Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The surface in $\mathbb{R}^3$ with equation $x^4+y^4+z^4+4xyz+1=0$ is smooth.

b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(x,y) \to +\infty$ for $|(x,y)| \to \infty$. Then f has a global minimum.

c) There exists a differentiable function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with at least 2023 strict local maxima.

d) A function $f(x,y)$ is integrable over $\Delta = \{(x,y) \in \mathbb{R}^2; x \geq 0, y \geq 0, x+y \leq 1\}$ if and only if $(u,v) \mapsto f\bigl(u(1-v), uv\bigr) \cdot u$ is integrable over the unit square $[0,1]^2$.

e) There exists a continuous vector field in $\mathbb{R}^2$ that is conservative and at every point of the unit circle nonzero and tangent to the unit circle.

f) Integrals of the differential 1-form $\frac{(x-y)\,\mathrm{d}x+(x+y)\,\mathrm{d}y}{x^2+y^2}$ in the region $\{(x,y) \in \mathbb{R}^2; x>0 \vee y>0\}$ are independent of path.

Question 2 (ca. 11 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = (1+6\,xy)\mathrm{e}^{-x^2-y^2}.$$

a) Determine all critical points $\mathbf{p}$ of f, their types using the 2nd-order partial derivatives test (Hesse matrix test), and the corresponding values $f(\mathbf{p})$.

Hint: There are 5 critical points. When computing 2nd-order derivatives it is best to use the identity $f_x(x,y) = 6y\,\mathrm{e}^{-x^2-y^2} - 2x\,f(x,y)$, and similarly for $f_y(x,y)$.

b) Does f have global extrema?

c) For $A = \{(x,y) \in \mathbb{R}^2; x>0, y>0\}$ compute the Lebesgue integral $\int_A f(x,y)\,\mathrm{d}^2(x,y)$.

Question 3 (ca. 7 marks)

Using the method of Lagrange multipliers, solve the optimization problem

$$\begin{array}{rl} \text{Maximize} & z \\ \text{subject to} & x^2+y^2+z^2 = x^3+y^3 = 9. \end{array}$$

Note: Required are (i) a proof that the optimization problem has a solution, (ii) the optimal objective value z^*, and (iii) all optimal solutions (x^*,y^*,z^*).

Question 4 (ca. 5 marks)

For $a > -1$ evaluate

$$\int_0^1 x^a (\ln x)^3\,\mathrm{d}x\,.$$

Hint: Start with $F(a) = \int_0^1 x^a\,\mathrm{d}x$, which is easy to evaluate directly, and show—carefully justifying each step—that F can be differentiated thrice under the integral sign.

Question 5 (ca. 7 marks)

Let K be the solid in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$x \geq 0, \quad x^2 + y^2 \leq 4, \quad 0 \leq z \leq x^2 - y^2.$$

Find the volume of K and the surface area of ∂K.

Solutions

1 a) False. The surface is the (-1)-level set of $g(x,y,z)=x^4+y^4+z^4+4xyz$, which has gradient $\nabla g(x,y,z)=(4x^3+4yz,4y^3+4xz,4z^3+4xy)$. At $(-1,-1,-1)$ we have $\nabla g=0$; since $g(-1,-1,-1)=-1$, this point is on the surface and hence a singular point. $\boxed{2}$

b) True. Since $\lim_{|(x,y)|\to\infty} f(x,y)=+\infty$, there exists $R>0$ such that $f(x,y)>f(0,0)$ for $|(x,y)|>R$. On the compact disk $\overline{\mathrm{B}_R(0,0)}$ te continuous function f attains a minimum m, which is $\leq f(0,0)$. Since $f(x,y)>f(0,0)\geq m$ for all points (x,y) outside the disk, this minimum is a global minimum.

c) True. For $g(x,y)=\sin x+\sin y$ we have

$$\nabla g(x,y)=(\cos x,\cos y)^{\mathsf{T}},$$
$$\mathbf{H}_g(x,y)=\begin{pmatrix}-\sin x & 0\\ 0 & -\sin y\end{pmatrix}.$$

All points $\mathbf{p}_{kl}=\left(\frac{(2k+1)\pi}{2},\frac{(2l+1)\pi}{2}\right)$, $k,l\in\mathbb{Z}$, are critical; those with k,l even have $\mathbf{H}_g(\mathbf{p}_{kl})=\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$, and hence provide strict local maxima. $\boxed{2}$

d) True. This is an instance of the change-of-variables theorem. In order to justify this, observe that $T(u,v)=\bigl(u(1-v),uv\bigr)$ has $\det \mathbf{J}_T(u,v)=\left|\begin{smallmatrix}1-v&-u\\v&u\end{smallmatrix}\right|=(1-v)u-v(-u)=u>0$ on $(0,1)^2$ and maps $(0,1)^2$ bijectively to Δ°. In order to see the latter, let $x=u(1-v)$, $y=uv$. If $(u,v)\in(0,1)^2$ then $x>0$, $y>0$, and $x+y=u<1$. Conversely, if $(x,y)\in\Delta$ then $u=x+y\in(0,1)$ and $v=\frac{y}{x+y}\in(0,1)$. $\boxed{2}$

e) False. On the unit circle the normalization $F/|F|$ of such a vector field, call it F, is continuous and tangent to the unit circle as well. Because of continuity the tangent direction cannot make a U-turn, and hence either $F(\cos t,\sin t)$ is a positive multiple of $(-\sin t,\cos t)$ for all $t\in[0,2\pi]$, or a negative multiple of $(-\sin t,\cos t)$ for all $t\in[0,2\pi]$. In the first case the line integral along the unit circle in its usual parametrization is positive, in the second case negative. This contradicts conservativity/exactness of the vector field/differential 1-form, which requires line integrals along closed curves to be zero.

f) True. The form, call it ω, is the sum of $\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}$, which is exact (an antiderivative is $\frac{1}{2}\ln(x^2+y^2)$), and the winding form. Since the winding form is locally exact, so is ω. Since $\bigl\{(x,y)\in\mathbb{R}^2; x>0\vee y>0\bigr\}$ is star shaped with centre, e.g., $(1,1)$, ω is exact (Poincaré's Lemma), and hence $\int_\gamma\omega$ is independent of path. $\boxed{2}$

Remarks: No marks were assigned for wrong answers and answers without justification.

a) Shockingly, almost half of the students falsely claimed that the surface is smooth. In all the surface has 4 singular points, viz. $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(-1,-1,-1)$. Since the suface is invariant under simultaneous sign changes of any two of the three variables x,y,z, it is clear that the first three points must also be singular, and it is not difficult to see that there are no further singular points.

b) Full marks were only assigned for a rigorous proof, which must essentially be the one given above. If students worked with the disk $\mathrm{B}_R(0,0)$ (which is open) instead of $\overline{\mathrm{B}_R(0,0)}$ and mentioned "closed", I haven't subtracted any marks. Many students provided invalid proofs, rephrasing just the statement to prove or relying on "this thing is obvious" and the like. Also it is false to assume that there exists a disk $\overline{\mathrm{B}_R(0,0)}$ such that $f(x_2,y_2) \geq f(x_1,y_1)$ for all points (x_2,y_2) outside the disk and all points (x_1,y_1) inside the disk.

c) For (correct) examples of such functions without any justification I have assigned only 0.5 marks. Many students used related examples such as $\sin x + \cos y$, in which case the maxima are located at points different from those above.

Several students used the example $f(x,y) = (x+y-1)(y-x)(y-2x)\cdots$ from one of our sample exams, but the argument presented there cannot work, since $g(x,y) = (x+y-1)(y-mx)$ has only a saddle point and no other extrema.

Other students used functions like $\sin(x+y)$, $\sin(xy)$, $y \sin x$, which don't have strict local maxima.

Two students discovered a nice example but didn't realize how easy the solution is: Use the function $g(x,y) = \sin^2 x + \sin^2 y$. Without any computation necessary, the maxima of this function are obviously at points (x,y) where $\sin x = \pm 1$ and $\sin y = \pm 1$, which are $\pi/2 \cdot (k,l)$ with k,l odd integers. Since these points form a discrete set, the maxima are strict, and there are of course infinitely many such points.

Another way to construct examples is to periodically repeat a function h with a single maximum. But in order for this to work, the supports of the translates of h should be disjoint. Essentially what one needs is that h has a strict maximum at $(0,0)$ and vanishes outside the unit disk. Such a function can be constructed by rotating a corresponding 1-variable function, but the differentiability requirement makes the solution nontrivial. (In Math285 we will see that even C^∞-functions with these properties exist.) For those answers, which never went into an actual construction, I have assigned 0.5 marks.

d) Many students recognized this as an instance of change-of-variables but were unable to fill in all details. A proof that T is bijective, which requires solving $x = u(1-v) \wedge y = uv$ for u,v was only given by few students. Also many students didn't notice that $\det \mathbf{J}_T(u,v)$ is involved and has to be computed.

e) Only few students noticed that a key property to be exploited is that the line integral of F along the unit circle must be zero. This was honored by 1 mark. Noone noticed that the second key property is that F is nonzero everywhere on the unit circle. (There exist conservative vector fields which are nonzero and tangent to the unit circle in all but two points. Think of related examples from Physics.)

f) Most students showed that the given 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ is locally exact, which requires to show $P_y = Q_x$. This was worth 1 mark. But many forgot that the shape of the region also plays a role in exactness. The 2nd mark was only assigned when it was mentioned that the region is star-shaped or simply connected.

$$\sum_{1} = 12$$

2 a) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f &= (1+6xy)\mathrm{e}^{-x^2-y^2} \\ f_x &= \bigl(6y-2x(1+6xy)\bigr)\mathrm{e}^{-x^2-y^2} = (6y-2x-12x^2y)\mathrm{e}^{-x^2-y^2} \\ f_y &= \bigl(6x-2y(1+6xy)\bigr)\mathrm{e}^{-x^2-y^2} = (6x-2y-12xy^2)\mathrm{e}^{-x^2-y^2} \\ yf_x - xf_y &= (6y^2-6x^2)\mathrm{e}^{-x^2-y^2}. \end{aligned}$$

Hence $\nabla f(x,y) = (0,0)$ implies $x = \pm y$.

Clearly $\mathbf{p}_0 = (0,0)$ is critical, and $f(\mathbf{p}_0) = 1$. $\boxed{1}$

Assuming $(x,y) \neq (0,0)$, we distinguish two mutually exclusive cases:

Case 1: $x = y$ Here $f_x = 0$ gives $4x - 12x^3 = 4x(1-3x^2) = 0$, $x = \pm\frac{1}{3}\sqrt{3}$. This yields the two critical points $\mathbf{p}_1 = \bigl(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\bigr)$, $\mathbf{p}_2 = \bigl(-\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\bigr)$. The corresponding values are $f(\mathbf{p}_1) = f(\mathbf{p}_2) = 3\,\mathrm{e}^{-2/3}$. $\boxed{2}$

Case 2: $x = -y$ Here $f_x = 0$ gives $-8x + 12x^3 = 4x(-2+3x^2) = 0$, $x = \pm\frac{\sqrt{2}}{\sqrt{3}} = \pm\frac{1}{3}\sqrt{6}$. This yields the two critical points $\mathbf{p}_3 = \bigl(\frac{1}{3}\sqrt{6}, -\frac{1}{3}\sqrt{6}\bigr)$, $\mathbf{p}_4 = \bigl(-\frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{6}\bigr)$. The corresponding values are $f(\mathbf{p}_3) = f(\mathbf{p}_4) = -3\,\mathrm{e}^{-4/3}$. $\boxed{2}$

For the determination of the types we compute the 2nd-order derivatives using the hint:

$$\begin{aligned} f_x &= 6y\,\mathrm{e}^{-x^2-y^2} - 2xf, \\ f_y &= 6x\,\mathrm{e}^{-x^2-y^2} - 2yf, \\ f_{xx} &= -12\,xy\,\mathrm{e}^{-x^2-y^2} - 2f - 2xf_x, \\ f_{xy} &= f_{yx} = 6(1-2xy)\mathrm{e}^{-x^2-y^2} - 2xf_y, \\ f_{yy} &= -12\,xy - 2f - 2yf_y. \end{aligned}$$

Since $f_x = f_y = 0$ at a critical point $\mathbf{p}$ and $f(\mathbf{p})$ has already been determined, this makes it easy to compute the corresponding Hesse matrices.

$$\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \mathbf{H}_f(\mathbf{p}_2) = \mathrm{e}^{-2/3}\begin{pmatrix} -10 & 2 \\ 2 & -10 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_3) = \mathbf{H}_f(\mathbf{p}_4) = \frac{1}{3}\,\mathrm{e}^{-2/3}\begin{pmatrix} 26 & -2 \\ -2 & 26 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_0)$ is indefinite (determinant < 0), the point $\mathbf{p}_0$ is a saddle point. $\boxed{1}$

Since $\mathbf{H}_f(\mathbf{p}_1) = \mathbf{H}_f(\mathbf{p}_2)$ is negative definite (determinant > 0, top-left entry < 0), the points $\mathbf{p}_1$, $\mathbf{p}_2$ are strict local maxima. $\boxed{1}$

Since $\mathbf{H}_f(\mathbf{p}_3) = \mathbf{H}_f(\mathbf{p}_4)$ is positive definite (determinant > 0, top-left entry > 0), the points $\mathbf{p}_3$, $\mathbf{p}_4$ are strict local minima. $\boxed{1}$

b) Yes. The points $\mathbf{p}_1$, $\mathbf{p}_2$ are the global maxima (with value $f(\mathbf{p}_1) = f(\mathbf{p}_2) = 3\,\mathrm{e}^{-2/3}$), and $\mathbf{p}_3$, $\mathbf{p}_4$ are the global minima (with value $f(\mathbf{p}_1) = f(\mathbf{p}_2) = -3\,\mathrm{e}^{-4/3}$).

The existence of a global maximum follows from from $\lim_{|(x,y)|\to\infty} f(x,y) = 0$ (clear from the exponential decrease of $(x,y) \mapsto \mathrm{e}^{-x^2-y^2}$) and the fact that f attains a positive value. Similarly, the existence of a global minimum follows from from $\lim_{|(x,y)|\to\infty} f(x,y) = 0$ and the fact that f attains a negative value. $\boxed{1}$

Since the global maxima/minima must be among the local maxima/minima, it is then easy to identify them.

c) We have

$$\begin{aligned}
\int_A f(x,y)\mathrm{d}^2(x,y) &= \int_A \mathrm{e}^{-x^2-y^2}\mathrm{d}^2(x,y) + 6\int_A xy\,\mathrm{e}^{-x^2-y^2}\,\mathrm{d}^2(x,y)\\
&= \frac{1}{4}\int_{\mathbb{R}^2} \mathrm{e}^{-x^2-y^2}\,\mathrm{d}^2(x,y) + 6\int_0^\infty x\,\mathrm{e}^{-x^2}\,\mathrm{d}x\int_0^\infty y\,\mathrm{e}^{-y^2}\,\mathrm{d}y\\
&= \pi/4 + 6\left(\left[-\frac{1}{2}\,\mathrm{e}^{-x^2}\right]_0^\infty\right)^2\\
&= \frac{\pi}{4} + 6\left(\frac{1}{2}\right)^2 = \frac{\pi+6}{4}.
\end{aligned} \qquad \boxed{2}$$

Remarks: Most students solved a) correctly.

In b) we haven't insisted on a full proof, since that was required in Question 1 b); you simply had to notice that f is continuous and satisfies $f(x,y) \to 0$ for $|(x,y)| \to \infty$. Marks for the identifications of the global extrema weren't assigned, because the values at the critical points are known from a).

Many students had problems with c). Very often polar coordinates $T(r,\theta) = (r\cos\theta, r\sin\theta)$ were used and Fubini's Theorem applied to the resulting integral $\int_B \dots \mathrm{d}^2(r,\theta)$, $B = T^{-1}(A) = (0,\infty)\times(0,\pi/2)$, which complicates matters. At the end one needs to evaluate $\int_0^\infty r^3\,\mathrm{e}^{-r^2}\,\mathrm{d}r$, which requires one step of integration by parts to reduce it to the known $\int_0^\infty r\,\mathrm{e}^{-r^2}\,\mathrm{d}r = 1/2$.

$$\sum_2 = 11$$

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3 The continuous function $f(x,y,z) = z$ attains a maximum on the set $S = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2 = 9, x^3+y^3 = 9\}$, which is closed (because it is defined by equality constraints involving continuous functions) and bounded (because it is contained in a sphere). This shows that the optimization problem has at least one solution. $\boxed{1}$

Setting $\mathbf{g} = (g_1, g_2)$ with $g_1(x,y,z) = x^2+y^2+z^2$, $g_2(x,y,z) = x^3+y^3$, the task is to minimize f on $\mathbb{R}^3$ under the constraint $\mathbf{g}(x,y,z) = (9,9)$.

$$\nabla f(x,y,z) = (0,0,1), \qquad \mathbf{J_g}(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ 3x^2 & 3y^2 & 0 \end{pmatrix}.$$

The Lagrange Multiplier Theorem is applicable in points $(x,y,z) \in S$ for which $\mathbf{J_g}(x,y,z)$ has rank 2. The 2×2 subdeterminants of $\mathbf{J_g}(x,y,z)$ are $6xy^2 - 6x^2y = 6xy(y-x)$, $-6x^2z$, $-6y^2z$. If $\operatorname{rank}\mathbf{J_g}(x,y,z) \le 1$ then all three subdeterminants are zero, which implies either $z = 0 \wedge (x = 0 \vee y = 0 \vee x = y)$ or $x = y = 0$. In the first case $z = 0$ gives $x^2 + y^2 = x^3 + y^3 = 9$, and each of the possibilities $x = 0$, $y = 0$, $x = y$ produces a contradiction. In the second case $x^3+y^3 = 0$, contradicting $x^3+y^3 = 9$. Thus rank $\mathbf{J_g} = 2$ for all points on S. $\boxed{1\frac{1}{2}}$

Hence every optimal solution must satisfy $\nabla f(x, y, z) = \lambda\nabla g_1(x, y, z) + \mu\nabla g_2(x, y, z)$ for some $\lambda, \mu \in \mathbb{R}$, giving the system of equations

$$\begin{aligned} 0 &= \lambda\, 2x + \mu\, 3x^2, \\ 0 &= \lambda\, 2y + \mu\, 3y^2, \\ 1 &= \lambda\, 2z, \\ x^2 + y^2 + z^2 &= 9, \\ x^3 + y^3 &= 9. \end{aligned} \qquad \boxed{2\tfrac{1}{2}}$$

The first two equations represent the linear system

$$\begin{pmatrix} 2x & 3x^2 \\ 2y & 3y^2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\lambda = 0$ is impossible in view of the 3rd equation, a solution must have $\begin{vmatrix} 2x & 3x^2 \\ 2y & 3y^2 \end{vmatrix} = 0$, which implies $x = 0 \vee y = 0 \vee x = y$. This gives 6 candidate points, viz.,

$$\left(0, 9^{1/3}, \pm\left(9 - 9^{2/3}\right)^{1/2}\right), \quad \left(9^{1/3}, 0, \pm\left(9 - 9^{2/3}\right)^{1/2}\right), \quad \left((9/2)^{1/3}, (9/2)^{1/3}, \pm\left((9 - 2(9/2)^{2/3})\right)^{1/2}\right).$$

$\boxed{3}$

Since $2(9/2)^{2/3} = 2^{1/3}9^{2/3} > 9^{2/3}$, we have $\left((9 - 2(9/2)^{2/3})\right)^{1/2} < \left(9 - 9^{2/3}\right)^{1/2}$, and hence the points on S maximizing z are

$$\left(0, 9^{1/3}, \left(9 - 9^{2/3}\right)^{1/2}\right), \quad \left(9^{1/3}, 0, \left(9 - 9^{2/3}\right)^{1/2}\right). \qquad \boxed{1}$$

Remarks: As usual, many students forgot to check the smoothness condition of the constraint surface C (in fact a curve in $\mathbb{R}^3$). If $\mathbf{J_g}(x, y, z)$ was computed and rank $\mathbf{J_g}(x, y, z) = 2$ for $(x, y, z) \in C$ was noted without proof, we have assigned 0.5 marks.

Several students copied the existence proof for a maximum from a previous sample exam with only 1 constraint, writing only that the sphere is closed and bounded. But the domain on which z needs to attain a maximum is the curve C.

It is possible (and simplifies the computation) to solve the problem with only one Lagrange multiplier, noting that maximizing z on C is equivalent to minimizing $x^2 + y^2$ on the surface $x^3+y^3 = 9$. Solutions of this kind we have generally accepted, except that in view of the simplified computation more marks were assigned for the theoretical part.

$$\sum_3 = 9$$

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4 $F(a)$ is defined for $a > -1$ and can be evaluated easily as follows:

$$F(a) = \int_0^1 x^a \, \mathrm{d}x = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1^{a+1}}{a+1} - \frac{0^{a+1}}{a+1} = \frac{1}{a+1}. \qquad \boxed{\tfrac{1}{2}}$$

(Strictly speaking, for $a < 0$ we should first integrate over $[\epsilon, 1]$, $\epsilon > 0$, which gives the value $\frac{1}{a+1} - \frac{\epsilon^{a+1}}{a+1}$, and then let $\epsilon \to 0$.)

Writing $f(a,x) = x^a$, we have $f_a(a,x) = \frac{\partial}{\partial a}\mathrm{e}^{(\ln x)a} = (\ln x)x^a$. In order to justify differentiation under the integral sign, we need for fixed $a_0 \in (-1,+\infty)$ an integrable upper bound $\Phi(x)$ for $|(\ln x)x^a| = -(\ln x)x^a$ on $(0,1)$ that is independent of a in some neighborhood of a_0. Choosing δ strictly between -1 and a_0, the intervall $(\delta,+\infty)$ is a neighborhood of a_0, and for $a \in (\delta,+\infty)$ we have

$$-(\ln x)x^a \le -(\ln x)x^\delta,$$

since $x \in (0,1)$. The function $\Phi_\delta(x) := -(\ln x)x^\delta$, $x \in (0,1)$, are integrable for $\delta > -1$, because $-\ln x$ for $x \downarrow 0$ grows more slowly than any (small) negative power of x, showing that $-(\ln x)x^\delta \le x^{\delta'}$ with δ' slightly smaller than δ but still > -1. $\boxed{2}$

Thus differentiation under the integral sign is justified and gives

$$F'(a) = \int_0^1 (\ln x)x^a\,\mathrm{d}x \quad \text{for } a > -1.$$

In the same way one proves

$$F''(a) = \int_0^1 (\ln x)^2 x^a\,\mathrm{d}x, \quad F'''(a) = \int_0^1 (\ln x)^3 x^a\,\mathrm{d}x \quad \text{for } a > -1, \qquad \boxed{1}$$

using that $(-1)^k(\ln x)^k$ for $x \downarrow 0$ for any positive integer k still grows more slowly than any (small) negative power of x.

