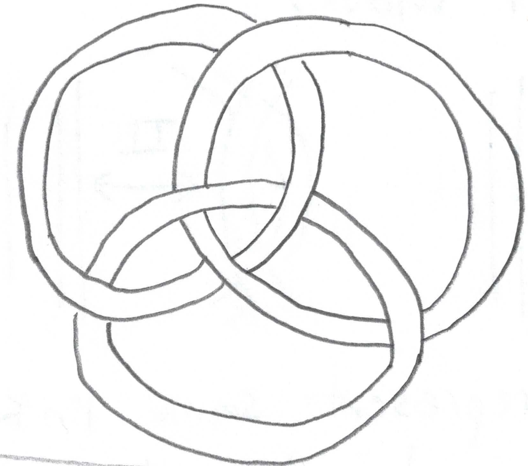
L1

Borromean rings



- no two linked
- can't be pulled apart
- coat of arms of the Borromeo family
  - many cultures

$\boxed{\text{Can be built from perfect circles? } \cancel{\text{No}}}$ ⇝ straighten toy

⇝ Knot theory:

geometry, topology, combinatorics

- 1867 ~~Kel~~ Lord Kelvin atoms = knots

knot = smooth embedding circle in 3-space

link = smooth embedding of disjoint circles in 3-space

⇝ can represent with diagram.

→ Why circle?

- When are two links the 'same'?

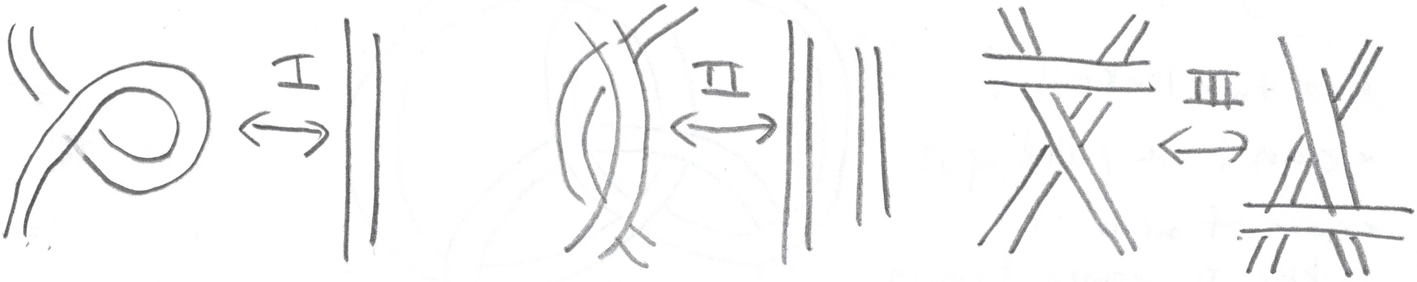
(orientation-preserving homeomorphism $(\mathbb{R}^3, L) \cong (\mathbb{R}^3, L')$.)

= continuous deformation of space sending L to L'.

→ Not obvious

- mistakes in tables

Reidemeister moves



Diagrams represent same link iff
one can be obtained from other via moves.

($\Leftarrow$) ✓

($\Rightarrow$) track projection onto plane.

[Trivial = equiv to n far apart circles]

Invariants

- assign quantities to objects invariant under equivalence

Clock arithmetic, congruence $a \equiv b \pmod{n}$ – (comp
"modular" $\Leftrightarrow a-b = kn$ x:

Brunnian link $\Leftrightarrow$ k-links such that any (k-1) subcollection is trivial

$\rightsquigarrow$ Can't be made of perfect circles.

