HOMOLOGICAL MIRROR SYMMETRY FOR NODAL STACKY CURVES

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Abstract. In this paper, we generalise the results of [35] to establish homological mirror symmetry for Deligne–Mumford stacks whose coarse moduli spaces are rings or chains of rational curves joined nodally, and where each irreducible component is allowed to have a generic stabiliser. The mirrors are $\mathbb{Z}$-graded open Riemann surfaces. As an application, we prove the Lekili–Ueda conjecture for invertible polynomials in the case of curves where the grading group on the B–side is allowed to be non-maximal.

1. Introduction

Whilst the topology of Riemann surfaces is very tractable and well-understood, the various flavours of Fukaya categories of such surfaces have rich and intricate structure. Correspondingly, homological mirror symmetry in this dimension has been an active and fruitful area of research in recent years. This has not only lead to new instances of homological mirror symmetry, but also interesting links to the representation theory of finite dimensional algebras.

Let Σ be a surface with non-empty boundary and choose Λ a collection of points on its boundary, called stops. Then, there exists the *partially wrapped Fukaya category*, which we denote by $\mathcal{W}(\Sigma;\Lambda)$ ([5],[50]). This gives a categorical resolution (smooth and proper) of the Fukaya category of Σ, $\mathcal{F}(\Sigma)$, and yields a sequence of categories

$$\mathcal{F}(\Sigma) \to \mathcal{W}(\Sigma;\Lambda) \to \mathcal{W}(\Sigma). \tag{1}$$

The last category in this sequence is the (fully) wrapped Fukaya category of the surface, and can be considered as the partially wrapped Fukaya category when $\Lambda = \emptyset$. The first functor above is full and faithful, and the second is given by localisation at the collection of Lagrangians supported near the stops.

Of particular interest to us is the seminal work of Haiden, Kontsevich, and Katzarkov, [27], who, amongst other things, give a combinatorial construction for the partially wrapped Fukaya category of a surface with a particular collection of stops. As part of the construction, the authors show that there exists a generating collection of Lagrangians of the partially wrapped Fukaya category whose endomorphism algebra is *formal* and *gentle*. This class of algebras has long been of interest to representation theorists ([4]), and this link with the symplectic geometry of surfaces provided a new tool in their study ([45], [2]). Converse to the construction in [27], it was shown in [36] how to construct a $\mathbb{Z}$-graded smooth surface with boundary, with a prescribed configuration of stops on this boundary, from a homologically smooth, $\mathbb{Z}$-graded gentle algebra.

On the algebro-geometric side of the correspondence, these algebras arose independently (and chronologically prior) in the work of Burban and Drozd, [10], who construct a categorical resolution of the derived category of perfect complexes on certain curves. This category is given as the derived category of coherent modules of a non-commutative sheaf of algebras, which they call the *Auslander sheaf*, and denote by $\mathcal{A}_\mathcal{C}$. Their main result is that this category has a tilting object whose endomorphism algebra is gentle. Moreover, there is a sequence of categories

$$\operatorname{perf}\mathcal{C} \to D^b(\mathcal{A}_\mathcal{C} - \mathrm{mod}) \to D^b\operatorname{Coh}(\mathcal{C}), \tag{2}$$

where the first functor is again full and faithful, and the second is given by localisation. It is still an open problem to find a B–side analogue to the work of [36]. Namely, to start with a given homologically smooth, $\mathbb{Z}$-graded gentle algebra and construct a B–model whose corresponding category has a tilting object whose endomorphism algebra is precisely the algebra we started with. The present paper provides new examples of such geometric realisations of gentle algebras, although does not include any examples whose corresponding graded marked surface on the A–side can have odd winding numbers. In particular, every line field considered in this paper comes from the projectivisation of a vector field, and so must have even winding number (cf. Section 4.1). For example, the gentle algebra considered in [11, Example 7.4] cannot arise as the endomorphism algebra of a tilting object for a curve considered here.

The fact that gentle algebras arise as the endomorphism algebras of generating objects on the A– and B–sides of homological mirror symmetry was first utilised in [35]. Here, the authors establish the conjecture in the case where $\mathcal{C}$ is a ring or chain of projective lines with n irreducible components, but where the nodes, and endpoints in the chain case, are allowed to be orbifold points. The irreducible components of these curves are examples of weighted projective lines in the sense of [23], and are referred to as balloons in [35]. The mirror surface is constructed by first considering the mirrors to the irreducible components, a cylinder with stops on its boundary, and then gluing these cylinders together in a way which is mirror to the stacky structure of the node where two irreducible components meet. The strategy of proof is to establish a derived equivalence between the categorical resolutions on the A– and B–sides by matching the corresponding gentle algebras. One can then establish HMS by matching localising subcategories under this equivalence, and then localising.

In this paper, we build on the strategy of [35] by allowing the irreducible components of the curves being considered to have non-trivial generic stabiliser. Such curves arise naturally in mirror symmetry as the B–model of invertible polynomials in two variables. In what follows, we have that $\mathbb{P}_{r_{i,-},r_{i,+}}$ is the orbifold $\mathbb{P}^1$ with orbifold points $(q_{i,-}, q_{i,+})$ such that $\mathrm{Aut}\ q_{i,-} \simeq \mu_{r_{i,-}}$ and $\mathrm{Aut}\ q_{i,+} \simeq \mu_{r_{i,+}}$ (in the chain case we are allowing $r_{1,-} = 0$, and/ or $r_{n,+} = 0$, so that the corresponding irreducible component is a (stacky) $\mathbb{A}^1$).

Theorem 1. *Let $\mathcal{C}$ be the Deligne–Mumford stack such that:*

- *The coarse moduli space of $\mathcal{C}$ is a ring or chain of n $\mathbb{P}^1$'s.*
- *Each irreducible component, $\mathcal{C}_i$, has underlying orbifold $\mathbb{P}_{r_{i,-},r_{i,+}}$ and generic stabiliser μ_{d_i} such that $r_{i,+}d_i = r_{i+1,-}d_{i+1}$ (we allow $r_{1,-}$ and/ or $r_{n,+} = 0$ in the case of a chain of curves).*
- *The node $q_i := |\mathcal{C}_i| \cap |\mathcal{C}_{i+1}|$ has isotropy group H_i and is presented as the quotient of $\mathrm{Spec}\, \mathbb{C}[x, y]/(xy)$ by H_i, where the action is given by*
$$h \cdot (x, y) = (\psi_{i,+}(h)x, \psi_{i+1,-}(h)y)$$
for some surjective $\psi_{i,+} : H_i \to \mu_{r_{i,+}}$ and $\psi_{i+1,-} : H_i \to \mu_{r_{i+1,-}}$.

Then
$$D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod}) \simeq \mathcal{W}(\Sigma; \Lambda)$$
is a quasi-equivalence of $\mathbb{Z}$-graded pre-triangulated A_∞-categories over $\mathbb{C}$, where Σ is a $\mathbb{Z}$-graded, b-punctured surface of genus g such that the genus, boundary components, and collection of stops, Λ, are determined by the $r_{i,\pm}$, d_i, and the local presentation of the nodes as the quotient by H_i.

Unlike in the orbifold case considered in [35], there is no canonical identification of the isotropy groups at the nodes. Even if one fixes $\psi_{i+1,-}$, it is possible to change the identification of H_i by an automorphism which pushes down to the identity by $\psi_{i+1,-}$, and this is an equivalent presentation of the node. The source of this non-uniqueness is that the generic stabilisers of the irreducible components are, strictly speaking, *torsors* for μ_{d_i}. The fact that there is no canonical identification of H_i can then be explained by the fact that there is no canonical identification of the generic stabiliser groups with μ_{d_i}. In order to work concretely with groups, a key ingredient in our argument is to choose a *gerbe structure* on the irreducible components, which one can heuristically think of as a 'principal $B\mu_{d_i}$-bundle' over $\mathbb{P}_{r_{i,-},r_{i,+}}$. This allows us to keep track of generic stabilisers, although it should be emphasised that our results are independent of the choice of specific gerbe structures. Gerbes have a long history in algebraic geometry, and were originally introduced by Giraud in the study of non-abelian cohomology [24]. Of particular interest to us is the *root stack* construction of Cadman and Abramovich, Graber, Vistoli ([12], [1]), as well as the toric Deligne–Mumford stack perspective provided in [9] and [18].

Remark 1.1. Note that our presentation agrees with the orbifold case when each $d_i = 1$ by observing that one can always arrange the action of $H_i \simeq \mu_{r_i}$ to be such that $\psi_{i+1,-} = \mathrm{id}$, and $\psi_{i,+} : \mu_{r_i} \xrightarrow{\wedge^{\kappa_i}} \mu_{r_i}$ for some $\kappa_i \in (\mathbb{Z}/r_i)^\times$.

Following [35], when referring to a specific configuration of points on the b boundary components of Σ, we will denote the partially wrapped Fukaya category by $\mathcal{W}(\Sigma; m_1, m_2, \ldots, m_b)$, where m_i is the number of stops on the i^{th} boundary component. When there are d boundary components with m stops, we shall notate this as $(m)^d$.

As part of the equivalence of Theorem 1, the respective localising subcategories are identified with each other. Moreover, one can match the characterisation of the category of perfect complexes under the inclusion (2) with the characterisation of the Fukaya category under the inclusion (1). This yields:

Theorem 2. *Let $\mathcal{C}$ and Σ be as in Theorem 1. Then*

$$\operatorname{perf} \mathcal{C} \simeq \mathcal{F}(\Sigma)$$
$$D^b \operatorname{Coh} \mathcal{C} \simeq \mathcal{W}(\Sigma),$$

are quasi-equivalences of $\mathbb{Z}$-graded pre-triangulated A_∞-categories over $\mathbb{C}$ in the case of a ring of curves. In the case of a chain of curves, there are quasi-equivalences of pre-triangulated A_∞-categories over $\mathbb{C}$

$$\operatorname{perf}_c \mathcal{C} \simeq \mathcal{F}(\Sigma; (r_{1,-})^{d_1}, (0)^{b-d_1-d_n}, (r_{n,+})^{d_n})$$
$$D^b \operatorname{Coh}(\mathcal{C}) \simeq \mathcal{W}(\Sigma; (r_{1,-})^{d_1}, (0)^{b-d_1-d_n}, (r_{n,+})^{d_n}),$$

where $\operatorname{perf}_c \mathcal{C}$ is the full subcategory of $\operatorname{perf} \mathcal{C}$ consisting of objects with proper support.

It should be emphasised that the choice of grading on the surface in the above theorems is a crucial piece of data. Changing it would change the grading of the endomorphism algebra of the generating Lagrangians, and, in general, would not yield a derived equivalent algebra. Moreover, taking $\operatorname{perf}_c \mathcal{C}$ in the case of a ring of curves is only necessary when $r_{1,-}$ and/ or $r_{n,+} = 0$. The category $\mathcal{F}(\Sigma; m_1, m_2, \ldots, m_n)$ is the *infinitesimally wrapped Fukaya category* of [42] (cf. [22]).

1.1. **Invertible polynomials.** The primary motivation for generalising the approach of [35] to allow for the irreducible components to have non-trivial generic stabiliser comes from invertible polynomials in two variables. To define an invertible polynomial, consider an $n \times n$ matrix A with non-negative integer entries a_{ij}. From this, we can define a polynomial $\mathbf{w} \in \mathbb{C}[x_1, \ldots, x_n]$ given by

$$\mathbf{w}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}.$$

If $\mathbf{w}$ is quasi-homogeneous, we can associate to it a *weight system* $(d_0, d_1, \ldots, d_n; h)$, where

$$\mathbf{w}(t^{d_1} x_1, \ldots, t^{d_n} x_n) = t^h \mathbf{w}(x_1, \ldots, x_n),$$

and $d_0 := h - d_1 - \cdots - d_n$. In [8], the authors define the transpose of $\mathbf{w}$, denoted by $\check{\mathbf{w}}$, to be the polynomial associated to A^T,

$$\check{\mathbf{w}}(\check{x}_1, \ldots, \check{x}_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} \check{x}_j^{a_{ji}},$$

and we call this the *Berglund–Hübsch transpose*. One can associate a weight system for $\check{\mathbf{w}}$, denoted by $(\check{d}_0, \check{d}_1, \dots, \check{d}_n; \check{h})$, in the same way as for $\mathbf{w}$. We call a polynomial $\mathbf{w}$ *invertible* if the matrix A is invertible over $\mathbb{Q}$, and if both $\mathbf{w}$ and $\check{\mathbf{w}}$ define isolated singularities at the origin.

A corollary of Kreuzer–Skarke's classification of quasi-homogeneous polynomials, [33], is that any invertible polynomial can be decoupled into the Thom–Sebastiani sum of *atomic* polynomials of the following three types:

- Fermat: $\mathbf{w} = x_1^{p_1}$,
- Loop: $\mathbf{w} = x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_n^{p_n} x_1$,
- Chain: $\mathbf{w} = x_1^{p_1} x_2 + x_2^{p_2} x_3 + \cdots + x_{n-1}^{p_{n-1}} x_n + x_n^{p_n}$.

The Thom–Sebastiani sums of polynomials of Fermat type are also called Brieskorn–Pham.

Remark 1.2. In this paper, we will use the word 'chain' to refer to both a chain of curves, as well as a curve corresponding to an invertible polynomial of chain type (which is in fact a ring of curves with two irreducible components). We hope that this distinction is clear from context.

To any invertible polynomial, one can associate its *maximal symmetry group*

$$\Gamma_{\mathbf{w}} := \{(t_1, \dots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} |\ \mathbf{w}(t_1 x_1, \dots, t_n x_n) = t_{n+1} \mathbf{w}(x_1, \dots, x_n)\}.$$

In general, this is a finite extension of $\mathbb{C}^*$, and is the group of diagonal transformations of $\mathbb{A}^n$ which keep $\mathbf{w}$ semi-invariant with respect to the character $(t_1, \dots, t_{n+1}) \mapsto t_{n+1}$. One can also consider certain admissible subgroups of finite index, $\Gamma \subseteq \Gamma_{\mathbf{w}}$ (see Section 7.1), and correspondingly one must then incorporate a dual group, $\check{\Gamma}$, into the A–side of the correspondence. This generalisation was studied by Krawitz ([32]), and homological Berglund–Hübsch–Krawitz mirror symmetry predicts:

Conjecture 1. *For any invertible polynomial* $\mathbf{w}$ *with admissible symmetry group* $\Gamma \subseteq \Gamma_{\mathbf{w}}$ *and corresponding dual group* $\check{\Gamma}$*, there is a quasi-equivalence*

$$\mathrm{mf}(\mathbb{A}^n, \Gamma, \mathbf{w}) \simeq \mathcal{FS}(\check{\mathbf{w}}, \check{\Gamma})$$

of pre-triangulated A_∞*-categories over* $\mathbb{C}$*.*

In the above, $\mathrm{mf}(\mathbb{A}^n, \Gamma, \mathbf{w})$ is the dg-category of Γ-equivariant matrix factorisations of $\mathbf{w}$ on $\mathbb{A}^n$, and $\mathcal{FS}(\check{\mathbf{w}}, \check{\Gamma})$ is the orbifold Fukaya–Seidel category of $(\mathbf{w}, \check{\Gamma})$. Presently, there does not exist a definition of such a category in full generality, although the work of [13] gives a definition in the context of invertible polynomials. In the maximally graded case where $\Gamma = \Gamma_{\mathbf{w}}$, this is known as homological Berglund–Hübsch mirror symmetry, and goes back to [51], [52]. Recently, there have been many results in the direction of establishing this conjecture. In the maximally graded case it has been proven in several cases – in particular, for Brieskorn–Pham polynomials in any number of variables in [19], and for Thom–Sebastiani sums of polynomials of type A and D in [20]. The conjecture is also established for all invertible polynomials in two variables in [26]. For each class of invertible polynomial, Kravets ([31]) establishes a full, strong, exceptional collection for $\mathrm{mf}(\mathbb{A}^n, \Gamma_{\mathbf{w}}, \mathbf{w})$ with $n \leq 3$. In the case of chain polynomials in any number of variables, Hirano and Ouchi ([28]) show that the category $\mathrm{mf}(\mathbb{A}^n, \Gamma_{\mathbf{w}}, \mathbf{w})$ has a full exceptional collection whose length is the Milnor number of $\check{\mathbf{w}}$. The full conjecture in the $\mathbb{Z}/2$-graded case of two variable invertible polynomials was established by Cho, Choa, and Jeong in [13]. Recently, progress on the conjecture was made in the case of maximally graded chain polynomials in [47], where the authors demonstrate that the exceptional collection in the category of matrix factorisations constructed in [3] satisfies a recursion relation with respect to the number of variables.

By the definition of $\check{\Gamma}$ (see Section 7.1), we have that it is a subgroup of $(\mathbb{C}^*)^n$ which keeps $\check{\mathbf{w}}$ *invariant*, and so, in particular, preserves its fibres when considering it as a map $\check{\mathbf{w}} : \mathbb{C}^n \to \mathbb{C}$. By considering the Milnor fibre to be $\check{\mathbf{w}}^{-1}(1) = \check{V}$, it therefore makes sense to define the equivariant Milnor fibre as the quotient stack $[\check{V}/\check{\Gamma}]$, although it should be emphasised that symplectic techniques in this setting are still in their infancy. Nevertheless, once an appropriate definition of the wrapped Fukaya category for such a quotient stack is made sense of, one expects:

Conjecture 2 ([37, Conjecture 1.4])**.** *For any invertible polynomial* $\mathbf{w}$ *with admissible symmetry group* $\Gamma \subseteq \Gamma_{\mathbf{w}}$ *and corresponding dual group* $\check{\Gamma}$*, there is a quasi equivalence*

$$\mathcal{W}([\check{V}/\check{\Gamma}]) \simeq \mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w} + x_0 x_1 \dots x_n).$$

of pre-triangulated A_∞*-categories over* $\mathbb{C}$*.*

Here, $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w} + x_0 x_1 \dots x_n)$ is the dg-category of Γ–equivariant matrix factorisations of $\mathbf{w} + x_0 \dots x_n$ on $\mathbb{A}^{n+1}$, where the action of Γ has been extended to $\mathbb{A}^{n+1}$ in a prescribed way ([37, Section 2]). In the maximally graded case, this conjecture was recently established in the case of $n \geq 3$ for all simple singularities in [38], and the case of Brieskorn–Pham polynomials of the form $x_1^2 + x_2^2 + \mathbf{w}$ in [39]. A $\mathbb{Z}/2$-graded equivalence was given for the Milnor fibre of any invertible polynomial in [21].

There is a trichotomy of cases depending on d_0, and in the case of $d_0 > 0$ (log general type), a generalisation of Orlov's theorem ([46, Theorem 3.11]) gives an equivalence

$$\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w} + x_0 x_1 \dots x_n) \simeq D^b \operatorname{Coh}(Z_{\mathbf{w},\Gamma}), \tag{3}$$

where

$$Z_{\mathbf{w},\Gamma} := \big[\big(\operatorname{Spec} \mathbb{C}[x_0, x_1, \dots, x_n]/(\mathbf{w} + x_0 x_1 \dots x_n) \setminus (\mathbf{0})\big)/\Gamma\big]. \tag{4}$$

The generalisation to the case where $\Gamma_{\mathbf{w}}$ is a finite extension of $\mathbb{C}^*$ is straightforward, and the extension to the setting of dg-categories was studied in [49], [30], [16]. In two variables every invertible polynomial is of log general type except for $x^2 + y^2$. This, however, corresponds to the well-understood HMS statement for $\mathbb{C}^*$.

Recall that the subcategory $\operatorname{perf} Z_{\mathbf{w},\Gamma} \subseteq D^b \operatorname{Coh}(Z_{\mathbf{w},\Gamma})$ is given by the objects which are Ext-finite, since $Z_{\mathbf{w},\Gamma}$ is a proper stack. On the symplectic side, it is clear that compact Lagrangians are Ext-finite; however, it is not known in general if non-compact Lagrangians are necessarily not. This is reasonable to expect though, and is certainly the case in all known circumstances. In the equivariant setting, it makes sense that the same statement should be true, since morphisms should be given by the $\check{\Gamma}$-invariant morphisms of the standard morphisms. Therefore, Conjecture 2 in the log general type case would imply an equivalence

$$\mathcal{F}([\check{V}/\check{\Gamma}]) \simeq \operatorname{perf} Z_{\mathbf{w},\Gamma}. \tag{5}$$

In the maximally graded case, the first instance of this was given in [34] for $x_1^3 + x_2^2$. The equivalence was subsequently establish in the cases of $\mathbf{w} = \sum_{i=1}^n x_i^{n+1}$ and $\mathbf{w} = x_1^2 + \sum_{i=2}^n x_i^{2n}$, both for any $n > 1$, in [37], and for all invertible polynomials in two variables in [25]. As an application of our main theorems, we show that Conjecture 2 and (5) hold for $n = 2$.