On the other hand, using $F(a) = \frac{1}{a+1}$ we have

$$\begin{aligned}\int_0^1 (\ln x)x^a\,\mathrm{d}x &= F'(a) = -\frac{1}{(a+1)^2},\\ \int_0^1 (\ln x)^2 x^a\,\mathrm{d}x &= F''(a) = \frac{2}{(a+1)^3},\\ \int_0^1 (\ln x)^3 x^a\,\mathrm{d}x &= F'''(a) = -\frac{6}{(a+1)^4}.\end{aligned} \qquad \boxed{1\tfrac{1}{2}}$$

Remarks: It seems that many students didn't understand the hint and tried more complicated solutions (such as integration by parts three times). Several students proved $F(a) = \frac{1}{a+1}$ and then stopped, which is worth only 0.5 marks. For the correct evaluation of the integral using integration by parts we have assigned 3/5 marks, because the most important topic assessed in this question is Feynmans technique of "differentiation under the integral sign". Those students who tried to justify it using Lebesgue's Bounded Convergence Theorem sometimes ran into problems—such as choosing δ on the wrong side of a_0 or neglecting the fact that $\ln x$ is negative in $(0,1)$.

$$\sum_4 = 5$$

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5 a) For $(x,y) \in \mathbb{R}^2$ the corresponding (x,y)-section $K_{x,y} = \{z \in \mathbb{R}; (x,y,z) \in K\}$ is nonempty iff $x^2 - y^2 \ge 0$, which under the assumption $x \ge 0$ simplifies to $-x \le y \le x$, and is an interval of length x^2-y^2 in that case. The set $S=\bigl\{(x,y)\in\mathbb{R}^2;x^2+y^2\leq 4,x\geq 0,-x\leq y\leq x\bigr\}$ (projection of K onto the (x,y)-plane) is a sector of the disk with radius $r=2$ centered at the origin and in polar coordinates given by $-\pi/4\leq\theta\leq\pi/4$.

$$\begin{aligned}
\Longrightarrow \operatorname{vol}_3(K)&=\int_K 1\,\mathrm{d}^3(x,y,z)=\int_S x^2-y^2\,\mathrm{d}^2(x,y)\\
&=\int\limits_{\substack{0\leq r\leq 2\\ -\pi/4\leq\theta\leq\pi/4}}\bigl((r\cos\theta)^2-(r\sin\theta)^2\bigr)r\,\mathrm{d}^2(r,\theta)\\
&=\left(\int_0^2 r^3\,\mathrm{d}r\right)\left(\int_{-\pi/4}^{\pi/4}\cos^2\theta-\sin^2\theta\,\mathrm{d}\theta\right)\\
&=\left[\frac{r^4}{4}\right]_0^2\left[\frac{1}{2}\sin(2\theta)\right]_{-\pi/4}^{\pi/4}=4\cdot 1=4.
\end{aligned}$$ $\boxed{3}$

b) The smooth part of ∂K consists of 3 parts, viz.,

$$\begin{aligned}
S_1&=\bigl\{(x,y,0);x^2+y^2<4,x>0,-x<y<x\bigr\},\\
S_2&=\bigl\{(x,y,x^2-y^2);(x,y)\in S^\circ\bigr\},\\
S_3&=\bigl\{(x,y,z)\in\mathbb{R}^3;x^2+y^2=4,x>0,-x<y<x,0<z<x^2-y^2\bigr\}.
\end{aligned}$$

The remaining parts are smooth curves and don't matter for surface integration.

Since S_1 is S embedded into $\mathbb{R}^3$, the area of S_1 is $2^2\pi/4=\pi$.

S_2 is the graph of the function $f(x,y)=x^2-y^2$, $(x,y)\in S^\circ$.

$$\begin{aligned}
\Longrightarrow \operatorname{vol}_2(S_2)&=\int_S\sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y)\\
&=\int_S\sqrt{1+4x^2+4y^2}\,\mathrm{d}^2(x,y) &&\text{since } \nabla f(x,y)=(2x,-2y)\\
&=\int\limits_{\substack{0\leq r\leq 2\\ -\pi/4\leq\theta\leq\pi/4}}\sqrt{1+4r^2}\,r\,\mathrm{d}^2(r,\theta)\\
&=\frac{\pi}{2}\int_0^2 r\sqrt{1+4r^2}\,\mathrm{d}r\\
&=\frac{\pi}{2}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^2=\frac{\pi}{24}\left(17\sqrt{17}-1\right).
\end{aligned}$$ $\boxed{2}$

A (regular, bijective) parametrization of S_3 is

$$\gamma(\theta,z)=\begin{pmatrix}2\cos\theta\\2\sin\theta\\z\end{pmatrix},\quad(\theta,z)\in\Omega$$

with $\Omega = \{(\theta, z) \in \mathbb{R}^2; -\pi/4 < \theta < \pi/4, 0 < z < 4\cos(2\theta)\}$ (using $\cos^2\theta - \sin^2\theta = \cos(2\theta)$).

$$\begin{aligned}
\mathbf{J}_\gamma(\theta, z) &= \begin{pmatrix} -2\sin\theta & 0 \\ 2\cos\theta & 0 \\ 0 & 1 \end{pmatrix}, \\
\mathbf{J}_\gamma(\theta, z)^\mathsf{T}\mathbf{J}_\gamma(\theta, z) &= \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \\
g^\gamma(\theta, z) &= \sqrt{\det\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}} = 2, \\
\mathrm{vol}_2(S_3) &= \int_\Omega 2\,\mathrm{d}^2(\theta, z) \\
&= 2\int_{-\pi/4}^{\pi/4} 4\cos(2\theta)\mathrm{d}\theta \\
&= 4\left[\sin(2\theta)\right]_{-\pi/4}^{\pi/4} = 4 \cdot 2 = 8. \qquad \boxed{2}
\end{aligned}$$

In all we obtain

$$\begin{aligned}
\mathrm{vol}_2(\partial K) = \mathrm{vol}_2(S_1) + \mathrm{vol}_2(S_2) + \mathrm{vol}_2(S_3) = \pi + \frac{\pi}{24}\left(17\sqrt{17} - 1\right) + 8 = \\
= \frac{\pi}{24}\left(17\sqrt{17} + 23\right) + 8. \qquad \boxed{1}
\end{aligned}$$

Remarks: The volume of K was computed correctly by most students. Some students thought that S is the half disk $r < 2$, $-\pi/2 < \theta < \pi/2$, which invalidates the whole computation.

In the surface area computation many students missed parts of ∂K, most often the vertical boundary S_3 but sometimes also the bottom S_1.

$$\sum_5 = 8$$

$$\sum_{\text{Final Exam}} = 12 + 11 + 9 + 5 + 8 = 45 = 40 + 5$$

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) For any $A \in \mathbb{R}$ the surface in $\mathbb{R}^3$ with equation $x^3 + y^3 + z^3 + A\,xyz = 1$ is smooth.

b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(x,y) \to 0$ for $|(x,y)| \to \infty$. Then f has a global extremum.

c) Suppose you start at the point $\mathbf{p} = (1,1)$ in the (x,y)-plane and follow the contour of $f(x,y) = xy^2 + x^2y$ through $\mathbf{p}$ in one of the two possible directions. After some time you reach a point that is closer to $(0,0)$ than $\mathbf{p}$.

d) There exists a function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with at least 2023 saddle points.

e) The set of all real numbers whose decimal expansion doesn't contain the digit 0 (i.e., only digits $1,2,3,4,5,6,7,8,9$ are allowed) has Lebesgue measure zero.

f) For any closed path γ in $\mathbb{R}^2$ and any choice of $a,b,c,d \in \mathbb{R}$ we have $\int_\gamma (ax+by)\,\mathrm{d}x + (cx+dy)\,\mathrm{d}y = \frac{c-b}{2}\int_\gamma x\,\mathrm{d}y - y\,\mathrm{d}x$.

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 + y^4 - 6\,x^2 - 4\,xy - 6\,y^2.$$

a) Which symmetry properties does f have? What can you conclude from this about the graph of f and the location/type of the critical points of f?

b) Determine all critical points of f and their types.

Hint: There are 9 critical points.

c) Does f have a global extremum?

Question 3 (ca. 7 marks)

Using the method of Lagrange multipliers, solve the optimization problem

$$\begin{array}{rl} \text{Maximize} & \zeta = xy + 6yz + 6zx \\ \text{subject to} & x^2 + y^2 + z^2 = 17. \end{array}$$

Note: Required are (i) a proof that the optimization problem has a solution, (ii) the optimal objective value ζ^*, and (iii) all optimal solutions (x^*, y^*, z^*).

Question 4 (ca. 6 marks)

Consider the function $F\colon (0,\infty) \to \mathbb{R}$ defined by

$$F(a) = \int_0^\infty \frac{\mathrm{d}x}{x^2 + a^2}.$$

a) Show that F is differentiable, and that $F'(a)$ can be obtained by differentation under the integral sign.

b) Using a), evaluate $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, $\quad a > 0$.

Hint: The integral defining $F(a)$ can be evaluated using the substitution $x = at$.

Question 5 (ca. 7 marks)

a) Find the mass of the solid K in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq 4x, \quad x^2 + y^2 \leq 16,$$

whose density is given by $\rho(x, y, z) = xyz^3$.

b) Find the area of the surface P in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$z = x^{3/2} + y^{3/2}, \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 2.$$

Solutions

1 a) True. The surface, call it S_A, is the 1-level set of $g_A(x,y,z)=x^3+y^3+z^3+A\,xyz$, which has gradient $\nabla g_A(x,y,z)=(3x^2+A\,yz,3y^2+A\,xz,3z^2+A\,xy)$.

For the proof suppose $(x,y,z)\in S_A$ satisfies $\nabla g(x,y,z)=(0,0,0)$. If $x=0$ then the 2nd and 3rd coordinate of $\nabla g(x,y,z)$ are $3y^2$ resp. $3z^2$, so that $y=z=0$ as well. But $(0,0,0)\notin S_A$, contradiction. Thus $x\neq 0$ and, by symmetry, $y\neq 0$ and $z\neq 0$. Further, $27x^2y^2z^2=(3x^2)(3y^2)(3z^2)=(-Ayz)(-Axz)(-Axy)=-A^3x^2y^2z^2$, which together with $xyz\neq 0$ gives $A=-3$ as the only possible exception. In this case $\nabla g(x,y,z)=(0,0,0)$ reduces to $x^2=yz$, $y^2=xz$, $z^2=xy$. From the 1st equation, $z=x^2/y$. Substituting this into the 2nd equation, $y^2=x^3/y$ and hence $x^3=y^3$, $x=y$. Then, by symmetry, $x=y=z$. But $g_{-3}(x,x,x)=x^3+x^3+x^3-3x^3=0$, and hence $(x,x,x)\notin S_{-3}$. This is the final contradiction. $\boxed{2}$

b) True. If f is the all-zero function, the statement is trivially true. Otherwise we may suppose w.l.o.g. that $f(x_0,y_0)>0$ for some $(x_0,y_0)\in\mathbb{R}^2$. By assumption, there exists $R>0$ such that $f(x,y)<f(x_0,y_0)$ for all points (x,y) with $|(x,y)|>R$. Since f is continuous, f attains a maximum on the closed disk $\mathrm{B}_R(0,0)$, say in (x_1,y_1). Since $(x_0,y_0)\in\mathrm{B}_R(0,0)$, we obtain $f(x,y)<f(x_0,y_0)\leq f(x_1,y_1)$ for all points (x,y) outside $\mathrm{B}_R(0,0)$. Thus the maximum in (x_1,y_1) is global.

Remark: It is not true that such a function f must have global extrema of both kinds, e.g., $f(x,y)=1/(1+x^2+y^2)$ has a global maximum but no global minimum.

c) False. We have

$$\nabla f(x,y)=\begin{pmatrix}y^2+2xy,\\ x^2+2xy\end{pmatrix},$$

$$\begin{vmatrix}y^2+2xy & x\\ x^2+2xy & y\end{vmatrix}=y^3+2xy^2-x^3-2x^2y=(y-x)(x^2+3xy+y^2).$$

Thus moving along the 2-contour from $(1,1)$ means moving in direction NW or SE (since $\nabla f(x,y)$ points to NE in the 1st quadrant and the coordinate axes, which are part of the 0-contour, cannot be reached). At a point on the 2-contour closest to $(0,0)$ (such a point exists by the usual continuity-compactness argument) the gradient $\nabla f(x,y)$ must be orthogonal to (x,y), which is the case only for points on the line $y=x$. But except for the starting point $(1,1)$, no such point can be reached. $\boxed{2}$

d) True. An example is

$$f(x,y)=(x+y-1)(y-x)(y-2x)\cdots(y-2023\,x).$$

For $m\in\{1,2,\dots,2023\}$ the intersection point of the lines $x+y=1$ and $y=mx$, viz. $\left(\frac{1}{m+1},\frac{m}{m+1}\right)$, is a saddle point of f. In order to see this, with m fixed it suffices to consider $g(x,y)=(x+y-1)(y-mx)=y^2-mx^2+(1-m)xy-y+mx$ instead.

$$\begin{aligned}
g_x&=-2mx+(1-m)y+m,\\
g_y&=2y+(1-m)x-1,\\
g_{xx}&=-2m,\\
g_{xy}&=1-m=g_{yx},\\
g_{yy}&=2.
\end{aligned}$$

One finds that $\nabla g\left(\frac{1}{m+1},\frac{m}{m+1}\right)=(0,0)$ (this also follows from the fact that the 0-contour of g or f is not smooth there), and $\det \mathbf{H}_g(x,y)=-4m-(1-m)^2<0$. $\boxed{2}$

e) True. Denote this set by S, and let $S_0=S\cap[0,1)$. Among the $10^k-10^{k-1}=9\cdot 10^{k-1}$ positive integers with exactly k decimal digits, 9^k don't involve the digit 0. Scaling by 10^{-k}, the set of real numbers in $[0,1)$ not involving the digit 0 in the first k digits after the decimal point has Lebesgue measure at most $9^k/(9\cdot 10^{k-1})=\left(\frac{9}{10}\right)^{k-1}$. Since $\left(\frac{9}{10}\right)^{k-1}\to 0$ for $k\to\infty$, we can conclude that S_0 has Lebesgue measure zero; cf. the corresponding argument for Cantor's Ternary Set. But then S, which is contained in a countable union of translates of S_0, must have Lebesgue measure zero as well. $\boxed{2}$

f) True. Using linearity of the line integral $\int_\gamma \omega$ as a function of ω, the equation can be rewritten as

$$\begin{aligned}&\int_\gamma\left(ax+by+\frac{c-b}{2}\,y\right)dx+\left(cx+dy-\frac{c-b}{2}\,x\right)\mathrm{d}y=0\\ \iff\quad &\int\left(ax+\frac{b+c}{2}\,y\right)\mathrm{d}x+\left(\frac{b+c}{2}\,x+dy\right)\mathrm{d}y=0.\end{aligned}$$

Denoting the latter integrand by $\omega=M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y$, we have $M_y=\frac{b+c}{2}=N_x$, i.e., ω is exact in $\mathbb{R}^2$ and hence $\int_\gamma\omega=0$. $\boxed{2}$

Remarks: No marks were assigned for answers without justification.

$$\sum_1 = 12$$

2 a) $f(-x,-y)=f(x,y)=f(y,x)$ for $(x,y)\in\mathbb{R}^2$ $\boxed{1}$
$\Longrightarrow$ G_f is symmetric with respect to the z-axis and the plane $x=y$. $\boxed{1}$
Alternatively, $f(y,x))=f(x,y)=f(-y,-x)$ for $(x,y)\in\mathbb{R}^2$, which says that G_f is symmetric with respect to the two planes $x=\pm y$ (and implies the symmetry with respect to the z-axis).
If (x_0,y_0) is a critical point of f, so are $(-x_0,-y_0)$, (y_0,x_0), and $(-y_0,-x_0)$, and all have the same type. $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned}f&=x^4+y^4-6\,x^2-4\,xy-6\,y^2\\ f_x&=4\,x^3-12\,x-4\,y\\ f_y&=4\,y^3-12\,y-4\,x,\\ f_x+f_y&=4(x^3+y^3)-16\,x-16\,y=4(x+y)\left(x^2-xy+y^2-4\right),\\ f_x-f_y&=4(x^3-y^3)-8x+8y=4(x-y)\left(x^2+xy+y^2-2\right).\end{aligned}$$

Then $\nabla f(x,y)=(0,0)$ if 2 of the 4 functions f_x,f_y,f_x+f_y,f_x-f_y vanish at (x,y).

Clearly $\mathbf{p}_0=(0,0)$ is critical. $\boxed{\tfrac12}$
Assuming $(x,y)\neq(0,0)$, we distinguish three mutually exclusive cases:

Case 1: $x = y$ Here $f_x + f_y = 0$ gives $x^2 - x^2 + x^2 = 4$, and hence $x = \pm 2$. This yields the two critical points $\mathbf{p}_1 = (2, 2)$, $\mathbf{p}_2 = (-2, -2)$. $\boxed{1}$

Case 2: $x = -y$ Here $f_x - f_y = 0$ gives $x^2 - x^2 + x^2 = 2$, and hence $x = \pm\sqrt{2}2$. This yields the two critical points $\mathbf{p}_3 = \big(\sqrt{2}, -\sqrt{2}\big)$, $\mathbf{p}_4 = \big(-\sqrt{2}, \sqrt{2}\big)$. $\boxed{1}$

Case 3: $x \neq \pm y$ Here we must have $x^2 - xy + y^2 = 4 \wedge x^2 + xy + y^2 = 2$. Adding/subtracting the two equations gives $2\,x^2 + 2\,y^2 = 6$, $-2xy = 2$, i.e., $x^2 + y^2 = 3 \wedge xy = -1$. $\implies x^2 + (-1/x)^2 = 3$, i.e., $x^4 - 3\,x^2 + 1 = 0$, $x^2 = \frac{1}{2}\big(3 \pm \sqrt{5}\big)$, $x = \pm\frac{1}{2}\big(1 \pm \sqrt{5}\big)$. This yields the four critical points

$$\mathbf{p}_5 = \left(\tfrac{1+\sqrt{5}}{2}, \tfrac{1-\sqrt{5}}{2}\right), \quad \mathbf{p}_6 = \left(\tfrac{1-\sqrt{5}}{2}, \tfrac{1+\sqrt{5}}{2}\right), \quad \mathbf{p}_7 = \left(\tfrac{-1+\sqrt{5}}{2}, \tfrac{-1-\sqrt{5}}{2}\right), \quad \mathbf{p}_8 = \left(\tfrac{-1-\sqrt{5}}{2}, \tfrac{-1+\sqrt{5}}{2}\right).$$

$\boxed{2}$

When determining the types of the critical points, by a) we need only test $\mathbf{p}_0$ and one from each set $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3, \mathbf{p}_4\}$, $\{\mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7, \mathbf{p}_8\}$. We have

$$\mathbf{H}_f(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12\,x^2 - 12 & -4 \\ -4 & 12\,y^2 - 12 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} -12 & -4 \\ -4 & -12 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 36 & -4 \\ -4 & 36 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_5) = \begin{pmatrix} 6 + 6\sqrt{5} & -4 \\ -4 & 6 - 6\sqrt{5} \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_0)$ is negative definite (determinant > 0, top-left entry < 0), the point $\mathbf{p}_0$ is a strict local maximum. $\boxed{\frac{1}{2}}$

Since $\mathbf{H}_f(\mathbf{p}_1)$, $\mathbf{H}_f(\mathbf{p}_3)$ are positive definite (determinant > 0, top-left entry > 0), the points $\mathbf{p}_1$, $\mathbf{p}_2, \mathbf{p}_3$, $\mathbf{p}_4$ are strict local minima. $\boxed{2}$

Since $\mathbf{H}_f(\mathbf{p}_5)$ is indefinite (determinant < 0), the points $\mathbf{p}_5$, $\mathbf{p}_6, \mathbf{p}_7$, $\mathbf{p}_8$ are saddle points. $\boxed{2}$

c) Yes. The points $\mathbf{p}_1$, $\mathbf{p}_2$ are global minima (with value $f(\mathbf{p}_i) = -32$).

The existence of a global minimum follows from $\lim_{|(x,y)|\to\infty} f(x, y) = +\infty$ using an argument analogous to that in the solution to Question 1 b). Indeed, from $x^2 + y^2 \geq 2xy$ we have $6x^2 + 4xy + 6y^2 \leq 8(x^2 + y^2)$ and $x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 \geq \frac{1}{2}(x^2 + y^2)^2$, and hence

$$f(x, y) \geq \frac{1}{2}\,r^4 - 8\,r^2, \quad r = |(x, y)|\,.$$

This clearly implies $\lim_{|(x,y)|\to\infty} f(x, y) = +\infty$. $\boxed{1\frac{1}{2}}$

Since the global minima must be among the local minima, in order to find them we only need to compare the values of f at $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. The proof is then finished by computing $f(\mathbf{p}_1) = f(\mathbf{p}_2) = -32$, $f(\mathbf{p}_3) = f(\mathbf{p}_4) = -8$.

Since $f(x, 0) = x^4 - 6\,x^2$ is unbounded from above, there is no global maximum. $\boxed{\frac{1}{2}}$

$$\sum_{2} = 14$$

3 The continuous function $f(x,y,z)=xy+6yz+6zx$ attains a maximum on the sphere $\mathrm{B}_{\sqrt{17}}(0,0,0)$, which is closed and bounded. This shows that the optimization problem has at least one solution. $\boxed{1}$

Setting $g(x,y,z)=x^2+y^2+z^2$, the task is to minimize f on $\mathbb{R}^3$ under the constraint $g(x,y,z)=17$.

$$\nabla f(x,y,z)=(y+6z,x+6z,6x+6y), \qquad \nabla g(x,y,z)=(2x,2y,2z).$$

Since $\nabla g(x,y,z)\neq(0,0,0)$ for all points on the sphere $\mathrm{B}_{\sqrt{17}}(0,0,0)$, the theorem on Langrange multipliers yields that every optimal solution must satisfy $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ for some $\lambda\in\mathbb{R}$. This gives the system of equations

$$\begin{aligned} y+6z&=\lambda x,\\ x+6z&=\lambda y,\\ 6x+6y&=\lambda z,\\ x^2+y^2+z^2&=17. \end{aligned} \qquad \boxed{3}$$

(For simplicity we have replaced λ by $\lambda/2$.)

The solutions (x,y,z,λ) of this system are precisely the unit eigenvectors of the matrix

$$\mathbf{A}=\begin{pmatrix}0&1&6\\1&0&6\\6&6&0\end{pmatrix}$$

together with the corresponding eigenvalues.

Since $\chi_{\mathbf{A}}(X)=X^3-73\,X-72=(X+1)(X^2-X-72)=(X+1)(X+8)(X-9)$, the eigenvalues of $\mathbf{A}$ are $\lambda_1=9$, $\lambda_2=-1$, $\lambda_3=-8$. This shows already that the eigenspaces of $\mathbf{A}$ are one-dimensional and that the above system has exactly 6 solutions. Next we compute the corresponding eigenvectors. Unit eigenvectors will be denoted by $\mathbf{u}_i$ and eigenvectors of length $\sqrt{17}$ by $\mathbf{v}_i$.

