show rigorously that $v(t)-u(t)\geq 0$ for $t\in[a,b]$).

18 a) Writing $\mathbf{A}=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, the assumption means $a^2+c^2=b^2+d^2=1$, $ab+cd=0$ (i.e., the columns of $\mathbf{A}$ are orthogonal and have unit length). Since $a^2+c^2=1$, there exists $\phi\in[0,2\pi)$ such that $a=\cos\phi$, $c=\sin\phi$. Further, since the vectors orthogonal to (a,c) form the line $\mathbb{R}(-c,a)$, there exists $\lambda\in\mathbb{R}$ such that $b=-\lambda c=-\lambda\sin\phi$, $d=\lambda a=\lambda\cos\phi$. From $1=b^2+d^2=\lambda^2\sin^2\phi+\lambda^2\cos^2\phi=\lambda^2$ we then get $\lambda=\pm1$, i.e., $\mathbf{A}=R(\phi)$ in the case $\lambda=1$ and $\mathbf{A}=S(\phi)$ in the case $\lambda=-1$.

b) Using the same way as in W18, we have

$$\begin{aligned}
R(\phi_1)R(\phi_2) &= \begin{pmatrix}\cos(\phi_1) & -\sin(\phi_1)\\ \sin(\phi_1) & \cos(\phi_1)\end{pmatrix}\begin{pmatrix}\cos(\phi_2) & -\sin(\phi_2)\\ \sin(\phi_2) & \cos(\phi_2)\end{pmatrix}\\
&= \begin{pmatrix}\cos(\phi_1)\cos(\phi_2)-\sin(\phi_1)\sin(\phi_2) & -\cos(\phi_1)\sin(\phi_2)-\sin(\phi_1)\cos(\phi_2)\\ \sin(\phi_1)\cos(\phi_2)+\cos(\phi_1)\sin(\phi_2) & -\sin(\phi_1)\sin(\phi_2)+\cos(\phi_1)\cos(\phi_2)\end{pmatrix}\\
&= \begin{pmatrix}\cos(\phi_1+\phi_2) & -\sin(\phi_1+\phi_2)\\ \sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2)\end{pmatrix}=R(\phi_1+\phi_2),\\
S(\phi_1)S(\phi_2) &= \begin{pmatrix}\cos(\phi_1) & \sin(\phi_1)\\ \sin(\phi_1) & -\cos(\phi_1)\end{pmatrix}\begin{pmatrix}\cos(\phi_2) & \sin(\phi_2)\\ \sin(\phi_2) & -\cos(\phi_2)\end{pmatrix}\\
&= \begin{pmatrix}\cos(\phi_1)\cos(\phi_2)+\sin(\phi_1)\sin(\phi_2) & \cos(\phi_1)\sin(\phi_2)-\sin(\phi_1)\cos(\phi_2)\\ \sin(\phi_1)\cos(\phi_2)-\cos(\phi_1)\sin(\phi_2) & \sin(\phi_1)\sin(\phi_2)+\cos(\phi_1)\cos(\phi_2)\end{pmatrix}\\
&= \begin{pmatrix}\cos(\phi_1-\phi_2) & -\sin(\phi_1-\phi_2)\\ \sin(\phi_1-\phi_2) & \cos(\phi_1-\phi_2)\end{pmatrix}=R(\phi_1-\phi_2).
\end{aligned}$$

Since $S(\phi_1)$, $S(\phi_2)$, are the reflections at the lines $L_i=\mathbb{R}\left(\cos\frac{\phi_i}{2},\sin\frac{\phi_i}{2}\right)$, $i=1,2$, which intersect at an angle of $\frac{\phi_1-\phi_2}{2}$, the preceding formula says that the composition of the two reflections at lines L_1,L_2 is equal to the rotation with angle twice the (oriented) angle between L_1 and L_2.

To calculate $R(\phi_1)S(\phi_2)$, first notice that $S(\phi)=R(\phi)\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)=R(\phi)S(0)$. Hence,

$$R(\phi_1)S(\phi_2)=R(\phi_1)R(\phi_2)S(0)=R(\phi_1+\phi_2)S(0)=S(\phi_1+\phi_2)$$

Because $S(0)^2 = \mathbf{I}$, $S(0) = S(0)^{-1}$, and $R(\phi) = S(\phi)S(0)^{-1} = S(\phi)S(0)$. In sum, $S(0)$ alters the signs in the second column of the matrix it multiplies, and thus alters between $R(\phi)$ and $S(\phi)$. As before,

$$S(\phi_1)R(\phi_2) = S(\phi_1)S(\phi_2)S(0) = R(\phi_1 - \phi_2)S(0) = S(\phi_1 - \phi_2)$$

c) The quadrangle is invariant under all rotations with angle an integral multiple of 90°. Since $R(\phi) = R(\phi + 2\pi)$, all such rotation matrices can be represented as $\{R(\frac{k\pi}{2}); k = 0,1,2,3\}$. As for reflection matrices, we need to find all axes of symmetry, which are $\mathbb{R}\left(\cos\frac{k\pi}{4}, \sin\frac{k\pi}{4}\right)$, $k \in \mathbb{Z}$. Since $S(\phi)$ has axis of symmetry $\mathbb{R}\left(\cos\frac{\phi}{2}, \sin\frac{\phi}{2}\right)$, all such reflection matrices are $\{S(\frac{k\pi}{2}); k = 0,1,2,3\}$. In all we obtain the 8 matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix},$$

where all combinations of signs are allowed.

d) The hexagon is invariant under all rotations with angle an integral multiple of 60°. So all such rotation matrices can be represented as $\{R(\frac{k\pi}{3}); 0 \le k \le 5\}$. As for reflection matrices, we need to find all axes of symmetry, which are $\phi = \frac{k\pi}{6}$, $k \in \mathbb{Z}$. So all such reflection matrices are $\{S(\frac{k\pi}{3}); 0 \le k \le 5\}$; cf. the computation in c). In all we obtain 12 matrices of the forms

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{2} & \pm\frac{1}{2}\sqrt{3} \\ \pm\frac{1}{2}\sqrt{3} & \pm\frac{1}{2} \end{pmatrix}.$$

For the first form all 4 sign combinations are allowed, for the second form only those combinations which have an odd number of 1's.

Remarks: Taking all $4+4+16$ matrices of the forms exhibited in c) and d) together (without any sign restrictions) gives the 24 symmetries of the regular 12-gon centered at $(0,0)$ with one vertex at $(1,0)$.

Parts c) and d) can also be solved by observing that the first column of the matrix must be a vertex of the corresponding polygon and the second column orthogonal to the first column and of length 1.

Part d) can also be solved by observing that the equilateral triangle of Exercise W11 d) and its mirror image with respect to the origin together form the hexagon. This implies that the 12 symmetries of the hexagon are those of the triangle and their negatives, i.e., the 12 symmetries are represented by the matrices $\pm\mathbf{A}$, where $\mathbf{A}$ is one of the 6 matrices determined in W11 d).

19 The curve $e(t) = R(\pi/6)^{-1}h(t+\pi/4)$, $t \in [-\pi/4, 3\pi/4]$, arises from $h(t)$ by a $-30°$-degree rotation, and a timeshift which has no influence on the corresponding non-parametric curve. (Since $h(t)$ is periodic with period 2π, we could also take the parameter domain of $e(t)$ as $[0, 2\pi]$.)

$$e(t) = R(-\pi/6)h(t+\pi/4) = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{2}\cos(t+\pi/4) + \frac{\sin(t+\pi/4)}{\sqrt{2}} \\ \frac{\sqrt{3}\sin(t+\pi/4)}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}\cos(t+\pi/4)}{\sqrt{2}} + \frac{\sqrt{3}\sin(t+\pi/4)}{\sqrt{2}} \\ -\frac{\cos(t+\pi/4)}{\sqrt{2}} + \frac{\sin(t+\pi/4)}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{3}\cos(t+\pi/4-\pi/4 \\ \sin(t+\pi/4-\pi/4) \end{pmatrix}, = \begin{pmatrix} \sqrt{3}\cos t \\ \sin t \end{pmatrix},$$

using $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. This is the standard ellipse with semi-axes $a = \sqrt{3}$, $b = 1$ parametrized in the usual way. The assertion follows.

20 a) *Case 1:* P is a finite set.
In this case there exists an index N such that $a_n \notin P$ for all $n = N, N+1, n+2, \ldots$. That is, there exists $n_1 > N$ such that $a_N < a_{n_1}$; there exists $n_2 > n_1$ such that $a_{n_1} < a_{n_2}$; and so on. Thus, writing $n_0 = N$ we have found a (strictly) increasing subsequence $a_{n_0}, a_{n_1}, a_{n_2}, \ldots$

Case 2: P is an infinite set.
Write the elements of P in increasing order, $p_1 < p_2 < p_3 < \cdots$. The definition of P then implies $a_{p_1} \geq a_{p_2} \geq a_{p_3} \geq \cdots$. Thus we have found a decreasing subsequence.

b) Consider a bounded sequence $a_1, a_2, a_3, \ldots$ of real numbers. By a), the sequence has either an increasing subsequence, which is bounded from above and hence converges to its supremum (viewing the sequence as a set); or it has a decreasing subsequence, which is bounded from below and hence converges to its infinum. Thus, in either case we have found a convergent subsequence of $a_1, a_2, a_3, \ldots$

Calculus III (Math 241)

W9 Prove the product rule for differentiating cross products,

$$\frac{d}{dt}\big(\mathbf{u}(t) \times \mathbf{v}(t)\big) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t),$$

by working directly with the corresponding coordinate functions and using results from Calculus I.

W10 Do Exercises 26 and 30 in [Ste21], Ch. 13.2.

W11 Consider the matrices

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}, \quad \phi \in [0, 2\pi).$$

a) Show that $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \to R(\phi)\mathbf{x}$ is the rotation around $(0,0)^\mathsf{T}$ with angle ϕ.

b) Show that $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \to S(\phi)\mathbf{x}$ is a reflection, and find its axis.

c) Show that $R(\phi)^{-1} = R(-\phi) = R(\phi)^\mathsf{T}$ and $S(\phi)^{-1} = S(\phi) = S(\phi)^\mathsf{T}$.

d) Determine all rotations/reflections of $\mathbb{R}^2$ that preserve the equilateral triangle Δ with vertices $(1,0)$, $\big(-\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}\big)$.

Hint: There are six such symmetries, three rotations and three reflections.

Solutions

9 Let $\mathbf{u}(t) = \big(u_1(t), u_2(t), u_3(t)\big)$, $\mathbf{v}(t) = \big(v_1(t), v_2(t), v_3(t)\big)$, $\mathbf{w}(t) = \mathbf{u}(t) \times \mathbf{v}(t) = \big(w_1(t), w_2(t), w_3(t)\big)$. Then according to the definition of cross products, $w_1(t) = u_2(t)v_3(t) - u_3(t)v_2(t)$, and the other coordinate functions of $\mathbf{w}(t)$ are obtained by cycling the indexes modulo 3.

$$\begin{aligned} w_1'(t) &= u_2'(t)v_3(t) + u_2(t)v_3'(t) - u_3'(t)v_2(t) - u_3(t)v_2'(t) \\ &= \underbrace{u_2'(t)v_3(t) - u_3'(t)v_2(t)}_{a_1(t)} + \underbrace{u_2(t)v_3'(t) - u_3(t)v_2'(t)}_{b_1(t)}. \end{aligned}$$

The function $a_1(t)$ is the first coordinate function of $\mathbf{u}'(t) \times \mathbf{v}(t)$, and $b_1(t)$ is the first coordinate function of $\mathbf{u}(t) \times \mathbf{v}'(t)$. This shows that the first coordinate functions on both sides of the identity are the same. Cycling indexes modulo 3 then gives the rest.

10

Ex. 26: The tangent line in parametric form at the curve point $\mathbf{r}(t)$ is

$$\mathbf{r}(t) + \mathbb{R}\,\mathbf{r}'(t) = \{\mathbf{r}(t) + \lambda\,\mathbf{r}'(t); \lambda \in \mathbb{R}\}.$$

Since $(0,0,1) = \mathbf{r}(0)$, we need to compute $\mathbf{r}'(0)$.

$$\begin{aligned} \mathbf{r}(t) &= \big(\ln(t+1), t\cos(2t), 2^t\big), \\ \mathbf{r}'(t) &= \left(\frac{1}{t+1}, \cos(2t) - 2t\sin(2t), \ln(2)2^t\right), \\ \mathbf{r}'(0) &= (1, 1, \ln 2) \end{aligned}$$

Hence the answer, written using column vectors, is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ \ln 2 \end{pmatrix} = \left\{ \begin{pmatrix} \lambda \\ \lambda \\ 1 + \lambda\ln 2 \end{pmatrix}; \lambda \in \mathbb{R} \right\}.$$

Ex. 30: The plane $\sqrt{3}\,x + y = 1$ has normal vector $(\sqrt{3}, 1, 0)$, which should be orthogonal to the tangent direction $\mathbf{r}'(t) = (-2\sin t, 2\cos t, \mathrm{e}^t)$. This gives the condition

$$\sqrt{3}(-2\sin t) + 2\cos t = 0, \quad \text{i.e.,} \quad \cos t = \sqrt{3}\sin t.$$

The unique solution in $[0, \pi]$ is $t = \pi/6$, and the corresponding curve point is

$$\mathbf{r}(\pi/6) = \big(2\cos(\pi/6), 2\sin(\pi/6), \mathrm{e}^{\pi/6}\big) = \big(\sqrt{3}, 1, \mathrm{e}^{\pi/6}\big).$$

11 From the lecture we know that the maps $R(\phi)$, $S(\phi)$ are linear, since they are given by a matrix-vector multiplication.

a) It is clear that $R(\phi)$ fixes $\mathbf{0} = (0,0)^\mathsf{T}$. Every other vector $\mathbf{x} \in \mathbb{R}^2$ can be represented as $\mathbf{x} = (r\cos\psi, r\sin\psi)^\mathsf{T}$, where $r = |\mathbf{x}|$ and $\psi \in [0, 2\pi)$. Since $R(\phi)(c\mathbf{x}) = cR(\phi)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, it suffices to check that $R(\phi)$ "rotates" vectors of length 1:

$$R(\phi)\mathbf{x} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi \\ \sin\phi\cos\psi + \cos\phi\sin\psi \end{pmatrix} = \begin{pmatrix} \cos(\phi+\psi) \\ \sin(\phi+\psi), \end{pmatrix}$$

where we have used the addition theorems for sin, cos.

b) Planar reflections s have a unique line L, called the "axis" that is fixed point-wise and determines s. We also say that s is the reflection at the line L. Lines M orthogonal to L are fixed set-wise, i.e. $s(M) = M$, but not point-wise. Reflections at lines through the origin are linear maps and are represented by the matrices $S(\phi)$. For a linear reflection there is a unique line L' (the line through the origin that is orthogonal to the axis L) whose points are mapped to their antipodal points, i.e. $s(\mathbf{x}) = -\mathbf{x}$. If $L = \mathbb{R}\mathbf{u}$, $L' = \mathbb{R}\mathbf{v}$ then s is given by $s(\lambda\mathbf{u} + \mu\mathbf{v}) = \lambda\mathbf{u} - \mu\mathbf{v}$ ($\lambda, \mu \in \mathbb{R}$). Another representation of s is

$$s(\mathbf{x}) = \mathbf{x} - \frac{2(\mathbf{x}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}. \tag{$*$}$$

In order to see this, check that $s(\mathbf{v}) = -\mathbf{v}$ and $s(\mathbf{x}) = \mathbf{x}$ for every vector orthogonal to $\mathbf{v}$. Since linear maps are determined by their effect on a basis of $\mathbb{R}^2$, the two conditions determine s uniquely. The axis of the reflection defined by $(*)$ is $(\mathbb{R}\mathbf{v})^\perp$.

Solution: Drawing a picture of the image vectors $S(\phi)\mathbf{e}_1$, $S(\phi)\mathbf{e}_2$ (the columns of $S(\phi)$) suggests that the axis of $\mathbf{x} \mapsto S(\phi)\mathbf{x}$ is spanned by $\mathbf{u} = \bigl(\cos(\phi/2), \sin(\phi/2)\bigr)^\mathsf{T}$, and the corresponding $\mathbf{v}$ is $\bigl(-\sin(\phi/2), \cos(\phi/2)\bigr)^\mathsf{T}$. We prove this rigorously without using $(*)$:

$$\begin{aligned}
S(\phi)\mathbf{u} &= \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi)\cos(\phi/2) + \sin(\phi)\sin(\phi/2) \\ \sin\phi\cos(\phi/2) - \cos(\phi)\sin(\phi/2) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi - \phi/2) \\ \sin(\phi - \phi/2) \end{pmatrix} && \text{(using the addition theorems)} \\
&= \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix} = \mathbf{u}, \\
S(\phi)\mathbf{v} &= \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix} \\
&= \begin{pmatrix} -\sin(\phi/2)\cos\phi + \cos(\phi/2)\sin(\phi) \\ -\sin(\phi/2)\sin\phi - \cos(\phi/2)\cos(\phi) \end{pmatrix} \\
&= \begin{pmatrix} \sin(\phi - \phi/2) \\ -\cos(\phi - \phi/2) \end{pmatrix} = -\mathbf{v}.
\end{aligned}$$

c) The formulas for $R(\phi)^{-1}$, $S(\phi)^{-1}$ are shown, e.g., using the formula for the inverse of a 2×2 matrix. Here is the first one:

$$\begin{aligned}
R(\phi)^{-1} &= \frac{1}{\cos^2\phi - (-\sin^2\phi)} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{pmatrix} = R(-\phi)
\end{aligned}$$

The identities $R(-\phi) = R(\phi)^\mathsf{T}$, $S(\phi) = S(\phi)^\mathsf{T}$ are obvious.

Remark: In particular we have that $R(\phi)$ and $S(\phi)$ are orthogonal matrices, i.e., their inverses coincide with their transposes. This can also be seen from the fact that the columns (or rows) of $R(\phi)$, $S(\phi)$ are orthogonal and have unit length. In fact all 2×2 orthogonal matrices are of the form $R(\phi)$ or $S(\phi)$, as can be easily shown.

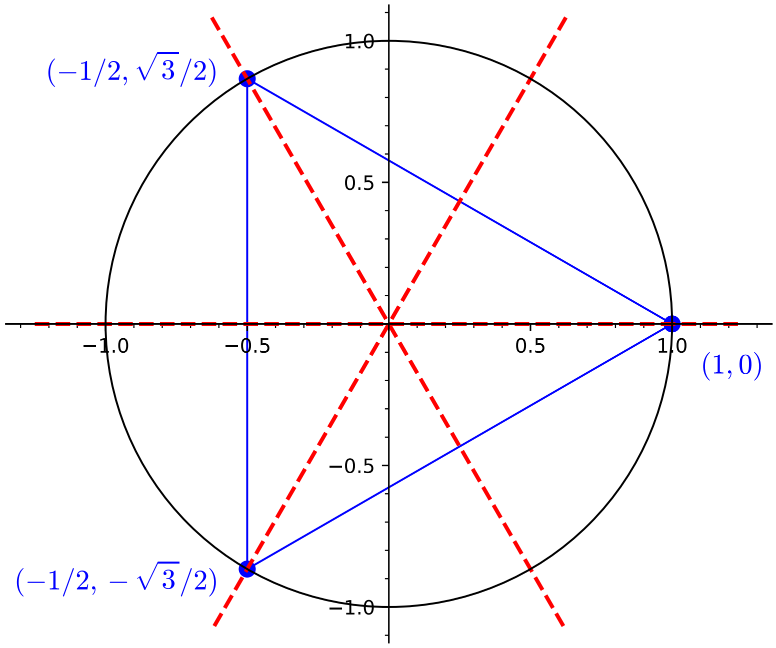


Figure 1: The three axes of symmetry of Δ

d) Though it wasn't stated in the exercise, it should be clear that only rotations with center $(0, 0)$ and reflections at a line through $(0, 0)$ can preserve Δ. Hence this question is about the matrices $R(\phi)$ and $S(\phi)$.

A rotation matrix $R(\phi)$ mapping Δ to itself must permute the vertices, so only $\phi \in \{0°, 120°, 240°\}$ is possible. For a reflection matrix $S(\phi)$ the same is true, so it must fix one vertex and interchange the other two vertices, giving again $\phi \in \{0°, 120°, 240°\}$. Since $\cos(2\pi/3) = -\frac{1}{2}$, $\sin(2\pi/3) = \frac{1}{2}\sqrt{3}$, the six matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}.$$

Calculus III (Math 241)

H21 Do Exercises 22, 34, 54, 58 and, optionally, Exercise 78 in [Ste21], Ch. 13.3.
In Exercise 54 you are also required to find a parametrization of the osculating circle of the curve at $(0,2,1)$.

H22 Do Exercise 73 in [Ste21], Ch. 13.3

H23 Do Exercises 32 and 36 in [Ste21], Ch. 13.4.

H24 *Optional Exercise*
Do Problem Plus No. 3 in [Ste21], Ch. 13.4, p. 930.

H25 *Optional Exercise*
The purpose of this exercise is to classify the solution sets in $\mathbb{R}^2$ of a quadratic equation

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0,$$

which are referred to as *conics*. The conic is said to be *central* if the system

$$\begin{array}{ccccccc} Ax & + & By & + & D & = & 0 \\ Bx & + & Cy & + & E & = & 0 \end{array}$$

has a unique solution (x_0,y_0) (called the *center* of the conic), and *degenerate* if

$$\Delta := \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0.$$

a) Show that a non-central, non-degenerate conic is a parabola (given by an equation $y^2+4ax=0$ with $a>0$ in a suitable coordinate system).

b) Show that the translation $x=x_0+x'$, $y=y_0+y'$ transforms a central conic into one with equation
$$Ax^2+2Bxy+Cy^2+\Delta/(AC-B^2)=0.$$

c) Show that a central, non-degenerate conic with at least one point is either an ellipse or a hyperbola.
Hint: In order to eliminate the xy term, consider rotations of the coordinate system, i.e., $x=(\cos\phi)x'-(\sin\phi)y'$, $y=(\sin\phi)x'+(\cos\phi)y'$. For a suitable choice of the angle ϕ the new equation will be $\lambda_1 x'^2+\lambda_2 y'^2+\Delta/(AC-B^2)=0$, where λ_1,λ_2 are the roots of $\lambda^2-(A+C)\lambda+AC-B^2=0$.

d) Briefly describe the possible forms of degenerate conics.

e) Work out the example

$$17x^2 - 12xy + 8y^2 + 12x - 16y - 12 = 0$$

and sketch this conic.

Due on Wed Oct 25, 7 pm

The mandatory exercises serve as midterm preparation and shouldn't be skipped. The optional exercises can be handed in until Wed Nov 1, 6 pm.

Solutions

21

Ex. 22

$$
\begin{aligned}
\mathbf{r}(t) &= \left(t, t, \tfrac{1}{2}t^2\right),\\
\mathbf{r}'(t) &= (1, 1, t),\\
\left|\mathbf{r}'(t)\right| &= \sqrt{t^2+2},\\
\implies \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{t^2+2}}(1,1,t),\\
\mathbf{T}'(t) &= -\frac{t}{(t^2+2)^{3/2}}(1,1,t) + \frac{1}{\sqrt{t^2+2}}(0,0,1) = \frac{1}{(t^2+2)^{3/2}}(-t,-t,2),\\
\implies \mathbf{N}(t) &= \frac{(-t,-t,2)}{|(-t,-t,2)|} = \frac{1}{\sqrt{2t^2+4}}(-t,-t,2),\\
\left|\mathbf{T}'(t)\right| &= \frac{\sqrt{2t^2+4}}{(t^2+2)^{3/2}} = \frac{\sqrt{2}}{t^2+2},\\
\implies \kappa(t) &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}}{(t^2+2)^{3/2}}
\end{aligned}
$$

Ex. 34 According to the lecture, the curvature at the point $(x, \ln x)$ is

$$
\kappa(x) = \frac{|p''(x)|}{(1+p'(x)^2)^{3/2}} = \frac{\left|-1/x^2\right|}{\left(1+(1/x)^2\right)^{3/2}} = \frac{x}{(x^2+1)^{3/2}}.
$$

Using Calculus I, one finds that $\kappa'(x)$ has a unique zero, viz. $x = 1/\sqrt{2}$. Since $\lim_{x\to\infty}\kappa(x) = \lim_{x\downarrow 0}\kappa(x) = 0$, this must be the (unique) global maximum of κ. The point on $y = \ln x$ with maximum curvature is therefore $\left(1/\sqrt{2}, \ln(1/\sqrt{2})\right) = \left(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\ln 2\right)$. The maximum curvature is $\frac{2}{3\sqrt{3}} \approx 0.385$.

Ex. 54 We have

$$
\begin{aligned}
f(t) &= (\ln t, 2t, t^2),\\
f'(t) &= (1/t, 2, 2t),\\
f''(t) &= (-1/t^2, 0, 2).
\end{aligned}
$$

For the normal plane at $(0,2,1) = f(1)$ we can take $f'(1) = (1,2,2)$ as normal vector and $x + 2(y-2) + 2(z-1) = 0$, i.e., $x + 2y + 2z = 6$ as equation.

For the osculating plane at $(0,2,1)$ we can take

$$
f'(1) \times f''(1) = \begin{pmatrix}1\\2\\2\end{pmatrix} \times \begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}4\\-4\\2\end{pmatrix}
$$

as normal vector, as well as $(2,-2,1)$, and $2x - 2(y-2) + z - 1 = 0$, i.e., $2x - 2y + z = -3$ as equation.

The curvature of f at $(0,2,1)$ is

$$\kappa(1) = \frac{|f'(1) \times f''(1)|}{|f'(1)|^3} = \frac{\sqrt{4^2+(-4)^2+2^2}}{\sqrt{1^2+2^2+2^2}^3} = \frac{6}{27} = \frac{2}{9},$$

and hence the radius of the osculating circle of f at $(0,2,1)$ is $9/2$.

For a parametrization of the osculating circle we also need $\mathbf{T}(1)$ and $\mathbf{N}(1)$, which is best computed by orthonormalizing $f'(1)$, $f''(1)$; cf. the lecture.

$$\mathbf{T}(1) = \frac{f'(1)}{|f'(1)|} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix},$$

$$\mathbf{u}_2' = f''(1) - \frac{f''(1)\cdot f'(1)}{f'(1)\cdot f'(1)} f'(1) = \begin{pmatrix}-1\\0\\2\end{pmatrix} - \frac{3}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}-4/3\\-2/3\\4/3\end{pmatrix} = \frac{2}{3}\begin{pmatrix}-2\\-1\\2\end{pmatrix},$$

$$\mathbf{N}(1) = \frac{\mathbf{u}_2'}{|\mathbf{u}_2'|} = \frac{1}{3}\begin{pmatrix}-2\\-1\\2\end{pmatrix}.$$

Alternatively, we can compute $\mathbf{T}(t)$, $\mathbf{T}'(t)$ and obtain $\mathbf{T}(1)$, $\mathbf{N}(1)$ by setting $t=1$ (and normalizing $\mathbf{T}'(1)$ to unit length).

$$\begin{aligned}
\left|f'(t)\right|^2 &= 1/t^2+4+4t^2 = \frac{1+4t^2+4t^4}{t^2} = \frac{(1+2t^2)^2}{t^2},\\
\mathbf{T}(t) &= \frac{t}{1+2t^2}\left(\frac{1}{t},2,2t\right) = \left(\frac{1}{1+2t^2},\frac{2t}{1+2t^2},\frac{2t^2}{1+2t^2}\right)\\
\mathbf{T}'(t) &= \frac{1}{(1+2t^2)^2}\left(-4t,2(1+2t^2)-(2t)(4t),4t(1+2t^2)-(2t^2)(4t)\right)\\
&= \frac{1}{(1+2t^2)^2}\left(-4t,2-4t^2,4t\right),\\
\mathbf{N}(t) &= \frac{\left(-2t,1-2t^2,2t\right)}{|(-2t,1-2t^2,2t)|} = \left(\frac{-2t}{1+2t^2},\frac{1-2t^2}{1+2t^2},\frac{2t}{1+2t^2}\right).
\end{aligned}$$

The 2nd computation is more elaborate, even if we spare computing $\mathbf{N}(t)$ and obtain $\mathbf{N}(1)$ using the observation that $\mathbf{T}'(1)$ is a positive multiple of $(-4,-2,4)$.

Hence the osculating circle of f in $(0,2,1)$ has center $(0,2,1)+\frac{9}{2}\mathbf{N}(1) = (0,2,1)+\frac{3}{2}(-2,-1,2) = \left(-3,\frac{1}{2},4\right)$. A parametrization of the osculating circle of f in $(0,2,1)$ therefore is

$$\begin{aligned}
c(t) &= \begin{pmatrix}-3\\1/2\\4\end{pmatrix} - \frac{9}{2}\cos t\,\mathbf{N}(1) + \frac{9}{2}\sin t\,\mathbf{T}(1)\\
&= \begin{pmatrix}-3\\1/2\\4\end{pmatrix} - \frac{3}{2}\cos t\begin{pmatrix}-2\\-1\\2\end{pmatrix} + \frac{3}{2}\sin t\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}-3+3\cos t+\frac{3}{2}\sin t\\1/2+\frac{3}{2}\cos t+3\sin t\\4-3\cos t+3\sin t\end{pmatrix}.
\end{aligned}$$

Ex. 58 A smart way to solve this exercise uses the fact that the osculating plane is spanned by $g'(t)$ and $g''(t)$, avoiding the computation of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. We have

$$\begin{aligned} g(t) &= (t^3, 3t, t^4), \\ g'(t) &= (3t^2, 3, 4t^3), \\ g''(t) &= (6t, 0, 12t^2). \end{aligned}$$

A normal vector for the osculating plane is therefore

$$\mathbf{n}(t) = g'(t) \times g''(t) = \begin{pmatrix} 3t^2 \\ 3 \\ 4t^3 \end{pmatrix} \times \begin{pmatrix} 6t \\ 0 \\ 12t^2 \end{pmatrix} = \begin{pmatrix} 36t^2 \\ -12t^4 \\ -18t \end{pmatrix}$$

Since the signs of the 1st and 2nd coordinate function of $\mathbf{n}(t)$ differ for $t \neq 0$, it is impossible to make $\mathbf{n}(t)$ a nonzero multiple of $(1,1,1)$. Hence the answer is No.

Ex. 78

a) For the function $F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ P(x) & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we mus have $P(0) = 0$ and $P(1) = 1$. For F' to be continuous, we must have $P'(0) = P'(1) = 0$. For the curvature $\kappa(x) = \frac{|F''(x)|}{(1+F'(x)^2)^{\frac{3}{2}}}$ to be continuous, we must have $P''(0) = P''(1) = 0$. As $P(x)$ is a polynomial of degree 5, we can write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 5cx + 2d$. To meet the conditions we have derived,

$$\begin{aligned} P(0) &= 0 \Rightarrow f = 0, & P(1) &= 1 \Rightarrow a+b+c+d+e+f = 1, \\ P'(0) &= 0 \Rightarrow e = 0, & P'(1) &= 0 \Rightarrow 5a+4b+3c+2d+e = 0, \\ P''(0) &= 0 \Rightarrow d = 0, & P''(1) &= 0 \Rightarrow 20a+12b+6c+2d = 0. \end{aligned}$$

From the above equations, we have $a = 6$, $b = -15$, $c = 10$, and $d = e = f = 0$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.

b) See Figure 1

Remark: The graph of F is symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2})$. This is equivalent to $1 - P(1-x) = P(x)$ for $x \in \mathbb{R}$ and follows from the uniqueness of the solution: The polynomial $Q(x) = 1 - P(1-x)$ also has degree ≤ 5 and satisfies $Q(0) = Q'(0) = Q''(0) = 0$, $Q(1) = 1$, $Q'(1) = Q''(1) = 0$. There is, however, exactly one polynomial of degree ≤ 5 having prescribed quadratic Taylor polynomials at two distinct points $x_0, x_1 \in \mathbb{R}$.

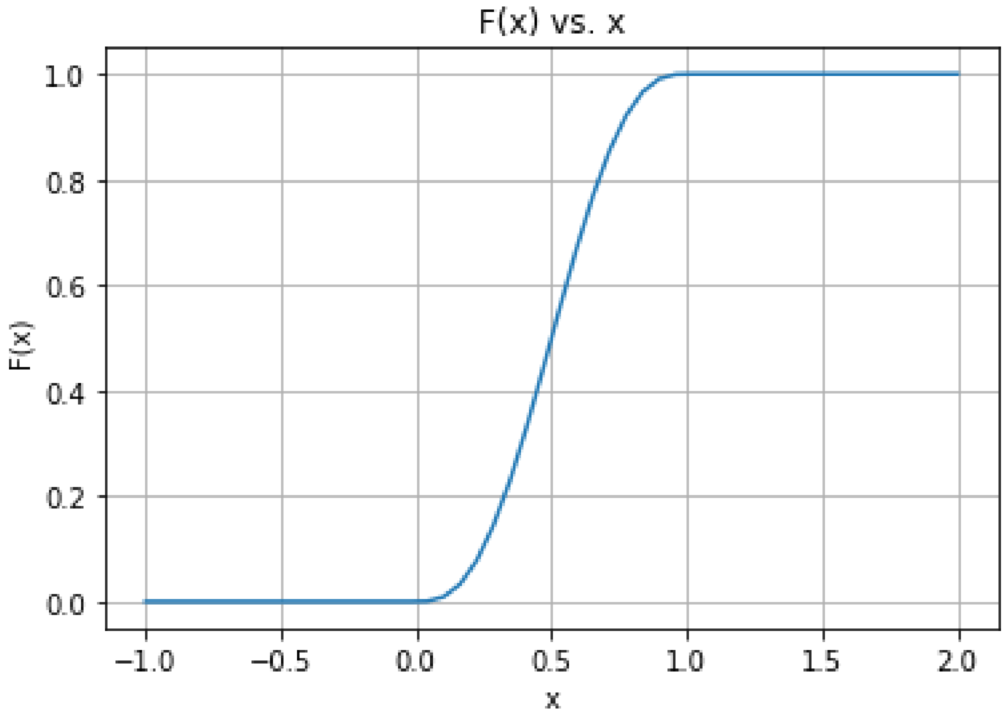


Figure 1: Graph of the function F in Exercise 78 b)

22 Ex. 73

$$\begin{aligned}
\mathbf{r}'(t) &= (-a\sin t, a\cos t, b),\\
\left|\mathbf{r}'(t)\right| &= \sqrt{a^2+b^2},\\
\mathbf{r}''(t) &= (-a\cos t, -a\sin t, 0),\\
\mathbf{r}'''(t) &= (a\sin t, -a\cos t, 0),\\
\mathbf{r}'(t)\times\mathbf{r}''(t) &= (ab\sin t, -ab\cos t, a^2),\\
\kappa(t) &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{a^2b^2+a^4}}{\sqrt{a^2+b^2}^3} = \frac{a}{a^2+b^2},\\
\tau(t) &= \frac{\mathbf{r}'''(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2} = \frac{a^2b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}.
\end{aligned}$$

23 Ex. 32 Assuming w.l.o.g. that the ball is thrown at time $t=0$ and using the approximation $g\approx 10\,\mathrm{m/s^2}$, we obtain

$$\begin{aligned}
&\mathbf{a}(t) = g(0,0,-1)+\mathbf{F}/m = (0,0,-10)+(4,0,0)/0.8 = (5,0,-10),\\
&\mathbf{v}(t) = \mathbf{v}(0)+\int_0^t \mathbf{a}(\tau)\mathrm{d}\tau = 30(0,-\cos 30^\circ,\sin 30^\circ)+t(5,0,-10) = (0,-15\sqrt{3},15)+(5t,0,-10t)\\
&\qquad = \left(5t,-15\sqrt{3},15-10t\right)\\
&\mathbf{r}(t) = \mathbf{r}(0)+\int_0^t \mathbf{v}(\tau)\mathrm{d}\tau = (0,0,0)+\left(\tfrac{5}{2}t^2,-15\sqrt{3}t,15t-5t^2\right) = \left(\tfrac{5}{2}t^2,-15\sqrt{3}t,15t-5t^2\right).
\end{aligned}$$

The ball lands at time $t_1>0$ determined by $15t_1-5t_1^2=0$, i.e., $t_1=3$.
$\Longrightarrow$ The ball lands at $\mathbf{r}(3)=\left(\frac{45}{2},-45\sqrt{3},0\right)\approx(22.5,-78,0)$ [m], i.e, 22.5 m eastward from

the physical y-axis and 78 m southward from the physical x-axis. (Since the earth's surface is not flat, these "lines" are in fact more like circles, viz. the meridian and circle of latitude through the chosen origin.) Its landing speed is $v(3) = |\mathbf{v}(3)| = \left|(15, -15\sqrt{3}, -15)\right| = 15\left|(1, -\sqrt{3}, -1)\right| = 15\sqrt{5} \approx 33.5$ [m/s]. Since the value used for the gravitational acceleration has only 1 significant digit, the final results can't have more and should also be rounded to 1 digit, i.e., answer "about 20 m", "about 80 m", "about 30 m/s", respectively. Intermediate computations may use more digits to reduce the overall error.

Ex. 36

(a) If the line is $L = \mathbf{a} + \mathbb{R}\mathbf{u}$, the velocity vectors and the acceleration vectors $\mathbf{a}(t)$ are contained in $\mathbb{R}\mathbf{u}$. Reason: If $\mathbf{r}(t), \mathbf{r}(t+h) \in L$ then $\mathbf{r}(t+h) - \mathbf{r}(t) \in \mathbb{R}\mathbf{u}$ and $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \in \mathbb{R}\mathbf{u}$. Letting $h \to 0$ then shows $\mathbf{v}(t) \in \mathbb{R}\mathbf{u}$ (because $\mathbb{R}\mathbf{u}$ is closed). Applying the same reasoning to $t \mapsto \mathbf{v}(t)$ shows $\mathbf{a}(t) \in \mathbb{R}\mathbf{u}$.

(b) The acceleration vector $\mathbf{a}(t)$ is perpendicular to the tangent direction in $\mathbf{r}(t)$, which is $\mathbb{R}\mathbf{v}(t)$. Reason: Constant speed means $|\mathbf{v}(t)| = v_0 \in \mathbb{R}$. Squaring gives that $\mathbf{v}(t) \cdot \mathbf{v}(t) = |\mathbf{v}(t)|^2 = v_0^2$ is constant and hence $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{v}(t) \cdot \mathbf{v}(t)\right] = 2\,\mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\,\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$.

24 (a) The height

$$y(t) = (v_0 \sin\alpha)t - \frac{g}{2}t^2 = -\frac{g}{2}t\left(t - \frac{2v_0 \sin\alpha}{g}\right)$$

is largest for $t = \frac{v_0 \sin\alpha}{g}$ with corresponding value

$$y\left(\frac{v_0 \sin\alpha}{g}\right) = \frac{g}{2}\left(\frac{v_0 \sin\alpha}{g}\right)^2 = \frac{v_0^2 \sin^2\alpha}{2g}.$$

The largest height is maximized for $\alpha = \pi/2$, in which case it takes the value $\frac{v_0^2}{2g}$ (i.e., half of the maximum range, which occurs for $\alpha = \pi/4$).

(b) Let now $R = v_0^2/g$, the maximum range possible given that the initial spead is v_0. The parabola $x^2 + 2Ry - R^2 = 0$ is determined by the three points $(-R, 0)$, $(0, R/2)$ and $(R, 0)$, which are hit by the projectile for $\alpha = 3\pi/4$, $\pi/2$ and $\pi/4$, respectively.

The task is to show that (x_0, y_0) is on the orbit of a projectile iff $0 \leq y_0 \leq \frac{R^2 - x_0^2}{2R}$. The orbit for $\alpha = \pi/2$ contains exactly the points $(0, y_0)$ with $0 \leq y_0 \leq R/2$. Other orbits do not contain points on the y-axis except for $(0,0)$. This proves the assertion for $x_0 = 0$. From now on we assume $x_0 \neq 0$ and $\alpha \neq \pi/2$.

Eliminating t from $x(t) = (v_0 \cos\alpha)t$, $y(t) = (v_0 \sin\alpha)t - \frac{g}{2}t^2$, we obtain

$$y(x) = (\tan\alpha)x - \frac{g}{2}\frac{x^2}{v_0^2 \cos^2\alpha} = (\tan\alpha)x - \frac{g}{2v_0^2}(1 + \tan^2\alpha)x^2.$$

The condition $y(x_0) = y_0$ for hitting the target gives for $a = \tan\alpha$ the quadratic equation

$$y_0 = ax_0 - \frac{gx_0^2}{2v_0^2}(1 + a^2) \iff a^2 - \frac{2v_0^2}{gx_0}a + \frac{2v_0^2 y_0}{gx_0^2} + 1 = 0 \iff a^2 - \frac{2R}{x_0}a + \frac{2Ry_0}{x_0^2} + 1 = 0.$$

The solution is

$$a = \frac{R}{x_0} \pm \sqrt{\frac{R^2}{x_0^2} - \frac{2Ry_0}{x_0^2} - 1} = \frac{R \pm \sqrt{R^2 - 2Ry_0 - x_0^2}}{x_0}$$

and is in $\mathbb{R}$ iff $x_0^2 + 2Ry_0 - R^2 \leq 0$. Since $\tan\alpha = a$ is always (uniquely) solvable for $\alpha \in [0, \pi]$, $\alpha \neq \pi/2$ ($a = 0$ is not possible, since $x_0^2 > 0$ and hence $\sqrt{R^2 - 2Ry_0 - x_0^2} < R$), the assertion follows.

We also see that points (x_0, y_0) on the parabola $x^2 + 2Ry - R^2 = 0$ are on a unique orbit and points under the parabola (except those on the y-axis) are on two different orbits.

(c) Writing $D = (d, 0)$, the target's curve is $\mathbf{r}(t) = (d, h - \frac{g}{2}t^2)$. The condition for a hit at time t_0 is

$$(v_0 \cos\alpha)t_0 = d \wedge (v_0 \sin\alpha)t_0 - \frac{g}{2}t_0^2 = h - \frac{g}{2}t_0^2 \wedge 0 \leq t_0 \leq \frac{2v_0 \sin\alpha}{g}.$$

Solving the first equation for t_0 and substituting the result into the second equation gives $(\tan\alpha)d = h$, i.e., $\tan\alpha = h/d$. This is exactly the condition that the gun focuses on the target before firing. Substituting the value of t_0 into the last condition transforms it into

$$\frac{d}{v_0 \cos\alpha} \leq \frac{2v_0 \sin\alpha}{g} \iff d \leq \frac{v_0^2 \sin(2\alpha)}{g} = R.$$

Thus the target will be hit regardless what the initial speed v_0 is, provided only that D is smaller than the range of the gun shot for the given angle of evelation α. This in turn is equivalent to a lower bound on v_0, viz.

$$v_0 \geq \sqrt{\frac{dg}{\sin(2\alpha)}} = \sqrt{\frac{dg(1 + \tan^2\alpha)}{2\tan\alpha}} = \sqrt{\frac{dg(1 + h^2/d^2)}{2h/d}} = \sqrt{\frac{g(h^2 + d^2)}{2h}}.$$

25 In matrix notation the equation is

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathsf{T}} \underbrace{\begin{pmatrix} A & B \\ B & C \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{x}} + 2\underbrace{\begin{pmatrix} D & E \end{pmatrix}}_{\mathbf{b}} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{F}_{c} = 0,$$

or, using the indicated abbreviations, $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} + 2\mathbf{b}^{\mathsf{T}}\mathbf{x} + c = 0$ (forgive me for changing F to c).

a) The matrix $\mathbf{A}$ must have rank 1 in this case, because $\mathbf{A} = \mathbf{0}$ ($A = B = C = 0$) implies $\Delta = 0$, i.e., that the conic is degenerate. (The case $A = B = C = 0$ turns the equation into a linear one and is often exluded right away. The corresponding "conics" are lines and the trivial affine subspaces $\mathbb{R}^2$, $\emptyset$ of $\mathbb{R}^2$.)

If $A = 0$ then necessarily $B = 0$ (otherwise $\operatorname{rk}\mathbf{A} = 2$), $C \neq 0$, and the equation reduces to $y^2 + (2D/C)x + (2E/C)y + F/C = 0$. If $A \neq 0$, we can assume $A = 1$ (by scaling the equation). Then $C = B^2$ and the quadratic part becomes $x^2 + 2Bxy + B^2y^2 = (x + By)^2$. The coordinate change $x' = \frac{B}{\sqrt{1+B^2}}x - \frac{1}{\sqrt{1+B^2}}y$, $y' = \frac{1}{\sqrt{1+B^2}}x + \frac{B}{\sqrt{1+B^2}}y$ or, in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{1+B^2}} \begin{pmatrix} B & -1 \\ 1 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

transforms the equation into $(1+B^2)y'^2+2D'x'+2E'y'+F'=0$ (in fact, $F'=F$) and, by a further scaling, into $y'^2+2D''x'+2E''y'+F''=0$.

Hence we may assume that the equation of a non-central, non-degenerate conic is of the form $y^2+2Dx+2Ey+F=0$. If $D=0$ (i.e., $A=B=D=0$) then this conic is degenerate. Otherwise the translation $y'=y+E$ (eliminating y), $x'=x+(F-E^2)/(2D)$ (eliminating the constant F) transforms it into $y'^2+2Dx'=0$. If $D>0$, this is of the required form with $a=D/2$. If $D<0$, a sign change $x''=-x'$ produces the required form with $a=-D/2$.

Since $\det(\mathbf{A})=AC-B^2$, we also see that the parabolic case is characterized by $\Delta\neq 0\wedge AC-B^2=0$.

b) We use the equivalent matrix form with $\mathbf{x}_0=(x_0,y_0)^\top$, $\mathbf{x}'=(x',y')^\top$.

$$\begin{aligned}\mathbf{x}^\top\mathbf{A}\mathbf{x}+2\mathbf{b}^\top\mathbf{x}+c&=(\mathbf{x}'+\mathbf{x}_0)^\top\mathbf{A}(\mathbf{x}'+\mathbf{x}_0)+2\mathbf{b}^\top(\mathbf{x}'+\mathbf{x}_0)+c\\&=\mathbf{x}'^\top\mathbf{A}\mathbf{x}'+2(\mathbf{x}_0^\top\mathbf{A}+\mathbf{b}^\top)\mathbf{x}'+\mathbf{x}_0^\top\mathbf{A}\mathbf{x}_0+2\mathbf{b}^\top\mathbf{x}_0+c\end{aligned}$$

The summand involving $\mathbf{x}'$ vanishes iff $\mathbf{A}\mathbf{x}_0+\mathbf{b}=(\mathbf{x}_0^\top\mathbf{A}+\mathbf{b}^\top)^\top=\mathbf{0}$, and in this case the new constant term is $c'=-\mathbf{x}_0^\top\mathbf{b}+2\mathbf{b}^\top\mathbf{x}_0+c=c+\mathbf{b}^\top\mathbf{x}_0$. By definition, the center solves this equation, and plugging in the coordinates of $\mathbf{x}_0=-\mathbf{A}^{-1}\mathbf{b}$ gives

$$c'=F-D\frac{CD-BE}{AC-B^2}-E\frac{-BD+AE}{AC-B^2}=\begin{vmatrix}A&B&D\\B&C&E\\D&E&F\end{vmatrix}/(AC-B^2),$$

as one can see by expanding the determinant along the last column. (Alternatively, observe that Δ is preserved by translations and hence, using $D'=E'=0$, $\Delta=(A'C'-B'^2)F'=(AC-B^2)F'=(AC-B^2)c'$.)

c) Translating as in b), we may assume that the conic has an equation without a linear term $Dx+Ey$. Non-degeneracy then implies that the constant term $\Delta/(AC-B^2)$ is nonzero, so that after scaling we may assume the equation is $Ax^2+2Bxy+Cy^2=1$. If $B=0$, we must have $AC\neq 0$ and hence the equation determines an ellipse or hyperbola—except for the case $A,C<0$, in which there is no solution at all ("empty conic"). Hence it suffices to find a rotation of $\mathbb{R}^2$ eliminating the xy-term. (Note that a rotation, being a linear transformation, doesn't reintroduce linear terms into the equation and leaves the constant term untouched.)