Theorem 3. *Let* $\mathbf{w}$ *be an invertible polynomial in two variables with admissible symmetry group* $\Gamma \subseteq \Gamma_{\mathbf{w}}$ *and corresponding dual group* $\check{\Gamma}$*. Then, the action of* $\check{\Gamma}$ *on* $\check{V}$ *is free, and there are quasi-equivalences*

$$\mathcal{F}(\check{V}/\check{\Gamma}) \simeq \operatorname{perf} Z_{\mathbf{w},\Gamma}$$
$$\mathcal{W}(\check{V}/\check{\Gamma}) \simeq D^b \mathrm{Coh}(Z_{\mathbf{w},\Gamma})$$

of $\mathbb{Z}$*-graded pre-triangulated* A_∞*-categories over* $\mathbb{C}$*.*

Remark 1.3. It should be reiterated that, although there is a trichotomy of cases depending on the weight d_0, all but one invertible polynomials in two variables are of log general type, and this exception is well-understood. We are therefore free to state Theorem 3 in the context of invertible polynomials of log general type without further assumptions.

1.2. **Structure of paper.** In Section 2, we recall the basic constructions of root stacks, both with and without section. In Section 3, we review the theory of Auslander orders over nodal (stacky) curves. In Section 4, we recall the construction of [27] of the partially wrapped Fukaya category. Sections 5 exposits the localisation argument on the A– and B–sides with the necessary alterations to our setting before proving Theorem 1. Section 6 characterises the category of perfect complexes on the B–side and the Fukaya category on the A–side before establishing Theorem 2. We provide applications in Section 7 and give first an example which does not arise as the Milnor fibre of an invertible polynomial before establishing Theorem 3.

1.3. **Conventions.** We work over $\mathbb{C}$ throughout. For a Deligne–Mumford (DM) stack $\mathcal{X}$ we write $x \in \mathcal{X}$ to mean $x : \operatorname{Spec}\mathbb{C} \to \mathcal{X}$, and let $|\mathcal{X}|$ be its underlying topological space. We define $\mathbb{G}_m$ to be the sheaf of invertible sections of $\mathcal{O}_{\mathcal{X}}$, and denote the bounded derived category of coherent sheaves, its full subcategory of perfect complexes, considered as pretriangulated dg-categories, as $D^b\operatorname{Coh}(\mathcal{X})$ and $\operatorname{perf}\mathcal{X}$, respectively. We refer to a DM stack with trivial generic stabiliser as an orbifold. For a sheaf of algebras $\mathcal{A}$, we denote the bounded derived category of finitely generated left modules, considered as a pretriangulated dg-category, as $D^b(\mathcal{A}-\mathrm{mod})$. All Fukaya categories are completed with respect to cones and direct summands.

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2. Root stacks

The notion of a root stack was introduced independently in [12] and [1]. There are two related notions of a root stack – the first is a way to 'insert stackiness' along an effective Cartier divisor, and the second defines a gerbe structure, which 'inserts stackiness' everywhere, and also keeps track of the generic stabiliser. In this section, we aim to give a self-contained account of the relevant theory in the context of how it will later be used.

Recall that the stack $[\mathbb{A}^1/\mathbb{C}^*]$ is the classifying stack of line bundles with section – this can be seen by considering a morphism to this stack as a principal $\mathbb{C}^*$-bundle with a global section of the corresponding associated line bundle. To define the root stack of a line bundle with section, consider X a scheme, $\mathscr{L}$ an invertible sheaf on X, $s \in \Gamma(X, \mathscr{L})$ a global section, and $r > 0$ an integer. Moreover, let $\theta_r : \left[\mathbb{A}^1/\mathbb{C}^*\right] \to \left[\mathbb{A}^1/\mathbb{C}^*\right]$ be the r^{th} power map on both $\mathbb{A}^1$ and $\mathbb{C}^*$.

Definition 1 ([12, Definition 2.2.1], [1, Appendix B.2]). *Define the stack $X_{(\mathscr{L},s,r)}$ to be the fibre product*

$$\begin{tikzcd} X_{(\mathscr{L},s,r)} \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \left[\mathbb{A}^1/\mathbb{C}^*\right] \arrow[d, "\theta_r"] \\ X \arrow[r, "(\mathscr{L},s)"] & \left[\mathbb{A}^1/\mathbb{C}^*\right] \end{tikzcd}$$

This is a Deligne–Mumford stack ([12, Theorem 2.3.3]), and is isomorphic to X away from the divisor $s^{-1}(0)$. By construction, $X_{(\mathscr{L},s,r)}$ comes with a line bundle N and a section $t \in \Gamma(X_{(\mathscr{L},s,r)}, N)$ such that $\varphi : N^{\otimes r} \xrightarrow{\sim} \mathrm{pr}_1^*\mathscr{L}$, and $\varphi(t^r) = \mathrm{pr}_1^* s$.

Concretely, we have that for X a scheme, an object of $X_{(\mathscr{L},s,r)}$ over a scheme S consists of a quadruple

$$(f, N, t, \varphi),$$

where $f : S \to X$ is a morphism, N is an invertible sheaf on S, $t \in \Gamma(S, N)$, and $\varphi : N^{\otimes r} \xrightarrow{\sim} f^*\mathscr{L}$ is an isomorphism such that $\varphi(t^r) = f^*s$. A morphism from $(f_1, N_1, t_1, \varphi_1)$ over S_1 to $(f_2, N_2, t_2, \varphi_2)$ over S_2 is a pair (h, ρ), where $h : S_1 \to S_2$ is a morphism of schemes such that $f_2 \circ f = f_1$, and $\rho : N_1 \xrightarrow{\sim} h^*N_2$ is an isomorphism such that

$$\begin{tikzcd} N_1^{\otimes r} \arrow[r, "\rho^r"] \arrow[d, "\varphi_1"] & h^*N_2^{\otimes r} \arrow[d, "h^*\varphi_2"] \\ f_1^*L \arrow[r, "\sim"] & h^*f_2^*L \end{tikzcd}$$

commutes, and the bottom isomorphism is canonical. The construction can be generalised to when X is a Deligne–Mumford stack.

For an effective Cartier divisor, we will also use the notation $X_{(D,r)}$ to mean $X_{(\mathcal{O}_X(D),1_D,r)}$, where 1_D is the tautological section vanishing along D. One can iterate this root construction, and for $\mathbb{D} = (D_1, \ldots, D_n)$ and $\vec{r} = (r_1, \ldots, r_n)$, we define $X_{\mathbb{D},\vec{r}}$ to be the root stack defined by iteratively applying the above construction. This is equivalent to the fibre product

$$\begin{tikzcd} X_{\mathbb{D},\vec{r}} \arrow[r] \arrow[d] & \left[\mathbb{A}^n/(\mathbb{C}^*)^n\right] \arrow[d, "\theta_{\vec{r}}"] \\ X \arrow[r] & \left[\mathbb{A}^n/(\mathbb{C}^*)^n\right], \end{tikzcd}$$

where $\theta_{\vec{r}} = \theta_{r_1} \times \theta_{r_2} \times \cdots \times \theta_{r_n}$, and $X \to \left[\mathbb{A}^n/(\mathbb{C}^*)^n\right]$ is given by the product of $(\mathcal{O}_X(D_i), 1_{D_i})_{i=1}^n$.

An important example for us will be the following:

Example 2.1 ([12, Lemma 2.3.1]). For $X = \mathbb{A}^1$, and $D = [0]$, there is an equivalence of categories $X_{(D,r)} \simeq \left[\mathbb{A}^1/\mu_r\right]$, where μ_r acts via its natural character.

In fact, Example 2.1 can be generalised ([12, Example 2.4.1], cf. [43, Theorem 10.3.10]) to any $X = \operatorname{Spec} A$ and $\mathscr{L} = \mathcal{O}_X$, with $s \in \Gamma(X, \mathcal{O}_X)$ such that $D = s^{-1}(0)$, yielding

$$X_{(D,r)} \simeq \left[\left(\operatorname{Spec} A[x]/(x^r - s)\right)/\mu_r\right],$$

where μ_r acts by $t \cdot x = t^{-1}x$, and $t \cdot a = ta$. In general, any root stack can be covered by such affine root stacks. For further exposition on root stacks of line bundles with section we refer to the original references [1], [12], as well as [43, Section 10.3].

The second flavour of root stack defines a gerbe over the original scheme (or stack), and we refer to [43, Chapter 12] for a definition and further discussion about generalities of gerbes. As already mentioned, one can think of a gerbe is a 'BG-bundle' over X for some group G, meaning that not only does the isotropy group of each point contain a copy of G, but the identification of this copy of G in the automorphism group of each point is a crucial part of the definition. In particular, an equivalence of gerbes is an equivalence of categories which is compatible with these identifications. Note that this means that two gerbes can be equivalent as stacks, but inequivalent as gerbes, in analogy with how two principal G-bundles can have diffeomorphic total spaces, but are not isomorphic G-bundles. For example, principal S^3 bundles over S^4 are classified by $\mathbb{Z} \oplus \mathbb{Z}$, and [15] establishes an explicit diffeomorphism between the total spaces of the bundles classified by $(1, 1)$ and $(2, 0)$. In what follows, we will restrict ourselves to the case at hand and only consider trivially banded gerbes, which are classified by $H^2(X, G)$.

Example 2.2. If one considers the topological setting, then a good example to have in mind is given by the observation that any principal S^1-bundle is in fact a $\mathbb{Z}$-gerbe, since $B\mathbb{Z} \simeq K(\mathbb{Z}, 1) \simeq S^1$. From this, we recover the usual classification of principal S^1-bundles as the cohomology class in $H^2(X, \mathbb{Z})$ corresponding to the Euler class.

To define a root stack of a line bundle (without section), consider $\mathscr{L} \in \operatorname{Pic} X$. Recall that such a line bundle is equivalent to a map $X \xrightarrow{\mathscr{L}} B\mathbb{C}^*$, and let $B\mathbb{C}^* \xrightarrow{\wedge^d} B\mathbb{C}^*$ be the d^{th} power map. Then, we have:

Definition 2 ([12, Definition 2.2.6], [1, Appendix B.1])**.** *The stack $X_{(\mathscr{L},d)}$ is defined to be the fibre product*

$$\begin{tikzcd} X_{(\mathscr{L},d)} \arrow[r, "pr_2"] \arrow[d, "pr_1"] & B\mathbb{C}^* \arrow[d, "\wedge^d"] \\ X \arrow[r, "\mathscr{L}"] & B\mathbb{C}^* \end{tikzcd}$$

The stack $X_{(\mathscr{L},d)}$ is a μ_d-gerbe over X, and, by construction, there is a line bundle $\mathscr{N} \in \operatorname{Pic} X_{(\mathscr{L},d)}$ such that

$$\mathscr{N}^{\otimes d} \simeq \mathrm{pr}_1^* \mathscr{L}.$$

Of course, there is also a corresponding iterated statement (see, for example [18, Proposition 6.9]), although we will not make use of it. We will mainly use the notation $X_{(\mathscr{L},d)} = \sqrt[d]{\mathscr{L}/X}$.

Perhaps a more geometric way to think of a root stack of a line bundle is given in [1, Appendix B.1]. Let $\mathscr{L}$ be a line bundle on a scheme X, and $\mathscr{L}^*$ be the total space minus the zero section (i.e. the principal $\mathbb{C}^*$-bundle associated to $\mathscr{L}$). Then,

$$\sqrt[d]{\mathscr{L}/X} = [\mathscr{L}^*/\mathbb{C}^*],$$

where $\mathbb{C}^*$ acts fibrewise with weight d. In particular, the usual description of the weighted projective stack $\mathbb{P}(d,d)$ is recovered as $\sqrt[d]{\mathcal{O}(-1)/\mathbb{P}^1}$, since $\mathcal{O}(-1)^* = \mathbb{A}^2 \setminus \{(0,0)\}$.

Remark 2.3. It should be noted that $X_{(\mathscr{L},d)}$ and $X_{(\mathscr{L},0,d)}$ are *not* equivalent. Indeed, as is demonstrated in [12, Example 2.4.3], the latter category is an infinitesimal thickening of the former.

The Kummer sequence

$$1 \to \mu_r \xrightarrow{\iota} \mathbb{G}_m \xrightarrow{\wedge^d} \mathbb{G}_m \to 1 \tag{6}$$

induces a long exact sequence on cohomology

$$\cdots \to H^1(X, \mathbb{G}_m) \xrightarrow{\partial} H^2(X, \mu_d) \xrightarrow{\iota_*} H^2(X, \mathbb{G}_m) \to \dots. \tag{7}$$

For a root stack $\sqrt[d]{\mathscr{L}/X}$, the corresponding class in $H^2(X, \mu_d)$ is the image of $\mathscr{L} \in H^1(X, \mathbb{G}_m) \simeq \operatorname{Pic} X$ under the connecting homomorphism. Conversely, a μ_d-gerbe is called *essentially trivial* if its corresponding class in $H^2(X, \mu_d)$ is in the image of the connecting homomorphism. In particular, in the case where $H^2(X, \mathbb{G}_m) = 0$, we make the identification

$$H^2(X, \mu_d) \simeq \operatorname{Pic} X / d \operatorname{Pic} X,$$

and so the cohomology class classifying the d^{th} root of $\mathscr{L}$ is given by the quotient of its corresponding class in the Picard group, namely its first Chern class. Moreover, in this case [18, Lemma 6.5] identifies $H^2(X, \mu_d) \simeq \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d, \operatorname{Pic} X)$, where a class $[\mathscr{L}] \in \operatorname{Pic} X / d \operatorname{Pic} X$ corresponds to the short exact sequence

$$0 \to \operatorname{Pic} X \to \operatorname{Pic} X \times_{\operatorname{Pic} X / d \operatorname{Pic} X} \mathbb{Z}/d \to \mathbb{Z}/d \to 0, \tag{8}$$

where the map $\operatorname{Pic} X \to \operatorname{Pic} X / d \operatorname{Pic} X$ is the projection, the map $\mathbb{Z}/d \to \operatorname{Pic} X / d \operatorname{Pic} X$ is given by $1 \mapsto [\mathscr{L}]$, and the first morphism of the extension is $\mathscr{L} \to (\mathscr{L}^{\otimes d}, 0)$.

For each μ_d-gerbe $\mathcal{X}$, there is an *underlying orbifold*. This is the stack which results from the stackification of the prestack whose objects are the same as the original stack, but whose isotropy groups are quotiented by μ_d. This process is known as *rigidification*, although we refer to Appendix C of [1] for the precise details. It suffices for us to observe that, in the case where the gerbe is the stack of roots of a line bundle on a scheme or orbifold, the map $\mathrm{pr}_1 : \sqrt[d]{\mathscr{L}/X} \to X$ is the rigidification map. In particular, for $\mathcal{X} = \sqrt[d]{\mathscr{L}/X}$ and D a Cartier divisor on X, by $\mathcal{O}_{\mathcal{X}}(D)$ we mean $\mathrm{pr}_1^*\mathcal{O}_X(D)$.

Example 2.4. The most basic example of a gerbe is given by considering $B\mu_d$ to be a μ_d-gerbe over a point.

Example 2.5. Consider an orbifold X and the trivial action of μ_d on X. Then the resulting quotient stack is given by $X \times B\mu_d$, and corresponds to the stack of d^{th} roots of $\mathcal{O}_X$, or indeed any line bundle on X whose d^{th} root exists in $\operatorname{Pic} X$.

Example 2.6. Consider the compactified moduli space of elliptic curves $\overline{\mathcal{M}}_{1,1} \simeq \mathbb{P}(4,6)$. This is a $\mathbb{Z}/2$-gerbe over $\mathbb{P}(2,3)$, where the $\mathbb{Z}/2$-torsor corresponds to the symmetry present in any lattice defining an elliptic curve. It can be constructed as the stack of square roots of any line bundle $\mathscr{L} \in \operatorname{Pic}\mathbb{P}(2,3) \simeq \mathbb{Z}$ such that $[\mathscr{L}] \in \operatorname{Pic}\mathbb{P}(2,3)/2\operatorname{Pic}\mathbb{P}(2,3) \simeq \mathbb{Z}/2$ is non-trivial. In this case, $\mathbb{P}(2,3)$ is the rigidification of the moduli space of elliptic curves.

Example 2.7. Consider the short exact sequence

$$1 \to K \to H \to G \to 1,$$

where K, H, and G are all finite abelian groups. Then $BH \to BG$ is a K-gerbe, so is classified by $H^2(BG,K) \simeq H^2(G,K)$, which recovers the usual classification of short exact sequences in terms of group cohomology.

Remark 2.8. Note that above, and in what follows, we are implicitly taking K to have the structure of a trivial G-module since we are only considering the case of trivially banded gerbes. For the remainder of the paper, we will only consider the cases where G and K are cyclic groups, and so we have $H^2(G,K) \simeq \operatorname{Ext}^1_{\mathbb{Z}}(G,K)$ by the universal coefficient theorem.

Example 2.9. In general, if $\mathcal{Y}$ is a K-gerbe over $\mathcal{X}$, and $x \in |\mathcal{X}|$ has isotropy group G_x, $x \in |\mathcal{Y}|$ has isotropy group H_x, then there is a short exact sequence

$$1 \to K \to H_x \to G_x \to 1.$$

We will exclusively deal with root stacks, both with and without section, over $\mathbb{P}^1$. To this end, consider $D_1 = [0] = q_-$ and $D_2 = [\infty] = q_+$ and $\vec{r} = (a,b)$. Then we define

$$\mathbb{P}_{a,b} := \mathbb{P}^1_{\mathbb{D},\vec{r}}$$

to be the weighted projective line with a stacky point of order a at q_- and of order b at q_+. Unless $\gcd(a,b) = 1$, then this is not a weighted projective space; however, if this is the case then we have

$$\mathbb{P}_{a,b} \simeq \mathbb{P}(a,b) := \left[\mathbb{A}^2 \setminus \{(0,0)\}/\mathbb{C}^*\right],$$

where $\mathbb{C}^*$ acts on $\mathbb{A}^2 \setminus \{(0,0)\}$ with weights a and b. In general, the space $\mathbb{P}_{a,b}$ can be realised as the quotient of $\mathbb{A}^2 \setminus \{(0,0)\}$ by $\mathbb{C}^* \times \mu_{\gcd(a,b)}$ ([18, Example 7.31]). Note that $H^2(\mathbb{P}_{a,b}, \mathbb{G}_m) = 0$, and so all gerbes whose underlying orbifold is $\mathbb{P}_{a,b}$ are essentially trivial.

Given a μ_d-gerbe over $\mathbb{P}_{a,b}$, $\mathcal{C}$, the structure of the gerbe at the points $q_\pm$ will be of central importance to us. Observe that there is a natural (surjective) map

$$H^2(\mathbb{P}_{a,b}, \mu_d) \to H^2([\mathbb{A}^1/\mu_a], \mu_d) \oplus H^2([\mathbb{A}^1/\mu_b], \mu_d) \tag{9}$$

which comes from the Mayer–Vietoris sequence, and this determines the Ext-class at $q_\pm$ which locally describes the gerbe. Explicitly, let $\mathcal{U}_- = [\mathbb{A}^1/\mu_a]$, suppose that $\mathcal{C} = \sqrt[d]{\mathcal{L}/\mathbb{P}_{a,b}}$, and that $\mathcal{L}|_{\mathcal{U}_-} \simeq \mathcal{O}_{\mathcal{U}_-}(nq_-)$ has class $\beta \in \operatorname{Pic}\mathcal{U}_- \simeq \mathbb{Z}/a$. Observe that $H^2([\mathbb{A}^1/\mu_a], \mu_d) \simeq \mathbb{Z}/\gcd(a,d)$, and that the reduction $\beta \bmod d$ yields an element $[\beta] \in \mathbb{Z}/\gcd(a,d)$ determining a short exact sequence

$$1 \to \mu_d \to H_- \to \mu_a \to 1, \tag{10}$$

classifying the gerbe on the patch $\mathcal{U}_-$, and corresponding to the d^{th} root of $\mathcal{O}_{\mathcal{U}_-}(nq_-)$. By construction, there exists a (not unique!) character χ_{d_-} of H_- such that H_- acts via $d\chi_{d_-}$ on the fibre of $\mathrm{pr}_1^*\mathcal{O}_{\mathcal{U}_-}(nq)$ at the origin, and which pulls back via the inclusion of μ_d to H_- to a unit in $\mathbb{Z}/d$. Therefore, as $\mathcal{N}|_{\mathcal{U}_-}$ we take the equivariant sheaf on $\mathbb{A}^1$ where H_- acts via χ_{d-} on the fibre at the origin. By construction, for any $\chi \in \widehat{H}_-$, there is a unique $k \in \{0, \dots, d-1\}$ and $j \in \{m, \dots, m+a-1\}$ such that H_- acts on the fibre of the sheaf

$$\mathrm{pr}_1^* \mathcal{O}_{\mathcal{U}_-}(jq) \otimes \mathcal{N}^{\otimes k} \tag{11}$$

at the origin with character χ. The local description of the gerbe on the patch $\mathcal{U}_+ = [\mathbb{A}^1/\mu_b]$ is analogous, giving the local description of the gerbe on the two patches of $\mathbb{P}_{a,b}$. Conversely, the description of a gerbe on $\mathbb{P}_{a,b}$ is given by the local description on $\mathcal{U}_\pm$, together with the information of how the two local descriptions get identified on the overlapping $\mathbb{C}^* = \mathcal{U}_+ \cap \mathcal{U}_-$.