$\underline{\lambda_1=9:}$

$$\mathbf{A}-9\,\mathbf{I}=\begin{pmatrix}-9&1&6\\1&-9&6\\6&6&-9\end{pmatrix}\to\begin{pmatrix}1&-9&6\\0&-80&60\\0&60&-45\end{pmatrix}\to\begin{pmatrix}1&-9&6\\0&-4&3\\0&0&0\end{pmatrix}$$

$\Longrightarrow \mathbf{u}_1=\pm\frac{1}{\sqrt{34}}(3,3,4)^\mathsf{T}$, $\mathbf{v}_1=\pm\frac{1}{\sqrt{2}}(3,3,4)^\mathsf{T}$; $\boxed{1}$

$\underline{\lambda_2=-1:}$

$$\mathbf{A}+\mathbf{I}=\begin{pmatrix}1&1&6\\1&1&6\\6&6&1\end{pmatrix}\to\begin{pmatrix}1&1&6\\0&0&-35\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&1&6\\0&0&1\\0&0&0\end{pmatrix}$$

$\Longrightarrow \mathbf{u}_2=\pm\frac{1}{\sqrt{2}}(1,-1,0)^\mathsf{T}$, $\mathbf{v}_2=\pm\frac{\sqrt{17}}{\sqrt{2}}(1,-1,0)^\mathsf{T}$; $\boxed{1}$

$\underline{\lambda_3=-8:}$

$$\mathbf{A}+8\,\mathbf{I}=\begin{pmatrix}8&1&6\\1&8&6\\6&6&8\end{pmatrix}\to\begin{pmatrix}1&8&6\\0&-63&-42\\0&-42&-28\end{pmatrix}\to\begin{pmatrix}1&8&6\\0&3&2\\0&0&0\end{pmatrix}$$

$\Longrightarrow \mathbf{u}_3 = \pm\frac{1}{\sqrt{17}}(2,2,-3)^\mathsf{T}$, $\mathbf{v}_3 = \pm(2,2,-3)^\mathsf{T}$. $\boxed{1}$

Since $f(3,3,4) = 153 > 0$, $f(1,-1,0) = -1 < 0$, $f(2,2,-3) = -68 < 0$, the maximum is attained at $(x^*,y^*,z^*) = \pm\frac{1}{\sqrt{2}}(3,3,4)^\mathsf{T}$, and $\zeta^* = 153/2$. $\boxed{2}$

$$\sum_3 = 9$$

4 a) Writing $f(x,a) = \frac{1}{x^2+a^2}$, we have

$$\int_0^\infty f_a(x,a)\,\mathrm{d}x = \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,\mathrm{d}x\,. \qquad \boxed{1}$$

Since $x^2+a^2 \geq 2xa$, we can bound the integrand as follows:

$$|f_a(x,a)| = \frac{2a}{(x^2+a^2)^2} \leq \frac{2a}{2xa(x^2+a^2)} = \frac{1}{x(x^2+a^2)} \leq \frac{1}{x(x^2+\delta^2)} =: \Phi(x), \qquad \boxed{2}$$

provided that $a \geq \delta > 0$. Since $\Phi(x)$ is independent of a and integrable over $[0,\infty)$, this shows that F is differentiable in (δ,∞) and $F'(a)$ can be obtained by differentation under the integral sign. Letting $\delta \downarrow 0$, we then obtain the assertion for the whole domain $(0,\infty)$. $\boxed{1}$

b) From a) we have

$$F'(a) = \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,\mathrm{d}x, \quad \text{i.e.,} \quad \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = -\frac{F'(a)}{2a}. \qquad \boxed{1}$$

On the other hand we have

$$\begin{aligned} F(a) &= \int_0^\infty \frac{\mathrm{d}x}{x^2+a^2} \\ &= a\int_0^\infty \frac{\mathrm{d}t}{a^2t^2+a^2} && \text{(Subst. } x = at,\ \mathrm{d}x = a\,\mathrm{d}t) \\ &= \frac{1}{a}\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \frac{1}{a}\left[\arctan t\right]_0^\infty = \frac{\pi}{2a}. && \boxed{1} \end{aligned}$$

It follows that

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = -\frac{-\pi/2a^2}{2a} = \frac{\pi}{4a^3}. \qquad \boxed{1}$$

$$\sum_4 = 7$$

5 a) The mass of K is

$$
\begin{aligned}
m &= \int_K xyz^3 \,\mathrm{d}^3(x,y,z) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} \int_{z=0}^{2\sqrt{x}} xyz^3 \,\mathrm{d}z \,\mathrm{d}^2(x,y) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} xy \left[\frac{z^4}{4}\right]_{z=0}^{2\sqrt{x}} \mathrm{d}^2(x,y) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} 4\,x^3y \,\mathrm{d}^2(x,y) \\
&= 4 \int\limits_{\substack{0\leq r\leq 4\\ 0\leq\theta\leq\pi/2}} (r\cos\theta)^3 r\sin\theta\, r \,\mathrm{d}^2(r,\theta) \\
&= 4\int_0^4 r^5\,\mathrm{d}r \int_0^{\pi/2} \cos^3\theta\sin\theta\,\mathrm{d}\theta \\
&= \frac{4^7}{6}\left[-\frac{1}{4}\cos^4\theta\right]_0^{\pi/2} = \frac{4^6}{6} = \frac{2^{11}}{3} = \frac{2048}{3}.
\end{aligned}
$$

$\boxed{3}$

b) Denoting the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, $(0,2)$ by Δ, the surface P is the graph of $f(x,y) = x^{3/2} + y^{3/2}$, $(x,y) \in \Delta$. Using the formula for such surfaces, or going the long way using the parametrization $\gamma(x,y) = \big(x, y, f(x,y)\big)$, we obtain the surface area as

$$
\begin{aligned}
A = \int_\Delta \sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y) &= \int_\Delta \sqrt{1+\left|\tfrac{3}{2}\left(\sqrt{x},\sqrt{y}\right)\right|^2}\,\mathrm{d}^2(x,y)\\
&= \int_\Delta \sqrt{1+\tfrac{9}{4}(x+y)}\,\mathrm{d}^2(x,y) \qquad \boxed{1}\\
&= \frac{1}{2}\int_0^2\int_0^{2-x}\sqrt{4+9x+9y}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{2}\int_0^2\left[\frac{2}{27}(4+9x+9y)^{3/2}\right]_{y=0}^{2-x}\mathrm{d}x\\
&= \frac{1}{27}\int_0^2 22^{3/2}-(4+9x)^{3/2}\,\mathrm{d}x\\
&= \frac{1}{27}\left(2\cdot 22^{3/2}-\left[\frac{2}{45}(4+9x)^{5/2}\right]_0^2\right)\\
&= \frac{2}{27}\,22^{3/2}-\frac{2}{27\cdot 45}\left(22^{5/2}-4^{5/2}\right)\\
&= \frac{64+46\cdot 22\sqrt{22}}{27\cdot 45}\\
&= \frac{64+1012\sqrt{22}}{1215} \qquad \boxed{3}
\end{aligned}
$$

$$\sum_5 = 7$$

$$\sum_{\text{Final Exam}} = 12+14+9+7+7 = 49 = 40+9$$

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The surface in $\mathbb{R}^3$ with equation $xyz + x + y - z = 2$ is smooth.

b) The function $f(x, y) = x^4 + y^4 + \sin(xy)$, $(x, y) \in \mathbb{R}^2$, has a global minimum.

c) Suppose $f\colon D \to \mathbb{R}$ is a differentiable function on the open unit disk $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$, all of whose gradients $\nabla f(x, y)$, $(x, y) \in D$, are contained in D. Then the range of f is an interval of length at most 2.

d) If $f\colon [0, \infty) \to \mathbb{R}$ is continuous, Lebesgue-integrable over $[0, \infty)$, and $f(0) = 0$, it follows that $\lim_{n\to\infty} \int_0^\infty f(x/n)\,\mathrm{d}x = 0$.

e) There exists a subset S of the open unit disk $D \subset \mathbb{R}^2$ whose set of accumulation points S' is equal to the unit circle (boundary of D).

f) For any path γ in $\mathbb{R}^3$ from $(2, 2, 1)$ to $(3, 0, 0)$ the line integral $\int_\gamma x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$ is zero.

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = 2\,x^3 - x^2y - 2\,x^2 - xy + y^2.$$

a) Determine all critical points of f and their types.

Hint: There are 3 critical points.

b) Does f have a global extremum?

c) Determine the 0-contour (level-zero set) C of f, and sketch C.

Hint: $f(x, y)$ admits a non-trivial factorization $f(x, y) = f_1(x, y)f_2(x, y)$, which for the corresponding 0-contours implies $C = C_1 \cup C_2$. One of the factors can be guessed by looking at the critical points of f.

d) Determine the extrema of f on the rectangle

$$R = \{(x, y) \in \mathbb{R}^2; 0 \le x \le 3, 0 \le y \le 2\}.$$

Question 3 (ca. 12 marks)

The sphere $x^2 + y^2 + z^2 = 9$ intersects the surface $4xy + yz + zx = 20$ in the first octant $O = \{(x, y, z) \in \mathbb{R}^3; x, y, z \ge 0\}$ in a curve C.

a) Show that C has no points on the boundary of O.

b) Show that there exist points on C with minimum, resp., maximum height (z-coordinate).

Hint: $(2, 2, 1) \in C$.

c) Using the method of Lagrange multipliers on the interior of O, determine all those points.

Hint: Don't forget to check for points on C where the Jacobi matrix of the vectorial constraint doesn't have full row rank. In fact there are no such points, but this requires a proof.

Question 4 (ca. 12 marks)

Consider the region K in $\mathbb{R}^3$ consisting of all points (x, y, z) in the first octant $(x, y, z \geq 0)$, inside the cylinder $x^2 + y^2 \leq 4$, and under the surface $z = xy$. Determine

a) the volume of K;

b) the surface area of the boundary ∂K;

c) the centroid ("center of mass") of K.

Solutions

1 a) False. The surface is a level set of $g(x,y,z) = xyz + x + y - z$, which has gradient $\nabla g(x,y,z) = (yz+1, xz+1, xy-1)$. If $\nabla g(x,y,z) = (0,0,0)$ then $x,y,z \neq 0$, $y = 1/x$, $z = -1/x$, $yz = -1/x^2 = -1$, i.e., $x = \pm 1$ and $(x,y,z) = (1,1,-1)$ or $(-1,-1,1)$. The first of these two points is on the surface, and hence the surface has a singular point. $\boxed{2}$

b) True. Since $x^2 + y^2 \leq \sqrt{2}\sqrt{x^4 + y^4}$ (Cauchy-Schwarz inequality), we have

$$f(x,y) \geq \frac{1}{2}(x^2+y^2)^2 - 1 \to +\infty \quad \text{for} \quad |(x,y)| \to \infty.$$

Hence there exists $R > 0$ such that $f(x,y) > 0$ for $|(x,y)| > R$. On the disk $x^2+y^2 \leq R$ the continuous function f attains a minimum, which must be $\leq f(0,0) = 0$. This minimum is evidently a global minimum. (Since $f(x,-x) = 2x^4 - \sin(x^2) < 0$ for small positive numbers x, the minimum is definitely not attained in $(0,0)$.) $\boxed{2}$

c) True. The range of f is an interval, since D is connected and f is continuous. $\boxed{\frac{1}{2}}$

Suppose $\mathbf{x}, \mathbf{y} \in D$. By the Mean Value Theorem (using that D is convex), $f(\mathbf{x}) - f(\mathbf{y}) = \nabla f(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{y})$ for some $\mathbf{z} \in D$ (more precisely, with $\mathbf{z}$ on the line segment connecting $\mathbf{x}$ and $\mathbf{y}$, which in the trivial case $\mathbf{x} = \mathbf{y}$ reduces to a single point), and hence

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{y})| &\leq |\nabla f(\mathbf{z})|\,|\mathbf{x} - \mathbf{y}| && \text{(Cauchy-Schwarz Inequality)} \\ &\leq |\mathbf{x} - \mathbf{y}| < 2. \end{aligned}$$

The assertion follows. (But note that "length exactly 2" is possible, e.g., take $f(x,y) = x$.) $\boxed{1\frac{1}{2}}$

d) False. The function sequence $f_n(x) = f(x/n)$, defined on $[0,\infty)$, converges point-wise to the all-zero function, but limit and integration can't be interchanged to obtain $\lim_{n\to\infty} \int_0^\infty f(x/n)\,\mathrm{d}x = 0$. In fact the change of variables $y = x/n$, $\mathrm{d}y = \mathrm{d}x\,/n$ gives $\int_0^\infty f(x/n)\,\mathrm{d}x = n\int_0^\infty f(y)\,\mathrm{d}y$, and the limit is $\pm\infty$ except when $\int_0^\infty f(y)\,\mathrm{d}y = 0$. $\boxed{2}$

e) True. Enumerate the rational numbers in $[0,2\pi)$ as $q_1, q_2, q_3, \dots$, let $\mathbf{p}^{(n)}$ be the unique point on ∂D such that the length of the arc from $(1,0)$ to $\mathbf{p}^{(n)}$ (following the mathematically positive direction) is equal to q_n, and set $\mathbf{x}^{(n)} = (1 - 1/n)\mathbf{p}^{(n)}$ for $n \in \mathbb{N}$. We claim that the sequence $\big(\mathbf{x}^{(n)}\big)$, or its range S, has $S' = \partial D$. Clearly no point (x,y) with $x^2 + y^2 > 1$ can be in S'. If $r = x^2 + y^2 < 1$, the disk around (x,y) with radius $(1-r)/2$ contains only finitely many points in S (it can contain only points $\mathbf{x}^{(n)}$ with $1 - 1/n < (1+r)/2$, i.e., $n < \frac{2}{1-r}$), showing that (x,y) cannot be in S' either. Now consider a point $(x,y) \in \partial D$ and a disk B of radius $\epsilon > 0$ around this point. Since $\{q_1, q_2, q_3, \dots\}' = [0,2\pi]$, there exist infinitely many n such that $\big|(x,y) - \mathbf{p}^{(n)}\big| < \epsilon/2$. Of these all but finitely many also have $\big|\mathbf{p}^{(n)} - \mathbf{x}^{(n)}\big| = 1/n < \epsilon/2$. Since B contains all points $\mathbf{x}^{(n)}$ with n satisfying both conditions, it contains infinitely many points of S, i.e., we have shown $(x,y) \in S'$. $\boxed{2}$

f) True. The form $x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$ is exact with antiderivative $f(x,y,z) = \frac{1}{2}\left(x^2+y^2+z^2\right)$, which is rotation-invariant and gives

$$\int_\gamma x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z = f(3,0,0) - f(2,2,1) = \frac{9}{2} - \frac{9}{2} = 0.$$

(The key point is that $(2,2,1)$ and $(3,0,0)$ have the same length. $\boxed{2}$

Remarks:

$$\sum_1 = 12$$

2 a) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f &= 2\,x^3 - x^2y - 2\,x^2 - xy + y^2,\\ f_x &= 6x^2 - 2xy - 4x - y, \\ f_y &= -x^2 - x + 2y, \end{aligned} \qquad \boxed{1}$$

$$\begin{aligned} \nabla f(x,y) = (0,0) &\implies y = \tfrac{1}{2}\left(x^2+x\right) \implies 6x^2 - x(x^2+x) - 4x - \tfrac{1}{2}\left(x^2+x\right) = 0\\ &\implies -x^3 + \tfrac{9}{2}\,x^2 - \tfrac{9}{2}\,x = 0\\ &\implies 2x^3 - 9x^2 + 9 = 2x(x-3)(x-3/2) = 0. \end{aligned}$$

This yields the three critical points

$$\mathbf{p}_1 = (0,0), \quad \mathbf{p}_2 = (3,6), \quad \mathbf{p}_3 = \left(\tfrac{3}{2}, \tfrac{15}{8}\right). \qquad \boxed{2}$$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x-2y-4 & -2x-1 \\ -2x-1 & 2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} -4 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_2) = \begin{pmatrix} 20 & -7 \\ -7 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 41/4 & -4 \\ -4 & 2 \end{pmatrix}.$$

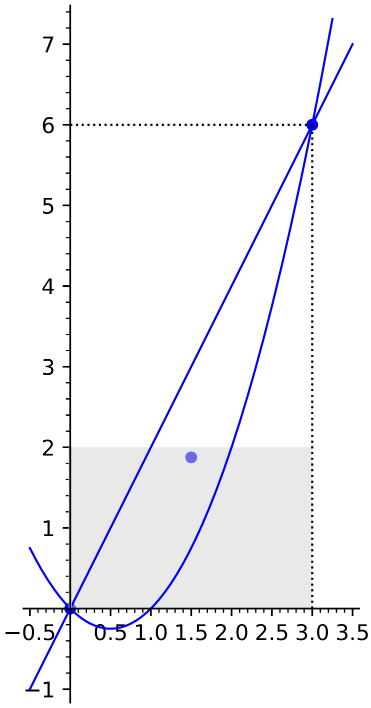
Since $\mathbf{H}_f(\mathbf{p}_1)$ has determinant $-8-1 = -9 < 0$, the point $\mathbf{p}_1$ is a saddle point. $\boxed{1}$
Since $\mathbf{H}_f(\mathbf{p}_2)$ has determinant $40 - 49 = -9 < 0$, the point $\mathbf{p}_2$ is a saddle point as well. $\boxed{1}$
Since $\mathbf{H}_f(\mathbf{p}_3)$ is positive definite ($f_{xx}(\mathbf{p}_3) = 41/4 > 0$, $\det \mathbf{H}_f(\mathbf{p}_3) = 41/2 - 16 = 9/2 > 0$), the point $\mathbf{p}_3$ is a strict local minimum. $\boxed{1}$
The corresponding value is $f(\mathbf{p}_3) = -\frac{81}{64}$.

b) No. Since global extrema must be local extrema, the only candidate is $\mathbf{p}_3$. Since $f(x,0) = 2\,x^3 - 2\,x^2 \to -\infty$ for $x \to -\infty$, the point $\mathbf{p}_3$ is not a global minimum. (Alternatively, use $f(-1,0) = -4 < -\frac{81}{64}$.) $\boxed{1}$

c) Intersection points of C_1 and C_2, if any, must be critical points of f, because C is not smooth there (except in the case $C_1 = C_2$). Thus it is reasonable to guess that C_1, C_2 pass through the saddle points $(0,0)$ and $(3,6)$. The simplest curve connecting the two points is the line $y = 2x$, and indeed $f(x,y)$ is divisible by $y - 2x$: $2x^3 - x^2y = -x^2(y-2x)$ and $y^2 - xy - 2x^2 = (y-2x)(y+x)$, and hence

$$f(x,y) = -x^2(y-2x) + (y+x)(y-2x) = (y-2x)(y-x^2+x). \qquad \boxed{1}$$

Thus C is the union of the line $y = 2x$ and the parabola $y = x^2 - x$.



$\boxed{1}$

Figure 1: The 0-contour of f (union of the blue curves), plotted together with the three critical points of f and the rectangle R (shaded)

d) Extrema located in R° must be critical points, and hence equal to $\mathbf{p}_3$. On the boundary ∂R we have

$$\begin{aligned} f(x,0) &= 2x^3 - 2x^2 = 2x^2(x-1),\\ f(0,y) &= y^2,\\ f(x,2) &= 2x^3 - 4x^2 - 2x + 4 = 2(x-1)(x-2)(x+1),\\ f(3,y) &= 36 - 12y + y^2 = (6-y)^2. \end{aligned}$$

On the left edge of R the values vary between 0 ($y=0$) and 4 ($y=2$), on the right edge between 36 ($y=0$) and 16 ($y=2$).

On the bottom edge the range is $\left[-\frac{8}{27}, 36\right]$, since $x \mapsto 2x^2(x-1)$ attains its minimum at $x = 2/3$ (the larger zero of $(2x^3-2x^2)' = 6x^2-4x$) with corresponding value $-8/27$), is decreasing in $\left[0, \frac{2}{3}\right]$ from 0 to $-8/27$ and increasing in $\left[\frac{2}{3}, 3\right]$ from $-8/27$ to 36.

On the top edge f the range is that of the cubic polynomial $g(x) = f(x,2)$ restricted to $[0,3]$. Since $g'(x) = 6x^2 - 8x - 2$ has roots $x_1 = \frac{2-\sqrt{7}}{3} < 0$, $x_2 = \frac{2+\sqrt{7}}{3} > 0$, the function g is decreasing in $[0,x_2] \subseteq [x_1,x_2]$ from 4 to $g(x_2)$ and increasing in $[x_2,3]$ from $g(x_2)$ to 16. The range is therefore $[g(x_2), 16]$.

It follows that $f(3,0) = 36$ is the unique maximum of f on R. $\boxed{2}$

It can be shown that

$$g(x_2) = 2\,\frac{-1+\sqrt{7}}{3}\,\frac{-4+\sqrt{7}}{3}\,\frac{5+\sqrt{7}}{3} = -\frac{4}{27}\left(7\sqrt{7}-10\right) > -\frac{81}{64},$$

and hence that the unique minimum of f on R is $f(\mathbf{p}_3) = f\left(\frac{3}{2}, \frac{15}{8}\right) = -\frac{81}{64}$. This amounts to the inequality $2^{16}7^3 < (2560 + 3^7)^2$, which requires computing with fairly large integers and is not really suitable for a closed-book exam.

However, there is an alternative argument using c). Consider the region

$$K = \left\{(x, y) \in \mathbb{R}^2; x^2 - x \leq y \leq 2x\right\}$$

enclosed by the two curves forming the 0-contour of f. Since K is closed and bounded, f attains a minimum on K. Since $\mathbf{p}_3 \in K$, $f(\mathbf{p}_3) = -\frac{81}{64} < 0$, and f vanishes on the boundary ∂K, the minimum must be in K°, and hence equal to $\mathbf{p}_3$. In particular, the values of f on the line segment $\left\{(x, 2); 1 \leq x \leq 2\right\}$ (the intersection of K and the line $y = 2$) must be larger than $-\frac{81}{64}$. This shows that the minimum of $g(x) = f(x, 2)$, viz. $x_2 = \frac{2+\sqrt{7}}{3}$, has $g(x_2) > -\frac{81}{64}$. The same argument also excludes a minimum on the bottom edge of R. $\boxed{2}$

Remarks:

$$\sum_2 = 13$$

---

3 a) Suppose C contains a point with $x = 0$. Then $y^2 + z^2 = 9$, $yz = 20$, and hence $(y - z)^2 = y^2 + z^2 - 2yz = -31 < 0$, contradiction. Then, by symmetry, C doesn't contain a point with $y = 0$ either. The case $z = 0$ is excluded in the same way: $x^2 + y^2 = 9$, $xy = 5$, implying $(x - y)^2 = x^2 + y^2 - 2xy = -1 < 0$, contradiction. $\boxed{1\frac{1}{2}}$.

b) Using the hint C is non-empty; C is closed as the intersection of two level sets of continuous functions and the closed set O; and C is bounded as a subset of a sphere. Hence the continuous function $f(x, y, z) = z$ attains a minimum and a maximum on C. $\boxed{1\frac{1}{2}}$

c) By a), C is contained in O°, so that we can apply the method of Lagrange multipliers to the objective function f and the two constraints $g_1(x, y, z) = x^2 + y^2 + z^2 - 9$, $g_2(x, y, z) = 4\,xy + yz + zx - 20$, all with domain O°, to find those points. For $\mathbf{g} = (g_1, g_2)^\mathsf{T}$ we have

$$\mathbf{J_g}(x, y, z) = \begin{pmatrix} 2x & 2y & 2z \\ 4y + z & 4x + z & x + y \end{pmatrix}.$$

Points on C where $\mathbf{J_g}$ has rank < 2 would need to be checked separately, but there are no such points as we now show:

$\operatorname{rank}(\mathbf{J_g}) < 2$ implies that all three 2×2 subdeterminants of $\mathbf{J_g}$ vanish. In particular we then have $x(4x + z) = y(4y + z)$, i.e., $4x^2 - 4y^2 + xz - yz = 0$, which can be factorized as $(x - y)\,[4(x + y) + z] = 0$. Since the 2nd factor is positive on O°, we must have $x = y$. Then one of the other subdeterminants gives $2x^2 = (4x + z)z$, i.e., $z^2 + 4xz - 2x^2 = 0$, $z = \left(-2 \pm \sqrt{6}\right)x$. Since $x, z > 0$, the plus sign must hold. Substituting $y = x$, $z = \left(-2 + \sqrt{6}\right)x$ into the two constraints, we then get

$$\begin{aligned} 2x^2 + \left(\left(-2 + \sqrt{6}\right)x\right)^2 &= \left(12 - 4\sqrt{6}\right)x^2 = 9, \\ 4x^2 + 2x\left(-2 + \sqrt{6}\right)x &= 2\sqrt{6}\,x^2 = 20. \end{aligned}$$

$\Longrightarrow 20\left(12-4\sqrt{6}\right)=9\cdot 2\sqrt{6} \Longrightarrow 240-98\sqrt{6}=0$. Since $\sqrt{6}$ is irrational, we have the desired contradiction. $\boxed{2}$

Thus the Lagrange multiplier condition applies to any minimum/maximum of f on C and yields the equations:

$$\begin{aligned}
\lambda x+\mu(4y+z)&=0,\\
\lambda y+\mu(4x+z)&=0,\\
\lambda z+\mu(x+y)&=1,\\
x^2+y^2+z^2&=9,\\
4xy+yz+zx&=20.
\end{aligned} \qquad \boxed{2}$$

Since $x,y,z>0$, the multipliers λ,μ must be nonzero. Then, from the first two equations we obtain as above $x=y$. $\boxed{1}$

This leaves the two equations $2x^2+z^2=9$, $4x^2+2xz=20$. Solving the 2nd equation for z and substituting the result into the 1st equation gives

$$\begin{aligned}
2\,x^2+\left(\frac{10-2x^2}{x}\right)^2=9 &\iff 2\,x^4+(10-2x^2)^2=9\,x^2\\
\iff 6\,x^4-49\,x^2+100=0 &\iff 6(x^2-4)(x^2-25/6)=0.
\end{aligned}$$

Thus $x=2\vee x=5/\sqrt{6}$, giving the two points $(2,2,1)$, $\left(\frac{5}{\sqrt{6}},\frac{5}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$. $\boxed{2}$

Thus $(2,2,1)$ is the unique point of maximum height 1 on C, and $\left(\frac{5}{\sqrt{6}},\frac{5}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$ is the unique point of minimum height $\frac{2}{\sqrt{6}}$ on C. $\boxed{1}$