In the following we work with representing matrices, since this better reveals the underlying geometry. We have

$$Ax^2+2Bxy+Cy^2=\begin{pmatrix}x\\y\end{pmatrix}^\top\begin{pmatrix}A&B\\B&C\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},\quad\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$$

Substituting the 2nd equation into the first we see that the representing matrix of the transformed quadratic form $A'x'^2+2B'x'y'+C'y'^2$ is

$$\begin{aligned}\begin{pmatrix}A'&B'\\B'&C'\end{pmatrix}&=\begin{pmatrix}\cos\phi&\sin\phi\\-\sin\phi&\cos\phi\end{pmatrix}\begin{pmatrix}A&B\\B&C\end{pmatrix}\begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix}\\&=\begin{pmatrix}A\cos^2\phi+2B\cos\phi\sin\phi+C\sin^2\phi&(C-A)\cos\phi\sin\phi+B(\cos^2\phi-\sin^2\phi)\\(C-A)\cos\phi\sin\phi+B(\cos^2\phi-\sin^2\phi)&A\sin^2\phi-2B\cos\phi\sin\phi+C\cos^2\phi\end{pmatrix}\\&=\begin{pmatrix}A\cos^2\phi+2B\cos\phi\sin\phi+C\sin^2\phi&\frac{1}{2}(C-A)\sin(2\phi)+B\cos(2\phi)\\\frac{1}{2}(C-A)\sin(2\phi)+B\cos(2\phi)&A\sin^2\phi-2B\cos\phi\sin\phi+C\cos^2\phi\end{pmatrix}.\end{aligned}$$

Since $\big(\sin(2\phi), \cos(2\phi)\big)$ can be chosen as any point on the unit circle, and hence to have any possible direction, we can make it orthogonal to $\big(\frac{1}{2}(C-A), B\big)$ to force $B' = 0$, proving the assertion.

That with this choice the diagonal elements A', C' are the roots of $\lambda^2 - (A+C)\lambda + AC - B^2 = 0$ is equivalent to $A' + C' = A + C \wedge A'C' = AC - B^2$. The first of these equations can be easily verified by inspecting the formulas for A', C' obtained above. The second equation follows by taking the determinant on both sides of the matrix equation and using the multiplicativity of the determinant function, as well as $\det R(\phi) = \det R(\phi)^\mathsf{T} = 1$.

d) As mentioned before we may assume $(A,B,C) \neq (0,0,0)$. If the conic is non-central (i.e., $\operatorname{rk} \mathbf{A} = 1$), an inspection of the solution of a) shows that the conic is equivalent to one with equation $y^2 + 2Ey + F = 0$ and hence equal to $\emptyset$, a line, or the union of two parallel lines. If the conic is central (i.e., $\operatorname{rk} \mathbf{A} = 2$), it is equivalent to one with equation $Ax^2 + 2Bxy + Cy^2 = 0$. Completing the square gives $(Ax+By)^2 + (AC-B^2)y^2 = 0$. For $AC - B^2 > 0$ the conic is a single point (the origin), and for $AC - B^2 < 0$ it consists of the union of two non-parallel lines meeting in its center.

e) Here

$$\mathbf{A} = \begin{pmatrix} 17 & -6 \\ -6 & 8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}, \quad c = -12.$$

Since $\operatorname{rk} \mathbf{A} = 2$, the conic is central with center

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 17 & -6 \\ -6 & 8 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ 8 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 8 & 6 \\ 6 & 17 \end{pmatrix} \begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the translation $(x,y) = (x',y') + (0,1)$ puts it into the form $17x^2 - 12xy + 8y^2 - 20 = 0$, which has center $(0,0)$. Since the constant term is nonzero, the conic is non-degenerate, and there is no need to check the condition $\Delta \neq 0$.

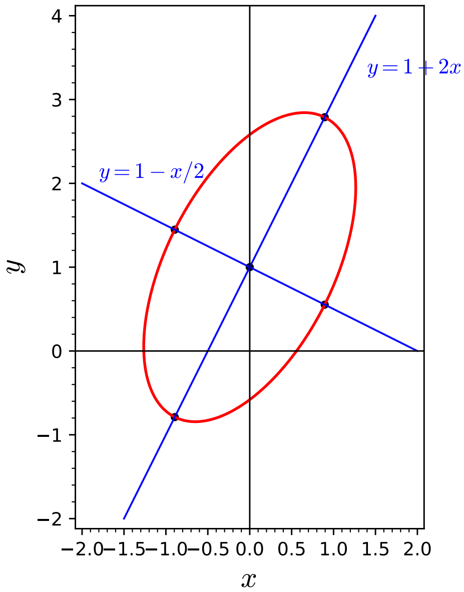
Next we eliminate the xy-term using the method developed in c). Since $\big(\frac{1}{2}(C-A), B\big) = (-9/2, -6) = -\frac{3}{2}(3,4)$, we can set $\big(\sin(2\phi), \cos(2\phi)\big) = \big(\frac{4}{5}, -\frac{3}{5}\big)$. Solving for $\cos\phi$, $\sin\phi$ (using the smaller of the two possible angles) gives $\cos\phi = \sqrt{\frac{1}{2}(1+\cos(2\phi)} = 1/\sqrt{5}$, $\sin\phi = 2/\sqrt{5}$. Thus the rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

puts the conic into the form $\lambda_1 x^2 + \lambda_2 y^2 = 20$, where $\lambda_, \lambda_2$ are the roots of $x^2 - (17+8)x + 17 \cdot 8 - (-6)^2 = x^2 - 25x + 100$, i.e., $\{\lambda_1, \lambda_2\} = \{5, 20\}$. To decide which λ is which, we compute

$$\begin{aligned} R(\phi)^\mathsf{T} \mathbf{A} R(\phi) &= \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 17 & -6 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 17 & -6 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 20 \end{pmatrix}. \end{aligned}$$

$\Longrightarrow \lambda_1 = 5$, $\lambda_2 = 20$ and the new conic has equation $x^2/4 + y^2 = 1$. In all we have found that the given conic is an ellipse with semi-axes $a = 2$, $b = 1$, which arises from the standard ellipse $x^2/4 + y^2 = 1$ by the rotation $\mathbf{x} \mapsto R(\phi)\mathbf{x}$ with angle $\phi = \arctan(2) \approx 63.435°$, followed by the translation $(x,y) \mapsto (x,y) + (0,1)$.



$y = 1 + 2x$
$y = 1 - x/2$
y
x

Calculus III (Math 241)

W12 Do Exercises 17 and 18 in [Ste21], Ch. 13.3.

W13 Do Exercises 42, 55, 60 in [Ste21], Ch. 13.3.

W14 Do the Exercises on the Frenet-Serret formulas in [Ste21], Ch. 13.3 (Exercises 63, 65, 71).

Solutions

12

Ex. 17 Writing $h(t) = (3\sin t, 4t, 3\cos t)$, we have $(0,0,3) = h(0)$, and the desired point is $P = h(t_0)$, where $t_0 > 0$ is determined by $\int_0^{t_0} |h'(s)|\,\mathrm{d}s = 5$. Since

$$\int_0^t |h'(s)|\,\mathrm{d}s = \int_0^t |(3\cos s, 4, -3\sin s)|\,\mathrm{d}s = \int_0^t \sqrt{9\cos^2 s + 16 + 9\sin^2 s}\,\mathrm{d}s = 5t,$$

we obtain $t_0 = 1$ and $P = h(1) = (3\sin 1, 4, 3\cos 1)$.

Ex. 18 From $\mathbf{r}(t) = (\frac{2}{t^2+1} - 1)\,\mathbf{i} + \frac{2t}{t^2+1}\,\mathbf{j} = \left(\frac{2}{t^2+1} - 1, \frac{2t}{t^2+1}\right)$ we have

$$\mathbf{r}'(t) = \left(\frac{-4t}{(t^2+1)^2}, \frac{-2t^2+2}{(t^2+1)^2}\right).$$

Since the initial point $(1,0)$ corresponds to $t = 0$, the arc length is

$$\begin{aligned}
s(t) &= \int_0^t |\mathbf{r}'(t)|\,\mathrm{d}t \\
&= \int_0^t \sqrt{\left(\frac{-4t}{(t^2+1)^2}\right)^2 + \left(\frac{-2t^2+2}{(t^2+1)^2}\right)^2}\,\mathrm{d}t \\
&= \int_0^t \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}}\,\mathrm{d}t \\
&= \int_0^t \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}\,\mathrm{d}t \\
&= \int_0^t \frac{2}{t^2+1}\,\mathrm{d}t \\
&= 2\arctan t.
\end{aligned}$$

Then $t = \tan\frac{s}{2}$, $s \in (-\pi, \pi)$. Substituting t with s, we have

$$\begin{aligned}
\mathbf{r}(t(s)) &= \left(\frac{2}{(\tan^2\frac{1}{2}s + 1} - 1, \frac{2\tan\frac{1}{2}s}{\tan^2\frac{1}{2}s + 1}\right) \\
&= \left(\frac{1 - \tan^2\frac{1}{2}s}{1 + \tan^2\frac{1}{2}s}, \frac{2\tan\frac{1}{2}s}{\sec^2\frac{1}{2}s}\right) \\
&= \left(\frac{1 - \tan^2\frac{1}{2}s}{\sec^2\frac{1}{2}s}, 2\tan\frac{1}{2}s\cos^2\frac{1}{2}s\right) \\
&= \left(\cos^2\frac{1}{2}s - \sin^2\frac{1}{2}s, 2\sin\frac{1}{2}s\cos\frac{1}{2}s\right) \\
&= (\cos s, \sin s).
\end{aligned}$$

From the reparametrization, we can see that the corresponding non-parametric curve is a unit circle. Note that $(-1,0)$ is not included in the curve, because $t = \tan\frac{s}{2}$ is undefined when $s = \pm\pi$. Rather we have $\lim_{t\to\pm\infty}\mathbf{r}(t) = (-1,0)$.

13

Ex. 42 The red curve is the curvature of the blue curve. This can be seen, e.g., from the fact that the curvature at inflection points must be zero.

Ex. 55 Since the centers of the two osculating circles are clearly on the ellipse's axes, it suffices to determine their radii or, equivalently, the curvature of the ellipse in $(2,0)$ and $(0,3)$. Using the parametrizaton $\mathbf{r}(t) = (2\cos t, 3\sin t, 0)$, we obtain

$$\begin{aligned}
\mathbf{r}'(t) &= (-2\sin t, 3\cos t, 0),\\
\mathbf{r}''(t) &= (-2\cos t, -3\sin t, 0),\\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= (0,0,6),\\
\kappa(t) &= \frac{|(0,0,6)|}{|(-2\sin t, 3\cos t, 0)|^3} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}
\end{aligned}$$

$\Longrightarrow \kappa(0) = 2/9$, $\kappa(\pi/2) = 3/4$, and the two osculating circles have radii $9/2$, $4/3$, centers $(-5/2, 0)$, $(0, 5/3)$, and equations

$$\left(x + \frac{5}{2}\right)^2 + y^2 = \left(\frac{9}{2}\right)^2, \quad x^2 + \left(y - \frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2.$$

The corresponding plot can be found in the answers section of [Ste16].

Ex. 60 The curve can be written as

$$\mathbf{r}(t) = \begin{pmatrix} 2\\ 1\\ 0 \end{pmatrix} + t\begin{pmatrix} 1\\ -1\\ 0 \end{pmatrix} + t^2\begin{pmatrix} 0\\ 0\\ 1/2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

$\Longrightarrow \mathbf{r}(t)$ is contained in the plane $E = (2,1,0) + \mathbb{R}(1,-1,0) + \mathbb{R}(0,0,1/2)$. (The same argument shows that any curve whose coordinate functions are polynomials of degree ≤ 2 is contained in a plane, or even in a line; cf. also our)

$$\begin{aligned}
\mathbf{r}'(t) &= \begin{pmatrix} 1\\ -1\\ 0 \end{pmatrix} + 2t\begin{pmatrix} 0\\ 0\\ 1/2 \end{pmatrix},\\
\mathbf{r}''(t) &= 2\begin{pmatrix} 0\\ 0\\ 1/2 \end{pmatrix}
\end{aligned}$$

Since $\mathbf{r}'(t)$, $\mathbf{r}''(t)$ span the same 2-dimensional subspace of $\mathbb{R}^3$ as $(1,-1,0)$, $(0,0,1/2)$, the osculating plane at every curve point $\mathbf{r}(t)$ is equal to E. (The same is true for any curve contained in some plane, provided only it has an osculating plane at every curve point.)

14

Ex. 63 From the lecture we know that $\mathbf{T}' = |\mathbf{T}'|\,\mathbf{N}$ and $\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|}$. If $\mathbf{T} = \mathbf{T}(s)$ is parametrized with respect to arc length, we have $|\mathbf{r}'(s)| = 1$ and hence $\kappa(s) = |\mathbf{T}'(s)|$. Substituting this in the first formula gives $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$, as asserted.

Ex. 65 (a) From $|\mathbf{B}(s)|^2 = 1$ we obtain, by differentiating both sides, $2\,\mathbf{B}(s) \cdot \mathbf{B}'(s) = 0$ and hence $\mathbf{B}(s) \perp \mathbf{B}'(s)$. (This fact does not require the curve to be parametrized with respect to arc length.)

(b) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ yields $\mathbf{B}' = \mathbf{T}' \times \mathbf{N} + \mathbf{T} \times \mathbf{N}'$. The first summand vanishes, since $\mathbf{T}'(s)$ is a multiple of $\mathbf{N}(s)$. It follows that $\mathbf{B}'(s) = \mathbf{T}(s) \times \mathbf{N}'(s)$, which is orthogonal to $\mathbf{T}(s)$. (Again this fact does not require the curve to be parametrized with respect to arc length.)

(c) Since $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}(s)$ form a basis of $\mathbb{R}^3$ (provided that is s is fixed), there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\mathbf{B}'(s) = \lambda_1 \mathbf{T}(s) + \lambda_2 \mathbf{N}(s) + \lambda_3 \mathbf{B}(s).$$

Taking the dot product with $\mathbf{T}$ on both sides of this equation and using (b) gives $\lambda_1 = 0$. Similarly, taking the dot product with $\mathbf{B}$ and using (a) gives $\lambda_3 = 0$. Thus $\mathbf{B}'(s) = \lambda_2 \mathbf{N}(s)$, and defining $\tau(s) = -\lambda_2$ (which depends on s) proves the claim.

Ex. 71 For simplicity we omit the argument s in the following computation:

$$\begin{aligned}
\mathbf{N}' &= \mathbf{B}' \times \mathbf{T} + \mathbf{B} \times \mathbf{T}' \\
&= (-\tau\,\mathbf{N}) \times \mathbf{T} + \mathbf{B} \times (\kappa\,\mathbf{N}) \\
&= -\tau\,\mathbf{N} \times \mathbf{T} + \kappa\,\mathbf{B} \times \mathbf{N} \\
&= \tau\,\mathbf{T} \times \mathbf{N} - \kappa\,\mathbf{N} \times \mathbf{B} \\
&= \tau\,\mathbf{B} - \kappa\,\mathbf{T}
\end{aligned}$$

Calculus III (Math 241)

H26 Do the applied project on Kepler's Laws in [Ste21], Ch. 13.4. p. 925/926.

H27 Do Exercises 48, 52, 62, 64, 66 in [Ste21]), Ch. 14.1.

H28

a) Show that a series $\sum_{n=0}^{\infty} c_n$ of complex numbers converges if it converges absolutely, i.e., $\sum_{n=0}^{\infty} |c_n|$ converges in $\mathbb{R}$.

Hint: Writing $c_n = a_n + \mathrm{i}\, b_n = (a_n, b_n)$ with $a_n, b_n \in \mathbb{R}$, show that the assumption implies that the real series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely.

b) Show that the length function in $\mathbb{C} \triangleq \mathbb{R}^2$ is multiplicative with respect to complex multiplication, i.e., $|zw| = |z|\,|w|$ for $z, w \in \mathbb{C}$

c) The complex exponential function can be defined by

$$\mathrm{e}^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for } z = x + \mathrm{i}\, y \in \mathbb{C}.$$

Show that this series converges for all $z \in \mathbb{C}$.

Hint: Use a) and b).

d) Show that the evaluation of the geometric series also holds in the complex setting:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z} \quad \text{for } z \in \mathbb{C} \text{ with } |z| < 1.$$

e) Evaluate $\sum_{n=0}^{\infty} \left(\frac{\mathrm{i}}{2}\right)^n$ and $\sum_{n=0}^{\infty} \left(\frac{1+\mathrm{i}}{2}\right)^n$, and graph the first few partial sums of these series together with their limit.

H29 *Optional Exercise*

For $D \subseteq \mathbb{R}^n$ prove the following facts; for the terminology see `lecture6-10_handout.pdf`, Slides 35–39.

a) "open balls" are open, "closed balls" are closed, and $\partial \mathrm{B}_r(\mathbf{a}) = \mathrm{S}_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n; |\mathbf{x}| = r\}$.

b) D is open iff $D \cap \partial D = \emptyset$.

c) D is closed iff $D \supseteq \partial D$.

d) D is open iff $\mathbb{R}^n \setminus D$ is closed.

e) The boundary of D and its complementary set are the same, $\partial D = \partial(\mathbb{R}^n \setminus D)$, and we have the disjoint decompositions $\overline{D} = D^\circ \uplus \partial D$, $\quad \mathbb{R}^n = D^\circ \uplus \partial D \uplus (\mathbb{R}^n \setminus D)^\circ$.

f) $\overline{D} = D \cup \partial D = D \cup D'$.

H30 *Optional Exercise*

a) Suppose $c = a + \mathrm{i}b \in \mathbb{C}$ is nonzero. Show without recourse to Euler's Identity (cf. `lecture18-20`) that the equation $z^2 = c$ has exactly two solutions in $\mathbb{C}$.

Hint: For $z = x + \mathrm{i}y$ the equation $z^2 = c$ is equivalent to $x^2 - y^2 = a \wedge 2xy = b$. Express $x^2 + y^2$ in terms of a, b.

b) Show (e.g., by completing the square) that a quadratic equation $Az^2 + Bz + C = 0$, $A, B, C \in \mathbb{C}$, $A \neq 0$, has (exactly) 2 solutions in $\mathbb{C}$ if $B^2 - 4AC \neq 0$ and 1 solution if $B^2 - 4AC = 0$.

c) Euler's Identity and the functional equation for $z \mapsto \mathrm{e}^z$ (cf. `lecture14-16_handout.pdf`, Slides 24–31) imply that the solutions of $z^n = 1$ in $\mathbb{C}$ (*n-th roots of unity*) have the form $\mathrm{e}^{2\pi \mathrm{i}k/n} = z_n^k$ with $k \in \{0, 1, \dots, n-1\}$, $z_n = \mathrm{e}^{2\pi\mathrm{i}/n}$, and form the vertices of a regular n-gon inscribed in the unit circle. Using the result of a), determine z_{24} in the form $u + \mathrm{i}v$ and sketch the solutions of $z^{24} = 1$ that are contained in the 1st quadrant.

Due on Wed Nov 1, 6 pm

Complex numbers (required for H28 and H30) will be discussed in Lecture 15 on Fri Oct 27. The optional exercises can be handed in until Wed Nov 8, 6 pm.

Solutions

26 In the following, **i**, **j**, **k** are not viewed as standard unit vectors (though this is what the notation suggests) but as a cartesian coordinate system adapted to the planet's motion. In the lecture we have used the notation $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ instead.

1. (a) $\mathbf{h} = \mathbf{r} \times \mathbf{r}' = ((r\cos\theta)\mathbf{i} + (r\sin\theta\mathbf{j})) \times ((r'\cos\theta - r\sin\theta\frac{d\theta}{dt})\mathbf{i} + (r'\sin\theta + r\cos\theta\frac{d\theta}{dt})\mathbf{j}) = (rr'\cos\theta\sin\theta + r^2\cos^2\theta\frac{d\theta}{dt} - rr'\cos\theta\sin\theta + r^2\sin^2\theta\frac{d\theta}{dt})\mathbf{k} = r^2\frac{d\theta}{dt}\mathbf{k}$

   (b) Since **k** was chosen as $\mathbf{h}/h$, we have $\mathbf{h} = h\mathbf{k}$. Comparing this with a) gives $h = r^2\frac{d\theta}{dt}$.

   (c) $A(t) = \frac{1}{2}\int_{\theta_0}^{\theta} |\mathbf{r}|^2\, d\theta = \frac{1}{2}\int_{\theta_0}^{\theta} r^2 \frac{d\theta}{dt} dt$ in polar coordinates. Thus, by the Fundamental Theorem of Calculus, $\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$

   (d) $\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{h}{2} =$ constant, since **h** is a constant vector and $h = |\mathbf{h}|$.

2. (a) Since $\frac{dA}{dt} = \frac{h}{2} =$ constant, $A(t) = A(0) + \frac{1}{2}ht$. Normalizing to $A(0) = 0$ gives $A(t) = \frac{1}{2}ht$. As $A(T) =$ Area of the ellipse $= \pi ab$, we must have $\frac{1}{2}hT = \pi ab$, i.e., $T = 2\pi ab/h$.

   (b) The equation
$$\frac{h^2}{GM} = ed$$
   is an instance of $e = l/p$ (the eccentricity e equals the semilatus rectum l divided by the focal parameter p), as discussed in the lecture. For this note that for a planetary orbit $l = h^2/GM$ (as shown in the lecture), and p equals d (the notation used in the textbook).

   It remains to show that the focal parameter equals b^2/a. We know that $a = \frac{l}{1-e^2} = \frac{ed}{1-e^2}$ (e.g., from the fact that the sum of the perihelion and aphelion distances, which are $\frac{l}{1+e}$ and $\frac{l}{1-e}$, respectively, equals $2a$) and $1 - e^2 = \frac{b^2}{a^2}$ (from $e = \sqrt{1 - \frac{b^2}{a^2}}$, which reflects the fact that in the standard representation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of an ellipse the foci are at $(\pm ea, 0)$ and their distance from $(0, b)$ is equal to a). Both equations together yield
$$\frac{h^2}{GM} = ed = \frac{b^2}{a}.$$

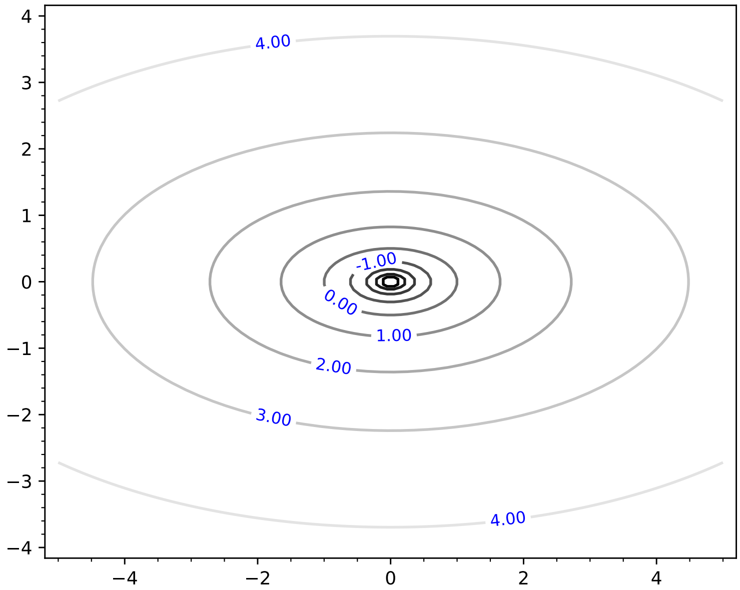
   (c) $T^2 = \frac{4\pi^2a^2b^2}{h^2} = 4\pi^2a^2b^2\frac{a}{GMb^2} = \frac{4\pi^2a^3}{GM}$

3. From Part 2, $T^2 = \frac{4\pi^2a^3}{GM}$. Since $T \approx 365.25\,\mathrm{d} \times (24 \times 60^2)\,\mathrm{s/d} \approx 3.1558 \cdot 10^7\,\mathrm{s}$. Therefore $a^3 = \frac{GMT^2}{4\pi^2} \approx \frac{(6.67\times10^{-11})(1.99\times10^{3}0)(3.1558\times10^7)^2}{4\pi^2} \approx 3.348 \cdot 10^{33}\,\mathrm{m}^3 \Rightarrow a \approx 1.496 \cdot 10^{11}\,\mathrm{m}$. Thus, the length of the major axis of the earth's orbit (that is, $2a$) is approximately $2.99 \cdot 10^{11}\,\mathrm{m} = 2.99 \cdot 10^8\,\mathrm{km}$.

4. We can adapt the equation $T^2 = \frac{2\pi^2a^3}{GM}$ from Part 2(c) with the earth at the center of the system; so T is the period of the satellite's orbit around the earth, M is the mass of the earth, and a is the length of the semimajor axis of the satellite's orbit (measured from the earth's center). Since we want the satellite to remain fixed above a particular point on the earth's equator, the orbit must be circular: $\frac{d\theta}{dt} =$ constant, since the same is true for points

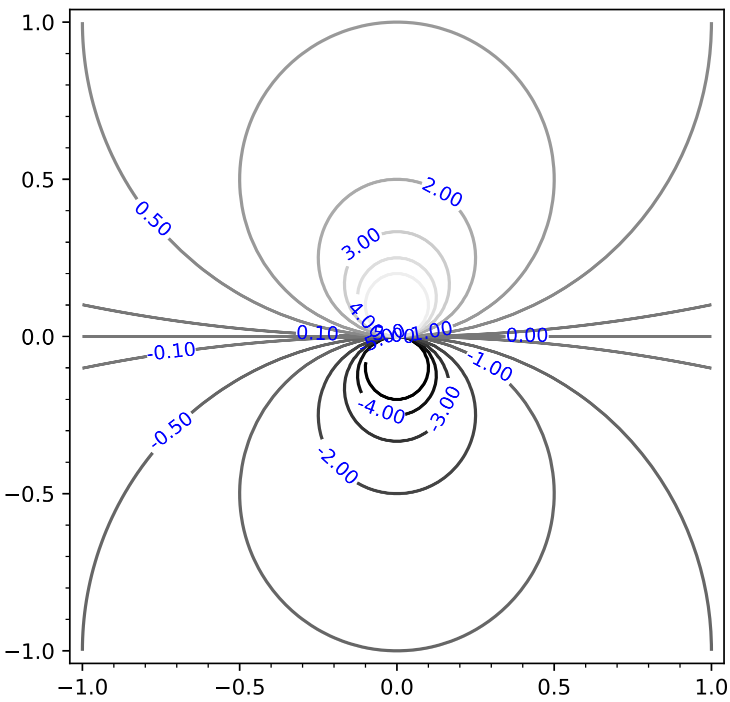
on the earth's equator, which together with 2(b) gives that r is constant. Moreover, T must coincide with the period of the earth's own rotation, i.e., $T = 24\,\mathrm{h} = 86,400\,\mathrm{s}$. The mass of the earth is $M = 5.98 \times 10^{24}\,\mathrm{kg}$, so $a = (\frac{T^2GM}{4\pi^2})^{\frac{1}{3}} \approx (\frac{(86400)^2(6.67\times10^{-11})(5.98\times10^{24})}{4\pi^2})^{\frac{1}{3}} \approx 4.23 \cdot 10^7\,\mathrm{m}$. Since the orbit is circular, its radius is a, and since the radius of the earth is approx. $6.37 \cdot 10^6\,\mathrm{m}$, the required altitude above the earth's surface for the satellite is approx. $4.23 \cdot 10^7\,\mathrm{m} - 6.37 \cdot 10^6\,\mathrm{m} \approx 3.59 \cdot 10^7\,\mathrm{m}$, or $35,900\,\mathrm{km}$.

27 Ex. 48 $f(x,y) = \ln(x^2+4y^2)$, $D = \mathbb{R}^2 \setminus \{(0,0)\}$



The contours are the same as for $g(x,y) = x^2+4y^2$ (except for the point $(0,0)$, but with different labeling.

Ex. 52 $f(x,y) = \frac{y}{x^2+y^2}$, $D = \mathbb{R}^2 \setminus \{(0,0)\}$



Ex. 62 $z = e^x \cos y$ matches (a) A, (b) IV.
Reasons: This function is periodic in y but not in x, a condition only satisfied by A and IV. Also, note that traces in $x = k$ are cosine curves with amplitude that increases as x increases.
Ex. 64 $z = \sin x - \sin y$ matches (a) E, (b) III.
Reasons: This function is periodic in both x and y. But it's not constant along lines $y = x + t$, so the contour map is III. Also notice that traces in $y = k$ are vertically shifted copies of the sine wave $z = \sin x$, so the graph must be E.
Ex. 66 $z = \frac{x-y}{1+x^2+y^2}$ matches (a) D, (b) V.
Reasons: This function satisfies $\lim_{|(x,y)|\to\infty} z(x,y) = 0$; in particular it is not periodic in any direction. The only graph that shows this behavior is D, which corresponds to V. The contour map V can also be uniquely identified by the property that distances between contours increase as we move away from the origin in any direction. This is a consequence of $\lim_{|(x,y)|\to\infty} z(x,y) = 0$. (Of course it is assumed here that the levels of the contours shown are in arithmetic progression.)

28 a) Limits of sequences in $\mathbb{R}^n$ were defined in Lecture 7 (see `lecture6-10_handout.pdf`, Slide 26) and the case of complex numbers, which are just points in $\mathbb{R}^2$, is the special case $n = 2$ thereof.

A sequence $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots$ of points in $\mathbb{R}^2$ converges to a point $\mathbf{a} \in \mathbb{R}^2$ if for every $\varepsilon > 0$ there is a response $N \in \mathbb{N}$ such that $|\mathbf{a}_k - \mathbf{a}| < \varepsilon$ for all $k > N$, and convergence of a series $\sum_{n=1}^{\infty} \mathbf{a}_n$ means the convergence of the associated sequence $\mathbf{s}_1 = \mathbf{a}_1$, $\mathbf{s}_1 = \mathbf{a}_1 + \mathbf{a}_2$, $\dots$, $\mathbf{s}_n = \mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_n$, $\dots$ of partial sums. As remarked above, this applies immediately to sequences $c_k = (a_k, b_k) = a_k + i b_k$ ($a_k, b_k \in \mathbb{R}$) of complex numbers, but in this setting we don't use bold type $\mathbf{c}_k$.

Now we switch back to indexing sequences by n (not k). For real series the corresponding assertion is proved in [Ste21], and we derive a) from this by showing that the absolute convergence of $\sum_{n=0}^{\infty} c_n$ implies the absolute convergence of the series formed by the real and imaginary parts of c_n. It should be noted that there is nothing special here about the case $n = 2$, and the result holds mutatis mutandis for series of points in $\mathbb{R}^n$. Only Part b), which relies on complex multiplication to define e^z, requires $n = 2$.

Writing $c_n = (a_n, b_n) = a_n + \mathrm{i} b_n$ with $a_n, b_n \in \mathbb{R}$, we have $s_n = c_1 + \dots + c_n = (a_1 + \dots + a_n, b_1 + \dots + b_n) = a_1 + \dots + a_n + \mathrm{i}(b_1 + \dots + b_n)$. Since the convergence of a sequence in $\mathbb{R}^2$ is equivalent to the convergence of its two coordinate sequences, we have $\sum_{n=1}^{\infty} c_n = (\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n) = \sum_{n=1}^{\infty} a_n + \mathrm{i} \sum_{n=1}^{\infty} b_n$, provided the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

Since $|a_n| \leq \sqrt{a_n^2 + b_n^2} = |c_n|$ and, similarly, $|b|_n \leq |c_n|$, the comparison test for non-negative series ([Ste21], Ch. 11.4) gives that the absolute convergence of $\sum_{n=1}^{\infty} c_n$ implies the absolute convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. Hence these series converge in $\mathbb{R}$ ([Ste21], Ch. 11.5, Th. 3), and so does $\sum_{n=1}^{\infty} c_n$ by the previous discussion.

b) We have

$$\begin{aligned}
|(a+\mathrm{i}b)(c+\mathrm{i}d)|^2 &= |ac - bd + \mathrm{i}(ad+bc)|^2 = (ac-bd)^2 + (ad+bc)^2 \\
&= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 = (a^2+b^2)(c^2+d^2) = |a+\mathrm{i}b|^2 \, |c+\mathrm{i}d|^2 \\
&= (|a+\mathrm{i}b| \, |c+\mathrm{i}d|)^2 .
\end{aligned}$$

c) $|zw| = |z|\,|w|$ implies in particular $|z^n| = |z|^n$ for all integers $n \geq 0$ (in fact, for all integers), and hence

$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!}.$$

Since the real exponential series converges everywhere in $\mathbb{R}$, the complex exponential series converges absolutely for all $z \in \mathbb{C}$, and an application of Part a) finishes the proof.

d) The summation formula for a finite geometric sum, which only uses field arithmetic, holds also for complex numbers. The formula gives

$$1+z+\cdots+z^n = \frac{1-z^{n+1}}{1-z} \to \frac{1}{1-z} \quad \text{for } n \to \infty,$$

since $z^{n+1} \to 0$ in $\mathbb{C}$ because of $\left|z^{n+1}\right| = |z|^{n+1} \to 0$ in $\mathbb{R}$, using $|z| < 1$. Continuity of the arithmetic operations in $\mathbb{C}$ is also required, but this is clear from their real representation $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, a_2+b_2)$, $(a_1,a_2)(b_1,b_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1)$.

e) We obtain

$$\sum_{n=0}^{\infty}\left(\frac{\mathrm{i}}{2}\right)^n = \frac{1}{1-\mathrm{i}/2} = \frac{2}{2-\mathrm{i}} = \frac{2(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})} = \frac{4+2\mathrm{i}}{5} = \frac{4}{5}+\frac{2}{5}\mathrm{i},$$
$$\sum_{n=0}^{\infty}\left(\frac{1+\mathrm{i}}{2}\right)^n = \frac{1}{1-\frac{1+\mathrm{i}}{2}} = \frac{2}{2-1-\mathrm{i}} = \frac{2}{1-\mathrm{i}} = \frac{2(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} = 1+\mathrm{i}.$$

Partial sums of these series are graphed in Figures 1 and 2.

29 a) For $\mathbf{b} \in \mathrm{B}_r(\mathbf{a})$ we have $|\mathbf{b}-\mathbf{a}| < r$ and hence $\delta := r - |\mathbf{b}-\mathbf{a}| > 0$. Since $|\mathbf{x}-\mathbf{b}| < \delta$ implies $|\mathbf{x}-\mathbf{a}| \leq |\mathbf{x}-\mathbf{b}| + |\mathbf{b}-\mathbf{a}| < \delta + |\mathbf{b}-\mathbf{a}| = r$, the ball $\mathrm{B}_\delta(\mathbf{b})$ is entirely contained in $\mathrm{B}_r(\mathbf{a})$. Thus $\mathrm{B}_r(\mathbf{a})$ is open.

For a convergent sequence $(\mathbf{x}^{(k)})$ in $\mathbb{R}^n$ that is contained in $\overline{\mathrm{B}_r(\mathbf{a})}$ we have $\left|\mathbf{x}^{(k)}-\mathbf{a}\right| \leq r$ for all k, and hence $\left|\lim_{k\to\infty}\mathbf{x}^{(k)}-\mathbf{a}\right| \leq r$, since the length function is continuous. Thus every limit point of $\overline{\mathrm{B}_r(\mathbf{a})}$ is contained in $\overline{\mathrm{B}_r(\mathbf{a})}$, showing that $\overline{\mathrm{B}_r(\mathbf{a})}$ is closed.

Consider $\mathbf{b} \in \mathrm{S}_r(\mathbf{a})$. Since translations $\mathbf{x} \mapsto \mathbf{x}-\mathbf{a}$ of $\mathbb{R}^n$ preserve the topology, we may assume w.l.o.g. $\mathbf{a} = \mathbf{0}$, i.e., $|\mathbf{b}| = r$. (This saves some writing. If you are sceptical, re-translate everything in the subsequent proof.) Then, by definition, $\mathbf{b} \notin \mathrm{B}_r(\mathbf{0})$, so every neighborhood of $\mathbf{b}$ contains a point of $\mathbb{R}^n \setminus \mathrm{B}_r(\mathbf{0})$, viz. $\mathbf{b}$. Further, we have $\lambda\mathbf{b} \in \mathrm{B}_r(\mathbf{0})$ for $0 \leq \lambda < 1$ (even for $|\lambda| < 1$). Since $|\mathbf{b}-\lambda\mathbf{b}| = |(1-\lambda)\mathbf{b}| = (1-\lambda)r < \varepsilon$ provided only that $\lambda > 1-\varepsilon/r$, we can find points of $\mathrm{B}_r(\mathbf{0})$ in every neighborhood of $\mathbf{b}$. This shows $\mathbf{b} \in \partial\mathrm{B}_r(\mathbf{0})$. On the other hand, if $|\mathbf{b}| < r$, we have seen above that an entire ball $\mathrm{B}_\delta(\mathbf{b})$ is contained in $\mathrm{B}_r(\mathbf{0})$, so that $\mathbf{b}$ can't be a boundary point of $\mathrm{B}_r(\mathbf{0})$. Similarly, if $|\mathbf{b}| > r$ then the ball $\mathrm{B}_\delta(\mathbf{b})$, $\delta := |\mathbf{b}| - r$, is contained in $\mathbb{R}^n \setminus \mathrm{B}_r(\mathbf{0})$, so that $\mathbf{b}$ can't be a boundary point of $\mathrm{B}_r(\mathbf{0})$ either. Alltogether this shows $\partial\mathrm{B}_r(\mathbf{0}) = \mathrm{S}_r(\mathbf{0})$.

b) If D is open then around $\mathbf{a} \in D$ some ball $\mathrm{B}_\delta(\mathbf{a})$, $\delta > 0$, is entirely contained in D. As this ball contains no point of $\mathbb{R}^n \setminus D$, we have $\mathbf{a} \notin \partial D$, and hence $D \cap \partial D = \emptyset$. Conversely, suppose $D \cap \partial D = \emptyset$ and consider $\mathbf{a} \in D$. Then $\mathbf{a} \notin \partial D$, so that there is some ball around $\mathbf{a}$ containing only points in D or only points in $\mathbb{R}^n \setminus D$. The latter is impossible, since $\mathbf{a} \in D$. Hence $\mathbf{a}$ is an interior point of D, and D is open.

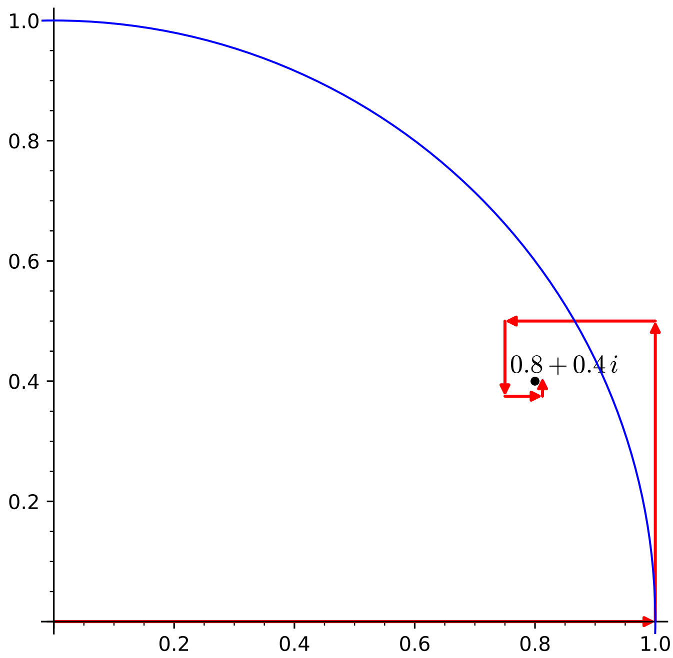


Figure 1: Illustration of $\displaystyle\sum_{n=0}^{5}\left(\frac{\mathrm{i}}{2}\right)^{n} \approx \frac{4}{5}+\frac{2}{5}\mathrm{i}$

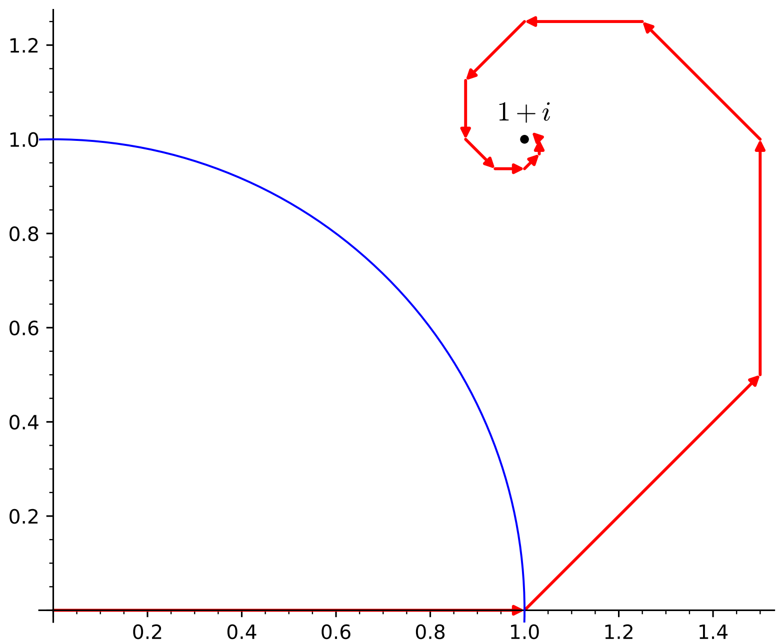


Figure 2: Illustration of $\displaystyle\sum_{n=0}^{11}\left(\frac{1+\mathrm{i}}{2}\right)^{n} \approx 1+\mathrm{i}$

c) Boundary points $\mathbf{b}$ of D are in particular limit points: Every ball $\mathrm{B}_\varepsilon(\mathbf{b})$ must contain a point of D. This is in particular true for any reciprocal integer $\varepsilon = 1/k$, so there exists $\mathbf{x}^{(k)} \in D$ such that $\left|\mathbf{b} - \mathbf{x}^{(k)}\right| < 1/k$. Then clearly $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{b}$, so that $\mathbf{b}$ is a limit point of D. Hence $\partial D \subseteq \overline{D}$, as asserted.

   With this preliminary observation we can prove the statement: If D is closed, we have $\partial D \subseteq \overline{D} = D$. Conversely, suppose $\partial D \subseteq D$ and let $\mathbf{a} \in \mathbb{R}^n \setminus D$. Then, as $\mathbf{a}$ is not a boundary point of D, there must exist some ball around $\mathbf{a}$ that doesn't contain a point of D. Then, obviously, $\mathbf{a}$ can't be a limit point of D, and hence D contains all its limit points. This shows that D is closed.

d) This follows from b), c) together with the observation that the boundaries of D and $\mathbb{R}^n \setminus D$ are the same, because the definition of "boundary point" is "symmetric" in D and $\mathbb{R}^n \setminus D$.

e) The truth of first statement has been observed in the proof of d).

   In order to prove $\overline{D} = D^\circ \uplus \partial D$, i.e., $\overline{D} = D^\circ \cup \partial D$ and $D^\circ \cap \partial D = \emptyset$, observe first that according to their definitions "interior point of D" and "boundary point of D" are clearly incompatible, i.e., $D^\circ \cap \partial D = \emptyset$. Further, we have $D^\circ \cup \partial D \subseteq \overline{D}$, since (every inner point and) every boundary point is a limit point; cf. the proof of c). Finally, suppose $\mathbf{a} \in \overline{D}$. Then there is a sequence $(\mathbf{x}^{(k)})$ in D with $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{a}$. This implies that every ball around $\mathbf{a}$ contains a point of D. Regarding points in $\mathbb{R}^n \setminus D$, either there is some ball around $\mathbf{a}$ containing no point of $\mathbb{R}^n \setminus D$, in which case $\mathbf{a} \in D^\circ$, or every ball around $\mathbf{a}$ contains a point of $\mathbb{R}^n \setminus D$, in which case $\mathbf{a} \in \partial D$. This shows $\overline{D} \subseteq D^\circ \cup \partial D$.

   In order to prove the last assertion, which means $\mathbb{R}^n = D^\circ \cup \partial D \cup (\mathbb{R}^n \setminus D)^\circ$ and that these three sets are mutually disjoint, consider the three properties (predicates) "there exists a ball $\mathrm{B}_\varepsilon(\mathbf{a})$ containing only points of D", "there exists a ball $\mathrm{B}_\varepsilon(\mathbf{a})$ containing only points of $\mathbb{R}^n \setminus D$", and "every ball $\mathrm{B}_\varepsilon(\mathbf{a})$ contains a point of D and a point of $\mathbb{R}^n \setminus D$. The corresponding three subsets of $\mathbb{R}^n$ (consisting of the points that have this property) are D°, $(\mathbb{R}^n \setminus D)^\circ$, and ∂D, respectively. Clearly every point in $\mathbb{R}^n$ has exactly one of these properties, and the assertion follows.

f) If $\mathbf{b} \in \partial D \setminus D$, every neighborhood of $\mathbf{b}$ must contain a point of D, i.e., $\mathbf{b} \in D'$. This shows $D \cup \partial D \subseteq D \cup D'$. Further, since $D \subseteq \overline{D}$ and obviously also $D' \subseteq \overline{D}$ (accumulation points are limit points), we have $D \cup D' \subseteq \overline{D}$. Finally, a limit point $\mathbf{a}$ of D is either in D or every neighborhood of $\mathbf{a}$ contains points in D and a point in $\mathbb{R}^n \setminus D$ (viz., $\mathbf{a}$), showing that $\overline{D} \subseteq D \cup \partial D$.

Remarks: a) and f) (or e)) together show that the closure of $\mathrm{B}_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n; |\mathbf{x} - \mathbf{a}| < r\}$ is indeed $\{\mathbf{x} \in \mathbb{R}^n; |\mathbf{x} - \mathbf{a}| \leq r\}$, justifying the notation $\overline{\mathrm{B}_r(\mathbf{a})}$ for this set.
The properties discussed in this exercise form only a selection of important properties of the topological concepts "open set", "closed set", "interior", "closure", "boundary", "accumulation point". For example, we have not discussed the behavior of open/closed sets under the basic set operations $\cup$ and $\cap$ (finite intersections and arbitrary unions of open sets are open as well, with a dual statement for closed sets), key properties of the operators $D \mapsto D^\circ$ and $D \mapsto \overline{D}$ such as $D^{\circ\circ} = D^\circ$ and $\overline{\overline{D}} = \overline{D}$ (which say that D° is always open and $\overline{D}$ is always closed), D° is the largest open set contained in D, $\overline{D}$ is the smallest closed set containing D, the connectedness of $\mathbb{R}^n$ ($\emptyset$ and $\mathbb{R}^n$ are the only subsets of $\mathbb{R}^n$ that are both open and closed), etc.