There is a strong link between the derived categories of root stacks and the representation theory of finite dimensional algebras. If one takes $a = b = 1$, then this relationship is classical, and is Beĭlinson's result ([7]) that

$$D^b(\mathbb{P}^1) \simeq D^b(\Lambda^{\mathrm{op}} - \mathrm{mod}),$$

where Λ is the path algebra of the Kronecker quiver. This was generalised in [23] to the situation $\mathbb{P}^1_{\mathbb{D},\vec{r}}$, where $\mathbb{D}$ is a finite collection of disjoint points with multiplicity one, and $\vec{r}$ is a tuple of positive integers. In particular, for $\mathbb{D} = (q_-, q_+)$ and $\vec{r} = (a, b)$ as above, it was shown that

$$D^b(\mathbb{P}_{a,b}) \simeq D^b(\Lambda^{\mathrm{op}}_{a,b} - \mathrm{mod}),$$

where $\Lambda_{a,b}$ is the path algebra of the quiver

$$\begin{tikzcd} \mathcal{O}(-aq_-) \arrow[r, "x"] \arrow[d, equal] & \mathcal{O}(-(a-1)q_-) \arrow[r, "x"] & \dots \arrow[r, "x"] & \mathcal{O}(-q_-) \arrow[r, "x"] & \mathcal{O} \arrow[d, equal] \\ \mathcal{O}(-bq_+) \arrow[r, "y"] & \mathcal{O}(-(b-1)q_-) \arrow[r, "y"] & \dots \arrow[r, "y"] & \mathcal{O}(-q_+) \arrow[r, "y"] & \mathcal{O}. \end{tikzcd} \tag{12}$$

As for sheaves on the gerbes constructed as the root stacks over orbifold curves, consider $\mathcal{C} = \sqrt[d]{\mathscr{L}/\mathbb{P}_{a,b}}$ for some $\mathscr{L} \in \operatorname{Pic}\mathbb{P}_{a,b}$. There are natural full and faithful functors

$$\Phi_i : \operatorname{Coh}\mathbb{P}^1_{a,b} \to \operatorname{Coh}\mathcal{C}$$
$$\mathcal{F} \mapsto \mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{N}^{\otimes i},$$

where $\mathrm{pr}_1 : \mathcal{C} \to \mathbb{P}_{a,b}$ is again the rigidification map. Taking the direct sum yields a special case of [29, Theorem 1.5], giving an equivalence

$$\operatorname{Coh}\mathcal{C} \simeq \big(\operatorname{Coh}\mathbb{P}_{a,b}\big)^{\oplus d}. \tag{13}$$

Note that is not just semi-orthogonal, but also orthogonal, and that the equivalence is at the level of abelian categories. Therefore, the derived category of coherent sheaves on a gerbe over a weighted projective line only depends on the generic stabiliser group and the underlying weighted projective line.

It is essentially because of (13) that our results are independent of the precise choice of gerbe structure on irreducible components. To elaborate, consider $\mathcal{C}$ to be a chain of curves with two irreducible components which has isotropy group H at their intersection; the general case proceeds inductively. One can construct $\mathcal{C}$ as the pushout

$$\begin{tikzcd} & \mathcal{C}_1 \\ \mathcal{C}_2 & BH, \arrow[l, hook', "\varphi"'] \arrow[u, hook] \end{tikzcd} \tag{14}$$

where $\varphi : BH \to \mathcal{C}_2$ is the composition of the autoequivalence of BH induced from the action of H on the node, followed by its inclusion into $\mathcal{C}_2$. Since the abelian (and hence derived) categories of $\mathcal{C}_1$ and $\mathcal{C}_2$ are independent of gerbe structures by (13), the only information required to understand the category of coherent sheaves of $\mathcal{C}$ is the autoequivalence of BH, and this is independent of the gerbe structure chosen, as well as the characters $\chi_{d_{1,+}}$ and $\chi_{d_{2,-}}$.

2.1. **Root stacks and stacky fans.** The theory of toric Deligne–Mumford stacks was initiated in [9], and the relationship with gerbes and root stacks was explored in [18]. For a more in-depth account we refer to these original sources.

Analogously to a toric variety, which contains an open, dense, torus, T, a toric Deligne–Mumford stack is defined to be a smooth, separated Deligne–Mumford stack with an open immersion of a Deligne–Mumford torus, $T \times BG$ for G a finite abelian group, such that the action of $T \times BG$ on itself extends to the whole stack ([18, Section 3]). In the case of invertible polynomials, we work with root stacks over $\mathbb{P}^1$ on the B–side, although these curves are naturally presented as a hypersurface in a quotient stack. Therefore, in order to be able to apply our theory, we must demonstrate an equivalence between the irreducible components of the curves arising in invertible polynomials, and root stacks over $\mathbb{P}^1$. To this end, recall that the data of a stacky fan is given by a triple $\mathbf{\Sigma} = (\Sigma, N, \beta)$, where:

- N is a finitely generated abelian group (not necessarily torsion-free),
- Σ is a fan in $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ with n rays such that the rays span $N_{\mathbb{Q}}$, and
- $\beta : \mathbb{Z}^n \to N$ is a morphism of groups such that the image of the i^{th} basis vector of $\mathbb{Z}^n$ in $N_{\mathbb{Q}}$ is on the i^{th} ray.

For simplicity, we will always assume that Σ is complete. From this data, one can construct a toric DM stack in analogy with the Cox construction for toric varieties ([14]) as follows. Let d be the rank of N, and choose a projective resolution

$$0 \to \mathbb{Z}^{\ell} \xrightarrow{Q} \mathbb{Z}^{d+\ell} \to N \to 0.$$

Then, choose a map $B : \mathbb{Z}^n \to \mathbb{Z}^{d+\ell}$ which lifts β. The cone of β, considered as a morphism of complexes $[0 \to \mathbb{Z}^n] \to [0 \to \mathbb{Z}^{\ell} \xrightarrow{Q} \mathbb{Z}^{d+\ell} \to 0]$, is given by the complex

$$0 \to \mathbb{Z}^{n+\ell} \xrightarrow{[BQ]} \mathbb{Z}^{d+\ell} \to 0.$$

We define $\mathrm{DG}(\beta) := \operatorname{coker}([BQ]^{\vee})$, and define the map

$$\beta^{\vee} : (\mathbb{Z}^n)^{\vee} \to \mathrm{DG}(\beta)$$

by the composition $(\mathbb{Z}^n)^{\vee} \hookrightarrow (\mathbb{Z}^{n+\ell})^{\vee} \to \mathrm{DG}(\beta)$. We then have $Z_{\mathbf{\Sigma}} = \mathbb{A}^n \setminus \{\mathbf{0}\}$ (since Σ is complete) is the quasi-affine variety associated to the fan. By defining $G_{\mathbf{\Sigma}} = \operatorname{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$, we get a morphism $G_{\mathbf{\Sigma}} \to (\mathbb{C}^*)^n$, and this induces an action of $G_{\mathbf{\Sigma}}$ on $Z_{\mathbf{\Sigma}}$ via the natural action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$. The resulting stack $\mathcal{X}(\mathbf{\Sigma}) := [Z_{\mathbf{\Sigma}}/G_{\mathbf{\Sigma}}]$ is called the toric Deligne–Mumford stack associated to $\mathbf{\Sigma}$.

Example 2.10 ([9, Example 3.5]). Let Σ be the complete fan in $\mathbb{Q}$, and

$$\beta : \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/2 =: N.$$

Then one can check that

$$\beta^{\vee} : (\mathbb{Z}^2)^{\vee} \xrightarrow{\begin{pmatrix} 4 & 6 \end{pmatrix}} \mathrm{DG}(\beta) \simeq \mathbb{Z},$$

and so the $\mathbb{C}^*$ action on $Z_{\mathbf{\Sigma}} = \mathbb{A}^2 \setminus \{(0,0)\}$ is $t \cdot (x, y) = (t^4 x, t^6 y)$, yielding $\mathcal{X}(\mathbf{\Sigma}) \simeq \mathbb{P}(4,6) \simeq \overline{\mathcal{M}}_{1,1}$.

Given a stacky fan $\mathbf{\Sigma} = (\Sigma, \beta, N)$, one can associate its rigidification $\mathbf{\Sigma}^{\text{rig}} = (\Sigma, \beta^{\text{rig}}, N/N_{\text{tor}})$ by defining $\beta^{\text{rig}} : \mathbb{Z}^n \to N/N_{\text{tor}}$ to be the composition of β and the quotient morphism $N \to N/N_{\text{tor}}$. The stack $\mathcal{X}(\mathbf{\Sigma}^{\text{rig}})$ is the DM stack associated to this stacky fan, and, by construction, comes with the rigidification map $\mathcal{X}(\mathbf{\Sigma}) \to \mathcal{X}(\mathbf{\Sigma}^{\text{rig}})$ induced from the injective morphism $\mathrm{DG}(\beta^{\text{rig}}) \to \mathrm{DG}(\beta)$.

Closed substacks corresponding to cones of the fan are defined in [9, Section 4]. We will restrict ourselves to the case of rays (one-dimensional cones), and recall the basic construction here. Let ρ_i be a ray of Σ, e_i the positive generator of the i^{th} summand of $\mathbb{Z}^n$, N_{ρ_i} the subgroup of N generated by $\beta(e_i)$, and $N(\rho_i) = N/N_{\rho_i}$ the quotient. This defines a surjection $N_{\mathbb{Q}} \to N(\rho_i)_{\mathbb{Q}}$, and the quotient fan Σ/ρ_i in $N(\rho_i)_{\mathbb{Q}}$ is defined as the image of the cones in Σ containing ρ_i under this surjection. The link of ρ_i is defined as $\operatorname{link}(\rho_i) = \{\tau |\ \tau + \rho_i \in \Sigma, \text{ and } \rho_i \cap \tau = 0\}$. Let ℓ be the number of rays in $\operatorname{link}(\rho_i)$. We define the closed substack associated to ρ_i as the triple $\boldsymbol{\Sigma/\rho_i} = (\Sigma/\rho_i, N(\rho_i), \beta(\rho_i))$, where

$$\beta(\rho_i) : \mathbb{Z}^\ell \to N(\rho_i)$$

is defined as the composition $\mathbb{Z}^\ell \hookrightarrow \mathbb{Z}^n \xrightarrow{\beta} N \to N/N_{\rho_i} = N(\rho_i)$. In particular, the divisor $\mathcal{D}_{\rho_i}$ corresponding to the ray ρ_i is $\mathcal{X}(\boldsymbol{\Sigma/\rho_i})$.

Of most importance to us is the fact that if $\mathcal{X}$ is a toric DM stack whose coarse moduli space is $\mathbb{P}^1$ or $\mathbb{P}^2$, then (amongst other things) [18, Theorem II] shows that there exists a stacky fan whose corresponding quotient stack is $\mathcal{X}$. Moreover, in the case that $\mathcal{X}$ is an orbifold, this fan is unique. This is far from true in the case where N has torsion, as is demonstrated in [18, Example 7.29]. There are several sources of non-uniqueness, although in our situation it is essentially equivalent to the fact that it is possible to choose multiple lifts of an element in $\mathbb{Z}/n$ to $\mathbb{Z}$.

From now on, we will restrict ourselves the the case of toric Deligne–Mumford stacks whose coarse moduli space is given by $\mathbb{P}^1$ or $\mathbb{P}^2$. Let $\mathcal{C} = \mathcal{X}(\boldsymbol{\Sigma})$ be a toric Deligne–Mumford stack whose rigidification is $\mathbb{P}_{a,b}$. Then, [18, Proposition 7.20] shows that there is a unique class in $\operatorname{Ext}^1_{\mathbb{Z}}(N_{\text{tor}}, \operatorname{Pic} \mathbb{P}_{a,b})$ such that the $\operatorname{Hom}(N_{\text{tor}}, \mathbb{C}^*)-$banded gerbe over $\mathbb{P}_{a,b}$ associated to this class is equivalent to $\mathcal{C}$. The proof of this proposition is constructive, and it is straightforward to determine the short exact sequence (8) from the data of a stacky fan. The main ingredient, however, which we will use in our application to invertible polynomials is [18, Theorem 7.24], which shows (as a special case) that if Σ is the complete fan in $\mathbb{Q}$ and

$$\beta : \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} a & -b \\ n_- & n_+ \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/d =: N,$$

then

$$X(\boldsymbol{\Sigma}) \simeq \sqrt[d]{\mathscr{L}/\mathbb{P}_{a,b}}$$

as toric DM stacks, where $\mathscr{L} = \mathcal{O}(q_-)^{n_-} \otimes \mathcal{O}(q_+)^{n_+}$.

Remark 2.11. It is important to emphasise that two inequivalent gerbes can be equivalent as toric DM stacks. This happens when the corresponding Ext-classes are isomorphic as sequences, but inequivalent as extensions – see [18, Remark 7.23] and [6, Proposition 6.2]. In particular, the above application of [18, Theorem 7.24] only makes a claim about toric DM stacks. By [18, Proposition 7.20], one can check when this equivalence is also an equivalence of gerbes, although this will not be necessary for our purposes.

Example 2.12. (cf. [9, Example 3.6]) Let Σ be the complete fan in $\mathbb{Q}$, $N = \mathbb{Z} \oplus \mathbb{Z}/3$, and

$$\beta_n : \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ n & 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/3.$$

Since Σ is complete, we have $Z_{\boldsymbol{\Sigma}_n} = \mathbb{A}^2 \setminus \{(0,0)\}$ for any n. In the case of $n \bmod 3 = 0$, we have

$$\beta_0^\vee : (\mathbb{Z}^2)^\vee \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}} \operatorname{DG}(\beta) \simeq \mathbb{Z} \oplus \mathbb{Z}/3,$$

and so $G_{\mathbf{\Sigma}_0} \simeq \mathbb{C}^* \times \mu_3$, and $\mathcal{X}(\mathbf{\Sigma}_0) \simeq \mathbb{P}^1 \times B\mu_3$. In the case where $n \bmod 3 \neq 0$, we have

$$\beta_n^\vee : (\mathbb{Z}^2)^\vee \xrightarrow{\begin{pmatrix}3 & 3\end{pmatrix}} \mathrm{DG}(\beta) \simeq \mathbb{Z},$$

and so $G_{\mathbf{\Sigma}_n} \simeq \mathbb{C}^*$, and $\mathcal{X}(\mathbf{\Sigma}_n) \simeq \mathbb{P}(3,3)$ *as toric DM stacks* for any such n. However, the class in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z})$ corresponding to n is given by the sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/3 \to 0,$$

and so n_1 and n_2 *do not* define equivalent gerbes unless $n_1 \equiv n_2 \bmod 3$. Moreover, this shows

$$\mathcal{X}(\mathbf{\Sigma}_n) \simeq \sqrt[3]{\mathcal{O}(1)/\mathbb{P}^1} \quad \text{for } n \bmod 3 = 1, \text{ and}$$
$$\mathcal{X}(\mathbf{\Sigma}_n) \simeq \sqrt[3]{\mathcal{O}(2)/\mathbb{P}^1} \quad \text{for } n \bmod 3 = 2.$$

This also demonstrates the non-uniqueness of the fan in the case where the DM stack is not an orbifold – taking β_n for any $n \in \mathbb{Z}$ yields a gerbe which is equivalent to the 3^{rd} root of $\mathcal{O}(n \bmod 3)$.

3. Auslander orders

In this section, we give a brief account of the theory of Auslander orders, as introduced in [10] and expanded upon in [35], before constructing the relevant generalisation.

Roughly speaking, an order is a non-commutative algebra which is finite over its centre. In [10], the notion of an *Auslander order* was introduced in studying non-commutative resolutions of the subcategory consisting of perfect complexes of the derived category of coherent sheaves on certain curves. Such an order is a sheaf of algebras, and a categorical resolution of the category of perfect complexes of the underlying curve is given by the derived category of finitely generated left modules of this sheaf, as discussed in the introduction.

Before moving on to the situation we are focusing on, it is instructive to review the non-stacky case, as in [10]. Let C be a chain or ring of $\mathbb{P}^1$'s joined nodally, and $\pi : \widetilde{C} \to C$ its normalisation (i.e. a disjoint union of $\mathbb{P}^1$'s). Let $\mathcal{I}$ be the ideal sheaf of the singular locus, and define the sheaf of $\mathcal{O}_C$-algebras

$$\mathcal{F} := \begin{pmatrix} \mathcal{I} \\ \mathcal{O}_C \end{pmatrix}. \tag{15}$$

One can then define the *Auslander sheaf* as

$$\mathcal{A}_C = \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{F}) = \begin{pmatrix} \widetilde{\mathcal{O}}_C & \mathcal{I} \\ \widetilde{\mathcal{O}}_C & \mathcal{O}_C \end{pmatrix}, \tag{16}$$

where $\widetilde{\mathcal{O}}_C = \pi_* \mathcal{O}_{\widetilde{C}}$. In [10], the authors study the category of finitely generated left $\mathcal{A}_C$ modules on the ringed space $(C, \mathcal{A}_C)$. Their main result is that $D^b(\mathcal{A}_C - \mathrm{mod})$ has a tilting object, and is also a categorical resolution of $\mathrm{perf}\, C$. They also show that $D^b \operatorname{Coh}(C)$ is equivalent to the localisation of $D^b(\mathcal{A}_C - \mathrm{mod})$ by a certain subcategory of torsion modules, yielding the sequence (2).

Remark 3.1. In [10], the authors work with triangulated categories, however, these categories have unique dg-enhancements by the work of [40], and we work with these enhancements.

In [35], the authors build on the construction of [10] to allow for the nodes to have stacky structure, meaning that the irreducible components are orbifold curves of the form $\mathbb{P}_{a,b}$, and where two irreducible components meet at an orbifold point. We further extend this approach to allow for the irreducible components to have non-trivial generic stabiliser, although the arguments in [35] carry over to our situation with only minor alterations.

Let $\mathcal{C}$ be as in Theorem 1 and choose a compatible gerbe structure on each irreducible component, meaning that the local model about $q_{i,\pm}$ is compatible with the maps $\psi_{i,\pm}$. This can always be done by taking the root of a line bundle on $\mathbb{P}_{r_{i,-}, r_{i,+}}$ whose restriction under (9) yields short exact sequences compatible with the action of the isotropy group at the nodes. Two compatible gerbe structures on an irreducible component will differ by how the two patches are identified on overlaps, but by (13), this does not affect our theory. To ease notation, we let $\mathbb{P}_i = \mathbb{P}_{r_{i,-},r_{i,+}}$ be the rigidified i^{th} irreducible component of $\mathcal{C}$. Let

$$\pi : \widetilde{\mathcal{C}} = \bigsqcup_{i=1}^{n} \widetilde{\mathcal{C}}_i \to \mathcal{C}$$

be the normalisation map, and H_i the isotropy group at the node $q_i = |\mathcal{C}_i| \cap |\mathcal{C}_{i+1}|$, and H_0 and H_n the isotropy groups of the points $q_{1,-}$ and $q_{n,+}$, respectively, in the chain case. At the points $q_{i,+}$ and $q_{i+1,-}$, there are, by construction, short exact sequences

$$1 \to \mu_{d_i} \to H_{i,+} \xrightarrow{\psi_{i,+}} \mu_{r_{i,+}} \to 1, \quad \text{and} \tag{17}$$

$$1 \to \mu_{d_{i+1}} \to H_{i+1,-} \xrightarrow{\psi_{i+1,-}} \mu_{r_{i+1,-}} \to 1. \tag{18}$$

There are (non-canonical) isomorphisms $H_i \simeq H_{i,+} \simeq H_{i+1,-}$, although by choosing the representatives of (17) and (18) such that $\psi_{i,+}$ (resp. $\psi_{i+1,-}$) are as in Theorem 1, one can take these identifications to be the identity. This yields the local model of q_i.

Remark 3.2. It should be emphasised that, even when it would make sense, we do not require that the short exact sequences (17), (18) are equivalent such that $H_{i,+} \simeq H_{i+1,-}$ via the identity map, only that the groups $H_{i,+}$ and $H_{i+1,-}$ can be identified with H_i.