Pf by contradiction

1. If a link consists of perfect circles, pairwise not-linked, then its trivial
2. Design an invariant of links
3. Borromean links are not trivial by 2.
(or Brunnian links)

$\boxed{2.5}$

If $\gcd(r,n)=1$

$\rightsquigarrow$

$ra \equiv rb \pmod n$

$\Rightarrow a \equiv b \pmod n$

Pf:

$ra \equiv rb \pmod n$

$\Leftrightarrow \exists k \in \mathbb{Z}: \quad r(a-b) = kn.$

$\Rightarrow r \mid kn$

(by asr) no prime factors of r divide n, so they all divide k, i.e., $r \mid k$

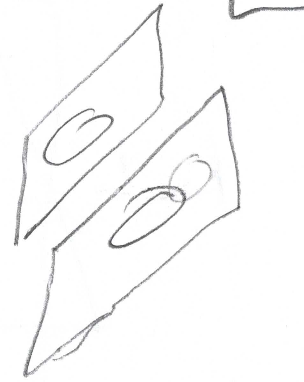
$\Leftrightarrow k/r$ is some integer s.

Thus $a-b = \frac{k}{r} n = sn$

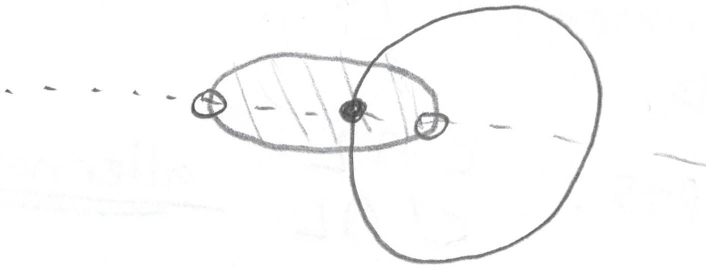
$\Rightarrow a \equiv b \mod n.$

1. Build movie to separate circles.

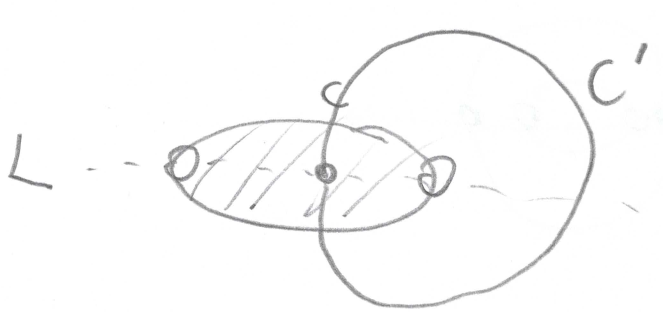
→ Wiggle circles so no two have parallel planes



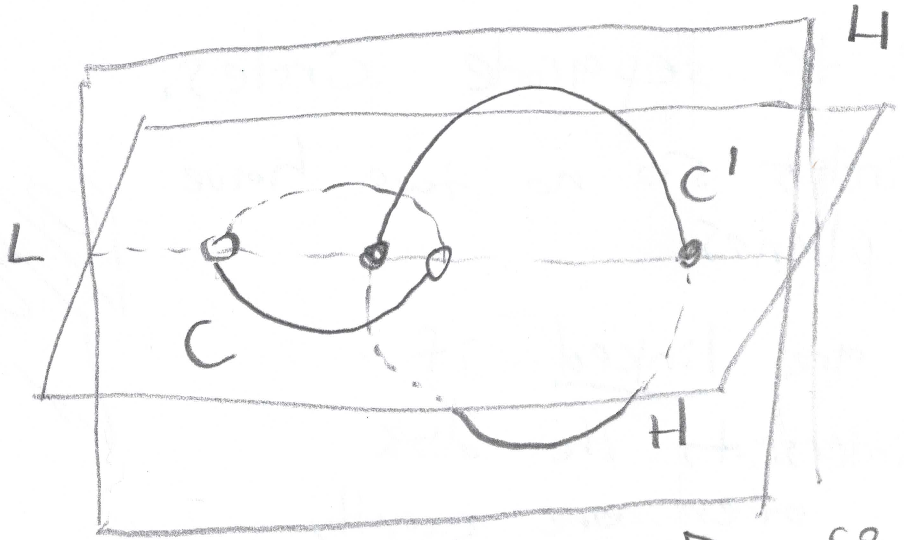
⇝ Two circles are <u>linked</u> if one of them intersects the disk spanned by the other one exactly <u>once</u>



If C' intersects D at <u>one</u> point, the pt lies in $D \subset H$ and $C' \subset D' \subset H'$ so lies on $L := H \cap H'$ line.