30 a) We have $a^2+b^2=(x^2-y^2)^2+(2xy)^2=x^4+2x^2y^2+y^4=(x^2+y^2)^2$ and hence $x^2+y^2=\sqrt{a^2+b^2}$. (Alternative proof: $z^2=c \Longrightarrow |z|^2=\left|z^2\right|=|c|$.)

$$\Longrightarrow x^2=\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right), \qquad y^2=\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)$$
$$\Longrightarrow x=\pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}, \qquad y=\pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}.$$

The signs are determined from the condition $2xy=b$, and so are $+/+$, $-/-$ for $b>0$ and $+/-$, $-/+$ for $b<0$. This shows that there are exactly two solutions z_1, z_2 and that $z_2=-z_1$; more precisely,

$$z_1=\pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}+\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\,\mathrm{i},$$
$$z_2=\mp\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}-\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\,\mathrm{i}.$$

The solution z_1, which is distinguished by $\operatorname{Im}(z_1)\geq 0$, is usually denoted by $\sqrt{c}$, and then $z_2=-\sqrt{c}$.

b) Completing the square transforms the equation into the equivalent

$$4A(Az^2+Bz+C)=(2Az+B)^2-(B^2-4AC)=0.$$

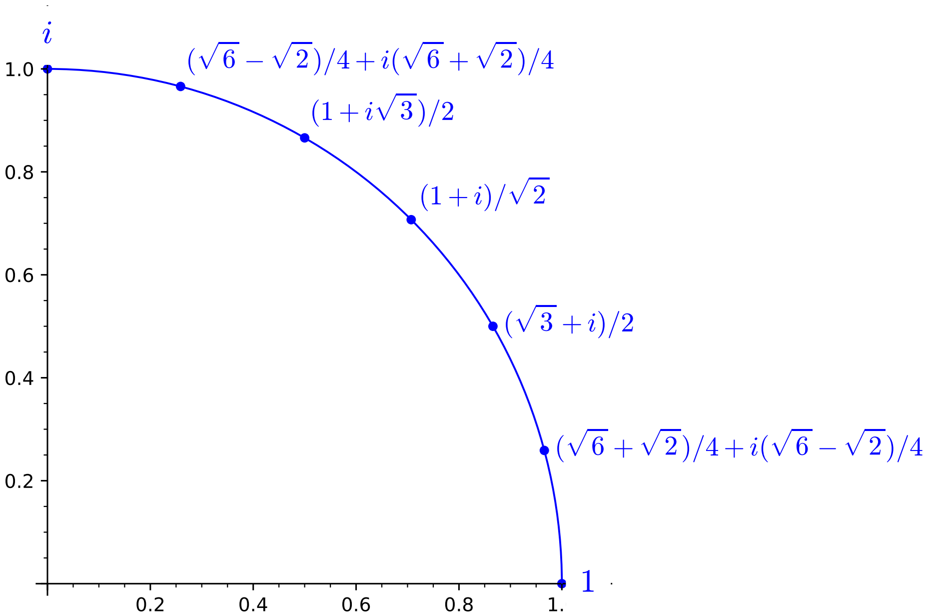
If $B^2-4AC\neq 0$ then from a) the solutions are $\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$, where $\sqrt{B^2-4AC}$ has the meaning introduced at the end of a). If $B^2-4AC=0$ then $-\dfrac{B}{2A}$ is the unique solution.

c) From plane geometry it is clear that $z_{12}=\mathrm{e}^{\pi\mathrm{i}/6}=\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$. Since $z_{24}^2=z_{12}$, we can apply a) with $a=\frac{\sqrt{3}}{2}$, $b=\frac{1}{2}$, $a^2+b^2=1$ to obtain

$$z_{24}=\sqrt{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)}+\sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)}\,\mathrm{i}=\frac{1}{2}\sqrt{2+\sqrt{3}}+\frac{1}{2}\sqrt{2-\sqrt{3}}\,\mathrm{i}.$$

Since $(1\pm\sqrt{3})^2=4\pm 2\sqrt{3}=2(2\pm\sqrt{3})$, this simplifies to

$$z_{24}=\frac{1+\sqrt{3}}{2\sqrt{2}}-\frac{1-\sqrt{3}}{2\sqrt{2}}\,\mathrm{i}=\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}\,\mathrm{i}.$$



i
$(\sqrt{6}-\sqrt{2})/4+i(\sqrt{6}+\sqrt{2})/4$
$(1+i\sqrt{3})/2$
$(1+i)/\sqrt{2}$
$(\sqrt{3}+i)/2$
$(\sqrt{6}+\sqrt{2})/4+i(\sqrt{6}-\sqrt{2})/4$
1
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.

Calculus III (Math 241)

W15 Do Exercise 29 in [Ste21], Ch. 13.4.

Note: You can find the figures (angles) in [Ste21], Appendix H, but the derivation of the figures is what counts.

W16 Do Exercise 46 in [Ste21], Ch. 13.4.

Solutions

15

Ex. 29 In an appropriate coordinate system the catapult is located at the origin and the city center on the positive x-axis. Adopting this setting and measuring lengths in m, the intersection points of the walls with the x-axis are $(100, 0)$ and $(600, 0)$ in the (x, y)-plane. Measuring time in s and normalizing the launch time to $t = 0$, the orbit is determined by

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(\tau)\mathrm{d}\tau, \quad \mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \mathbf{a}(\tau)\mathrm{d}\tau = 80(\cos\alpha, 0, \sin\alpha) + \int_0^t (0, 0, -g)\mathrm{d}\tau,$$

where α denotes the angle of elevation of the catapult. Integrating two times gives

$$\begin{aligned} \mathbf{v}(t) &= (80\cos\alpha, 0, 80\sin\alpha - gt), \\ \mathbf{r}(t) &= \big((80\cos\alpha)t, 0, (80\sin\alpha)t - \tfrac{g}{2}t^2\big). \end{aligned}$$

Thus $x(t) = (80\cos\alpha)t$, $z(t) = (80\sin\alpha)t - \frac{g}{2}t^2$. Eliminating t and using $1/\cos^2\alpha = 1 + \tan^2\alpha$ gives

$$z(x) = 80\sin\alpha\,\frac{x}{80\cos\alpha} - \frac{g}{2}\,\frac{x^2}{(80\cos\alpha)^2} = (\tan\alpha)x - \frac{g(1+\tan^2\alpha)}{12800}\,x^2.$$

The conditions for placing the fired stones inside the city are $z(100) > 15 \wedge z(600) < 15$. This gives the following two quadratic inequalities for $a = \tan\alpha$:

$$\begin{aligned} \frac{25g}{32}\,a^2 - 100a + \frac{25g}{32} + 15 > 0, \\ \frac{225g}{8}\,a^2 - 600a + \frac{225g}{8} + 15 < 0. \end{aligned}$$

The solution set of the first inequality is $[x_1, x_2]$ with

$$x_{1/2} = \frac{16}{25g}\left(100 \pm \sqrt{10,000 - 4\,\frac{25g}{32}\left(\frac{25g}{32} + 15\right)}\right).$$

The solution set of the second inequality is $\mathbb{R} \setminus [x_3, x_4]$ with

$$x_{3/4} = \frac{4}{225g}\left(600 \pm \sqrt{360,000 - 4\,\frac{225g}{8}\left(\frac{225g}{8} + 15\right)}\right).$$

It turns out that $x_1 < x_3 < x_4 < x_2$ (cf. the subsequent computation of the actual values), and hence that the addmissible angles are those in the two intervals $[\arctan(x_1), \arctan(x_3)]$ and $[\arctan(x_4), \arctan(x_2)]$ with the endpoints excluded. Using $g \approx 9.81$ and a calculator gives

$$13.0^\circ < \alpha < 36.1^\circ \quad \text{or} \quad 55.3^\circ < \alpha < 85.5^\circ.$$

16 Ex. 46

(a) The stated equation can be rewritten as $\mathbf{v}'(t) = \frac{m'(t)}{m(t)}\,\mathbf{v}_e$.

$$\Longrightarrow\ \mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{v}'(t)\,\mathrm{d}t = \mathbf{v}(0) + \left(\int_0^t \frac{m'(t)}{m(t)}\,\mathrm{d}t\right)\mathbf{v}_e = \mathbf{v}(0) + [\ln m(t)]_0^t\,\mathbf{v}_e$$
$$= \mathbf{v}(0) + (\ln m(t) - \ln m(0))\,\mathbf{v}_e = \mathbf{v}(0) + \ln\frac{m(t)}{m(0)}\mathbf{v}_e = \mathbf{v}(0) - \ln\frac{m(0)}{m(t)}\mathbf{v}_e$$

(b) Since $\mathbf{v}(t)$ and $\mathbf{v}_e$ have opposite direction, the condition is

$$-2\,\mathbf{v}_e = \mathbf{v}(0) - \ln\frac{m(0)}{m(t)}\mathbf{v}_e = -\ln\frac{m(0)}{m(t)}\mathbf{v}_e, \quad \text{i.e.,} \quad \ln\frac{m(0)}{m(t)} = 2.$$

$\Longrightarrow \frac{m(t)}{m(0)} = \mathrm{e}^{-2}$, and the fraction of the fuel the rocket needs to burn is $1 - \mathrm{e}^{-2} \approx 86.5\,\%$.

Calculus III (Math 241)

H31 Find the limit, if it exists, or show that the limit does not exist.

a) $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$;

b) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$;

c) $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+4y^3+9z^4}$;

d) $\lim_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} \frac{1}{xy}$.

H32 Suppose $f = (f_1, f_2): D \to \mathbb{R}^2$, $D \subseteq \mathbb{R}^2$, and $(x_0, y_0) \in D'$. Prove: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \mathbf{a} = (a_1, a_2)$ iff for every $\varepsilon > 0$ there exists a response $\delta > 0$ such that $|f_1(x,y) - a_1| < \varepsilon$ and $|f_2(x,y) - a_2| < \varepsilon$ hold for all $(x,y) \in D$ with $|x - x_0| < \delta$, $|y - y_0| < \delta$.

Note: This exercise shows that the topology of $\mathbb{R}^2$ may be based on squares with sides parallel to the coordinate axes (instead of disks). The result generalizes mutatis mutandis to maps $f = (f_1, \dots, f_m): D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$.

H33

a) Do Exercise 38 in [Ste21], Ch. 14.2.

b) Sketch the maximal domain D of h and compute for boundary points (x_0, y_0) of D the limit $\lim_{(x,y)\to(x_0,y_0)} h(x,y)$. Does $\lim_{|(x,y)|\to\infty} h(x,y)$ exist?

H34 Following the example of the squaring map $s(x,y) = (x^2 - y^2, 2xy)$ in the lecture, show without resort to partial derivatives that $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \begin{pmatrix} x+y \\ xy \end{pmatrix}$$

is differentiable per se (i.e., in every point $(x,y) \in \mathbb{R}^2$), and determine the differential $\mathrm{d}F(x,y)$ and the Jacobi matrix $\mathbf{J}_F(x,y)$.

H35 *Optional Exercise*

This exercise is a continuation of H33

a) Show that there is a unique isolated point $(x_0, y_0) \in D$ with $\mathrm{d}h(x_0, y_0) = 0$, and draw the contour of h through this point.

b) Show that h attains a maximum value k_0 on D and determine all points $(x,y) \in D$ with $h(x,y) = k_0$ (i.e., the k_0-contour of h) algebraically.

c) Make an informative contour map for h.

Hints: h has the form $h(x,y) = \phi(xy)$ for some one-variable function $\phi\colon (-\infty, 1)$. For c) it is a good idea to decide first for which levels k the k-contours have points in all four quadrants of $\mathbb{R}^2$.

Due on Wed Nov 8, 6 pm

The optional exercise can be handed in until Wed Nov 15, 6 pm.

Solutions

31 a) We show that the limit is zero. Using $|\sin y| \le |y|$, which is true for all $y \in \mathbb{R}$, we can estimate as follows:

$$\left|\frac{x^2 \sin^2 y}{x^2+2y^2}\right| \le \frac{x^2y^2}{x^2+2y^2} \le \frac{x^2y^2}{x^2+y^2} \le \frac{\frac{1}{2}(x^2+y^2)^2}{x^2+y^2} = \frac{1}{2}(x^2+y^2) < \varepsilon$$

if $|(x,y)| = \sqrt{x^2+y^2} < \sqrt{2\varepsilon}$. Hence $\delta = \sqrt{2\varepsilon} > 0$ can serve as a response for $\varepsilon > 0$ in the proof that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 \sin^2 y}{x^2+2y^2} = 0$.

b) Denoting the corresponding function by $f(x,y)$, we have

$$f(x,0) = f(0,y) = 0, \quad f(y^4,y) = \frac{y^4y^4}{(y^4)^2+y^8} = \frac{1}{2}.$$

Since $(0,0)$ is an accumulation point of the curve $x = y^4$ ("we can continuously walk along this curve to the origin") and of the coordinate axes as well, the limit does not exist.

c) The function vanishes on the y-axis (except for the origin, where it is not defined), and on the line $\mathbb{R}(1,1,1)$ has the limit

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\ x=y=z}} \frac{xy+yz}{x^2+4y^3+9z^4} = \lim_{x\to 0} \frac{2x^2}{x^2+4x^3+9x^4} = \lim_{x\to 0} \frac{2}{1+4x+9x^2} = 2.$$

Hence the limit does not exist.

d) We show that $\lim\limits_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} \dfrac{1}{xy} = +\infty$. Given $R > 0$, the inequality $\frac{1}{xy} > R$ is equivalent to $xy < 1/R$ and is certainly fulfilled if $x < 1/\sqrt{R} \wedge y < 1/\sqrt{R}$. Hence $(x,y) \mapsto 1/(xy)$ maps the square $(0,1/\sqrt{R}) \times (0,1/\sqrt{R})$ into $(R,+\infty)$. The same is then true of the quarter disk $\left\{(x,y) \in \mathbb{R}^2; x^2+y^2 < 1/R, x > 0, y > 0\right\}$, so that $\delta = 1/\sqrt{R}$ can serve as response for R.

32 $\Longrightarrow$: Given $\varepsilon > 0$, there exists $\delta > 0$ such that $(x,y) \in D \wedge \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ implies $\sqrt{(f_1(x,y)-a_1)^2+(f_2(x,y)-a_2)^2} < \varepsilon$. The premise of this implication is certainly true if $|x-x_0| < \delta/2$, $|y-y_0| < \delta/2$, so that in this case the conclusion holds as well. Since the conclusion implies $|f_1(x,y)-a_1| < \varepsilon$, $|f_2(x,y)-a_2| < \varepsilon$, the number δ provides a response to ε in the new formulation.

$\Longleftarrow$: Given $\varepsilon > 0$, let $\delta > 0$ be a response to $\varepsilon/2$ in the new formulation and suppose $(x,y) \in \mathrm{B}_\delta(x_0,y_0) \cap D$. Then $|x-x_0| \le \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$, and similarly $|y-y_0| < \delta$. Hence $|f_1(x,y)-a_1| < \varepsilon/2$, $|f_2(x,y)-a_2| < \varepsilon/2$ by our choice of δ, and

$$|f(x,y)-\mathbf{a}| = \sqrt{(f_1(x,y)-a_1)^2+(f_2(x,y)-a_2)^2} < \sqrt{\varepsilon^2/4+\varepsilon^2/4} = \varepsilon/\sqrt{2} < \varepsilon.$$

Thus δ can serve as a response to ε in the definition of $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \mathbf{a}$.

Remarks: Geometrically speaking, the two properties are equivalent because every disk in $\mathbb{R}^2$ (with positive radius) contains a square with sides parallel to the coordinate axes and

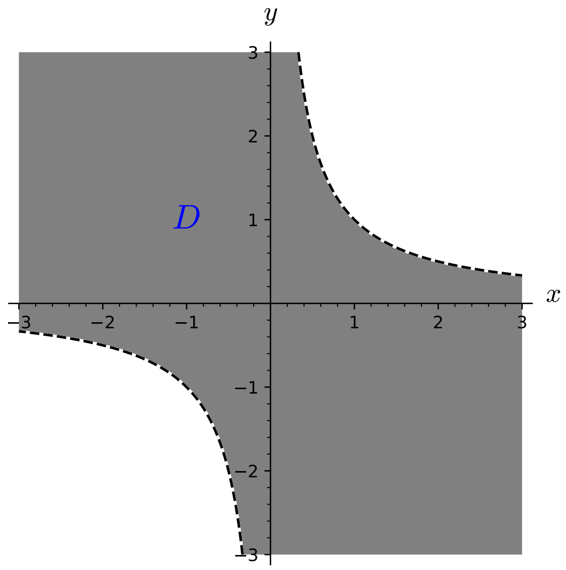
the same center as the disk and, conversely, every square with sides parallel to the coordinate axes contains a disk with the same center. The generalization to higher dimensions uses balls/(hyper)cubes in place of disks/squares.
The solution readily gives as a by-product the following fact mentioned in the lecture: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \mathbf{a} \iff \lim_{(x,y)\to(x_0,y_0)} f_1(x,y) = a_1 \wedge \lim_{(x,y)\to(x_0,y_0)} f_2(x,y) = a_2$; similarly for $m > 2$.

33 a)

$$h(x,y) = \frac{1-xy}{1+x^2y^2} + \ln\frac{1-xy}{1+x^2y^2}$$

has maximal domain of definition $D = \{(x,y) \in \mathbb{R}^2; xy < 1\}$.

b) D is the region between the two branches of the hyperbola $xy = 1$. The plot below shows the part of D contained in the square $[-3,3]^2$.



Since $(x,y) \to (x_0,y_0)$ (from within D) implies $xy \uparrow 1$, we have

$$\lim_{(x,y)\to(x_0,y_0)} h(x,y) = \lim_{u\uparrow 1}\left(\frac{1-u}{1+u^2} + \ln\frac{1-u}{1+u^2}\right) = -\infty.$$

$\lim_{|(x,y)|\to\infty} h(x,y)$ does not exist, since every non-empty contour of h is unbounded (which in turn is due to the fact that h is constant on each hyperbola $xy = k$). Looking at two particular such contours, say $k_1 = 0$ and $k_2 = 1$, then shows that ouside every (large) disk $\mathrm{B}_R(0,0)$ the function h attains the values 0 and 1. This contradicts the existence of the limit.

34 For (x,y), $\mathbf{h} = (h_1,h_2) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ we have

$$\begin{aligned} F(x+h_1,y+h_2) &= \begin{pmatrix} x+h_1+y+h_2 \\ (x+h_1)(y+h_2) \end{pmatrix} = \begin{pmatrix} x+y+h_1+h_2 \\ xy+yh_1+xh_2+h_1h_2 \end{pmatrix} \\ &= \begin{pmatrix} x+y \\ xy \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} 0 \\ h_1h_2 \end{pmatrix} \\ &= F(x,y) + \mathbf{A}\mathbf{h} + R(\mathbf{h}) \end{aligned}$$

with $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ y & x \end{smallmatrix}\right)$ and $R(\mathbf{h}) = \left(\begin{smallmatrix} 0 \\ h_1 h_2 \end{smallmatrix}\right)$. If (x,y) is fixed, $\mathbf{A}$ is a fixed matrix, and we then have

$$\frac{|R(\mathbf{h})|}{|\mathbf{h}|} = \frac{|h_1 h_2|}{\sqrt{h_1^2 + h_2^2}} = \underbrace{\frac{|h_1|}{\sqrt{h_1^2 + h_2^2}}}_{\leq 1} \cdot |h_2| \to 0 \quad \text{if } \mathbf{h} = (h_1, h_2) \to \mathbf{0} \text{ in } \mathbb{R}^2.$$

This means $R(\mathbf{h}) = \mathrm{o}(\mathbf{h})$ for $\mathbf{h} \to \mathbf{0}$ and shows that F is differentiable in (x,y) with

$$\mathrm{d}F(x,y)\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{h} \mapsto \mathbf{A}\mathbf{h} = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \quad \text{and} \quad \mathbf{J}_F(x,y) = \mathbf{A} = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}.$$

35 The function h has the form $h(x,y) = \phi(xy)$ with

$$\phi(u) = \frac{1-u}{1+u^2} + \ln \frac{1-u}{1+u^2}, \quad u < 1.$$

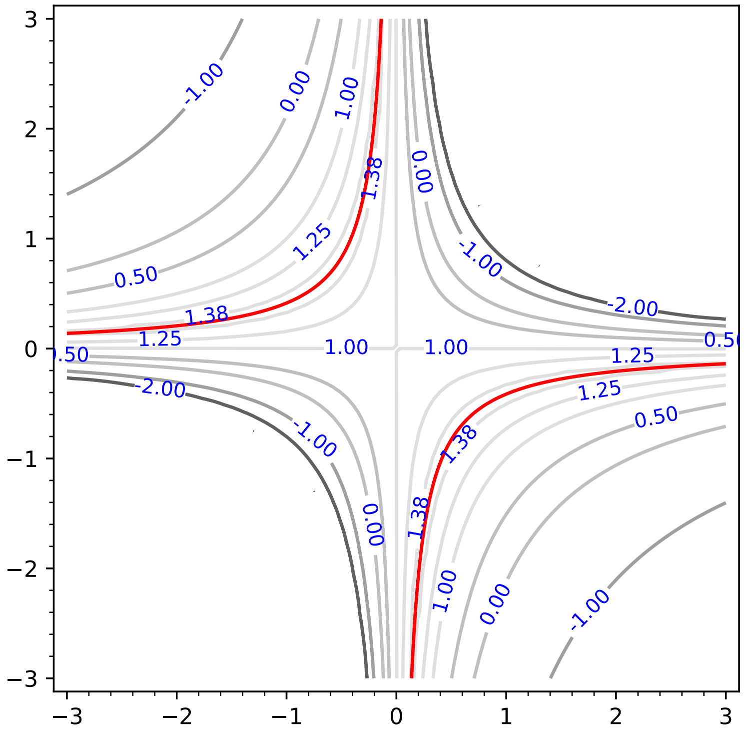
a) $h_x(x,y) = y\phi'(xy)$, $h_y(x,y) = x\phi'(xy)$
$\Longrightarrow h_x(x,y) = h_y(x,y) = 0$ iff $(x,y) = (0,0)$ or $\phi'(xy) = 0$. Points (x,y) with $\phi'(xy) = 0$ comprise a whole hyperbola, cf. b), and are not isolated. Therefore the unique such point is the origin $(0,0)$. Since $h(0,0) = 1$, th contour in question is the 1-contour. Since $\phi(u) = 1$ precisely for $u \in \{0, -1\}$, the 1-contour consists of the coordinate axes ($xy = 0$) and the two branches of the hyperbola $xy = -1$; see the contour map in c).

b) Clearly the range of h is equal to the range of ϕ and the points $(x,y) \in D$ with $h(x,y) = k$ are those on all hyperbolas $xy = u$ with $\phi(u) = k$ ($u = 0$ is permitted). Hence we can decide the question by determining the maxmimum of ϕ.

$$\phi'(u) = \frac{-(1+u^2) - 2u(1-u)}{(1+u^2)^2} \left(1 + \frac{1+u^2}{1-u}\right) = \frac{(u^2 - 2u - 1)(u^2 - u + 2)}{(1+u^2)^2}$$

The first factor of the numerator has zeros $1 \pm \sqrt{2}$, of which only $1 - \sqrt{2}$ belongs to the domain of ϕ. Since the second factor of the numerator and the denominator are everywhere positive, the sign of $\phi'(u)$ is that of $u^2 - 2u - 1$, and it follows that ϕ is strictly increasing in $(-\infty, 1-\sqrt{2})$ and strictly decreasing in $(1-\sqrt{2}, 1)$. Therefore, ϕ has a maximum in $1 - \sqrt{2}$, and this maximum is the only one.
$\Longrightarrow k_0 = \phi(1-\sqrt{2})$ is the maximum value of h on D and is attained precisely at the points of the hyperbola $xy = 1 - \sqrt{2}$. Numerically, we have $1 - \sqrt{2} \approx -0.4142$ and $k_0 \approx 1.395$.

c) By b) and since $\lim_{u \to -\infty} \phi(u) = \lim_{u \uparrow 1} \phi(u) = -\infty$, the function ϕ takes every value $k \in (-\infty, k_0)$ exactly twice. Hence for such k the k-contour of h is the union of two hyperbolas $xy = u_1$ and $xy = u_2$ (4 branches alltogether). For $k > 1$ (equivalently, $u_1, u_2 < 0$) the branches are located in the 2nd and 4th quadrant (2 branches in each quadrant); for $k < 1$ (equivalently, $u_1 < 0 < u_2$) there is one branch in each quadrant. The subsequent contour plot of h shows the k_0-contour in red without inline label.



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3

Calculus III (Math 241)

W17 Find the limit, if it exists, or show that the limit does not exist.

a) $\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$;

b) $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$;

c) $\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$;

d) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

W18 Using the ε-δ definition of the limit $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ for functions $f\colon D\to\mathbb{R}^m$, $D\subseteq\mathbb{R}^n$, and $\mathbf{x}_0\in D'$, show that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})=\mathbf{a}\wedge\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})=\mathbf{b}\quad\text{implies}\quad\mathbf{a}=\mathbf{b}.$$

W19 a) Prove Euler's identity $e^{i\phi}=\cos\phi+i\sin\phi$.
Hint: $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$, etc.

b) Prove the functional equation for the complex exponential function: $e^{z+w}=e^z e^w$ for $z,w\in\mathbb{C}$.

Hint: For two absolutely convergent series $\sum_{k=0}^{\infty} c_k$, $\sum_{l=0}^{\infty} d_l$ the identity

$$\left(\sum_{k=0}^{\infty} c_k\right)\left(\sum_{l=0}^{\infty} d_l\right)=\sum_{n=0}^{\infty}(c_0d_n+c_1d_{n-1}+\cdots+c_nd_0)\quad\text{holds.}$$

c) For $z=x+iy$ show that $\mathrm{Re}(e^z)=e^x\cos y$, $\mathrm{Im}(e^z)=e^x\sin y$.

d) Show that the range of the complex exponential function is $\mathbb{C}\setminus\{0\}$ and that $e^{z+2\pi i}=e^z$ for $z\in\mathbb{C}$.

Solutions

17 a) The function $f(x,y) = \mathrm{e}^{-xy}\cos(x+y)$ is continuous in $\mathbb{R}^2$, so we have

$$\lim_{(x,y)\to(1,-1)} \mathrm{e}^{-xy}\cos(x+y) = f(1,-1) = \mathrm{e}^{-1(-1)}\cos(1-1) = \mathrm{e}.$$

b) Again denoting the function by $f(x,y)$, we have $f(x,0) = 0$ and $f(x,x) = \frac{5}{2}\cos^2 x \to \frac{5}{2}$ for $x \to 0$. This shows that the limits $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y)$ along the lines $L = \mathbb{R}(1,0)$ and $\mathbb{R}(1,1)$ are 0 and $\frac{5}{2}$, respectively, and hence that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Note: It's slightly easier to use the y-axis $\mathbb{R}(0,1)$ in place of $\mathbb{R}(1,1)$, since $f(0,y) = 5$ is constant and hence trivially $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in\mathbb{R}(0,1)}} f(x,y) = \lim_{y\to 0} f(0,y) = 5$.

c) This limit reduces to one considered in the lecture, viz.

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} &= \lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} \\ &= \lim_{(x',y)\to(0,0)} \frac{x'y}{x'^2+y^2}. \qquad \text{(Subst. } x' = x-1\text{)}\end{aligned}$$

As shown in the lecture, the limit does not exist.

d) Here we can estimate as follows

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2}\right| + \left|\frac{y^3}{y^2}\right| = |x|+|y|$$

Since $\lim_{(x,y)\to(0,0)}\big(|x|+|y|\big) = 0$, it follows that $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$ as well.

An explicit response to a given $\varepsilon > 0$ is $\delta = \varepsilon/2$, since $|(x,y)| < \varepsilon/2$ implies $|x| < \varepsilon/2$ and $|y| < \varepsilon/2$ and hence $\left|\frac{x^3-y^3}{x^2+y^2} - 0\right| \leq |x|+|y| < \varepsilon/2+\varepsilon/2 = \varepsilon$.

18 Suppose $\mathbf{a} \neq \mathbf{b}$. Then $|\mathbf{a}-\mathbf{b}| > 0$, and for $\varepsilon = \frac{1}{2}|\mathbf{a}-\mathbf{b}|$ there exist responses $\delta_1, \delta_2 > 0$ such that $\mathbf{x} \in D \wedge 0 < |\mathbf{x}-\mathbf{x}_0| < \delta_1$ implies $|f(\mathbf{x})-\mathbf{a}| < \varepsilon$ and $\mathbf{x} \in D \wedge 0 < |\mathbf{x}-\mathbf{x}_0| < \delta_2$ implies $|f(\mathbf{x})-\mathbf{b}| < \varepsilon$. For $\delta = \min\{\delta_1,\delta_2\}$ we then have that $\mathbf{x} \in D \wedge 0 < |\mathbf{x}-\mathbf{x}_0| < \delta$ implies $|f(\mathbf{x})-\mathbf{a}| < \varepsilon \wedge |f(\mathbf{x})-\mathbf{b}| < \varepsilon$. The premise of this implication can be made true, since $\mathbf{x}_0$ is an accumulation point of D. The conclusion, however, is always false:

$$2\varepsilon = |\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}-f(\mathbf{x})| + |f(\mathbf{x})-\mathbf{b}|,$$

and hence at least one of $|f(\mathbf{x})-\mathbf{a}|$, $|f(\mathbf{x})-\mathbf{b}|$ must be $\geq \varepsilon$. This contradiction shows that $\mathbf{a} \neq \mathbf{b}$ is false, i.e., we must have $\mathbf{a} = \mathbf{b}$.

19 a)

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\phi} &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}\phi)^n}{n!} \\ &= 1 + \mathrm{i}\phi - \frac{\phi^2}{2!} - \mathrm{i}\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + \mathrm{i}\frac{\phi^5}{5!} - \frac{\phi^6}{6!} - \mathrm{i}\frac{\phi^7}{7!} + \frac{\phi^8}{8!} + \mathrm{i}\frac{\phi^9}{9!} - \frac{\phi^{10}}{10!} - \cdots \\ &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots + \mathrm{i}\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots\right) \\ &= \cos\phi + \mathrm{i}\sin\phi,\end{aligned}$$

using the well-known power series representations of cos, sin.

b)

$$\begin{aligned}
\mathrm{e}^z\mathrm{e}^w &= \left(\sum_{k=0}^{\infty}\frac{z^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{w^l}{l!}\right) = \sum_{n=0}^{\infty}\left(\frac{z^0w^n}{0!n!}+\frac{z^1w^{n-1}}{1!(n-1)!}+\cdots+\frac{z^nw^0}{n!0!}\right) && \text{(using the hint)}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{z^kw^{n-k}}{k!(n-k)!}\right),\\
\mathrm{e}^{z+w} &= \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}z^kw^{n-k} && \text{(using the Binomial Theorem)}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{n!}\frac{n!}{k!(n-k)!}z^kw^{n-k}\right)
\end{aligned}$$

A comparison shows that the summands of the two series coincide, and hence $\mathrm{e}^z\mathrm{e}^w = \mathrm{e}^{z+w}$ for all $z, w \in \mathbb{C}$.

c) Using a) and b), we have $\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = \mathrm{e}^x\cos y + \mathrm{i}(\mathrm{e}^x\sin y)$.
$\implies \mathrm{Re}(\mathrm{e}^z) = \mathrm{e}^x\cos y$, $\mathrm{Im}(\mathrm{e}^z) = \mathrm{e}^x\sin y$

d) We know that every point in $\mathbb{C}\setminus\{0\} = \mathbb{R}^2\setminus\big\{(0,0)\big\}$ has a polar coordinate representation $z = (r\cos\phi, r\sin\phi)$ with $r > 0$, $\phi\in\mathbb{R}$. Setting $r = \mathrm{e}^x$ (i.e., $x = \ln r$), we obtain $z = (\mathrm{e}^x\cos\phi, \mathrm{e}^x\sin\phi) = \mathrm{e}^{x+\mathrm{i}\phi} = \exp(x+\mathrm{i}\phi)$. On the other hand, since $\mathrm{e}^x \neq 0$ for $x\in\mathbb{R}$ and $\cos y$, $\sin y$ have no common zero, the number 0 is not in the range of exp. This proves the first assertion. The second assertion follows from c) and the 2π-periodicity of cos, sin; more precisely, $\mathrm{e}^{z+2\pi\mathrm{i}} = \mathrm{e}^z\mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^z\big(\cos(2\pi)+\mathrm{i}\sin(2\pi)\big) = \mathrm{e}^z\cdot 1 = \mathrm{e}^z$ (the key property being $\mathrm{e}^{2\pi\mathrm{i}} = 1$).

Calculus III (Math 241)

H36 Do Exercises 2 and 6 in [Ste21], Ch. 14.3.

H37 (Continuation of Exercise W21 from Worksheet 7)

a) Show that the k-contours of u intersecting the x-axis at a point $\neq (1,0)$ in the right half-plane have levels $k > 2$ and a vertical tangent at the intersection points. Compute the intersection points for $k = 2.5$ and $k = 3$.

   Hint: Use the gradient of u to determine the tangent directions of the contours.

b) Draw the contours of u for the levels $k = 0$, 1, 1.5, 1.95, 2, 2.5, 3. Based on this try to describe, as accurately as possible, the shape of the k-contour of u for a general level k.

c) Find the locus of all points $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ where the corresponding contour of u has a horizontal tangent, and insert this curve into the contour plot in b).

   Hint: Use polar coordinates.

H38 This exercise continues the Example on Slide 46 ff of `lecture17-20_handout.pdf`.

a) Show that the restriction of $g(u,v) = (u+v, u^2+v^2, u^3+v^3)$ maps $\{\{(u,v) \in \mathbb{R}^2; u > v\}$ bijectively onto $S \setminus C$.

b) Show that S can be represented as graph of a function $z = h(x,y)$, compute the tangent plane to G_h in $(x,y,z) \in S$ according to our earlier definition, and verify that both definitions yield the same tangent planes.

   Hint: Eliminate u,v from $x = u+v$, $y = u^2+v^2$, $z = u^3+v^3$.

H39 Do Exercises 40, 43, 62 in [Ste21], Ch. 14.6.

Due on Wed Nov 15, 6 pm

Solutions

36 Ex. 2

a) $\frac{\partial f}{\partial v}(v,t_0)$ represents the rate of change of the wave height with respect to v if the time $t = t_0$ is fixed. (Assuming that wave heights are measured in feet and the wind blows since t_0 hours, this means that increasing/decreasing the wind speed from v_0 knots to v knots increases/decreases the wave height approximately by $\frac{\partial f}{\partial v}(v_0,t_0)\cdot(v-v_0)$ feet, provided $v-v_0$ is small.) Similarly, $\frac{\partial f}{\partial t}(v_0,t)$ represents the rate of change of the wave height with respect to t if the wind speed $v = v_0$ is fixed.

b) A reasonable approximation for $\frac{\partial f}{\partial v}(v,t)$ is $\frac{f(v+h,t)-f(v,t)}{h}$ for small (positive or negative) values of h. If $f(v\pm h,t)$ is available, we can use $\frac{f(v+h,t)-f(v-h,t)}{2h}$ instead, which corresponds to averaging $\frac{f(v+h,t)-f(v,t)}{h}$ and $\frac{f(v-h,t)-f(v,t)}{-h}$. Of course this applies mutatis mutandis to $\frac{\partial f}{\partial t}(v,t)$ as well.

Taking the smallest h for which this data is available gives

$$f_v(80,15) \approx \frac{f(100,15)-f(60,15)}{40} = \frac{11.0-4.9}{40} = 0.1525,$$
$$f_t(80,15) \approx \frac{f(80,20)-f(80,10)}{10} = \frac{8.6-6.4}{10} = 0.22$$

c) From the table one can conjecture that $\lim_{t\to+\infty}\frac{\partial f}{\partial t}(v,t) = 0$ for every fixed v.

Ex. 6 Let $(x_1,1)$ and $(x_2,1)$ be the intersection points of the 8-contour and the 12-contour, respectively, of f with the line $y=1$. From the picture we have $x_2-x_1\approx 1.5$, and a reasonable estimate for $f_x(2,1)$ is then

$$f_x(2,1) \approx \frac{f(x_2,1)-f(x_1,1)}{x_2-x_1} \approx \frac{12-8}{1.5} = \frac{8}{3}.$$

Similarly, using the same contours to estimate $f_y(2,1)$ we obtain

$$f_y(2,1) \approx \frac{8-12}{2} = -2.$$

37 a) Since $u(x,y) = x\left(1+\frac{1}{x^2+y^2}\right)$, the 0-contour has equation $x=0$ and hence is the y-axis with the origin removed. From the symmetry properties of u it is then clear that the k-contours with $k>0$ precisely fill the (open) right half-plane $\{(x,y)\in\mathbb{R}^2; x>0\}$.

For $k>0$ the k-contour of u has equation

$$y^2 = \frac{x}{k-x} - x^2 = \frac{x^3-kx^2+x}{k-x} \iff y = \pm\sqrt{\frac{x^3-kx^2+x}{k-x}}. \tag{1}$$

Since $x^3-kx^2+x = x(x^2-kx+1)$ has roots 0 and $\frac{1}{2}(k\pm\sqrt{k^2-4})$, a k-contour intersects the x-axis iff $k\geq 2$. In b) we have seen that the 2-contour contains the point $(1,0)$, where ∇u vanishes.

From W21 c) we have $u_x(x,0) = 1-1/x^2$, $u_y(x,0)=0$. It follows that $\nabla(x,0)$ is a nonzero multiple of $\mathrm{e}_1 = (1,0)$ for all points on the positive x-axis except $(1,0)$. A k-contour with

$k > 2$ intersecting the x-axis in $(x,0)$ must have a vertical tangent at $(x,0)$, since the tangent direction must be orthogonal to $\nabla(x,0)$.

For $k = 3$ the intersection points are $\left(\frac{3\pm\sqrt{5}}{2},0\right) \approx (2.62,0)$, $(.38,0)$, and for $k = 2.5$ they are $(2,0)$ and $(\frac{1}{2},0)$.

b) It appears from the contour plot that k-contours with levels $k < 2$ (after adjoining the origin as a further point) are connected and located to the left of the 2-contour and outside its loop, while k-contours with levels $k > 2$ consist of two connected components, one being located to the right of the 2-contour and the other inside its loop. Moreover, it appears that the k-contour has the line line $x = k$ as a vertical asymptote if $y \to \pm\infty$, and that the k-contours with $(0,0)$ adjoined have a vertical tangent in $(0,0)$.

For $0 < k < 2$ we have $x^3 - kx^2 + x > 0$ for all $x > 0$ and hence the square root in (1) is defined precisely for $0 < x < k$. Together with a) and the facts that $y \simeq \pm\sqrt{x/k}$ for $x \downarrow 0$, $y \to \pm\infty$ for $x \uparrow k$ the assertions follow in this case.

For $k > 2$ the square root in (1) is defined for $0 \le x \le \frac{1}{2}\left(k - \sqrt{k^2-4}\right)$ and $\frac{1}{2}\left(k + \sqrt{k^2-4}\right) \le x < k$. Hence for $k > 2$ the contour has two branches, a closed curve contained in the region bounded by the loop of the 2-contour with vertical tangents at $(0,0)$ and $\left(\frac{1}{2}\left(k - \sqrt{k^2-4}\right),0\right)$, and an inifnite branch with vertical tangent in $\left(\frac{1}{2}\left(k + \sqrt{k^2-4}\right),0\right)$ and asymptote $x = k$ for $y \to \pm\infty$.

More accurate information can be obtained with some effort. As an example we consider the question which k-contours have horizontal tangents and where. The 1.95-contour was included in the exercise because it has 4 points with horizontal tangents; the 1- and 1.5-contour, which otherwise look similar, don't have such points.

The condition for a horizontal tangent in (x,y) is $u_x(x,y) = 0 \wedge u_y(x,y) \neq 0$. The latter just excludes the coordinate axes, while the former can be rewritten as

$$(x^2+y^2)^2 + y^2 - x^2 = 0.$$

Substituting into this equation the condition $x^2 + y^2 = \frac{x}{k-x}$ for $(x,y) \in \mathrm{N}_u(k)$ we obtain

$$\left(\frac{x}{k-x}\right)^2 + \frac{x}{k-x} - 2x^2 = 0,$$

which is equivalent to

$$x(2x^3 - 4kx^2 + 2k^2x - k) = 0.$$

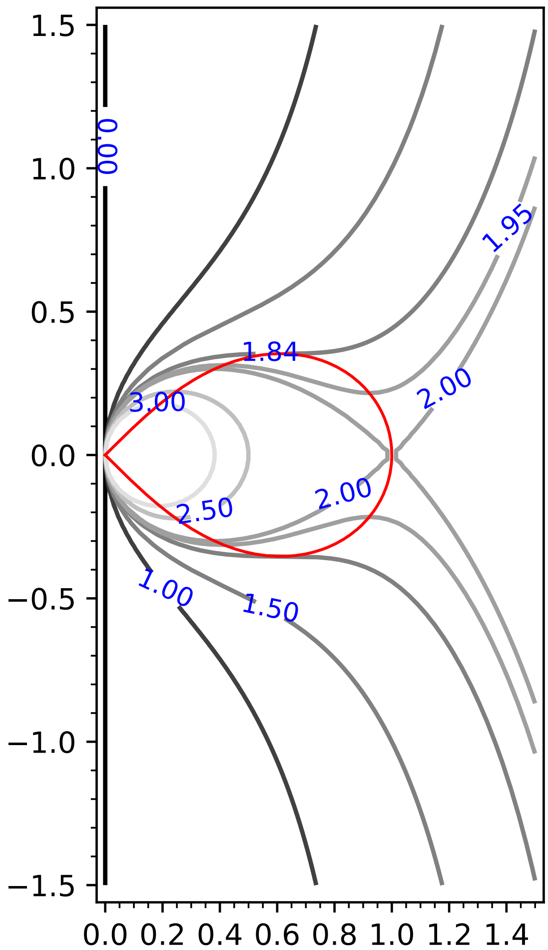
An analysis of the cubic factor reveals that it has a local maximum at $\left(\frac{k}{3}, \frac{8k^3}{27} - k\right)$ and a local minimum at $(k,-k)$ (and hence an inflection point at $\left(\frac{2k}{3}, \frac{4k^3}{27} - k\right)$). From this we can see that the k-contour has points with horizontal tangents iff $\frac{8k^3}{27} - k \ge 0$, i.e., $k \ge (3/2)\sqrt{3/2} \approx 1.84$. The extremal case, where there is only one pair of points with horizontal tangents with x-coordinate $k/3 = (1/2)\sqrt{3/2}$ is indicated in the subsequent plot for e).

It turns out that the horizontal tangents of the 2-contour have x-coordinate $\frac{3-\sqrt{5}}{2} \approx 0.38$, the same as the intersection point of the bounded (ellipse-like) component of the 3-contour with the x-axis. Is this a pure coincidence?

c) The condition for a horizontal tangent of the contour of u through (x,y) is $u_x = 0 \wedge u_y \neq 0$. From W21 c), $u_x(x,y) = 0$ is equivalent to

$$1 + \frac{y^2 - x^2}{(x^2+y^2)^2} = 0 \iff (x^2+y^2)^2 = x^2 - y^2.$$

Setting $x = r\cos\phi$, $y = r\sin\phi$ this becomes $r^4 = r^2\cos^2\phi - r^2\sin^2\phi = r^2\cos(2\phi)$, i.e., $r = \sqrt{\cos(2\phi)}$, $r = 0$ being excluded because $u(0,0)$ is not defined, and $r = 1$ being excluded because $u_y(\pm 1, 0) = 0$. Points on the curve $r = \sqrt{\cos(2\phi)}$ have $-\pi/4 < \phi < \pi/4$ or $-5\pi/4 < \phi < 3\pi/4$, giving it the shape of an ∞. The subsequent plot shows the part located in the right half plane $x > 0$.



38 a) Let $\Omega = \{\{(u,v) \in \mathbb{R}^2; u > v\}$, and suppose $g(u,v) = (x,y,z)$. Then $u+v = x$, $u^2 + v^2 = y$, $u^3 + v^3 = z$. $\implies x^2 = u^2 + 2uv + v^2 = y + 2uv$, and u,v solve the quadratic $t^2 - xt + (x^2-y)/2 = 0$. Points $(x,y,z) \in C$ have $y = x^2/2$, in which case the discriminant $\Delta = x^2 - 4(x^2-y)/2 = 2y - x^2$ is zero, so that there is only the solution $u = v = x/2$. Together with $g(u,v) = g(v,u)$ this shows $g(\Omega) = S \setminus C$. Injectivity follows from the fact that a quadratic has at most two solutions (at most one pair u,v of solutions with $u < v$).

b) According to a), the set of points (x,y) for which there exists z such that $(x,y,z) \in S$ (the projection of S onto the (x,y) plane) is $D = \{(x,y) \in \mathbb{R}^2; y \geq x^2/2\}$ (corresponding to $\Delta \geq 0$).

For $(x,y,z) \in S$ we have

$$\begin{aligned} z &= u^3 + v^3 = (u+v)(u^2+v^2) - uv^2 - u^2v = (u+v)(u^2+v^2) - uv(u+v) \\ &= xy - \frac{x^2-y}{2}x = -\frac{1}{2}x^3 + \frac{3}{2}xy. \end{aligned}$$

This shows that S is the graph of $h\colon D \to \mathbb{R}$, $h(x,y) = -\frac{1}{2}x^3 + \frac{3}{2}xy$.
Now suppose $(x,y,z) = g(u,v) \in S \setminus C$. The tangent plane to G_h in $\big(x,y,h(x,y)\big)$ has parametric form

$$\begin{pmatrix} x \\ y \\ h(x,y) \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ h_x(x,y) \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ h_y(x,y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ -\frac{1}{2}x^3 + \frac{3}{2}xy \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2}x^2 + \frac{3}{2}y \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ \frac{3}{2}x \end{pmatrix}.$$

On the other hand, using the parametrization $g(u,v)$, the tangent plane is

$$\begin{pmatrix} u+v \\ u^2+v^2 \\ u^3+v^3 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 2u \\ 3u^2 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 2v \\ 3v^2 \end{pmatrix}.$$

Since both planes have the same starting point, they are equal iff their direction spaces are equal. The column space of $\mathbf{J}_g(u,v)$ remains unchanged if we multiply it on the right with an invertible 2×2 matrix. The obvious choice to transform $\mathbf{J}_g(u,v)$ into the Jacobi matrix of the other parametrization is

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 2u & 2v \\ 3u^2 & 3v^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2u & 2v \end{pmatrix}^{-1} &= \frac{1}{2(v-u)} \begin{pmatrix} 1 & 1 \\ 2u & 2v \\ 3u^2 & 3v^2 \end{pmatrix} \begin{pmatrix} 2v & -1 \\ -2u & 1 \end{pmatrix} = \frac{1}{2(v-u)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 6u^2v - 6uv^2 & 3v^2 - 3u^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3uv & \frac{3}{2}(u+v) \end{pmatrix}. \end{aligned}$$

Since $-3uv = -\frac{3}{2}(x^2-y) = -\frac{3}{2}x^2 + \frac{3}{2}y$ and $\frac{3}{2}(u+v) = \frac{3}{2}x$, this is indeed the Jacobi matrix of the other parametrization. Hence both planes are equal.
Note: When writing the exercise, I had in b) only the special point $(1,1,1)$ in mind, for which we have computed the tangent plane in the lecture. The generalization could then be posed as an optional problem. But this restriction somehow hasn't made it into the statement of the exercise.

39 Ex. 40 $\nabla z(x,y) = (-0.01x, -0.02y)$, $\nabla z(60,40) = (-0.6, -0.8)$

(a) Since $\frac{\partial z}{\partial y}(60,40) = -0.8 < 0$, I will descend when walking north and ascend when walking south. The corresponding rate (i.e., the slope) is 0.8 (meter per meter).

(b) The northwest directional derivative is, up to a positive factor, $\nabla z(60,40) \cdot (-1,1) = 0.6 - 0.8 = -0.2$. Hence I will start to descend in this case, at a rate of $\frac{0.2}{|(-1,1)|} = \sqrt{2} \approx 0.41$.

(c) The slope is largest in direction $(-6,-8)$, or $(-3,-4)$, and the corresponding slope is $|(-0.6,-0.8)| = 1$. This means I will start to climb at a rate of 1 (meter per meter), and at an angle of $45°$.