Recall that the ideal sheaf of a closed substack is the sheaf which pulls back to the ideal sheaf of the preimage in any atlas. As such, we define

$$\mathcal{I} = \bigoplus_i \pi_{i*}\mathcal{O}_{\widetilde{\mathcal{C}}_i}(-q_{i,-} - q_{i,+})$$

for a ring of curves, and analogously for a chain. Here $\pi_i : \widetilde{\mathcal{C}}_i \to \mathcal{C}$ is again the natural projection. We let $\mathcal{F}$ be as in (15) and $\mathcal{A}_{\mathcal{C}}$ be as in (16). For any integers j, m, and $k \in \{0, \dots, d_i - 1\}$, we define distinguished $\mathcal{A}_{\mathcal{C}}$-modules

$$\mathcal{P}_i(j,m,k) = \begin{pmatrix} \pi_{i*}\big(\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,-} + mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k}\big) \\ \pi_{i*}\big(\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,-} + mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k}\big) \end{pmatrix}.$$

For fixed integers j, m, and $0 \le k \le d_i - 1$, let $\mathbf{Exc}_i(j,m,k)$ be the collection

$$\begin{tikzcd} \mathcal{P}_i(j,m,k) \arrow[r, "x_i"] \arrow[d, equal] & \mathcal{P}_i(j+1,m,k) \arrow[r, "x_i"] & \dots \arrow[r, "x_i"] & \mathcal{P}_i(j+r_{i,-}-1,m,k) \arrow[r, "x_i"] & \mathcal{P}_i(j+r_{i,-},m,k) \arrow[d, equal] \\ \mathcal{P}_i(j,m,k) \arrow[r, "y_i"] & \mathcal{P}_i(j,m+1,k) \arrow[r, "y_i"] & \dots \arrow[r, "y_i"] & \mathcal{P}_i(j,m+r_{i,+}-1,k) \arrow[r, "y_i"] & \mathcal{P}_i(j,m+r_{i,+},k) \end{tikzcd} \tag{19}$$

Note that by the decomposition (13), we have that $\mathcal{P}_i(j,m,k)$ is orthogonal to $\mathcal{P}_{i'}(j',m',k')$ unless $k = k'$. With this, it follows directly from the proof of [35, Lemma 1.2.1] that the modules $\mathcal{P}_i(j,m,k)$ are exceptional, and $\mathbf{Exc}_i(j,m,k)$ is an exceptional collection for any fixed j, m, and $k \in \{0, \dots, d_i - 1\}$. In the case of $d_i = 1$ we omit k from the notation.

As in the non-stacky and orbifold cases we also define simple modules at each node, given by

$$\mathcal{S}_q = \begin{pmatrix} 0 \\ \mathcal{O}_q \end{pmatrix}.$$

Fixing an identification of the isotropy group of the node $q_i = |\mathcal{C}_i| \cap |\mathcal{C}_{i+1}|$ with H_i (for i counted modulo n in the ring case), let $\psi_{i,+}$ and $\psi_{i+1,-}$ be as in Theorem 1 and fit into the short exact sequences (17) and (18), respectively. We have that locally, around q_i, we can view $\mathcal{A}_{\mathcal{C}}$–modules as equivariant H_i modules on $\operatorname{Spec} \mathbb{C}[x,y]/(xy) = \operatorname{Spec} S$, where the H_i action is given by $h \cdot (x,y) = (\psi_{i,+}(h)x, \psi_{i+1,-}(h)y)$. We fix the $\mu_{r_{i+1,-}}$ action on the fibre of the sheaf $\mathcal{O}_{\mathbb{P}_{i+1}}(-q_{i+1,-})$

at $q_{i+1,-}$ to be via its natural character, and similarly for the action of $\mu_{r_{i,+}}$ on the fibre of the sheaf $\mathcal{O}_{\mathbb{P}_i}(-q_{i,+})$ at $q_{i,+}$. Moreover, we define the character corresponding to the weight of the action of H_i on the fibre of $\mathcal{O}_{\widetilde{\mathcal{C}}_{i+1}}(-q_{i+1,-})$ to be the character induced from the natural character of $\mu_{r_{i+1,-}}$ under the dual of $\psi_{i+1,-}$, and we call this $\chi_{r_{i+1,-}}$. We define $\chi_{r_{i,+}}$ similarly as the character of H_i induced by the natural character under the dual of $\psi_{i,+}$. For the chosen gerbe structure, choose $\chi_{d_{i,\pm}}$ such that $d_{i,\pm}\chi_{d_{i,\pm}} = \chi_{r_{i,\pm}}$ as in Section 2.

Since H_i is diagonalisable (is isomorphic to a closed subgroup of an algebraic torus), we have an eigenspace decomposition of an H_i-equivariant S-module M as

$$M = \bigoplus_{\chi \in \widehat{H}_i} M_\chi,$$

where $\widehat{H}_i$ is the group of characters of H_i. Furthermore, for any $\chi \in \widehat{H}_i$ there is a twisting operation $M \mapsto M\{\chi\}$ which identifies the γ-eigenspace of $M\{\chi\}$ with the $(\chi+\gamma)$-eigenspace of M.

For a chain (resp. ring) of nodal stacky curves, consider a tuple of characters $\boldsymbol{\chi} = (\chi_0, \dots, \chi_{n+1}) \in \widehat{H}_0 \oplus \cdots \oplus \widehat{H}_{n+1}$ (resp. $\boldsymbol{\chi} = (\chi_1, \dots, \chi_n) \in \widehat{H}_1 \oplus \cdots \oplus \widehat{H}_n$). We call such a tuple *admissible* if there exists a line bundle $\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,-} + mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k}$ on each $\widetilde{\mathcal{C}}_i$ such that H_{i-1} acts on the fibre at $q_{i,-}$ with character χ_{i-1} and H_i on the fibre at $q_{i,+}$ with character χ_i. Denote by $\widehat{H}_{\mathrm{ad}}$ the set of admissible characters. It is not true that $\widehat{H}_{\mathrm{ad}}$ contains any tuple of characters; however, for any character $\chi \in \widehat{H}_i$ there is a tuple in $\widehat{H}_{\mathrm{ad}}$ such that $\chi_i = \chi$. For each admissible $\boldsymbol{\chi}$, we define the sheaf $\mathcal{M}\{\boldsymbol{\chi}\}$ by gluing the line bundles of the above form together at the nodes.

Consider the map $p : \mathcal{C} \to C$, where C is the coarse moduli space of the stacky curve, i.e. is a chain or ring of $\mathbb{P}^1$ joined nodally. Following [44], we call a sheaf $\mathcal{E}$ on $\mathcal{C}$ an *generator* of $\mathrm{QCoh}(\mathcal{C})$ with respect to p if the natural map

$$p^*(p_*\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\mathcal{E}, \mathcal{G})) \otimes \mathcal{E} \to \mathcal{G}$$

is a surjection for any $\mathcal{G}$.

Lemma 3.3. *The sheaf*

$$\bigoplus_{\boldsymbol{\chi} \in \widehat{H}_{ad}} \mathcal{M}\{\boldsymbol{\chi}\}$$

is a generator of QCoh($\mathcal{C}$) with respect to p.

Proof. Let x be a point of $\mathcal{C}$, considered as a map $x : \operatorname{Spec} \mathbb{C} \to \mathcal{C}$. Let G_x be its isotropy group, and denote by $\tilde{x} : BG_x \to \mathcal{C}$ the corresponding natural map. Then, [44, Theorem 5.2] stipulates that if $\mathcal{E}$ is a locally free sheaf such that $\tilde{x}^*\mathcal{E}$ contains every irreducible representation of G_x for every geometric point x, then $\mathcal{E}$ is a generator of $\mathrm{QCoh}(\mathcal{C})$ with respect to p.

From the fact that for each $\chi \in \widehat{H}_i$ there is a $\boldsymbol{\chi} \in \widehat{H}_{\mathrm{ad}}$ such that $\chi_i = \chi$, it is clear that the fibre of $\bigoplus_{\boldsymbol{\chi}} \mathcal{M}\{\boldsymbol{\chi}\}$ at any nodal point (as well as at $q_{1,-}$ and $q_{n,+}$ in the chain case) contains every irreducible representation of H_i. Since χ_{d_i} pushes down to a generator of $\mathbb{Z}/d_i$, the fibre of $\bigoplus_{\boldsymbol{\chi}} \mathcal{M}\{\boldsymbol{\chi}\}$ at a point whose isotropy group is μ_{d_i} contains every irreducible representation of μ_{d_i}, and this establishes the claim. □

To calculate the morphisms between the modules $\mathcal{S}_{q_i}$, and their twists $\mathcal{S}_{q_i}\{\chi\}$ for $\chi \in \widehat{H}_i$, with the $\mathcal{P}_i(j,m,k)$, we can work locally in the patch $U = \operatorname{Spec} S$, as above, and consider H_i equivariant $\mathcal{A}_U$-modules. We begin by observing that, as in the non-stacky and orbifold cases, the only relevant Ext-class is given by the short exact sequence of $\mathcal{A}_U$-modules

$$0 \to \begin{pmatrix} I \\ I \end{pmatrix} \to \begin{pmatrix} I \\ \mathcal{O}_U \end{pmatrix} \to \mathcal{S}_{q_i} \to 0, \tag{20}$$

and that this class is H_i-equivariant. Therefore, we have morphisms

$$\mathrm{Ext}^1(S_{q_i}, \mathcal{P}_i(j,m,0)) = a_i(m,0)$$
$$\mathrm{Ext}^1(S_{q_i}, \mathcal{P}_{i+1}(j,m,0)) = b_i(j,0)$$

for any $m \equiv -1 \bmod r_{i,+}$, and $j \equiv -1 \bmod r_{i+1,-}$, respectively. Consider $\mathcal{M}\{\boldsymbol{\chi}\}$ such that the character at q_i is χ_i. It is clear that we have

$$\mathcal{S}_{q_i} \otimes \mathcal{M}\{\boldsymbol{\chi}\} \simeq \mathcal{S}_{q_i}\{\chi_i\}.$$

In particular, as in (11), we have that for each $\chi \in \widehat{H}_i$, and any $m_i,\ j_i,\ m_{i+1},\ j_{i+1} \in \mathbb{Z}$, there exists $m \in \{m_i, \ldots, m_i + r_{i,+} - 1\},\ k_+ \in \{0, \ldots d_i - 1\}$ and $j \in \{j_{i+1}, \ldots, j_{i+1} + r_{i+1,-} - 1\},\ k_- \in \{0, \ldots, d_{i+1} - 1\}$ such that H_i acts on the fibres of the sheaves

$$\mathcal{O}_{\widetilde{\mathcal{C}}_i}(mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k_+},$$
$$\mathcal{O}_{\widetilde{\mathcal{C}}_{i+1}}(jq_{i+1,-}) \otimes \mathcal{N}_{i+1}^{\otimes k_-}$$

at $q_{i,+}$ and $q_{i+1,-}$, respectively, with character χ.

By twisting the sequence (20) by $\mathcal{M}\{\boldsymbol{\chi}\}$, we obtain morphisms

$$\begin{aligned} &\mathrm{Ext}^1(S_{q_i}\{\chi\}, \mathcal{P}_i(j_i, m_0 + m, k_+)) = \mathbb{C} \cdot a_i(m, k_+), \quad \text{and} \\ &\mathrm{Ext}^1(S_{q_i}\{\chi\}, \mathcal{P}_{i+1}(j_0 + j, m_{i+1}, k_-)) = \mathbb{C} \cdot b_i(j, k_-), \end{aligned} \tag{21}$$

for each $\chi \in \widehat{H}_i$, where $m_0 \in \{m_i, \ldots, m_i + r_{i,+} - 1\}$ is a distinguished element such that $m_0 \equiv -1 \bmod r_{i,+}$, and (m, k_+) as above solves

$$-m\chi_{r_{i,+}} + k_+\chi_{d_i,+} = \chi, \tag{22}$$

$j_0 \in \{j_{i+1}, \ldots, j_{i+1} + r_{i+1,-} - 1\}$ is a distinguished element such that $j_0 =\equiv -1 \bmod r_{i+1,-}$, and (j, k_-) as above solves

$$-j\chi_{r_{i+1-}} + k_-\chi_{d_{i+1},-} = \chi. \tag{23}$$

Now, we have constructed a full, strong exceptional collection consisting of the objects:

- For any fixed $j_i,\ m_i \in \mathbb{Z}$, and each irreducible component, being a μ_{d_i}-gerbe over $\mathbb{P}_i$, the collections
$$\bigoplus_{k=0}^{d_i-1} \mathbf{Exc}_i(j_i, m_i, k),$$
- For each node $q_i = |\mathcal{C}_i| \cap |\mathcal{C}_{i+1}|$, the objects
$$\mathcal{S}_{q_i}\{\chi_k\} \quad \text{for each } \chi_k \in \widehat{H}_i.$$

The endomorphism algebra of this collection is generated by the morphisms $x_i,\ y_i$ in (19), as well as the morphisms given by (21). The relations are $ya = 0$ and $xb = 0$ whenever the composition is possible. The proof of this, as well as the claim that the collection is full and strong, can be seen from following through the proof of [35, Theorem 1.2.3] *mutatis mutandis* (cf. [10, Theorem 9]). Of course, the resulting category $D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod})$ only depends on the parameters stated in Theorem 1, ultimately for the same reason as $D^b\,\mathrm{Coh}(\mathcal{C})$ does.

4. The partially wrapped Fukaya category

In this section, we briefly recall the construction of graded symplectic surfaces by gluing columns of cylinders, as described in [25, Section 3.2]. We then recount the strategy of [27] for constructing the partially wrapped Fukaya category of a graded symplectic surface before describing the collections of generating objects for surfaces glued from columns of cylinders.

4.1. **Gluing annuli.** Let $A(\ell, r; d)$ denote d annuli, each with r ordered marked points, $p^+_{rk}, \dots p^+_{r(k+1)-1}$, on the first boundary component, and ℓ ordered marked points, $p^-_{\ell k}, \dots, p^-_{\ell(k+1)-1}$, on the second boundary component, which have been placed in a column. Here $k \in \{0, \dots, d-1\}$ refers to which annulus the marked points are on, where we count top-to-bottom. We visualise this as d disjoint rectangles which have been placed in a column, each rectangle has top and bottom identified, and with the left boundary components containing the points $p^-_{\ell k+i}$ and the right boundary components containing the points p^+_{rk+i}. The reasoning for the labelling is that we would like to keep track of where the marked points are on each individual annulus, as well as where each marked point is on the left (respectively right) side of the column of annuli with respect to the ordering $p^-_0, \dots, p^-_{d_i\ell_i-1}$ (respectively $p^+_0, \dots, p^+_{d_ir_i-1}$).

Given a collection of annuli

$$A(\ell_1, r_1; d_1), \dots, A(\ell_n, r_n; d_n),$$

such that $r_id_i = \ell_{i+1}d_{i+1}$, and corresponding permutations $\sigma_i \in \mathfrak{S}_{d_ir_i}$, we can glue these annuli together in the following way. For each $j \in \{0, \dots, d_ir_i - 1\}$, we glue a small neighbourhood around the stop p^+_j in $A(\ell_i, r_i; d_i)$ to a small neighbourhood around the stop $p^-_{\sigma_i(j)}$ in $A(\ell_{i+1}, r_{i+1}; d_{i+1})$ by attaching a strip. We call such a gluing *circular* if $A(\ell_n, r_n; d_n)$ is glued back to $A(\ell_1, r_1; d_1)$, and in this case we count $i \bmod n$ – see Figure 1 for an example. Otherwise, we call a gluing *linear*, and take $i \in \{1, \dots, n\}$. In the case of linear gluing we refer to the left boundary components of $A(\ell_1, r_1, d_1)$ and the right boundary components of $A(\ell_n, r_n, d_n)$ as the left and right distinguished boundary components, respectively.

For each i, the number of boundary components arising from gluing the i^{th} and $(i+1)^{\text{st}}$ columns can be computed as follows. Consider the permutations

$$\tau_{r_i} = \big(0, r_i - 1, r_i - 2, \dots, 1\big)\big(r_i, 2r_i - 1, 2r_i - 2, \dots, r_i + 1\big) \dots \big((d_i - 1)r_i, m_ir_i - 1, \dots, (d_i - 1)r_i + 1\big)$$

and

$$\tau_{\ell_i} = \big(0, 1, \dots, \ell_{i+1} - 1\big)\big(\ell_{i+1}, \dots, 2\ell_{i+1} - 1\big) \dots \big((d_{i+1} - 1)\ell_{i+1}, \dots, d_{i+1}\ell_{i+1} - 1\big).$$

The number of boundary components between the i^{th} and $(i+1)^{\text{st}}$ columns will then be given by the number of cycles in the decomposition of $\sigma_i^{-1}\tau_{\ell_{i+1}}\sigma_i\tau_{r_i} \in \mathfrak{S}_{m_ir_i}$. Note that if $d_i = d_{i+1}$, then we have $\tau_{r_i} = \tau^{-1}_{\ell_{i+1}}$, and we simply get the commutator. When there is no risk of confusion we will simply refer to the surface which has been constructed as Σ.

To compute the homology groups of Σ, one can construct a ribbon graph

$$\Gamma(\ell_1, \dots, \ell_n; r_1, \dots, r_n; m_1, \dots, m_n; \sigma_1, \dots, \sigma_n) \subseteq \Sigma, \tag{24}$$

on to which the surface deformation retracts. To do this, let there be a topological disc $\mathbb{D}^2$ for each of the annuli. For each disc, attach a strip which has one end on the top, and the other end on the bottom. Then, attach a strip which connects two discs if there is a strip which connects the corresponding annuli. These strips must be attached in such a way as to respect the cyclic ordering given by the gluing permutation. One can then deformation retract this onto a ribbon graph, whose cyclic ordering at the nodes is induced from the ordering of the strips on each annulus. If there is no ambiguity, we will refer to this graph as $\Gamma(\Sigma)$.

Since the embedding of $\Gamma(\Sigma)$ in to Σ induces an isomorphism on homology, the homology groups of Σ can be easily computed. Namely, since the graph is connected, we have $H_0(\Sigma) = \mathbb{Z}$. For circular gluing we have $\chi(\Sigma) = V - E = \mathrm{rk} H_0(\Sigma) - \mathrm{rk} H_1(\Sigma) = -\sum_{i=1}^n r_id_i$, and for linear gluing we have $\chi(\Sigma) = -\sum_{i=1}^{n-1} r_id_i$, yielding $H_1(\Sigma) = \mathbb{Z}^{(1-\chi)}$ in both cases. A basis for the first homology of the graph is given by an integral cycle basis, and so the basis of the first homology for Σ is given by

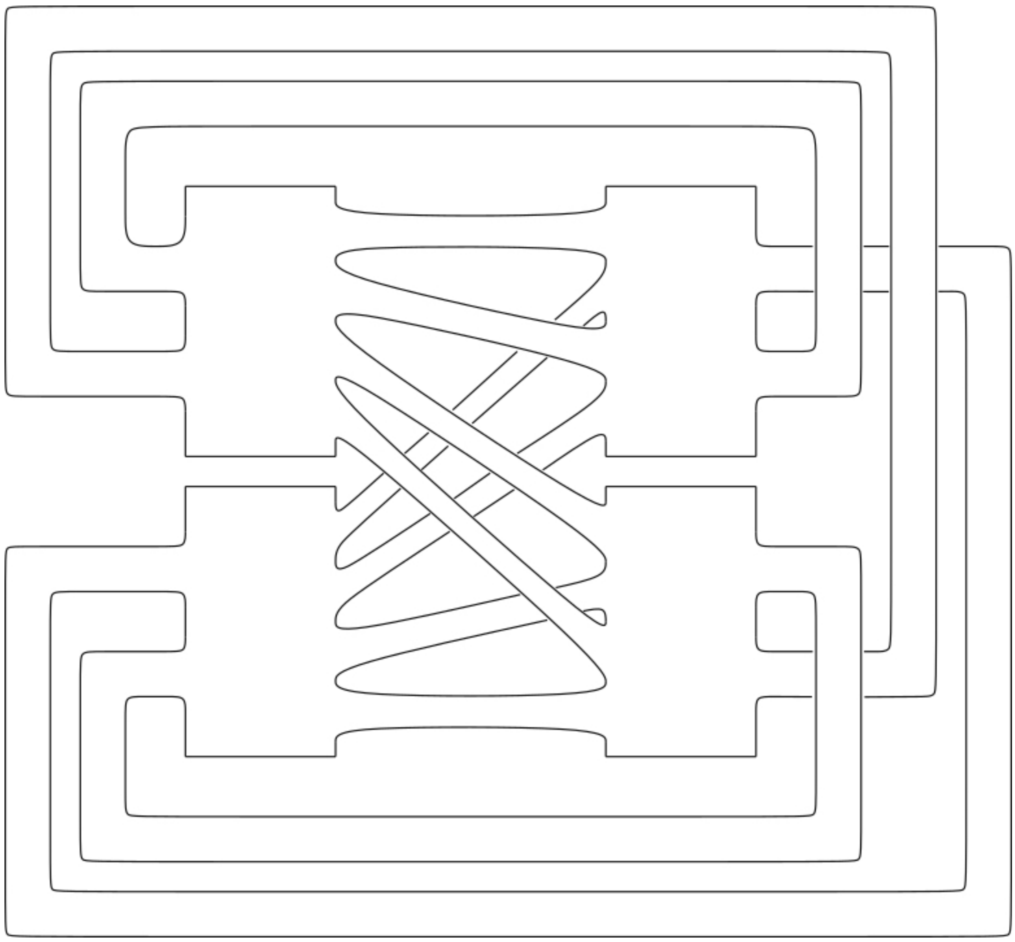

Figure 1. A genus 5 surface with 4 boundary components constructed by gluing $A(2,4;2)$ to $A(4,2;2)$ via the permutations $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 1 & 3 & 5 & 7 \end{smallmatrix}\right)$ and $\sigma_2 = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 1 \end{smallmatrix}\right)$.

loops which retract onto these cycles.