Remarks:

$$\sum_3 = 11$$

---

4 a) The volume doesn't change if we include/omit (part of) the boundary ∂K, since ∂K has volume zero. Hence it doesn't matter whether we use strict or weak inequalities in the following computation:

$$\begin{aligned}
\operatorname{vol}(K)&=\int_K 1\,\mathrm{d}^3(x,y,z)=\int_{\substack{x^2+y^2\leq 4\\ x,y\geq 0}}\int_{z=0}^{xy}\mathrm{d}z\,\mathrm{d}^2(x,y)=\int_{\substack{x^2+y^2\leq 4\\ x,y\geq 0}} xy\,\mathrm{d}^2(x,y) && \boxed{1}\\
&=\int_{\substack{0<r<2\\ 0<\theta<\pi/2}} r(r\cos\theta)(r\sin\theta)\,\mathrm{d}^2(r,\theta) && \boxed{1}\\
&=\left(\int_0^2 r^3\,\mathrm{d}r\right)\left(\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta\right)=\frac{1}{2}\left[\frac{r^4}{4}\right]_0^2\left(\int_0^{\pi/2}\sin(2\theta)\mathrm{d}\theta\right)\\
&=2\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/2}=2. && \boxed{1}
\end{aligned}$$

b) ∂K is the disjoint union of the 2-dimensional surfaces

$$\begin{aligned}
S_1 &= \{(x,y,z)\in\mathbb{R}^3; x>0, y>0, x^2+y^2<4, z=0\}, && \text{("bottom")}\\
S_2 &= \{(x,y,z)\in\mathbb{R}^3; x>0, y>0, x^2+y^2=4, 0<z<xy\}, && \text{("side")}\\
S_3 &= \{(x,y,z)\in\mathbb{R}^3; x>0, y>0, x^2+y^2<4, z=xy\}, && \text{("top")}
\end{aligned}$$

and lower-dimensional parts (2 line segments, a quarter circle, and part of the curve $z=xy\wedge x^2+y^2=4$), which don't contribute to the surface area. $\boxed{1}$

The surface area of S_1 is $A_1=\frac{1}{4}(2^2\pi)=\pi$. $\boxed{\frac{1}{2}}$

S_2 is the range of $\gamma_(\theta,z)=(2\cos\theta,2\sin\theta,z)$, $0<\theta<\pi/2$, $0<z<(2\cos\theta)(2\sin\theta)=2\sin(2\theta)$. We obtain

$$\begin{aligned}
\mathbf{J}_\gamma(\theta,z) &= \begin{pmatrix}-2\sin\theta & 0\\ 2\cos\theta & 0\\ 0 & 1\end{pmatrix},\\
\mathbf{J}_\gamma(\theta,z)^\mathsf{T}\mathbf{J}_\gamma(\theta,z) &= \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},\\
\sqrt{g^\gamma(\theta,z)} &= 2,\\
A_2 &= 2\int_{\theta=0}^{\pi/2}\int_{z=0}^{2\sin(2\theta)}\mathrm{d}z\,\mathrm{d}\theta = 4\int_0^{\pi/2}\sin(2\theta)\mathrm{d}\theta = 4.
\end{aligned}$$ $\boxed{1\frac{1}{2}}$

S_3 is the range of $\gamma(x,y)=(x,y,xy)$, $x>0$, $y>0$, $x^2+y^2<4$ or, alternatively, the graph of the function $f(x,y)=xy$ with the same domain. We obtain

$$\begin{aligned}
\mathbf{J}_\gamma(x,y) &= \begin{pmatrix}1 & 0\\ 0 & 1\\ y & x\end{pmatrix},\\
\mathbf{J}_\gamma(x,y)^\mathsf{T}\mathbf{J}_\gamma(x,y) &= \begin{pmatrix}1+y^2 & xy\\ xy & 1+x^2\end{pmatrix},\\
g^\gamma(x,y) &= (1+y^2)(1+x^2)-(xy)^2 = 1+x^2+y^2, \\
&\qquad\qquad (\text{equal to } 1+|\nabla f(x,y)|^2)\\
\sqrt{g^\gamma(x,y)} &= \sqrt{1+x^2+y^2},\\
A_3 &= \int_{\substack{x^2+y^2<4\\ x,y>0}}\sqrt{1+x^2+y^2}\,\mathrm{d}^2(x,y)
\end{aligned}$$ $\boxed{1\frac{1}{2}}$

$$\begin{aligned}
&= \int_{\substack{0<r<2\\ 0<\theta<\pi/2}} r\sqrt{1+r^2}\,\mathrm{d}^2(r,\theta) = \frac{\pi}{2}\int_0^2 r\sqrt{1+r^2}\,\mathrm{d}r\\
&= \frac{\pi}{2}\left[\frac{1}{3}(1+r^2)^{3/2}\right]_0^2 = \frac{\pi}{6}\left(5\sqrt{5}-1\right).
\end{aligned}$$ $\boxed{1}$

In all we have

$$\mathrm{vol}_2(S_3) = A_1+A_2+A_3 = \pi+4+\frac{\pi}{6}\left(5\sqrt{5}-1\right) = 4+\frac{5\left(\sqrt{5}+1\right)}{6}\,\pi.$$ $\boxed{\frac{1}{2}}$

c) Since K is symmetric w.r.t. the plane $x = y$, it is clear that the centroid $\mathbf{s}$ has the form (x_0, x_0, z_0). $\boxed{\frac{1}{2}}$

$$\begin{aligned}
x_0 &= \frac{1}{\mathrm{vol}(K)} \int_K x \,\mathrm{d}^3(x,y,z) = \frac{1}{2} \int_{\substack{x^2+y^2<4\\ x,y>0}} \int_{z=0}^{xy} x \,\mathrm{d}z \,\mathrm{d}^2(x,y)\\
&= \frac{1}{2} \int_{\substack{x^2+y^2<4\\ x,y>0}} x^2 y \,\mathrm{d}^2(x,y)\\
&= \frac{1}{2} \int_{\substack{0<r<2\\ 0<\theta<\pi/2}} r^4 \cos^2\theta \sin\theta \,\mathrm{d}^2(r,\theta) = \frac{1}{2} \left(\int_0^2 r^4 \,\mathrm{d}r\right) \left(\int_0^{\pi/2} \cos^2\theta \sin\theta \,\mathrm{d}\theta\right)\\
&= \frac{1}{2} \left[\frac{r^5}{5}\right]_0^2 \left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{2}\,\frac{32}{5}\,\frac{1}{3} = \frac{16}{15}
\end{aligned} \qquad \boxed{1\tfrac{1}{2}}$$

$$\begin{aligned}
z_0 &= \frac{1}{\mathrm{vol}(K)} \int_K z \,\mathrm{d}^3(x,y,z) = \frac{1}{2} \int_{\substack{x^2+y^2<4\\ x,y>0}} \int_{z=0}^{xy} z \,\mathrm{d}z \,\mathrm{d}^2(x,y)\\
&= \frac{1}{2} \int_{\substack{x^2+y^2<4\\ x,y>0}} \left[z^2/2\right]_0^{xy} \mathrm{d}^2(x,y) = \frac{1}{4} \int_{\substack{x^2+y^2<4\\ x,y>0}} x^2 y^2 \,\mathrm{d}^2(x,y)\\
&= \frac{1}{4} \int_{\substack{0<r<2\\ 0<\theta<\pi/2}} r^5 \cos^2\theta \sin^2\theta \,\mathrm{d}^2(r,\theta) = \frac{1}{4} \left(\int_0^2 r^5 \,\mathrm{d}r\right) \left(\int_0^{\pi/2} \cos^2\theta \sin^2\theta \,\mathrm{d}\theta\right)\\
&= \frac{1}{4}\,\frac{64}{6}\,\frac{1}{4} \int_0^{\pi/2} \sin^2(2\theta)\mathrm{d}\theta = \frac{2}{3}\,\frac{\pi}{4} = \frac{\pi}{6}.
\end{aligned} \qquad \boxed{2}$$

For the 2nd-to-last equality we have used $\sin^2(2\theta)+\cos^2(2\theta) = 1$ and $\int_0^{\pi/2} \sin^2(2\theta)\mathrm{d}\theta = \int_0^{\pi/2} \cos^2(2\theta)\mathrm{d}\theta$ (by symmetry of cos, sin).

It follows that $\mathbf{s} = \left(\frac{16}{15}, \frac{16}{15}, \frac{\pi}{6}\right)$.

Remarks:

$$\sum_4 = 13$$

$$\sum_{\text{Final Exam}} = 40 + 9$$

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The surface in $\mathbb{R}^3$ with equation $x^2y + y^2z + z^2x = 1$ is smooth.

b) Suppose you start at the point $(1, 1)$ in the (x, y)-plane and follow (and continuously adjust) the direction of steepest ascent of $f(x, y) = \frac{x-y}{x+y}$. After some time (provided you won't get tired) you will cross the x-axis at a point $(x_0, 0)$ with $x_0 > 2$.

c) If $f\colon [0, 1] \to \mathbb{R}$ is continuous and $f(0) = 0$ then $\lim_{n\to\infty} \int_0^1 f(x^n)\,\mathrm{d}x = 0$.

d) The equation $x^2 + xy + y^2 = 3$ defines a circle, which is symmetric to the line $y = x$.

e) There exists a subset D of the upper half plane $\{(x, y) \in \mathbb{R}^2; y > 0\}$ whose set of accumulation points is equal to the real axis (x-axis).

f) If γ is a closed path in $\mathbb{R}^3$ satisfying $\int_\gamma x\,\mathrm{d}y + y\,\mathrm{d}z + z\,\mathrm{d}x = 0$, we must have $\int_\gamma x\,\mathrm{d}z + y\,\mathrm{d}x + z\,\mathrm{d}y = 0$ as well.

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^4 - x^3y - xy + y^2.$$

a) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f?

b) Determine all critical points of f and their types.
Hint: There are 5 critical points.

c) Does f have a global extremum?

d) Determine the extrema of f on the unit square $Q = \{(x, y) \in \mathbb{R}^2; 0 \le x, y \le 1\}$.

Question 3 (ca. 12 marks)

The sphere $x^2 + y^2 + z^2 = 9$ intersects the surface $xy + yz + zx = 8$ in the first octant $O = \{(x, y, z) \in \mathbb{R}^3; x, y, z \ge 0\}$ in a curve C.

a) Show that C has no points on the boundary of O.

b) Show that there exist points on C with minimum, resp., maximum height (z-coordinate).
Hint: $(2, 2, 1) \in C$.

c) Using the method of Lagrange multipliers on the interior of O, determine all those points.
Note: Don't forget to check for points on C where the Jacobi matrix of the vectorial constraint doesn't have full row rank. In fact there are no such points, but this requires a proof.

d) At the point $(2,2,1)$ the curve C admits locally a parametrization $\gamma(x) = \bigl(x, h(x), k(x)\bigr)$ with functions $h, k\colon (2-\epsilon, 2+\epsilon) \to \mathbb{R}$. Determine $h'(2)$ and $k'(2)$.

Question 4 (ca. 12 marks)

Consider the transformation

$$T(s,t,u) = (us\cos t, us\sin t, us + ut)$$

from the region $U = \bigl\{(s,t,u) \in \mathbb{R}^3; 0 < s < t < 2\pi, 0 < u < 1\bigr\}$ to the region $V = T(U) \subset \mathbb{R}^3$, and the "helicoid"

$$S = \bigl\{(s\cos t, s\sin t, s+t); 0 < s < t < 2\pi\bigr\}$$

bounding V from above.

a) Show that $T\colon U \to V$ is a diffeomorphism (i.e., T is one-to-one, and both T and T^{-1} are differentiable).

b) Determine the volume of V.

c) Express the surface area of S as an ordinary 1-dimensional Riemann integral.

Solutions

1 a) True. The surface is a level set of $g(x,y,z) = x^2y + y^2z + z^2x$, which has gradient $\nabla g(x,y,z) = (2xy + z^2, 2yz + x^2, 2zx + y^2)$. If $\nabla g(x,y,z) = (0,0,0)$ then $8x^2y^2z^2 = (2xy)(2yz)(2zx) = (-x^2)(-y^2)(-z^2) = -x^2y^2z^2$, and hence $xyz = 0$. By symmetry, we can assume $x = 0$. Then $g_x = 0$ gives $z = 0$, and $g_z = 0$ gives $y = 0$. But the point $(0,0,0)$ isn't on the surface, and hence $\nabla g(x,y,z) = (0,0,0)$ has no solution on the surface. $\boxed{2}$

b) False. We have

$$\begin{aligned} f_x &= \frac{1(x+y) - 1(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}, \\ f_y &= \frac{(-1)(x+y) - 1(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}, \end{aligned}$$

and hence gradients $\nabla f(x,y)$ in the open first quadrant point south-eastern and are orthogonal to (x,y). Hence, following the direction of steepest ascent (i.e., the gradient) you will move south-eastern from $(1,1)$, turn more and more southward, and cross the x-axis at a point closer to the origin than when following the gradient at $(1,1)$ all the time, i.e., closer than the point $(2,0)$. $\boxed{2}$

c) True. The function sequence $f_n(x) = f(x^n)$, defined on $[0,1]$, converges point-wise to the all-zero function, except possibly for $x = 1$, where the limit is $f(1)$. For this observe that $x^n \to 0$ and hence, since f is continuous, $f(x^n) \to f(0) = 0$ for $0 \leq x < 1$. The functions f_n are continuous, hence Lebesgue-integrable, and bounded by a constant $M > 0$ independently of n. (Any bound for the continuous function f works also for f_n. Thus $\Phi(x) = M$, $x \in [0,1]$, serves as an integrable bound for (f_n) and allows us to apply Lebesgue's dominated convergence theorem to conclude that $\lim_{n\to\infty} \int_0^1 f(x^n)\,\mathrm{d}x = \lim_{n\to\infty} \int f_n(x)\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0$. $\boxed{2}$

d) False. It is an ellipse with semi-axes $a = \sqrt{6}$ on the line $y = -x$, and $b = \sqrt{2}$ on the line $y = x$. This can be seen by diagonalizing the corresponding symmetric matrix, which is $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and has eigenvalues $3/2$ and $1/2$. But to prove the statement false, it suffices to observe that $(1,1)$ and $\left(-\sqrt{3}, \sqrt{3}\right)$ satisfy the equation and have different distance from the center $(0,0)$. $\boxed{2}$

e) True. Enumerate the rational numbers as $q_1, q_2, q_3, \dots$ and define $\mathbf{x}^{(n)} = (q_n, 1/n)$ for $n \in \mathbb{N}$. We claim that the sequence $\left(\mathbf{x}^{(n)}\right)$, or its range D, has $D' = \mathbb{R} \times \{0\}$. Clearly no point (x,y) with $y < 0$ can be in D'. If $y > 0$, the disk around (x,y) with radius $y/2$ contains only finitely many points in D (it can contain only points $\mathbf{x}^{(n)}$ with $n < 2/y$), showing that (x,y) cannot be in D' either. Now consider a point $(x,0)$ and a disk B of radius $\epsilon > 0$ around this point. Since $\mathbb{Q}' = \mathbb{R}$, there exist infinitely many n such that $|x - q_n| < \epsilon/\sqrt{2}$. Of these all but finitely many also have $1/n < \epsilon/\sqrt{2}$. Since B contains all points $\mathbf{x}^{(n)}$ with n satisfying both conditions, it contains infinitely many points of D, i.e., we have shown $(x,0) \in D'$. $\boxed{2}$

f) True. The sum of the two integrands is $(y+z)\,\mathrm{d}x + (x+z)\,\mathrm{d}y + (x+y)\,\mathrm{d}z$, which is exact in $\mathbb{R}^3$. Hence the sum of the two integrals is zero (since it is the integral of the sum). So, if one integral is zero, the other must be too. $\boxed{2}$

Remarks: No marks were assigned for answers without justification.

$$\sum_1 = 12$$

2 a) $f(-x,-y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$
$\Longrightarrow$ The graph of f is symmetric with respect to the z-axis. $\boxed{1}$
The contours of f are point-symmetric with respect to the origin. $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f_x &= 4\,x^3 - 3\,x^2y - y, \\ f_y &= -x^3 - x + 2\,y, \end{aligned} \qquad \boxed{1}$$

$$\begin{aligned} \nabla f(x,y) = (0,0) &\Longrightarrow y = \tfrac{1}{2}\left(x^3+x\right) \Longrightarrow 4x^3 - \left(3x^2+1\right)\tfrac{1}{2}\left(x^3+x\right) = 0 \\ &\Longrightarrow 3x^5 - 4x^3 + x = 0 \\ &\Longrightarrow x(x^2-1)(3x^2-1) = 0. \end{aligned}$$

$\Longrightarrow$ The critical points of f are

$$\begin{gathered} \mathbf{p}_1 = (0,0), \quad \mathbf{p}_2 = (1,1), \quad \mathbf{p}_3 = (-1,-1), \\ \mathbf{p}_4 = \left(\tfrac{1}{3}\sqrt{3}, \tfrac{2}{9}\sqrt{3}\right), \quad \mathbf{p}_5 = \left(-\tfrac{1}{3}\sqrt{3}, -\tfrac{2}{9}\sqrt{3}\right). \end{gathered} \qquad \boxed{2\tfrac{1}{2}}$$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 - 6xy & -3x^2-1 \\ -3x^2-1 & 2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{2/3}) = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{4/5}) = \begin{pmatrix} 8/3 & -2 \\ -2 & 2 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$ has determinant $-1 < 0$, the point $\mathbf{p}_1$ is a saddle point. $\boxed{\tfrac{1}{2}}$
Since $\mathbf{H}_f(\mathbf{p}_{2/3})$ has determinant $12-16 = -4 < 0$, the points $\mathbf{p}_2$, $\mathbf{p}_3$ are saddle points. $\boxed{1}$
Since $\mathbf{H}_f(\mathbf{p}_{4/5})$ is positive definite ($f_{xx}(\mathbf{p}_{4/5}) = 8/3 > 0$, $\det \mathbf{H}_f(\mathbf{p}_{4/5}) = 16/3 - 4 = 4/3 > 0$), the points $\mathbf{p}_4$, $\mathbf{p}_5$ are strict local minima. $\boxed{1}$
The corresponding value is $f(\mathbf{p}_{4/5}) = -1/27$.

c) No. This follows, e.g., from $f(x,0) = x^4 \to +\infty$ for $x \to \pm\infty$, $f(x,2x) = -x^4 + 2x^2 \to -\infty$ for $x \to \pm\infty$. $\boxed{1}$

d) Extrema located in Q° must be critical points, and hence equal to $\mathbf{p}_4$. On the boundary ∂Q we have

$$\begin{aligned} f(x,0) &= x^4, \\ f(0,y) &= y^2, \\ f(x,1) &= x^4 - x^3 - x + 1 = (x^3-1)(x-1), \\ f(1,y) &= 1 - 2y + y^2 = (1-y)^2. \end{aligned}$$

One sees that the values on ∂Q vary between 0 and 1, with 1 attained at $(1,0)$, $(0,1)$, and 0 attained at $(0,0)$, $(1,1)$. Comparing these with $f(\mathbf{p}_4) = -1/27$ shows that f on Q attains its minimum at $\mathbf{p}_4$ and two maxima at $(1,0)$, $(0,1)$. $\boxed{3}$

Remarks:

$\sum_2 = 12$

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3 a) Suppose C contains a point with $x = 0$. Then $y^2 + z^2 = 9$, $yz = 8$, and hence $(y-z)^2 = y^2 + z^2 - 2yz = -7 < 0$, contradiction. Then, by symmetry, C doesn't contain a point with $y = 0$ or $z = 0$ either. $\boxed{1}$

b) Using the hint, C is non-empty; C is closed as the intersection of two level sets of continuous functions and the closed set O; and C is bounded as a subset of a sphere. Hence the continuous function $f(x,y,z) = z$ attains a minimum and a maximum on C. $\boxed{2}$

c) By a), C is contained in O°, so that we can apply the method of Lagrange multipliers to the objective function f and the two constraints $g_1(x,y,z) = x^2 + y^2 + z^2 - 9$, $g_2(x,y,z) = xy + yz + zx - 8$, all with domain O°, to find those points. For $\mathbf{g} = (g_1, g_2)^\mathsf{T}$ we have

$$\mathbf{J_g}(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ y+z & x+z & x+y \end{pmatrix}.$$

Points on C where $\mathbf{J_g}$ has rank < 2 would need to be checked separately, but there are no such points as we now show:

$\operatorname{rank}(\mathbf{J_g}) < 2$ implies that all three 2×2 subdeterminants of $\mathbf{J_g}$ vanish. In particular we then have $x(x+z) = y(y+z)$, i.e., $x^2 - y^2 + xz - yz = 0$, which can be factorized as $(x-y)(x+y+z) = 0$. Since the 2nd factor is positive on O°, we must have $x = y$. By symmetry (or using the other two subdeterminants), we also have $x = z$ and $y = z$, i.e., $x = y = z$. But, since $3x^2 = 9$ and $3x^2 = 8$ are mutually exclusive, C doesn't contain a point with $x = y = z$. $\boxed{2}$

Thus the Lagrange multiplier condition applies to any minimum/maximum of f on C and yields the equations:

$$\begin{aligned} \lambda x + \mu(y+z) &= 0, \\ \lambda y + \mu(x+z) &= 0, \\ \lambda z + \mu(x+y) &= 1, \\ x^2 + y^2 + z^2 &= 9, \\ xy + yz + zx &= 8. \end{aligned}$$ $\boxed{2}$

Since $x, y, z > 0$, the multipliers λ, μ must be nonzero. Then, from the first two equations we obtain as above $x = y$. $\boxed{1}$

This leaves the two equations $2x^2 + z^2 = 9$, $x^2 + 2xz = 8$. Solving the 2nd equation for z and substituting the result into the 1st equation gives

$$2\,x^2 + \left(\frac{8 - x^2}{2x}\right)^2 = 9 \iff 8\,x^4 + (8 - x^2)^2 = 36\,x^2$$
$$\iff 9\,x^4 - 52\,x^2 + 64 = 0 \iff 9(x^2 - 4)(x^2 - 16/9) = 0.$$

Thus $x = 2 \vee x = 4/3$, giving the two points $(2,2,1)$, $\left(\frac{4}{3},\frac{4}{3},\frac{7}{3}\right)$. $\boxed{2}$

Thus $(2,2,1)$ is the unique point of minimal height 1 on C, and $\left(\frac{4}{3},\frac{4}{3},\frac{7}{3}\right)$ is the unique point of maximal height $\frac{7}{3}$ on C. $\boxed{1}$

d) The vector $\gamma'(2) = \big(1, h'(2), k'(2)\big)$ gives the tangent direction to C in $(2,2,1)$. The tangent direction is orthogonal to $\nabla g_1(2,2,1) = (2,2,2)$ and $\nabla g_2(2,2,1) = (3,3,4)$ and hence equal to $\mathbb{R}(1,-1,0)$. Thus $h'(2) = -1$, $k'(2) = 0$. $\boxed{1}$

Remarks: It is possible to solve the question without Lagrange multipliers, noting that $(x+y+z)^2 = x^2+y^2+z^2+2(xy+xz+yz) = 9+2\cdot 8 = 25$ and hence C is the intersection of the sphere and the plane $x+y+z=5$. But then an independent argument must be given that the maximum and minimum must satisfy $x=y$. On can also use Lagrange multipliers with the constraints $x^2+y^2+z^2=9$ and $x+y+z=5$. If no Lagrange multipliers were used at all, 2 marks were subtracted (the marks for the set of 5 equations). The statement of the question clearly says that Lagrange multipliers must be used.

$$\sum_{3} = 12$$

4 a) Suppose $T(s,t,u) = T(s',t',u')$, i.e.,

$$(us\cos t, us\sin t, us+ut) = (u's'\cos t', u's'\sin t', u's'+u't').$$

Looking at the first two coordinates, which for fixed s,u, resp., s', u' parametrize a circle of radius $us > 0$, we obtain $t = t'$ since $t,t' \in (0,2\pi)$, and $us = u's'$ since $u,s,u',s' > 0$). Then, looking at the last coordinate we find $ut = u't' = u't$ and hence $u = u'$ (since $t > 0$). Thus $us = u's' = us'$, implying $s = s'$. $\boxed{2}$

Clearly T is continuously differentiable with

$$\mathbf{J}_T(s,t,u) = \begin{pmatrix} u\cos t & -us\sin t & s\cos t \\ u\sin t & us\cos t & s\sin t \\ u & u & s+t \end{pmatrix}. \qquad \boxed{1}$$

For the differentiability of T^{-1} it suffices to show that $\mathbf{J}_T(s,t,u)$ is invertible on U. $\boxed{1}$