Ex. 43

(a) $\nabla(au+bv) = \big((au+bv)_x, (au+bv)_y\big) = (au_x+bv_x, au_y+bv_y) = a(u_x,u_y)+b(v_x,v_y) = a\nabla u + b\nabla v$

(b) $\nabla(uv) = \big((uv)_x, (uv)_y\big) = (u_x v + uv_x, u_y v + uv_y) = v(u_x,u_y)+u(v_x,v_y) = v\nabla u + u\nabla v$

(c) $\nabla\left(\frac{u}{v}\right) = \left(\left(\frac{u}{v}\right)_x, \left(\frac{u}{v}\right)_y\right) = \left(\frac{u_x v - uv_x}{v^2}, \frac{u_y v - uv_y}{v^2}\right) = \frac{1}{v^2}\big(v(u_x,u_y)-u(v_x,v_y)\big) = \frac{1}{v^2}\big(v\nabla u - u\nabla v\big)$

(d) $\nabla(u^n) = \big((u^n)_x, (u^n)_y\big) = (nu^{n-1}u_x, nu^{n-1}u_y) = nu^{n-1}\nabla u$

Ex. 62 Both the ellipsoid and the shifted sphere contain the point $(1,1,2)$, as is easily checked. Generally, since gradients are orthogonal to level surfaces, the tangent plane of $\mathrm{N}_f(k)$ at a point (x_0,y_0,z_0) with $f(x_0,y_0,z_0)=k$ and $\nabla f(x_0,y_0,z_0) \neq (0,0,0)$ has equation $\nabla f(x_0,y_0,z_0)\cdot(x-x_0, y-y_0, z-z_0) = 0$. It follows that we only need to check that $f(x,y,z) = 3x^2+2y^2+z^2$ and $g(x,y,z) = x^2+y^2+z^2-8x-6y-8z$ have linearly dependent (and nonzero) gradients at $(1,1,2)$:

$$\nabla f(1,1,2) = (6x, 4y, 2z)^{\mathsf{T}}\Big|_{(1,1,2)} = (6,4,4),$$
$$\nabla g(1,1,2) = (2x-8, 2y-6, 2z-8)^{\mathsf{T}}\Big|_{(1,1,2)} = (-6,-4,-4).$$

These are indeed linearly dependent.

Calculus III (Math 241)

W20 Show that the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g(0,0) = 0$ and

$$g(x,y) = \frac{xy^2}{x^2+y^2} \quad \text{for} \quad (x,y) \neq (0,0)$$

has directional derivatives at $(0,0)$ in every direction but is not differentiable at $(0,0)$.

W21 *From a previous midterm*

Consider the function $u\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$u(x,y) = x + \frac{x}{x^2+y^2}.$$

a) Which symmetry properties does u have?

b) Describe the behaviour of $u(x,y)$ for $(x,y) \to (0,0)$, $x \geq 0$.
Hint: Use polar coordinates.

c) Show that u is differentiable and determine the differential $\mathrm{d}u$.

d) Show that there exists exactly one point (x_0,y_0) with $x_0 \geq 0$ at which $\mathrm{d}u$ vanishes.

e) Sketch the contour of u through (x_0,y_0).

Solutions

20 From the lecture we know that g is continuous at $(0,0)$.

Since g vanishes on the coordinate axes, the partial derivatives $\frac{\partial g}{\partial x}(0,0)$ and $\frac{\partial g}{\partial y}(0,0)$ are zero.

The restriction of g to a line $y = mx$, $m \neq 0$, is

$$g(x,mx) = \frac{x(mx)^2}{x^2+(mx)^2} = \frac{m^2x}{1+m^2}.$$

For $\mathbf{u} = (1,m)$ this gives

$$g_{\mathbf{u}}(0,0) = \lim_{t\to 0} \frac{g\big(t(1,m)\big)-g(0,0)}{t} = \lim_{t\to 0} \frac{g(t,tm)-0}{t} = \lim_{t\to 0} \frac{m^2t}{(1+m^2)t} = \frac{m^2}{1+m^2}.$$

Hence all directional derivatives in $(0,0)$ exist. (In fact more is true: The portion of G_g above the line $y = mx$ is a line, viz. $\mathbb{R}\left(1,m,\frac{m^2}{1+m^2}\right)$ and hence has constant slope.)

The differential, provided it exists, must be zero (i.e., the all-zero map), since $\frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$. The corresponding approximation property, $g(\mathbf{h})-g(\mathbf{0}) = g(h_1,h_2) = \mathrm{o}(\mathbf{h})$ is not fulfilled, however, as reasoning with other directional derivatives shows: Taking $h_1 = h_2 = h$, for example, gives

$$g(h,h) = \frac{h^3}{2h^2} = \frac{h}{2} \neq \mathrm{o}(h) \quad \text{for } h \to 0.$$

Hence g is not differentiable at $(0,0)$.

Alternatively one can use the formula $f_{\mathbf{u}}(\mathbf{x}) = \mathrm{d}f(\mathbf{x})(\mathbf{u})$ from the lecture, which shows that for a differentiable function at $\mathbf{x}$ with vanishing differential all directional derivatives in $\mathbf{x}$ must be zero.

Note: It is *not* sufficient to show that the partial derivatives g_x and g_y don't have a limit for $(x,y) \to (0,0)$. This can be seen already in the univariate case: There exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$, for which the derivative is discontinuous at some point x_0, and it can be shown that in such a case $\lim_{x\to x_0} f'(x)$ does not exist. A concrete example is $f(x) = x^2\sin(1/x)$ for $x \neq 0$, $f(0) = 0$, which exhibits this behaviour at $x_0 = 0$.

Question: The formula for $g_{(1,m)}(0,0)$ derived above shows that $g_{(1,m)}(0,0)$ tends to 1 for $m \to \pm\infty$, i.e., if the line $\mathbb{R}(1,m)$, which has equation $y = mx$, tends to the vertical coordinate axis $x = 0$. On the other hand we have $g_{(0,1)}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$. This apparent discontinuity of slopes seems to contradict the continuity of g at points $(0,y)$ near the origin. How to resolve this?

21 a) As in the lecture, we denote the graph of u by G_u, i.e.,

$$\mathrm{G}_u = \big\{\big(x,y,u(x,y)\big); (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)\big\}.$$

$u(x,-y) = u(x,y) \Longrightarrow \mathrm{G}_u$ is symmetric with respect to the (x,z)-plane.
$u(-x,y) = -u(x,y) \Longrightarrow \mathrm{G}_u$ is symmetric with respect to the y-axis.

As a consequence of the second symmetry property, it suffices to investigate u in the right half-plane $\big\{(x,y) \in \mathbb{R}^2; x \geq 0\big\}$. This point-of-view is adopted in the following subquestions.

b) In terms of $x = r\cos\phi$, $y = r\sin\phi$ we have

$$u(x,y) = r\cos\phi + \frac{r\cos\phi}{r^2} = \cos\phi\left(r + \frac{1}{r}\right).$$

For $x > 0$ we have $\cos\phi > 0$ and hence, since $1/r \to +\infty$ for $r \to 0$, in every neighborhood of $(0,0)$ points (x,y) with arbitrarily large values $u(x,y)$.

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x\geq 0}} u(x,y)$ doesn't exist even in the improper sense, since every disk around $(0,0)$ contains points $(0,y)$ with $y > 0$, at which $u(0,y) = 0$.

c) The partial derivatives of u are

$$u_x = 1 + \frac{1\cdot(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}, \quad u_y = -\frac{2xy}{(x^2+y^2)^2}$$

Since these are continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$, the function u is differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$ with differential

$$\mathrm{d}u(x,y) = \left(1 + \frac{y^2-x^2}{(x^2+y^2)^2}\right)\mathrm{d}x - \frac{2xy}{(x^2+y^2)^2}\,\mathrm{d}y,$$

i.e.,

$$\mathrm{d}u(x,y)(\mathbf{h}) = \left(1 + \frac{y^2-x^2}{(x^2+y^2)^2}\right)h_1 - \frac{2xy}{(x^2+y^2)^2}\,h_2 \quad \text{for } \mathbf{h} = (h_1,h_2) \in \mathbb{R}^2.$$

d) The differential $\mathrm{d}u(x_0,y_0)$ vanishes (i.e., is zero) iff $u_x(x_0,y_0) = u_y(x_0,y_0) = 0$.

Looking at u_y we see that then either $x_0 = 0$ or $y_0 = 0$. In the first case there is no solution, since $u_x(0,y_0) = 1 + 1/y_0^2 > 0$. In the second case there are two solutions $(x_0,y_0) = (\pm 1, 0)$, and one solution, viz. $(x_0,y_0) = (1,0)$, with $x_0 \geq 0$.

e) Since $u(1,0) = 2$, the contour in question is the 2-contour. It has equation

$$x + \frac{x}{x^2+y^2} = 2 \iff \frac{x}{x^2+y^2} = 2 - x \iff x^2 + y^2 = \frac{x}{2-x}$$
$$\iff y = \pm\sqrt{\frac{x}{2-x} - x^2} = \pm\sqrt{\frac{x - 2x^2 + x^3}{2-x}} = \pm\sqrt{\frac{x}{2-x}}\cdot|x-1|$$

which gives it approximately the shape of $y = \pm|x-1|$ at the point $(x_0,y_0) = (1,0)$ and the shape of $y = \pm\sqrt{x/2}$ near the point $(0,0)$.

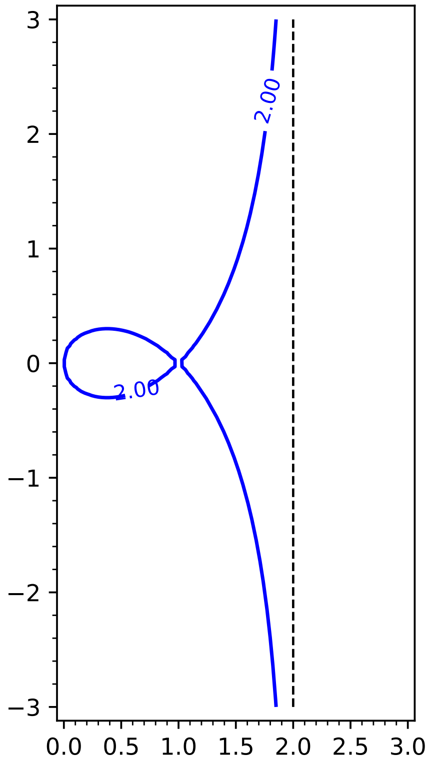
Points (x,y) on the 2-contour satisfy $0 < x < 2$. Hence we can set $x = t^2$ and obtain the parametrization

$$\gamma(t) = \left(t^2, \frac{t(t^2-1)}{\sqrt{2-t^2}}\right), \quad t \in \left(-\sqrt{2}, \sqrt{2}\right),$$

which includes $\gamma(0) = (0,0)$, where u is not defined. From this parametrization it is easy to plot the 2-contour; see below.

Since $\gamma(\pm 1) = (1,0)$, it has a double point at $(1,0)$. We will see soon in the lecture that contours can exhibit such behaviour only at points where the differential vanishes; cp. with d).

The tangent directions in $\gamma(-1) = (1,0)$ and $\gamma(1) = (1,0)$ are $(-1,1)$ and $(1,1)$, respectively. In $\gamma(0) = (0,0)$ the curve γ has a vertical tangent, and for $t \to \pm\infty$ it approaches the line $y = 2$.



Calculus III (Math 241)

H40 *Partially from a previous midterm*

The period of a simple pendulum is

$$T = 2\pi\sqrt{l/g},$$

where l is the length and g is the gravitational constant.

a) If we compute T by taking $\pi = 3$ ($|\text{error}| < 0.15$), $l = 40\,\text{cm}$ ($|\text{error}| < 0.5\,\text{cm}$) and $g = 10\,\text{ms}^{-2}$ ($|\text{error}| < 0.2\,\text{ms}^{-2}$), find the approximately possible error in T.

b) Using the Mean Value Theorem, make the error estimate in a) rigorous.

H41 Do Exercise 76 in [Ste21], Ch. 14.6, p. 1008.

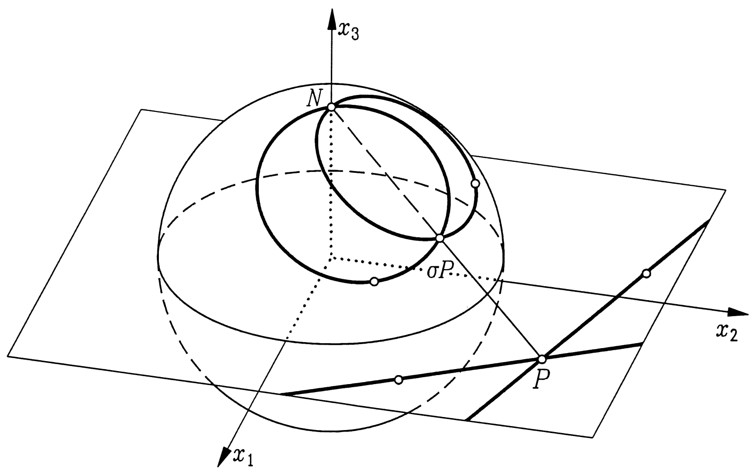
H42 Suppose $H = \{(x,y) \in \mathbb{R}^2; x > 0\}$.

a) Determine the inverse function g of the polar coordinate function $f\colon (0,+\infty)\times(-\pi/2,\pi/2) \to H$, $(r,\phi) \mapsto (r\cos\phi, r\sin\phi)$.

b) Using the representation of g determined in a), compute the Jacobi matrix $\mathbf{J}_g(x,y)$ for $(x,y) \in S$.

c) Verify that for $(x,y) = f(r,\phi)$ the matrix $\mathbf{J}_g(x,y)$ is the inverse of $\mathbf{J}_f(r,\phi)$.

H43 The map $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$ that assigns to a point $P = (x_1,x_2) \triangleq (x_1,x_2,0)$ the second intersection point of the line NP, $N = (0,0,1)$, with the unit sphere $\mathrm{S}^2 = \mathrm{S}_1(\mathbf{0}) \subset \mathbb{R}^3$ is called *stereographic projection*.



a) Derive an explicit formula for $\sigma(x_1,x_2)$ and show that σ maps $\mathbb{R}^2$ bijectively onto $\mathrm{S}^2 \setminus \{N\}$.

b) Show that σ is differentiable and determine $\mathbf{J}_\sigma(x_1,x_2)$.

c) Show that σ is conformal. (Further it can be shown that σ maps lines and circles of the plane $\mathbb{R}^2$ to circles on the sphere S^2.)

d) Find the point (x_1,x_2) satisfying $\sigma(x_1,x_2) = \left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$, and use σ to determine the tangent plane T of S^2 in $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$ in parametric form; compare with the example in `lecture17-20_handout.pdf`, Slide 59.

H44 *Optional Exercise*

The map $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined by

$$F(x,y) = \left(x + \frac{x}{x^2+y^2}, y - \frac{y}{x^2+y^2} \right)$$

is known as *Zhukovskii* transformation, named after the Russian Physicist and Engineer N.E. ZHUKOVSKII (1847–1921).

a) Express $F(x,y) = F(x+\mathrm{i}y)$ as a complex function of $z = x+\mathrm{i}y$ and use this to give a "complex" proof of the conformality of F in $\mathbb{C} \setminus \{0, \pm 1\}$.

b) For $w \in \mathbb{C}$ determine the number of $z \in \mathbb{C}$ satisfying $F(z) = w$

c) Sketch the images under F of the circles $x^2 + y^2 = r^2$ for $r \in \{1,2,3\}$, and of the half-lines $y = mx$, $x > 0$, for $m \in \{0, 1/\sqrt{3}, 1\}$.

   Notes: It is advisable to use polar coordinates and for the second part the substitution $r = \mathrm{e}^s$. Your sketch should show the angles of intersection between the image curves.

d) What is the geometric meaning of the points $(\pm 2, 0)$ for the curves in c) ?

e) Plot the images of circles through $(1,0)$ for various choices of the radius and center. (Include at least circles passing through $(-1,0)$, circles containing/not containing $(-1,0)$ in their interior, and circles not on one of the coordinate axes.) What do you observe? Can you prove your experimental observations?

Due on Wed Nov 22, 7:30 pm

The optional exercise can be handed in until Wed Nov 29, 6 pm.

Solutions

40 a) The differential of T is

$$\mathrm{d}T = T_\pi\,\mathrm{d}\pi + T_l\,\mathrm{d}l + T_g\,\mathrm{d}g = 2\sqrt{\frac{l}{g}}\,\mathrm{d}\pi + \frac{\pi}{\sqrt{lg}}\,\mathrm{d}l - \pi\sqrt{\frac{l}{g^3}}\,\mathrm{d}g.$$

$\Longrightarrow$ The error is approximately

$$\begin{aligned}\Delta T &\approx 2\sqrt{\frac{40}{1000}}\Delta\pi + \frac{3}{\sqrt{40\cdot 1000}}\Delta l - 3\sqrt{\frac{40}{1000^3}}\Delta g\\ &= \frac{2}{5}\Delta\pi + \frac{3}{200}\Delta l - \frac{6}{10\,000}\Delta g,\end{aligned}$$

where cm and s have been chosen as units for the computation and hence the error is measured in seconds. Substituting $\Delta\pi \approx 0.15$, $|\Delta l| \le 0.5\,[\mathrm{cm}]$, $\Delta g \approx -20\,[\mathrm{cm\,s^{-2}}]$ shows that the largest in absolute value possible error is positive and approximately equal to

$$\frac{30}{500} + \frac{3}{400} + \frac{120}{10\,000} = \frac{795}{10\,000} = 0.0795\,[\text{seconds}].$$

In our computation we have assumed that π and g are known and hence the signs of $\Delta\pi$ and Δg are known as well. If we don't assume this, the result is still the same (except that ΔT may now be negative), because the summands in ΔT corresponding to π and g happen to have the same sign and hence the largest value of $|\Delta T|$ is obtained by giving the third summand the same sign, i.e., for $\Delta l = 0.5$.

b) By the Mean Value Theorem,

$$\Delta T = T_\pi(\tilde\pi,\tilde l,\tilde g)\Delta\pi + T_l(\tilde\pi,\tilde l,\tilde g)\Delta l + T_g(\tilde\pi,\tilde l,\tilde g)\Delta g$$

for some point $(\tilde\pi,\tilde l,\tilde g)$ on the line segment between (π,l,g) and $(\pi+\Delta\pi, l+\Delta l, g+\Delta g)$. Using $|\Delta\pi| \le 0.15$, $|\Delta l| \le 0.5\,[\mathrm{cm}]$, $|\Delta g| \le 20\,[\mathrm{cm\,s^{-2}}]$ and the maximum values of the partial derivatives in the resulting uncertainty region $2.85 \le \pi \le 3.15$, $39.5 \le l \le 40.5\,[\mathrm{cm}]$, $980 \le g \le 1020\,[\mathrm{cm\,s^{-2}}]$, we obtain

$$|\Delta T| \le 2\sqrt{\frac{40.5}{980}}\cdot 0.15 + \frac{3.15}{\sqrt{39.5\cdot 980}}\cdot 0.5 + 3.15\sqrt{\frac{40.5}{980^3}}\cdot 20 \approx 0.0821\,[\text{seconds}].$$

Here we have adopted the viewpoint of a (dull) applied mathematician who has never heard about π (except that it is approximately equal to 3) and the gravitational constant g (except that it is approximately equal to 10). In his mind the errors, within the stated bounds, could have both signs.

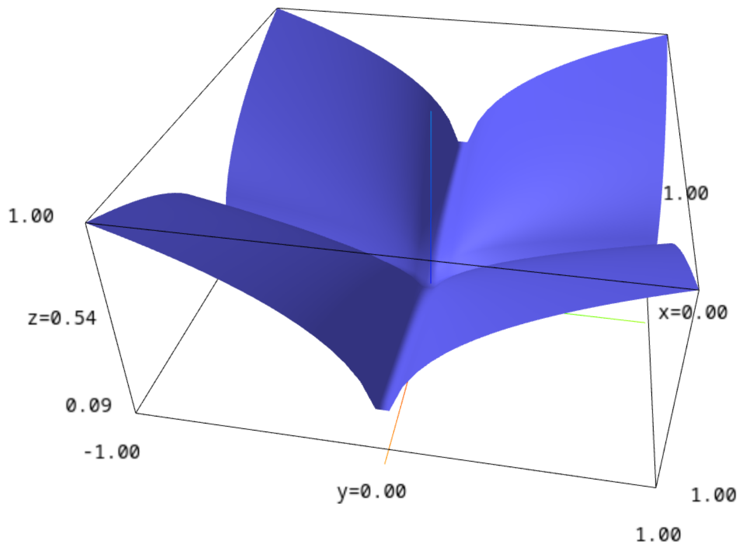
41 a) As composition $h \circ g$ of the continuous functions $g(x,y) = xy$ and $h(t) = \sqrt[3]{t}$, which has domain $\mathbb{R}$, the function f is continuous. Since f vanishes on the coordinate axes, f is partially differentiable at $(0,0)$ with $f_x(0,0) = f_y(0,0) = 0$.

In polar coordinates we have

$$f(x,y) = f(r\cos\phi, r\sin\phi) = \sqrt[3]{r^2\cos\phi\sin\phi} = r^{2/3}\sqrt[3]{\tfrac{1}{2}\sin(2\phi)}.$$

On a line $\mathbb{R}(\cos\phi_0,\sin\phi_0)$ different from the coordinate axes, i.e. $\phi_0 \notin \left\{0,\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$, the function f looks like $r \mapsto c\,r^{2/3}$, $c \neq 0$, which is not differentiable at $r = 0$. This shows that the corresponding directional derivative of f at $(0,0)$ doesn't exist.

b) The function f is graphed below. You can see the non-smoothness of the graph at the coordinate axes in the x-y plane, which is implied by a).



42 a) *Remark:* The domain of the polar coordinate map is usually taken as the larger set $S = \mathbb{R}^2 \setminus \{(x,0); x \leq 0\}$ ("slotted plane"), but this complicates matters, since $\phi \mapsto \tan\phi$ cannot be inverted on $(-\pi,\pi)$. The solution given below is for this more general case. When restricted to T, the inverse is simply $g(x,y) = \left(\sqrt{x^2+y^2}, \arctan(y/x)\right)$, as the proof shows.

Under the given restrictions on r,ϕ,x,y, the system $x = r\cos\phi$, $y = r\sin\phi$ is equivalent to $r = \sqrt{x^2+y^2}$, $\frac{y}{x} = \frac{\sin\phi}{\cos\phi} = \tan\phi$, provided that $x \neq 0$, i.e., $\phi \neq \pm\pi/2$. Since $(-\pi,\pi)$ splits into three subintervals on which $\phi \mapsto \tan\phi$ is defined, we need to distinguish three cases when inverting it. For the boundaries $\phi = \pm\pi/2$, i.e., the y-axis, the limits of the corresponding branches can be taken from either side. This gives

$$g(x,y) = \begin{cases} \left(\sqrt{x^2+y^2}, \arctan(y/x)\right) & \text{if } x > 0, \\ \left(\sqrt{x^2+y^2}, \arctan(y/x)+\pi\right) & \text{if } x < 0,\, y > 0, \\ \left(\sqrt{x^2+y^2}, \arctan(y/x)-\pi\right) & \text{if } x < 0,\, y < 0, \\ (y, \pi/2) & \text{if } x = 0,\, y > 0, \\ (-y, -\pi/2) & \text{if } x = 0,\, y < 0. \end{cases}$$

b) For $(x,y) \in S$ with $x > 0$ we obtain from a)

$$
\begin{aligned}
\frac{\partial}{\partial x}\sqrt{x^2+y^2} &= \frac{x}{\sqrt{x^2+y^2}},\\
\frac{\partial}{\partial y}\sqrt{x^2+y^2} &= \frac{y}{\sqrt{x^2+y^2}},\\
\frac{\partial}{\partial x}\arctan(y/x) &= \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2},\\
\frac{\partial}{\partial y}\arctan(y/x) &= \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2},\\
\mathbf{J}_g(x,y) &= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}.
\end{aligned}
$$

The computation is also valid for the other two regions and in fact also for the y-axis. This can be verified with some effort, but it is easier to invoke the theorem about local invertibility of a C^1-map $f\colon D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$, in points $\mathbf{x} \in D$ where $\mathbf{J}_f(\mathbf{x})$ is invertible. This theorem guarantees that the corresponding inverse g is also a C^1-map. Since in the case under consideration f is globally invertible, the inverse g must be a C^1-map as well (since C^1 is a local property).

c) As derived in the lecture, we have

$$\mathbf{J}_f(r,\phi) = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}$$

and hence

$$\mathbf{J}_f(r,\phi)^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\phi & r\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -(\sin\phi)/r & (\cos\phi)/r \end{pmatrix}.$$

This is the same as $\mathbf{J}_g(r\cos\phi, r\sin\phi)$, as is easily checked.

43 a) Parametrizing the line NP as $tP + (1-t)N = (tx_1, tx_2, 1-t)$, $t \in \mathbb{R}$, we obtain for the intersection points the equation $(tx_1)^2 + (tx_2)^2 + (1-t)^2 = 1$, which is equivalent to $(x_1^2 + x_2^2 + 1)t^2 - 2t = 0$. This quadratic has 2 solutions: $t = 0$, which corresponds to the "north pole" N of the unit sphere, and $t = 2(x_1^2 + x_2^2 + 1)^{-1}$, which gives the 2nd intersection point.

$$\Longrightarrow \sigma(x_1,x_2) = \frac{1}{x_1^2+x_2^2+1}\begin{pmatrix} 2x_1 \\ 2x_2 \\ x_1^2+x_2^2-1 \end{pmatrix}$$

σ is bijective, because the equation $\sigma(x_1,x_2) = (y_1,y_2,y_3)$ can be uniquely solved for x_1, x_2: $x_1 = y_1/t = y_1/(1-y_3)$, $x_2 = y_2/(1-y_3)$.

Remark: For understanding the map σ it is instructive to check a few common points and their images. The origin of $\mathbb{R}^2$, for example, is projected to the "south pole" of S^2, the points on the unit circle S^1 of $\mathbb{R}^2$ are fixed by σ, and a glance at the 3rd component function σ_3 (which has the same sign as $x_1^2 + x_2^2 - 1$) reveals that σ maps the unit disk of $\mathbb{R}^2$ to the "southern" hemisphere of S^2 and the "outer ring" $x_1^2 + x_2^2 > 1$ to the "northern" hemisphere.

b) We compute some partial derivatives and infer the rest by symmetry.

$$\frac{\partial\sigma_1}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{2x_1}{x_1^2+x_2^2+1}\right)=\frac{2(x_1^2+x_2^2+1)-(2x_1)(2x_1)}{(x_1^2+x_2^2+1)^2}=\frac{2(-x_1^2+x_2^2+1)}{(x_1^2+x_2^2+1)^2},$$
$$\frac{\partial\sigma_1}{\partial x_2}=\frac{-2x_1(2x_2)}{(x_1^2+x_2^2+1)^2}=\frac{-4x_1x_2}{(x_1^2+x_2^2+1)^2},$$
$$\frac{\partial\sigma_3}{\partial x_1}=\frac{\partial}{\partial x_1}\left(1-\frac{2}{x_1^2+x_2^2+1}\right)=\frac{4x_1}{(x_1^2+x_2^2+1)^2},$$

giving

$$\mathbf{J}_\sigma(x_1,x_2)=\frac{2}{(x_1^2+x_2^2+1)^2}\begin{pmatrix}-x_1^2+x_2^2+1 & -2x_1x_2\\ -2x_1x_2 & x_1^2-x_2^2+1\\ 2x_1 & 2x_2\end{pmatrix}.$$

Since the partial derivatives of σ are rational (hence continuous) functions without singularities, σ is differentiable everywhere.

c) With a little effort it can be checked that the columns of $\mathbf{J}_\sigma(x_1,x_2)$ are orthogonal and have the same length $2(x_1^2+x_2^2+1)^{-1}$. This is equivalent to

$$\mathbf{J}_\sigma(x_1,x_2)^{\mathsf{T}}\mathbf{J}_\sigma(x_1,x_2)=\frac{4}{(x_1^2+x_2^2+1)^2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

and implies that σ is conformal and preserves intersection angles of any curves; cf. Worksheet 8, Exercise W34 c).

One can show that the image of any line L in $\mathbb{R}^2$ under σ is a circle. Here is a geometric proof of this fact: Together with N the line L spans a plane E, and the intersection of E with S^2 is precisely $\sigma(L)\cup\{N\}$, because E is covered by the lines through N that meet L and one further line through N parallel to L and to $\mathbb{R}^2\times\{0\}$. If you accept the geometric fact that planes intersect spheres in circles, this shows that lines in $\mathbb{R}^2$ are mapped to circles on S^2 containing N. Furher one can show that σ maps circles in $\mathbb{R}^2$ to circles on S^2 not containing N. Since σ is conformal, the angles of intersection between the lines (circles) are the same as between their images.

d) Let $Q=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$. Then the point $(x_1,x_2)\triangleq(x_1,x_2,0)$ is the intersection point of the line NQ and the plane $x_3=0$:

$$N+\lambda(Q-N)=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}\frac{1}{3}-0\\ \frac{2}{3}-0\\ \frac{2}{3}-1\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}\implies\lambda=3,\ x_1=1,\ x_2=2.$$

(You can check that indeed $\sigma(1,2)=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$.)

The formula for the tangent plane of a parametric surface derived in `lecture21-24` then gives $T=Q+\left\{\mathbf{J}_\sigma(1,2)\mathbf{h};\mathbf{h}\in\mathbb{R}^2\right\}$ ("Q plus the column space of $\mathbf{J}_\sigma(1,2)$"). Discarding the factor $2(x_1^2+x_2^2+1)^{-2}$, we obtain

$$T=\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{pmatrix}+\mathbb{R}\begin{pmatrix}4\\-4\\2\end{pmatrix}+\mathbb{R}\begin{pmatrix}-4\\-2\\4\end{pmatrix}.$$

Since the linear subspace ("direction space") associated with T has $(1,2,2)$ as a normal vector, it is the same as that in the example on Slide 58, and T is parallel to the tangent plane in the example. Because the example differs from the present setting only by a scale factor of 3, this is the correct result and validates our computation, in case you had any doubt. (The key point is that any parametrization whatsoever can be used to compute tangent planes of parametric surfaces. But this is not easy to prove rigorously.)

44 a) Writing $z=(x,y)=x+\mathrm{i}y=x+y\mathrm{i}$, we have

$$F(z)=x+\frac{x}{x^2+y^2}+\left(y-\frac{y}{x^2+y^2}\right)\mathrm{i}=x+y\mathrm{i}+\frac{x-y\mathrm{i}}{x^2+y^2}=z+\frac{\bar{z}}{|z|^2}=z+\frac{1}{z}.$$

$\Longrightarrow$ The map F is holomorphic (complex differentiable) with $F'(z)=1-\frac{1}{z^2}$. In the lecture we have seen that this implies

$$\mathbf{J}_F(z)=\mathbf{J}_F(x,y)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{where } a,b \text{ are determined by } F'(z)=a+b\mathrm{i}.$$

Another representation is $a=u_x(x,y)$, $b=v_x(x,y)$, where u and v deonote the real and imaginary part of F (the functions discussed in Q1 of Midterm 2 this year/last year).

Since

$$\mathbf{J}_F(z)^{\mathsf{T}}\mathbf{J}_F(z)=\begin{pmatrix} a^2+b^2 & \\ & a^2+b^2 \end{pmatrix},$$

we have that F is conformal in all points z with $F'(z)\neq 0$. (This holds, more generally, for any holomorphic function.) Since $1-\frac{1}{z^2}=0 \iff z\pm 1$, F is conformal in $\mathbb{C}\setminus\{0,\pm 1\}$.

b) $F(z)=z+1/z=w \iff z^2-wz+1=0 \iff z=\frac{1}{2}(w\pm\sqrt{w^2-4})$. Since a nonzero complex number has exactly two square roots, all $w\in\mathbb{C}$ except ± 2 have two preimages under F, and 2 and -2 have one preimage (viz., 1 and -1, respectively).

c) In polar coordinates we have

$$F(x,y)=F(r\cos\phi,r\sin\phi)=\left(\left(r+\frac{1}{r}\right)\cos\phi,\left(r-\frac{1}{r}\right)\sin\phi\right).$$

$\Longrightarrow$ The image of the circle $x^2+y^2=r^2$, $r\neq 1$, is the ellipse $x^2/a^2+y^2/b^2=1$ with semi-axes $a=r+\frac{1}{r}$, $b=\left|r-\frac{1}{r}\right|$. (The ellipses corresponding to r and $1/r$ are the same.)

For $r=1$ the parametrization reduces to $x=2\cos\phi$, $y=0$, so that the image of the unit circle is the line segment of the x-axis between $(-2,0)$ and $(2,0)$.

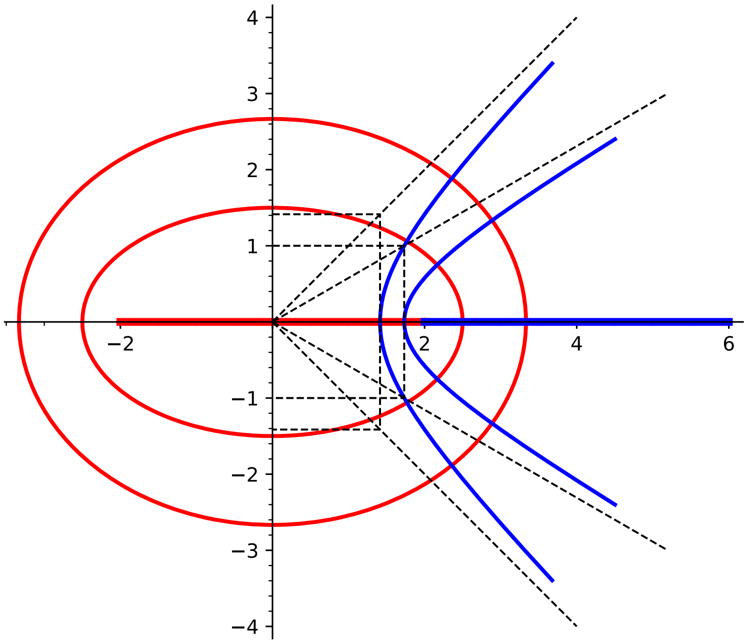
Further, setting $r=\mathrm{e}^s$, we have

$$F(x,y)=\left((\mathrm{e}^s+\mathrm{e}^{-s})\cos\phi,(\mathrm{e}^s-e^{-s})\sin\phi\right)=\left(2\cosh(s)\cos\phi,2\sinh(s)\sin\phi\right).$$

$\Longrightarrow$ The image of the half-line $\phi=$ const., $\phi\notin\mathbb{Z}(\pi/2)$ is a branch of the hyperbola $x^2/a^2-y^2/b^2=1$ with semi-axes $a=2|\cos\phi|$, $b=2|\sin\phi|$. For $-\pi/2<\phi<\pi/2$ the branch is in the right half-plane $x>0$. (The branches corresponding to ϕ and $-\phi$ are the same.) For $\phi=0$ ($\triangleq m=0$) it reduces to the half-line $\{(x,0);x\geq 2\}$ and for $\phi=\pi/2$ (not required) to the y-axis $\{(0,y);y\in\mathbb{R}\}$. The angles corresponding to $m=1/\sqrt{3}$ and $m=1$ are $\phi=\pi/6$ and $\phi=\pi/4$, respectively.

The data for the ellipses/hyperbolas required in c) is

	$r=2$	$r=3$	$m=1/\sqrt{3}$	$m=1$
a	$5/2$	$10/3$	$\sqrt{3}$	$\sqrt{2}$
b	$3/2$	$8/3$	1	$\sqrt{2}$



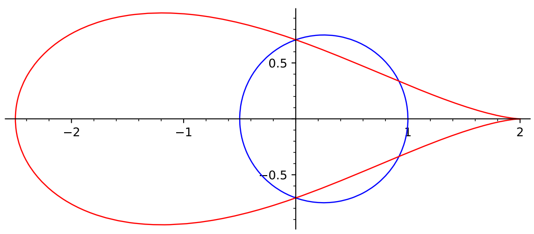
$\boxed{3}$

Since F is conformal, the ellipses and hyperbolas (including the degenerate ones) intersect in right angles except for those through the points $(\pm 2, 0) = F(\pm 1, 0)$.

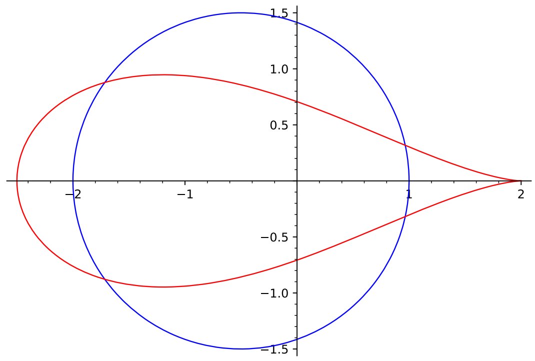
d) The image of the circle $x^2 + y^2 = r^2$, $r \neq 1$, is the ellipse $x^2/a^2 + y^2/b^2 = 1$ with semi-axes $a = r + \frac{1}{r}$, $b = \left| r - \frac{1}{r} \right|$. Since $a^2 - b^2 = (r + 1/r)^2 - (r - 1/r)^2 = 4$, all these ellipses have their foci in $\sqrt{a^2 - b^2} = \pm 2$.

The hyperbolas $x^2/a^2 - y^2/b^2 = 1$ with semi-axes $a = 2\,|\cos\phi|$, $b = 2\,|\sin\phi|$, which arise as images of the lines $\phi = \text{const.}$, have their foci in ± 2 as well, since in this case $a^2 + b^2 = 4$.

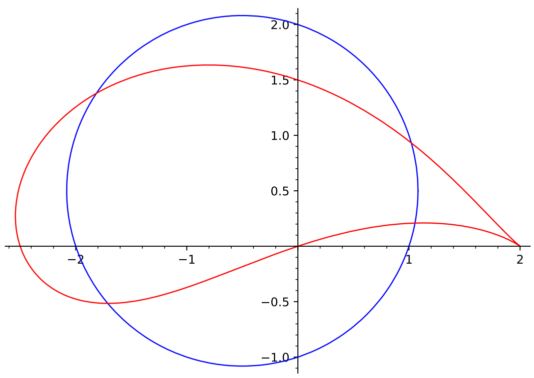
e) In the following plots the circles are drawn in blue and their images under F in red.



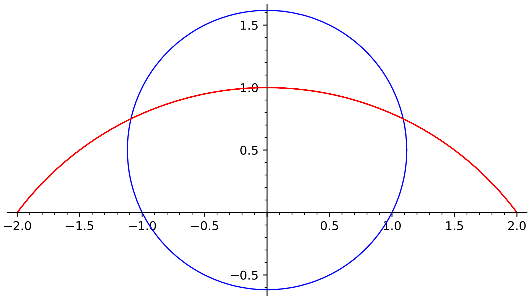
(a) $\mathbf{c} = (0.25, 0)$



(b) $\mathbf{c} = (-0.5, 0)$



(c) $\mathbf{c} = (-0.5, 0.5)$



(d) $\mathbf{c} = (0, 0.5)$

Calculus III (Math 241)

W22 Do Exercises 26, 28 in [Ste21], Ch. 14.5.

W23 *From a previous midterm*

The height and mass ("weight") of a basketball player are known to be $h_0 = 2.00\,\mathrm{m}$ and $m_0 = 100\,\mathrm{kg}$, respectively, with a possible error of $0.5\,\mathrm{cm}$, respectively, $0.5\,\mathrm{kg}$.

a) Find the linear approximation of the *body mass index* $B(m,h) = m/h^2$ near (m_0, h_0).

b) Using the Mean Value Theorem, state a (tight) rigorous upper bound for the (absolute) error when the body mass index of the basketball player is calculated from the known data.

W24 Suppose $f: D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$ is differentiable.

a) Let $\gamma\colon I \to D$ be a differentiable curve in D and $\beta(t) = f(\gamma(t))$, $t \in I$, the corresponding image curve. Show that β is differentiable as well and satisfies $\beta'(t) = \mathbf{J}_f(\gamma(t))\gamma'(t)$ for $t \in I$.

b) For $\mathbf{x}_0 \in D$ the columns of $\mathbf{J}_f(\mathbf{x}_0)$ can be interpreted as tangent vectors to certain curves. What are these curves, and what is the relation with a) ?

c) f is said to be *conformal* in $\mathbf{x}_0 \in D$ if f preserves the angles between any two smooth curves γ_1, γ_2 through $\mathbf{x}_0$. Show that f is conformal in $\mathbf{x}_0$ if $\mathbf{J}_f(\mathbf{x}_0)^\mathsf{T}\mathbf{J}_f(\mathbf{x}_0) = \lambda \mathrm{I}_n$ for some $\lambda > 0$.

d) Show that the squaring map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ is conformal in $\mathbb{C} \setminus \{0\}$ and sketch the images of the lines $x = k$, $y = k$ for $k = 0, 1, 2, 3, 4, 5$.

Solutions

22 Ex. 26

$$\begin{pmatrix}\frac{\partial T}{\partial p} & \frac{\partial T}{\partial q} & \frac{\partial T}{\partial r}\end{pmatrix} = \begin{pmatrix}\frac{\partial T}{\partial u} & \frac{\partial T}{\partial v}\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial p} & \dfrac{\partial u}{\partial q} & \dfrac{\partial u}{\partial r}\\ \dfrac{\partial v}{\partial p} & \dfrac{\partial v}{\partial q} & \dfrac{\partial v}{\partial r}\end{pmatrix}$$

$$= \left(-\frac{2v}{(2u+v)^2}, \frac{2u}{(2u+v)^2}\right)\begin{pmatrix} q\sqrt{r} & p\sqrt{r} & \frac{pq}{2\sqrt{r}} \\ \sqrt{q}r & \frac{pr}{2\sqrt{q}} & p\sqrt{q}\end{pmatrix}$$

At the indicated point $(p,q,r) = (2,1,4)$ we have $(u,v) = (4,8)$ and hence

$$\begin{pmatrix}\frac{\partial T}{\partial p} & \frac{\partial T}{\partial q} & \frac{\partial T}{\partial r}\end{pmatrix} = \begin{pmatrix}-\frac{1}{16} & \frac{1}{32}\end{pmatrix}\begin{pmatrix}2 & 4 & \frac{1}{2}\\ 4 & 4 & 2\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{8} & \frac{1}{32}\end{pmatrix}$$

Ex. 28:

$$\begin{pmatrix}\frac{\partial P}{\partial x} & \frac{\partial P}{\partial y}\end{pmatrix} = \begin{pmatrix}\frac{\partial P}{\partial u} & \frac{\partial P}{\partial v} & \frac{\partial P}{\partial w}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}\end{pmatrix}$$

$$\frac{1}{\sqrt{u^2+v^2+w^2}}\begin{pmatrix}u & v & w\end{pmatrix}\begin{pmatrix}\mathrm{e}^y & x\mathrm{e}^y\\ y\mathrm{e}^x & \mathrm{e}^x\\ y\mathrm{e}^{xy} & x\mathrm{e}^{xy}\end{pmatrix}$$

At $(x,y) = (0,2)$ we have $(u,v,w) = (0,2,1)$ and hence

$$\begin{pmatrix}\frac{\partial P}{\partial x} & \frac{\partial P}{\partial y}\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}0 & 2 & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^2 & 0\\ 2 & 1\\ 2 & 0\end{pmatrix} = \begin{pmatrix}\frac{6}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}.$$

23 a) We use m and kg as units of measurement.

$$\begin{aligned}\mathrm{d}B &= h^{-2}\mathrm{d}m - 2mh^{-3}\mathrm{d}h\\ \Longrightarrow B(m,h) &\approx \frac{100}{2^2} + \frac{1}{2^2}\Delta m - \frac{2\cdot 100}{2^3}\Delta h\\ &= 25 + \frac{1}{4}\Delta m - 25\,\Delta h \quad [\mathrm{kg\,m^{-2}}],\end{aligned}$$

where $\Delta m = m - m_0$, $\Delta h = h - h_0$.

b) The Mean Value Theorem gives that the error $\Delta B = B(m,h) - B(100,2)$ is equal to $h'^{-2}\Delta m - 2m'h'^{-3}\Delta h$ for some $m' \in [99.5, 100.5]$ and $h' \in [1.995, 2.005]$.

From this, using mm as the unit of measurement, we obtain

$$\begin{aligned}\Longrightarrow |\Delta B| &\leq \frac{0.5}{1.995^2} + \frac{2\cdot 100.5\cdot 0.005}{1.995^3}\\ &\approx 0.252199269727261 \quad [\mathrm{kg\,m^{-2}}].\end{aligned}$$

(This should be compared with the corresponding non-rigorous estimate, which is $|\Delta B| \leq \frac{1}{8} + \frac{1}{8} = 0.25$ $[\mathrm{kg\,m^{-2}}]$.)

24 a) For a curve $\gamma\colon I \to \mathbb{R}^n$ we have $\gamma'(t) = \mathbf{J}_\gamma(t)$ (since the entries of both are the partial derivatives). This applies also to β and gives

$$\beta'(t) = \mathbf{J}_\beta(t) = \mathbf{J}_{f\circ\gamma}(t) = \mathbf{J}_f\big(\gamma(t)\big)\mathbf{J}_\gamma(t) = \mathbf{J}_f\big(\gamma(t)\big)\gamma'(t).$$

b) If $\gamma(t_0) = \mathbf{x}_0$ and $\gamma'(t_0) = \mathbf{e}_j$ (the j-th standard unit vector in $\mathbb{R}^n$, then according to a) we have $\beta'(t_0) = \mathbf{J}_f(\mathbf{x}_0)\mathbf{e}_j$, the j-th column of $\mathbf{J}_f(\mathbf{x}_0)$. The obvious choice for γ having the required properties is $\gamma(t) = \mathbf{x}_0 + t\mathbf{e}_j$, $t_0 = 0$ (called "coordinate lines" in the lecture), so that we can take $\beta(t) = f(\mathbf{x}_0 + t\mathbf{e}_j)$. For this curve β, whose domain can be taken as $(-\delta,\delta)$ for some $\delta > 0$, the tangent vector $\beta'(0)$ equals the j-th column of $\mathbf{J}_f(\mathbf{x}_0)$. Since

$$\beta'(0) = \lim_{t\to 0}\tfrac{1}{t}\big(\beta(t) - \beta(0)\big) = \lim_{t\to 0}\tfrac{1}{t}\big(f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)\big),$$

it is a vectorial analogue of the partial derivative w.r.t. x_j for real-valued functions, and is denoted by $\frac{\partial f}{\partial x_j}$ as well.

c) We may assume that $\gamma_1(t_0) = \gamma_2(t_0) = \mathbf{x}_0$. (Actually the parameter values corresponding to the intersection point $\mathbf{x}_0$ do not matter at all. Only the tangent vectors matter, which are determined up to positive scalar multiples.) The angle ϕ between γ_1, γ_2 at $\mathbf{x}_0$ is then determined by

$$\cos\phi = \frac{\gamma_1'(t_0)^\mathsf{T}\gamma_2'(t_0)}{\sqrt{\gamma_1'(t_0)^\mathsf{T}\gamma_1'(t_0)}\sqrt{\gamma_2'(t_0)^\mathsf{T}\gamma_2'(t_0)}}.$$

For the angle ϕ' between the corresponding image curves $\beta_1 = f\circ\gamma_1$, $\beta_2 = f\circ\gamma_2$ at the point $f(\mathbf{x}_0)$ we have, setting $\mathbf{A} = \mathbf{J}_f(\mathbf{x}_0)$ and using a),

$$\begin{aligned}\cos\phi' &= \frac{\big(\mathbf{A}\gamma_1'(t_0)\big)^\mathsf{T}\mathbf{A}\gamma_2'(t_0)}{\sqrt{\big(\mathbf{A}\gamma_1'(t_0)\big)^\mathsf{T}\mathbf{A}\gamma_1'(t_0)}\sqrt{\big(\mathbf{A}\gamma_2'(t_0)\big)^\mathsf{T}\mathbf{A}\gamma_2'(t_0)}}\\ &= \frac{\gamma_1'(t_0)^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{A}\gamma_2'(t_0)}{\sqrt{\gamma_1'(t_0)^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{A}\gamma_1'(t_0)}\sqrt{\gamma_2'(t_0)^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{A}\gamma_2'(t_0)}},\end{aligned}$$

But $\mathbf{A}^\mathsf{T}\mathbf{A} = \lambda\mathrm{I}_n$, and hence the factors $\mathbf{A}^\mathsf{T}\mathbf{A}$ in the numerator and denominator cancel, giving $\cos\phi = \cos\phi'$ and finally $\phi = \phi'$.