A $\mathbb{Z}$-grading of the surface is given by a homotopy class of (unorientated) line field, as explained in [48, Section 13(c)], and a Lagrangian is gradable with respect to this line field if and only if its winding number is zero. Note that in the case where the winding number with respect to a given line field vanishes on each embedded Lagrangian in a basis of the first homology of Σ, the homotopy class of the line field is unique. In the remainder of this paper, we will only consider the case where the line field used to grade the surface is given by the horizontal line field on each annulus, and is parallel to the boundary components on each attaching strip. With this, we see that the line field comes from the projectivisation of a vector field by the same proof as in [36, Lemma 4.1.1]. With such a description of a surface, it is possible to determine when two surfaces are graded symplectomorphic ([36, Corollary 1.2.6]); however, in order to do this one must (in many cases) compute the Arf invariant. Whilst this is theoretically simple, it is computationally intractable to do in generality without imposing restrictions on the form of the gluing permutations being considered, as in [36, Section 4.3], or [25, Section 4.3].

4.2. **Computation of the partially wrapped Fukaya category.** Once we have constructed the surfaces in question, our approach to mirror symmetry involves computing the partially wrapped Fukaya category via the method given in [27].

Given a surface with non-empty boundary, Σ, and a collection of stops on its boundary Λ, [27, Section 3] shows that if $\{L_i\}$ is a collection of pairwise disjoint and non-isotopic Lagrangians such that $\Sigma \setminus \big(\sqcup_i L_i \big)$ is topologically a union of discs, each of which with exactly one marked point on its boundary, then the L_i generate $\mathcal{W}(\Sigma; \Lambda)$. Moreover, it is also shown that the total endomorphism algebra of the generators is formal, and can be described as the algebra of a quiver with monomial relations. A connection to the representation theory of finite dimensional algebras is given by the observation that the endomorphism algebra of such a generating collection of objects is *gentle*.

To construct the partially wrapped Fukaya category, it was shown that there exists a ribbon graph dual to the collection of Lagrangians. This graph has an $n-$valent vertex at the centre of each $2n$-gon cut out by the Lagrangians, and the half edges connect two vertices if that edge is dual to a Lagrangian on the boundary of both of the corresponding discs. The cyclic ordering is induced from the orientation of the surface. From this, it was shown in [27, Theorem 3.1] that the partially wrapped Fukaya category is given as the global sections of a constructible cosheaf of A_∞-categories on the ribbon graph. In particular, for each n-valent vertex at the centre of a $2n$-gon, there is a fully faithful inclusion functor

$$\mathcal{W}(\mathbb{D}^2; n+1) \to \mathcal{W}(\Sigma; \Lambda), \tag{25}$$

where $\mathcal{W}(\mathbb{D}^2; n+1)$ is the partially wrapped Fukaya category of the disc with $n+1$ stops on its boundary.

The two prototypical examples from which our strategy is built are the disc with m points on its boundary, as well as the cylinder with a stops on one boundary, and b stops on the other. Consider the disc with m stops on its boundary, and $m-1$ Lagrangians, $L_1, \dots, L_{m-1}$ supported near these stops, as in Figure 2. The morphisms between Lagrangians is given by the Reeb flow along the boundary of the disc in the anticlockwise direction. Let $a_i : L_i \to L_{i+1}$ be such a morphism, and observe that $a_{i+1}a_i = 0$ for $i = 1, \dots, m-2$. It is clear that the endomorphism algebra of the direct sum $\bigoplus_{i=1}^{m-1} L_i$ is the A_{m-1} quiver with relations given by disallowing any composition.

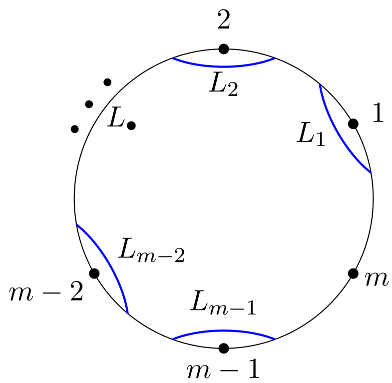

FIGURE 2. A collection of generating Lagrangians for $\mathbb{D}^2$ with m stops. The Reeb flow is in the counterclockwise direction.

There are two key facts about the collection of Lagrangians $L_1, \dots, L_{m-1}$. The first is that the Lagrangian L_m is quasi-isomorphic to the twisted complex

$$L_1[m-2] \longrightarrow L_2[m-3] \longrightarrow \dots \longrightarrow L_{m-2}[-1] \longrightarrow L_{m-1}. \tag{26}$$

This is first observed in [27, Section 3.3], and will be important later in our localisation argument. The second key observation is that the complement $\mathbb{D}^2 \setminus \big(\sqcup_{i=1}^{m-1} L_i \big)$ is a collection of topological discs, each with exactly one marked point on the boundary. Therefore, the collection $\{L_1, \dots, L_{m-1}\}$ generates the partially wrapped Fukaya category $\mathcal{W}(\mathbb{D}^2; m)$.

The second fundamental example which forms the cornerstone of our strategy is the annulus, A, with a stops on one boundary component, and b on the other. A generating collection of Lagrangians on such an annulus is given in Figure 3, and its corresponding quiver in Figure 4. Observe that the quiver algebra of the generators for the single annulus with a stops on one boundary component and b on the other matches precisely the quiver algebra of the exceptional collection of $\mathbb{P}_{a,b}$ given in (12). This establishes that

$$\mathcal{W}(A; a, b) \simeq D^b \operatorname{Coh}(\mathbb{P}_{a,b}),$$

and this observation is at the heart of our strategy.

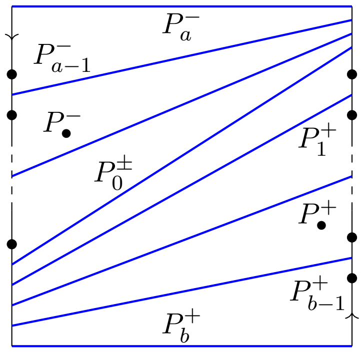

FIGURE 3. A collection of generating Lagrangians for $A(a, b; 0)$. Top and bottom identified.

$$\begin{tikzcd}
P_0^- \arrow[r, "x_1"] \arrow[d, equal] & P_1^- \arrow[r, "x_2"] & \dots \arrow[r, "x_{a-1}"] & P_{a-1}^- \arrow[r, "x_a"] & P_a^- \arrow[d, equal] \\
P_0^+ \arrow[r, "y_1"] & P_1^+ \arrow[r, "y_2"] & \dots \arrow[r, "y_{b-1}"] & P_{b-1}^+ \arrow[r, "y_b"] & P_b^+
\end{tikzcd}$$

FIGURE 4. Quiver for $A(a, b; 0)$.

4.2.1. *Circular Gluing.* Here we compute the partially wrapped Fukaya category for columns of annuli glued circularly, as in Section 4.1. To begin with, we add two stops on each attaching strip – one on the top, and one on the bottom. We will refer to this collection as Λ. On the k^{th} annulus in the i^{th} column we have a collection of Lagrangians $\boldsymbol{P}_{i,k}$ of the same form as in Figure 3. This collection consists of the objects

$$\{P^+_{i,0,k}, P^+_{i,1,k}, \dots, P^+_{i,r_i,k}, P^-_{i,0,k}, \dots P^-_{i,\ell_i,k}\}.$$

The morphisms within this collection are of the same form as in Figure 4. For each attaching strip, we consider a Lagrangian which spans it in such a way that the two stops are in the clockwise direction of its endpoints. We label the Lagrangian which spans the attaching strip beginning at the neighbourhood around the j^{th} stop between the i^{th} and $(i+1)^{\text{st}}$ columns by $S_{i,j}$. Here $j \in \{0, \dots, r_i m_i - 1\}$ and $i \in \mathbb{Z}/n$.

As well as the morphisms within each collection $\boldsymbol{P}_{i,k}$, if we write $j = k_+ r_i + c_+$ and $\sigma(j) = k_- \ell_{i+1} + c_-$ for $k_+ \in \{0, \dots, d_i - 1\}$, $c_+ \in \{0, \dots, r_i - 1\}$, $k_- \in \{0, \dots d_{i+1} - 1\}$, and $c_- \in \{0, \dots, \ell_{i+1} - 1\}$, we also have morphisms

$$a_{i,j} : S_{i,j} \to P^+_{i,c_+,k_+}$$
$$b_{i,j} : S_{i,j} \to P^-_{i+1,\ell_{i+1}-1-c_-,k_-}.$$

By construction, the complement of this collection of Lagrangians is the disjoint union of hexagons, each with exactly one stop on its boundary. Therefore, we have that the collection of Lagrangians consisting of all of the $\boldsymbol{P}_{i,k}$, as well as the $S_{i,j}$ is a generating collections of Lagrangians for $\mathcal{W}(\Sigma; \Lambda)$.

4.2.2. *Linear Gluing.* The case of linear gluing is almost identical to that of circular gluing; however, the first and last columns are now no longer glued to each other. Due to this, we include the stops on the distinguished boundary components in Λ, although we allow the number of stops on the distinguished boundary components to be empty. In dividing the surface into topological discs for the computation of the partially wrapped Fukaya category, observe that a topological disc with a

Lagrangian $S_{i,j}$ on its boundary is a hexagon, as in the circular gluing case, and a quadrilateral otherwise. The generating collection is again given by all of the $\boldsymbol{P}_{i,k}$, as well as the $S_{i,j}$. See Figure 5 for an example, where its corresponding quiver is given in Figure 6.

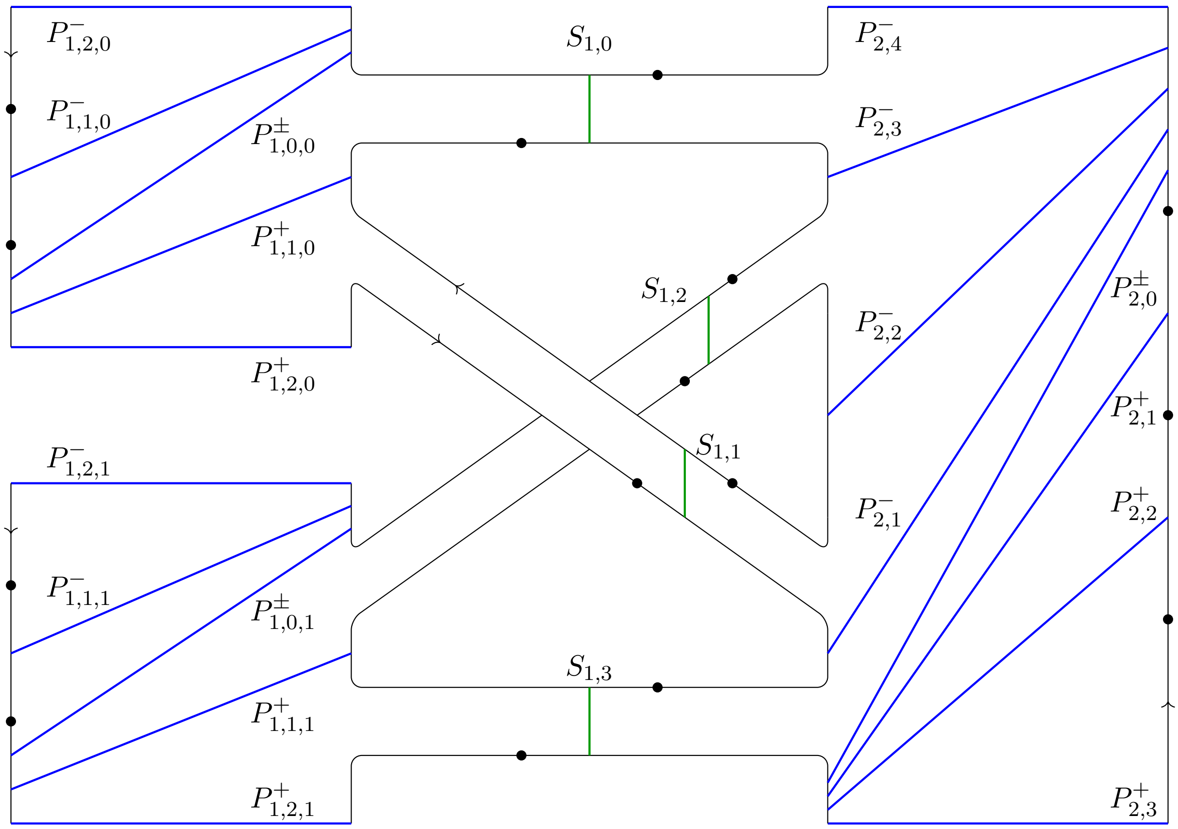

FIGURE 5. Generating collections of Lagrangians for circular gluing of $A(2,2;2)$ to $A(4,3;1)$ via $\sigma_1 : (0,1,2,3) \mapsto (0,2,1,3)$. Top and bottom of each annulus is identified.

$$\begin{tikzcd}
& P^-_{1,0,0} \arrow[r, "x_{1,1,0}"] \arrow[d, equal] & P^-_{1,1,0} \arrow[r, "x_{1,2,0}"] & P^-_{1,2,0} \arrow[d, equal] & P^-_{1,0,1} \arrow[r, "x_{1,1,1}"] \arrow[d, equal] & P^-_{1,1,1} \arrow[r, "x_{1,2,1}"] & P^-_{1,2,1} \arrow[d, equal] \\
& P^+_{1,0,0} \arrow[r, "y_{1,1,0}"] & P^+_{1,1,0} \arrow[r, "y_{1,2,0}"] & P^+_{1,2,0} & P^+_{1,0,1} \arrow[r, "y_{1,1,1}"] & P^+_{1,1,1} \arrow[r, "y_{1,2,1}"] & P^+_{1,2,1} \\
& S_{1,0} \arrow[u, "a_{1,0}"] \arrow[d, "b_{1,0}"] & S_{1,1} \arrow[u, "a_{1,1}"] \arrow[drr, "b_{1,1}"] & & S_{1,2} \arrow[u, "a_{1,2}"] \arrow[dll, "b_{1,2}"'] & S_{1,3} \arrow[u, "a_{1,3}"] \arrow[d, "b_{1,3}"] & \\
P^-_{2,4} \arrow[d, equal] & P^-_{2,3} \arrow[l, "x_{2,4}"] & P^-_{2,2} \arrow[l, "x_{2,3}"] & & P^-_{2,1} \arrow[ll, "x_{2,2}"] & P^-_{2,0} \arrow[l, "x_{2,1}"] \arrow[d, equal] & \\
P^+_{2,3} & P^+_{2,2} \arrow[l, "y_{2,3}"] & & P^+_{2,1} \arrow[ll, "y_{2,2}"] & & P^+_{2,0} \arrow[ll, "y_{2,1}"] &
\end{tikzcd}$$

FIGURE 6. Quiver describing the endomorphism algebra of the generating collection of Figure 5. Relations given by $xb = 0$ and $ya = 0$.

5. Localisation

As described in the introduction, there are natural localisation functors on the A– and B–sides, given by the second functors in (1) and (2), respectively. The strategy to establishing Theorem 2 is to show that the quasi-equivalence in Theorem 1 intertwines localisation on both sides, although this argument is also required to prove Theorem 1 in the case of a chain of curves with $r_{1,-}$ and/ or $r_{n,+}$ is zero. In this section, we describe the localisation functors on the A– and B–sides before establishing Theorem 1.

5.1. **Localisation on the B–side.** As in the non-stacky case, we consider the functor

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, -) : \mathcal{A}_{\mathcal{C}} - \mathrm{mod} \to \mathrm{Coh}\,\mathcal{C}, \tag{27}$$

and construct a subcategory $\mathcal{T}$ on which this functor vanishes. We again work locally, and so the analysis follows from the non-stacky case by working equivariantly, as is demonstrated in the orbifold case [35, Section 3.2]. Note that this functor is exact since $\mathcal{F}$ is a summand of $\mathcal{A}_{\mathcal{C}}$, so is locally projective.

In order to construct $\mathcal{T}$, we define the modules

$$\widetilde{\mathcal{S}}_i(j,k)^{\pm} = \begin{pmatrix} \pi_{i*}(\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,\pm}) \otimes \mathcal{N}_i^{\otimes k})|_q \\ \pi_{i*}(\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,\pm}) \otimes \mathcal{N}_i^{\otimes k})|_q \end{pmatrix},$$

where $q = \pi_i(q_{i,\pm})$. These modules fit into the short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{P}_i(j-1,m,k) \longrightarrow \mathcal{P}_i(j,m,k) \longrightarrow \widetilde{\mathcal{S}}_i(j,k)^- \longrightarrow 0 \\ 0 \longrightarrow \mathcal{P}_i(j,m-1,k) \longrightarrow \mathcal{P}_i(j,m,k) \longrightarrow \widetilde{\mathcal{S}}_i(m,k)^+ \longrightarrow 0, \end{gathered} \tag{28}$$

and have support at $\pi_i(q_{\pm})$. When the point $q_{i,\pm}$ is not a node (i.e. the $q_{1,-}$ and $q_{n,+}$ in the chain case) we set $\mathcal{E}_i^{\pm}(j,k) = \widetilde{\mathcal{S}}_i(j,k)^{\pm}$. If $q_{i,\pm}$ is a node, then we have natural inclusions

$$\begin{gathered} \mathcal{S}_{q_i}\{\chi_+\} \hookrightarrow \widetilde{\mathcal{S}}_i(m,k)^+ \\ \mathcal{S}_{q_{i-1}}\{\chi_-\} \hookrightarrow \widetilde{\mathcal{S}}_i(j,k)^-, \end{gathered}$$

where χ_+ (resp. χ_-) is the character through which H_i (resp. H_{i-1}) acts on the fibre of the sheaf $\mathcal{O}_{\widetilde{\mathcal{C}}_i}(mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k}$ (resp. $\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,-}) \otimes \mathcal{N}_i^{\otimes k}$) at $q_{i,+}$ (resp. $q_{i,-}$). We then define $\mathcal{E}_i(j,k)^{\pm}$ to fit into the short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{S}_{q_i}\{\chi_+\} \longrightarrow \widetilde{\mathcal{S}}_i(m,k)^+ \longrightarrow \mathcal{E}_i(m,k)^+ \longrightarrow 0, \\ 0 \longrightarrow \mathcal{S}_{q_{i-1}}\{\chi_-\} \longrightarrow \widetilde{\mathcal{S}}_i(j,k)^- \longrightarrow \mathcal{E}_i(j,k)^- \longrightarrow 0. \end{gathered}$$

As in the orbifold case, we find that the objects $\mathcal{E}_i(j,k)^{\pm}$ are exceptional unless $\pi_i(q_{i,\pm})$ is a smooth point with no stacky structure. At nodes, this follows from the presentation as a quotient of $xy = 0$ by H_i. In this presentation, the relevant Ext-groups are the H_i-invariant classes of the Ext-groups computed on $xy = 0$, and it is shown in [35, Lemma 3.2.1] that these groups vanish. In the case where the point is a smooth point with non-trivial stacky structure, we find that the objects $\mathcal{E}_i(j,k)^{\pm}$ are exceptional from the locally projective resolution (28). We define $\mathcal{T}$ to be the subcategory formed by direct sums of all the objects $\mathcal{E}_i(j,k)^{\pm}$ supported at the nodes.

With this we have that $\mathcal{T} \subseteq D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod})$ is a Serre subcategory, and identifies

$$\begin{gathered} \mathrm{Coh}\,\mathcal{C} \simeq \mathcal{A}_{\mathcal{C}} - \mathrm{mod}/\mathcal{T} \\ D^b\,\mathrm{Coh}(\mathcal{C}) \simeq D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod})/\langle \mathcal{T} \rangle. \end{gathered}$$

To see this, note that the derived equivalence follows from the equivalence of abelian categories by [41]. The equivalence of abelian categories is given for non-stacky curves in [10, Theorem 4.8], and the present situation follows from this. As explained in the orbifold case, [35, Proposition 3.2.3], one must check that certain adjunctions are equivalences, and this boils down to checking the statement locally at nodes. One can then use the presentation at a node as the quotient of $xy = 0$ by H_i, and the argument follows from the non-stacky case.

5.2. **Localisation on the A–side.** Part of the utility of the construction in [27] is that it not only provides a categorical resolution of the compact Fukaya category of a surface, but also gives an explicit description of a map

$$\mathcal{W}(\Sigma;\Lambda) \to \mathcal{W}(\Sigma;\Lambda'),$$

where Λ' is obtained from Λ by removing stops. This map is given by taking the quotient of the partially wrapped Fukaya category by the category generated by Lagrangians which are supported near the stops being removed. In particular, by removing all of the stops in the case of circular gluing, one recovers a map to the wrapped Fukaya category of the surface. In the case of linear gluing, the situation is analogous, however, the stops on the distinguished boundary components are not removed. It is in this context that the quasi-isomorphism (26) and the functor (25) show their utility by giving the Lagrangian supported near a stop to be removed in terms of the generating Lagrangians on a disc.