Since $L \subset H$ and contains pt in D, it intersects C at exactly <u>two</u> points



- C' intersects H in D, so there is a second intersection pt on L but outside D.

Two pairs of pts: $C \cap L$, $C' \cap L$ alternating!

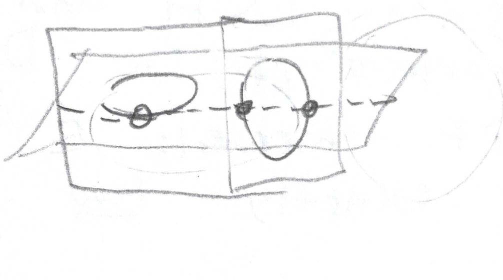
$\rightsquigarrow$ C intersects D' at one point too.

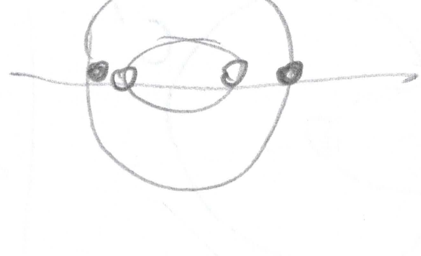
Linked $\Rightarrow$ Alternating pts

Alternating $\Rightarrow$ Linked ?

Goes both ways!

Yes, if C and C' not linked, one misses the inside of disk of the other one, and either < 4 pts of $C \cup C'$ on L or 4 pts do not alternate.





- Construct spherical "domes"

For a circle C with center c and radius r we have a dome

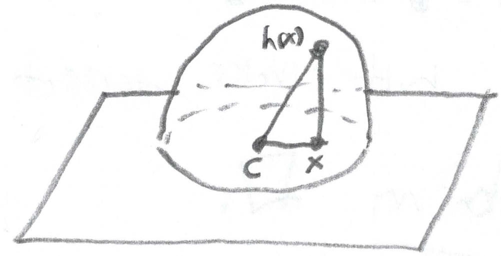
4th dim!

$$S = \{ (x, h(x)) \in \mathbb{R}^3 \times \mathbb{R} \mid x \in D \}$$

graph ~~of~~

$h : D \to \mathbb{R}$

$h(x) := \sqrt{r^2 - |x-c|^2}$



$|x-c|^2 + |h(x) - 0|^2 = r^2$

eqn of sphere / pythag thm

projecting to 3-space
$(x,t) \mapsto x$ leaves us with D.

Claim: If two disjoint circles C, C' are not linked, their domes S, S' do not intersect.

Pf: Show: if S and S' are intersects, C and C' intersect.

Let $(x_0, t_0) \in S \cap S'$.

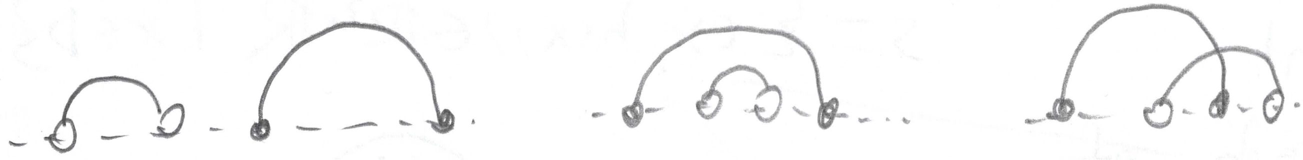
$(x_0, t_0) \in S \Rightarrow x_0 \in D$

$(x_0, t_0) \in S' \Rightarrow x_0 \in D'$.

Hence x_0 lies in L and on $D \cap D'$ with "lifting functions" h and h'.

Restricting to L, L h and h' give perfect half-circles (~~that~~ does not work with ellipses!)

Since the half circles above $D \cap L$ & $D' \cap L$ intersect, their endpoints on L alternate:



half circles intersect iff end pts alternate.