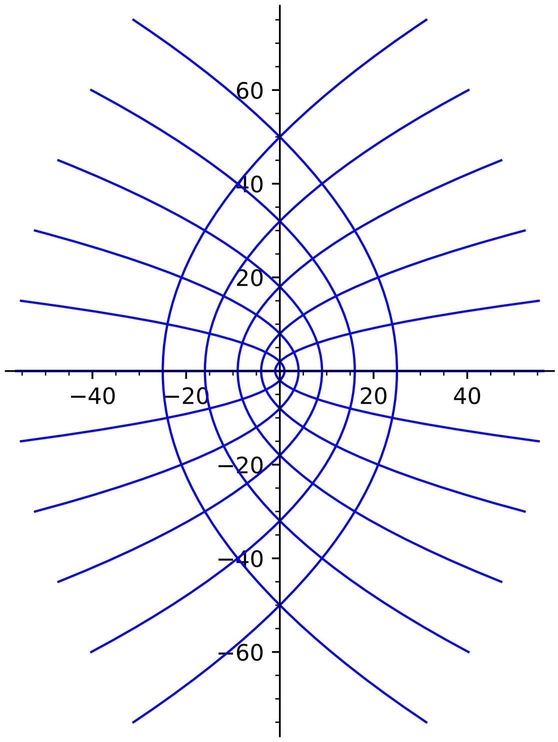
d) The real version of the squaring map is $f(x,y) = (x^2 - y^2, 2xy)$, $(x,y) \in \mathbb{R}^2$.

$$\Longrightarrow \mathbf{J}_f(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}, \quad \mathbf{J}_f(x,y)^\mathsf{T}\mathbf{J}_f(x,y) = \begin{pmatrix} 4x^2 + 4y^2 & 0 \\ 0 & 4x^2 + 4y^2 \end{pmatrix} = 4(x^2 + y^2)\mathrm{I}_2.$$

By c), the squaring map is conformal in $\mathbb{R}^2 \setminus \big\{(0,0)\big\} \triangleq \mathbb{C}\setminus\{0\}$.

The x-axis is mapped to $\big\{(x^2,0); x \in \mathbb{R}\big\}$, i.e., to the non-negative x-axis, and the y-axis is mapped to $\big\{(-y^2,0); y \in \mathbb{R}\big\}$, i.e., to the non-positive x-axis.

For $k \neq 0$ the image of a horizontal line $y = k$ is $\beta(t) = (t^2 - k^2, 2kt)$, $t \in \mathbb{R}$, which can be reparametrized to $x = \big(\frac{y}{2k}\big)^2 - k^2$. This gives a family of parabolas symmetric to the x-axis with vertices in $(-k^2, 0)$ and open to the right. Similarly, the image of $x = k$ is $\beta(t) = (k^2 - t^2, 2kt)$, $t \in \mathbb{R}$, or $x = k^2 - \big(\frac{y}{2k}\big)^2$, giving a family of parabolas symmetric to the x-axis with vertices in $(0, k^2)$ and open to the left. Since the squaring map is conformal, the members of the two families intersect at right angles, as illustrated by the picture.



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Calculus III (Math 241)

H45 Do Exercises 34, 38 and 44 in [Ste21], Ch. 14.5.

H46 Do Exercise 77 in [Ste21], Ch. 14.3.
Hint: Establish first that for any C^2-function f of a single variable the functions $(x,t) \mapsto f(x \pm at)$ solve the wave equation. This implies the results in Exercises 77 (c), (d); 77 (a), (b) can be done similarly.

H47
a) Do Exercise 78 in [Ste21], Ch. 14.3.
b) Prove that linear combinations and partial derivatives of harmonic functions (for simplicity, in 2-variables) are again harmonic.
c) *Optional Exercise:* Which matrices $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ have the property that the variable change $v(x,y) = u(ax+by, cx+dy)$ preserves harmonic functions, i.e., $\Delta u = 0$ implies $\Delta v = 0$?

H48 Do Exercise 79 in [Ste21], Ch. 14.3.

H49 Do Exercise 55 in [Ste21], Ch. 14.5.

H50 *Optional Exercise*
Stereographic projection (cf. Homework 8, Exercise H43) can be generalized to any dimension, i.e., projecting $\mathbb{R}^n \times \{0\}$ from the "north pole" $N = (0,\dots,0,1) \in \mathbb{R}^{n+1}$ to the unit sphere $\mathrm{S} = \{\mathbf{y} \in \mathbb{R}^{n+1}; |\mathbf{y}| = 1\}$ in $\mathbb{R}^{n+1}$ yields a differentiable bijection from $\mathbb{R}^n$ onto $\mathrm{S}^n \setminus \{N\}$.

a) Write down the component functions of σ for $n = 1$ and $n = 3$.
b) Use the map σ for $n = 1$ to enumerate all solutions (a,b,c) in positive integers of the equation

$$a^2 + b^2 = c^2. \qquad \text{(so-called } \textit{Pythagorean Triples}\text{)}$$

Hint: "Primitive" solutions with $\gcd(a,b,c) = 1$ correspond to points with rational coordinates on the unit circle S^1.

Due on Wed Nov 29, 6 pm

The optional exercises can be handed in until Wed Dec 6, 6 pm.

Solutions

45 Ex. 34 Let $F(x,y)=\mathrm{e}^y\sin x-x-xy$. We can solve $F(x,y)=0$ for y (i.e., write the 0-contour of F as a function $y=y(x)$) locally at any point (x,y) satisfying $F_y(x,y)=\mathrm{e}^y\sin x-x\neq 0$. (The "bad" points, where this condition is violated, are those on the y-axis and on the curves $y=\ln\frac{x}{\sin x}$, $k\pi<x<(k+1)\pi$, $k\in\mathbb{Z}$.)
If applicable, the function $y(x)$ is known to be differentiable, and applying the chain rule gives

$$0=\frac{\mathrm{d}}{\mathrm{d}x}F\big(x,y(x)\big)=\big(F_x(x,y)\quad F_y(x,y)\big)\begin{pmatrix}1\\ y'(x)\end{pmatrix}=F_x(x,y)+F_y(x,y)y'(x)=0,$$

where $y=y(x)$ in the argument of F_x, F_y. This gives the formula $y'=-F_x/F_y$ presented in [Ste16].
Here we get

$$y'(x)=-\frac{F_x(x,y)}{F_y(x,y)}=-\frac{\mathrm{e}^y\cos x-1-y}{\mathrm{e}^y\sin x-x}.$$

Ex. 38 Let $F(x,y,z)=yz+x\ln y-z^2$. Applying the chain rule to $F\big(x,y,z(x,y)\big)=F\big(G(x,y)\big)=0$ with $G(x,y)=\big(x,y,z(x,y)\big)^{\mathsf{T}}$ gives

$$\mathbf{J}_F\mathbf{J}_G=\big(F_x\quad F_y\quad F_z\big)\begin{pmatrix}1&0\\0&1\\\frac{\partial z}{\partial x}&\frac{\partial z}{\partial y}\end{pmatrix}=0.$$

(Arguments have been omitted.) This yields the formulas $\frac{\partial z}{\partial x}=-F_x/F_z$, $\frac{\partial z}{\partial y}=-F_y/F_z$ presented in the lecture. They are valid locally near points (x,y,z) where $F_z(x,y,z)=y-2z\neq 0$. (The "bad" points form a plane in this case.)

$$\begin{aligned}\frac{\partial z}{\partial x}(x,y)&=-\frac{F_x(x,y,z)}{F_z(x,y,z)}=-\frac{\ln y}{y-2z},\\ \frac{\partial z}{\partial y}(x,y)&=-\frac{F_y(x,y,z)}{F_z(x,y,z)}=-\frac{z+x/y}{y-2z}=-\frac{x+yz}{y^2-2yz},\qquad z=z(x,y).\end{aligned}$$

Ex. 44

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}I}{\mathrm{d}t}R+I\frac{\mathrm{d}R}{\mathrm{d}t}\Longrightarrow\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{-0.01\,\mathrm{Vs}^{-1}-(0.08\,\mathrm{A})(0.03\,\Omega\mathrm{s}^{-1})}{400\,\Omega}=-\frac{0.0124}{400}\mathrm{As}^{-1}=31\,\mu\mathrm{As}^{-1}.$$

46 For $u(x,t)=f(x+at)$ we have $u_x(x,t)=f_x(x+at)$, $u_{xx}(x,t)=f_{xx}(x+at)$, $u_t(x,t)=f_t(x+at)a$, $u_{tt}(x,t)=\frac{\partial}{\partial t}[f_t(x+at)a]=f_{tt}(x+at)a^2=a^2u_{xx}(x,t)$. Thus $u(x,t)$ solves the wave equation. For $v(x,t)=f(x-at)$ the same is true (with virtually the same proof).
Since sums of solutions of the wave equation are again solutions (cf. Worksheet 8, Exercise W32), the results in Exercises 77 (c), (d) follow. In (a) we can use the formula $\sin x\sin y=\frac{1}{2}\big(\cos(x-y)-\cos(x+y)\big)$ to write

$$\sin(kx)\sin(akt)=\frac{1}{2}\left[\big(\cos\big(k(x-at)\big)-\cos\big(k(x+at)\big)\right]$$

as a sum of such functions, and in (b) we can use partial fractions to write

$$\frac{t}{a^2t^2-x^2}=\frac{1}{2a}\left(\frac{1}{at-x}+\frac{1}{at+x}\right)$$

as a sum of such functions. The settles the remaining parts of Exercise 77.

47 a) **Ex. 78**

(a) $u = u(x,y) = x^2 + y^2$, $u_x = 2x$, $u_{xx} = 2$, $u_y = 2y$, $u_{yy} = 2$. Since $u_{xx} + u_{yy} = 4 \neq 0$, the function does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$, $u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$, $u_{yy} = -2$. Since $u_{xx} + u_{yy} = 0$, the function is a solution to Laplace's Equation.

(c) $u = x^3 + 3xy^2$, $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$, $u_y = 6xy$, $u_{yy} = 6x$. Since $u_{xx} + u_{yy} = 12x \neq 0$, the function is not a solution to Laplace's Equation.

(d) $u = \ln \sqrt{x^2 + y^2}$, $u_x = \frac{x}{x^2+y^2}$, $u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $u_y = \frac{y}{x^2+y^2}$, $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$. Since $u_{xx} + u_{yy} = 0$, the function is a solution to Laplace's Equation.

(e) $u = \sin x \cosh y + \cos x \sinh y$, $u_x = \cos x \cosh y - \sin x \sinh y$, $u_{xx} = -\sin x \cosh y - \cos x \sinh y$, $u_y = \sin x \sinh y + \cos x \cosh y$, $u_{yy} = \sin x \cosh y + \cos x \sinh y$. Since $u_{xx} + u_{yy} = 0$, the function is a solution to Laplace's Equation.

(f) $u = \exp -x \cos y - \exp -y \cos x$, $u_x = -\exp -x \cos y + \exp -y \sin x$, $u_{xx} = \exp -x \cos y + \exp -y \cos x$, $u_y = -\exp -x \sin y + \exp -y \cos x$, $u_{yy} = -\exp -x \cos y - \exp -y \cos x$. Since $u_{xx} + u_{yy} = 0$, the function is a solution to Laplace's Equation.

Remark: In b), d), e), f) one can also find a holomorphic function on $\mathbb{C}$ (or, in d), an appropriate subset of $\mathbb{C}$) whose real or imaginary part is the given function and quote the result from the lecture that such functions are harmonic: $x^2 - y^2 = \mathrm{Re}(z^2)$; $\ln \sqrt{x^2+y^2} = \mathrm{Re}(\log z)$ (the complex logarithm); the summands in e), f) are itself harmonic and corresponding complex functions are in e) $\sin x \cosh y = \mathrm{Re}(\sin z)$, $\cos x \sinh y = \mathrm{Im}(\sin z)$ (to see this, expand $\sin z = \sin(x + y\mathrm{i})$ using the addition theorems) and in f) $\mathrm{e}^{-x} \cos y = \mathrm{Re}(\mathrm{e}^{-z})$, $\mathrm{e}^{-y} \cos x = \mathrm{Re}(\mathrm{e}^{\mathrm{i}z})$.

b) (i) The Laplace operator is linear, i.e., it satisfies $\Delta(u+v) = \Delta u + \Delta v$ and $\Delta(au) = a\,\Delta(u)$ for functions $u, v \colon D \to \mathbb{R}$ and $a \in \mathbb{R}$. This implies that sums and scalar multiples of solutions of $\Delta u = 0$ are again solutions, and generalizes via induction to linear combinations.

(ii) $\Delta u = 0 \implies \Delta u_x = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx} = (u_{xx} + u_{yy})_x = 0$; similarly, $\Delta u_y = 0$.

Remarks: Part (ii) explains in a conceptual way that $\Delta(\mathrm{e}^x \cos y) = 0$ implies $\Delta(\mathrm{e}^x \sin y) = 0$. In the lecture I had said that $\Delta(\mathrm{e}^x \sin y) = 0$ can be proved in the same way as $\Delta(\mathrm{e}^x \cos y) = 0$.

In Part (ii) we have tacitly assumed that every harmonic function is C^3, which is true. In fact every harmonic function is C^∞, as shown in the theory of harmonic functions as a consequence of the so-called *mean value property*.

Properties (i), (ii) generalize to harmonic functions in n variables. For simplicity we have considered only 2-variable functions.

c) Using the chain rule, we obtain (suppressing the obvious arguments)

$$
\begin{aligned}
v_x(x,y) &= a\,u_x(ax+by, cx+dy) + c\,u_y(ax+by, cx+dy),\\
v_y(x,y) &= b\,u_x(ax+by, cx+dy) + d\,u_y(ax+by, cx+dy),\\
v_{xx}(x,y) &= a^2\,u_{xx}(\dots) + ac\,u_{xy}(\dots) + ca\,u_{yx}(\dots) + c^2\,u_{yy}(\dots),\\
v_{yy}(x,y) &= b^2\,u_{xx}(\dots) + bd\,u_{xy}(\dots) + db\,u_{yx}(\dots) + d^2\,u_{yy}(\dots).
\end{aligned}
$$

Together with Clairaut's Theorem this gives

$$\Delta v(x,y) = (a^2+b^2)u_{xx}(\dots) + 2(ac+bd)u_{xy}(\dots) + (c^2+d^2)u_{yy}(\dots).$$

If $a^2+b^2 = c^2+d^2$ and $ab+cd = 0$, i.e., the rows of $\mathbf{A}$ are orthogonal and have the same length, we get $\Delta v(x,y) = (a^2+b^2)\Delta u(ax+by, cx+dy)$, and hence that $\Delta u \equiv 0$ implies $\Delta v \equiv 0$.

Remarks: The conditions are equivalent to $\mathbf{A}$ being a scaled orthogonal matrix, i.e., of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ (rotation matrix scaled by $\sqrt{a^2+b^2}$) or $\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$ (reflection matrix scaled by $\sqrt{a^2+b^2}$).

If you have difficulties to understand this, use that $v = u \circ L$ with $L(x,y) = \mathbf{A}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Since L is linear, the chain rule gives $(v_x, v_y) = \nabla v = (\nabla u)\mathbf{A} = (u_x, u_y)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, i.e., $v_x = a\,u_x + c\,u_y$, $v_y = b\,u_x + d\,y_y$.

48 Ex. 79

$$\begin{aligned}
u_x(x,y,z) &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}, \\
u_{xx}(x,y,z) &= -(x^2+y^2+z^2)^{-3/2} - x(-\tfrac{3}{2})(x^2+y^2+z^2)^{-5/2}(2x) \\
&= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.
\end{aligned}$$

By symmetry, it follows that

$$\begin{aligned}
u_{yy}(x,y,z) &= \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}}, \\
u_{zz}(x,y,z) &= \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}},
\end{aligned}$$

and hence that $u_{xx}+u_{yy}+u_{zz} = 0$.

49 Ex. 55 Setting $g(r,\theta) = f(r\cos\theta, r\sin\theta)$, we have $g = f \circ P$ with $P(r,\theta) = (r\cos\theta, r\sin\theta)$, the polar coordinate map. The chain rule gives

$$\begin{aligned}
\mathbf{J}_g(r,\theta) &= \mathbf{J}_f(r\cos\theta, r\sin\theta)\mathbf{J}_P(r,\theta) \\
&= \big(f_x(r\cos\theta, r\sin\theta), f_y(r\cos\theta, r\sin\theta)\big)\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\
&= \big(f_x\cos\theta + f_y\sin\theta, f_x(-r\sin\theta) + f_y(r\cos\theta)\big),
\end{aligned}$$

where the arguments of f_x, f_y have been omitted. Thus we have

$$\begin{aligned}
g_r &= f_x\cos\theta + f_y\sin\theta, \\
g_\theta &= f_x(-r\sin\theta) + f_y(r\cos\theta).
\end{aligned}$$

This can also be proved by applying the Chain Rule as stated in [Ste21] (Case 2).

Iterating and using $f_x \triangleq f_x(r\cos\theta, r\sin\theta) = (f_x \circ P)(r,\theta)$, $f_y \triangleq (f_y \circ P)(r,\theta)$, we obtain

$$\begin{aligned}
g_{rr} &= (f_{xx}\cos\theta + f_{yx}\sin\theta)\cos\theta + (f_{yx}\cos\theta + f_{yy}\sin\theta)\sin\theta \\
&= f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta, \\
g_{\theta\theta} &= \big(f_{xx}(-r\sin\theta) + f_{yx}(r\cos\theta)\big)(-r\sin\theta) + f_x(-r\cos\theta) \\
&\quad + \big(f_{yx}(-r\sin\theta) + f_{yy}(r\cos\theta)\big)(r\cos\theta) + f_y(-r\sin\theta) \\
&= f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\cos\theta\sin\theta + f_{yy}r^2\cos^2\theta - f_x r\cos\theta - f_y r\sin\theta
\end{aligned}$$

From this it follows that

$$\begin{aligned}
g_{rr} + \frac{1}{r^2}g_{\theta\theta} &= f_{xx} + f_{yy} - \frac{f_x\cos\theta + f_y\sin\theta}{r} \\
&= f_{xx} + f_{yy} - \frac{1}{r}g_r.
\end{aligned}$$

Thus we have $f_{xx} + f_{yy} = g_{rr} + \frac{1}{r^2}g_{\theta\theta} + \frac{1}{r}g_r$, as claimed in the exercise.

50 a) Reasoning as in W37 shows that the equation

$$|tP + (1-t)N|^2 = |(tx_1, \dots, tx_n, (1-t)|^2 = 1$$

has the solutions $t = 0$ and $t = \frac{2}{|\mathbf{x}|^2+1}$. Thus

$$\sigma(\mathbf{x}) = \sigma(x_1, \dots, x_n) = \frac{1}{x_1^2 + \dots + x_n^2 + 1}\begin{pmatrix} 2x_1 \\ \vdots \\ 2x_n \\ x_1^2 + \dots + x_n^2 - 1 \end{pmatrix}$$

in general.

For $n = 1$ we have

$$\sigma(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right).$$

b) If (a,b,c) is a Pythagorean triple then $(x,y) = \left(\frac{a}{c}, \frac{b}{c}\right)$ satisfies $x,y > 0$ and $x^2 + y^2 = 1$. If $\gcd(a,b,c) = 1$ then a/c and b/c are in lowest terms (a common divisor $m > 1$ of a, c would also divide b in view of $a^2 + b^2 = c^2$, and similarly for a common divisor $m > 1$ of b, c) and hence (a,b,c) is uniquely determined by (x,y). Now suppose that $\left(\frac{a}{c}, \frac{b}{d}\right)$ is any rational point on S^1 with $a,b,c,d > 0$ and $\gcd(a,c) = \gcd(b,d) = 1$. Then $(a/c)^2 + (b/d)^2 = 1$ implies $a^2d^2 + b^2c^2 = c^2d^2$. From this we can conclude $c^2 \mid d^2$, since $\gcd(c^2, a^2) = 1$. Similarly, we have $d^2 \mid c^2$ and hence $c^2 = d^2$, $c = d$. In all this shows that the correspondence $(a,b,c) \to \left(\frac{a}{c}, \frac{b}{c}\right)$ between primitive Pythagorean triples and rational points (x,y) on the unit circle S^1 with $x,y > 0$ is one-to-one.

If (a,b,c) is a Pythagorean triple and $m = \gcd(a,b,c) > 1$, then $a = ma'$, $b = mb'$, $c = mc'$ and $a'^2 + b'^2 = c'^2$. Moreover, the triple (a',b',c') is primitive. This shows that all Pythagorean triples are obtained by enumerating the primitive Pythagorean triples and taking for each primitive triple (a',b',c') all triples (ma',mb',mc'), $m = 1,2,\dots$

It remains to determine the primitive Pythagorean triples. Since stereographic projection for $n=1$,

$$\sigma(t)=\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right)=(x,y),$$

maps rational numbers to rational points on the unit cirle, and vice versa except for $(0,1)$ (the latter from $t=x/(1-y)$), the rational points (x,y) on S^1 with $x,y>0$ are exactly of the form

$$\sigma(m/n)=\left(\frac{2m/n}{(m/n)^2+1},\frac{(m/n)^2-1}{(m/n)^2+1}\right)=\left(\frac{2mn}{m^2+n^2},\frac{m^2-n^2}{m^2+n^2}\right)$$

with relatively prime integers $m>n>0$. It follows that the primitive Pythagorean triples (a,b,c) are enumerated by

$$(a,b,c)=(2mn,m^2-n^2,m^2+n^2),\quad m>n>0,\ \gcd(m,n)=1,\ m\not\equiv n \pmod 2.$$

The first few such triples are

(m,n)	$(2,1)$	$(3,2)$	$(4,1)$	$(4,3)$
(a,b,c)	$(4,3,5)$	$(12,5,13)$	$(8,15,17)$	$(24,7,25)$

We have met a non-primitive Pythagorean triple in H38, Ex. 34 (c), viz. $(8,6,10)$.

Calculus III (Math 241)

W25 Do Exercises 23, 34, 36 in [Ste21], Ch. 14.7.

W26 Do Exercise 5 in [Ste21], Ch. 14.8.

W27 Using Lagrange Multipliers, do Exercise 50 in [Ste21], Ch. 14.7.

Solutions

25 Ex. 23 We have

$$\nabla f(x,y) = (2x - 4y, 8y - 4x) = (0,0)$$

iff $y = x/2$. Thus the critical points of f are those on the line $y = x/2$. Further, we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix},$$

which has rank 1 at all points (not only the critical points). Since $D = \det \mathbf{H}_f(x,y)$ in the notation of [Ste16], we have in particular $D = 0$ at all critical points. Thus the Hesse matrix doesn't help in deciding the nature of the critical points.

However, it is very easy to decide this directly. Since

$$f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x-2y)^2 + 2 \geq 2 \quad \text{with equality iff } x = 2y,$$

the points on the line $y = x/2$ are (the) global minima of f.

Ex. 34 $f(x,y) = x + y - xy$,

1. First find the critical points in D: $f_x = 1 - y$, $f_y = 1 - x$, so the only critical point is $(1,1)$. We have $(1,1) \in D$, and $f(1,1) = 1$.

2. Then compute extrema on the boundary.

   (a) $f(0,y) = y$, $y \in [0,2]$. $f_{\max} = f(0,2) = 2$; $f_{\min} = f(0,0) = 0$.

   (b) $f(x,0) = x$, $x \in [0,4]$. $f_{\max} = f(4,0) = 4$; $f_{\min} = f(0,0) = 0$.

   (c) $f(x, 2 - \frac{1}{2}x) = \frac{1}{2}(x - \frac{3}{2})^2 + \frac{7}{8}$, $x \in [0,4]$. $f_{\max} = f(4,0) = 4$; $f_{\min} = f(\frac{3}{2}, \frac{5}{4}) = \frac{7}{8}$.

Comparing the results, we find that the maximum value of f on D is 4, and the minimum value is 0.

Ex. 36 $f(x,y) = x^2 + xy + y^2 - 6y$, $D = [-3,3] \times [0,5]$

1. $f_x = 2x + y$, $f_y = x + 2y - 6$; the unique solution of $2x + y = 0 \wedge x + 2y = 6$ is $x = -2$, $y = 4$. This gives one critical point $(-2,4)$, which is in D and has $f(-2,4) = -12$.

2. On the boundary we have

   (a) $f(-3,y) = y^2 - 9y + 9$, $y \in [0,5]$; $f_{\max} = f(-3,0) = 9$, $f_{\min} = f(-3,9/2) = -45/4 = -11.25$;

   (b) $f(3,y) = y^2 - 3y + 9$, $y \in [0,5]$; $f_{\max} = f(3,5) = 19$, $f_{\min} = f(3,3/2) = 27/4 = 6.75$;

   (c) $f(x,0) = x^2$, $x \in [-3,3]$; $f_{\min} = f(0,0) = 0$, $f_{\max} = f(\pm 3, 0) = 9$;

   (d) $f(x,5) = x^2 + 5x - 5$, $x \in [-3,3]$; $f_{\max} = f(3,5) = 19$, $f_{\max} = f(-5/2,5) = -45/4$.

Comparing the values, we find that the maximum value of f on D is 19, and the minimum value is -12.

26 Ex. 5 $f(x,y) = xy$, $g(x,y) = 4x^2 + y^2 - 8$;
$\nabla g(x,y) = (8x, 2y) \neq (0,0)$ for all $(x,y) \in \mathbb{R}^2$ on the ellipse $4x^2 + y^2 = 8$. Hence the extrema of f on the ellipse must satisfy $\nabla f(x,y) = (y,x) = \lambda(8x, 2y)$ for some $\lambda \in \mathbb{R}$. This gives the system $y = 8\lambda x \wedge x = 2\lambda y \wedge 4x^2 + y^2 = 8$. Substituting the 2nd equation into the first gives $y = 16\lambda^2 y$. Since $y = 0$ does not yield a solution, it follows that $16\lambda^2 = 1$, i.e., $\lambda = \pm\frac{1}{4}$, $y = \pm 2x$, which leads to the 4 solutions $(x,y) = (\pm 1, \pm 2)$. $\Longrightarrow$ f has the two maxima $f(1,2) = f(-1,-2) = 2$ and the two minima $f(1,-2) = f(-1,2) = -2$.

That f attains a maximum and a minimum on the ellipse in the first place, follows from continuity of f and compactness of an ellipse.

A way to see this without Lagrange multipliers is to use the inequality $8 = 4x^2 + y^2 = (2x-y)^2 + 4xy \geq 4xy$ on the ellipse, which shows $xy \leq 2$ with equality iff $y = 2x$, and the inequality $8 = 4x^2 + y^2 = (2x+y)^2 - 4xy \geq -4xy$ on the ellipse, which shows $xy \geq -2$ with equality iff $y = -2x$. Substituting the equations $y = \pm 2x$ into the equation for the ellipse gives the points determined above.

27 Ex. 50 This is the optimization problem "minimize $2xy + 2xz + 2yz$ subject to $xyz = 1000$" with domain $D = (\mathbb{R}^3)^+ = \big\{(x,y,z) \in \mathbb{R}^3; x > 0, y > 0, z > 0\big\}$ and feasible region $S = \big\{(x,y,z) \in D; xyz = 1000\big\}$. Writing $f(x,y,z) = 2xy + 2xz + 2yz$, $g(x,y,z) = xyz$, we have $\nabla f = (2y+2z, 2x+2z, 2y+2x)$, $\nabla g = (yz, xz, zy) \neq (0,0,0)$ for all $(x,y,z) \in D$, and hence a necessary condition for a minimum in $(x,y,z) \in D$ is the Lagrange multiplier condition $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some $\lambda \in \mathbb{R}$. This gives the system of equations

$$\begin{aligned} 2y + 2z &= \lambda yz, \\ 2x + 2z &= \lambda xz, \\ 2x + 2y &= \lambda xy, \\ xyz &= 1000. \end{aligned}$$

Multiplying the 1st equation by x, the 2nd equation by y and subtracting gives $2z(x-y) = 0$, and hence $x = y$ since $z \neq 0$. By symmetry, we then obtain $x = y = z$, and from $xyz = 1000$ further $x = y = z = 10$ (and $\lambda = 2/5$), so that the optimal box is a cube of surface area $f(10,10,10) = 600$.

However, it remains to show that the optimization problem has a solution, because otherwise $(10,10,10)$ could just be a local extremum or no extremum at all. If one variable, say x, is < 1, we must have $yz > 1000$, and hence $f(x,y,z) > 2000$. If one variable, say again x, is > 1000 then one of y, z must be < 1, and hence again $f(x,y,z) > 2000$. This shows that the minimum of f on the closed and bounded set $\big\{(x,y,z) \in S; 1 \leq x,y,z \leq 1000\big\}$, which exists since f is continuous and is ≤ 600, is a minimum on the whole feasible region S.

Calculus III (Math 241)

H51 Do Exercises 24, 38, 40 in [Ste21], Ch. 14.7.

H52 Do Exercise 56 in [Ste21], Ch. 14.7.

H53 Use Lagrange multipliers to find the extreme values of the function subject to the given constraint (two answers suffice):

a) $f(x,y,z) = e^{xyz}, \quad 2x^2 + y^2 + z^2 = 24$;
b) $f(x,y,z) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1), \quad x^2+y^2+z^2 = 12$;
c) $f(x,y,z) = x^4+y^4+z^4; \quad x^2+y^2+z^2 = 1$;
d) $f(x_1,x_2,\dots,x_n) = x_1+x_2+\dots+x_n; \; x_1^2+x_2^2+\dots+x_n^2 = 1$.

H54 Do Exercise 58 (and optionally Exercises 56, 60) in [Ste21], Ch. 14.8.

H55 This exercise fills the gap in our derivation of the criterion for a local minimum/maximum of a 2-variable function.

a) Show that in the case $A = f_{xx}(x_0,y_0) = 0$ the Hesse quadratic form $q(h_1,h_2) = (h_1,h_2)\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)(h_1,h_2)^\mathsf{T} = Ah_1^2 + 2Bh_1h_2 + Ch_2^2$ of f at (x_0,y_0) either has rank ≤ 1 (the *rank* of a quadratic form being that of the associated symmetric matrix) or can be transformed into ${h'_1}^2 - {h'_2}^2$ using a linear change of variables.

b) Consider a linear map $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{h} \mapsto \mathbf{A}\mathbf{h} = \mathbf{h}'$. Assuming that $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is invertible, show that there exist constants $C_1, C_2 > 0$ such that

$$C_1\,|\mathbf{h}| \leq |\mathbf{A}\mathbf{h}| \leq C_2\,|\mathbf{h}| \quad \text{for all } \mathbf{h} \in \mathbb{R}^2.$$

(This property holds, more generally, for invertible matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$.)

c) Using a), show that any function $g(\mathbf{h}) = g(h_1,h_2)$ that is $\mathrm{o}(|\mathbf{h}|^2)$ for $\mathbf{h} \to \mathbf{0}$ is also $\mathrm{o}(|\mathbf{h}'|^2)$, and conversely.

H56 *Optional Exercise*

The *Fundamental Theorem of Algebra (FTA)* asserts that every polynomial $p(X) = p_0 + p_1X + p_2X^2 + \dots + p_dX^d \in \mathbb{C}[X]$ of degree $d \geq 1$ (i.e., $p_d \neq 0$) has a root in $\mathbb{C}$. This exercise guides you through of proof of FTA by contradiction.

a) Using a similar argument as in the solution of Optimization Problem 4 in `lecture24-27_handout.pdf`, show that $\mathbb{C} \to \mathbb{R}$, $z \mapsto |p(z)|$ attains a minimum.

b) Assuming that FTA is false, show that there exists a polynomial $q(X) \in \mathbb{C}[X]$ of the form

$$q(X) = 1 + q_m X^m + q_{m+1} X^{m+1} + \cdots + q_d X^d, \quad 1 \le m \le d,\ q_m q_d \neq 0,$$

which satisfies $|q(z)| \geq 1$ for all $z \in \mathbb{C}$.

c) Let $z_0 \in \mathbb{C}$ satisfy $q_m z_0^m = -1$. (Why does such a complex number z_0 exist?) Show that in

$$q(rz_0) = 1 - r^m + q_{m+1}(rz_0)^{m+1} + \cdots + q_d(rz_0)^d = 1 - r^m + R(r), \quad r > 0,$$

the remainder $R(r)$ satisfies $|R(r)| < r^m$ for small positive numbers r.

d) Derive a contradiction from b) and c).

Due on Wed Dec 6, 6 pm

The optional exercises can be handed in until Wed Dec 13, 6 pm. Parts b) and c) of H55 are also considered as optional, but should be handed in together with Part a) on Dec 6.

Solutions

51 Ex. 24 We have

$$\begin{aligned}
f_x(x,y) &= 2xy\,\mathrm{e}^{-x^2-y^2} + x^2y\,\mathrm{e}^{-x^2-y^2}(-2x) = 2xy(1-x^2)\mathrm{e}^{-x^2-y^2},\\
f_y(x,y) &= x^2\,\mathrm{e}^{-x^2-y^2} + x^2y\,\mathrm{e}^{-x^2-y^2}(-2y) = x^2(1-2y^2)\mathrm{e}^{-x^2-y^2},\\
\nabla f(x,y) = 0 &\iff 2xy(1-x^2) = x^2(1-2y^2) = 0.
\end{aligned}$$

$\Longrightarrow$ The critical points of f are the points on the y-axis ($x=0$) and the four points $\left(\pm 1, \pm 1/\sqrt{2}\right)$ (the solutions of $1-x^2 = 1-2y^2-0$).
Setting $r = \sqrt{x^2+y^2}$, we have $|f(x,y)| \leq r^3\,\mathrm{e}^{-r^2} \to 0$ for $r\to\infty$. This shows $\lim_{|(x,y)|\to\infty} f(x,y) = 0$, and hence that f, being continuous, attains a global maximum and minimum on $\mathbb{R}^2$; cf. the discussion of Problem 4 of the lecture slides on optimization. Since such points must be critical points of f, and $f(x,0)=0$, $f\left(\pm 1, 1/\sqrt{2}\right) > 0$, $f\left(\pm 1, -1/\sqrt{2}\right) < 0$, the global maxima must be at $\left(\pm 1, 1/\sqrt{2}\right)$ and the global minima at $\left(\pm 1, -1/\sqrt{2}\right)$.
Further we have

$$\begin{aligned}
\mathbf{H}_f(x,y) &= \mathrm{e}^{-x^2-y^2}\begin{pmatrix} 4x^4y - 10x^2y + 2y & 4x^3y^2 - 2x^3 - 4xy^2 + 2x \\ 4x^3y^2 - 2x^3 - 4xy^2 + 2x & 4x^2y^3 - 6x^2y \end{pmatrix},\\
\mathbf{H}_f(0,y) &= \mathrm{e}^{-y^2}\begin{pmatrix} 2y & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Since for points on the y-axis the Hesse matrix has rank 1, it can't be used to decide on the nature of these points.
However, since f vanishes on the coordinate axes, $f \geq 0$ in the upper half plane $y>0$, and $f\leq 0$ in the lower half plane $y<0$, it is clear that f has a non-strict local minimum in all points $(0,y)$ with $y>0$, a non-strict local maximum in all points $(0,y)$ with $y<0$, and no local extremum in $(0,0)$. The origin is not a saddle point, since the restriction of f to any line $y = mx$, $m>0$, has no local extremum in $(0,0)$. Alternatively, this follows from $\det \mathbf{H}_f(0,0) = 0$ (according to the definition of "saddle point" in the lecture), or $f(x,mx) = mx^3\,\mathrm{e}^{-(1+m^2)x^2}$, $m>0$, which changes sign at $x=0$.
Ex. 38 $f(x,y) = 4x+6y-x^2-y^2$, $D = [0,4]\times[0,5]$

1. First find the critical points in D: $f_x = 4-2x$, $f_y = 6-2y$, so the only critical point is $(2,3)$, with value $f(2,3) = 13$.

2. Then compute extrema on the boundary.

   (a) $f(x,0) = 4x-x^2$, which has range $[0,4]$ when restricted to $[0,4]$.

   (b) $f(0,y) = 6y-y^2$, which has range $[0,9]$ when restricted to $[0,5]$.

   (c) $f(x,5) = 4x+5-x^2$, which has range $[5,9]$ when restricted to $[0,4]$.

   (d) $f(4,y) = 6y-y^2$, which has range $[0,9]$ when restricted to $[0,5]$.

It follows that $f(2,3) = 13$ is the unique maximum of f, and the minima of f are $f(0,0)=0$, $f(4,0)=0$.
Ex. 40 $f(x,y) = x^3-3x-y^3+12y$

1. First find critical points in D: $f_x = 3x^2 - 3$, $f_y = -3y^2 + 12$, so the critical points of f are $(\pm 1, \pm 2)$, of which $(1,2)$ and $(-1,2)$ are in D, and $f(1,2) = 14$, $f(-1,2) = 18$.

2. Then compute extrema on the boundary.

   (a) $f(-2,y) = -y^3 + 12y - 2$, $y \in [-2,3]$, $f_{\max} = f(-2,2) = 14$; $f_{\min} = f(-2,-2) = -18$.

   (b) $f(2,y) = -y^3 + 12y + 2 = f(-2,y) + 4$, $y \in [2,3]$, $f_{\max} = f(2,2) = 18$, $f_{\min} = f(2,3) = 11$.

   (c) $f(x,3) = x^3 - 3x + 9$, $x \in [-2,2]$. $f_{\max} = f(-1,3) = f(2,3) = 11$, $f_{\min} = f(-2,3) = f(1,3) = 7$.

   (d) $f(x,x) = 9x$, $x \in [-2,2]$. $f_{\max} = f(2,2) = 18$, $f_{\min} = f(-2,-2) = -18$.

Comparing the results, we find that the maximum value of f on D is 18, and the minimum value is -18.

52 Ex. 56

(a) Let x be the length (in m) of north and south walls, y the length of east and west walls, and z the height of the building. The loss of heat is then given by

$$f(x,y,z) = 2yz * 10 + 2xz * 8 + xy + xy * 5 = 20yz + 16xz + 6xy$$

Since the volume $V = xyz = 4000$, the heat loss function can be rewritten as

$$f(x,y) = 6xy + \frac{80000}{x} + \frac{64000}{y},$$

whose domain is given by $(x,y) | x \geq 30, y \geq 30, xy \leq 1000$.

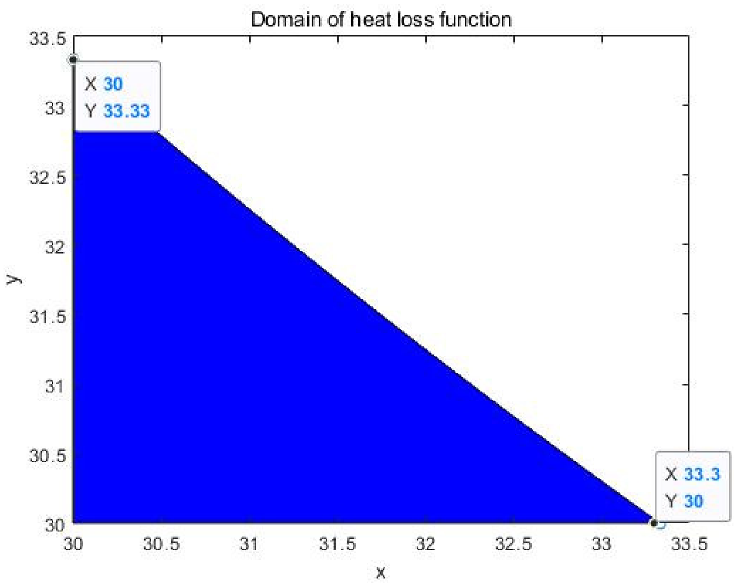


Figure 1: Domain of the heat loss function $f(x,y)$

(b) $f_x = 6y - \frac{80000}{x^2}$, $f_y = 6x - \frac{64000}{y^2}$. The only critical point of f is $\left(\sqrt[3]{\frac{50000}{3}}, \frac{4}{5}\sqrt[3]{\frac{50000}{3}}\right) \approx (25.54, 20.43)$, which is out of the domain.
For the boundary:

(i) $f(30,y) = 180y + \frac{64000}{y} + \frac{8000}{3}$ for $30 \le y \le 33\frac{1}{3}$; since $y \mapsto 180y + \frac{64000}{y}$ has its unique minimum at $y_0 = \sqrt{\frac{64000}{180}} = \frac{80}{3\sqrt{2}} < 30$, the minimum on this part of the boundary is $f(30,30) = 10200$.

(ii) $f(x,30) = 180x + \frac{80000}{x} + \frac{6400}{3}$ for $30 \le x \le 33\frac{1}{3}$; the minimum of $x \mapsto 180x + \frac{80000}{x}$ is at $x_0 = \sqrt{\frac{80000}{180}} = \frac{20}{3}\sqrt{10} < 30$, so the minimum on this part of the boundary is again $f(30,30) = 10200$.

(iii) $f(x, 1000/x) = 64x + \frac{80000}{x} + 6000$ for $30 \le x \le 33\frac{1}{3}$; the minimum of $x \mapsto 64x + \frac{80000}{x} + 6000$ is at $x_0 = \sqrt{\frac{80000}{64}} = 25\sqrt{2} > 33\frac{1}{3}$, so the minimum on this part of the boundary is $f(33\frac{1}{3}, 30) = \frac{31600}{3} \approx 10533$.

Comparing the results, we find the minimum heat loss to be 10200, achieved at $x = y = 30\,[\mathrm{m}]$, and $z = \frac{40}{9} \approx 4.44\,[\mathrm{m}]$.

(c) Consider the critical point in (b), $(x,y) = \left(\sqrt[3]{\frac{50000}{3}}, \frac{4}{5}\sqrt[3]{\frac{50000}{3}}\right)$.
$f_{xx} = \frac{160000}{x^3}$,$f_{yy} = \frac{128000}{y^3}$, and $f_{xy} = 6 \Longrightarrow D = f_{xx}f_{yy} - f_{xy}^2 = 108 > 0$, and $f_{xx} > 0$. Hence the heat loss function has the local minimum $f\left(\sqrt[3]{\frac{50000}{3}}, \frac{4}{5}\sqrt[3]{\frac{50000}{3}}\right) \approx 9395.7$, which is smaller than the minimum value computed in (b). Therefore, when $x \approx 25.54\,[\mathrm{m}]$, $y \approx 20.43\,[\mathrm{m}]$, and $z \approx 7.66\,[\mathrm{m}]$, heat loss is minimized.

53 In all cases f is continuous, and the surface S defined by the constraint is compact (an ellipsoid in a), and a sphere in b), c), d)). Hence there exists at least one maximum and one minimum.

a) Since the exponential function is strictly increasing, the problem is equivalent to finding the extrema of $f_1(x,y,z) = xyz$ under the constraint $g(x,y,z) = 2x^2 + y^2 + z^2 = 24$. We have $\nabla f_1 = (yz, xz, xy)$, $\nabla g = (4x, 2y, 2z) \neq (0,0,0)$ for all points on the constraint surface. Thus the condition $\nabla f = \lambda \nabla g$ can be applied and yields $yz = 4\lambda x$, $xz = 2\lambda y$, $xy = 2\lambda z$. It follows that $xyz = 4\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$, and hence $(x,y,z) = \left(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$.
$\Longrightarrow f_1$ has minima $f_1\left(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2}\right) = -16$ at the four points with an odd number of minus signs, and maxima $f_1\left(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2}\right) = 16$ at the four points with an even number of minus signs. For f then the same is true with ± 16 replaced by $\mathrm{e}^{\pm 16}$.

b) The Lagrange function is

$$L(x,y,z,\lambda) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1) + \lambda(x^2+y^2+z^2-12),$$

and there are, as before, no exceptional points to consider.

Find the partial derivatives and let them all be 0 :

$$\begin{cases} L_x = \dfrac{2x}{x^2+1} + 2\lambda x = 0 \\ L_y = \dfrac{2y}{y^2+1} + 2\lambda y = 0 \\ L_z = \dfrac{2z}{z^2+1} + 2\lambda z = 0 \\ L_\lambda = x^2 + y^2 + z^2 - 12 = 0 \end{cases}$$

We can see that for any of x, y, z, say x, either $x = 0$ or $x \neq 0 \wedge \lambda = -\frac{1}{x^2+1}$. Since $x = y = z = 0$ is not possible, we have the following cases.

(i) None of x, y, z equals 0. Then $\lambda = \frac{1}{x^2+1} = \frac{1}{y^2+1} = \frac{1}{z^2+1)} \Longrightarrow x^2 = y^2 = z^2 \Longrightarrow x^2 = y^2 = z^2 = 4 \Longrightarrow f(x,y,z) = f(\pm 2, \pm 2, \pm 2) = 3\ln 5 = \ln 125$.

(ii) Exactly one of x, y, z equals 0, say $x = 0$. Then $\lambda = \frac{1}{y^2+1} = \frac{1}{z^2+1} \Longrightarrow y^2 = z^2 = 6 \Longrightarrow f(x,y,z) = f(0, \pm\sqrt{6}, \pm\sqrt{6}) = 2\ln 7 = \ln 49$.

(iii) Exactly two of x, y, z are 0, say $x = y = 0$. Then $z^2 = 12$ and $f(x,y,z) = f(0,0,\pm\sqrt{12}) = \ln 13$

In conclusion, we have $f_{max} = 3\ln 5$ attained at the 8 points in (i), and $f_{min} = \ln 13$ attained at the 2 points in (iii).

c) We have $\nabla f = 4(x^3, y^3, z^3)$, $\nabla g = 2(x,y,z)$. Since ∇g doesn't vanish at any point of the unit sphere in $\mathbb{R}^3$, the condition $\nabla f = \lambda \nabla g$ can be applied and yields $x^3 = \lambda x$, $y^3 = \lambda y$, $z^3 = \lambda z$. Eliminating λ gives $x^3 y = xy^3$, $x^3 z = xz^3$, $y^3 z = yz^3$. The solutions on the unit sphere are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ (two of x, y, z are zero), $\left(\pm 1\sqrt{2}, \pm 1/\sqrt{2}, 0\right)$, $\left(\pm 1\sqrt{2}, 0, \pm 1/\sqrt{2}\right)$, $\left(0, \pm 1\sqrt{2}, \pm 1/\sqrt{2}\right)$ (if only one of x, y, z, say z, is zero then the 1st equation can be divided by xy to yield $x^2 = y^2$), and $\left(\pm 1\sqrt{3}, \pm 1\sqrt{3}, \pm 1/\sqrt{3}\right)$, with all combinations of signs being allowed. Clearly the minima/maxima are then

$$f\left(\pm 1\sqrt{3}, \pm 1\sqrt{3}, \pm 1/\sqrt{3}\right) = 1/3,$$
$$f(\pm 1, 0, 0) = f(0, \pm 1, 0) = f(0, 0, \pm 1) = 1.$$

d) We have $\nabla f = (1, 1, \ldots, 1)$, $\nabla g = 2(x_1, x_2, \ldots, x_n)$. Since ∇g doesn't vanish at any point of the unit sphere in $\mathbb{R}^n$, the condition $\nabla f = \lambda \nabla g$ can be applied and yields $x_1 = x_2 = \cdots = x_n$. Thus $x_1 = x_2 = \cdots = x_n = \pm 1/\sqrt{n}$ (with all of the same sign), and the minimum/maximum is

$$f\left(-\frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}, \ldots, -\frac{1}{\sqrt{n}}\right) = -\sqrt{n},$$
$$f\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) = \sqrt{n}.$$

54 Ex. 56 The problem can be rewritten as "find the maximum and minimum of the function $V(x,y,z) = xyz$ subject to $xy + yz + xz = 750$ and $x + y + z = 50$.