We have

$$\begin{aligned}
\det \mathbf{J}_T(s,t,u) &= u^2 \begin{vmatrix} \cos t & -s\sin t & s\cos t \\ \sin t & s\cos t & s\sin t \\ 1 & 1 & s+t \end{vmatrix} \\
&= u^2 \begin{vmatrix} \cos t & -s\sin t & s\cos t \\ \sin t & s\cos t & s\sin t \\ 1 & 1 & s \end{vmatrix} + u^2 \begin{vmatrix} \cos t & -s\sin t & 0 \\ \sin t & s\cos t & 0 \\ 1 & 1 & t \end{vmatrix} \\
&= u^2 \begin{vmatrix} \cos t & -s\sin t & 0 \\ \sin t & s\cos t & 0 \\ 1 & 1 & t \end{vmatrix} = u^2 t \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} = u^2 st \neq 0
\end{aligned}$$

for $(s,t,u) \in U$. This implies that $\mathbf{J}_T(s,t,u)$ is invertible on U. $\boxed{2}$

b) Applying the change-of-variables formula to T (valid on account of a)) gives

$$\begin{aligned}
\mathrm{vol}_2(V) &= \int_{T(U)} 1\,\mathrm{d}^3\mathbf{y} = \int_U |\mathbf{J}_T(s,t,u)|\,\mathrm{d}^3(s,t,u) && \boxed{1}\\
&= \int_0^{2\pi}\int_0^t\int_0^1 u^2 st\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{3}\int_0^{2\pi}\int_0^t st\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{6}\int_0^{2\pi} t^3\,\mathrm{d}t\\
&= \frac{1}{6}\left[t^4/4\right]_0^{2\pi} = \frac{16}{24}\,\pi^4 = \frac{2}{3}\,\pi^4. && \boxed{2}
\end{aligned}$$

c) The (non-standard) helicoid is parametrized by $\gamma(s,t) = (s\cos t, s\sin t, s+t)$, $0 < s < t < 2\pi$.

$$\begin{aligned}
\mathbf{J}_\gamma(t) &= \begin{pmatrix} \cos t & -s\sin t\\ \sin t & s\cos t\\ 1 & 1\end{pmatrix},\\
\mathbf{J}_\gamma(t)^\mathsf{T}\mathbf{J}_\gamma(t) &= \begin{pmatrix} 2 & 1\\ 1 & 1+s^2\end{pmatrix},\\
g_\gamma(s,t) &= 2(1+s^2) - 1 = 1 + 2s^2,\\
\sqrt{g_\gamma(s,t)} &= \sqrt{1+2s^2}, && \boxed{2}\\
\mathrm{vol}_2(S) &= \int\limits_{\substack{(s,t)\in\mathbb{R}^2\\ 0<s<t<2\pi}} \sqrt{1+2s^2}\,\mathrm{d}^2(s,t)\\
&= \int_0^{2\pi}\int_0^t \sqrt{1+2s^2}\,\mathrm{d}s\,\mathrm{d}t\\
&= \int_0^{2\pi}\int_s^{2\pi} \sqrt{1+2s^2}\,\mathrm{d}t\,\mathrm{d}s && \text{(Fubini)}\\
&= \int_0^{2\pi} (2\pi - s)\sqrt{1+2s^2}\,\mathrm{d}s\,. && \boxed{2}
\end{aligned}$$

Remarks:

$$\sum_4 = 13$$

$$\sum_{\text{Final Exam}} = 40 + 9$$

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The function $f(x,y) = \dfrac{\sin(x)\cos(y)}{1+x^2+y^2}$, $(x,y) \in \mathbb{R}^2$ attains a global maximum and a global minimum.

b) The function $g(x,y) = xy(x^2+2y^2-3)$, $(x,y) \in \mathbb{R}^2$ has at least 9 critical points.

c) Suppose you start at the point $(1,0)$ and move 0.5 units in the (x,y)-plane following (and continuously adjusting) the direction of steepest ascent of $h(x,y) = \dfrac{xy}{x^2+y^2}$. Afterwards you are closer to the y-axis than before.

d) If C^2-functions $f(u,v)$ and $g(x,y)$ are related by $g(x,y) = f(ax+by, cx+dy)$ ($a,b,c,d \in \mathbb{R}$, $ad-bc \neq 0$) then $g_{xx}g_{yy} - g_{xy}^2 = (ad-bc)(f_{uu}f_{vv} - f_{uv}^2)$.

e) The line integral of $(\sin y + y\sin x)\,\mathrm{d}x + (x\cos y - \cos x)\,\mathrm{d}y$ along the quarter circle $x^2+y^2=1$, $x \geq 0$, $y \geq 0$ (in the mathematically positive direction) is zero.

f) If P,Q are C^2-functions on $\mathbb{R}^2$ such that both $P\,\mathrm{d}x + Q\,\mathrm{d}y$ and $Q\,\mathrm{d}x - P\,\mathrm{d}y$ are exact, then P and Q solve Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$.

Question 2 (ca. 9 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 - 3x^2y + x^2 + 2y^2 - y.$$

a) Which symmetry property does f have? What can you conclude from this about the graph of f and the contours of f?

b) Determine the gradient $\nabla f(x,y)$ and the Hesse matrix $\mathbf{H}_f(x,y)$.

c) Determine all critical points of f and their types.
Hint: There are exactly 3 critical points.

d) Does f have a global extremum?

Question 3 (ca. 9 marks)

Consider the surface S in $\mathbb{R}^3$ with equation

$$xz - y^2 + 2y + 2 = 0.$$

a) Show that S is smooth.

b) Using the method of Lagrange multipliers, determine the point(s) on S that minimize(s) the distance from the origin $(0,0,0)$, and the corresponding distance d. (It need not be proved that at least one such point exists.)

c) Determine an equation for the tangent plane to S in the point $(2,-2,3)$.

d) The surface S is a (central) quadric. Determine its center and type.

Question 4 (ca. 6 marks)

Evaluate the integral

$$\int_0^1 \frac{t^{1010}+t-2}{\ln t}\,\mathrm{d}t\,.$$

Hint: Consider $F(x)=\displaystyle\int_0^1 \frac{t^x+t-2}{\ln t}\,\mathrm{d}t$, $x\in(-1,\infty)$. Show first that F is well-defined and can be differentiated under the integral sign.

Question 5 (ca. 6 marks)

a) Let K be the solid in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$x\geq 0,\quad y\geq 0,\quad x+y\leq 2,\quad 0\leq z\leq 1+x^2+2y.$$

Find the volume of K.

b) Let P be the surface in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$z=x^2+y^2\leq 1.$$

Find the area of P.

Solutions

1 a) True. The function f is continuous, satisfies $\lim_{|(x,y)|\to\infty} f(x,y) = 0$ (since $|f(r\cos\phi, r\sin\phi)| \leq \frac{1}{1+r^2}$ in polar coordinates), and takes both positive and negative values. This implies the existence of a global maximum and a global minimum, as the following argument, e.g. for the minimum, shows.

We have $f(\pi/2,\pi) = \frac{-1}{1+5\pi^2/4} < 0$. Since $\lim_{|(x,y)|\to\infty} f(x,y) = 0$, there exists $R > 0$ such that $f(x,y) > \frac{-1}{1+\pi^2/4}$ for all points (x,y) with $|(x,y)| > R$. On the disk $\overline{\mathrm{B}_R(0,0)} = \{(x,y); x^2+y^2 \leq R^2\}$, which is compact (i.e., closed and bounded) and contains $(\pi/2,0)$, the continuous function f attains a minimum value, which must be $\leq \frac{-1}{1+\pi^2/4}$, and hence also $\leq f(x,y)$ for every point $(x,y) \in \mathbb{R}^2 \setminus \overline{\mathrm{B}_R(0,0)}$. $\boxed{2}$

b) True. The 0-contour of g is the union of the lines $x = 0$, $y = 0$, and the ellipse $x^2 + 2y^2 = 3$. The 5 points where two of these curves intersect, viz. $(0,0)$, $(\pm a, 0)$, $(0, \pm b)$ with $a = \sqrt{3}$, $b = \sqrt{3/2}$ (the semi-axes of the ellipse) must be critical, since the 0-contour isn't smooth there. Further, the coordinate axes divide the solid ellipse into 4 compact regions K_1, K_2, K_3, K_4, on which g (which is continuous) must attain both a maximum and a minimum. One of these is zero and attained on the boundary of K_i, but the other is nonzero and attained in the inerior of K_i. Since such points are critical, there are at least 4 further critical points. $\boxed{2}$

Remark: In fact there are exactly 9 critical points.

c) True. The direction of steepest ascent in (x,y) is that of the gradient $\nabla h(x,y)$. We have

$$\begin{aligned} h_x &= \frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}, \\ h_y &= \frac{x(x^2-y^2)}{(x^2+y^2)^2} && \text{(by symmetry)} \end{aligned}$$

Thus $\nabla h(1,0) = (0,1)$, implying that you move from $(1,0)$ north and enter the 1st quadrant. In the sector of the 1st quadrant specified by $0 < \theta < 45°$ in polar coordinates we have $h_x < 0$, $h_y > 0$, i.e., gradients point in the direction NW there. Thus you will approach the x-axis as long as you don't cross the line $y = x$. But this impossible, since the distance between $(1,0)$ and the line is $\sqrt{2}/2 > 0.5$. $\boxed{2}$

d) False. The correct relation is $g_{xx}g_{yy} - g_{xy}^2 = (ad-bc)^2(f_{uu}f_{vv} - f_{uv}^2)$. This can be proved as follows:

$$\begin{aligned} g_x &= f_u a + f_v c, \\ g_y &= f_u b + f_v c, \\ g_{xx} &= f_{ux}a + f_{vx}c = (f_{uu}a + f_{uv}c)a + (f_{vu}a + f_{vv}c)c = a^2 f_{uu} + 2ac\, f_{uv} + c^2 f_{vv}, \\ g_{xy} &= f_{uy}a + f_{vy}c = (f_{uu}b + f_{uv}d)a + (f_{vu}b + f_{vv}d)c = ab\, f_{uu} + (ad+bc)f_{uv} + cd\, f_{vv}, \\ g_{yx} &= g_{xy}, \\ g_{yy} &= f_{uy}b + f_{vy}d = (f_{uu}b + f_{uv}d)b + (f_{vu}b + f_{vv}d)d = b^2 f_{uu} + 2bd\, f_{uv} + d^2 f_{vv}, \end{aligned}$$

which just says

$$\begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Taking the determinant on both sides proves the claim. $\boxed{2}$

That the relation with the un-squared determinant $ad - bc$ can't be true, can also be seen as follows: Suppose f has a strict local minimum in $(0,0)$ and $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc = -1$. Then we would have $g_{xx}g_{yy} - g_{xy}^2 = -(f_{uu}f_{vv} - f_{uv}^2) < 0$ at $(x, y) = (0, 0)$, which corresponds to $(u, v) = (0, 0)$, and hence g would have a saddle point in $(0, 0)$. But (bijective) linear coordinate changes clearly preserve the type of critical points; contradiction. (If you want a concrete counterexample, take $f(u, v) = u^2 + v^2$, $g(x, y) = f(x, -y) = x^2 + y^2$, which satisfy $f_{uu}f_{vv} - f_{uv}^2 = g_{xx}g_{yy} - g_{xy}^2 = 4$ everywhere but $a = 1$, $d = -1$, $b = c = 0$, and hence $ad - bc = -1$.)

e) False. Since $\frac{\partial}{\partial y}(\sin y + y \sin x) = \cos y + \sin x = \frac{\partial}{\partial x}(x \cos y - \cos x)$, the given form ω is exact in $\mathbb{R}^2$ (which is simply connected). An anti-derivative f of ω is obtained by the usual method (or can just be guessed): $f_x = \sin y + y \sin x \implies f(x, y) = x \sin y - y \cos x + h(y) \implies f_y = x \cos y - \cos x + h'(y) = x \cos y - \cos x \iff h'(y) = 0$, i.e., we can take $f(x, y) = x \sin y - y \cos x$. Then the Fundamental Theorem for Line Integrals gives

$$\int_C \omega = f(0, 1) - f(1, 0) = 0 \cdot \sin 1 - 1 \cdot \cos 0 - (1 \cdot \sin 0 - 0 \cdot \cos 1) = -1 \neq 0. \qquad \boxed{2}$$

f) True. Exactness implies $P_y = Q_x$ and $Q_y = -P_x$. Hence $P_{yy} = Q_{xy} = Q_{yx} = (-P_x)_x = -P_{xx}$, using Clairaut's Theorem and the linearity of partial derivatives. Similarly, $Q_{xx} = P_{yx} = P_{xy} = (-Q_y)_y = -Q_{yy}$. $\boxed{2}$

$$\sum_1 = 12$$

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2 a) $f(-x, y) = f(x, y)$ for $(x, y) \in \mathbb{R}^2$ $\boxed{1}$

This implies $(x, y, z) \in \mathrm{G}_f \iff (-x, y, z) \in \mathrm{G}_f$, i.e., the graph of f is symmetric with respect to the (y, z)-plane in $\mathbb{R}^3$, and the contours of f are symmetric with respect to the y-axis in $\mathbb{R}^2$. $\boxed{1}$

b) We compute

$$\begin{aligned} f(x, y) &= x^4 - 3x^2 y + x^2 + 2y^2 - y, \\ \nabla f(x, y) &= (4x^3 - 6xy + 2x, -3x^2 + 4y - 1) \qquad \boxed{1} \\ &= \big(x(4x^2 - 6y + 2), -3x^2 + 4y - 1\big), \\ \mathbf{H}_f(x, y) &= \begin{pmatrix} 12x^2 - 6y + 2 & -6x \\ -6x & 4 \end{pmatrix}. \qquad \boxed{1} \end{aligned}$$

c) The system $f_x = f_y = 0$ is equivalent to

$$\big(x = 0 \wedge -3x^2 + 4y - 1 = 0\big) \vee \big(4x^2 - 6y + 2 = 0 \wedge -3x^2 + 4y - 1 = 0\big).$$

The first alternative has the solution $(x,y) = \left(0, \frac{1}{4}\right)$.
The second alternative is a linear system of equations for x^2, y, which can be solved, e.g., by Gaussian eliminiation:

$$\left[\begin{array}{rr|r} 4 & -6 & -2 \\ -3 & 4 & 1 \end{array}\right] \to \left[\begin{array}{rr|r} 4 & -6 & -2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

The solution is $y = 1$, $x^2 = (-2 + 6x_2)/4 = 1$, so that $(x,y) = (\pm 1, 1)$.
In all there are three critical points, viz.,

$$\mathbf{p}_1 = \left(0, \tfrac{1}{4}\right), \quad \mathbf{p}_2 = (1,1), \quad \mathbf{p}_3 = (-1,1). \qquad \boxed{1\tfrac{1}{2}}$$

Further we have

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 4 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_2) = \begin{pmatrix} 8 & -6 \\ -6 & 4 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 8 & 6 \\ 6 & 4 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$ is positive definite, $\mathbf{p}_1$ is a local minimum of f. $\boxed{\tfrac{1}{2}}$

Since $\det \mathbf{H}_f(\mathbf{p}_2) = \det \mathbf{H}_f(\mathbf{p}_3) = 8 \cdot 4 - (\pm 6)^2 = -4 < 0$, the points $\mathbf{p}_2$, $\mathbf{p}_3$ are saddle points of f. $\boxed{1}$

d) No. A global extremum must be a critical point. Since saddle points are not even local extrema, the only remaining possibility is that $\mathbf{p}_1$ is a global minimum. But

$$\begin{aligned} f(\mathbf{p}_1) &= 2\left(\tfrac{1}{4}\right)^2 - \tfrac{1}{4} = -\tfrac{1}{8}, \\ f(2,3) &= 2^4 - 3 \cdot 2^2 \cdot 3 + 2^2 + 2 \cdot 3^2 - 3 = -1 < -\tfrac{1}{8}, \end{aligned}$$

and hence $\mathbf{p}_1$ is not a global minimum. $\boxed{2}$

$$\sum_2 = 9$$

---

3 a) S is a level set of $g(x,y,z) = xz - y^2 + 2y$, which has $\nabla g(x,y,z) = (z, -2y+2, x)$. Evidently, the only point at which ∇g vanishes is $(0,1,0)$, but $(0,1,0)$ is not on S. Hence S is smooth. $\boxed{1}$

b) Let $f(x,y,z) = x^2+y^2+z^2$. The task is equivalent to solving the optimization problem "minimize $x^2 + y^2 + z^2$ subject to $g(x,y,z) = -2$".

Since S has no singular points, the theorem on Langrange multipliers is applicable everywhere and yields that every minimum must satisfy $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some $\lambda \in \mathbb{R}$. Since $\nabla f(x,y,z) = (2x, 2y, 2z)$ is a multiple of (x,y,z) (the factor 2 can be discarded), this gives the system of equations

$$\begin{aligned} x &= \lambda z, \\ y &= \lambda(-2y+2), \\ z &= \lambda x, \\ xz - y^2 + 2y &= -2. \end{aligned} \qquad \boxed{2}$$

The 1st and 3rd equation give $x = \lambda^2 x$, i.e., $x = 0 \vee \lambda = \pm 1$.

$\underline{x = 0}$: From the 3rd equation $z = 0$, and then from the 4th equation $y^2 - 2y - 2 = 0$, i.e., $y = 1 \pm \sqrt{3}$. This gives the two points $\bigl(0, 1 \pm \sqrt{3}, 0\bigr)$. $\boxed{1}$

$\underline{\lambda = 1}$: Here $x = z$, the 2nd equation gives $y = 2/3$ and the 4th equation $x^2 = xz = 4/9 - 4/3 - 2 < 0$. Thus there is no solution in this case.

$\underline{\lambda = -1}$: Here $x = -z$, the 2nd equation gives $y = 2$, and the 4th equation $x^2 = -xz = -2^2 + 2 \cdot 2 + 2 = 2$. This gives the two points $\bigl(\pm\sqrt{2}, 2, \mp\sqrt{2}\bigr)$. $\boxed{1}$

The distance of these points from the origin is $2\sqrt{2}$. Since obviously $\sqrt{3} - 1 < 2\sqrt{2}$, the unique point on S minimizing the distance from the origin is $\bigl(0, 1 - \sqrt{3}, 0\bigr)$, and $d = \sqrt{3} - 1$. $\boxed{1}$

c) The tangent plane to S in $(2, -2, 3)$ has equation $\nabla g(2, -2, 3) \cdot (x - 2, y + 2, z - 3) = 0$. Since $\nabla g(2, -2, 3) = (3, 6, 2)$, this gives $3(x - 2) + 6(y + 2) + 2(z - 3) = 0$, i.e., $3x + 6y + 2z = 0$. (This is also clear from the requirements that $\nabla g(2, -2, 3)$ must be a normal vector of the plane and the point $(2, -2, 3)$ must be on the plane.) $\boxed{1}$

d) Rewriting the equation for S as $xz - (y - 1)^2 + 3 = 0$ shows that the center is $(0, 1, 0)$. $\boxed{\frac{1}{2}}$

S is thus affinely equivalent to the quadric $xz - y^2 + 3 = 0$. The further coordinate change $x = x' + z'$, $y = y'$, $z = x' - z'$ turns the latter into $x'^2 - z'^2 - y'^2 + 3 = 0$ or, after dropping primes and normalizing such that the right-hand side is positive, $-x^2 + y^2 + z^2 = 3$, which reveals that S is a hyperboloid of one sheet. $\boxed{1\frac{1}{2}}$

$$\sum_3 = 9$$

4 The integrand

$$f(x, t) = \frac{t^x + t - 2}{\ln t}$$

is not defined for $t \in \{0, 1\}$. The singularity at $t = 1$ can be removed, since the numerator is zero for $t = 1$ and, e.g., l'Hospital's Rule can be applied: $\lim_{t \to 1} f(x, t) = \lim_{t \to 1} \frac{x t^{x-1} + 1}{1/t} = x + 1$. The singularity at $t = 0$ cannot be removed. However, since $\int_0^1 t^x \,\mathrm{d}x = \frac{1}{x+1}$ exists for $x > -1$, it is clear that $\int_0^1 f(x, t) \,\mathrm{d}t$ exists for $x > -1$ as well. $\boxed{1}$

Differentiating under the integral sign gives

$$F'(x) = \int_0^1 f_x(x, t) \,\mathrm{d}t = \int_0^1 \frac{(\ln t) t^x}{\ln t} \,\mathrm{d}t = \left[\frac{t^{x+1}}{x + 1}\right]_0^1 = \frac{1}{x + 1}. \qquad \boxed{1}$$

It can be justified as follows: For small $\delta > 0$ and $0 < t < 1$ we have

$$t^x = \mathrm{e}^{x \ln t} \leq \begin{cases} 1 & \text{if } x \geq 0, \\ t^{-1+\delta} & \text{if } -1 + \delta < x < 0. \end{cases}$$

Thus $t^x = |t^x| \leq t^{-1+\delta} =: \Phi(t)$ for all $x \in (-1 + \delta, \infty)$ and $t \in (0, 1)$, which provides an integrable bound independent of x on account of $\int_0^1 t^{-1+\delta} = \bigl[t^\delta/\delta\bigr]_0^1 = 1/\delta$. $\boxed{2}$

It follows that $F(x) = \ln(x+1) + C$ for some $C \in \mathbb{R}$. The constant C can be determined from

$$F(0) = \int_0^1 \frac{t-1}{\ln t}\,\mathrm{d}t,$$
$$F(1) = \int_0^1 \frac{2t-2}{\ln t}\,\mathrm{d}t = 2\,F(0),$$

which gives $\ln(2) + C = 2C$, i.e., $C = \ln 2$, and $F(x) = \ln(x+1) + \ln 2 = \ln(2x+2)$.

$$\implies \int_0^1 \frac{t^{1010}+t-2}{\ln t}\,\mathrm{d}t = F(1010) = \ln(2022). \qquad \boxed{2}$$

$$\sum_4 = 6$$

5 a) The volume of K is

$$V = \int_K 1\,\mathrm{d}^3(x,y,z) = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{1+x^2+2y} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \qquad \boxed{1}$$
$$= \int_0^2\int_0^{2-x} 1+x^2+2y\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^2 \big[(1+x^2)y+y^2\big]_{y=0}^{2-x}\,\mathrm{d}x$$
$$= \int_0^2 (1+x^2)(2-x)+(2-x)^2\,\mathrm{d}x$$
$$= \int_0^2 6-5x+3x^2-x^3\,\mathrm{d}x$$
$$= \big[6x-\tfrac{5}{2}\,x^2+x^3-\tfrac{1}{4}\,x^4\big]_0^2 = 12-10+8-4 = 6. \qquad \boxed{2}$$

b) Denoting the unit disk in $\mathbb{R}^2$ by D, the surface P is the graph of $f(x,y) = x^2+y^2$, $(x,y) \in D$. Using the formula for such surfaces, or going the long way using the parametrization $\gamma(x,y) = \big(x,y,f(x,y)\big)$, we obtain the surface area as

$$A = \int_D \sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y) = \int_D \sqrt{1+|(2x,2y)|^2}\,\mathrm{d}^2(x,y)$$
$$= \int_D \sqrt{1+4x^2+4y^2}\,\mathrm{d}^2(x,y) \qquad \boxed{1}$$
$$= \int_{\substack{0\leq r\leq 1\\ 0\leq\phi\leq 2\pi}} r\sqrt{1+4r^2}\,\mathrm{d}^2(r,\phi) = 2\pi\int_0^1 r\sqrt{1+4r^2}\,\mathrm{d}r \qquad \boxed{1}$$
$$= \frac{\pi}{6}\big[(1+4r^2)^{3/2}\big]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right). \qquad \boxed{1}$$

$$\sum_5 = 6$$

$$\sum_{\text{Final Exam}} = 42$$

Question 1 (ca. 10 marks)

Decide whether the following statements are true or false, and justify your answers.

a) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable in $(0,0)$ and $f(x,x) = f(x,x^2) = 2021$ for all $x \in \mathbb{R}$. Then we must have $\nabla f(0,0) = (0,0)$.

b) Under the assumptions made in a), the function f satisfies $f(x,y) = 2021$ for all points (x,y) in some neighborhood of $(0,0)$.

c) If a C^1-function $f\colon \mathbb{R}^2 \to \mathbb{R}$ has a minimum in (x_0,y_0) subject to the constraint $y^2 = x^3$, there exists $\lambda \in \mathbb{R}$ such that $f_x(x_0,y_0) = 3\lambda\, x_0^2$, $f_y(x_0,y_0) = -2\lambda\, y_0$.

d) Suppose you start at the point $\mathbf{p} = (1,2)$ in the (x,y)-plane and follow the contour of $f(x,y) = xy^2 - x^2y$ through $\mathbf{p}$ in one of the two possible directions. After a while you are closer to the x-axis than before.

e) The line integral of $(x-2y)\,\mathrm{d}y + (y-2x)\,\mathrm{d}x$ along the quarter circle $x^2+y^2 = 43\cdot 47$, $x \geq 0$, $y \geq 0$ (in the mathematically positive direction) is zero.

f) The differential 1-form $\dfrac{x-2y}{x^2+y^2}\,\mathrm{d}x + \dfrac{\boldsymbol{y+2x}}{x^2+y^2}\,\mathrm{d}y$ on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is exact.