For circular gluing, we will define the object $E^+_{i,j}$ (resp. $E^-_{i+1,j}$) to be Lagrangian supported near the stop on the bottom (resp. top) of the attaching strip beginning at the neighbourhood of the j^{th} stop on a boundary component between the i^{th} and $(i+1)^{\text{st}}$ columns. By again writing $j = k_+r_i+c_+$ and $\sigma_i(j) = k_-\ell_{i+1} + c_-$ for $k_+ \in \{0,\dots,d_i-1\}$, $c_+ \in \{0,\dots,r_i-1\}$, $k_- \in \{0,\dots d_{i+1}-1\}$, and $c_- \in \{0,\dots,\ell_{i+1}-1\}$, we have by (26) and (25)

$$E^+_{i,j} \simeq \{S_{i,j}[3] \to P^+_{i,c_+,k_+}[2] \to P^+_{i,c_++1,k_+}[1]\}$$
$$E^-_{i+1,j} \simeq \{S_{i,j}[3] \to P^-_{i+1,\ell_{i+1}-1-c_-,k_-}[2] \to P^-_{i+1,\ell_{i+1}-c_-,k_-}[1]\}.$$

In the case of linear gluing we have the same iterated cones in for the hexagonal regions, as well as the cones

$$E^-_{1,j} \simeq \{P^-_{i,j}[2] \to P^-_{i,j+1}[1]\}$$
$$E^+_{n,j} \simeq \{P^+_{i,j}[2] \to P^+_{i,j+1}[1]\}.$$

Proof of Theorem 1. In order to prove the statement, it suffices to match the generators of the categories in question. We begin with the case of a ring of curves, or a chain where both $r_{1,-}, r_{n,+} > 0$. On the B–side, we fix an exceptional collection such that $j_i = 0$ and $m_i = -1$. We again label the characters in $\widehat{H}_i$ such that $\chi_{k_+r_{i,+}+m}$ is the character of $\mathcal{O}_{\widetilde{\mathcal{C}}_i}(mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k_+}$. On the A–side we construct the candidate mirror as follows: For each irreducible component $\mathcal{C}_i$ of $\mathcal{C}$, being a μ_{d_i}-gerbe over $\mathbb{P}_{r_{i,-},r_{i,+}}$, we consider a column of annuli $A(r_{i,-}, r_{i,+}; d_i)$. Let j, k_- solve

$$-j\chi_{r_{i+1,-}} + k_-\chi_{d_{i+1},-} = \chi_{k_+r_{i,+}+m},$$

as in (23). We then define the permutation σ_i to be given by

$$k_+r_{i,+} + m \mapsto k_-r_{i+1,-} + (-j) \bmod r_{i+1,-}.$$

Let Σ be the surface constructed in this way, and let $\mathcal{W}(\Sigma;\Lambda)$ be its partially wrapped Fukaya category, as described in Section 4.2. The identification of the generation objects on both sides is given by:

$$P^-_{i,j,k} \longleftrightarrow \mathcal{P}_i(j,-1,k)$$
$$P^+_{i,m,k} \longleftrightarrow \mathcal{P}_i(0,m-1,k)$$
$$S_{i,j} \longleftrightarrow \mathcal{S}_i\{\chi_j\}[-1].$$

From this, we can see that the endomorphism algebras of the two exceptional collections which generate their respective categories are equivalent, which establishes the claim in this case.

To complete the proof in the case of a ring of curves, where either or both of $r_{1,-}, r_{n,+} = 0$, we must utilise [35, Proposition 3.2.2], which, suitably reworded to our context, states that under the above equivalence, we have a correspondence

$$\{\mathcal{A}_{\mathcal{C}} - \text{modules} \quad \mathcal{E}_i^-(j,k)\} \longleftrightarrow \{E^-_{i,r_{i,-}k+j}[-1]\} \tag{29}$$

$$\{\mathcal{A}_{\mathcal{C}} - \text{modules} \quad \mathcal{E}_i^+(m,k)\} \longleftrightarrow \{E^+_{i,r_{i,+}k+m-1}[-1]\}. \tag{30}$$

The proof of the alteration of the statement to our situation follows directly from the proof of the original statement. Namely, we let **Exc** be the direct sum of the objects in the exceptional collection in $D^b(\mathcal{A}_{\mathcal{C}} - \text{mod})$ described in Section 3 and A its endomorphism algebra. One can describe the right A-modules of the form $\mathrm{RHom}(\mathbf{Exc}, -)$ corresponding to the objects $\mathcal{E}_i^-(j,k)$ and $\mathcal{E}_i^+(m,k)$ in the equivalence

$$\mathrm{RHom}(\mathbf{Exc}, -) : D^b(\mathcal{A}_{\mathcal{C}} - \text{mod}) \xrightarrow{\sim} D^b(\text{mod} - A),$$

and see that they match with the objects in $D^b(\text{mod} - A)$ calculated using the presentations of $E^-_{i,j}$ (resp. $E^+_{i,j-1}$) given in Section 5.2.

Now, consider the case of $r_{n,+} > r_{1,-} = 0$, and define the stack $\overline{\mathcal{C}}$ to be the same curve as $\mathcal{C}$, but where $\mathcal{C}_1 = \mathbb{P}_{1,r_{1,+}}$. Namely, we have $\mathcal{C} = \overline{\mathcal{C}} \setminus \{q_{1,-}\}$. Since $\mathcal{A}_{\overline{\mathcal{C}}}$ is isomorphic near $q_{1,-}$ to the matrix algebra over $\mathcal{O}$, we have that the restriction functor

$$\mathcal{A}_{\overline{\mathcal{C}}} - \text{mod} \to \mathcal{A}_{\mathcal{C}} - \text{mod}$$

identifies $\mathcal{A}_{\mathcal{C}}$−mod with the quotient of $\mathcal{A}_{\overline{\mathcal{C}}}$−mod by the Serre subcategory generated by $\bigoplus_{k=0}^{d_1-1} \mathcal{E}_1(0,k)^-$ (i.e. $\bigoplus_{k=0}^{d_1-1} \begin{pmatrix} \mathcal{O}_{q_{1,-}} \\ \mathcal{O}_{q_{1,-}} \end{pmatrix} \otimes \mathcal{N}_1^{\otimes k}$). By the main result of [41], this yields a derived equivalence

$$D^b(\mathcal{A}_{\overline{\mathcal{C}}} - \text{mod}) / \langle \bigoplus_{k=0}^{d_1-1} \mathcal{E}_1(0,k)^- \rangle \simeq D^b(\mathcal{A}_{\mathcal{C}} - \text{mod}).$$

From the first part of the proof, there is a graded surface $(\Sigma, \overline{\Lambda})$ such that

$$\mathcal{W}(\Sigma; \overline{\Lambda}) \simeq D^b(\mathcal{A}_{\overline{\mathcal{C}}} - \text{mod}).$$

Now, since $\mathcal{E}_1^-(0,k)$ is identified with $E^-_{1,k}[-1]$ in (29), removing the stops on the left distinguished boundary corresponds to localising $D^b(\mathcal{A}_{\overline{\mathcal{C}}} - \text{mod})$ by the category generated by $\bigoplus_{k=0}^{d_1-1} \mathcal{E}_1(0,k)^-$, which yields the result. The cases of $r_{1,-} > r_{n,+} = 0$ or $r_{0,-} = r_{n-1,+} = 0$ are analogous. □

Remark 5.1. Choosing different values for m_i and j_i in the above theorem corresponds to changing the identification of the cylinders on the A–side by a cyclic reordering. This yields homeomorphic mirrors, since cyclically changing the identification of an individual annulus, and/ or reordering the annuli in a column, does not change the number of cycles, or their length, in the cycle decomposition determining the topology of the surface.

6. Characterisation of perfect derived categories

In order to establish the statement about perfect objects in Theorem 2, one must show that the compact Fukaya category and derived category of perfect complexes, considered as full subcategories of $\mathcal{W}(\Sigma; \Lambda)$ and $D^b(\mathcal{A}_{\mathcal{C}} - \text{mod})$, respectively, are identified with each other under the quasi-equivalence of Theorem 1. The aim of this section is to characterise perfect complexes on the A– and B–sides of the correspondence before establishing Theorem 2.

6.1. **The derived category of perfect complexes.** As in the localisation argument, our strategy closely follows that of [10, Theorem 2] for the non-stacky case. Let $\mathcal{C}$ be a ring or chain of curves with $r_{1,-}, r_{n,+} > 0$, $\mathcal{F}$ as in Section 3, and consider the functor

$$\begin{gathered} \operatorname{perf} \mathcal{C} \to D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod}) \\ G \mapsto \mathcal{F}_{\mathcal{C}} \otimes_{\mathcal{O}_{\mathcal{C}}} G. \end{gathered}$$

In the non-stacky case, it is shown that this functor is full and faithful in [10, Theorem 2 (5)], and this result is generalised to the orbifold case in [35, Proposition 4.1.3]. As in these cases, one can again identify the essential image of $\operatorname{perf} \mathcal{C}$ in $D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod})$ as the subcategory which is both left and right orthogonal to the category $\mathcal{T}$ defined in Section 5.1. The proof of the this follows verbatim from the proof of [35, Proposition 4.1.3 (i)] after replacing μ_r by H, an extension of μ_r by μ_d. In the case of a ring of curves with $r_{n,+} > r_{1,-} = 0$, the category of compactly supported perfect complexes on $\mathcal{C}$ is identified with the category which is both left and right orthogonal to $\overline{\mathcal{T}}$, where this category is formed by the objects of $\mathcal{T}$, together with $\mathcal{E}_1(k)^-$ for $0 \leq k \leq d_1 - 1$. To prove this, observe that $\mathcal{E}_1^-(0,k) \simeq \begin{pmatrix} \mathcal{O}_{q_{1,-}}(k\chi_{d_{1,-}}) \\ \mathcal{O}_{q_{1,-}}(k\chi_{d_{1,-}}) \end{pmatrix}$ near $q_{1,-}$, and so a module in $\operatorname{perf} \overline{\mathcal{C}}$ is left or right orthogonal to $\bigoplus_{k=0}^{d_1-1} \mathcal{E}_1^-(0,k)$ if and only if its support does not contain $q_{1,-}$. Then, the rest of the proof in this case follows as in [35, Proposition 4.1.3 (ii)]. The cases when $r_{1,-} > r_{n,+} = 0$ and $r_{1,-} = r_{n,+} = 0$ are considered similarly.

6.2. **Characterisation of the Fukaya category.** On the A–side of the correspondence, the characterisation of the Fukaya category as a subcategory of the partially wrapped Fukaya category remains unchanged from [35, Section 4.2]. We briefly recall the argument here, and refer to loc. cit. for the proof.

Let $\mathcal{T}_i$ be the collection of Lagrangians supported near the stops on the i^{th} boundary component. It is shown ([35, Proposition 4.2.1]) that $\mathcal{T}_i^\perp = {}^\perp \mathcal{T}_i$ corresponds to those Lagrangians in $\mathcal{W}(\Sigma;\Lambda)$ *not* ending on the i^{th} boundary component. One direction of this argument is clear: if there is a Lagrangian which is either compact, or does not end on the i^{th} boundary component, then the intersection with the geometric representatives of Lagrangians supported near the stops can be taken to be empty. In the other direction, one shows that if a Lagrangian *does* end on a boundary component, then there is necessarily a non-trivial morphism at the level of cohomology between this Lagrangian and a Lagrangian in $\mathcal{T}_i$. In the case where just one endpoint of the Lagrangian lies on the i^{th} boundary component there is a chain level morphism between the Lagrangian and a Lagrangian in $\mathcal{T}_i$ which is of rank one, so the differential vanishes. In the case where both endpoints lie on the i^{th} boundary component, the chain level morphism complex between the Lagrangian and a Lagrangian in $\mathcal{T}_i$ is either rank one or two. In the rank one case we again have that the differential must vanish, and in the rank two case one shows that the differential vanishes by a covering argument. This shows that any Lagrangian with at least one endpoint on the i^{th} boundary component cannot belong to $\mathcal{T}_i^\perp$. Checking that a Lagrangian with at least one endpoint on the i^{th} boundary component cannot belong to ${}^\perp\mathcal{T}_i$ is done in the same way.

By summing over the boundary components of Σ we define $\mathcal{T} = \bigoplus_i \mathcal{T}_i$. Then, [35, Corollary 4.2.2] shows:

- In the case of a ring of curves, the subcategory $\mathcal{F}(\Sigma) \subseteq \mathcal{W}(\Sigma;\Lambda)$ coincides with $\mathcal{T}^\perp = {}^\perp \mathcal{T}$, where $\mathcal{T}$ is the category generated by the objects $E_{i,j}^\pm$.
- In the case of a chain of curves with $r_{1,-}, r_{n,+} > 0$, the subcategory $\mathcal{F}(\Sigma; (r_{1,-})^{d_1}, (0)^{b-d_1-d_n}, (r_{n,+})^{d_n}) \subseteq \mathcal{W}(\Sigma;\Lambda)$ coincides with $\mathcal{T}^\perp = {}^\perp \mathcal{T}$, where $\mathcal{T}$ is the category generated by $E_{i,j}^+$ for $i \in \{1,\ldots,n-1\}$ and $E_{i,j}^-$ for $i \in \{2,\ldots,n\}$.

Proof of Theorem 2. In the case of a ring of curves, or a chain where $r_{1,-}, r_{n,+} > 0$, the theorem follows from the observation that the generating objects of the category $\mathcal{T}$ on both sides of the correspondence are identified under the equivalence given in Theorem 1. In the case where $r_{n,+} >$

$r_{1,-}=0$ we again consider $\overline{\mathcal{C}}$ such that $\mathcal{C}=\overline{\mathcal{C}}\setminus\{q_{1,-}\}$. Then, the statement follows from using the characterisation of $\operatorname{perf}\mathcal{C}\subseteq D^b(\mathcal{A}_{\overline{\mathcal{C}}}-\mathrm{mod})\simeq\mathcal{W}(\Sigma;\overline{\Lambda})$ as the category which is both left and right perpendicular to $\overline{\mathcal{T}}$. □

7. Applications

Before demonstrating our main application of invertible polynomials, we first consider an example which does not arise in this context.

Example 7.1. For an example outside of the framework of invertible polynomials, consider a ring of curves with two irreducible components given by

$$\mathcal{C}_1=\sqrt[4]{\mathcal{O}(q_{1,-}+2q_{1,+})/\mathbb{P}_{2,4}}$$
$$\mathcal{C}_2=\sqrt[2]{\mathcal{O}(q_{2,+})/\mathbb{P}_{8,4}},$$

where the presentation at $q_i\in|\mathcal{C}_1|\cap|\mathcal{C}_2|$ for $i=1,2$ is given by the action of $H_1=\mu_2\times\mu_8$

$$(\zeta^i,\eta^j)\cdot(x,y)=(\zeta^i\eta^{2j}x,\eta^j y)$$

at q_1, and the action of $H_2=\mu_8$ on the node q_2 is given by

$$t\cdot(x,y)=(t^2x,t^4y),$$

where the x coordinate is on $\mathcal{C}_2$ here. Letting $\mathcal{U}_{1,\pm}=\mathbb{P}_{2,4}\setminus\{q_{1,\mp}\}$, we have that isotropy group of the node $q_{1,+}$ is determined by the class in $H^2(\mathcal{U}_{1,+},\mu_4)$ given by the restriction $\mathcal{O}(q_{1,-}+2q_{1,+})|_{\mathcal{U}_{1,+}}$. This yields the short exact sequence

$$1\mapsto\mu_4\xrightarrow{\varphi_{1,+}}\mu_2\times\mu_8\xrightarrow{\psi_{1,+}}\mu_4\to 1,$$

where $\varphi_{1,+}$ is the map $\lambda\mapsto(\zeta^{-1},\eta^2)$, and $\psi_{1,+}$ is given by the map $(\zeta^i,\eta^j)\mapsto\zeta^i\eta^{2j}$. The short exact sequence characterising the isotropy group of the gerbe at $q_{2,-}$ is split, and so we have $\chi_{r_{2,-}}=(0,1)\in\mathbb{Z}/2\oplus\mathbb{Z}/8=\widehat{H}_1$, and take $\chi_{d_{2,-}}=(1,0)\in\widehat{H}_1$. Since $\chi_{r_{1,+}}=(1,2)$, we that the weight of H_1 on the fibre of $\mathcal{O}_{\widetilde{\mathcal{C}}_1}(q_{1,-}+2q_{1,+})$ at $q_{1,+}$ is $(0,-4)\in\widehat{H}_1$, and we take $\chi_{d_{1,+}}=(0,-1)\in\widehat{H}_1$.

At the node q_2, the isotropy group at $q_{1,-}$ is determined by the class of $\mathcal{O}(q_{1,-}+2q_{1,+})|_{\mathcal{U}_{1,-}}$ in $H^2(\mathcal{U}_{1,-},\mu_4)$, yielding the non-split exact sequence

$$1\to\mu_4\to\mu_8\xrightarrow{\wedge^4}\mu_2\to 1,$$

where the first map is the inclusion. The short exact sequence corresponding to the gerbe structure at $q_{2,+}$ is given by

$$1\mapsto\mu_2\to\mu_8\xrightarrow{\wedge^2}\mu_4\to 1.$$

At the node q_2 we have $\chi_{r_{2,+}}=2\in\mathbb{Z}/8=\widehat{H}_2$, $\chi_{r_{1,-}}=4$, and take $\chi_{d_{1,-}}=\chi_{d_{2,+}}=-1$. In order to compute the endomorphism algebra of the exceptional collection given in Section 3, we order the characters of H_1 such that $\chi_{4k_{1,+}+c_{1,+}}$ is the character $k_{1,+}\chi_{d_{1,+}}-c_{1,+}\chi_{r_{1,+}}$. Similarly, we order the characters of H_2 such that $\chi_{4k_{2,+}+c_{2,+}}$ is the character $k_{2,+}\chi_{d_{2,+}}-c_{2,+}\chi_{r_{2,+}}$.

To calculate the endomorphism algebra of the exceptional collection given in Section 3, fix $j_i=0$ and $m_i=-1$ for $i=0,1$. Then, there are morphisms

$$\operatorname{Ext}^1(\mathcal{S}_{q_1}\{\chi_{4k_{1,+}+c_{1,+}}\},\mathcal{P}_1(0,c_{1,+}-1,k_{1,+}))=\mathbb{C}\cdot a_1(c_{1,+},k_{1,+}),$$
$$\operatorname{Ext}^1(\mathcal{S}_{q_1}\{\chi_{4k_{1,+}+c_{1,+}}\},\mathcal{P}_2((7+k_{1,+}+2c_{1,+})\bmod 8),-1,c_{1,+}\bmod 2))=\mathbb{C}\cdot b_1(c_{2,-},k_{2,-}).$$

At the node q_2, we have morphisms

$$\mathrm{Ext}^1(\mathcal{S}_{q_2}\{\chi_{4k_{2,+}+c_{2,+}}\}, \mathcal{P}_2(0, c_{2,+}-1, k_{2,-}) = \mathbb{C}\cdot a_2(c_{2,+}, k_{2,+}),$$

$$\mathrm{Ext}^1(\mathcal{S}_{q_2}\{\chi_k\}, \mathcal{P}_1((2+\lfloor\frac{2c_{2,+}+k_{2,+}}{4}\rfloor) \bmod 2, -1, (2c_{2,+}+k_{2,+}) \bmod 4) = \mathbb{C}\cdot b_2(c_{1,-}, k_{1,-}).$$

Based on this, we can construct the mirror by gluing together $A(2,4;4)$ to $A(8,4;2)$ via the permutations

$$\sigma_1(4k_{1,+}+c_{1,+}) = 8(c_{1,+} \bmod 2) + (-k_{1,+}-2c_{1,+}) \bmod 8$$
$$\sigma_2(4k_{2,+}+c_{2,+}) = 2(2c_{2,+}+k_{2,+} \bmod 4) + (-\lfloor\frac{2c_{2,+}+k_{2,+}}{4}\rfloor) \bmod 2.$$

The cycle decompositions of determining the boundary components and their winding numbers are

$$\sigma_1^{-1}\tau_{\ell_2}\sigma_1\tau_{r_1} = (0\ 13\ 8\ 7\ 2\ 15\ 10\ 5)(1\ 14\ 9\ 4\ 3\ 12\ 11\ 6)$$
$$\sigma_2^{-1}\tau_{\ell_1}\sigma_2\tau_{r_2} = (0\ 1\ 2\ 3)(4\ 5\ 6\ 7),$$

yielding two boundary components with winding number -16, and two with winding number -8. The Euler characteristic is -24, and so the genus of the surface is 9.