The Lagrange function is

$$L(x,y,z,\lambda_1,\lambda_2) = xyz + \lambda_1(xy+yz+xz-750) + \lambda_2(x+y+z-50).$$

Since

$$\mathbf{J}_g(x,y,z) = \begin{pmatrix} y+z & x+z & x+y \\ 1 & 1 & 1 \end{pmatrix}$$

has rank 2 except for $x=y=z$, but there is no such point satisfying the constraints ($3x = 50 \wedge 3x^2 = 750$ has no solution), there are no exceptional points to consider.
Find the partial derivatives and let them all be 0:

$$\begin{cases} L_x = yz + \lambda_1(y+z) + \lambda_2 = 0 \\ L_y = xz + \lambda_1(x+z) + \lambda_2 = 0 \\ L_z = xy + \lambda_1(x+y) + \lambda_2 = 0 \\ L_{\lambda_1} = xy+yz+xz-750 = 0 \\ L_{\lambda_2} = x+y+z-50 = 0 \end{cases}$$

Subtracting the first equation from the second, we get $z(x-y)+\lambda_1(x-y) = (z+\lambda_1)(x-y) = 0$ and, by symmetry, also $(x+\lambda_1)(y-z) = (y+\lambda_1)(x-z) = 0$.

(i) x,y,z are distinct.
In this case the equations above yield $x+\lambda_1 = y+\lambda_1 = z+\lambda_1 = 0$, which contradicts the assumption.

(ii) Exactly two of x,y,z are equal, say $x=y$.
In this case $L_x = L_y = L_z = 0$ can be solved for λ_1, λ_2. Substituting $x=y$ into the constraints, we get the system

$$\begin{cases} x^2 + 2xz = 750 \\ 2x + z = 50 \end{cases}$$

The solutions are $(x,z) = \left(\frac{50\pm5\sqrt{10}}{3}, \frac{50\mp10\sqrt{10}}{3}\right)$. Since there are only two solutions, the corresponding points (x,x,z) must be the unique maximum and the unique minimum. Computing $V(x,y,z) + xyz$ for the two points gives

$$f_{max} = f\left(\frac{50-5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}, \frac{50+10\sqrt{10}}{3}\right) = \frac{87500+2500\sqrt{10}}{27} \approx 3534,$$

$$f_{min} = f\left(\frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50-10\sqrt{10}}{3}\right) = \frac{87500-2500\sqrt{10}}{27} \approx 2948.$$

The extremal volumes are measured in cm^3.

Ex. 58

(a) See Figure 2

(b) The Lagrange function is

$$L(x,y,z,\lambda_1,\lambda_2) = z + \lambda_1(4x-3y+8z-5) + \lambda_2(x^2+y^2-z^2).$$

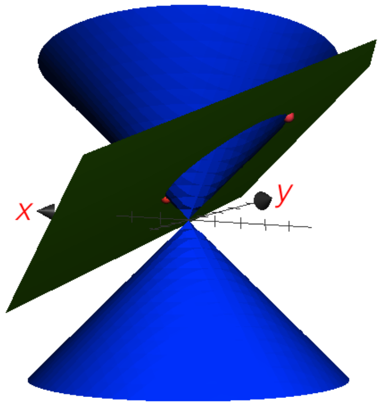


Figure 2: Cone, plane, and ellipse

The Jacobi matrix

$$\mathbf{J}_g(x,y,z) = \begin{pmatrix} 4 & -3 & 8 \\ 2x & 2y & -2z \end{pmatrix}$$

has rank 2 unless $(x,y,z) = c(4,-3,-8)$ for some $c \in \mathbb{R}$. Substituting this into the 2nd constraint gives $16c^2 + 9c^2 - 64c^2 = 0 \Longrightarrow c = 0 \Longrightarrow (x,y,z) = (0,0,0)$. But $(0,0,0)$ does not satisfy the first constraint. Hence there are no exceptional points to consider.

Find the partial derivatives and let them all be 0 :

$$\begin{cases} L_x = 4\lambda_1 + 2\lambda_2 x = 0 \\ L_y = -3\lambda_1 + 2\lambda_2 y = 0 \\ L_z = 1 + 8\lambda_1 - 2\lambda_2 z = 0 \\ L_{\lambda_1} = 4x - 3y + 8z - 5 = 0 \\ L_{\lambda_2} = x^2 + y^2 - z^2 = 0 \end{cases}$$

Solutions of this system must have $\lambda_2 \neq 0$, giving

$$\begin{cases} x = -\dfrac{2\lambda_1}{\lambda_2} \\ y = \dfrac{3\lambda_1}{2\lambda_2} \\ z = \dfrac{8\lambda_1 + 1}{2\lambda_2} \end{cases}$$

Subsituting this into the last two equations gives

$$\begin{cases} -4\dfrac{2\lambda_1}{\lambda_2} - 3\dfrac{3\lambda_1}{2\lambda_2} + 8\dfrac{8\lambda_1 + 1}{2\lambda_2} = 5 \\ (-\dfrac{2\lambda_1}{\lambda_2})^2 + (\dfrac{3\lambda_1}{2\lambda_2})^2 - (\dfrac{8\lambda_1 + 1}{2\lambda_2})^2 = 0 \end{cases}$$

From the second equation we can obtain λ_1 and then from the first equation λ_2. The solutions are $\lambda_1 = -\frac{1}{13}$, $\lambda_2 = \frac{1}{2}$ and $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = -\frac{1}{2}$. In the first case we obtain $x = \frac{4}{13}$, $y = -\frac{3}{13}$, $z = \frac{5}{13}$, and the second case $x = -\frac{4}{3}$, $y = 1$, $z = \frac{5}{3}$. Thus, the highest point on the ellipse is $(-\frac{4}{3}, 1, \frac{5}{3})$, and the lowest point is $(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13})$.

Ex. 60 Writing $g(x,y,z) = (x^2 - y^2 - z, x^2 + z^2 - 4)$, the feasible region of this optimization problem is $S = \{(x,y,z) \in \mathbb{R}^3; g(x,y,z) = (0,0)\}$.

$$\mathbf{J}_g(x,y,z) = \begin{pmatrix} 2x & -2y & -1 \\ 2x & 0 & 2z \end{pmatrix}.$$

First we show that $\mathbf{J}_g(x,y,z)$ has rank 2 on S everywhere. The condition $x^2 + z^2 = 4$ gives that at least one of x,z must be nonzero. Hence the rank is clearly 2 unless $y = 0$. If $y = 0$ then $x^2 = z$ and $\mathbf{J}_g(x,y,z) = \left(\begin{smallmatrix} 2x & 0 & -1 \\ 2x & 0 & 2x^2 \end{smallmatrix}\right)$. This matrix has rank 2 unless $x = 0$ (seen, e.g., by using one step of Gaussian elimination), but there are no points on S satisfying $x = y = 0$. This proves the claim.

Hence any extremum of f on S must satisfy the Lagrange multiplier condition

$$\nabla f(x,y,z) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 2x & -2y & -1 \\ 2x & 0 & 2z \end{pmatrix}$$

Together with the defining equations for S this gives the system

$$\begin{aligned} 2\lambda_1 x + 2\lambda_2 x &= 1, \\ -2\lambda_1 y &= 1, \\ -\lambda_1 + 2\lambda_2 z &= 1, \\ x^2 - y^2 - z &= 0, \\ x^2 + z^2 - 4 &= 0 \end{aligned}$$

Now one can either solve this system directly using a CAS or solve for λ_1, λ_2 first (which is easy, because the corresponding subsystem is linear) and then invoke the CAS.
I've done it using the 2nd method and obtained $\lambda_1 = -\frac{1}{2y}$, $\lambda_2 = \frac{1}{2x} + \frac{1}{2y}$, $x - 2xy + 2xz + 2yz = 0$ (which represents the condition for solvabilty of the linear system for λ_1, λ_2). Solving the system $x^2 - y^2 - z = x^2 + z^2 - 4 = x - 2xy + 2xz + 2yz = 0$ using, e.g., SageMath yields 4 real solutions, viz.

No.	x	y	z	$x+y+z$
1	-1.65287821811	-1.96419410004	-1.1260523322	-4.74312465035
2	1.89517799034	1.71834681828	0.638983878847	4.25250868746
3	-1.5028	0.968872257187	1.31969407266	0.785766329844
4	-0.992512479201	1.64967668315	-1.73635235732	-1.07918815337

The 1st and 2nd give the minimum and maximum of f on S.

55 a) If the rank of $\left(\begin{smallmatrix} 0 & B \\ B & C \end{smallmatrix}\right)$ is 2, we must have $B \neq 0$ and $q(h_1,h_2) = 2Bh_1h_2 + Ch_2^2 = (2Bh_1 + Ch_2)h_2 = h_1'h_2'$ with $h_1' = 2Bh_1 + Ch_2$, $h_2' = h_2$. Since $B \neq 0$, this represents an admissible (i.e., invertible) change of variables. The change of variables $h_1' = h_1'' + h_2''$, $h_2' = h_1'' - h_2''$ turns $h_1'h_2'$ into $h_1''^2 - h_2''^2$, so that the composition $2Bh_1 + Ch_2 = h_1'' + h_2''$, $h_2 = h_1'' - h_2''$, i.e., $h_1 = \frac{1-C}{2B}h_1'' + \frac{1+C}{2B}h_2''$, $h_2 = h_1'' - h_2''$ has the required property.

Remark: That q is indefinite can already be seen from $q(h_1,h_2) = (2Bh_1 + Ch_2)h_2$, because in the case $B \neq 0$ the restriction $q(h_1,1) = 2Bh_1 + C$ attains both positive and negative values.

b) Replacing $\mathbf{h}$ by $\mathbf{A}^{-1}\mathbf{h}$ turns the first inequality into $C_1\left|\mathbf{A}^{-1}\mathbf{h}\right| \leq |\mathbf{h}|$, i.e., $\left|\mathbf{A}^{-1}\mathbf{h}\right| \leq \frac{1}{C_1}|\mathbf{h}|$, which is an instance of the second inequality (for $\mathbf{A}^{-1}$ in place of $\mathbf{A}$). Hence it suffices to prove the second inequality. Writing $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$\begin{aligned}
|\mathbf{A}\mathbf{h}|^2 &= (ah_1+bh_2)^2+(ch_1+dh_2)^2 = (a^2+c^2)h_1^2+(b^2+d^2)h_2^2+2(ab+cd)h_1h_2 \\
&\leq (a^2+c^2)h_1^2+(b^2+d^2)h_2^2+2(|ab|+|cd|)\,|h_1h_2| \\
&\leq C(h_1^2+h_2^2)
\end{aligned}$$

with $C = \max\left\{a^2+c^2, b^2+d^2\right\}+|ab|+|cd|$, since $2\,|h_1h_2| \leq h_1^2+h_2^2$. Thus $|\mathbf{A}\mathbf{h}| \leq C_2\,|\mathbf{h}|$ holds for all $\mathbf{h} \in \mathbb{R}^2$ if set $C_2 = \sqrt{C}$.

c) From b) we have $C_1^2 \leq |\mathbf{h}'|^2 / |\mathbf{h}|^2 \leq C_2^2$ for all $\mathbf{h} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

$$\implies \quad \frac{|g(\mathbf{h})|}{|\mathbf{h}'|^2} \leq \frac{1}{C_1^2}\frac{|g(\mathbf{h})|}{|\mathbf{h}|^2} \quad \text{and} \quad \frac{|g(\mathbf{h})|}{|\mathbf{h}|^2} \leq C_2^2\frac{|g(\mathbf{h})|}{|\mathbf{h}'|^2} \quad \text{for } \mathbf{h} \neq \mathbf{0},$$

from which the equivalence

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|g(\mathbf{h})|}{|\mathbf{h}|^2} = 0 \iff \lim_{\mathbf{h}\to\mathbf{0}} \frac{|g(\mathbf{h})|}{|\mathbf{h}'|^2} = 0$$

follows.

56 a) First we show that $\lim_{|z|\to\infty}|p(z)| = +\infty$. For $z \neq 0$ we have

$$p(z) = z^d\left(p_d+\frac{p_{d-1}}{z}+\cdots+\frac{p_0}{z^d}\right).$$

Since the 2nd factor, call it $f(z)$, tends to $p_d \neq 0$ for $|z| \to \infty$, there exists $R > 0$ such $|f(z)| > \frac{1}{2}|p_d|$ for $|z| > R$. Hence $|p(z)| > \frac{1}{2}|p_d|\,|z|^d$ for $|z| > R$, which together with $d \geq 1$ clearly shows that $\lim_{|z|\to\infty}|p(z)| = +\infty$

The rest of the argument is virtually the same as in Optimization Problem 4: Since $\lim_{|z|\to\infty}|p(z)| = +\infty$, there exists $R > 0$ such that $|p(z)| > |p(0)|$ for $|z| > R$. This implies that the minimum of $z \mapsto |p(z)|$ on the disk $|z| \leq R$, which exists in view of the continuity of $z \mapsto |p(z)|$ and must be $\leq |p(0)|$, is a global minimum (i.e., a minimum on $\mathbb{C}$).

b) Suppose $p(X)$, of degree ≥ 1, has no root in $\mathbb{C}$. By a), $z \mapsto |p(z)|$ attains a minimum, say in z_0. Then $p(z_0) \neq 0$, and the polynomial $q(X) := \frac{1}{p(z_0)}\,p(X+z_0)$, which satisfies $q(0) = 1$, is such that $z \mapsto |q(z)|$ attains a minimum in $z = 0$. Thus $q(X)$, which clearly has the same degree d as $p(X)$, has the required property, provided we let m be the index of the first nonzero coefficient among $q_1, q_2, \dots, q_d$. (It is possible that $m = d$.)

c) $q_m z_0^m = -1$ is equivalent to $z_0^m = -1/q_m$ and solvable in $\mathbb{C}$, because every complex number has an m-th root in $\mathbb{C}$. In fact there are m distinct solutions. Writing $-1/q_m = r\mathrm{e}^{\mathrm{i}\phi}$ in polar form, one solution is $z_0 = \sqrt[m]{r}\,\mathrm{e}^{\mathrm{i}\phi/m}$, and the remaining solutions are $z_k = z_0\,\mathrm{e}^{2\pi\mathrm{i}k/m}$, $1 \leq k \leq m-1$. These form a regular m-gon, which is inscribed into the circle $|z| = \sqrt[m]{r}$.

d) By c), there exists $r \in (0,1)$ such that $|R(r)| < r^m$. Then

$$|q(rz_0)| = |1 - r^m + R(r)| \leq |1 - r^m| + |R(r)| = 1 - r^m + |R(r)| < 1,$$

contradicting c).

Since $q_m(rz_0)^m = q_m r^m z_0^m = -r^m$, the number $q(rz_0)$ admits the indicated representation. We have

$$\begin{aligned} |R(r)| &= \left| \sum_{k=m+1}^{d} q_k r^k z_0^k \right| \leq \sum_{k=m+1}^{d} r^k \left| q_k z_0^k \right| \\ &= r^{m+1} \left(\left| q_{m+1} z_0^{m+1} \right| + r \left| q_{m+2} z_0^{m+2} \right| + \cdots + r^{d-m-1} \left| q_d z_0^d \right| \right). \end{aligned}$$

Since the 2nd factor tends to $\left| q_{m+1} z_0^{m+1} \right|$ for $r \downarrow 0$, this is certainly smaller than r^m for small positive r. (In fact we even have $R(r) = \mathrm{O}\left(r^{m+1}\right) = \mathrm{o}\left(r^m\right)$ for $r \downarrow 0$.)

Note: This proof is essentially the one given by Charles Feffermann in *An Easy Proof of the Fundamental Theorem of Algebra*, American Mathematical Monthly 74-7(1967), pp. 854–855. It is a variant of the elementary proof of Argand (1806).

ZJU-UIUC Institute
Prof. Thomas Honold

Fall Semester 2023
Worksheet 10

Calculus III (Math 241)

W28 Do Exercise 35 in [Ste21], Ch. 14.8.

W29 Determine the type of each quadric surface Q_a in the family

$$y^2 + xz + x - y - z = a, \quad a \in \mathbb{R}.$$

Hint: Q_a is central; the center can be found by rewriting the equation in a way similar to "completing the square".

W30 Do Exercises 20, 38, 40, 68, 78 in [Ste21], Ch. 15.2.

Solutions

28 Ex. 35

(a) $f(x,y) = x$ has gradient $\nabla f(x,y) = (1,0)$, and $g(x,y) = y^2 + x^4 - x^3$ has gradient $\nabla g(x,y) = (4x^3 - 3x^2, 2y)$. The critical points of g are $(0,0)$ and $(3/4,0)$. Of these only $(0,0)$ is on the curve $y^2 + x^4 - x^3 = 0$. It follows that in all curve points $\neq (0,0)$ we can apply the theorem on Lagrange multipliers and conclude that a local extremum of f on the curve must satisfy

$$(1,0) = \lambda(4x^3 - 3x^2, 2y) \quad \text{for some } \lambda \in \mathbb{R}.$$

The second equation, $2\lambda y = 0$, forces $y = 0$, since $\lambda = 0$ is impossible, and further $x = 1$, since $y^2 + x^4 - x^3 = x^3(x-1) = 0$ and $(x,y) \neq (0,0)$. Thus the only solution is $(x,y) = (1,0)$ (and $\lambda = 1$). The point $(1,0)$, however, is not a minimum of f on the curve, since it has x-coordinate 1 whereas $(0,0)$ has x-coordinate 0. (In fact, the point $(1,0)$ is the unique maximum of f on the curve; see the plot below.)

(b) Since $y^2 + x^4 - x^3 = y^2 + x^3(x-1) = 0$ implies $0 \leq x \leq 1$, $f(0,0) = 0$ is the unique minimum of f on the curve; cf. also the plot below. Since $\nabla f(0,0) = (1,0)$, $\nabla g(0,0) = (0,0)$, the equation $\nabla f(0,0) = \lambda \nabla g(0,0)$ doesn't hold for any λ.

(c) Because the minimum value is attained at a critical point of g (visible as a cusp of the curve).

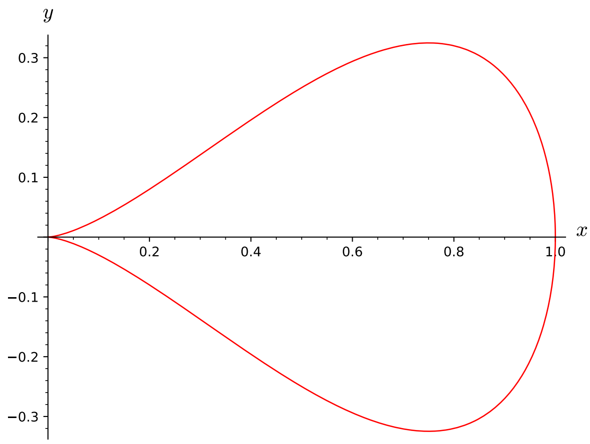


Figure 1: The curve $y^2 + x^4 - x^3 = 0$

29 A clever way to solve this exercise is as follows: The equation can be rewritten as

$$\left(y - \frac{1}{2}\right)^2 + (x-1)(z+1) = a - \frac{3}{4}.$$

This shows that the quadric Q_a (the one with parameter a) is central with center $(x,y,z) = \left(1, \frac{1}{2}, -1\right)$ (independent of a) and translation-equivalent to $y^2 + xz = a - \frac{3}{4}$. Using a further variable change, viz. $x = x' + z'$, $z = x' - z'$, $y = y'$, the latter is transformed into $y^2 + x^2 - z^2 = a - \frac{3}{4}$.

$$\implies Q_a \text{ is a } \begin{cases} \text{cone} & \text{if } a = \frac{3}{4}, \\ \text{hyperboloid of one sheet} & \text{if } a > \frac{3}{4}, \\ \text{hyperboloid of two sheets} & \text{if } a < \frac{3}{4}. \end{cases}$$

Of course, the standard method, discussed in the lecture, can also be used. The equation defining Q_a is equivalent to

$$q_a(x,y,z) = y^2 + xz + x - y - z - a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\mathsf{T}} \underbrace{\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2 \underbrace{\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}}_{\mathbf{b}}^{\mathsf{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - a = 0.$$

Since $\operatorname{rk} \mathbf{A} = 3$, Q_a is central with center $\mathbf{v} = (v_1, v_2, v_3)$ determined by

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The solution is $\mathbf{v} = \left(1, \frac{1}{2}, -1\right)$, so that Q_a is equivalent to the quadric with equation

$$y^2 + xz + q_a(\mathbf{v}) = y^2 + xz + \left(\tfrac{1}{2}\right)^2 + 1(-1) + 1 - \tfrac{1}{2} - (-1) - a = y^2 + xz + \tfrac{3}{4} - a = 0.$$

Then we use the algorithm for transforming $\mathbf{A}$ into Sylvester canonical form:

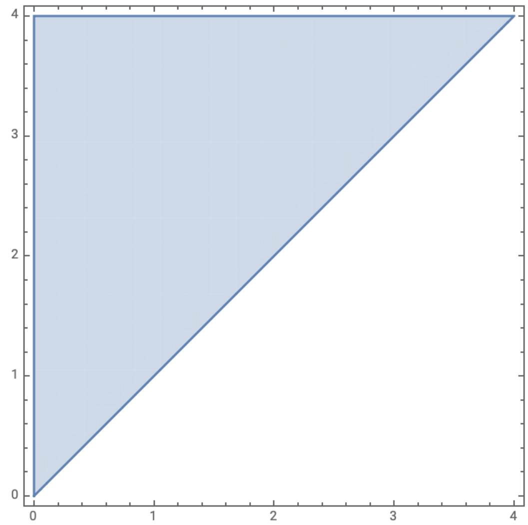
$$\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \xrightarrow[C1=C1+C3]{R1=R1+R3} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \xrightarrow{R3=R3-\frac{1}{2}R1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \xrightarrow{C3=C3-\frac{1}{2}C1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}$$
$$\xrightarrow[C3=2C3]{R3=2R3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\implies Q_a$ is equivalent to the quadric with equation $x^2 + y^2 - z^2 + \frac{3}{4} - a = 0$, which is the same as obtained above.

30 Ex. 20 The domain of integration is
The iterated integrals that can be used for evaluating the double integral are

$$\iint_D y^2 e^{xy} dA = \int_{x=0}^{4} \int_{y=x}^{4} y^2 e^{xy} \,\mathrm{d}y \,\mathrm{d}x = \int_{y=0}^{4} \int_{x=0}^{y} y^2 e^{xy} \,\mathrm{d}x \,\mathrm{d}y.$$

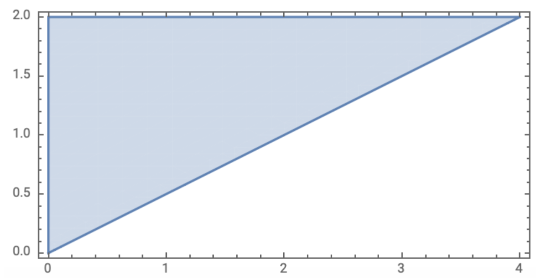
Since $y^2 \mathrm{e}^{xy}$ is easier to integrate with respect to x (for which an antiderivative is immediate) than with respect to y (for which one needs to use partial integration twice), we use the 2nd



iterated integral:

$$\begin{aligned}
\iint\limits_D y^2 e^{xy}\, dA &= \int_0^4 \int_0^y y^2 e^{xy}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^4 \left[y^2 \cdot \frac{1}{y} e^{xy} \right]_0^y \mathrm{d}y \\
&= \int_0^4 \left(y e^{y^2} - y \right) \mathrm{d}y \\
&= \left[\frac{1}{2} e^{y^2} - \frac{y^2}{2} \right]_0^4 \\
&= \frac{1}{2} \left(e^{16} - 17 \right).
\end{aligned}$$

Ex. 38 The volume is equal to the integral of the function $f(x,y) = \sqrt{4-y^2}$ over the region $D = \{(x,y) \in \mathbb{R}^2; 0 \le y \le 2, 0 \le x \le 2y\}$.



$$
\begin{aligned}
\iint_D \sqrt{4-y^2}\,\mathrm{dA} &= \int_0^2 \int_0^{2y} \sqrt{4-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^2 \left[x\sqrt{4-y^2}\right]_0^{2y} \mathrm{d}y \\
&= \int_0^2 2y\sqrt{4-y^2}\,\mathrm{d}y
\end{aligned}
$$

Making the substitution for $u = 4-y^2$, $\mathrm{d}u = -2y\,\mathrm{d}y$, we have further

$$
\begin{aligned}
\int_0^2 2y\sqrt{4-y^2}\,\mathrm{d}y &= \int_4^0 \sqrt{u}\,\mathrm{d}(-u) \\
&= \int_0^4 u^{1/2}\,\mathrm{d}u \\
&= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^4 \\
&= \frac{16}{3}.
\end{aligned}
$$

Thus the desired volume is $\frac{16}{3}$.

Ex. 40 The volume is twice the value of the integral of the function $f(x,y) = \sqrt{r^2-y^2}$ over the region $D = \left\{(x,y) \in \mathbb{R}^2; x^2+y^2 \leq r^2\right\}$, which is the disk of radius r centered at origin.

$$
\begin{aligned}
\iint_D \sqrt{4-y^2}\,\mathrm{dA} &= \int_{y=-r}^{r} \int_{x=-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{y=-r}^{r} \sqrt{r^2-y^2}\,[x]_{x=-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\,\mathrm{d}y \\
&= 2\int_{-r}^{r} (r^2-y^2)\,\mathrm{d}y \\
&= 4r^3 - 4\int_0^r y^2\,\mathrm{d}y = 4r^3 - \tfrac{4}{3}r^3 = \tfrac{8}{3}r^3
\end{aligned}
$$

Thus the desired volume is $\frac{16}{3}r^3$.

Ex. 68 Divide the region D into 2 parts split at the x-axis:

$$
\begin{aligned}
\iint_D y\,\mathrm{dA} &= \int_{y=-1}^{0} \int_{x=-1}^{y-y^3} y\,\mathrm{d}x\,\mathrm{d}y + \int_{y=0}^{1} \int_{x=\sqrt{y}-1}^{y-y^3} y\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-1}^{0} (y^2-y^4+y)\,\mathrm{d}y + \int_0^1 (y^2-y^4-y^{3/2}+y)\,\mathrm{d}y \\
&= 2\int_0^1 y^2-y^4\,\mathrm{d}y - \int_0^1 y^{3/2}\,\mathrm{d}y = 2\left(\tfrac{1}{3}-\tfrac{1}{5}\right) - \tfrac{2}{5} = -\tfrac{2}{15}
\end{aligned}
$$

Ex. 78 The region D is the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$. Using linearity, we have

$$\begin{aligned}\iint_D 2 + x^2y^3 - y^2 \sin x \, \mathrm{dA} &= 2 \iint\limits_D 1 \, \mathrm{dA} + \iint\limits_D x^2y^3 \, \mathrm{dA} - \iint\limits_D y^2 \sin x \, \mathrm{dA} \\ &= 2\mathbf{I_1} + \mathbf{I_2} - \mathbf{I_3}, \quad \text{say.}\end{aligned}$$

Notice that the integrand in $\mathbf{I_2}$, viz. x^2y^3, is even w.r.t. x and odd w.r.t. y. Moreover, the region D is symmetric w.r.t. both x and y. This means if we integrate over y first (i.e., use the order $\mathrm{d}y\,\mathrm{d}x$), we will get 0 for the inner integral, and hence $\mathbf{I_2} = 0$. Similarly, we have $\mathbf{I_3} = 0$ since $y^2 \sin x$ is even w.r.t. y and odd w.r.t. x. Thus, we have

$$\iint_D 2 + x^2y^3 - y^2 \sin x \, \mathrm{dA} = 2\mathbf{I_1} = 2 \, \mathrm{vol}(D) = 4.$$

Calculus III (Math 241)

H57 Do Exercise 42 in [Ste21], Ch. 14.7.

H58 Do Exercises 46, 48 in [Ste21], Ch. 14.7, using the machinery developed for determining extrema of multivariable functions. Afterwards think of alternative, more direct solutions.

H59

a) Show that a non-degenerate quadric Q in $\mathbb{R}^3$ ("quadric surface") has a tangent plane in every point $\mathbf{x}_0 \in Q$.

Hint: Q is the level-0 surface of $f(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + 2\,\mathbf{b}^\mathsf{T}\mathbf{x} + c$ with $\mathbf{A} \in \mathbb{R}^{3\times 3}$ symmetric, $\mathbf{b} \in \mathbb{R}^3$, $c \in \mathbb{R}$. Non-degeneracy of Q is equivalent to $\det\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^\mathsf{T} & c \end{pmatrix} \neq 0$; cf. lecture. The task is to show that this implies $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$ for all $\mathbf{x}_0 \in Q$.

b) Determine the type of the quadric Q with equation

$$x^2 + y^2 + z^2 + xy + xz + yz + x + y + z = 0.$$

H60 For $b > 1$ evaluate $\displaystyle\int_1^b \frac{\mathrm{d}x}{x}$ without recourse to the Fundamental Theorem of Calculus.

Hint: Use upper and lower Darboux sums for partitions of $[1, b]$ that are in geometric progression; cf. the example on Slide 13 f in `lecture29-31_handout.pdf`.

H61 Do Exercises 22, 38, 42, 46, and optionally Exercise 54 in [Ste21], Ch. 15.6.

H62 *Optional Exercise*

a) Determine the non-degenerate types of space quadrics that contain a line; cf. Exercise 53 in [Ste21], Chapter 12.6.

b) Determine the non-degenerate types of space quadrics that are disjoint from a plane.

Hint: Since affine coordinate changes preserve both properties, it suffices to check the 5 affine normal forms $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 - z^2 = 1$, $x^2 - y^2 - z^2 = 1$, $z = x^2 + y^2$, $z = x^2 - y^2$ derived in the classification theorem.

H63 *Optional Exercise*

In the lecture we have seen that Dirichlet's function $f\colon [0,1] \to \mathbb{R}$, defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$, is not Riemann integrable.

a) Show that $g\colon [0,1] \to \mathbb{R}$, defined by $g(x) = 1/q$ if $x = p/q \in \mathbb{Q}$, $q \in \mathbb{N}$, $\gcd(p,q) = 1$, and $g(x) = 0$ if $x \notin \mathbb{Q}$ is Riemann integrable with $\int_0^1 g(x)\,\mathrm{d}x = 0$.

Hint: For $N \in \mathbb{N}$ consider the partition $P_N = \left\{0, \frac{1}{N^3}, \frac{2}{N^3}, \dots, \frac{N^3-1}{N^3}, 1\right\}$ and estimate the number of subintervals containing a number x with $g(x) > 1/N$.

b) Show that f is discontinuous everywhere, and g is continuous at irrational numbers and discontinuous at rational numbers.

Due on Wed Dec 13, 6 pm

The optional exercises can be handed in until Wed Dec 20, 7 pm.

Solutions to Selected Exercises

57 Ex. 42 The partial derivatives of $f(x,y) = 3xe^y - x^3 - e^{3y}$ are $f_x(x,y) = 3e^y - 3x^2$ and $f_y(x,y) = 3xe^y - 3e^{3y}$. From $f_x = 0$ we have $e^y = x^2$. Plugging it into $f_y = 0$ we have $3x^3 - 3x^6 = 0 \implies x = 0$ or $x = 1$. Because $e^y = x^2$, x cannot be 0. Thus, we have the only critical point $(1,0)$, and there can be at most one local (or global) extremum.
Then we check that the critical point is a local maximum:

$$\mathbf{H}_f(x,y) = \begin{pmatrix} -6x & 3\,e^y \\ 3\,e^y & 3x\,e^y - 9\,e^{3y} \end{pmatrix}, \quad \mathbf{H}_f(1,0) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$$

Since $f_{xx} = -6 < 0$ and $f_{xx}f_{yy} - f_{xy}^2 = (-6)^2 - 3^2 = 27 > 0$ at $(1,0)$, the critical point must be a local maximum. (The corresponding Hesse quadratic form $q(h_1,h_2) = (h_1,h_2)\left(\begin{smallmatrix} -6 & 3 \\ 3 & -6 \end{smallmatrix}\right)(h_1,h_2)^\mathsf{T} = -6h_1^2 + 6h_1h_2 - 6h_2^2$ is negative definite.)
It is easy to see that the critical point is not an absolute maximum. We can just take $y = 0$ and $f(x,y)$ becomes the single variable function $f(x,0) = 3x - x^3 - 1$. Since, e.g., $f(-4,0) = 51 > f(1,0) = 1$, the critical point is not an absolute maximum.

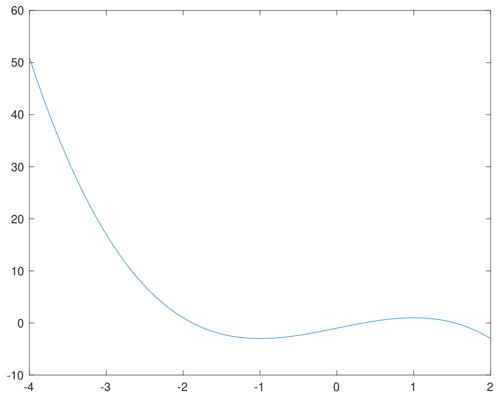


Figure 1: $f(x,0)$

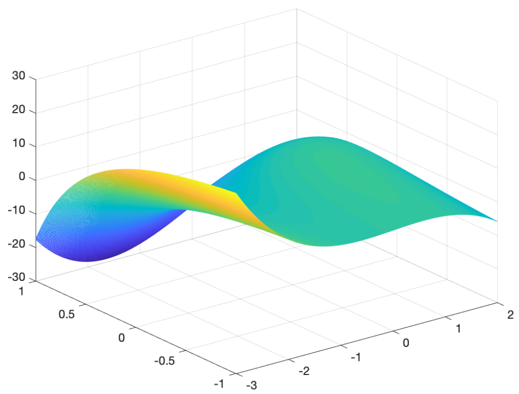


Figure 2: $f(x,y)$

58 Ex. 46 For a point (x,y,z) on the surface S, the squared distance from the origin is

$$x^2 + y^2 + z^2 = x^2 + 9 + xz + z^2 =: f(x,z).$$

Since there are exactly two points on S with given x- and z-coordinates, viz. $(x,\pm y,z)$, minimizing $f(x,z)$ over $\mathbb{R}^2$ solves the problem.

$$\nabla f(x,z) = (2x+z, x+2z) = (0,0) \iff x = z = 0.$$

Hence the only candidates are $(0,\pm 3,0)$.
Since $\nabla f(x,z) = 0$ is only a necessary condition for an extremum, it is still possible that there is no point on S closest to the origin. But if there are such points, they must be critical and hence equal to $(0,\pm 3,0)$.
Since S is closed (if $\big((x_n,y_n,z_n)\big)$ is a convergent sequence of points on S, we have $y_n^2 = 9 + x_nz_n$ for all n and hence also $y^2 = 9 + xz$ for $(x,y,z) = \lim_{n\to\infty}(x_n,y_n,z_n)$) and nonempty ($(0,\pm 3,0) \in$

S, for example), there must be a point on the surface closest to the origin. This follows from the multivariable analog of the Extreme Value Theorem, because the (continuous) length function on $\mathbb{R}^2$ attains a minimum on the closed and bounded sets $S \cap \mathrm{B}_R(0,0,0)$, which for sufficiently large R (in our case $R \geq 3$) are non-empty.
$\Longrightarrow$ The points on S closest to the origin are $(0,\pm 3,0)$.
Solution with Lagrange Multipliers: The problem is to minimize $f(x,y,z) = x^2+y^2+z^2$ under the constraint $g(x,y,z) = y^2 - xz - 9 = 0$. Since $\nabla g(x,y,z) = (-z,2y,-x) \neq (0,0,0)$ for $(x,y,z) \in S$ (the origin $(0,0,0)$ is not on S because of the "-9"), every putative minimum must satisfy the Lagrange multiplier condition $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$. This gives the 4 equations

$$\begin{aligned} 2x &= -\lambda z, \\ 2y &= 2\lambda y, \\ 2z &= -\lambda x, \\ y^2 &= 9 + xz. \end{aligned}$$

If $y \neq 0$ then $\lambda = 1$ and $4x = -2z = x$, which implies $x = z = 0$, $y^2 = 9$. Thus we obtain again the candidates $(0,\pm 3,0)$. If $y = 0$ then from the 4th equation x and z must have different sign. The 1st and 3rd equation give $x = \pm z$, leading to the candidates $(\pm 3,0,\mp 3)$ (and $\lambda = 2$). Since the last two points have distance $\sqrt{18} > 3$ from the origin, only $(0,\pm 3,0)$ remain as candidates. The remaining part of the proof is the same as above, i.e., one needs to show that a point on S closest to the origin indeed exists.
Ex. 48 Denoting the three numbers by x,y,z, we have $z = 12 - x - y$ and hence

$$x^2+y^2+z^2 = x^2+y^2+(12-x-y)^2 = 2x^2+2y^2+2xy-24x-24y+144 =: f(x,y).$$

Since $z > 0$, we mus have $x+y < 12$, and the task is to minimize f over the open triangle $\Delta = \big\{(x,y) \in \mathbb{R}^2; x > 0, y > 0, x+y < 12\big\}$.

$$\nabla f(x,y) = (4x+2y-24, 4y+2x-24) = (0,0) \iff (x,y) = (4,4)$$

Hence the solution is $(x,y,z) = (4,4,4)$, provided we can show that there is a solution at all, i.e., f attains a minimum on Δ.
The boundary of Δ consists of three line segments, viz., $S_1 = \big\{(x,0); 0 \leq x \leq 12\big\}$, $S_2 = \big\{(0,y); 0 \leq y \leq 12\big\}$, $S_3 = \big\{(x,12-x); 0 \leq x \leq 12\big\}$, and we have

$$\begin{aligned} f(x,0) &= x^2+(12-x)^2 = 2x^2-24x+144 = 2(x-6)^2+72 \geq 72, \\ f(0,y)) &= y^2+(12-y)^2 \geq 72, \\ f(x,12-x) &= x^2+(12-x)^2 \geq 72. \end{aligned}$$

$\Longrightarrow$ The minimum of f on the closed and bounded set $\overline{\Delta}$, which is $\leq f(4,4) = 48$, is attained in its interior Δ, and hence equal to $f(4,4) = 48$. Thus $(x,y,z) = (4,4,4)$ is indeed the unique solution, having the minimal sum of squares $x^2+y^2+z^2 = 48$ subject to the constraints $x,y,z > 0$ and $x+y+z = 12$.
Solution with Lagrange Multipliers: The problem is to minimize $f(x,y,z) = x^2+y^2+z^2$ under the constraint $g(x,y,z) = x+y+z-12 = 0$. Since $\nabla g(x,y,z) = (1,1,1) \neq \mathbf{0}$, the Lagrange multiplier condition is satisfied and gives that a putative minimum (x,y,z) must be a scalar multiple of $(1,1,1)$. Hence $x = y = z = 4$ is the only candidate. That the minimum indeed exists is proved as in the previous exercise, using that the plane $x+y+z = 12$ is closed.

Both exercises have alternative solutions avoiding multivariable differential calculus. In Ex. 44 we can use the estimate $f(x,z) = (x+z/2)^2 + 3z^2/4 + 9 \geq 9$ with equality iff $(x,z) = (0,0)$, and in Ex. 46 we can use the Cauchy-Schwarz Inequality:

$$12 = x+y+z = (x,y,z)\cdot(1,1,1) \leq \sqrt{x^2+y^2+z^2}\sqrt{3},$$

which is equivalent to $x^2+y^2+z^2 \geq 144/3 = 48$. Equality holds iff (x,y,z) is a scalar multiple of $(1,1,1)$, i.e., $x=y=z=4$.
Ex. 46 can also be seen as a geometric problem: On the solid triangle with vertices $(12,0,0)$, $(0,12,0)$, $(0,0,12)$, which is the part of the plane $x+y+z=12$ that contained in the first octant of $\mathbb{R}^3$, find the point that is closest to the origin.

59 a) Using the known form of the gradients of a quadratic form and a linear form, we get

$$\nabla f(\mathbf{x}_0) = 2\mathbf{A}\mathbf{x}_0 + 2\mathbf{b} = 2(\mathbf{A}\mathbf{x}_0+\mathbf{b}) = \mathbf{0} \iff \mathbf{A}\mathbf{x}_0 = -\mathbf{b}.$$

Substituting this into the equation for Q gives

$$\mathbf{x}_0^\mathsf{T}(-\mathbf{b}) + 2\,\mathbf{b}^\mathsf{T}\mathbf{x}_0 + c = \mathbf{b}^\mathsf{T}\mathbf{x}_0 + c = 0.$$

Hence $\mathbf{x}_0$ is a singular point of Q (i.e., a point on Q without a tangent plane) precisely if $\mathbf{A}\mathbf{x}_0 + \mathbf{b} = \mathbf{0} \wedge \mathbf{b}^\mathsf{T}\mathbf{x}_0 + c = 0$. Both equations are in turn equivalent to the 4×4 system

$$\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^\mathsf{T} & c \end{pmatrix}\begin{pmatrix} \mathbf{x}_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}.$$

If Q is non-degenerate, the coefficient matrix is invertible and hence there is no such point $\mathbf{x}_0$.

Remark: Conversely, if Q has no singular point, we cannot conclude that Q is non-degenerate, because the 4×4 system may have a nonzero solution with last coordinate 0, i.e., $\mathbf{A}\mathbf{x}_0 = \mathbf{0} \wedge \mathbf{b}^\mathsf{T}\mathbf{x}_0 = 0$.

b) Multiplying the equation by 2 it becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^\mathsf{T} \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\mathbf{b}}^\mathsf{T} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Since $\mathbf{A}$ has rank 3 (as follows, e.g., by computing $\det(\mathbf{A}) = 8+1+1-3\cdot 2 = 4 \neq 0$), Q is central with center $\mathbf{v} = (v_1, v_2, v_3)$ determined by

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The solution is $\mathbf{v} = -\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$, and the "centered form" of the equation for Q is

$$\left(x+\tfrac{1}{4}\right)^2 + \left(y+\tfrac{1}{4}\right)^2 + \left(z+\tfrac{1}{4}\right)^2 + \left(x+\tfrac{1}{4}\right)\left(y+\tfrac{1}{4}\right) + \left(x+\tfrac{1}{4}\right)\left(z+\tfrac{1}{4}\right) + \left(y+\tfrac{1}{4}\right)\left(z+\tfrac{1}{4}\right) = \tfrac{6}{16}.$$

$\Longrightarrow Q$ is translation-equivalent to $x^2+y^2+z^2+xy+xz+yz=\frac{6}{16}$, or $(x,y,z)\mathbf{A}(x,y,z)^\top=\frac{12}{16}$. The type of this quadric is determined by the Sylvester canonical form of $\mathbf{A}$, which is obtained as follows:

$$\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\longrightarrow\begin{pmatrix}2&0&0\\0&\frac{3}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{3}{2}\end{pmatrix}\longrightarrow\begin{pmatrix}2&0&0\\0&\frac{3}{2}&0\\0&0&\frac{8}{6}\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

$\Longrightarrow Q$ is affinely equivalent to the quadric with equation $x^2+y^2+z^2=\frac{12}{16}$, i.e., an ellipsoid.

60 For $N\in\mathbb{N}$ consider the partition $P=\{1,q,q^2,\dots,q^{N-1},b\}$ of $[1,b]$ where $q:=\sqrt[N]{b}$, i.e, for $1\leq i\leq N$ the i-th subinterval of P is $[x_{i-1},x_i]=[q^{i-1},q^i]$. Since $x\mapsto 1/x$ attains its maximum in $[q^{i-1},q^i]$ at q^{i-1} and its minimum in q^i, we have

$$\begin{aligned}\overline{S}(1/x;P)&=\sum_{i=1}^{N}\frac{q^i-q^{i-1}}{q^{i-1}}=N(q-1),\\ \underline{S}(1/x;P)&=\sum_{i=1}^{N}\frac{q^i-q^{i-1}}{q^i}=N(1-1/q),\\ \overline{S}(1/x;P)-\underline{S}(1/x;P)&=N(q-2+1/q).\end{aligned}$$

Moreover,

$$\begin{aligned}\lim_{N\to\infty}N(q-2+1/q)&=\lim_{N\to\infty}N\left(b^{1/N}-2+b^{-1/N}\right)=\lim_{N\to\infty}\frac{b^{1/N}-2+b^{-1/N}}{1/N}=\lim_{x\downarrow 0}\frac{b^x-2+b^{-x}}{x}\\&=\lim_{x\downarrow 0}\frac{\ln(b)b^x-\ln(b)b^{-x}}{1}=0,\end{aligned}$$

where we have used l'Hospital's Rule. This shows that $x\mapsto 1/x$ is Riemann-integrable over $[1,b]$ with

$$\int_1^b\frac{\mathrm{d}x}{x}=\lim_{N\to\infty}N\left(\sqrt[N]{b}-1\right)=\lim_{N\to\infty}\frac{b^{1/N}-1}{1/N}=\frac{\mathrm{d}}{\mathrm{d}x}b^x\bigg|_{x=0}=\ln b.$$

(Or apply l'Hospital's Rule a second time.)