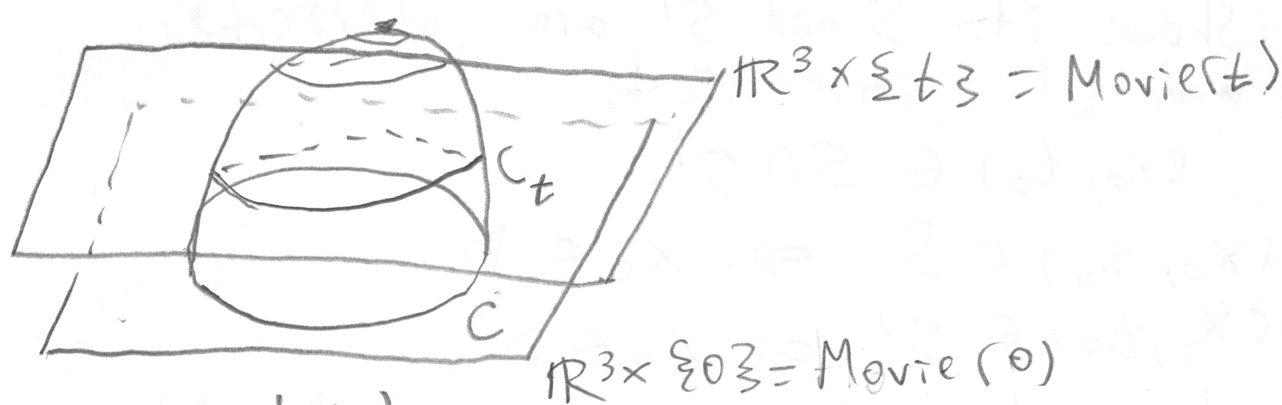
$\rightsquigarrow$ Claim $\square$.

Domes let us create a movie separating the circles if we consider the 4th coordinate as time.

At $t=0$, we can identify $(\mathbb{R}^3, L)$ with $\mathbb{R}^3 \times \{0\}, L$

Now we continuously increase t;

$\rightsquigarrow$ each circle shrinks to a point and disappears.

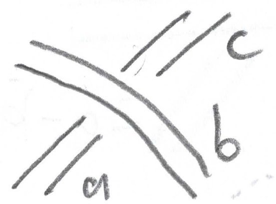


$\rightsquigarrow$ when circles shrinks, center c and plane H do not change.

By disjoint dome claim, they stay disjoint and thus pairwise non-linked.

$\rightsquigarrow$ stop shrinking when circle is small enough to not intersect other's plane. | · finish when all disks are disjoint

2. For $n \geq 2$ a Fox n-labeling of a link diagram labels each arc by an integer mod n s.t. for each crossing we have the crossing relation:



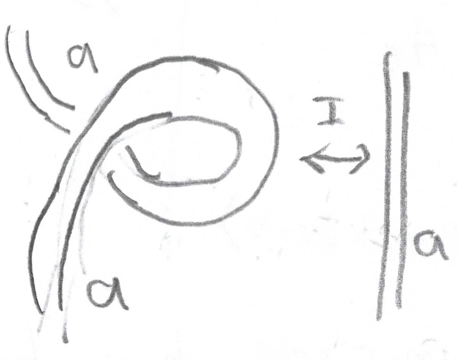
$$a + c \equiv 2b \pmod{n}.$$

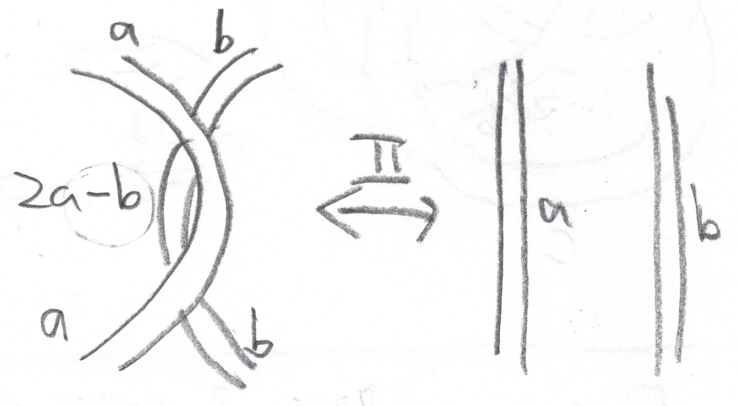
$\rightsquigarrow$ n "trivial" n-labelings (all same label)

Example: a link with two far away components has at least n^2-labelings.