Putting this all together, Theorem 1 yields

$$D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod}) \simeq \mathcal{W}(\Sigma_{9,4}; (8)^2, (16)^2).$$

Theorem 2 yields

$$\mathrm{perf}\,\mathcal{C} \simeq \mathcal{F}(\Sigma_{9,4})$$
$$D^b\,\mathrm{Coh}(\mathcal{C}) \simeq \mathcal{W}(\Sigma_{9,4}).$$

This surface does not arise as the Milnor fibre of an invertible polynomial; however, it is interesting to observe that it is graded symplectomorphic to a surface considered in [36]. Specifically, it is shown that the open surface obtained by gluing $A(8,16;1)$ to $A(16,8;1)$ by the permutations $\sigma_0(x) = -x \bmod 16$ and $\sigma_1(x) = -x \bmod 8$ is mirror to the ring of orbifolds whose irreducible components are given by $\mathbb{P}_{8,16}$ and $\mathbb{P}_{16,8}$, where the structure of the node is given by the action of G on $xy=0$ by $t\cdot(x,y) = (tx, ty)$ for $G = \mu_8$ or μ_{16}. This curve is denoted by $C(8,16;1,1) = \mathcal{C}_{\mathrm{orb}}$, and [35, Theorem A] establishes the existence of a surface Σ such that

$$D^b(\mathcal{A}_{\mathcal{C}_{\mathrm{orb}}} - \mathrm{mod}) \simeq \mathcal{W}(\Sigma; (8)^2, (16)^2),$$

and [35, Theorem B] yields

$$\mathrm{perf}\,\mathcal{C}_{\mathrm{orb}} \simeq \mathcal{F}(\Sigma),$$
$$D^b\,\mathrm{Coh}(\mathcal{C}_{\mathrm{orb}}) \simeq \mathcal{W}(\Sigma).$$

In this case, we can deduce that the surfaces $\Sigma_{9,4}$ and Σ are *graded symplectomorphic* by [36, Corollary 1.2.6] (The Arf invariant doesn't need to be checked in this case). Therefore, there are derived equivalences

$$D^b(\mathcal{A}_{\mathcal{C}_{\mathrm{orb}}} - \mathrm{mod}) \simeq D^b(\mathcal{A}_{\mathcal{C}} - \mathrm{mod}),$$
$$D^b\,\mathrm{Coh}(\mathcal{C}_{\mathrm{orb}}) \simeq D^b\,\mathrm{Coh}(\mathcal{C}).$$

7.1. **Invertible polynomials.** In this section, we establish Theorem 3 by firstly applying Theorems 1 and 2 to the curves appearing as the B–model of invertible polynomials in two variables. We then show that the surfaces constructed are graded symplectomorphic to $\check{V}/\check{\Gamma}$.

To begin with, recall the definition of invertible polynomials and the maximal symmetry group, as defined in Section 1.1. For simplicity, we will restrict ourselves to the case of two variable invertible polynomials, although much of the following is true in generality ([37, Section 2], [17, Section 1], [32, Section 3]).

Since $\mathbf{w}$ is quasi-homogeneous, there is an injective map

$$\begin{aligned}\varphi : \mathbb{C}^* &\to \Gamma_{\mathbf{w}} \\ t &\mapsto (t^{d_1}, t^{d_2}),\end{aligned}$$

and this fits in to the short exact sequence

$$1 \to \mathbb{C}^* \xrightarrow{\varphi} \Gamma_{\mathbf{w}} \to \ker\, \chi_{\mathbf{w}}/\langle j_{\mathbf{w}}\rangle \to 1, \tag{31}$$

where $j_{\mathbf{w}} = (e^{2\pi\sqrt{-1}\frac{d_1}{h}}, e^{2\pi\sqrt{-1}\frac{d_2}{h}})$ generates the cyclic group $\mathrm{im}(\varphi) \cap \ker \chi_{\mathbf{w}}$, and is called the grading element. We call a subgroup $\Gamma \subseteq \Gamma_{\mathbf{w}}$ of finite index containing $\varphi(\mathbb{C}^*)$ *admissible.* For each Γ we denote $\chi := \chi_{\mathbf{w}}|_{\Gamma}$, and define $\overline{\Gamma} = \ker \chi$. Note that, by construction, $\langle j_{\mathbf{w}}\rangle \subseteq \overline{\Gamma}$, and $[\ker \chi_{\mathbf{w}} : \overline{\Gamma}] < \infty$. Moreover, such subgroups of finite index containing the group generated by the grading element are in bijection with finite index subgroups $\Gamma \subseteq \Gamma_{\mathbf{w}}$ containing $\mathrm{im}(\varphi)$.

Given an admissible subgroup $\Gamma \subseteq \Gamma_{\mathbf{w}}$ of index ℓ, one can define the dual group as

$$\check{\Gamma} = \mathrm{Hom}(\ker \chi_{\mathbf{w}}/\overline{\Gamma}, \mathbb{C}^*) \subseteq \ker \chi_{\check{\mathbf{w}}} \cap \mathrm{SL}_2(\mathbb{C}).$$

This differs from the definition given in [32, Definition 3], although is equivalent by [17, Proposition 3]. The group $\check{\Gamma}$ acts naturally on $\mathbb{A}^2$ through its inclusion in $\ker \chi_{\check{\mathbf{w}}}$, and in each case, $\check{\Gamma} = \mu_\ell$ acts on $\mathbb{A}^2$ by

$$\xi \cdot (x, y) = (\xi x, \xi^{-1} y). \tag{32}$$

This can be checked directly, or deduced from the fact that $\check{\Gamma}$ is a diagonal matrix in $\mathrm{SL}_2(\mathbb{C})$, and so its two entries must be inverses of each other. Clearly, the only fixed point of this action is the origin, which is not a point in the Milnor fibre, and so the quotient of the Milnor fibre by $\check{\Gamma}$ is again a manifold.

7.1.1. *Loop polynomials.* For a loop polynomial $\mathbf{w} = x^p y + y^q x$, where we take $p \geq q$, we consider $\mathbf{W} = x^p y + y^q x + xyz$, and the corresponding stack

$$Z_{\mathbf{w}_{\mathrm{loop}},\Gamma} := [(\mathbf{W}^{-1}(0) \setminus \{\mathbf{0}\})/\Gamma],$$

where we take the action of Γ to be given by its inclusion to $\Gamma_{\mathbf{w}}$. Let $\ell = [\Gamma_{\mathbf{w}} : \Gamma]$ and identify $\Gamma \simeq \mathbb{C}^* \times \mu_{\frac{d}{\ell}}$, where $d = \gcd(p-1, q-1)$. The stack $Z_{\mathbf{w}_{\mathrm{loop}},\Gamma}$ has a natural interpretation as a codimension one closed substack in the toric DM orbifold $[(\mathbb{A}^3 \setminus \{\mathbf{0}\})/\Gamma]$. The unique stacky fan describing this DM orbifold is readily checked to be given by the data of

$$\beta : \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} \frac{p-1}{\ell} & \frac{1-q}{\ell} & 0 \\ 0 & q-1 & -1 \end{pmatrix}} \mathbb{Z}^2 =: N,$$

and each column corresponds to a ray of the fan Σ. The maximal cones of the fan are given by the span of any two rays. In general, this is a quotient of weighted projective space by $\mu_{\frac{d}{\ell}}$.

Remark 7.2. It is worth noting that we have made a choice in the identification $\Gamma \simeq \mathbb{C}^* \times \mu_{\frac{d}{\ell}}$, and thus how Γ acts on $\mathbb{A}^3 \setminus \{\mathbf{0}\}$; however, the above fan is independent of this choice. Choosing a different identification of Γ corresponds to choosing different change-of-basis matrices in the Smith normal form decomposition of $[BQ]^\vee$ used to calculate its cokernel.

With this description, one can see that $\mathcal{C}_1 = \{y = 0\} \subseteq Z_{\mathbf{w}_{\mathrm{loop}},\Gamma}$ is the closed substack of $[(\mathbb{A}^3 \setminus \{\mathbf{0}\})/\Gamma]$ corresponding to the ray $\rho_2 = \frac{1-q}{\ell} e_1 + (q-1)e_2$, and similarly that $\mathcal{C}_3 = \{x = 0\}$ is the closed substack corresponding to the ray $\rho_1 = \frac{p-1}{\ell} e_1$. The quotient fan $\boldsymbol{\Sigma}/\boldsymbol{\rho_2}$ is given by the complete fan in $\mathbb{Q}$, and

$$\beta(\rho_2) : \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} p-1 & -1 \\ \frac{1-p}{\ell} & 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/(\frac{q-1}{\ell}) =: N(\rho_2)$$

This is a $\mu_{\frac{q-1}{\ell}}$-gerbe over $\mathbb{P}_{p-1,1}$, and [18, Theorem 7.24] establishes that there is an isomorphism of toric DM stacks

$$\mathcal{C}_1 \simeq \sqrt[\frac{q-1}{\ell}]{\mathcal{O}(-\frac{p-1}{\ell}q_{1,-})/\mathbb{P}_{p-1,1}}.$$

Similarly, we have an isomorphism of toric DM stacks

$$\mathcal{C}_3 \simeq \sqrt[\frac{p-1}{\ell}]{\mathcal{O}(\frac{q-1}{\ell}q_{3,+})/\mathbb{P}_{1,q-1}}.$$

The curve $\mathcal{C}_2$ is always an orbifold, and can be identified with $\mathcal{C}_2 \simeq \mathbb{P}_{\frac{q-1}{\ell},\frac{p-1}{\ell}}$.

Remark 7.3. It is worth reiterating that we are not claiming that the gerbe structure of $\mathcal{C}_1$ and $\mathcal{C}_3$ are given as above, only that there is an isomorphism of DM stacks. Due to this, there is some freedom in the identifications, and we have chosen these for later convenience.

The majority of the analysis in studying the modules over the Auslander sheaf is at $q_3 = |\mathcal{C}_3| \cap |\mathcal{C}_1|$, which corresponds to the point $[0:0:1] \in |\mathcal{X}|$. This node is presented as the quotient of $xy = 0$ by the action of $\mu_{\frac{(p-1)(q-1)}{d}} \times \mu_{\frac{d}{\ell}}$ given by

$$(t,\xi)\cdot(x,y) = (t^{\frac{p-1}{d}}\xi^{-n}x, t^{\frac{q-1}{d}}\xi^{m}y),$$

where m, n are Bézout coefficients solving

$$m(p-1) + n(q-1) = d. \tag{33}$$

Therefore the gerbe structure of the point $q_{3,+}$ is determined by the cohomology class in $\mathbb{Z}/\gcd(q-1, \frac{p-1}{\ell}) \simeq H^2([\mathbb{A}^1/\mu_{q-1}], \mu_{\frac{p-1}{\ell}})$ corresponding to the mod $\frac{p-1}{\ell}$ reduction of $\frac{(\ell-1)(q-1)}{\ell} \in \mathbb{Z}$. Similarly, we have that the gerbe at $q_{1,-} \in |\mathcal{C}_1|$ is classified by the cohomology class in $\mathbb{Z}/\gcd(p-1, \frac{q-1}{\ell}) \simeq H^2([\mathbb{A}^1/\mu_{p-1}], \mu_{\frac{q-1}{\ell}})$ corresponding to the mod $\frac{q-1}{\ell}$ reduction of $\frac{p-1}{\ell} \in \mathbb{Z}$. The corresponding short exact sequences at $q_{3,+}$ and $q_{1,-}$ are

$$1 \to \mu_{\frac{p-1}{\ell}} \xrightarrow{\varphi_{3,+}} \mu_{\frac{(p-1)(q-1)}{d}} \times \mu_{\frac{d}{\ell}} \xrightarrow{\psi_{3,+}} \mu_{q-1} \to 1, \text{ and} \tag{34}$$

$$1 \to \mu_{\frac{q-1}{\ell}} \xrightarrow{\varphi_{1,-}} \mu_{\frac{(p-1)(q-1)}{d}} \times \mu_{\frac{d}{\ell}} \xrightarrow{\psi_{1,-}} \mu_{p-1} \to 1, \tag{35}$$

respectively. Here $\lambda_\pm$, η, and ξ are

$$\lambda_+ = e^{2\pi\sqrt{-1}\frac{\ell}{p-1}}, \quad \lambda_- = e^{2\pi\sqrt{-1}\frac{\ell}{q-1}},$$
$$\eta = e^{2\pi\sqrt{-1}\frac{d}{(p-1)(q-1)}}, \quad \xi = e^{2\pi\sqrt{-1}\frac{\ell}{d}},$$

and $\varphi_{3,+}$ is the map $\lambda_+ \mapsto (\eta^{-n\frac{(q-1)\ell}{d}}, \xi^{-1})$, $\psi_{3,+}$ is $(\eta^a, \xi^b) \mapsto \eta^{\frac{p-1}{d}a}\xi^{-nb}$, $\varphi_{1,-}$ is $\lambda_- \mapsto (\eta^{m\frac{(p-1)\ell}{d}}, \xi^{-1})$, $\psi_{1,-}$ is $(\eta^a, \xi^b) \mapsto \eta^{\frac{q-1}{d}a}\xi^{mb}$, where m, n are again the Bézout coefficients of (33).

From this description, we have that the group H_3 acts on the fibre of $\mathcal{O}_{\widetilde{\mathcal{C}}_3}(-q_{3,+})$ at $q_{3,+}$ with weight $\chi_{r_{3,+}} = (\frac{p-1}{d}, -n) \in \mathbb{Z}/(\frac{(p-1)(q-1)}{d}) \oplus \mathbb{Z}/(\frac{d}{\ell}) \simeq \widehat{H}_3$ for m, n solving (33), and similarly $\mathcal{O}_{\widetilde{\mathcal{C}}_1}(-q_{1,-})$ at $q_{1,-}$ is acted on with weight $\chi_{r_{1,-}} = (\frac{q-1}{d}, m)$. The character with which H_3 acts on the fibre of $\mathcal{N}_3$ is (non-uniquely) determined by the condition that $\frac{p-1}{\ell}\chi_{d_{3,+}} = \frac{1-q}{\ell}\chi_{r_{3,+}}$, and maps to a unit in $\mathbb{Z}/(\frac{p-1}{\ell})$ under the dual of $\varphi_{3,+}$. The natural choice for this is $\chi_{d_{3,+}} = -\chi_{r_{1,-}}$, and similarly we choose $\chi_{d_{1,-}} = \chi_{r_{3,+}}$.

In $\widehat{H}_3$, we label the characters such that $\chi_{k_+(q-1)+i} = -i\chi_{r_{3,+}} + k_+\chi_{d_{3,+}}$ for $k_+ \in \{0, \dots, \frac{p-1}{\ell} - 1\}$ and $i \in \{0, \dots, q-2\}$. This is the B–side version of labelling the stops on the right side of the left column of cylinders top-to-bottom. With this ordering, the sheaf on $\widetilde{\mathcal{C}}_1$ whose fibre at $q_{1,-}$ is acted on by H_3 with character $\chi_{k_+(q-1)+i}$ is given by

$$\mathcal{O}_{\widetilde{\mathcal{C}}_1}(jq_{1,-}) \otimes \mathcal{N}_1^{\otimes k_-},$$

where $j \in \{0, \ldots, p-2\}$ and $k_- \in \{0, \ldots, \frac{q-1}{\ell} - 1\}$ solves

$$-j\chi_{r_{1,-}} + k_-\chi_{d_{1,-}} = -i\chi_{r_{3,+}} + k_+\chi_{d_{3,+}}. \tag{36}$$

A solution to this is readily checked to be given by

$$\begin{aligned} k_- &= -i \bmod \frac{q-1}{\ell} \\ j &= k_+ - \frac{p-1}{\ell}\lfloor \frac{-i\ell}{q-1} \rfloor \bmod p-1. \end{aligned} \tag{37}$$

Fixing $m_i = -1$ and $j_i = 0$ as in the proof of Theorem 1, one computes

$$\begin{aligned} &\mathrm{Ext}^1(\mathcal{S}_{q_3}\{-i\chi_{r_{3,+}} + k_+\chi_{d_{3,+}}\}, \mathcal{P}_3(0, (i-1) \bmod q-1, k_+)) = \mathbb{C}\cdot a(i, k_+), \quad \text{and} \\ &\mathrm{Ext}^1(\mathcal{S}_{q_3}\{-i\chi_{r_{3,+}} + k_+\chi_{d_{3,+}}\}, \mathcal{P}_1((j-1) \bmod p-1, -1, k_-)) = \mathbb{C}\cdot b(j, k_-) \end{aligned}$$

for j, k_- as in (37).

Consider now the nodes $q_1 = |\mathcal{C}_1| \cap |\mathcal{C}_2|$ and $q_2 = |\mathcal{C}_2| \cap |\mathcal{C}_3|$. The structure of these nodes is far more simple, and at q_1 we have the node is presented as the quotient of $xy = 0$ by the action of $\mu_{\frac{q-1}{\ell}}$ given by

$$t \cdot (x, y) = (x, ty),$$

and analogously for q_2. Therefore, one has $\widehat{H}_1 \simeq \mathbb{Z}/(\frac{q-1}{\ell})$ and $\widehat{H}_2 \simeq \mathbb{Z}/(\frac{p-1}{\ell})$, and $\chi_{r_{2,-}}$ and $\chi_{r_{2,+}}$ are the identity in $\mathbb{Z}/(\frac{q-1}{\ell})$ and $\mathbb{Z}/(\frac{p-1}{\ell})$, respectively. The character with which H_1 acts on the fibre of $\mathcal{N}_1$ at $q_{1,+}$ (resp. on $\mathcal{N}_3$ at $q_{3,-}$) is any unit of $\widehat{H}_1$ (resp. $\widehat{H}_2$), and so we choose $\chi_{d_{1,+}}$ to be the identity and $\chi_{d_{3,-}}$ to be minus the identity in their respective character groups. With this, the morphisms between objects in the exceptional collection supported at q_1 are readily checked to be

$$\begin{aligned} &\mathrm{Ext}^1(\mathcal{S}_{q_1}\{c\}, \mathcal{P}_1(0, -1, c)) = \mathbb{C}\cdot a(0, c) \\ &\mathrm{Ext}^1(\mathcal{S}_{q_1}\{c\}, \mathcal{P}_2((-1-c) \bmod \frac{q-1}{\ell}, -1)) = \mathbb{C}\cdot b(-c), \end{aligned}$$

and similarly for the morphisms between objects supported at q_2.

As the mirror to $\mathcal{C}$, we take the surface given by gluing $A(p-1, 1; \frac{q-1}{\ell})$, $A(\frac{q-1}{\ell}, \frac{p-1}{\ell}; 1)$ and $A(1, q-1; \frac{p-1}{\ell})$ via the permutations $\sigma_1 = \mathrm{id} \in \mathfrak{S}_{\frac{q-1}{\ell}}$, $\sigma_2 = \mathrm{id} \in \mathfrak{S}_{\frac{p-1}{\ell}}$, and $\sigma_3 \in \mathfrak{S}_{\frac{(p-1)(q-1)}{\ell}}$ is given by

$$k_+(q-1) + i \mapsto k_-(p-1) + (-j) \bmod p-1$$

for i, j solving (37). From this, it is clear that one boundary component with winding number $-2\frac{q-1}{\ell}$ arises from σ_1, and similarly that one boundary component with winding number $-2\frac{p-1}{\ell}$ arises from σ_2. The number of boundary components, and their winding numbers, arising from σ_3 is given by the number of cycles, and their respective lengths, of $\sigma_3^{-1}\tau_{\ell_1}\sigma_3\tau_{r_3}$. This permutation is given by

$$k_+(q-1) + i \mapsto (q-1)\Big((k_+ - 1) \bmod \frac{p-1}{\ell}\Big) + \Big(i - 1 + \frac{q-1}{\ell}\Big\lfloor \frac{(k_+ - 1)\ell}{p-1} \Big\rfloor\Big) \bmod q-1$$

and so there are $\gcd(q-1, \frac{p+q-2}{\ell}) = \gcd(p-1, \frac{p+q-2}{\ell})$ cycles, each of length $\frac{(p-1)(q-1)}{\gcd(\ell(q-1), p+q-2)}$. Therefore, $\gcd(q-1, \frac{p+q-2}{\ell})$ boundary components arise from this gluing, and each has winding number $-2\frac{(p-1)(q-1)}{\gcd(\ell(q-1), p+q-2)}$.