61 **Ex. 22** The region E is shown in Fig. 3 (with the part of ∂E on the plane $y=3x$ shaded). Analytically, the region is $E=\left\{(x,y,z)\in\mathbb{R}^3;0\leq x\leq 1;3x\leq y\leq 3;0\leq z\leq\sqrt{9-y^2}\right\}$.

$$\begin{aligned}\iiint_E z\,\mathrm{dV}&=\int_{x=0}^{1}\int_{y=3x}^{3}\int_{z=0}^{\sqrt{9-y^2}}z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\&=\int_{x=0}^{1}\int_{y=3x}^{3}\frac{1}{2}(9-y^2)\,\mathrm{d}y\,\mathrm{d}x=\frac{1}{2}\int_{x=0}^{1}\left[9y-\tfrac{1}{3}y^3\right]_{y=3x}^{3}\mathrm{d}x\\&=\frac{1}{2}\int_0^1(18-27x+9x^3)\,\mathrm{d}x\\&=\frac{1}{2}\left(18-\frac{27}{2}+\frac{9}{4}\right)\\&=\frac{27}{8}\end{aligned}$$

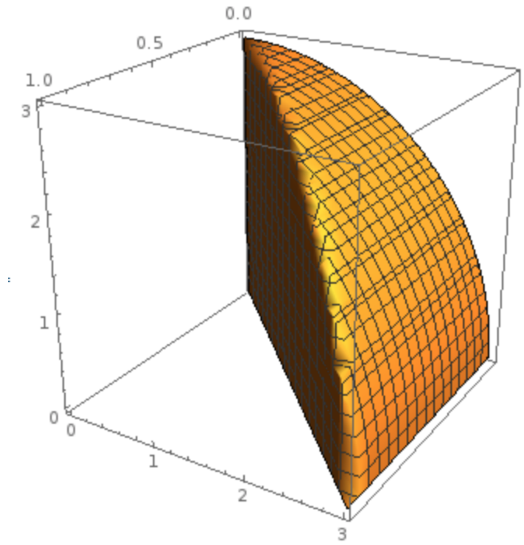


Figure 3: The region of Exercise 22

Ex. 38 Analytically, the corresponding region is

$$\left\{(x,y,z)\in\mathbb{R}^3; 0\le x\le 1, 0\le y\le 1-x, 0\le z\le 1-x^2\right\}.$$

There are a total of 6 different orders. In what follows we denote the given integral by $\mathbf{I}$.
If we integrate w.r.t. y first, we have

$$\begin{aligned}\mathbf{I} &= \int_{x=0}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{1-x} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\ &= \int_{z=0}^{1}\int_{x=0}^{\sqrt{1-z}}\int_{y=0}^{1-x} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\,.\end{aligned}$$

If we integrate w.r.t. z first, we have

$$\begin{aligned}\mathbf{I} &= \int_{y=0}^{1}\int_{x=0}^{1-y}\int_{z=0}^{1-x^2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x^2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

Finally, if we integrate w.r.t. x first, we have for x the two conditions $x\le 1-y$, $x\le\sqrt{1-z}$, which are equivalent to $x\le\min\left\{1-y,\sqrt{1-z}\right\}$. Hence

$$\begin{aligned}\mathbf{I} &= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{\min\left\{1-y,\sqrt{1-z}\right\}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= \int_{y=0}^{1}\int_{z=0}^{1}\int_{x=0}^{\min\left\{1-y,\sqrt{1-z}\right\}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\end{aligned}$$

We can make the representation "min-free" by splitting the middle interval into two intervals. The minimum is equal to $1-y$ if $1-y \leq \sqrt{1-z}$, and to $\sqrt{1-z}$ otherwise. Solving $1-y \leq \sqrt{1-z}$ for y,z gives $y \geq 1-\sqrt{1-z}$ and $z \leq 1-(1-y)^2 = 2y-y^2$, respectively, so that

$$\mathbf{I} = \left[\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y}\right] f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \left[\int_0^1 \int_0^{2y-y^2} \int_0^{1-y} + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}}\right] f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y.$$

Ex. 42 The solution is similar to Ch.15.3, Ex.66. Using linearity, we can express the integral in terms of those with integrands $(x,y,z) \mapsto z^3$, $(x,y,z) \mapsto \sin y$, and $(x,y,z) \mapsto 1$. Since z^3 is odd w.r.t. z and $\sin y$ is odd w.r.t. y, the first two integrals are zero. Hence we obtain

$$\iiint_B (z^3 + \sin y + 3)\,\mathrm{dV} = 3\iiint_B 1\,\mathrm{dV} = 3\,\mathrm{vol}(B) = 4\pi$$

Ex. 46

$$m = \iiint_E y\,\mathrm{dV} = \int_{y=0}^1 \int_{z=0}^{1-y} \int_{x=0}^{1-y-z} y\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_{y=0}^1 y \int_{\substack{x+z\leq 1-y \\ x,z\geq 0}} \mathrm{d}^2(x,z)\,\mathrm{d}y$$
$$= \int_0^1 y(1-y)^2/2\,\mathrm{d}y = \frac{1}{2}\int_0^1 y - 2y^2 + y^3\,\mathrm{d}y = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24},$$
$$\bar{x} = \frac{1}{m}\iiint_E xy\,\mathrm{dV} = 24\int_{y=0}^1 y \int_{x=0}^{1-y} x \int_{z=0}^{1-x-y} \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$
$$= 24\int_{y=0}^1 y \int_{x=0}^{1-y} x - x^2 - xy\,\mathrm{d}x\,\mathrm{d}y = 24\int_{y=0}^1 y\left[(1-y)x^2/2 - x^3/3\right]\mathrm{d}y$$
$$= 4\int_0^1 y(1-y)^3\,\mathrm{d}y = 4\int_0^1 y - 3y^2 + 3y^3 - y^4\,\mathrm{d}y = 4\left[\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right] = \frac{1}{5}.$$

By symmetry, we then also have $\bar{z} = 1/5$.

$$\bar{y} = 24\iiint_E y^2\,\mathrm{dV} = 24\int_0^1 y^2(1-y)^2/2\,\mathrm{d}y = 12\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{2}{5}.$$

$\Longrightarrow$ The center of mass is $\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

Ex. 54

(a)

$$\begin{aligned}
m &= \iiint\limits_E \rho(x,y,z)\,\mathrm{dV} \\
&= \int_{x=0}^{1}\int_{y=3x}^{3}\int_{z=0}^{\sqrt{9-y^2}} (x^2+y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{1}\int_{y=3x}^{3} (x^2+y^2)\sqrt{9-y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{y=0}^{3}\int_{x=0}^{y/3} (x^2+y^2)\sqrt{9-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{y=0}^{3} \sqrt{9-y^2}\,\left[\tfrac{1}{3}x^3+xy^2\right]_{x=0}^{y/3}\mathrm{d}y = \frac{28}{81}\int_{y=0}^{3} y^3\sqrt{9-y^2}\,\mathrm{d}y \\
&= \frac{28}{81}\cdot\frac{162}{5} = \frac{56}{5} = 11.2
\end{aligned}$$

The last step was derived with the help of Matlab, but $\int y^3\sqrt{9-y^2}\,\mathrm{d}y$ can also be found by standard methods.

(b) The x-coordinate of the center of mass is

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\iiint\limits_E x\,\rho(x,y,z)\,\mathrm{dV} \\
&= \int_0^1\int_{3x}^{3}\int_0^{\sqrt{9-y^2}} x(x^2+y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{855\,\pi}{7168} \approx 0.375\,.
\end{aligned}$$

Similarly, we have $\bar{y} = \frac{45\pi}{64} \approx 2.208$ and $\bar{z} = \frac{15}{16} = 0.9375$. Thus, the center of mass is approximately equal to $(0.375, 2.21, 0.938)$.

(c)

$$\begin{aligned}
I_z &= \iiint\limits_E (x^2+y^2)\,\rho(x,y,z)\,\mathrm{dV} \\
&= \int_0^1\int_{3x}^{3}\int_0^{\sqrt{9-y^2}} (x^2+y^2)^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{10464}{175} \approx 59.8
\end{aligned}$$

Also the integrals in b), c) can be evaluated by hand with a little effort.

62 a) Spheres, elliptic paraboloids and hyperboloids of two sheets clearly don't contain lines. We prove this formally for the hyperboloid, call it Q: Suppose $L = \mathbf{p} + \mathbb{R}\mathbf{u} = (x,y,z) + \mathbb{R}(u_1,u_2,u_3)$. Then $L \subseteq S$ iff

$$(x+tu_1)^2 - (y+tu_2)^2 - (z+tu_3)^2 = 1 \quad \text{for all } t \in \mathbb{R},$$

$$\textit{i.e.,}\quad x^2 - y^2 - z^2 + 2t(xu_1 - yu_2 - zu_3) + t^2(u_1^2 - u_2^2 - u_3^2) = 1 \quad \text{for all } t \in \mathbb{R}.$$

This is only possible if $x^2-y^2-z^2=1$ (i.e., $\mathbf{p}\in S$), $xu_1-yu_2-zu_3=u_1^2-u_2^2-u_3^2=0$. Eliminating u_3 from the last two equations gives $z^2(u_1^2-u_2^2)=z^2u_3^2=(xu_1-yu_2)^2$, or $(z^2-x^2)u_1^2+2xyu_1u_2-(z^2+y^2)u_2^2=0$. The discriminant of this quadratic form in (u_1,u_2) is

$$(2xy)^2+4(z^2-x^2)(z^2+y^2)=4(z^4-x^2z^2+z^2y^2)=4z^2(z^2-x^2+y^2)=-4z^2.$$

Hence, if $z\neq 0$ then there is only the solution $u_1=u_2=0$, leading to $\mathbf{u}=(u_1,u_2,u_3)=(0,0,0)$, a contradiction. If $z=0$, we must have $x^2-y^2=1$, $xu_1=yu_2$, which gives $(1+y^2)u_1^2=x^2u_1^2=y^2u_2^2$ and hence $y^2u_3^2=y^2(u_1^2-u_2^2)=-u_1^2$. This forces $u_1=0$, $u_2^2+u_3^2=0$, so that again $\mathbf{u}=\mathbf{0}$, a contradiction.

The other two quadrics contain lines, e.g., $x^2+y^2-z^2=1$ is satisfied for $x=\pm 1$, $y=\pm z$, saying that the lines $(\pm 1,0,0)+\mathbb{R}(0,1,\pm 1)$ are contained in the canonical hyperboloid of one sheet, and $z=x^2-y^2$ is satisfied for $z=0$, $x=\pm y$, saying that the lines $\mathbb{R}(1,\pm 1,0)$ are contained in the canonical hyperbolic paraboloid.

In fact one can show that through every point of the latter two quadrics there pass exactly two lines that are entirely contained in the quadric; cf. H**??**

b) Again it is clear that spheres, elliptic paraboloids and hyperboloids of two sheets are disjoint from some plane. (For $x^2+y^2+z^2=1$ take the plane $x=2$, for $z=x^2+y^2$ the plane $z=-1$, and for $x^2-y^2-z^2=1$ the plane $x=0$.) For the remaining two quadric types this is not true, as we will show.

In a) we have seen that a hyperboloid of one sheet and a hyperbolic paraboloid contain 2 lines L_1,L_2 through one of its points. These lines span a plane E. Every plane not parallel to E intersects E in a line, which must meet at least one of L_1 and L_2. Thus the assertion is true for all planes not parallel to E. For planes parallel to E it will be true if we can show that the two types of quadrics contain a further line that is not parallel to E. This follows from the much stronger results in H**??**, but in order to make the present proof independent of H**??**, we find such a line directly:

For $x^2+y^2-z^2=1$ we can take $y=1\wedge x=z$, i.e., $L_3=(0,1,0)+\mathbb{R}(1,0,1)$, which is not parallel to the planes $x=\pm 1$ spanned by the lines found earlier.

For $z=x^2-y^2$ we consider the point $(1,1,0)$. A line $(1,1,0)+\mathbb{R}(u_1,u_2,u_3)$ through this point is contained in the quadric if $tu_3=(1+tu_1)^2-(1+tu_2)^2$ for all $t\in\mathbb{R}$, i.e., $u_3=2u_1-2u_2\wedge u_1^2-u_2^2=0$, yielding the lines $(1,1,0)+\mathbb{R}(1,1,0)=\mathbb{R}(1,1,0)$ and $(1,1,0)+\mathbb{R}(1,-1,4)$. The second line is not parallel to the plane $z=0$ spanned by the lines found earlier.

63 a) Since g is positive, it suffices to exhibit, for any given $\varepsilon>0$, a partition P such that $\overline{S}(g;P)<\varepsilon$. (In fact, just like for Dirichlet's function, all lower Darboux sums $\underline{S}(g;P)$ are zero.)

We have $g(x)>1/N$ iff x is rational with denominator $q<N$. Since $x\in[0,1]$, this leaves no more than N^2 possibilities for x if $N>1$. (In fact there are much less, since, leaving $x\in\{0,1\}$ aside, for a fixed denominator $q\in\{2,\dots,N-1\}$ there are at most $q-1$ choices for the numerator p, and hence the total number of choices is $\leq 2+\sum_{q=2}^{N-1}(q-1)=2+\frac{(N-2)(N-1)}{2}$.) Each such x affects the values of g in at most two subintervals determined by P_N, implying that

$$\overline{S}(g;P_N) = \frac{1}{N^3}\sum_{i=1}^{N^3} \sup\left\{g(x); \frac{i-1}{N^3} \leq x \leq \frac{i}{N^3}\right\} \leq \frac{1}{N^3}\left(2N^2\cdot 1 + (N^3-2N^2)\cdot\frac{1}{N}\right)$$
$$< \frac{3N^2}{N^3} = \frac{3}{N}.$$

Hence, in order to achieve $\overline{S}(g;P_N) < \varepsilon$ it suffices to choose $N = \lceil 3/\varepsilon \rceil$.

b) First we consider Dirichlet's function. For $x_0 \in [0,1]$, we claim that in a proof of the continuity of f at x_0 there is no response to $\varepsilon = 1$. Indeed, if $x_0 \in \mathbb{Q}$ then there are irrational numbers x arbitrarily close to x_0 (e.g., $x = x_0 + \frac{\sqrt{2}}{N}$ with N large), and for these we have $f(x) - f(x_0) = -1$, $|f(x) - f(x_0)| = 1$; if $x_0 \notin \mathbb{Q}$, there are rational numbers x arbitrarily close to x_0 (e.g., take N large, $i \in \{1,\dots,N\}$ such that $\frac{i-1}{N} < x < \frac{i}{N}$, and set $x = \frac{i}{N}$), and for these we have again $|f(x) - f(x_0)| = 1$.

Next we consider the function g. Here for $x_0 = p/q \in \mathbb{Q}$ there exists no response to $\varepsilon = 1/q$ (cf. the argument for f), and hence g is discontinuous at every $x_0 \in \mathbb{Q}$.

Now suppose $x_0 \notin \mathbb{Q}$. Given $\varepsilon > 0$ choose $N \in \mathbb{N}$ with $1/N < \varepsilon$. There are only finitely many (viz., no more than N^2) rational numbers in $[0,1]$ with denominator $\leq N$. Let a be the one closest to x_0 and set $\delta = |a - x_0|$. Then $\delta > 0$ and for $x \in (x_0 - \delta, x_0 + \delta)$ we must have $g(x) \leq \frac{1}{N+1}$ (this holds for irrational x as well), so that δ can serve as response to ε.

Calculus III (Math 241)

W31 Suppose $A_k \subset \mathbb{R}^n$, $k \in \mathbb{N}$, are measurable sets with $A_{k+1} \subseteq A_k$ for all k. Show that $\bigcap_{k=1}^{\infty} A_k$ is measurable and

$$\mathrm{vol}\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mathrm{vol}(A_k).$$

Hint: Work with $B_k = A_1 \setminus A_k$ and use the result from the lecture for unions of ascending sequences of measurable sets.

W32
a) Formulate the Monotone Convergence Theorem for integration over a subset $A \subseteq \mathbb{R}^n$.

b) Do Problem Plus 5 in [Ste21], p. 1121.
Hint: The function $f(x,y) = \frac{1}{1-xy}$ is Lebesgue integrable over $[0,1]^2$, and the double integral can be evaluated by expanding $f(x,y)$ into an infinite series and applying the Monotone Convergence Theorem.

W33 *From an old final exam*

Suppose $F\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) := \int_0^1 \frac{e^{xt}}{1+t}\,dt \quad \text{for} \quad x \in \mathbb{R}.$$

a) Show that F is differentiable.

b) Show that F solves the differential equation $\quad y' + y = (e^x - 1)/x \quad$ on $\mathbb{R} \setminus \{0\}$.

W34 *From an old final exam*

Let $\quad K = \left\{(x,y,z) \in \mathbb{R}^3;\ \sqrt{x^2+y^2} \leq z(1-z)\right\}$.

a) Show that K is compact (i.e., closed and bounded).

b) Determine the volume $\mathrm{vol}(K)$.

Solutions

31 Since $\chi_{B_k} = \chi_{A_1} - \chi_{A_k}$ is the difference of two integrable functions, B_k is measurable with $\mathrm{vol}(B_k) = \int \chi_{B_k} = \int \chi_{A_1} - \int \chi_{A_k} = \mathrm{vol}(A_1) - \mathrm{vol}(A_k)$. The sets B_k satisfy $B_{k+1} \supseteq B_k$ for all k, and

$$\bigcup_{k=1}^{\infty} B_k = A_1 \setminus \bigcap_{k=1}^{\infty} A_k, \quad \text{or} \quad \bigcap_{k=1}^{\infty} A_k = A_1 \setminus \bigcup_{k=1}^{\infty} B_k.$$

Since $B_k \subseteq A_1$, the sequence $\big(\mathrm{vol}(B_k)\big)_{k\in\mathbb{N}}$ is bounded by $\mathrm{vol}(A_1)$, and hence $\bigcup_{k=1}^{\infty} B_k$ is measurable with $\mathrm{vol}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mathrm{vol}(B_k)$; cf. the lecture. It follows that $\bigcap_{k=1}^{\infty} A_k$ is measurable as well, and

$$\begin{aligned}\mathrm{vol}\left(\bigcap_{k=1}^{\infty} A_k\right) &= \mathrm{vol}(A_1) - \mathrm{vol}\left(\bigcup_{k=1}^{\infty} B_k\right) = \mathrm{vol}(A_1) - \lim_{k\to\infty} \mathrm{vol}(B_k) \\ &= \lim_{k\to\infty}\big(\mathrm{vol}(A_1) - \mathrm{vol}(B_k)\big) = \lim_{k\to\infty} \mathrm{vol}(A_k).\end{aligned}$$

32 a) Suppose that $f_1, f_2, f_3, \dots$ are real-valued (or $\overline{\mathbb{R}}$-valued) functions with common domain $A \subseteq \mathbb{R}^n$. (Alternatively, the domains of all functions f_k should contain A.) Suppose further that

(i) all f_k are integrable over A;

(ii) $f_k(\mathbf{x}) \leq f_{k+1}(\mathbf{x})$ for all k and all $\mathbf{x} \in A$;

(iii) there exists a bound $R \in \mathbb{R}$ such that $\int_A f_k(\mathbf{x})\, \mathrm{d}^n\mathbf{x} \leq R$ for all k.

Then the limit function $f\colon A \to \overline{\mathbb{R}}$, $\mathbf{x} \mapsto \lim_{k\to\infty} f_k(x)$ is (finite almost everywhere and) integrable over A, and we have

$$\lim_{k\to\infty} \int_A f_k(\mathbf{x}) \mathrm{d}^n\mathbf{x} = \int_A f(\mathbf{x}) \mathrm{d}^n\mathbf{x}.$$

This follows by applying the Monotone Convergence Theorem, as stated in the lecture, to the functions F_k and F, which are defined as f_k and f, respectively, on A and as zero on $\mathbb{R}^n \setminus A$ ("trivial extension").

b) The expansion

$$f(x,y) = \frac{1}{1-xy} = \sum_{k=0}^{\infty} (xy)^k$$

holds for all points $(x,y) \in [0,1]^2$, provided we set $f(1,1) = +\infty$. (Alternatively, we can think of f as being defined for all points in the unit square except at the upper right corner $(1,1)$. Since single points have Lebesgue measure zero, this doesn't affect the integrals involved in the following argument.)

Now we apply the Monotone Convergence Theorem in a) with $A = [0,1]^2$ and $f_k(x,y) = \sum_{i=0}^{k} (xy)^i$. Condition (i) is satisfied, since f_k is continuous, and we have

$$\int_{[0,1]^2} f_k(x,y)\, \mathrm{d}^2(x,y) = \sum_{i=0}^{k} \int_{[0,1]^2} x^i y^i\, \mathrm{d}^2(x,y) = \sum_{i=0}^{k} \frac{1}{(i+1)^2};$$

cf. the lecture. Condition (ii) is satisfied, since $(xy)^i \geq 0$; and Condition (iii), since $\sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

$$\begin{aligned}
\implies \int_0^1 \int_0^1 \frac{1}{1-xy} \,\mathrm{d}x \,\mathrm{d}y &= \int_{[0,1]^2} \frac{1}{1-xy} \,\mathrm{d}^2(x,y) = \int_{[0,1]^2} \left(\lim_{k\to\infty} f_k(x,y) \right) \mathrm{d}^2(x,y) \\
&= \lim_{k\to\infty} \int_{[0,1]^2} f_k(x,y) \,\mathrm{d}^2(x,y) && \text{(by a))} \\
&= \lim_{k\to\infty} \sum_{i=0}^{k} \frac{1}{(i+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

Note: In terms of infinite series, the Monotone Convergence Theorem says

$$\int \sum_{k=0}^{\infty} f_k(\mathbf{x}) \,\mathrm{d}^n\mathbf{x} = \sum_{k=0}^{\infty} \int f_k(\mathbf{x}) \,\mathrm{d}^n\mathbf{x},$$

provided that all f_k are integrable, $f_k(\mathbf{x}) \geq 0$ for all k and $\mathbf{x}$, and the series on the right (which is a series of non-negative real numbers) converges in $\mathbb{R}$.

33 a) We provide two solutions. The first solution computes $F'(x)$ in a direct way, thereby showing its special form ("differentiating under the integral sign"). The second solution uses the machinery of parameter integrals developed in the lecture on Thu Dec 14.

First solution: We have

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_0^1 \frac{\mathrm{e}^{(x+h)t}}{1+t} \,\mathrm{d}t - \int_0^1 \frac{\mathrm{e}^{xt}}{1+t} \,\mathrm{d}t \right) = \int_0^1 \frac{\mathrm{e}^{(x+h)t} - \mathrm{e}^{xt}}{h} \frac{1}{1+t} \,\mathrm{d}t$$

For $h \to 0$ the first factor of the integrand becomes just the partial derivative $\frac{\partial}{\partial x} \mathrm{e}^{xt} = t\,\mathrm{e}^{xt}$. But we cannot yet conclude from this that $F'(x) = \int_0^1 \frac{t\,\mathrm{e}^{xt}}{t+1} \,\mathrm{d}t$, since this requires interchanging limit and integration.

The Mean Value Theorem of Calculus I gives $\frac{\mathrm{e}^{(x+h)t} - \mathrm{e}^{xt}}{h} = t\mathrm{e}^{\xi t}$ with $\xi = \xi(h,t)$ between x and $x+h$ (t is viewed as a constant here). Hence we have

$$\frac{F(x+h)-F(x)}{h} - \int_0^1 \frac{t\,\mathrm{e}^{xt}}{t+1} \,\mathrm{d}t = \int_0^1 \frac{t\,\mathrm{e}^{\xi t} - t\,\mathrm{e}^{xt}}{1+t} \,\mathrm{d}t = \int_0^1 \frac{t\,\mathrm{e}^{xt}}{1+t} \left(\mathrm{e}^{(\xi - x)t} - 1 \right) \mathrm{d}t$$

Since $|(\xi - x)t| \leq |h|$, the last integrand can be estimated as

$$\left| \frac{t\,\mathrm{e}^{xt}}{1+t} \left(\mathrm{e}^{(\xi-x)t} - 1 \right) \right| \leq \begin{cases} \mathrm{e}^{x} \left(\mathrm{e}^{|h|} - 1 \right) & \text{if } x > 0, \\ \mathrm{e}^{|h|} - 1 & \text{if } x \leq 0. \end{cases}$$

For the integral, which is over an interval of length 1, we then obtain the same bound, and letting $h \to 0$ shows that $F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$ exists and is equal to $\int_0^1 \frac{t\,\mathrm{e}^{xt}}{t+1} \,\mathrm{d}t$.

Second solution: We have $F(x) = \int_0^1 f(x,t) \,\mathrm{d}t$ with

$$f \colon \mathbb{R} \times [0,1] \to \mathbb{R}, \quad (x,t) \mapsto \frac{\mathrm{e}^{xt}}{1+t}.$$

The partial derivative

$$\frac{\partial}{\partial x} f(x,t) = \frac{t \cdot \mathrm{e}^{xt}}{1+t}$$

is continuous on $\mathbb{R} \times [0,1]$ as function of 2 variables, and the domain of integration $[0,1]$ is compact.

$\Longrightarrow F$ is differentiable; cf. the discussion of parameter integrals in the lecture.

b) Differentiating under the integral sign (cf. the lecture or the justification in the first solution above) gives

$$\begin{aligned}
F'(x) &= \int_0^1 \frac{\partial}{\partial x} f(t,x)\,\mathrm{d}t = \int_0^1 \frac{t \cdot \mathrm{e}^{xt}}{1+t}\,\mathrm{d}t. \\
\Longrightarrow F(x) + F'(x) &= \int_0^1 \left(\frac{\mathrm{e}^{xt}}{1+t} + \frac{t \cdot \mathrm{e}^{xt}}{1+t} \right) \mathrm{d}t = \int_0^1 \frac{(1+t)\mathrm{e}^{xt}}{1+t}\,\mathrm{d}t \\
&= \int_0^1 \mathrm{e}^{xt}\,\mathrm{d}t \\
&= \left[\frac{\mathrm{e}^{xt}}{x} \right]_0^1 \\
&= \frac{\mathrm{e}^x - 1}{x} \quad \text{for} \quad x \neq 0.
\end{aligned}$$

34 a) K is closed because it's defined using '$\leq$' and the functions involved are continuous; K is bounded, since $(x,y,z) \in K$ implies $z(1-z) \geq 0$ and hence $0 \leq z \leq 1$, and then further $x^2 + y^2 \leq 1$, i.e., $|x|, |y| \leq 1$.

b) The set K is rotation-invariant with respect to the z-axis; the z-section $K_z = \{(x,y) \in \mathbb{R}^2; (x,y,z) \in K\}$ is empty for $z < 0$ or $z > 1$ and is a circle of radius $r(z) = z(1-z)$ for $0 \leq z \leq 1$.

$$\begin{aligned}
\Longrightarrow \operatorname{vol}(K) &= \pi \int_0^1 r(z)^2\,\mathrm{d}z \\
&= \pi \int_0^1 z^2(1-z)^2\,\mathrm{d}z \\
&= \pi \int_0^1 z^2 - 2z^3 + z^4\,\mathrm{d}z \\
&= \pi \left[\frac{z^3}{3} - \frac{z^4}{2} + \frac{z^5}{5} \right]_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) \\
&= \frac{\pi}{30}.
\end{aligned}$$

Calculus III (Math 241)

H64 For $x \in \mathbb{R}$ let $\quad F(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{-t^2} \cos(xt)\,\mathrm{d}t$

a) Show that F solves the initial value problem $y' = -\frac{1}{2}xy$, $y(0) = \sqrt{\pi}$.

b) Show that $\quad F(x) = \sqrt{\pi}\,\mathrm{e}^{-x^2/4}$.
Hint: Consider $G(x) = \mathrm{e}^{x^2/4}F(x)$ and show that G is constant.

H65 Do Exercises 40, 42, 50 d) in [Ste21], Ch. 15.3.

H66 Do Exercises 38, 48, 50 in [Ste21], Ch. 15.8.
Hint: There is a quick solution of the last exercise using the change-of-variables formula for rotation-invariant functions given in the lecture slides.

H67 Do Exercise 57 in [Ste21], Ch. 15.1. Does the answer contradict Fubini's general theorem stated in the lecture?
Hint: There is no need to employ a CAS for this exercise—just try to evaluate the iterated integrals using your knowledge from Calculus I/II.

H68 *Optional Exercise*
Do Problem Plus 6 in [Ste21], p. 1121.

H69 *Optional Exercise*
This exercise exhibits a non-measurable bounded subset of $\mathbb{R}$.

a) Show that the relation $\sim$ on $\mathbb{R}$ defined by $x \sim y :\iff x - y \in \mathbb{Q}$ is an equivalence relation; further show that there exists a system V of representatives for the equivalence classes of $\sim$ (i.e., V contains exactly one element from each equivalence class) that is contained in $[0,1]$.

b) Let $\mathbb{Q} \cap [-1,1] = \{q_1, q_2, \dots\}$ and $A := \bigcup_{k=1}^{\infty}(q_k + V)$.
Assuming that V is measurable, show that $\operatorname{vol}(A) = \sum_{k=1}^{\infty} \operatorname{vol}(q_k + V)$ and conclude that $\operatorname{vol}(A) = \operatorname{vol}(V) = 0$.
Hint: $A \subseteq [-1,2]$

c) Show that A contains $[0,1]$ and hence that $\operatorname{vol}(A) \geq 1$.

The contradiction between b) and c) shows that V is not measurable.

H70 *Optional Exercise*

Suppose $f\colon D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, is a map and $\mathbf{a} \in D$ for a), respectively, $\mathbf{a} \in D^\circ$ for b).

a) Show that f is continuous in $\mathbf{a}$ iff for every sequence $(\mathbf{x}^{(k)})$ of points in D with $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{a}$ we have

$$\lim_{k\to\infty} f\big(\mathbf{x}^{(k)}\big) = f(\mathbf{a}) = f\left(\lim_{k\to\infty} \mathbf{x}^{(k)}\right).$$

b) Show that f is partially differentiable in $\mathbf{a}$ with respect to x_1, say, iff for every sequence (t_k) in $\mathbb{R}$ with $\lim_{k\to\infty} t_k = 0$ the sequence $\left(\frac{f(\mathbf{a}+t_k\mathbf{e}_1)-f(\mathbf{a})}{t_k}\right)_{k\in\mathbb{N}}$ converges in $\mathbb{R}$.

Can you identify the topological property of $\mathbb{R}^n$ which is responsible for this characterization of continuity/differentiability in terms of sequences?

Due on Wed Dec 20, 7 pm

The optional exercises can be handed in until Wed Dec 27, 6 pm. Polar coordinates for two- and three-dimensional integrals (required for H65, H66) and the general change-of-variables formula (required for H68) will be discussed in the lecture on Fri Dec 15.

Solutions

64 $F(x)$ is well-defined, since $\left|\mathrm{e}^{-t^2}\cos(xt)\right| \leq \mathrm{e}^{-t^2}$ and the integral $\int_{-\infty}^{+\infty}\mathrm{e}^{-t^2}\,\mathrm{d}t$ exists.

a) Differentiating under the integral sign, we have

$$F'(x) = \int_{-\infty}^{+\infty}\mathrm{e}^{-t^2}\big(-t\sin(xt)\big)\,\mathrm{d}t = -\int_{-\infty}^{+\infty} t\mathrm{e}^{-t^2}\sin(xt)\,\mathrm{d}t\,. \qquad (\star)$$

This is justified, because for an arbitrary real number $R > |x|$ we have $\left|x\mathrm{e}^{-t^2}\sin(xt)\right| \leq R\mathrm{e}^{-t^2}$, which is integrable and independent of x. Thus in the theorem on parameter integrals we can take the bound as $\Phi(t) = R\mathrm{e}^{-t^2}$. The theorem then yields that $(\star)$ holds for $|x'| < R$ and thus in particular for the given x.

Further, partial integration gives

$$\int_{-\infty}^{+\infty} t\mathrm{e}^{-t^2}\sin(xt)\,\mathrm{d}t = \Big[-\tfrac{1}{2}\mathrm{e}^{-t^2}\sin(xt)\Big]_{-\infty}^{+\infty} + \tfrac{1}{2}\int_{-\infty}^{+\infty}\mathrm{e}^{-t^2}x\cos(xt)\,\mathrm{d}t = \frac{x}{2}F(x).$$

Thus $F'(x) = -\frac{1}{2}xF(x)$, as claimed. From the lecture we know that $F(0) = \int_{-\infty}^{+\infty}\mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{\pi}$.

b) $G'(x) = (x/2)\mathrm{e}^{x^2/4}F(x) + \mathrm{e}^{x^2/4}F'(x) = (x/2)\mathrm{e}^{x^2/4}F(x) + \mathrm{e}^{x^2/4}(-x/2)F(x) = 0$
$\Longrightarrow G(x) \equiv G(0) = \mathrm{e}^0 F(0) = \sqrt{\pi} \Longrightarrow F(x) = \sqrt{\pi}\mathrm{e}^{-x^2/4}$

65 **Ex. 40** The domain of integration for the 2-dimensional integral is $D = \{(x,y) \in \mathbb{R}^2; 0 \leq y \leq a, -\sqrt{a^2-y^2} \leq x \leq \sqrt{a^2-y^2}\}$, which is the half disk given in polar coordinates by $\{(r,\theta); 0 < r < a \wedge 0 < \theta < \pi\}$ (with its boundary and center stripped off).

$$\begin{aligned}
\Longrightarrow \int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 2x + y\,\mathrm{d}x\,\mathrm{d}y &= \int\limits_{\substack{0<r<a\\0<\theta<\pi}} (2r\cos\theta + r\sin\theta)r\,\mathrm{d}^2(r,\theta)\\
&= \int\limits_{\substack{0<r<a\\0<\theta<\pi}} r^2(2\cos\theta + \sin\theta)\,\mathrm{d}^2(r,\theta)\\
&= \left(\int_0^a r^2\,\mathrm{d}r\right)\left(\int_0^\pi 2\cos\theta + \sin\theta)\mathrm{d}\theta\right)\\
&= \tfrac{1}{3}a^3\,[2\sin\theta - \cos\theta]_0^\pi = \tfrac{2}{3}a^3.
\end{aligned}$$

Ex. 42

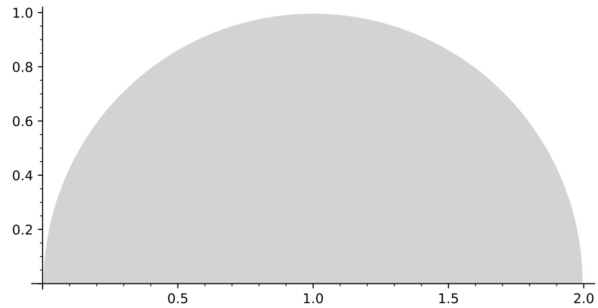


Figure 1: The region $\{(x,y) \in \mathbb{R}^2; y^2 < 2x - x^2 \wedge y > 0\}$, which corresponds to $\{(r,\theta) \in \mathbb{R}^2; 0 < r < 2\cos\theta \wedge 0 < \theta < \pi/2\}$

The (open) domain D of integration for the 2-dimensional integral is given by the inequalities $y^2 < 2x - x^2 \wedge y > 0$, which (since the first inequality is equivalent to $(x-1)^2 + y^2 \leq 1$) forms a half circle of radius 1 with center $(1,0)$; see Figure 1. In polar coordinates D is given by $\{(r,\theta); 0 < r < 2\cos\theta \wedge 0 < \theta < \pi/2\}$.

$$\begin{aligned}
\Longrightarrow \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x &= \int_D \sqrt{x^2+y^2}\,\mathrm{d}^2(x,y) \\
&= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi/2} \left[\tfrac{1}{3} r^3\right]_0^{2\cos\theta} \mathrm{d}\theta \\
&= \int_0^{\pi/2} \tfrac{8}{3} \cos^3\theta\,\mathrm{d}\theta \\
&= \tfrac{8}{3} \int_0^{\pi/2} (1 - \sin^2\theta)\cos\theta\,\mathrm{d}\theta \\
&= \tfrac{8}{3} \left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} \\
&= \frac{16}{9}.
\end{aligned}$$

Ex. 50 a), b), c) where done in the lecture. Using the Lebesgue integral, it is not necessary to make the detour involving improper Riemann integrals indicated in the statement of a).

d) We apply the change-of-variables theorem from the lecture: $x = \sqrt{2}t = T(t)$ is a diffeomorphism from $\mathbb{R}$ to itself with Jacobian $T'(t) = \sqrt{2}$. Hence the change-of-variables theorem can be applied and gives

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-T(t)^2/2} \left|T'(t)\right| \mathrm{d}t = \sqrt{2} \int_{\mathbb{R}} \mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$$

This says that $f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$ satisfies $\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1$. The function f is the probability density function of the normalized *normal distribution*.

66 Ex. 38 Place the center of the sphere at $(0,0,0)$, and suppose that the two planes in spherical coordinates are given by $\theta = 0$ and $\theta = \pi/6$, i.e., the planes are vertical and intersect in the z-

axis. Then in spherical coordinates the volume is given by

$$\begin{aligned} V &= \int\limits_{\substack{0<r<a\\0<\theta<\pi/6\\0<\phi<\pi}} r^2 \sin\phi \, \mathrm{d}^2(r,\theta,\phi) \\ &= \int_0^{\pi/6} \mathrm{d}\theta \int_0^{\pi} \sin\phi \, \mathrm{d}\phi \int_0^a r^2 \, \mathrm{d}r \\ &= \frac{\pi}{6} \cdot 2 \cdot \frac{1}{3} a^3 = \frac{1}{9} \pi a^3. \end{aligned}$$

Ex. 48 We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho = 3960\,\mathrm{mi}$ $\theta = 360^\circ - 73.60^\circ = 286.40^\circ$ $\phi = 90^\circ - 45.50^\circ = 44.50^\circ$	$\rho = 3960\,\mathrm{mi}$ $\theta = 360^\circ - 118.25^\circ = 241.75^\circ$ $\phi = 90^\circ - 34.06^\circ = 55.94^\circ$

Now we change the above to cartesian coordinates using $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho \cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the earth). In particular:

$$\text{Montréal} : (783.67, -2662.67, 2824.47) \quad \text{Los Angeles} : (-1552.80, -2889.91, 2217.84)$$

To find the angle γ between these two vectors we use the dot product:

$$\begin{aligned} \cos\gamma &= \frac{(783.67, -2662.67, 2824.47)}{(3960)^2} \approx 0.8126 \\ &\Rightarrow \gamma \approx 0.6223\,\mathrm{rad}. \end{aligned}$$

The great circle distance between the cities is $s = \rho\gamma \approx (3960)(0.6223) \approx 2462\,\mathrm{mi}$.
Note: We hope you are not confused by the usage of r in the course versus ρ in [Ste12/16] for the first spherical coordinate ("radius"). This exercise is probably the only instance where we are following the textbook's convention. The lecturer thinks that using ρ is not a good idea because it conflicts with the usage of Greek letters for angles.
Ex. 50 This triply iterated integral (or improper triple Riemann integral) is the same as the Lebesgue integral

$$\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}\, \mathrm{e}^{-(x^2+y^2+z^2)} \, \mathrm{d}^3(x,y,z),$$

which can be evaluated using spherical coordinates or the formula for integrals of rotation-invariant functions on $\mathbb{R}^3$ directly:

$$
\begin{aligned}
\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}\,\mathrm{e}^{-(x^2+y^2+z^2)}\,\mathrm{d}^3(x,y,z) &= \int\limits_{\substack{r>0\\0<\theta<2\pi\\0<\phi<\pi}} r\mathrm{e}^{-r^2} r^2 \sin\phi\,\mathrm{d}^3(r,\theta,\phi)\\
&= \int_0^\infty r^3\mathrm{e}^{-r^2}\,\mathrm{d}r \int_0^{2\pi} 1\,\mathrm{d}\theta \int_0^\pi \sin\phi\,\mathrm{d}\phi\\
&= 4\pi\int_0^\infty r^3\mathrm{e}^{-r^2}\,\mathrm{d}r && \text{(cf. lecture slides)}\\
&= 2\pi\int_0^\infty t\,\mathrm{e}^{-t}\,\mathrm{d}t && \text{(substitution } r=t^{1/2},\ \mathrm{d}r=\tfrac{1}{2}t^{-1/2})\\
&= 2\pi\Gamma(2) = 2\pi\cdot 1\cdot\Gamma(1) = 2\pi.
\end{aligned}
$$

The integral $\int_0^\infty t\,\mathrm{e}^{-t}\,\mathrm{d}t$ can be evaluated without resort to the Gamma function, of course.

67 The first double integral has the value $1/2$. Knowing this, the second double integral, which is obviously the negative of the first (interchange the variable names x,y), must have value $-1/2$.
Here is the evaluation of the first integral:

$$
\begin{aligned}
\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^1 \frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_0^1\left[-\frac{x}{(x+y)^2}+\frac{1}{x+y}\right]_{y=0}^1\mathrm{d}x = \int_0^1\left(-\frac{x}{(x+1)^2}+\frac{1}{x+1}+\frac{1}{x}-\frac{1}{x}\right)\mathrm{d}x\\
&= \int_0^1\frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}
\end{aligned}
$$

If $(x,y)\mapsto\frac{x-y}{(x+y)^3}$ were integrable over $[0,1]^2$ then so were $(x,y)\mapsto\frac{|x-y|}{(x+y)^3}$. However, we have

$$
\begin{aligned}
\int\limits_{[\varepsilon,1]\times[0,1]} \frac{|x-y|}{(x+y)^3}\,\mathrm{d}^2(x,y) &= \int_\varepsilon^1\int_0^1 \frac{|x-y|}{(x+y)^3}\,\mathrm{d}y\,\mathrm{d}x && \text{(Little Fubini)}\\
&= \int_\varepsilon^1 \int_{y=0}^x \frac{x-y}{(x+y)^3}\,\mathrm{d}y + \int_{y=x}^1 \frac{y-x}{(x+y)^3}\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_\varepsilon^1 \left[-\frac{x}{(x+y)^2}+\frac{1}{x+y}\right]_{y=0}^x + \left[\frac{x}{(x+y)^2}-\frac{1}{x+y}\right]_{y=x}^1\mathrm{d}x\\
&= \int_\varepsilon^1\left(-\frac{1}{4x}+\frac{1}{2x}+\frac{1}{x}-\frac{1}{x}+\frac{x}{(x+1)^2}-\frac{1}{x+1}-\frac{1}{4x}+\frac{1}{2x}\right)\mathrm{d}x\\
&= \int_\varepsilon^1 \frac{1}{2x}-\frac{1}{(x+1)^2}\,\mathrm{d}x = \left[\frac{1}{2}\ln x+\frac{1}{x+1}\right]_\varepsilon^1 = \frac{1}{2}-\frac{\ln\varepsilon}{2}-\frac{1}{\varepsilon+1}.
\end{aligned}
$$

For $\varepsilon\downarrow 0$ this tends to $+\infty$, saying that the 2-dimensional Lebesgue integral $\int_{[0,1]^2}\frac{|x-y|}{(x+y)^3}\,\mathrm{d}^2(x,y)$ does not exist. Hence Fubini's general theorem is not applicable.

68 Let $x = \frac{u-v}{\sqrt{2}}$ and $y = \frac{u+v}{\sqrt{2}}$. We know the region of integration in the xy-plane, so to find its image in the uv-plane we get u and v in terms of x and y, and then use the methods of change of variables in multiple integrals. In matrix terms we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

using the fact that the inverse of a rotation matrix $R(\phi)$ is $R(-\phi)$ (here $\phi = \pi/4$). Since $T(u,v) = (x,y)$ is a rotation, its Jacobian is equal to 1. The inverse T^{-1} maps the unit square $[0,1]^2$ to the parallelogram spanned by $T^{-1}(\mathbf{e}_1) = \frac{1}{\sqrt{2}}(1,-1)^{\mathsf{T}}$ and $T^{-1}(\mathbf{e}_2) = \frac{1}{\sqrt{2}}(1,1)^{\mathsf{T}}$, which is another square with sidelength 1 and depicted in Figure 2. (In fact, since the image is obtained by rotating the unit square by -45°, no computation is necessary; just draw the picture.) In Figure 2 it is also indicated how the edges of the unit square are mapped by T, but this is not needed in the sequel.

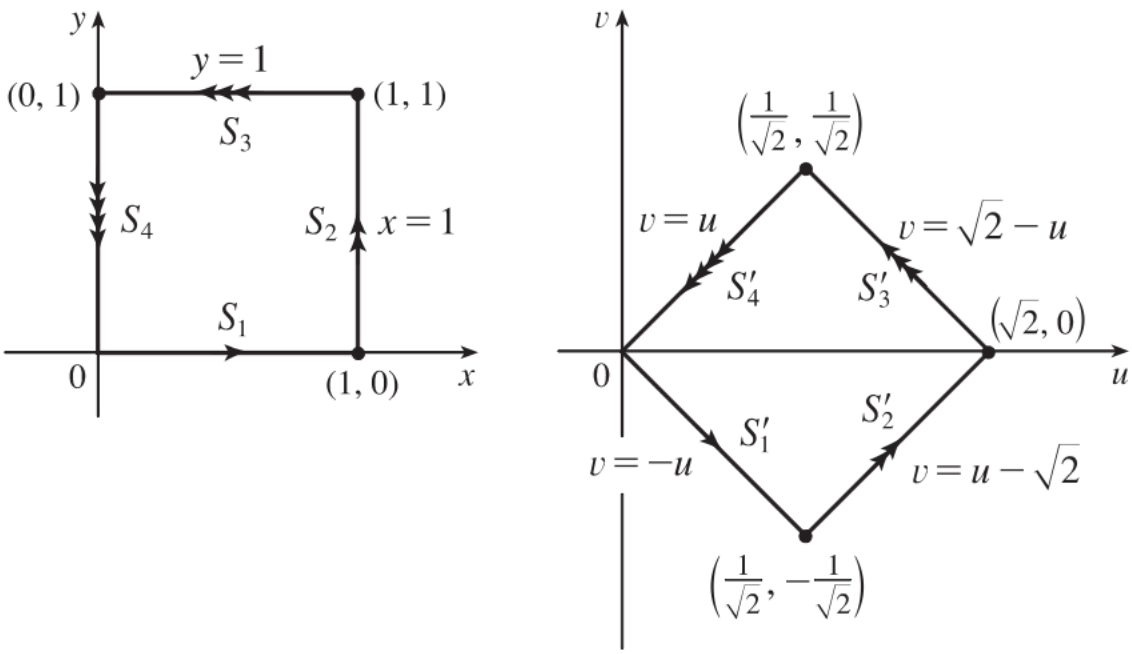


Figure 2: The unit square $[0,1]^2$ and its image under T^{-1}

From the diagram, we see that we must evaluate two integrals: one over the region

$$\left\{ (u,v); 0 \le u \le \frac{1}{\sqrt{2}}, -u \le v \le u \right\},$$

and the other over

$$\left\{ (u,v); \frac{1}{\sqrt{2}} \le u \le \sqrt{2}, -\sqrt{2}+u \le v \le \sqrt{2}-u \right\}.$$

We obtain

$$
\begin{aligned}
&\int_0^1 \int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{1}{1-(\frac{1}{\sqrt{2}}(u+v))(\frac{1}{\sqrt{2}}(u-v))}\,\mathrm{d}v\,\mathrm{d}u + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{1}{1-(\frac{1}{\sqrt{2}}(u+v))(\frac{1}{\sqrt{2}}(u-v))}\,\mathrm{d}v\,\mathrm{d}u \\
&= \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{2}{2-u^2+v^2}\,\mathrm{d}v\,\mathrm{d}u + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{2}{2-u^2+v^2}\,\mathrm{d}v\,\mathrm{d}u \\
&= 2\left[\int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u}\mathrm{d}u + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-\sqrt{2}+u}^{\sqrt{2}-u}\mathrm{d}u\right] \\
&= 4\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{u}{\sqrt{2-u^2}}\right]\mathrm{d}u + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\right]\mathrm{d}u\right]
\end{aligned}
$$

Now let $u = \sqrt{2}\sin\theta$, so $\mathrm{d}u = \sqrt{2}\cos\theta\,\mathrm{d}\theta$ and the limits change to 0 and $\pi/6$ (in the first integral) and $\pi/6$ and $\pi/2$ (in the second integral). Continuing:

$$
\begin{aligned}
&\int_0^1 \int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y \\
&= 4\int_0^{\pi/6} \frac{1}{\sqrt{2-2\sin^2\theta}} \arctan\frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\,(\sqrt{2}\cos\theta\,\mathrm{d}\theta) \\
&+4\int_{\pi/6}^{\pi/4} \frac{1}{\sqrt{2-2\sin^2\theta}} \arctan\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\,(\sqrt{2}\cos\theta\,\mathrm{d}\theta) \\
&= 4\int_0^{\pi/6} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta} \arctan\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\,\mathrm{d}\theta + 4\int_{\pi/6}^{\pi/4} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta} \arctan\frac{\sqrt{2}(1-\sin\theta)}{\sqrt{2}\cos\theta}\,\mathrm{d}\theta \\
&= 4\int_0^{\pi/6} \arctan\tan\theta\,\mathrm{d}\theta + 4\int_{\pi/6}^{\pi/2} \arctan\frac{1-\sin\theta}{\cos\theta}\,\mathrm{d}\theta.
\end{aligned}
$$

But (following the hint),

$$
\begin{aligned}
\frac{1-\sin\theta}{\cos\theta} &= \frac{1-\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)} \\
&= \frac{1-(2\sin^2(1/2(\pi/2-\theta)))}{2\sin(1/2(\pi/2-\theta))\cos(1/2(\pi/2-\theta))} \\
&= \tan(1/2(\pi/2-\theta)).
\end{aligned}
$$

Continuing:

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y\\
&= 4\int_0^{\pi/6} \arctan\tan\theta\,\mathrm{d}\theta + 4\int_{\pi/6}^{\pi/2} \arctan\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\mathrm{d}\theta\\
&= 4\int_0^{\pi/6} \theta\,\mathrm{d}\theta + 4\int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\mathrm{d}\theta\\
&= 2\frac{\pi^2}{36} + \pi\left(\frac{\pi}{2}-\frac{\pi}{6}\right) - \left(\frac{\pi^2}{4}-\frac{\pi^2}{36}\right)\\
&= \frac{\pi^2}{6}.
\end{aligned}$$

69 a) The relation $\sim$ is reflexive (since $x-x=0\in\mathbb{Q}$), symmetric (since $x-y\in\mathbb{Q}$ implies $y-x=-(x-y)\in\mathbb{Q}$), and transitive (since $x-y\in\mathbb{Q}$, $y-z\in\mathbb{Q}$ imply $x-z=(x-y)+(y-z)\in\mathbb{Q}$). Hence it is an equivalence relation.