Question 2 (ca. 8 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^3y - xy^3 - x^2y + y^3.$$

a) Which symmetry property does f have? What can you conclude from this about the graph of f and the contours of f?

b) Determine all critical points of f and their types.

Hint: There are 6 critical points.

c) Does f have a global extremum?

Question 3 (ca. 8 marks)

Consider the cubic surface S in $\mathbb{R}^3$ with equation

$$x^3 + y^3 + z^3 + 3xyz = 6.$$

a) Show that S is smooth.

b) Show that there is a point on S that minimizes the distance from the origin $(0,0,0)$.

c) Determine all points on S with the property in b) and their distance from $(0,0,0)$.

d) The plane $3x - 2y - z = 0$ intersects S in a curve C. Determine the tangent line to C in $(1,1,1)$ in parametric form.

Question 4 (ca. 6 marks)

Consider the function $F\colon [0,2\pi] \to \mathbb{R}$ defined by

$$F(x) = \int_0^1 \frac{\ln(t^2 - 2t\cos x + 1)}{t}\,\mathrm{d}t\,.$$

a) Show that F is differentiable and $F'(x)$ can be obtained by differentation under the integral sign.
b) Using a), show $F'(x) = \pi - x$ for $x \in [0, 2\pi]$.
c) Determine the function F from b) and $F(0)$.

Hint: In b) the identities $\sin x = \cos(\pi/2 - x)$, $1 - \cos x = 2\sin^2(x/2)$, $\sin x = 2\sin(x/2)\cos(x/2)$ may be helpful, and in c) the logarithm series $\ln(1-t) = = -t - t^2/2 - t^3/3 - t^4/4 - \cdots$.

Question 5 (ca. 8 marks)

Consider the set K of all points $(x, y, z) \in \mathbb{R}^3$ satisfying the inequalities $x \geq 0$ and $y^2 + z^2 \leq x - x^3$.

a) Show that K is compact (i.e., closed and bounded), and sketch K graphically.
b) Determine the volume $\mathrm{vol}(K)$.
c) The surface area $\mathrm{vol}_2(\partial K)$ of the boundary ∂K can be expressed as $\pi \int_0^1 \sqrt{P(x)}\,\mathrm{d}x$ with a polynomial $P(x)$ of degree 4. Find such a polynomial.
d) Bound $\mathrm{vol}_2(\partial K)$ numerically from above and from below (the sharper the bounds, the better).
e) Determine the centroid ("center of mass") of K.

Solutions

1 a) True. Using the chain rule, the functions $g_1(t) = f(t,t)$ and $g_2(t) = f(t,t^2)$ are differentiable in $t = 0$ with derivatives

$$\begin{aligned} g_1'(0) &= \nabla f(0,0) \cdot (1,1) = f_x(0,0) + f_y(0,0), \\ g_2'(0) &= \nabla f(0,0) \cdot (1,0) = f_x(0,0). \end{aligned}$$

Since $g_1(t) = g_2(t) = 2021$ are constant, we must have $g_1'(0) = g_2'(0) = 0$, and hence $f_x(0,0) = f_y(0,0) = 0$, i.e., $\nabla f(0,0) = (0,0)$. $\boxed{2}$

b) False. The functions $f(x,y) = 2021 + (y-x)(y-x^2)$, which satisfies $f(x,y) \neq 2021$ for all (x,y) that are neither on the line $y = x$ nor on the curve $y = x^2$ (clearly every neighborhood of $(0,0)$ contains such points), provides a counterexample. $\boxed{2}$

c) False. The assertion holds only for points on the curve $y^2 = x^3$ where the gradient of $g(x,y) = x^3 - y^2$, viz. $\nabla g = (3x^2, -2y)$, is nonzero. At $(0,0)$ the gradient of g vanishes, and indeed the function $f(x,y) = x$ provides a counterexample: Since $y^2 = x^3$ implies $x \geq 0$, $f(0,0) = 0$ is the unique minimum of f subject to $y^2 = x^3$, but $\nabla f(0,0) = (1,0)$ is not a scalar multiple of $\nabla g(0,0) = 0$. $\boxed{2}$

d) False. We have $\nabla f(x,y) = (y^2 - 2xy, 2xy - x^2) = \big(y(y-2x), x(2y-x)\big)$. Hence $\nabla f(1,2) = (0,3)$ and the contour of f through $(1,2)$, which is the 2-contour, has a horizontal tangent in $(1,2)$. Since gradients on the left/right of $y = 2x$ point northeast/northwest, respectively, the shape of the contour must be convex. Thus, no matter which direction you choose, you will end up further away from the x-axis. (It is not possible for the contour to escape from the sectors determined by $y > 2x$ and $x < y < 2x$ in the first quadrant, since the lines $x = 0$ and $y = x$ are itself contours (part of the 0-contour, which also contains the line $y = 0$) and since the direction of the move is away from the other sector boundary. $\boxed{2}$

Alternatively, one can show that $xy^2 - x^2y = xy(y-x) = 2$ and $y > 0$ imply $y \geq 2$ with equality iff $(x,y) = (1,2)$. (Clearly $y > x > 0$ under this assumption; hence $2/y = x(y-x) \leq y^2/4$, i.e., $y^3 \geq 8$ and $y \geq 2$; equality holds iff $x = y/2 = 1$.) Since the move can't leave the upper half plane, this also proves the claim.

e) True. Since $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ with $P(x,y) = y - 2x$, $Q(x,y) = x - 2y$ and $P_y = 1 = Q_x$, ω is exact in $\mathbb{R}^2$ (which is simply connected). A function $f(x,y)$ satisfying $\omega = \mathrm{d}f$ is readily computed by the usual method: $f_y = Q = x - 2y \Longrightarrow f = xy - y^2 + h(x) \Longrightarrow f_x = y + h'(x) = P = y - 2x$ iff $h'(x) = -2x \Longrightarrow h(x) = -x^2 + C \Longrightarrow f(x,y) = xy - x^2 - y^2 + C$. For simplicity we take $C = 0$.

With $a = \sqrt{43 \cdot 47} = \sqrt{2021}$, the starting and end point of the quarter circle Q, using the indicated orientation, are $(a,0)$ and $(0,a)$, respectively. The Fundamental Theorem for Line Integrals gives

$$\int_Q \omega = f(0,a) - f(a,0) = -a^2 - (-a^2) = 0. \qquad \boxed{2}$$

There are several other alternative solutions. One can compute $\int_Q \omega$ directly from the definition of line integrals, using the parametrization $\gamma(t) = (a\cos t, a\sin t)$, $t \in$

$[0, \pi/2]$, which leads to

$$\begin{aligned}\int_Q \omega &= \int_Q (y-2x)\,\mathrm{d}x + (x-2y)\,\mathrm{d}y\\ &= \int_0^{\pi/2} (a\sin t - 2a\cos t)(-a\sin t) + (a\cos t - 2a\sin t)a\cos t\,\mathrm{d}t\\ &= a^2\int_0^{\pi/2} \cos^2 t - \sin^2 t\,\mathrm{d}t\\ &= a^2\int_0^{\pi/2}\cos(2t)\,\mathrm{d}t = a^2\left[\tfrac12\sin(2t)\right]_0^{\pi/2} = a^2\left(\tfrac12\sin\pi - \tfrac12\sin 0\right) = 0.\end{aligned}$$

Or one can use, after observing that ω is exact and hence $\int_\gamma \omega$ is independent of path, another path from $(a,0)$ to $(0,a)$ to compute $\int_Q \omega$. The best choice seems to be the path from $(a,0)$ via $(0,0)$ to $(0,a)$ along the coordinate axes, using $\gamma_1(t) = (a-t, 0)$, $\gamma_2(t) = (0,t)$, $t \in [0,a]$, which gives

$$\begin{aligned}\int_Q \omega &= \int_0^a \bigl(0 - 2(a-t)\bigr)(-1)\,\mathrm{d}t + \int_0^a (0-2t)1\,\mathrm{d}t\\ &= \int_0^a 2a - 4t\,\mathrm{d}t = 2a^2 - 4\cdot a^2/2 = 0.\end{aligned}$$

f) [In the printed examination paper the numerator of $Q(x,y)$ (the coefficient function of $\mathrm{d}y$) was incorrectly stated as $y-2x$. That the sign should be taken as $+$ was announced at the beginning of the examination, and earlier on BB without surrounding context.]

False. We have

$$\omega := \frac{x-2y}{x^2+y^2}\,\mathrm{d}x + \frac{y+2x}{x^2+y^2}\,\mathrm{d}y = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2+y^2} + 2\,\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} = \omega_1 + 2\,\omega_2, \quad \text{say.}$$

The form ω_1 is exact with anti-derivative $f(x,y) = \frac12\ln(x^2+y^2)$ and the form ω_2, which is the winding form, is not exact. Since linear combinations of exact forms are exact, ω cannot be exact (otherwise $\omega_2 = \frac12\,\omega - \frac12\,\omega_1$ would be exact). $\boxed{2}$

Alternatively, for the closed curve $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ we have

$$\int_\gamma \omega = \int_0^{2\pi} \frac{\cos t - 2\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\sin t + 2\cos t}{\cos^2 t + \sin^2 t}\cos t\,\mathrm{d}t = \int_0^{2\pi} 2\,\mathrm{d}t = 4\pi \neq 0,$$

and hence ω cannot be exact.

Remarks: Most students had great difficulties with this more theoretical type of questions, and hardly any student received more than 50 % of the marks for Question 1.

a) One can also argue with the directional derivatives of $f_{\mathbf{u}}(0,0)$ in the two tangent directions $(1,1)$ and $(1,0)$ of the 2021-contour of f and observe that $\nabla f(0,0)$ must be orthogonal to both. Although this is not directly covered by the theorem on contours and the gradient—in the proof we had assumed that the contour in the respective point is smooth, it was honored by 2 marks if the tangent directions were explicitly stated.

b) Most students who answered "False" only observed that the values of f at points not on the two curves $y = x$ and $y = x^2$ are undetermined. This was penalized by 0.5 marks, because it is not clear that there actually exists a counterexample. For example, setting $f(x,x) = f(x,x^2) = 2021$ for $x \in \mathbb{R}$ and $f(x,0) = 0$ for $x \in \mathbb{R} \setminus \{0\}$ ensures that f is non-constant in any neighborhood of $(0,0)$, but such a function can't be differentiable at $(0,0)$.

c) As in b), 0.5 marks were subtracted for answers that only stated that the Lagrange multiplier rule doesn't apply at $(0,0)$.

d) Here many students correctly obtained $\nabla f(1,2) = (0,3)$ (honored by 1 mark), but then falsely claimed that the move is northward (following the gradient) or horizontal (following the tangent of the contour).

Some students answered the question by solving the quadratic $y\,x^2 - y^2x + 2 = 0$ with parameter $y \neq 0$, which has discriminant $y^4 - 8y = y(y^3 - 8)$ and hence is solvable iff $y < 0 \vee y \geq 2$. Clearly this implies that the answer is "False".

e) Most students who answered this question correctly used one of the first two suggested solutions. Often it was overlooked that the differential form is written in the unusual order $\omega = Q\,\mathrm{d}y + P\,\mathrm{d}x$ (similar to writing the winding form as $x\,\mathrm{d}y - y\,\mathrm{d}x$), and as a consequence $\frac{1}{2}\,x^2 + \frac{1}{2}\,y^2 - 2xy$ obtained as anti-derivative. This was penalized only by 0.5 marks, because it doesn't change the final result.

Several students applied the Fundamental Theorem for Line Integrals directly to the quarter circle or used Green's Theorem to conclude that the line integral is zero, which is wrong because the the quarter circle is not closed.

f) Here most students observed that $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is not simply-connected and then falsely concluded that the given differential form cannot be exact; cp. the differential form ω_1 in the suggested solution. In this case only 0.5 marks were assigned.

$$\sum_1 = 12$$

2 a) $f(x,-y) = -f(x,y)$ for $(x,y) \in \mathbb{R}^2$ $\boxed{1}$

This says that the graph of f is symmetric with respect to the x-axis in $\mathbb{R}^3$, and that reflection at the x-axis in $\mathbb{R}^2$ maps the k-contour of f to the $-k$-contour ($k \in \mathbb{R}$). $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f &= x^3y - xy^3 - x^2y + y^3, \\ f_x &= 3x^2y - y^3 - 2xy = y(3x^2 - y^2 - 2x), \\ f_y &= x^3 - 3xy^2 - x^2 + 3y^2. \end{aligned}$$

The factorization of f_x shows that $\nabla f(x,y) = (0,0)$ is equivalent to

$$\bigl(y = 0 \wedge x^3 - x^2 = 0\bigr) \vee \bigl(y^2 = 3x^2 - 2x \wedge x^3 - 3x(3x^2 - 2x) - x^2 + 3(3x^2 - 2x) = 0\bigr).$$

The first alternative has the solutions $(x, y) = (0, 0)$, $(1, 0)$. The second equation in the second alternative simplifies to $-8x^3 + 14x^2 - 6x = -8x(x-1)(x-3/4) = 0$. Hence the second alternative has the solutions $(x, y) = (0, 0)$, $(1, \pm 1)$, $(3/4, \pm\sqrt{3}/4)$. Thus there are exactly six critical points, viz.,

$$\mathbf{p}_1 = (0,0),\ \ \mathbf{p}_2 = (1,0),\ \ \mathbf{p}_3 = (1,1),\ \ \mathbf{p}_4 = (1,-1),\ \ \mathbf{p}_5 = \left(\tfrac{3}{4}, \tfrac{\sqrt{3}}{4}\right),\ \ \mathbf{p}_6 = \left(\tfrac{3}{4}, -\tfrac{\sqrt{3}}{4}\right).$$

$\boxed{3}$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6xy - 2y & 3x^2 - 3y^2 - 2x \\ 3x^2 - 3y^2 - 2x & -6xy + 6y \end{pmatrix}$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{H}_f(\mathbf{p}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 4 & -2 \\ -2 & 0 \end{pmatrix}, \qquad \mathbf{H}_f(\mathbf{p}_4) = \begin{pmatrix} -4 & -2 \\ -2 & 0 \end{pmatrix},$$
$$\mathbf{H}_f(\mathbf{p}_5) = \begin{pmatrix} \frac{5}{8}\sqrt{3} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8}\sqrt{3} \end{pmatrix}, \qquad \mathbf{H}_f(\mathbf{p}_6) = \begin{pmatrix} -\frac{5}{8}\sqrt{3} & -\frac{3}{8} \\ -\frac{3}{8} & -\frac{3}{8}\sqrt{3} \end{pmatrix}.$$

Since $f(0, y) = y^3$, there is no extremum in $\mathbf{p}_1$. $\boxed{\tfrac{1}{2}}$

Since $\mathbf{H}_f(\mathbf{p}_i)$, $i = 2, 3, 4$, has negative determinant, the points $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ are saddle points. $\boxed{1\tfrac{1}{2}}$

Since $\det \mathbf{H}_f(\mathbf{p}_5) = \det \mathbf{H}_f(\mathbf{p}_6) = \frac{9}{16} > 0$, the matrix $\mathbf{H}_f(\mathbf{p}_5)$ is positive definite, and $\mathbf{H}_f(\mathbf{p}_6)$ is negative definite.
$\Longrightarrow$ $\mathbf{p}_5$ is a strict local minimum (with value $f(\mathbf{p}_5) = -\frac{3\sqrt{3}}{128}$), and $\mathbf{p}_6$ is a strict local maximum (with value $f(\mathbf{p}_6) = \frac{3\sqrt{3}}{128}$). $\boxed{1}$

c) No.

Since $f(0, y) = y^3$, the function f attains arbitrarily small and arbitrarily large values, and hence has no global extremum. $\boxed{1}$

Remarks: Most students obtained 8 marks for this question, failing only to answer the question about the contours in a); the contours are definitely not symmetric with respect to the x-axis!

Regarding the point $(0, 0)$, it wasn't required to show that there is no extremum at this point. Computing $\mathbf{H}_f(0, 0)$ or stating that $(0, 0)$ is a saddle point (which it is according to some definitions) was considered as sufficient. The explicit values $f(\mathbf{p}_5)$, $f(\mathbf{p}_6)$ weren't required for a full score, but could be used in the answer of c) by comparing them with other values of f.

The function f can be factorized as $f(x, y) = y(x - y)(x + y)(x - 1)$. Since the 0-contour is the union of the four lines $y = 0$, $y = x$, $y = -x$, and $x = 1$, which intersect in $(0, 0)$, $(1, 0)$, $(1, 1)$, $(1, -1)$, these four points must be critical (the 0-contour is not smooth there). Further, there must be critical points in the interiors of the two triangles with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, respectively, $(0, 0)$, $(1, 0)$, $(1, -1)$, since f attains both a maximum and a minimum on the closed triangles, of which only one can be (and in fact is) at the boundary. This observation can be refined to obtain all required results except for the exact location of the two local extrema.

$$\sum_{2} = 9$$

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3 a) S is a level set of $g(x,y,z) = x^3 + y^3 + z^3 + 3xyz$, which has $\nabla g(x,y,z) = 3(x^2 + yz, y^2 + xz, z^2 + xy)$. At a point (x,y,z) with $g_x = g_y = g_z = 0$ we have $x^2 = -yz$, $y^2 = -xz$, $z^2 = -xy$ and (multiplying these equations) $x^2y^2z^2 = -x^2y^2z^2$. Hence one of x, y, z must be zero, which by symmetry can be assumed to be x. It follows that $g_y = y^2$, $g_z = z^2$, and hence y, z are zero as well. Since $(0,0,0) \notin S$, the surface S has no singular point and is smooth. $\boxed{2}$

b) Consider a point on S, e.g., $P = (1,1,1)$. The length function $(x,y,z) \mapsto |(x,y,z)|$ is continuous and attains a minimum $|(x_0,y_0,z_0)| = d_0$ on the set

$$K = \{(x,y,z) \in S; x^2 + y^2 + z^2 \leq 3\},$$

which is closed, bounded, and non-empty (since $P \in K$). Since $d_0 \leq \sqrt{3}$ and all points in $S \setminus K$ have length $> \sqrt{3}$, the point $P_0 = (x_0,y_0,z_0)$ has the required property. $\boxed{1}$

c) Let $f(x,y,z) = x^2+y^2+z^2$. The task is equivalent to solving the optimization problem "minimize $x^2 + y^2 + z^2$ subject to $g(x,y,z) = 6$".

Since S has no singular points, the theorem on Langrange multipliers is applicable everywhere and yields that every minimum must satisfy $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some $\lambda \in \mathbb{R}$. Since $\nabla f(x,y,z) = (2x, 2y, 2z)$ is a multiple of (x,y,z) and $\nabla g(x,y,z)$ a multiple of $(x^2 + yz, y^2 + xz, z^2 + xy)$ (the factors 2 and 3 can be discarded), this gives the system of equations

$$\begin{aligned} x &= \lambda(x^2 + yz), \\ y &= \lambda(y^2 + zx), \\ z &= \lambda(z^2 + xy), \\ x^3 + y^3 + z^3 + 3xyz &= 6. \end{aligned} \qquad \boxed{2}$$

Eliminating λ from the first two equations gives $x(y^2 + zx) = y(x^2 + yz)$, which can be rewritten as $y^2(x-z) = x^2(y-z)$. Using the cyclic symmetry $x \mapsto y \mapsto z \mapsto x$ of the system then gives the three equations

$$\begin{aligned} y^2(x-z) &= x^2(y-z), \\ z^2(y-x) &= y^2(z-x), \\ x^2(z-y) &= z^2(x-y). \end{aligned}$$

Multiplying the three equations gives

$$x^2y^2z^2(x-z)(y-x)(z-y) = x^2y^2z^2(y-z)(z-x)(x-y) = -x^2y^2z^2(x-z)(y-x)(z-y).$$

Thus one of x, y, z, $x-z$, $y-x$, $z-y$ must be zero. $\boxed{1}$

If $x = 0$, say, the first equation of the original system becomes $0 = \lambda yz$. Since clearly $\lambda \neq 0$, one of y, z must be zero as well. By symmetry, this accounts for the points $\bigl(\sqrt[3]{6}, 0, 0\bigr)$, $\bigl(0, \sqrt[3]{6}, 0\bigr)$, $\bigl(0, 0, \sqrt[3]{6}\bigr)$. $\boxed{1}$

If $x = y$, say, we can assume that this quantity is $\neq 0$ (otherwise we are in the previous case). Then, e.g., the equation $z^2(y - x) = y^2(z - x)$ gives $z = x$, i.e., $x = y = z$. This accounts for the point $(1, 1, 1)$. $\boxed{1}$

Since $\sqrt{3} < \sqrt[3]{6}$ (to see this, raise the numbers to their 6th powers, which are $\sqrt{3}^6 = 3^3 = 27$, $\sqrt[3]{6}^6 = 6^2 = 36$), the unique point on S closest to $(0, 0, 0)$ is $(1, 1, 1)$, and its distance from $(0, 0, 0)$ is $\sqrt{3}$. $\boxed{1}$

d) The tangent line L to C in $P = (1, 1, 1)$ is the intersection of the tangent plane to S in P, viz. $x + y + z = 3$ (since it has normal vector $\nabla g(1, 1, 1)$, which is a multiple of $(1, 1, 1)$) and the plane $3x - 2y - z = 0$. A nonzero vector orthogonal to $(1, 1, 1)$ and $(3, -2, -1)$ is $(1, 4, -5)$, so that $L = (1, 1, 1) + \mathbb{R}(1, 4, -5)$. $\boxed{2}$

Remarks: In a) several students stated without proof that $x^2 + yz = y^2 + xz = z^2 + xy = 0$ implies $x = y = z = 0$. This is not obvious and was penalized by 1 mark.

In b) a careful argument was required to obtain 1 mark. I have subtracted 0.5 marks if continuity of the distance function was missing, and not assigned any marks if the set K was defined as a ball (instead of the intersection of S with a ball), or wasn't closed (using strict inequality in the definition), or wasn't bounded (e.g., setting $K = S$).

In c) several students concluded falsely from the system of equations for x, y, z, λ that $x = y = z$ and hence the solution must be $(1, 1, 1)$, $d = \sqrt{3}$. This was penalized by 3 marks. More precisely, the system of equations was worth 2 marks, any meaningful computation towards the solution (such as eliminating λ) 1 further mark, the 4 candidate points 2 further marks, and the decision which distance is the smallest 1 further mark. For the latter the inequality $\sqrt{3} < \sqrt[3]{6}$ was required (serving also as proof that the distance from $\bigl(\sqrt[3]{6}, 0, 0\bigr)$ to $(0, 0, 0)$ is $\sqrt[3]{6} = 6^{1/3}$, and not $6^{2/3}$ as a few students claimed).