<u>Claim</u> If two diagrams represent equivalent links, they have the same number of n-labelings.

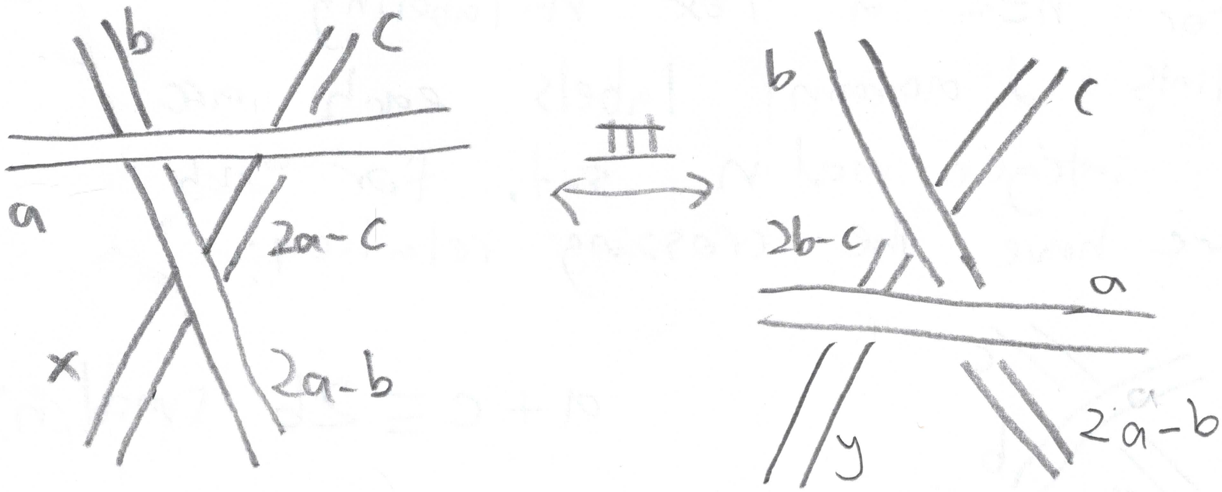
Pf: Check Reidemeister moves!
Relations are forced





$a + b \equiv 2a$

$\Rightarrow b \equiv a$



x and y are forced:

$x \equiv 2(2a-b)-(2a-c) \equiv 2a-2b+c$ before move

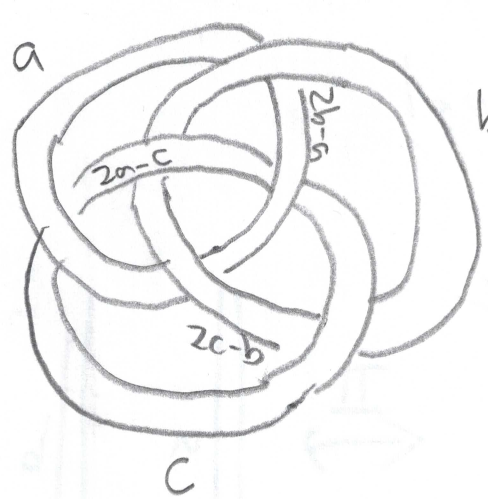
$y \equiv 2a-(2b-c) \equiv 2a-2b+c$ after move

Claim ✓

3.

Exercise: ◯◯◯ has n^3 n-labelings

For Borromean rings all labelings are trivial if $n \geq 3$ odd.



$2(2b-a) \equiv (2a-c)+c$

$2(2c-b) \equiv (2b-a)+a$

$2(2a-c) \equiv (2c-b)+b$

$\Longleftrightarrow$ $4a \equiv 4b \equiv 4c$

$\xRightarrow{n \text{ odd}}$ $a \equiv b \equiv c$

$(\equiv 2a-c \equiv 2b-a \equiv 2c-b)$

27, 125

3, 5

Open: • Does 3 copies of any non-circle shape have Borromean rings?

• Can we make a movie with constant circle size?