From the general theory of equivalence relations we know that $\mathbb{R}$ is partitioned into equivalence classes. Choosing exactly one element from each equivalence class, we obtain our set V. Since $v\sim v-\lfloor v\rfloor$, we can assume $V\subseteq[0,1]$.

b) Suppose $\mathrm{vol}(V)=c>0$. Since the sets q_k+V are mutually disjoint ($k\neq l$ and $q_k+v=q_l+v'$ implies $v-v'=q_l-q_k\in\mathbb{Q}\setminus\{0\}$, which contradicts the system-of-representatives property) and the Lebesgue measure is translation-invariant, we have

$$\mathrm{vol}\left(\bigcup_{k=1}^{n}(q_k+V)\right)=\sum_{k=1}^{n}\mathrm{vol}(q_k+V)=nc\quad\text{for every } n\in\mathbb{N}.$$

Choosing n with $nc>3$ gives a contradiction, since $\bigcup_{k=1}^{n}(q_k+V)\subset A\subseteq[-1,2]$.

Hence we must have $\mathrm{vol}(V)=0$, and σ-additivity of the Lebesgue measure then gives that A is measurable with

$$\mathrm{vol}(A)=\sum_{k=1}^{\infty}\mathrm{vol}(q_k+V)=0.$$

c) Every $x\in[0,1]$ has a representation $x=q+v$ for some $q\in\mathbb{Q}$ and $v\in V$. Since $0\leq x,v\leq 1$, we $q=x-v\in[-1,1]$ and hence $q=q_k$ for some k.
$\Longrightarrow x\in q_k+V\subset A$. Thus we have $[0,1]\subseteq A$ and $\mathrm{vol}(A)\geq\mathrm{vol}\big([0,1]\big)=1$.

70 a) $\Longrightarrow$: Suppose that f is continuous in $\mathbf{a}$. Then for any $\varepsilon>0$ there exists a response $\delta>0$ such that $\mathbf{x}\in D\wedge|\mathbf{x}-\mathbf{a}|<\delta$ implies $|f(\mathbf{x})-f(\mathbf{a})|<\varepsilon$.

Now let $(\mathbf{x}^{(k)})$ be a sequence in D with $\lim_{k\to\infty}\mathbf{x}^{(k)}=\mathbf{a}$. Then for the given δ there is a response $K\in\mathbb{N}$ such that $k>K$ implies $\left|\mathbf{x}^{(k)}-\mathbf{a}\right|<\delta$. From the above we then have $\left|f(\mathbf{x}^{(k)})-f(\mathbf{a})\right|<\varepsilon$ for $k>K$, i.e., K also serves as a response for ε in the proof of $\lim_{k\to\infty}f\big(\mathbf{x}^{(k)}\big)=f(\mathbf{a})$.

$\Longleftarrow$: Suppose that f is discontinuous in $\mathbf{a}$. Then there is an $\varepsilon>0$ that has no response, i.e., for every $\delta>0$ there exists $\mathbf{x}\in D$ with $|\mathbf{x}-\mathbf{a}|<\delta\wedge|f(\mathbf{x})-f(\mathbf{a})|\geq\varepsilon$. Specializing to $\delta = 1/k$, $k \in \mathbb{N}$, there exists $\mathbf{x}^{(k)} \in D$ such that $\left|\mathbf{x}^{(k)} - \mathbf{a}\right| < 1/k$ but $\left|f(\mathbf{x}^{(k)}) - f(\mathbf{a})\right| \geq \varepsilon$. The sequence $(\mathbf{x}^{(k)})$ is easily seen to converge to $\mathbf{a}$, but $\big(f(\mathbf{x}^{(k)})\big)$ does not converge to $f(\mathbf{a})$. This contradiction proves " $\Longleftarrow$".

b)
The crucial fact used in the proof of " $\Longleftarrow$" is that the balls $\mathrm{B}_{1/k}(\mathbf{a})$, $k \in \mathbb{N}$, form a basis of neighborhoods of $\mathbf{a}$, i.e., every ball $\mathrm{B}_{\delta}(\mathbf{a})$, $\delta > 0$, contains a ball $\mathrm{B}_{1/k}(\mathbf{a})$, $k \in \mathbb{N}$. (For this choose k such that $1/k < \delta$.) Since there are only countably many balls of the form $\mathrm{B}_{1/k}(\mathbf{a})$, we can describe continuity in terms of sequences.

ZJU-UIUC Institute
Prof. Thomas Honold

Fall Semester 2023
Worksheet 12

Calculus III (Math 241)

W35 Do Exercises 23, 27 in [Ste21], Ch. 15.8.

W36 Do Exercises 16, 20, 26 in [Ste21], Ch. 15.9.

W37 Do Exercises 18, 20 in [Ste21], Ch. 16.2

Solutions

35 Ex. 23

$$\iiint_B (x^2+y^2+z^2)^2\,\mathrm{dV} = \int\limits_{x^2+y^2+z^2\le 25} (x^2+y^2+z^2)^2\,\mathrm{d}^3(x,y,z) = \int\limits_{\substack{0<r<5\\0<\theta<2\pi\\0<\phi<\pi}} r^4(r^2\sin\phi)\,\mathrm{d}^2(r,\theta,\phi)$$

$$= \int_0^5 r^6\,\mathrm{d}r \int_0^{2\pi} 1\,\mathrm{d}\theta \int_0^{\pi} \sin\phi\,\mathrm{d}\phi = \frac{5^7}{7}\cdot 2\pi\cdot 2 = \frac{312500\,\pi}{7}$$

Ex. 27

$$\iiint_E x\,\mathrm{e}^{x^2+y^2+z^2}\,\mathrm{dV} = \int\limits_{\substack{x^2+y^2+z^2\le 1\\x,y,z\ge 0}} x\,\mathrm{e}^{x^2+y^2+z^2}\,\mathrm{d}^3(x,y,z) = \int\limits_{\substack{0<r<1\\0<\theta<\pi/2\\0<\phi<\pi/2}} r\cos\theta\sin\phi\,\mathrm{e}^{r^2}\,r^2\sin\phi\,\mathrm{d}^3(r,\theta,\phi)$$

$$= \int_0^1 r^3\mathrm{e}^{r^2}\,\mathrm{d}r \int_0^{\pi/2}\cos\theta\,\mathrm{d}\theta \int_0^{\pi/2}\sin^2\phi\,\mathrm{d}\phi = \frac{1}{2}\cdot 1\cdot\frac{\pi}{4} = \frac{\pi}{8},$$

since $\int_0^1 r^3\mathrm{e}^{r^2}\,\mathrm{d}r = \int_0^1 (r^2)(r\mathrm{e}^{r^2})\,\mathrm{d}r = \left[r^2\,\tfrac{1}{2}\mathrm{e}^{r^2}\right]_0^1 - \int_0^1 2r\,\tfrac{1}{2}\mathrm{e}^{r^2}\,\mathrm{d}r = \tfrac{\mathrm{e}}{2} - \left[\tfrac{1}{2}\mathrm{e}^{r^2}\right]_0^1 = \tfrac{1}{2}$ and $\int_0^{\pi/2}\sin^2\phi\,\mathrm{d}\phi = \int_0^{\pi/2}\cos^2\phi\,\mathrm{d}\phi = \tfrac{1}{2}\int_0^{\pi/2} 1\,\mathrm{d}\phi = \tfrac{\pi}{4}$.

36 Ex. 16

$$x = u+vw,\; y = v+wu,\; z = w+uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix}1 & w & v\\ w & 1 & u\\ v & u & 1\end{vmatrix} = 1 + 2uvw - u^2 - v^2 - w^2$$

Ex. 20

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\sqrt{2} & -\sqrt{2/3}\\ \sqrt{2} & \sqrt{2/3}\end{vmatrix} = \frac{4}{\sqrt{3}},$$
$$x^2 - xy + y^2 = 2u^2 + 2v^2$$

Hence the elliptic disk $x^2 - xy + y^2 \le 2$ is the image of the circular disk $u^2+v^2\le 1$. Thus,

$$\iint_R (x^2-xy+y^2)\,\mathrm{d}A = \iint\limits_{u^2+v^2\le 1} (2u^2+2v^2)\frac{4}{\sqrt{3}}\,\mathrm{d}^2(u,v) = \int_0^{2\pi}\int_0^1 \frac{8}{\sqrt{3}}\,r^3\,\mathrm{d}r\,\mathrm{d}\theta = \frac{4\pi}{\sqrt{3}} = \frac{4\pi\sqrt{3}}{3}.$$

Ex. 26 Letting $u = x+y$ and $v = x-y$, we have $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix} = -\frac{1}{2},$$

and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 3$, $v = 0$, and $v = 2$. Thus,

$$\begin{aligned}
\iint_R (x+y)e^{x^2-y^2}\,\mathrm{d}A &= \int\limits_{[0,3]\times[0,2]} ue^{uv}\left|-\frac{1}{2}\right|\,\mathrm{d}^2(u,v) \\
&= \frac{1}{2}\int_0^3\int_0^2 ue^{uv}\,\mathrm{d}v\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^3 \left[e^{uv}\right]_{v=0}^{v=2}\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^3 (e^{2u}-1)\,\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{1}{2}e^{2u}-u\right]_0^3 \\
&= \frac{1}{2}(\frac{1}{2}e^6-3-\frac{1}{2}) \\
&= \frac{1}{4}(e^6-7).
\end{aligned}$$

37 Ex. 18

1. C_1: $(0,0,0)\to(1,0,1)$:
   Let $x = t$, $y = 0$, $z = t$, $0 \le t \le 1$. Then $\mathrm{d}x = \mathrm{d}z = \mathrm{d}r$, and $\mathrm{d}y = 0$. So $\int_{C_1}(y+z)\ \mathrm{d}x + (x+z)\ \mathrm{d}y + (x+y)\ \mathrm{d}z = \int_0^1 2t\ \mathrm{d}r = 1$.

2. C_2: $(1,0,1)\to(0,1,2)$:
   Let $x = t$, $y = 1-t$, $z = 2-t$, $0 \le t \le 1$. Then $\mathrm{d}x = \mathrm{d}r$, and $\mathrm{d}y = \mathrm{d}z = -\mathrm{d}r$. So $\int_{c_2}(y+z)\ \mathrm{d}x + (x+z)\ \mathrm{d}y + (x+y)\ \mathrm{d}z = \int_1^0 -2t\,\mathrm{d}r = 1$.

3. Combining C_1 and C_2, we get $\int_c(y+z)\ \mathrm{d}x + (x+z)\ \mathrm{d}y + (x+y)\ \mathrm{d}z = 1+1 = 2$.

Remark by TH: The solution above is correct but in a way non-standard. Here is the computation of $\int_{C_2}(y+z)\ \mathrm{d}x + (x+z)\ \mathrm{d}y + (x+y)\ \mathrm{d}z$ using the terminology of the lecture: First we choose a parametrization of C_2, e.g.,

$$\gamma(t) = \begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} + t\left[\begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix}\right] = \begin{pmatrix}1\\0\\1\end{pmatrix} + t\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}1-t\\t\\1+t\end{pmatrix}, \quad t\in[0,1].$$

Then the integral is computed as

$$\begin{aligned}
\int_{C_2}(y+z)\ \mathrm{d}x + (x+z)\ \mathrm{d}y + (x+y)\ \mathrm{d}z &= \int_0^1 \begin{pmatrix} y(t)+z(t)\\ x(t)+z(t)\\ x(t)+y(t)\end{pmatrix}\cdot\begin{pmatrix}x'(t)\\y'(t)\\z'(t)\end{pmatrix}\mathrm{d}r \\
&= \int_0^1 \begin{pmatrix}1+2t\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}\mathrm{d}r = \int_0^1 2-2t\,\mathrm{d}r = 1.
\end{aligned}$$

Ex. 20

1. C_1: Positive. Vectors attached to points on C_1 form an acute angle with the tangent direction of C_1 at that point. So the dot products $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)$, $t \in [a, b]$ say, are all positive, and hence $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\,\mathrm{d}r$ is positive as well.

2. C_2: Negative. Half of the vectors attached to points on C_2 form an acute angle with the tangent direction of C_2 at that point and half an obtuse angle, but those forming an obtuse angle are larger in length (and the angle is "more obtuse"). Hence the integral over the 2nd quartercircle of C_2 (in the given orientation) is negative and larger in absolute value than the integral over the 1st quartercircle.

Calculus III (Math 241)

H71 Do Exercise 30 in [Ste21], Ch. 15.8.

H72 Do Exercises 17, 25 in [Ste21], Ch. 15.9.

Note: You can check your figures against those in [Ste21], Appendix H, but the computation is what counts.

H73 Do Exercises 38, 44 in [Ste21], Ch. 16.2.
For Ex. 38 you may need to study the material on line integrals with respect to arc length in Ch. 16.2 (look for the paragraph "Line Integrals in Space").

H74 Do some of the even-numbered Exercises 3–10 in [Ste21], Ch. 16.3.

Note: You need not do all the exercises, but include at least one of each type (i.e., one where **F** is conservative and one where it is not).

H75 Do Exercises 24, 26, 32, 34 in [Ste121], Ch. 16.3 (three answers suffice).

H76 *Optional Exercise*

a) Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a closed path with $(0,0) \notin \gamma\big([a,b]\big)$. In the lecture it is stated that the oriented area A enclosed by the normalized curve

$$\beta\colon [a,b] \to \mathbb{R}^2,\ t \mapsto \frac{\gamma(t)}{|\gamma(t)|}$$

satisfies

$$A = \frac{1}{2}\int_\gamma \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}.$$

Prove this formula.

Hint: Use *Leibniz's sector formula* for $\beta(t) = \big(u(t), v(t)\big)$, viz. $A = \frac{1}{2}\int_\beta x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}\int_a^b uv' - vu'\,\mathrm{d}t$. You can derive this formula by switching in Formula 4 on p. 670 of [Ste16], Ch. 10.4 from polar to cartesian coordinates.

b) Graph the trefoil curve

$$\gamma(t) = \big(\cos t + 2\cos(2t), -\sin t + 2\sin(2t)\big), \quad t \in [0, 2\pi]$$

together with the winding numbers $n(\gamma;\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^2 \setminus \gamma\big([0,2\pi]\big)$.

c) Compute the winding number $n(\gamma;\mathbf{0})$ using the definition as a line integral.

Hint: The resulting ordinary integral can be evaluated using the substitution $t = \tan(s/2)$, which transforms the integrand into a rational function.

Due on Wed Dec 27, 6 pm

The optional exercise should also be handed in on Wed Dec 27.

Solutions

71 The average distance is

$$\frac{1}{\frac{4}{3}a^3\pi}\int_{x^2+y^2+z^2\le a^2}\sqrt{x^2+y^2+z^2}\,\mathrm{d}^3(x,y,z)=\frac{1}{\frac{4}{3}a^3\pi}\times 4\pi\int_0^a r\,r^2\,\mathrm{d}r=\frac{a^4\pi}{\frac{4}{3}a^3\pi}=\frac{3a}{4}.$$

72 **Ex. 17** The linear transformation

$$T\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}2u+v\\u+2v\end{pmatrix}=\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

maps $(0,0)$ to $(0,0)$, $(1,0)$ to $(2,1)$, $(0,1)$ to $(1,2)$ (think of these as column vectors) and hence the triangle Δ with vertices $(0,0)$, $(1,0)$, $(0,1)$ to R.

$$\begin{aligned}\Longrightarrow \iint_R x-3y\,\mathrm{dA}&=\int_R x-3y\,\mathrm{d}^2(x,y)=\int_\Delta T(u,v)\,|\det \mathbf{J}_T(u,v)|\,\mathrm{d}^2(u,v)\\&=\int_\Delta 2u+v-3(u+2v)\left|\det\begin{pmatrix}2&1\\1&2\end{pmatrix}\right|\mathrm{d}^2(u,v)=3\int_\Delta -u-5v\,\mathrm{d}^2(u,v)\\&=-3\int_{u=0}^1\int_{v=0}^{1-u}u+5v\,\mathrm{d}v\,\mathrm{d}u=-3\int_0^1\left[uv+\tfrac{5}{2}v^2\right]_{v=0}^{1-u}\mathrm{d}u\\&=-3\int_0^1 u(1-u)+\tfrac{5}{2}(1-u)^2\,\mathrm{d}u=-3\int_0^1\tfrac{3}{2}u^2-4u+\tfrac{5}{2}\,\mathrm{d}u\\&=-3\left(\tfrac{1}{2}-2+\tfrac{5}{2}\right)=-3\end{aligned}$$

Ex. 25 Change-of-variables gives

$$\iint_R\frac{x-2y}{3x-y}\,\mathrm{dA}=\int_{T^{-1}(R)}\frac{u}{v}\det(T)\mathrm{d}^2(u,v),$$

where $T\colon\mathbb{R}^2\to\mathbb{R}^2$, $(u,v)\to(x,y)$ is the linear map determined by $(u,v)=(x-2y,3x-y)$, i.e., the inverse of $(x,y)\mapsto(x-2y,3x-y)$. The map T has matrix $\left(\begin{smallmatrix}1&-2\\3&-1\end{smallmatrix}\right)^{-1}=\frac{1}{5}\left(\begin{smallmatrix}-1&2\\-3&1\end{smallmatrix}\right)$ and determinant $\frac{1}{5}$. (Generally, $\det(\mathbf{A}^{-1})=1/\det\mathbf{A}$, as follows from $\det(\mathbf{A})\det(\mathbf{A}^{-1})=\det(\mathbf{A}\mathbf{A}^{-1})=\det(\mathbf{I})=1$.) Moreover, T^{-1} maps the 4 lines bounding R as follows:

$x-2y=0$	$x-2y=4$	$3x-y=1$	$3x-y=8$
$x=0$	$x=4$	$y=1$	$y=8$

From this it is clear that $T^{-1}(R)=[0,4]\times[1,8]$ is a rectangle with sides parallel to the coordinate axes.

$$\Longrightarrow\iint_R\frac{x-2y}{3x-y}\,\mathrm{dA}=\frac{1}{5}\int_{u=0}^4\int_{v=1}^8\frac{u}{v}\,\mathrm{d}^2(u,v)=\frac{1}{5}\int_0^4 u\,\mathrm{d}u\int_1^8\frac{\mathrm{d}v}{v}=\frac{8\ln 8}{5}$$

73 **Ex. 38** $\rho(t)=\left(\sqrt{x^2+y^2+z^2}\right)^2=t^2+\cos^2 t+\sin^2 t=t^2+1$
$\mathrm{d}s=\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,\mathrm{d}t=\sqrt{2}\,\mathrm{d}t$
$m=\int_C\rho(x,y,z)\,\mathrm{d}s=\int_0^{2\pi}(t^2+1)\sqrt{2}\,\mathrm{d}t=\sqrt{2}(\frac{8}{3}\pi^3+2\pi)$
$\bar{x}=\frac{1}{m}\int_C x\rho(x,y,z)\,\mathrm{d}s=\frac{1}{m}\int_0^{2\pi}(t^3+t)\sqrt{2}\,\mathrm{d}t=\frac{6\pi^3+3\pi}{4\pi^2+3}$

$\bar{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,\mathrm{d}s = \frac{1}{m}\int_0^{2\pi}\cos t\,(t^2+1)\sqrt{2}\,\mathrm{d}t = \frac{6}{4\pi^2+3}$
$\bar{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,\mathrm{d}s = \frac{1}{m}\int_0^{2\pi}\sin t\,(t^2+1)\sqrt{2}\,\mathrm{d}t = \frac{-6\pi}{4\pi^2+3}$
So the mass is $\sqrt{2}(\frac{8}{3}\pi^3+2\pi)$, and the center of mass is $(\frac{6\pi^3+3\pi}{4\pi^2+3}, \frac{6}{4\pi^2+3}, \frac{-6\pi}{4\pi^2+3})$.
Ex. 44 The line segment is parametrized by $\gamma(t) = (2,0,0)+t(0,1,5) = (2,t,5t)$, $t \in [0,1]$.
$\Longrightarrow$ The work W done by the electric field is

$$\begin{aligned} W &= K\int_0^1 \frac{\gamma(t)}{|\gamma(t)|^3}\gamma'(t)\,\mathrm{d}t = K\int_0^1 \frac{(2,t,5t)\cdot(0,1,5)}{(4+26t^2)^{3/2}}\,\mathrm{d}t = K\int_0^1 \frac{26t}{(4+26t^2)^{3/2}}\,\mathrm{d}t \\ &= -K\left[\left(4+26t^2\right)^{-1/2}\right]_0^1 = K\left(\frac{1}{2}-\frac{1}{\sqrt{30}}\right). \end{aligned}$$

74 In what follows, we write $\mathbf{F} = (P,Q)$. Since the domain of F ($\mathbb{R}^2$ in the first three exercises, the first quadrant of $\mathbb{R}^2$ in the last exercise) is simply-connected in each case, the condition $P_y = Q_x$ is sufficient for the existence of an anti-derivative f.
Ex. 4 $\mathbf{F}(x,y) = (y^2-2x, 2xy)$
$P_y = 2y = Q_x \Longrightarrow \mathbf{F}$ is conservative; $f_x = y^2-2x \Longrightarrow f(x,y) = xy^2-x^2+g(y) \Longrightarrow f_y = 2xy+g'(y) = 2xy \Longrightarrow g'(y) = 0$. Hence we can take $f(x,y) = xy^2-x^2$.
Ex. 6 $\mathbf{F}(x,y) = (y\mathrm{e}^x, \mathrm{e}^x+\mathrm{e}^y)$
$P_y = \mathrm{e}^x = Q_x \Longrightarrow \mathbf{F}$ is conservative. An anti-derivative found by the same method is $f(x,y) = y\mathrm{e}^x+\mathrm{e}^y$.
Ex. 8 $\mathbf{F}(x,y) = (3x^2-2y^2, 4xy+3)$
$P_y = -4y \neq 4y = Q_x$ (except for points on the x-axis) $\Longrightarrow \mathbf{F}$ is not conservative.
Ex. 10 $F(x,y) = (\ln y+y/x, \ln x+x/y)$
$P_y = 1/y+1/x = Q_x \Longrightarrow \mathbf{F}$ is conservative. An anti-derivative is $f(x,y) = x\ln y+y\ln x$.

75 Ex. 24

(a) $f_x = \sin y \Longrightarrow f(x,y,z) = x\sin y+g(y,z)$
$f_y = x\cos y+g_y(y,z) = x\cos y+\cos z \Rightarrow g_y(y,z) = \cos z \Rightarrow g(y,z) = y\cos z+h(z)$
$f_z = g_z(y,z) = -y\sin z+h'(z) = -y\sin z \Longrightarrow h'(z) = 0 \Longrightarrow h(z) = C$
$f(x,y,z) = x\sin y+y\cos z+C$
Let $C = 0$ we have $f(x,y,z) = x\sin y+y\cos z$

(b) $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = f(\mathbf{r}(\frac{\pi}{2}))-f(\mathbf{r}(0)) = f(1,\frac{\pi}{2},\pi)-f(0,0,0) = 1-\frac{\pi}{2}$

Ex. 26

1. Since $\frac{\partial}{\partial y}(1-y\mathrm{e}^{-x}) = -\mathrm{e}^{-x} = \frac{\partial}{\partial x}\mathrm{e}^{-x}$ and $\mathbb{R}^2$ is star-shaped, the differential 1-form $\omega = (1-y\mathrm{e}^{-x})\,\mathrm{d}x+\mathrm{e}^{-x}\,\mathrm{d}y$ is exact on $\mathbb{R}^2$. (Alternatively, the vector field $\mathbf{F}(x,y) = (1-y\mathrm{e}^{-x}, \mathrm{e}^{-x})$, $(x,y) \in \mathbb{R}^2$, is conservative.)

2. One such path is $(0,1) \to (1,1) \to (1,2)$ along straight line segments (parametrized in the obvious way) and gives

$$\begin{aligned} \int_C (1-y\mathrm{e}^{-x})\,\mathrm{d}x+\mathrm{e}^{-x}\,\mathrm{d}y &= \int_0^1 (1-1\mathrm{e}^{-x})\,\mathrm{d}x+\int_0^1 \mathrm{e}^{-1}\,\mathrm{d}y \\ &= 1-\int_0^1 \mathrm{e}^{-x}\,\mathrm{d}x+\mathrm{e}^{-1} = 1-(1-\mathrm{e}^{-1})+\mathrm{e}^{-1} = 2\mathrm{e}^{-1}. \end{aligned}$$

Alternatively, $f(x,y) = x + ye^{-x}$ satisfies $\mathrm{d}f = \omega$, and hence the fundamental theorem for line integrals gives

$$\int_C (1 - ye^{-x})\,\mathrm{d}x + e^{-x}\,\mathrm{d}y = f(1,2) - f(0,1) = 1 + 2e^{-1} - (0 + 1e^{-0}) = 2e^{-1}.$$

In this case both alternatives require roughly the same effort but for other integrands/paths of integration using the fundamental theorem for line integrals can be much cheaper.

Ex. 32 In this graph, any closed path will have attached some vectors pointing in the direction of the curve and some pointing against it. Integration along the path could then yield the value 0 Although we can't be certain since we aren't given any numbers, this vector field is likely to be conservative.
Another argument in support of "$\mathbf{F}$ is conservative" is the following: One can imagine a function $f = f(x,y)$ which has $\mathbf{F}$ as its gradient field, a critical point in $(0,0)$, valleys in directions NW and SE from $(0,0)$ and ridges in directions SW and NE from $(0,0)$.
Yet another argument (perhaps the most convincing, since it reverse-engineers the construction in [Ste12/16]) is this: $\mathbf{F}$ has an apparent symmetry, viz. $\mathbf{F} \circ R(\phi) = R(-\phi) \circ \mathbf{F}$ for any 2×2 rotation matrix $R(\phi)$. This can also be written as $R(\phi) \circ \mathbf{F} \circ R(\phi) = \mathbf{F}$ and is satisfied by all linear reflections of $\mathbb{R}^2$; see the solution to H13, from which $R(\phi_1)S(\phi_2)R(\phi_1) = S(\phi_2)$ is immediate. If we assume that $\mathbf{F}$ is indeed a reflection, it must be the one at the plane $\phi = 3\pi/8$, since $F(1,0)$ has direction $\mathbb{R}(-1,1)$ (and $F(1,0)$ direction $\mathbb{R}(1,1)$, which is consistent with this). Thus we arrive at

$$\mathbf{F}(x,y) = \left(\frac{-x+y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right) = \begin{pmatrix} x & y \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} S(3\pi/4),$$

and we see from this that $\mathbf{F}$ is conservative, because it satisfies the necessary condition in Poincaré's Lemma and $\mathbb{R}^2$ is simply connected. (More generally, what is the condition for linear maps $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ to determine a conservative vector field?)
Ex. 34 Because $\mathbf{F} = \operatorname{grad} f$ means that $\mathbf{F}$ is conservative, any line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ equals the difference between the values of f at the end point and the starting point of C.

1. In particular we have $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ if $x - 2y$ attains the same value at $\mathbf{a}$ and $\mathbf{b}$. Thus we can take, e.g., as C_1 any curve from $(0,0)$ to $(2,1)$.

2. Since $\sin(\pi/2) - \sin 0 = 1$, we can take as C_2 any curve from $\mathbf{a} = (a_1, a_2)$ to $\mathbf{b} = (b_1, b_2)$ satisfying $a_1 - 2a_2 = 0$ and $b_1 - 2b_2 = \pi/2$. For example, $\mathbf{a} = (0,0)$ and $\mathbf{b} = (\pi, \pi/4)$ are legitimate choices.

76 First a few words about the meaning and derivation of Leibniz's sector formula.
For a C^1-curve $\gamma\colon [a,b] \to \mathbb{R}^2$ Leibniz's sector formula gives the oriented area of the "sector" enclosed by the line segments $[\mathbf{0}, \gamma(a)]$, $[\mathbf{0}, \gamma(b)]$ and the curve. (At least for small parameter intervals $[a,b]$ and if the angle $\phi(t)$ of the curve in polar coordinates is strictly increasing/decreasing in $[a,b]$ then this region looks like a sector of a circle.)
For small $\Delta t > 0$ the oriented area A of the "sector" enclosed by the line segments $[\mathbf{0}, \gamma(t)]$, $[\mathbf{0}, \gamma(t + \Delta t)]$ and the curve (restricted to $[t, t + \Delta t]$) is approximately equal to that of the triangle with vertices $(0,0)$, $\gamma(t)$, $\gamma(t + \Delta t)$, which is given by

$$A = \frac{1}{2} \begin{vmatrix} x(t) & x(t+\Delta t) \\ y(t) & y(t+\Delta t) \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x(t) & x(t+\Delta t) - x(t) \\ y(t) & y(t+\Delta t) - y(t) \end{vmatrix}, \qquad \text{where } \gamma(t) = \big(x(t), y(t)\big).$$

Since γ is differentiable, we have $x(t+\Delta t)-x(t)=x'(\tau_1)\Delta t$ for some $\tau_1 \in (t,t+\Delta t)$ by the Mean Value Theorem, and similarly for $y(t)$, so that

$$A=\frac{1}{2}\begin{vmatrix}x(t) & x'(\tau_1)\Delta t\\ y(t) & y'(\tau_2)\Delta t\end{vmatrix}=\frac{\Delta t}{2}\begin{vmatrix}x(t) & x'(\tau_1)\\ y(t) & y'(\tau_2)\end{vmatrix}=\Delta t\,\frac{x(t)y'(\tau_2)-y(t)x'(\tau_1)}{2}.$$

Summing these quantities for a partition $a=t_0<t_1<\cdots<t_N=b$ of the parameter interval, $t=t_{i-1}$, $\Delta t=t_i-t_{i-1}$, letting the mesh size tend to zero, and using continuity of $x'(t)$, $y'(t)$, we see that the oriented area swept out by the radius vector of the curve γ is given by $\frac{1}{2}\int_a^b xy'-yx'\,\mathrm{d}t$. If γ is closed and has no double points (i.e., looks like a deformed circle), this area is equal to the oriented area enclosed by the curve. This is true even if the origin is in the unbounded component of $\mathbb{R}^2\setminus\gamma\bigl([a,b]\bigr)$), because points in the unbounded region are swept out twice with opposite orientation. (Another way to see this is to translate the curve until the origin falls into the bounded component and use the easily proved fact that $\frac{1}{2}\int_a^b xy'-yx'\,\mathrm{d}t$ is translation-invariant.)

a) From $u=\frac{x}{\sqrt{x^2+y^2}}$, $v=\frac{y}{\sqrt{x^2+y^2}}$ we get

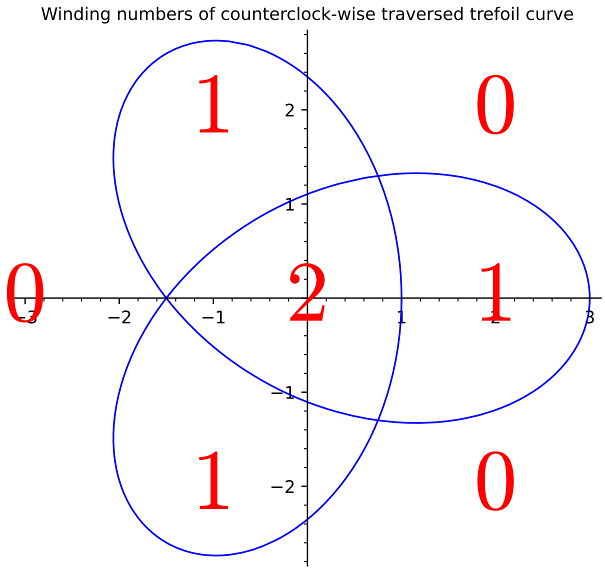
$$\begin{aligned}u'&=\frac{x'}{\sqrt{x^2+y^2}}-\frac{x(xx'+yy')}{(x^2+y^2)^{3/2}},\\ v'&=\frac{y'}{\sqrt{x^2+y^2}}-\frac{y(xx'+yy')}{(x^2+y^2)^{3/2}},\end{aligned}$$

and hence

$$\begin{aligned}\int_a^b uv'-vu'\,\mathrm{d}t&=\int_a^b \frac{xy'}{x^2+y^2}-\frac{xy(xx'+yy')}{(x^2+y^2)^2}-\frac{yx'}{x^2+y^2}+\frac{yx(xx'+yy')}{(x^2+y^2)^2}\,\mathrm{d}t\\ &=\int_a^b\frac{xy'-yx'}{x^2+y^2}\,\mathrm{d}t,\end{aligned}$$

as asserted.

b)



c) We have

$$
\begin{aligned}
x(t)^2+y(t)^2 &= |\gamma(t)|^2 = (\cos t + 2\cos(2t))^2 + (-\sin t + 2\sin(2t))^2 \\
&= \cos^2 t + \sin^2 t + 4\cos^2(2t) + 4\sin^2(2t) + 4\cos t\cos(2t) - 4\sin t\sin(2t) \\
&= 5 + 4\cos(3t), \\
x(t)y'(t) - y(t)x'(t) &= (\cos t + 2\cos(2t))(-\cos t + 4\cos(2t) - (-\sin t + 2\sin(2t))(-\sin t - 4\sin(2t)) \\
&= -\cos^2 t - \sin^2 t + 8\cos^2(2t) + 8\sin^2(2t) + 2\cos t\cos(2t) - 2\sin t\sin(2t) \\
&= 7 + 2\cos(3t),
\end{aligned}
$$

and hence

$$
n(\gamma,\mathbf{0}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{7+2\cos(3t)}{5+4\cos(3t)}\,\mathrm{d}t\,.
$$

Substituting $\phi = 3t$ and using that cos has period 2π and is an even function gives

$$
\begin{aligned}
n(\gamma,\mathbf{0}) &= \frac{1}{6\pi}\int_0^{6\pi} \frac{7+2\cos\phi}{5+4\cos\phi}\,\mathrm{d}\phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{7+2\cos\phi}{5+4\cos\phi}\,\mathrm{d}\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{7+2\cos\phi}{5+4\cos\phi}\,\mathrm{d}\phi \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{7+2\cos\phi}{5+4\cos\phi}\,\mathrm{d}\phi.
\end{aligned}
$$

The substitution $s = \tan(\phi/2)$, $\cos\phi = \frac{1-s^2}{1+s^2}$, $\sin\phi = \frac{2s}{1+s^2}$, $\mathrm{d}\phi = \frac{2}{1+s^2}\,\mathrm{d}s$ transforms such an integral into one over a rational function, which can be solved by standard methods (e.g., partial fractions).

Here we get

$$\begin{aligned}
\int_0^\pi \frac{7+2\cos\phi}{5+4\cos\phi}\,\mathrm{d}\phi &= \int_0^\infty \frac{7+2\frac{1-s^2}{1+s^2}}{5+4\frac{1-s^2}{1+s^2}}\frac{2}{1+s^2}\,\mathrm{d}s = 2\int_0^\infty \frac{9+5s^2}{(9+s^2)(1+s^2)}\,\mathrm{d}s \\
&= 2\int_0^\infty \frac{9/2}{9+s^2}+\frac{1/2}{1+s^2}\,\mathrm{d}s = \int_0^\infty \frac{1}{1+(s/3)^2}+\frac{1}{1+s^2}\,\mathrm{d}s \\
&= [3\arctan(s/3)+\arctan(s)]_0^\infty = \frac{3\pi}{2}+\frac{\pi}{2} = 2\pi
\end{aligned}$$

and hence $n(\gamma,\mathbf{0}) = 2$, as it should be.

Calculus III (Math 241)

W38 Consider the differential 1-forms

$$\omega_1 = (x^2 - yz)\,\mathrm{d}x + (y^2 - xz)\,\mathrm{d}y - xy\,\mathrm{d}z,$$
$$\omega_2 = \omega_1 + 2xy\;\mathrm{d}z\,.$$

a) Determine a potential function (antiderivative) of ω_1.

b) Is ω_2 locally exact?

c) Compute the integrals $\int_\gamma \omega_1$ and $\int_\gamma \omega_2$ for the family of helixes $\gamma(t) = \gamma_c(t) = (\cos t, \sin t, ct)$, $t \in [0, 2\pi]$ ($c \in \mathbb{R}$).

W39 a) Do Exercises 38, 40 in [Ste21], Ch. 16.3.

b) Repeat Exercises 38, 40 for

i) $\{(x,y,z) \in \mathbb{R}^3; (x,y) \neq (0,0)\}$ ("$\mathbb{R}^3$ minus a line");

ii) $\{(x,y,z) \in \mathbb{R}^3; (x,y,z) \neq (0,0,0)\}$ ("$\mathbb{R}^3$ minus a point").

Note: Justifications are required but can be informal.

W40 Do Exercises 41, 46 in [Ste21], Ch. 16.6.

W41 Use the stereographic projection

$$\sigma(x_1, x_2) = \frac{1}{x_1^2 + x_2^2 + 1} \begin{pmatrix} 2x_1 \\ 2x_2 \\ x_1^2 + x_2^2 - 1 \end{pmatrix}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

(cf. Homework 8, Exercise H44) to compute the surface area of the unit sphere S^2 in $\mathbb{R}^3$.

Solutions

38 a) $f_x = x^2 - yz \Longrightarrow f(x,y,z) = \frac{1}{3}x^3 - xyz + g(y,z)$
$f_y = -xz + g_y(y,z) = y^2 - xz \Longrightarrow g_y(y,z) = y^2 \Longrightarrow g(y,z) = \frac{1}{3}y^3 + h(z)$
$f_z = -xy + g_z = -xy + h'(z) = -xy \Longrightarrow h'(z) = 0 \Longrightarrow h(z) = C$
Let $C = 0$, then $f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - xyz$.

b) $\omega_2 = (x^2 - yz)\,\mathrm{d}x + (y^2 - xz)\,\mathrm{d}y + xy\,\mathrm{d}z$
$\partial_z(x^2 - yz) = -y$, but $\partial_x(xy) = y \neq -y$, so ω_2 is not locally exact.

c) (i) $\int_\gamma \omega_1 = f(\gamma(2\pi)) - f(\gamma(0)) = f(1,0,2c\pi) - f(1,0,0) = 0$

(ii) $\int_\gamma \omega_2 = \int_\gamma \omega_1 + \int_\gamma 2xy\ \mathrm{d}z = 2\int_0^{2\pi}(\cos t \sin t)c\ \mathrm{d}t = c\int_0^{2\pi}\sin(2t)\,\mathrm{d}t$
$= c\left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} = 0$

39 a) **Ex. 38** $U_1 = \{(x,y) \in \mathbb{R}^2; 1 < |x| < 2\}$ is open, because it is defined via a continuous function and strict inequalities. (Here is a direct proof: Suppose $(x,y) \in U_1$ and let $r := \min\{|x|-1, 2-|x|\}$. Then $r > 0$ and the ball $\mathrm{B}_r(x,y)$ is entirely contained in U_1. (Draw a picture!) U_1 is not connected, because it is the union of two disjoint open sets (the vertical strips $-2 < x < -1$ and $1 < x < 2$), and hence not simply-connected either.

Ex. 40 $U_2 = \{(x,y) \in \mathbb{R}^2; (x,y) \neq (2,3)\}$ ("$\mathbb{R}^2$ minus a point") is open (e.g., since the complementary set, a single point, is closed) and connected (one can move from any $\mathbf{a} \in U_2$ to any $\mathbf{b} \in U_2$ along a polygonal path consisting of at most two straight line segments; two line segments are needed if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c} = (2,3)$ are collinear), but not simply connected. For a proof of the latter quote the result from the lecture that $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected and the invariance of topological properties under translation, or show that for the closed curve $\gamma(t) = (2 + \cos t, 3 + \sin t)$, $t \in [0, 2\pi]$ and the (closed) differential 1-form $\omega = \frac{(x-2)\,\mathrm{d}y - (y-3)\,\mathrm{d}x}{x^2+y^2}$ the line integral $\int_\gamma \omega$ has the value $2\pi \neq 0$.

b) i) $U_3 = \{(x,y,z) \in \mathbb{R}^3; (x,y) \neq (0,0)\} = \mathbb{R}^3 \setminus \mathbb{R}(0,0,1)$, as well as $\mathbb{R}^3 \setminus L$ for any other line L in $\mathbb{R}^3$, is open and connected but not simply-connected.

U_3 is open, because any point $\mathbf{a} \in U_3$ has distance $d > 0$ to L and hence the ball $\mathrm{B}_d(\mathbf{a})$ is contained in U_3. U_3 is connected, because we can move from $\mathbf{a} \in U_3$ to $\mathbf{b} \in U_3$ entirely within U_3 using a path composed of at most two straight line segments. (One straight line segment, viz. $[\mathbf{a}, \mathbf{b}]$ is enough except in the case $a_1 + b_1 = a_2 + b_2 = 0$, in which we can use $[\mathbf{a}, \mathbf{c}]$ followed by $[\mathbf{c}, \mathbf{b}]$ for some other point $\mathbf{c} \in U_3$.) But U_3 is not simply-connected, since, e.g., $\gamma(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$, is not freely homotopic in U_3 to a point path, because it winds around L. (If it where, then projecting the corresponding homotopy onto the x-y plane via $(x,y,z) \mapsto (x,y)$ would yield a free homotopy in $\mathbb{R}^2 \setminus \{(0,0)\}$ from the unit circle to a point path, which doesn't exist; cf. the lecture.)

ii) $U_4 = \mathbb{R}^3 \setminus \{(0,0,0)\}$, as well as $\mathbb{R}^3 \setminus \mathbf{p}$ for any other point $\mathbf{p} \in \mathbb{R}^3$, is open and simply-connected. Graphically, when contracting a closed path in U_4 to a point one can avoid hitting the origin $(0,0,0)$.

But a rigorous proof of this fact is quite tricky on account of the "space-filling" capabilities of continuous curves. One way is to use the (easily established) fact that any closed curve γ in U_4 is homotopic to a closed polygonal path $\pi\colon [0,1] \to \mathbb{R}^3$. The cone determined by π and $(0,0,0)$ (union of all lines connecting $(0,0,0)$ to a point on π is the union of finitely many planes and hence cannot be all of $\mathbb{R}^3$. It follows that there exists a point $\mathbf{a} \in \mathbb{R}^3$ such that no line through $\mathbf{a}$ and a point of π contains $(0,0,0)$. Hence $H(t,s) = (1-s)\pi(t) + s(0,0,0)$, $(s,t) \in [0,1]^2$, defines a free homotopy in U_4 from π to a point curve (the image of H is the cone determined by π and $\mathbf{a}$), and the given curve γ is freely homotopic in U_4 to a point curve as well.

40 **Ex. 41** An immersion parametrizing the surface S is

$$\gamma(x,y) = \begin{pmatrix} x \\ y \\ (1-x-2y)/3 \end{pmatrix}, \qquad (x,y) \in \mathrm{B}_{\sqrt{3}}(0,0).$$

We obtain

$$\begin{aligned} \mathbf{J}_\gamma(x,y) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1/3 & -2/3 \end{pmatrix}, \\ \mathbf{J}_\gamma(x,y)^\mathsf{T} \mathbf{J}_\gamma(x,y) &= \begin{pmatrix} 10/9 & 2/9 \\ 2/9 & 13/9 \end{pmatrix}, \\ \sqrt{g^\gamma(x,y)} &= \sqrt{\frac{130-4}{81}} = \frac{1}{3}\sqrt{14}, \\ \mathrm{vol}_2(S) &= \int_{x^2+y^2<3} \sqrt{g^\gamma(x,y)} \mathrm{d}^2(x,y) = \frac{1}{3}\sqrt{14} \cdot 3\pi = \sqrt{14}\,\pi. \end{aligned}$$

Alternatively, instead of computing $\sqrt{g^\gamma(x,y)}$ directly, we could have used the formula $\sqrt{g^\gamma(x,y)} = \sqrt{1 + |\nabla f(x,y)|^2}$ with $f(x,y) = (1-x-2y)/3$ for $(x,y) \in \mathrm{B}_{\sqrt{3}}(0,0)$, because S is the graph of f. This gives again $\sqrt{g^\gamma(x,y)} = \sqrt{1 + |(-1/3,-2/3)|^2} = \sqrt{1+5/9} = \sqrt{14}/3$.

Ex. 46 The surface S is the graph of $f(y,z) = z^2 + y$, $(y,z) \in [0,2]^2$, and therefore has area

$$\begin{aligned} \mathrm{vol}_2(S) &= \int_{[0,2]^2} \sqrt{1 + |\nabla f(y,z)|^2}\, \mathrm{d}^2(y,z) = \int_{[0,2]^2} \sqrt{1 + |(1,2z)|^2}\, \mathrm{d}^2(y,z) \\ &= \int_{[0,2]^2} \sqrt{2+4z^2}\, \mathrm{d}^2(y,z) = 2\int_0^2 \sqrt{2+4z^2}\, \mathrm{d}z = 4\int_0^2 \sqrt{z^2 + \tfrac{1}{2}}\, \mathrm{d}z \\ &= 4\left[\frac{1}{2}\left(z\sqrt{z^2+\tfrac{1}{2}} + \tfrac{1}{2}\operatorname{arsinh}\left(\sqrt{2}\,z\right)\right)\right]_0^2 = 6\sqrt{2} + \operatorname{arsinh}\left(2\sqrt{2}\right) \approx 10.25 \end{aligned}$$

Here the inverse hyperbolic function $\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ (sometimes denoted erroneously by "arcsinh") was used.

Alternatively, we can use the parametrization $\gamma(y,z) = (z^2+y, y, z)^\mathsf{T}$, $(y,z) \in [0,2]^2$, directly and obtain

$$\begin{aligned}
\mathbf{J}_\gamma(y,z) &= \begin{pmatrix} 1 & 2z \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
g^\gamma(x,z) &= \begin{vmatrix} 2 & 2z \\ 2z & 1+4z^2 \end{vmatrix} = 2(1+4z^2) - 4z^2 = 2+4z^2, \\
\sqrt{g^\gamma(x,z)} &= \sqrt{2+4z^2},
\end{aligned}$$

from which onwards the computation will be identical.

41 From the solution to W35 on Worksheet 9 we know that $\sigma\colon \mathbb{R}^2 \to \mathrm{S}^2\setminus\{N\}$ is bijective.

$$\mathbf{J}_\sigma(x_1,x_2) = \frac{2}{(x_1^2+x_2^2+1)^2}\begin{pmatrix} -x_1^2+x_2^2+1 & -2x_1x_2 \\ -2x_1x_2 & x_1^2-x_2^2+1 \\ 2x_1 & 2x_2 \end{pmatrix}.$$

Since the columns of $\mathbf{J}_\sigma(x_1,x_2)$ are orthogonal and nonzero, $\mathbf{J}_\sigma(x_1,x_2)$ has rank 2 everywhere, and hence σ is an immersion. Moreover, the Gram determinant $g^\sigma(x_1,x_2)$ is easy to compute. We obtain

$$\begin{aligned}
\mathbf{J}_\sigma^\mathsf{T}\mathbf{J}_\sigma &= \begin{pmatrix} \frac{4}{(x_1^2+x_2^2+1)^2} & 0 \\ 0 & \frac{4}{(x_1^2+x_2^2+1)^2} \end{pmatrix}, \\
g^\sigma = \det(\mathbf{J}_\sigma^\mathsf{T}\mathbf{J}_\sigma) &= \frac{16}{(x_1^2+x_2^2+1)^4}, \\
\sqrt{g^\sigma} &= \frac{4}{(x_1^2+x_2^2+1)^2}, \\
\mathrm{vol}_2(\mathrm{S}^2) &= \int_{\mathrm{S}^2\setminus N} 1\,\mathrm{d}S = \int_{\mathbb{R}^2} \sqrt{g^\sigma(x_1,x_2)}\,\mathrm{d}^2(x_1,x_2) \\
&= \int_{\mathbb{R}^2} \frac{4}{(x_1^2+x_2^2+1)^2}\,\mathrm{d}^2(x_1,x_2) = \int_{(r,\phi)\in(0,+\infty)\times(0,2\pi)} \frac{4r}{(r^2+1)^2}\,\mathrm{d}^2(r,\phi) \\
&= 4\pi\int_0^\infty \frac{2r}{(r^2+1)^2}\,\mathrm{d}r = 4\pi\left[-\frac{1}{r^2+1}\right]_0^\infty = 4\pi.
\end{aligned}$$