In d) several students eliminated z from the two given equations, but this yields only the projection of C onto the x-y plane and cannot be used to compute tangent lines.

$$\sum_3 = 11$$

4 The assertion in a) is slightly incorrect: The function F is differentiable in $[0, 2\pi]$, but differentiation under the integral sign yields $F'(x)$ only for $x \in (0, 2\pi)$. At the end points we have $F'(0) = \pi$, $F'(2\pi) = -\pi$ (cf. the reasoning in b) below and also the subsequent plot), but $\int_0^1 f_x(x, t)\,\mathrm{d}t = \int_0^1 \frac{0}{t^2 - 2t + 1}\,\mathrm{d}t = 0$.

a) The integrand

$$f(x, t) = \frac{\ln(t^2 - 2t\cos x + 1)}{t} = \frac{\ln\bigl[(t - \cos x)^2 + \sin^2 x\bigr]}{t}$$

is defined for $(x, t) \in [0, 2\pi] \times (0, 1)$, since the ony way to make $(t - \cos x)^2 + \sin^2 x$ equal to zero is $t - \cos x = \sin x = 0$, which has no solution in $(x, t) \in [0, 2\pi] \times (0, 1)$. By the same token, f is continuously differentiable on $(x, t) \in [0, 2\pi] \times (0, 1)$. (This wouldn't be the case if we include the boundaries of the rectangle.) $\boxed{1}$

Thus we can apply the theorem on the differentiation of parameter integrals with $X \times T := (0, 2\pi) \times [0, 1]$ (the special case with compact domain of integration T and a continuous 2-variable function $f_x(x, t)$). $\boxed{1}$

It follows that F is differentiable on $(0, 2\pi)$ with

$$F'(x) = \int_0^1 f_x(x,t)\,\mathrm{d}t = \int_0^1 \frac{2\sin x}{t^2 - 2t\cos x + 1}\,\mathrm{d}t = 2\sin x \int_0^1 \frac{\mathrm{d}t}{t^2 - 2t\cos x + 1}. \quad \boxed{1}$$

b) For $0 < x < 2\pi$, $x \neq \pi$ we have

$$\begin{aligned} F'(x) &= 2\sin x \int_0^1 \frac{\mathrm{d}t}{(t-\cos x)^2 + \sin^2 x} = \frac{2}{\sin x}\int_0^1 \frac{\mathrm{d}t}{\left(\frac{t-\cos x}{\sin x}\right)^2 + 1} \\ &= 2\int_{-\frac{\cos x}{\sin x}}^{\frac{1-\cos x}{\sin x}} \frac{\mathrm{d}s}{s^2+1} \qquad \text{Subst. } s = \tfrac{t-\cos x}{\sin x},\ \mathrm{d}s = \tfrac{\mathrm{d}t}{\sin x} \\ &= 2\left[\arctan s\right]_{-\frac{\cos x}{\sin x}}^{\frac{1-\cos x}{\sin x}} = 2\left(\arctan\frac{1-\cos x}{\sin x} + \arctan\frac{\cos x}{\sin x}\right). \quad \boxed{2} \end{aligned}$$

Using the various hints,

$$\begin{aligned} \frac{1-\cos x}{\sin x} &= \frac{1-\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}, \\ \frac{\cos x}{\sin x} &= \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} = \tan\left(\frac{\pi}{2}-x\right), \end{aligned}$$

and hence

$$F'(x) = 2\left(\frac{x}{2} + \frac{\pi}{2} - x\right) = \pi - x. \quad \boxed{1}$$

Since F' is continuous on $[0, 2\pi]$, this also holds for $x = 0, \pi, 2\pi$, but there is quite a subtlety involved in the proof: For $x = \pi$ we can use continuity of F' on $(0, 2\pi)$, which the said theorem about parameter integrals implies, but for $x = 0, 2\pi$ one needs to show the existence of $F'(x)$ in the first place. One can appeal to the (nontrivial) fact that the existence of $\lim_{x\downarrow 0} F'(x)$ (known from the representation $F'(x) = \pi - x$ for $0 < x < 2\pi$) together with the continuity of F at 0 imply $F'(0) = \lim_{x\downarrow 0} F'(x) = \lim_{x\downarrow 0}(\pi - x) = \pi$, and similarly for $F'(2\pi)$, but the continuity of F at 0, 2π still needs to be shown. For the latter one can apply the theorem about the continuity of parameter integrals: The inequalities $1 \geq t^2 - 2t\cos x + 1 \geq t^2 - 2t + 1$ $(0 \leq t \leq 2\cos x)$, which hold in particular for $t \in [0, 1]$ and x sufficiently close to 0 or 2π, imply

$$0 \geq \frac{\ln(t^2 - 2t\cos x + 1)}{t} \geq \frac{\ln(t^2 - 2t + 1)}{t} = \frac{2\ln(1-t)}{t} \qquad (t \neq 1).$$

Since the right-hand side is independent of x and integrable over $[0, 1]$ (since it can be continuously extended into $t = 0$ and tends slowly to $-\infty$ at $t = 1$), the theorem can be applied with $\Phi(t) := -\frac{2\ln(1-t)}{t}$.

c) From b) we have $F(x)=\pi x-x^2/2+C$ for some constant C. Moreover,

$$C=F(0)=\int_0^1\frac{\ln(t^2-2t+1)}{t}\,\mathrm{d}t=2\int_0^1\frac{\ln(1-t)}{t}\,\mathrm{d}t=-2\int_0^1 1+\frac{t}{2}+\frac{t^2}{3}+\frac{t^3}{4}+\cdots\,\mathrm{d}t\,.$$

$\boxed{1}$

The partial sums $f_n(t)=\sum_{k=0}^n\frac{t^k}{k+1}$ of the integrand satisfy $f_n(t)\leq f_{n+1}(t)$ for $t\in[0,1]$ and are integrable over $[0,1]$ with

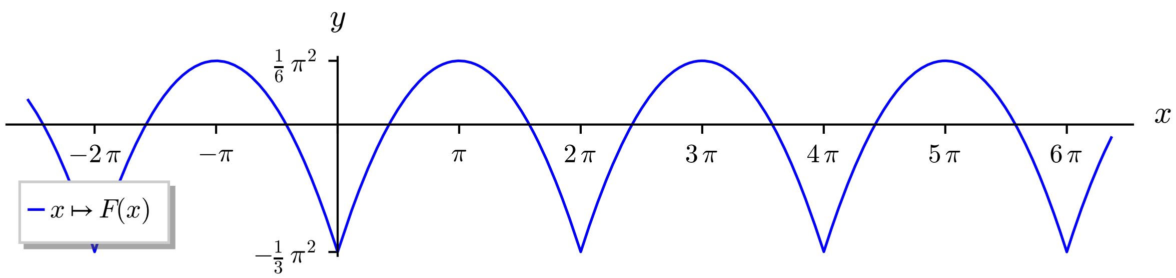
$$\int_0^1 f_n(t)=\left[\sum_{k=0}^n\frac{t^{k+1}}{(k+1)^2}\right]_0^1=\sum_{k=0}^n\frac{1}{(k+1)^2}=\sum_{k=1}^{n+1}\frac{1}{k^2}. \qquad \boxed{1}$$

Since the series $\sum_{k=1}^\infty\frac{1}{k^2}$ converges in $\mathbb{R}$ (to $\pi^2/6$), we have an upper bound for $\int_0^1 f_n(t)$, which doesn't depend on n. Hence we can apply the Monotone Convergence Theorem to (f_n) and conclude that

$$C=-2\int_0^1\lim_{n\to\infty}f_n(t)\,\mathrm{d}t=-2\lim_{n\to\infty}\int_0^n f_n(t)\,\mathrm{d}t=-2\lim_{n\to\infty}\sum_{k=1}^{n+1}\frac{1}{k^2}=-2\sum_{k=1}^{\infty}\frac{1}{k^2}=-\frac{\pi^2}{3},$$

and $F(x)=\pi x-x^2/2-\pi^2/3$ for $x\in[0,2\pi]$. $\boxed{1}$

The subsequent plot shows the obvious extension of F to $\mathbb{R}$, which inherits the 2π-perodicity from $x\mapsto\cos x$. Note that the extension is not differentiable at integral multiples of 2π, because the (one-sided) derivatives of the original function at 0 and 2π are not equal.



Remarks: Though this is a nice exercise, it is not really suitable for a final examination, I have to admit. Also, it would have been better to define F with domain $(0,2\pi)$ and use $F(\pi)$ instead of $F(0)$ for the determination of the constant C, thereby avoiding the incorrect statement in a) and the subtlety in b). Therefore all marks assigned for Question 4 are on top of what is required for a full final exam score.

Only few students provided something substantial to be counted in a). Some students had the correct idea that one needs to bound x away from zero (and also from 2π) in order to obtain an integrable bound $\Phi(t)$ for $|f_x(x,t)|$, but then weren't able to get all details correct (quite understandable on account of the incorrect statement regarding $x=0,\,2\pi$). However, many more students estimated $|f_x(x,t)|$ by $2/(t-1)^2$ and claimed that this function is integrable, which is nonsense. The domain of integration is $[0,1]$ and at $t=1$ this function behaves like $1/t^2$ at $t=0$. Now you should remember from Calculus II that $\int_0^1\frac{\mathrm{d}t}{t^2}$ doesn't exist.

In b) I have subtracted 1 mark if the final justification using the various trigonometric identities was missing. Thus things like "arctan $\frac{1-\cos x}{\sin x} = x/2$" weren't accepted without further explanation.

In c) many students were able to evaluate the constant C in terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but didn't justify interchanging sum and integration and failed to produce the actual value $\pi^2/6$ of this series (with many incorrect guesses such as $3/2$, $\mathrm{e}/2$, or $2\ln 2$). In such a case 1.5 marks were assigned (1 mark penalty for the missing justification, 0.5 marks penalty for missing out $\pi^2/6$).

$$\sum_4 = 9$$

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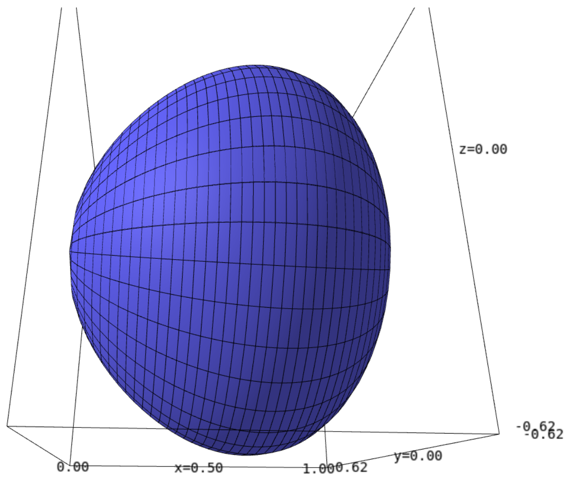
5 a) K is closed, because it is defined by weak inequalities $g(x, y, z) \leq c$ involving one or more continuous functions g.

The function $[0, \infty) \to \mathbb{R}$, $x \mapsto x - x^3$ is non-negative only for $0 \leq x \leq 1$ and has the maximum $f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$. For $(x, y, z) \in K$ we have $x - x^3 = y^2 + z^2 \geq 0$ and hence $0 \leq x \leq 1$. This in turn gives $y^2 + z^2 \leq \frac{2}{3\sqrt{3}} < 1$, showing that K is contained in $[0, 1]^3$. In particular K is bounded.



$\boxed{2}$

b) The boundary surface of K is obtained by rotating in $\mathbb{R}^3$ the curve $z = \sqrt{x - x^3}$, $0 \leq x \leq 1$, around the x-axis. Thus, if $0 < x < 1$, the x-section $K_x = \{(y, z); (x, y, z) \in K\}$ is a circle of radius $r(x) = \sqrt{x - x^3}$. Cavalieri's Principle gives

$$\mathrm{vol}(K) = \int_0^1 \pi\, r(x)^2 \,\mathrm{d}x = \pi \int_0^1 x - x^3 \,\mathrm{d}x = \pi \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{4}. \qquad \boxed{2}$$

c) ∂K is parametrized by $\gamma(x, t) = \left(x, \sqrt{x - x^3}\cos t, \sqrt{x - x^3}\sin t\right)$, $(x, t) \in [0, 1] \times$

$[0, 2\pi]$.

$$\mathbf{J}_\gamma(x,t) = \begin{pmatrix} 1 & 0 \\ \frac{1-3x^2}{2\sqrt{x-x^3}}\cos t & -\sqrt{x-x^3}\sin t \\ \frac{1-3x^2}{2\sqrt{x-x^3}}\sin t & \sqrt{x-x^3}\cos t \end{pmatrix}, \qquad \boxed{1}$$

$$\mathbf{J}_\gamma(x,t)^\mathsf{T}\mathbf{J}_\gamma(x,t)^\mathsf{T} = \begin{pmatrix} 1+\frac{(1-3x^2)^2}{4(x-x^3)} & 0 \\ 0 & \sqrt{x-x^3} \end{pmatrix},$$

$$\begin{aligned} g^\gamma(x,t) &= \left(1+\frac{(1-3x^2)^2}{4(x-x^3)}\right)(x-x^3) = \frac{1}{4}\big(4(x-x^3)+(1-3x^2)^2\big) \\ &= \frac{1}{4}\left(9x^4-4x^3-6x^2+4x+1\right), \\ \sqrt{g^\gamma(x,t)} &= \frac{1}{2}\sqrt{9x^4-4x^3-6x^2+4x+1}, \\ \mathrm{vol}_2(\partial K) &= \int_{\substack{0\leq x\leq 1\\ 0\leq t\leq 2\pi}} \sqrt{g^\gamma(x,t)}\,\mathrm{d}^2(x,t) = \frac{1}{2}\int_{x=0}^{1}\int_{t=0}^{2\pi}\sqrt{9x^4-4x^3-6x^2+4x+1}\,\mathrm{d}t\,\mathrm{d}x \\ &= \pi\int_0^1 \sqrt{9x^4-4x^3-6x^2+4x+1}\,\mathrm{d}x\,. \qquad \boxed{2} \end{aligned}$$

Thus $P(x) = 9x^4-4x^3-6x^2+4x+1$ has the required property.

Alternatively, one can use the formula for the area of a surface of revolution derived in the lecture: The rotating curve is $\alpha(x) = \big(x, r(x)\big) = \big(x, \sqrt{x-x^3}\big)$, $x \in [0,1]$, so that

$$\mathrm{vol}_2(\partial K) = 2\pi\int_0^1 r(x)\sqrt{1+r'(x)^2}\,\mathrm{d}x = 2\pi\int_0^1 \sqrt{x-x^3}\sqrt{1+\frac{(1-3x^2)^2}{4(x-x^3)}}\,\mathrm{d}x\,.$$

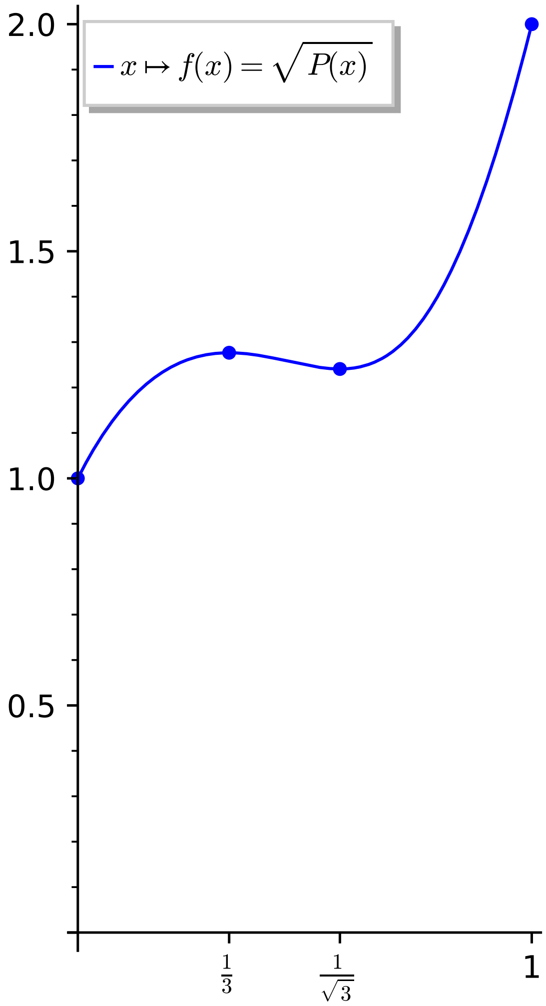
This is the same as above.

d) The integrand $f(x) = \sqrt{P(x)}$ satisfies $f(0)=1$, $f(1)=2$,

$$f'(x) = \frac{P'(x)}{2\sqrt{P(x)}} = \frac{36x^3-12x^2-12x+4}{2\sqrt{P(x)}} = \frac{36\left(x-\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)}{2\sqrt{P(x)}}.$$

Thus f is increasing in $\left[0,\frac{1}{3}\right]$, decreasing in $\left[\frac{1}{3},\frac{1}{\sqrt{3}}\right]$, and increasing in $\left[\frac{1}{\sqrt{3}},1\right]$. Since $P\left(\frac{1}{3}\right) = \frac{44}{27} < 2$, $P\left(\frac{1}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}} > 1$, we have $f(x)\in[1,2]$ for $x\in[0,1]$ and hence $\pi \leq \mathrm{vol}_2(\partial K) \leq 2\pi$. $\boxed{2}$

A much better lower bound can be obtained by using the fact that balls have minimum surface area among all bodies with fixed volume. Thus $\mathrm{vol}_2(\partial K)$ is lower-bounded by the surface area of the ball of volume $\mathrm{vol}(K)=\pi/4$, whose radius r satisfies $\frac{4}{3}r^3\pi = \frac{\pi}{4}$, i.e., $r = \sqrt[3]{\frac{3}{16}} = \frac{3^{1/3}}{2^{4/3}}$. This gives $\mathrm{vol}_2(\partial K) \geq 4\pi r^2 = \left(\frac{3}{2}\right)^{2/3}\pi \approx 4.12$. The true value of $\mathrm{vol}_2(\partial K)$, rounded to two decimal digits, is 4.17. A better upper bound can be easily obtained by taking into account the shape of the graph of f; cf. subsequent plot. One good upper bound is $\mathrm{vol}_2(\partial K) \leq \frac{4\pi}{3} \approx 4.19$.



e) Since K is symmetric w.r.t. the x-axis, it is clear that $\mathbf{s}$ has the form $(x_0, 0, 0)$. $\boxed{\frac{1}{2}}$

$$x_0 = \frac{1}{\mathrm{vol}(K)} \int_K x \, \mathrm{d}^3(x,y,z) = \frac{4}{\pi} \int_0^1 x \, \mathrm{vol}_2(K_x) \, \mathrm{d}x = \frac{4}{\pi} \int_0^1 x \, \pi(x - x^3) \, \mathrm{d}x$$
$$= 4 \int_0^1 x^2 - x^4 \, \mathrm{d}x = 4 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{8}{15}. \qquad \boxed{1\tfrac{1}{2}}$$

Remarks: In a) I have subtracted 1 mark if the sketch of K was missing, but generally accepted a good (or not too bad) sketch, together with, e.g., $0 \leq x \leq 1$ as proof of compactness.

In b) and e) several students used the squared radius $r(x)^2 = x - x^3$ in the integrals, i.e., $\int_0^1 \pi(x - x^3)^2 \, \mathrm{d}x$ and $\int_0^1 \pi x(x - x^3)^2 \, \mathrm{d}x$, for which I have assigned half of the marks.

In c) quite a few students obtained $P(x) = 4(x - x^3)$ or other wrong polynomials, which mostly couldn't be honored by any marks

d) was solved (or at least partially solved) only by very few students. Notably, 2 students produced exactly the solution I had in mind (see above). An interesting, slightly weaker but acceptable lower bound obtained by another 2–3 students is the following:

From $P(x) = (3x^2 - 1)^2 + 4(x - x^3)$ and $x \geq x^3$ for $x \in [0,1]$ we get

$$\begin{aligned}
\operatorname{vol}_2(\partial K) &\geq \pi \int_0^1 \sqrt{(3x^2-1)^2}\,\mathrm{d}x = \pi \int_0^1 \left|3x^2 - 1\right|\mathrm{d}x \\
&= \pi \left(\int_0^{1/\sqrt{3}} 1 - 3x^2\,\mathrm{d}x + \int_{1/\sqrt{3}}^1 3x^2 - 1\,\mathrm{d}x \right) \\
&= \pi \left(\left[x - x^3\right]_0^{1/\sqrt{3}} + \left[x^3 - x\right]_{1/\sqrt{3}}^1 \right) \\
&= 2\pi\,(x - x^3)\big|_{x=1/\sqrt{3}} = \frac{4\pi}{3\sqrt{3}} \approx 0.77\,\pi.
\end{aligned}$$

The trivial bound $\operatorname{vol}_2(\partial K) > 0$ wasn't accepted, of course.

$$\sum_{5} = 11$$

$$\sum_{\text{Final Exam}} = 52$$

The target score of the final exam is set at 40 marks. Thus, roughly, for obtaining a full score, solving the rather difficult Questions 4 and 5 d) was not required.

Question 1 (ca. 9 marks)

Decide whether the following statements are true or false, and justify your answers briefly.

a) The surface in $\mathbb{R}^3$ with equation $xyz = x + y + z$ is smooth.

b) The function $f(x, y) = (2019 + x^4 + y^4)\mathrm{e}^{-x^2-y^2}$, $(x, y) \in \mathbb{R}^2$, has a global maximum but no global minimum.

c) If $\mathbf{x}^*$ solves the optimization problem "minimize $f(\mathbf{x})$ subject to $g_1(\mathbf{x}) = g_2(\mathbf{x}) = 0$" (where $f, g_1, g_2 \colon \mathbb{R}^3 \to \mathbb{R}$ are C^1-functions) and $\nabla g_1(\mathbf{x}^*)$, $\nabla g_2(\mathbf{x}^*)$ are both nonzero, then there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(\mathbf{x}^*) = \lambda_1 \nabla g_1(\mathbf{x}^*) + \lambda_2 \nabla g_2(\mathbf{x}^*)$.

d) In $\mathbb{R}^2$ the line integral $\int_\gamma (ax + by)\,\mathrm{d}x + (cx + dy)\,\mathrm{d}y$ $(a, b, c, d, \in \mathbb{R})$ is independent of path if and only if $b = c$.

e) Let $D = \mathbb{R}^2 \setminus S$, where $S = \{(x, 0); -1 \leq x \leq 1\}$ is the line segment connecting $(-1, 0)$ and $(1, 0)$. Then every vector field $F = (f_1, f_2)$ on D satisfying $(f_1)_y = (f_2)_x$ is a gradient field.

f) Let $D = \mathbb{R}^3 \setminus S$, where now S denotes the line segment connecting $(-1, 0, 0)$ to $(1, 0, 0)$. Then every vector field $F = (f_1, f_2, f_3)$ on D satisfying $\operatorname{curl}(F) \equiv \mathbf{0}$ is a gradient field.

Question 2 (ca. 7 marks)

Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) := xy(x + y - 1).$$

a) Determine all critical points of f and their types.

b) Does f have a global extremum?

c) A polynomial $\ell(x, y) = ax + by + c$ with $(a, b) \neq (0, 0)$ may be called a *line polynomial*, since $\ell(x, y) = 0$ defines a line L in $\mathbb{R}^2$. Show that the product $g = \ell_1\ell_2\ell_3$ of 3 line polynomials whose corresponding lines L_1, L_2, L_3 are in general position (i.e., determine a triangle in $\mathbb{R}^2$) must have at least 4 critical points.

Question 3 (ca. 10 marks)

Consider the quadric surface S in $\mathbb{R}^3$ with equation

$$xy - xz - yz = 3.$$

a) Determine the type of S.

b) Show that there is a point on S that minimizes the distance to the origin $(0, 0, 0)$.

c) Determine all points on S with the property in b) and their distance from $(0, 0, 0)$.

d) Show that the point $\mathbf{q} = (-3, 1, 3)$ is on S, and determine an equation $a_1x + a_2y + a_3z = b$ for the tangent plane of S in $\mathbf{q}$.

Question 4 (ca. 7 marks)

Consider the function $F\colon [0,\infty) \to \mathbb{R}$ defined by

$$F(x) = \int_0^\infty \frac{\mathrm{e}^{-x(1+t^2)}}{1+t^2}\,\mathrm{d}t\,.$$

a) Show that F is continuous, $F'(x)$ exists for $x > 0$ and can be obtained by differentiation under the integral sign, and $\lim_{x\to\infty} F(x) = 0$.

b) Show that

$$F'(x) = -\frac{\mathrm{e}^{-x}}{\sqrt{x}}\,I \quad \text{with} \quad I = \int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s\,.$$

c) Show that

$$F(x) = F(0) - \int_0^x \frac{\mathrm{e}^{-t}}{\sqrt{t}}\,I\,\mathrm{d}t = \frac{\pi}{2} - 2I\int_0^{\sqrt{x}} \mathrm{e}^{-s^2}\,\mathrm{d}s\,.$$

d) Letting $x \to \infty$, conclude from a) and c) that $I = \frac{1}{2}\sqrt{\pi}$.

Question 5 (ca. 7 marks)

Consider the solid K in $\mathbb{R}^3$ defined by

$$K = \bigl\{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 \leq z \leq 1 + x\bigr\}.$$

a) Show that K is compact (i.e., closed and bounded).

b) Determine the volume $\operatorname{vol}(K)$.

Hint: Show that the projection of K onto the x-y plane is a disk, and use polar coordinates relative to the center of this disk.

c) Explain how to compute the surface area $\operatorname{vol}_2(\partial K)$.

Note: An explicit figure for $\operatorname{vol}_2(\partial K)$ is not required, but you should simplify the task of computing the surface area as far as possible.

Question 6 (ca. 4 marks)

Let Δ be the (solid) triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(0,1)$. Evaluate the integral

$$\int_\Delta \frac{3x+y}{x+3y}\,\mathrm{d}^2(x,y).$$

Hint: Use the obvious change of variables and the fact that the integral over the interior of Δ has the same value.

Solutions

1 a) True. The surface is a level set of the function $f(x,y,z)=xyz-x-y-z$, which has

$$\nabla f(x,y,z)=(yz-1,xz-1,xy-1).$$

Suppose $\nabla f(x,y,z)=(0,0,0)$. Then x,y,z are nonzero, and from $yz=1=xz$ we obtain $x=y$, and then by symmetry $x=y=z$. The points on the surface satisfying this condition are those with $x^3=3x$, i.e., $(0,0,0)$, $(\sqrt{3},\sqrt{3},\sqrt{3})$, $(-\sqrt{3},-\sqrt{3},-\sqrt{3})$. The corresponding gradients are $(-1,-1,-1)$, $(2,2,2)$, $(2,2,2)$. Since these are nonzero, there is no point (x,y,z) on the surface satisfying $\nabla f(x,y,z)=(0,0,0)$. $\boxed{1\frac{1}{2}}$

b) True. Since $x\mapsto e^x$ grows faster than any polynomial, we have $\lim_{|(x,y)|\to\infty} f(x,y)=0$. Together with $f(x,y)>0$ for all $(x,y)\in\mathbb{R}^2$, this clearly implies that f has no global minimum. On the other hand, if $R>0$ is such that $f(x,y)<f(0,0)=2019$ for all (x,y) outside $\overline{\mathrm{B}_R(0,0)}$, then a maximum of f on $\overline{\mathrm{B}_R(0,0)}$, which exists since f is continuous, must be a global maximum. $\boxed{1\frac{1}{2}}$

c) False. The condition on $\nabla g_1(\mathbf{x}^*)$, $\nabla g_2(\mathbf{x}^*)$ in the Lagrange Multiplier Theorem is that these vectors should be linearly independent (not only both nonzero), and there are indeed counterexamples to the present statement. $\boxed{1\frac{1}{2}}$

d) True. Writing $\omega=f\,\mathrm{d}x+g\,\mathrm{d}y$, the equation $b=c$ is equivalent to $f_y=g_x$, which guarantees that ω has an anti-derivative (by Poincaré's Lemma or, if you like, since $\mathbb{R}^2$ is simply-connected) and hence that $\int_\gamma\omega$ is independent of path. Conversely, if $\int_\gamma\omega$ is independent of path then ω has an anti-derivative F and $f_y=F_{xy}=F_{yx}=g_x$ by Clairaut's Theorem. $\boxed{1\frac{1}{2}}$

e) False. For the closed curve $\gamma(t)=(2\cos t,2\sin t)$, $t\in[0,2\pi]$, we have $\int_\gamma\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{x^2+y^2}=2\pi\neq 0$, and hence $F(x,y)=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$, $(x,y)\in D$, is not a gradient field. But F satisfies $(f_1)_y=(f_2)_x$; cf. lecture. $\boxed{1\frac{1}{2}}$

f) True. D is simply connected, since every closed path in D can be contracted continuously to a point without touching S. (This is proved in the same way as for $\mathbb{R}^3\setminus\{(0,0,0)\}$.) It follows that every differential 1-form $\omega=f_1\,\mathrm{d}x+f_2\,\mathrm{d}y+f_3\,\mathrm{d}z$ on D which satisfies the conditions in Poincaré's Lemma is exact. These conditions amount to $\operatorname{curl}(F)\equiv 0$, and "$\omega$ is exact" amounts to "F is a gradient field". $\boxed{1\frac{1}{2}}$

$$\sum_1 = 9$$

2 a) Local extrema (if any) must be critical points.

$$\begin{aligned}
f&=x^2y+xy^2-xy,\\
f_x&=2xy+y^2-y,\\
f_y&=x^2+2xy-x=2xy+x^2-x\\
\nabla f(x,y)=0 &\iff 2xy+x^2-x=0\wedge x^2-x=y^2-y
\end{aligned}$$

The 2nd equation can be rewritten as $(x-y)(x+y) = x^2-y^2 = x-y$ and is equivalent to $x = y \lor x+y = 1$.

Case 1: $\underline{x = y}$ The first equation becomes $3x^2 - x = x(3x-1) = 0$ and yields two critical points, viz. $\mathbf{p}_1 = (0,0)$, $\mathbf{p}_2 = \left(\frac{1}{3}, \frac{1}{3}\right)$. $\boxed{1}$

Case 2: $\underline{x + y = 1}$ Here the first equation becomes $2x(1-x)+x^2-x = 3x(1-x) = 0$ and yields two further critical points, viz. $\mathbf{p}_3 = (1,0)$, $\mathbf{p}_4 = (0,1)$. $\boxed{1}$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2y & 2x+2y-1 \\ 2x+2y-1 & 2x \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_2) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{H}_f(\mathbf{p}_4) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$, $\mathbf{H}_f(\mathbf{p}_3)$, $\mathbf{H}_f(\mathbf{p}_4)$ have determinant $-1 < 0$, the points $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{p}_4$ are saddle points. $\boxed{1}$

Since $\mathbf{H}_f(\mathbf{p}_2)$ is positive definite ($f_{xx}(\mathbf{p}_1) = \frac{2}{3} > 0$, $\det \mathbf{H}_f(\mathbf{p}_2) = \left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{3} > 0$), the point $\mathbf{p}_2$ is a (strict) local minimum. $\boxed{1}$

b) No. Since global extrema are in particular local extrema, the only candidate for a global extremum is $\mathbf{p}_2$. Since $f(\mathbf{p}_2) = \frac{1}{27} + \frac{1}{27} - \frac{1}{9} = -\frac{1}{27}$ and, e.g., $f(-1,-1) = (-1)(-1)(-3) = -3 < -\frac{1}{27}$, the point $\mathbf{p}_2$ is not a global minimum (and, of course, not a global maximum either). $\boxed{1}$

c) The 3 intersection points $\mathbf{q}_1 = L_2 \cap L_3$, $\mathbf{q}_2 = L_1 \cap L_3$, $\mathbf{q}_3 = L_1 \cap L_2$ are critical, since

$$g_x = (\ell_1\ell_2\ell_3)_x = (\ell_1)_x\ell_2\ell_3 + \ell_1(\ell_2)_x\ell_3 + \ell_1\ell_2(\ell_3)_x$$

vanishes whenever two of ℓ_1, ℓ_2, ℓ_3 vanish (and similarly for g_y), and this is the case (precisely) for $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. $\boxed{1}$

Further, on the solid triangle Δ determined by $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ we have either $g(x,y) \geq 0$ or $g(x,y) \leq 0$ with equality iff $(x,y) \in \partial\Delta$. Since Δ is closed and bounded and g is continuous, g attains both a maximum and a minimum on Δ. One of these must be in the interior Δ° and hence yield a 4th critical point $\mathbf{q}_4$. $\boxed{1}$

$$\sum_2 = 7$$

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3 a) The type of S can be determined by transforming the representing symmetric matrix $\mathbf{A}$ into canonical form using the algorithm outlined in the lecture. Since the type of S is invariant under scaling $\mathbf{A}$ with a positive constant, we can also use $2\mathbf{A}$, which has integer entries:

$$2\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \xrightarrow[R1=R1+R2]{C1=C1+C2} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix} \xrightarrow[R3=R3+R1]{R2=R2-\frac{1}{2}R1} \begin{pmatrix} 2 & 1 & -2 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
$$\xrightarrow[C3=C3+C1]{C2=C2-\frac{1}{2}C1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since the right-hand side of $xy - xz - yz = 3$ is positive, the canonical form of the quadric is $x^2 - y^2 - z^2 = 1$, which therefore is a hyperboloid of two sheets. $\boxed{2}$

b) Consider a point on S, e.g., $P = (1, 3, 0)$. The length function $(x, y, z) \mapsto |(x, y, z)|$ is continuous and attains a minimum $|(x_0, y_0, z_0)| = d_0$ on the set

$$C = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq 10, xy - xz - yz = 3\},$$

which is closed, bounded, and non-empty (since $P \in C$). Since $d_0 \leq \sqrt{10}$ and all points in $S \setminus C$ have length $> \sqrt{10}$, the point $P_0 = (x_0, y_0, z_0)$ has the required property. $\boxed{1\frac{1}{2}}$

c) Define $f, g\colon \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = xy - xz - yz - 3$. Then the task is to minimize f on $\mathbb{R}^3$ under the constraint $g(x, y, z) = 0$.

$$\nabla f(x, y, z) = (2x, 2y, 2z), \qquad \nabla g(x, y, z) = (y - z, x - z, -x - y).$$

Since $\nabla g(x, y, z) = (0, 0, 0)$ implies $x = y = z$ and no point on S has this property, the theorem on Langrange multipliers is applicable everywhere and yields that every minimum must satisfy $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for some $\lambda \in \mathbb{R}$. This gives the system of equations

$$\begin{aligned} y - z &= 2\lambda x, \\ x - z &= 2\lambda y, \\ -x - y &= 2\lambda z, \\ xy - xz - yz &= 3. \end{aligned} \qquad \boxed{2}$$

Eliminating λ from the first two equations gives $y(y-z) = x(x-z)$, which is equivalent to $y^2 - x^2 = yz - xz$ and further to $x = y \vee z = x + y$.

Case 1 $\underline{x = y}$ Eliminating λ from the 2nd and 3rd equation gives $(x-z)z = -2x^2$, and the 4th equation becomes $x^2 - 2xz = 3$. Solving the latter for z and substituting the result into the former gives

$$\begin{aligned} & \left(x - \frac{x^2 - 3}{2x}\right) \frac{x^2 - 3}{2x} = -2x^2 \\ \iff\quad & (2x^2 - x^2 + 3)(x^2 - 3) = -8x^4 \\ \iff\quad & 9x^4 = 9 \\ \iff\quad & x = \pm 1 \end{aligned}$$

The corresponding values of z are

$$z = \frac{1 - 3}{\pm 2} = \mp 1.$$

This yields the two candidate points $P_0 = (1, 1, -1)$ and $P_1 = (-1, -1, 1)$, which are antipodal with respect to $(0, 0, 0)$.

Case 2 $\underline{z = x + y}$ Substituting $z = x + y$ into the 4th equation gives $xy - (x + y)^2 = -x^2 - xy - y^2 = 1$. But $x^2 + xy + y^2 = \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 \geq 0$, contradiction. Hence this case doesn't occur.

$\Longrightarrow$ The points on S minimizing the distance to $(0, 0, 0)$ are $P_0 = (1, 1, -1)$ and $P_1 = (-1, -1, 1)$. The minimal distance is $\sqrt{3}$. $\boxed{2}$

Alternative solution: For $(x, y, z) \in S$ the Cauchy-Schwarz Inequality gives

$$3 = \begin{pmatrix} x \\ z \\ y \end{pmatrix} \cdot \begin{pmatrix} y \\ -x \\ -z \end{pmatrix} \leq \sqrt{x^2 + z^2 + y^2}\sqrt{y^2 + (-x)^2 + (-z)^2} = x^2 + y^2 + z^2$$

with equality iff $(x, z, y) = \lambda(y, -x, -z)$ for some $\lambda \in \mathbb{R}$. This vector equation is equivalent to the scalar equations $x = \lambda y \wedge z = -\lambda x \wedge y = -\lambda z$ and implies $x = \lambda y = -\lambda^2 z = \lambda^3 x$. Since $(0, 0, 0) \notin S$, we must have $\lambda^3 = 1$, i.e., $\lambda = 1$ and $(x, y, z) = (x, x, -x)$. Together with $|(x, y, z)| = 3$ this forces $(x, y, z) = (1, 1, -1)$ or $(-1, -1, 1)$, which are indeed solutions. Thus we have again found that the points $P_0, P_1 \in S$ are at minimal distance $\sqrt{3}$ from the origin, and are the only such points.

d) $(-3)1 - (-3)3 - 1 \cdot 3 = 3 \Longrightarrow Q \in S$. $\boxed{+1}$
Since S is a level set of the function g in c), a normal vector for the tangent plane E in Q is $\nabla g(Q) = \nabla g(-3, 1, 3) = (-2, -6, 2)$ and an equation for E is

$$-2(x + 3) - 6(y - 1) + 2(z - 3) = 0 \iff x + 3y - z = -3. \qquad \boxed{1\tfrac{1}{2}}$$

$$\sum_3 = 9 + 1$$

---

4 a) We have $F(x) = \int_0^\infty f(x, t)\,\mathrm{d}t$ with

$$|f(x, t)| = \frac{\mathrm{e}^{-x(1+t^2)}}{1 + t^2} \leq \frac{1}{1 + t^2} =: \Phi(t),$$

independently of x. Since $\Phi(t)$ is integrable over $\mathbb{R}$ (with integral $[\arctan(t)]_0^\infty = \pi/2$), the function F is continuous. $\boxed{+1}$.

We have
$$\left|\frac{\partial f(x, t)}{\partial x}\right| = \mathrm{e}^{-x(1+t^2)} \leq \mathrm{e}^{-r(1+t^2)} = \Phi_r(t) \quad \text{for } x > r > 0.$$

Since $\Phi_r(t)$ is integrable over $\mathbb{R}$, the function F can be differentiated under the integral sign. $\boxed{+1}$.

We have

$$\lim_{x\to\infty} f(x, t) = \lim_{x\to\infty} \frac{\mathrm{e}^{-x(1+t^2)}}{1 + t^2} = 0 \quad \text{for every } t \in \mathbb{R}.$$
$$\Longrightarrow \lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_0^\infty f(x, t)\,\mathrm{d}t = \int_0^\infty \lim_{x\to\infty} f(x, t)\,\mathrm{d}t = \int_0^\infty 0\,\mathrm{d}t = 0,$$

provided we can interchange the limit with integration. Since $|f(x,t)| \leq \Phi(t)$ independently of x (see above), this is justified. $\boxed{+1}$

b) From a),

$$\begin{aligned} F'(x) &= \int_0^\infty \frac{\partial f(x,t)}{\partial x} = \int_0^\infty -\mathrm{e}^{-x(1+t^2)}\,\mathrm{d}t = -\mathrm{e}^{-x}\int_0^\infty \mathrm{e}^{-xt^2}\,\mathrm{d}t \\ &= -\frac{e^{-x}}{\sqrt{x}}\int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = -\frac{e^{-x}}{\sqrt{x}}\,I. \qquad \text{(Subst. } t = s/\sqrt{x},\ \mathrm{d}t = \mathrm{d}s/\sqrt{x}\text{)} \end{aligned}$$

$\boxed{1\tfrac{1}{2}}$

c) Since F is continuous in $[0,\infty)$ and differentiable in $(0,\infty)$, the Fundamental Theorem of Calculus applies and yields

$$\begin{aligned} F(x) &= F(0) + \int_0^x F'(t)\,\mathrm{d}t = \pi/2 - I\int_0^x \frac{\mathrm{e}^{-t}}{\sqrt{t}}\,\mathrm{d}t \\ &= \pi/2 - 2I\int_0^{\sqrt{x}} \mathrm{e}^{-s^2}\,\mathrm{d}s\,. \qquad \text{(Subst. } t = s^2,\ \mathrm{d}t = 2s\,\mathrm{d}s\text{)} \end{aligned}$$

$\boxed{1\tfrac{1}{2}}$

d) Letting $x \to \infty$ in c) gives

$$\lim_{x\to\infty} F(x) = \pi/2 - 2I\int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \pi/2 - 2I^2.$$

Together with a) this implies $\pi/2 - 2I^2 = 0$, i.e., $I^2 = \pi/4$ and, since clearly $I > 0$, further $I = \sqrt{\pi/4} = \sqrt{\pi}/2$. $\boxed{1}$

$$\sum_4 = 4 + 3$$

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5 a) K is closed, because it is defined by weak inequalities $g(x,y,z) \leq c$ involving one or more continuous functions g. $\boxed{+\tfrac{1}{2}}$

For $(x,y,z) \in K$ we have $x^2 + y^2 \leq 1 + x$, which is equivalent to $\left(x - \frac{1}{2}\right)^2 + y^2 \leq \frac{5}{4}$ and shows that $|x| \leq \frac{1+\sqrt{5}}{2}$, $|y| \leq \frac{\sqrt{5}}{2}$, $|z| \leq 1 + |x| \leq \frac{3+\sqrt{5}}{2}$. This shows that K is bounded. $\boxed{1}$

b) The projection of K to the x-y plane is the disk $\left(x-\frac{1}{2}\right)^2+y^2\leq\frac{5}{4}$; cf. a).

$$\Longrightarrow \operatorname{vol}(K)=\int\limits_{\left(x-\frac{1}{2}\right)^2+y^2\leq\frac{5}{4}} 1+x-x^2-y^2\,\mathrm{d}^2(x,y) \qquad \boxed{1}$$

$$=\int\limits_{\substack{0\leq r\leq\sqrt{5}/2\\0\leq\phi\leq 2\pi}} r\left(\tfrac{3}{2}+r\cos\phi-\left(\tfrac{1}{2}+r\cos\phi\right)^2-r^2\sin^2\phi\right)\mathrm{d}^2(r,\phi)$$

(Subst. $x=\frac{1}{2}+r\cos\phi$, $y=r\sin\phi$)

$$=\int_0^{2\pi}\int_0^{\sqrt{5}/2} r\left(\tfrac{5}{4}-r^2\right)\mathrm{d}r\,\mathrm{d}\phi$$

$$=2\pi\left[\tfrac{5}{8}r^2-\tfrac{1}{4}r^4\right]_0^{\sqrt{5}/2}=2\pi\left(\tfrac{25}{32}-\tfrac{25}{64}\right)=\tfrac{25}{32}\,\pi. \qquad \boxed{2}$$

c) ∂K is the disjoint union of the surfaces

$$S_1=\left\{(x,y,z)\in\mathbb{R}^3;x^2+y^2=z<1+x\right\},$$
$$S_2=\left\{(x,y,z)\in\mathbb{R}^3;x^2+y^2<z=1+x\right\},$$
$$S_3=\left\{(x,y,z)\in\mathbb{R}^3;x^2+y^2=z=1+x\right\}.$$

S_3 is 1-dimensional, as the parametrization

$$\gamma(t)=\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\cos t,\frac{\sqrt{5}}{2}\sin t,\frac{3}{2}+\frac{\sqrt{5}}{2}\cos t\right),\quad t\in[0,2\pi]$$

shows. Hence $\operatorname{vol}_2(S_3)=0$. $\boxed{\tfrac{1}{2}}$

S_1 is the graph of $f(x,y)=x^2+y^2$, $(x,y)\in D$, where D is the disk $\left(x-\frac{1}{2}\right)^2+y^2\leq\frac{5}{4}$.

$$\Longrightarrow \operatorname{vol}_2(S_1)=\int_D\sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y)$$

$$=\int\limits_{\left(x-\frac{1}{2}\right)^2+y^2\leq\frac{5}{4}}\sqrt{1+4x^2+4y^2}\,\mathrm{d}^2(x,y) \qquad \boxed{1}$$

$$=\int\limits_{\substack{0\leq r\leq\sqrt{5}/2\\0\leq\phi\leq 2\pi}} r\sqrt{1+4\left(\tfrac{1}{2}+r\cos\phi\right)^2+4(r\sin\phi)^2}\,\mathrm{d}^2(r,\phi)$$

$$=\int\limits_{\substack{0\leq r\leq\sqrt{5}/2\\0\leq\phi\leq 2\pi}} r\sqrt{2+4r\cos\phi+4r^2}\,\mathrm{d}^2(r,\phi) \qquad \boxed{+\tfrac{1}{2}}$$

S_2 is the graph of $g(x,y)=1+x$, $(x,y)\in D$, and has area

$$\operatorname{vol}_2(S_2)=\int_D\sqrt{1+|\nabla g(x,y)|^2}\,\mathrm{d}^2(x,y)=\int_D\sqrt{2}\,\mathrm{d}^2(x,y)=\sqrt{2}\operatorname{vol}_2(D)=\sqrt{2}\,\frac{5}{4}\,\pi=\frac{5\pi}{2\sqrt{2}}.$$

$\boxed{1}$

It follows that $\operatorname{vol}_2(\partial K) = \frac{5\pi}{2\sqrt{2}} + \operatorname{vol}_2(S_1)$. $\boxed{\frac{1}{2}}$

$$\sum_5 = 7 + 1$$

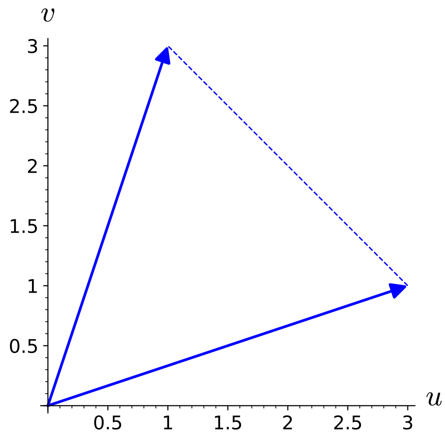
6 We use the change of variables

$$T(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3x+y \\ x+3y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (x,y) \in \Delta.$$

The change-of-variables formula says

$$\int_\Delta \frac{3x+y}{x+3y} \det \mathbf{J}_T(x,y)\,\mathrm{d}^2(x,y) = \int_{T(\Delta)} \frac{u}{v}\,\mathrm{d}^2(u,v).$$

Since T is linear, we have $\mathrm{d}T(x,y) = T$, $\mathbf{J}_T(x,y) = \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, and $\det \mathbf{J}_T(x,y) = 3^2 - 1^2 = 8$.
$T(\Delta)$ is the triangle "spanned" by the columns of $\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, i.e., has verices $(0,0)$, $(3,1)$, $(1,3)$; cf. the picture. The lines bounding $T(\Delta)$ have equations $v = u/3$ (bottom), $v = 3u$ (top left), and $v = 4 - u$ (top right).
Since u/v is easier to integrate with respect to u (correct me, if I'm wrong), we evaluate the integral in the order $\mathrm{d}u\,\mathrm{d}v$.



$$\begin{aligned}
\int_\Delta \frac{3x+y}{x+3y}\,\mathrm{d}^2(x,y) &= \frac{1}{8}\int_{T(\Delta)} \frac{u}{v}\,\mathrm{d}^2(u,v) && \boxed{1}\\
&= \frac{1}{8}\left(\int_{v=0}^{1}\int_{u=v/3}^{3v} \frac{u}{v}\,\mathrm{d}u\,\mathrm{d}v + \int_{v=1}^{3}\int_{u=v/3}^{4-v} \frac{u}{v}\,\mathrm{d}u\,\mathrm{d}v\right) && \boxed{1}\\
&= \frac{1}{8}\left(\int_{v=0}^{1} \frac{1}{v}\left[\frac{u^2}{2}\right]_{v/3}^{3v}\mathrm{d}v + \int_{v=1}^{3} \frac{1}{v}\left[\frac{u^2}{2}\right]_{v/3}^{4-v}\mathrm{d}v\right)\\
&= \frac{1}{16}\left(\int_0^1 \frac{9v^2 - v^2/9}{v}\,\mathrm{d}v + \int_1^3 \frac{(4-v)^2 - v^2/9}{v}\,\mathrm{d}v\right)\\
&= \frac{1}{16}\left(\int_0^1 \frac{80}{9}v\,\mathrm{d}v + \int_1^3 \frac{16}{v} - 8 + \frac{8}{9}v\,\mathrm{d}v\right)\\
&= \frac{1}{16}\left(\frac{40}{9} + 16\ln 3 - 16 + \frac{8}{9}\frac{3^2-1^2}{2}\right) = \ln 3 - \frac{1}{2}. && \boxed{2}
\end{aligned}$$

$$\sum_6 = 4$$

$$\sum_{\text{Final Exam}} = 40 + 5$$