Putting this all together, we have that the surface constructed, call it $\Sigma_{\mathbf{w}_{\text{loop}},\Gamma}$, has $2+\gcd(q-1,\frac{p+q-2}{\ell})$ components, and Euler characteristic given by

$$-\chi(\Sigma_{\mathbf{w}_{\text{loop}},\Gamma}) = \frac{q-1}{\ell} + \frac{p-1}{\ell} + \gcd(q-1,\frac{p+q-2}{\ell})\frac{(p-1)(q-1)}{\ell\gcd(q-1,\frac{p+q-2}{\ell})} = \frac{pq-1}{\ell}.$$

Therefore, the genus is

$$g(\Sigma_{\mathbf{w}_{\text{loop}},\Gamma}) = \frac{1}{2\ell}(pq-1-\gcd(\ell(q-1),p+q-2)).$$

Applying Theorem 1 yields a quasi-equivalence

$$D^b(\mathcal{A}_{\mathcal{C}}-\text{mod}) \simeq \mathcal{W}\left(\Sigma_{\mathbf{w}_{\text{loop}},\Gamma}; 2\frac{p-1}{\ell}, \left(2\frac{(p-1)(q-1)}{\gcd(\ell(q-1),p+q-2)}\right)^{\gcd(q-1,\frac{p+q-2}{\ell})}, 2\frac{q-1}{\ell}\right),$$

and then Theorem 2 establishes quasi-equivalences

$$D^b\operatorname{Coh}(Z_{\mathbf{w}_{\text{loop}},\Gamma}) \simeq \mathcal{W}(\Sigma_{\mathbf{w}_{\text{loop}},\Gamma})$$
$$\operatorname{perf} Z_{\mathbf{w}_{\text{loop}},\Gamma} \simeq \mathcal{F}(\Sigma_{\mathbf{w}_{\text{loop}},\Gamma}).$$

In the case of $\Gamma = \Gamma_{\mathbf{w}}$, we observe that the graded surface constructed on the A–side is graded symplectomorphic to the Milnor fibre of the transpose invertible polynomial. To see this, we note that the above gluing is the same as the gluing permutation of [25, Section 3.2.1], although where the identification of the cylinders in $A(p-1,1;\frac{q-1}{\ell})$ here have been rotated $-\frac{2\pi}{p-1}$ degrees. It was established in loc. cit. that the surface glued in this way is graded symplectomorphic to the Milnor fibre of $\check{\mathbf{w}}$ by comparing the corresponding ribbon graphs. Building on this strategy, we establish a graded symplectomorphism $\check{V}/\check{\Gamma} \simeq \Sigma_{\mathbf{w}_{\text{loop}},\Gamma}$ by first making a topological identification via the quotient ribbon graphs, and then deducing that the grading structures match by elimination.

Recall the description of $\check{V}$ as $\check{\mathbf{w}}_{\varepsilon}^{-1}(-\delta)$ for $0<\delta\ll\varepsilon$ given in [26, Section 3], where

$$\check{\mathbf{w}}_{\varepsilon} = \check{\mathbf{w}} - \varepsilon\check{x}\check{y} = \check{x}\check{y}(\check{x}^{p-1}+\check{y}^{p-1}-\varepsilon) = \check{x}\check{y}\check{w}.$$

Firstly, observe that the Morsification chosen is $\check{\Gamma}$-equivariant, and so taking the quotient commutes with Morsifying. Moreover, since the quotient map is an unramified cover, the deformation retract which takes $\check{V}$ to its ribbon graph also commutes with the quotient map. With respect to the classification of critical points in [26, Section 3.1], we refer to neck regions which form by smoothing critical points of type (i) as neck regions of type (i), and the corresponding node in the ribbon graph as a node of type (i). We refer similarly to neck regions and nodes of type (ii) and (iii). We index the nodes of type (i) and (ii) according to the $\check{x}$ and $\check{y}$ argument of the corresponding critical points, respectively. Then, the l^{th} node of type (i) is identified with the $(l+\frac{p-1}{\ell})^{\text{th}}$ node of type (i) under the action of $\check{\Gamma}$. Similarly, the m^{th} node of type (ii) is identified with the $(m-\frac{q-1}{\ell})^{\text{th}}$ node of type (ii). This partitions the nodes of the ribbon graph.

To understand how $\check{\Gamma}$ partitions the edges, recall that part of the basis for the first homology group of $\check{V}$ is given by the Lagrangians ${}^{l}V_{\check{y}\check{w}}$ (resp. ${}^{m}V_{\check{x}\check{w}}$ and $V_{\check{x}\check{y}}$), which were defined as the waist curves which form in the l^{th} neck region of type (i) (resp. the m^{th} neck region of type (ii), and the neck region of type (iii)) upon smoothing. Since Morsification commutes with the action of $\check{\Gamma}$, the Lagrangians ${}^{l}V_{\check{y}\check{w}}$ and ${}^{l+\frac{p-1}{\ell}}V_{\check{y}\check{w}}$ become identified in the quotient, and therefore so to do the edges of the ribbon graph onto which these Lagrangians deformation retract. The analogous statement for the Lagrangians ${}^{m}V_{\check{x}\check{w}}$ is also true, and so we see that two loops of the graph are identified with each other when the corresponding nodes are.

To understand the action of $\check{\Gamma}$ on the remaining edges, recall that two nodes are connected by an edge if there is a vanishing cycle which passes through both corresponding neck regions. The cyclic ordering of the nodes is determined by the argument of the Lagrangian away from the neck regions which it connects – see, for example, [26, Figure 8]. From this, it is clear that edges between the node of type (iii) and nodes of type (i) (resp. type (ii)) are identified in the quotient when the corresponding nodes of type (i) (resp. type (ii)) are. All that remains is to understand the action of $\check{\Gamma}$ on edges which connect the nodes of type (i) and (ii). For this, recall ([26, Section 3.5]) that the remaining vanishing cycles which form a basis of the first homology of $\check{V}$ are given by ${}^{l,m}V_0$ for $l \in \{0, \ldots, p-2\}$, $m \in \{0, \ldots, q-2\}$, and these are the Lagrangians which pass through the l^{th} neck region of type (i) and the m^{th} neck region of type (ii). By analysing the action of $\check{\Gamma}$ on the $\check{x}$ and $\check{y}$ projections of the Milnor fibre, as given in [26, Section 3.3], we see that ${}^{l,m}V_0$ gets identified with ${}^{l+\frac{p-1}{\ell}, m-\frac{q-1}{\ell}}V_0$ as it enters the $\{\check{w} = \varepsilon\}$ component the Milnor fibre. Away from the neck regions which connect it to the smoothings of the $\{\check{x} = 0\}$ and $\{\check{y} = 0\}$ components, $\{\check{w} = \varepsilon\}$ is an unramified cover of $\{\{u + v = \varepsilon\} \setminus (B_\delta(\varepsilon, 0) \cup B_\delta(0, \varepsilon))\} \subseteq \mathbb{C}^2$, and so the Lagrangians ${}^{l,m}V_0$ and ${}^{l+\frac{p-1}{\ell}, m-\frac{q-1}{\ell}}V_0$ get identified in the component $\{\check{w} = \varepsilon\}$. Therefore, the edge of the ribbon graph connecting the l^{th} node of type (i) with the m^{th} node of type (ii) gets identified with the edge connecting the $(l + \frac{p-1}{\ell})^{\text{th}}$ node of type (i) with the $(m - \frac{q-1}{\ell})^{\text{th}}$ node of type (ii) – see Figure 7 for an example. Note that this identifies the cyclic ordering of the two nodes in a non-trivial way. Moreover, the pushforward of the basis of the first homology for the ribbon graph of $\check{V}$ given by the deformation retract of vanishing cycles spans the first homology of the quotient ribbon graph. Therefore, the pushforward of vanishing cycles spans the first homology of $\check{V}/\check{\Gamma}$. It should be emphasised, however, that we are making no attempt to precisely describe a basis of Lagrangians on $\check{V}/\check{\Gamma}$; we only claim that the vanishing cycles span the first homology of $\check{V}/\check{\Gamma}$. In general, two Lagrangians ${}^{l,m}V_0$ and ${}^{l+\frac{p-1}{\ell}, m-\frac{q-1}{\ell}}V_0$ are not isotopic in the quotient, but are related by Dehn twists around the waist curves of the cylinders through which they both pass.

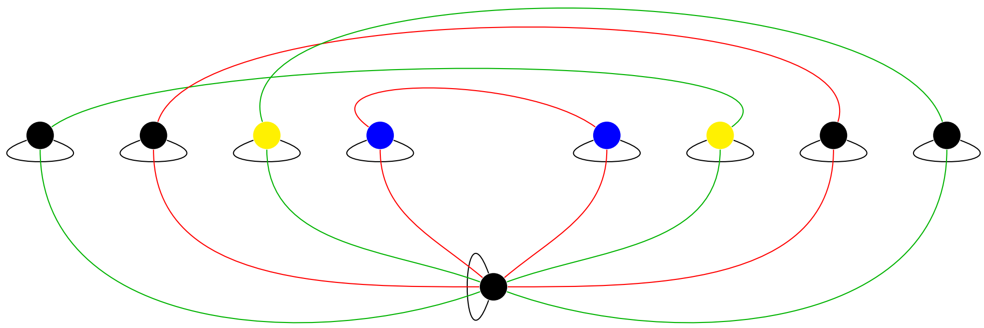

FIGURE 7. Part of the ribbon graph corresponding to $\check{V}$ for $\check{\mathbf{w}} = \check{x}^5\check{y} + \check{y}^5\check{x}$. For clarity, we have only drawn the edges which form the cycles onto which the vanishing cycles ${}^{i,-i}V_0$ for $i \in \{0, 1, 2, 3\}$ deformation retract. In the quotient of $\check{V}$ by $\check{\Gamma} = \mu_2$, the two red cycles and two green cycles are identified, and the representatives of the nodes are given by the blue and yellow nodes (recall $\arg \check{x} = -\arg \check{y}$), together with the node of type (iii). In the case of $\check{\Gamma} = \mu_4$, all coloured cycles are identified, and the blue nodes, as well as the node of type (iii), are taken as the representative in the quotient.

Since the cyclic ordering at nodes is identified in a non-trivial way, one must choose a representative of each equivalence class of nodes to work with a specific ordering. By convention, we will choose the nodes of type (i) corresponding to the neck regions which arise from smoothing the critical points with argument $\arg \check{x} \in \{0, \frac{2\pi}{p-1}, \ldots, \frac{2\pi(p-1-\ell)}{\ell(p-1)}\}$, and similarly we choose the nodes of type (ii) to correspond to the smoothing of the critical points of type (ii) with $\arg \check{y} \in \{0, -\frac{2\pi}{q-1}, \ldots, -\frac{2\pi(q-1-\ell)}{\ell(q-1)}\}$. Figures 8 and 9 show the cases of $\check{V}/\check{\Gamma}$ for $\check{V}$ the Milnor fibre of $\check{x}^5\check{y} + \check{y}^5\check{x}$ and $\check{\Gamma} = \mu_2,\ \mu_4$, respectively. From this, we see that the surface corresponding to this quotient ribbon graph is given by gluing $A(p-1,1;\frac{q-1}{\ell})$, $A(\frac{q-1}{\ell},\frac{p-1}{\ell};1)$ and $A(1,q-1;\frac{p-1}{\ell})$ via the permutations $\sigma_1 = \mathrm{id} \in \mathfrak{S}_{\frac{q-1}{\ell}}$, $\sigma_2 = \mathrm{id} \in \mathfrak{S}_{\frac{p-1}{\ell}}$, and $\sigma_3 \in \mathfrak{S}_{\frac{(p-1)(q-1)}{\ell}}$, where σ_3 is given by

$$k_+(q-1)+i \mapsto \big((-i) \bmod \frac{q-1}{\ell}\big)(p-1) + (p-2-k_+ + \frac{p-1}{\ell}\lfloor \frac{-i\ell}{q-1} \rfloor).$$

As in the maximally graded case, this only differs from the gluing given for $\Sigma_{\mathbf{w}_{\mathrm{loop}},\Gamma}$ by changing the identification of the cylinders in the column $A(p-1,1;\frac{q-1}{\ell})$.

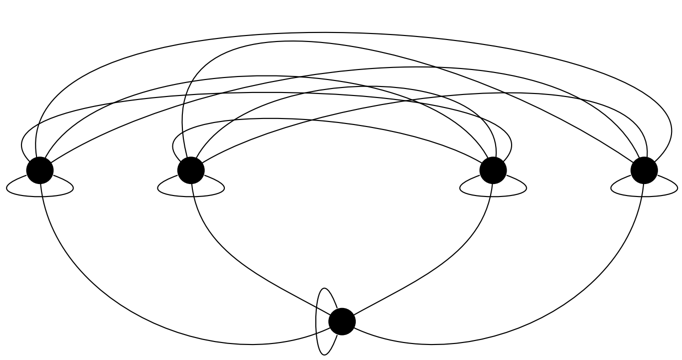

FIGURE 8. Ribbon graph corresponding to $\check{V}/\mu_2$ for $\check{\mathbf{w}} = \check{x}^5\check{y} + \check{y}^5\check{x}$.

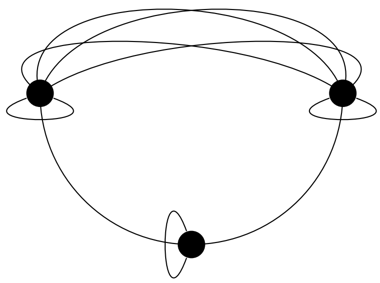

FIGURE 9. Ribbon graph corresponding to $\check{V}/\mu_4$ for $\check{\mathbf{w}} = \check{x}^5\check{y} + \check{y}^5\check{x}$.

To identify the line field used to grade $\check{V}/\check{\Gamma}$, observe that the pushforward of any vanishing cycle in $\check{V}$ is gradable with respect to the line field which is horizontal on cylinders and parallel to the edges of the attaching strips, which we denote by η. Indeed, for the waist curves to be gradable, the only possible line field is the one which is horizontal on cylinders. To see that the pushforward of the Lagrangians ${}^{l,m}V_0$ are gradable with respect to η, observe that the pushforward of such a Lagrangian deformation retracts onto a cycle of the quotient ribbon graph which passes through three nodes, one each of type (i), (ii), and (iii). Therefore, the pushforward Lagrangian is characterised by which attaching strips it passes through, as well as some number of Dehn twists about the waist curves in the cylinders which the attaching strips connect, and any such Lagrangian is gradable with respect to η. By the uniqueness (up to homotopy) of the line field with respect to which the pushforward of the vanishing cycles of $\check{V}$ are all gradable, the line field on $\check{V}/\check{\Gamma}$ is homotopic to η. This completes the proof of Theorem 3 in the case of loop polynomials.

7.1.2. *Chain polynomials.* For a chain polynomial $\mathbf{w} = x^py + y^q$ we consider $\mathbf{W} = x^py + y^q + xyz$, and $\Gamma \subseteq \Gamma_{\mathbf{w}}$ of index ℓ with identification $\Gamma \simeq \mathbb{C}^* \times \mu_{\frac{d}{\ell}}$, where $d := \gcd(p, q-1)$. We define the corresponding stack

$$Z_{\mathbf{w}_{\mathrm{chain}},\Gamma} := [\big(\mathbf{W}^{-1}(0) \setminus \{\mathbf{0}\}\big)/\Gamma],$$

where Γ acts by its inclusion into $\Gamma_{\mathbf{w}}$. This stack has two irreducible components – the first is $\mathcal{C}_2 = \{x^p + y^{q-1} + xz = 0\} \simeq \mathbb{P}_{\frac{(p-1)(q-1)}{\ell},\frac{q-1}{\ell}}$, and the second we identify with a $\mu_{\frac{q-1}{\ell}}$-gerbe over $\mathbb{P}_{1,p-1}$ as follows: We identify $\mathcal{C}_1$ as the closed substack of $Z_{\mathbf{w}_{\mathrm{chain}},\Gamma}$ corresponding to the divisor $\{y = 0\}$. Analogously to the loop case, we see that the quotient stack $[(\mathbb{A}^3 \setminus \{\mathbf{0}\})/\Gamma]$ corresponds to the stacky fan given by the data of a morphism

$$\beta : \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 1-q & 1 \\ 0 & \frac{(p-1)(q-1)}{\ell} & -\frac{p}{\ell} \end{pmatrix}} \mathbb{Z}^2 =: N,$$

and the rays of the fan Σ correspond to the column vectors. The maximal cones of the fan are given by the span of any two rays. In general, this is a quotient of weighted projective space by $\mu_{\frac{d}{\ell}}$.

With this description, we see that $\mathcal{C}_1$ is the closed substack corresponding to the ray $\rho_2 = (1-q)e_1 + \frac{(p-1)(q-1)}{\ell}e_2$, and so $\mathcal{C}_1$ is given by the quotient fan consisting of the complete fan in $\mathbb{Q}$, $N = \mathbb{Z} \oplus \mathbb{Z}/(\frac{q-1}{\ell})$, and

$$\beta : \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} p-1 & -1 \\ -\frac{p}{\ell} & 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/(\frac{q-1}{\ell}).$$

Again, by [18, Theorem 7.24], we see that there is an equivalence of toric DM stacks

$$\mathcal{C}_1 \simeq \sqrt[\frac{q-1}{\ell}]{\mathcal{O}(-\frac{p}{\ell}q_{1,-})/\mathbb{P}_{p-1,1}}.$$

As in the loop case, the computation of the morphisms in the exceptional collection is done locally. To this end, consider a local presentation of the node $q_2 = |\mathcal{C}_2| \cap |\mathcal{C}_1| = [0:0:1]$. This is given by the quotient of $xy = 0$ by the action of $\mu_{\frac{(p-1)(q-1)}{\ell}}$ given by

$$t \cdot (x, y) = (tx, t^{q-1}y).$$

This yields $\chi_{r_{1,-}} = q-1$ and $\chi_{r_{2,+}} = 1$. Therefore, the presentation of the gerbe $\mathcal{C}_1$ at $q_{1,-}$ is determined by the class of $\frac{p}{\ell} \bmod \frac{q-1}{\ell} \in \mathbb{Z}/\gcd(p-1, \frac{q-1}{\ell}) \simeq H^2([\mathbb{A}^1/\mu_{p-1}], \mu_{\frac{q-1}{\ell}})$. This gives the short exact sequence

$$1 \to \mu_{\frac{q-1}{\ell}} \hookrightarrow \mu_{\frac{(p-1)(q-1)}{\ell}} \xrightarrow{\wedge^{q-1}} \mu_{p-1} \to 1.$$

The action of H_2 on $\mathcal{N}_1$ at $q_{1,-}$ is such that $\frac{q-1}{\ell}\chi_{d_{1,-}} = \frac{p}{\ell}\chi_{r_{1,-}}$ in $\mathbb{Z}/(\frac{(p-1)(q-1)}{\ell}) = \widehat{H}_2$, and a natural choice for this character is $\chi_{d_{1,-}} = 1$. We order the characters in $\widehat{H}_2$ such that $\chi_c = -c$. With this ordering, the sheaf on $\widetilde{\mathcal{C}}_1$ whose fibre at $q_{1,-}$ is acted on by H_2 with character χ_c is given by

$$\mathcal{O}_{\widetilde{\mathcal{C}}_1}(jq_{1,-}) \otimes \mathcal{N}_1^{\otimes k_-},$$

where

$$\begin{aligned} k_- &= -c \bmod \frac{q-1}{\ell} \\ j &= -\frac{p}{\ell}\lfloor \frac{-c\ell}{q-1} \rfloor \bmod p-1. \end{aligned} \tag{38}$$

From this, one can see that we have the following morphisms in the exceptional collection:

$$\begin{aligned} &\mathrm{Ext}^1(\mathcal{S}_{q_2}\{\chi_c\}, \mathcal{P}_2(0, c-1)) = \mathbb{C} \cdot a_2(c) \\ &\mathrm{Ext}^1(\mathcal{S}_{q_2}\{\chi_c\}, \mathcal{P}_1((-1-j) \bmod p-1, -1, k_-)) = \mathbb{C} \cdot b_2(-j, k_-) \end{aligned}$$

for j, k_- as in (38).

As in the loop case, the analysis of the node $q_1 = |\mathcal{C}_1| \cap |\mathcal{C}_2|$ is determined by the choice of $\chi_{d_{1,-}}$. In particular, we have $\widehat{H}_1 = \mathbb{Z}/(\frac{q-1}{\ell})$, $\chi_{r_{2,-}} = 1$, and take $\chi_{d_{1,+}} = -1$. We again order the elements of $\widehat{H}_2$ such that $\chi_c = -c$, and with this we have the following morphisms in the exceptional collection:

$$\begin{aligned} &\mathrm{Ext}^1(\mathcal{S}_{q_1}\{\chi_c\}, \mathcal{P}_1(0, -1, c)) = \mathbb{C} \cdot a_1(0, c) \\ &\mathrm{Ext}^1(\mathcal{S}_{q_1}\{\chi_c\}, \mathcal{P}_2((c-1) \bmod \frac{(p-1)(q-1)}{\ell}, -1)) = \mathbb{C} \cdot b_1(c). \end{aligned}$$

To construct the mirror to this curve, we glue together two columns, $A(p-1,1;\frac{q-1}{\ell})$ and $A(\frac{q-1}{\ell},\frac{(p-1)(q-1)}{\ell};1)$ via the permutation $\sigma_1 = \mathrm{id} \in \mathfrak{S}_{\frac{q-1}{\ell}}$ gluing the first column to the second, and the permutation $\sigma_2 \in \mathfrak{S}_{\frac{(p-1)(q-1)}{\ell}}$ given by

$$c \mapsto k_-(p-1) + (-j) \bmod p-1$$

for k_-, j as in (38) gluing the second column back to the first. From this, it is clear that there is one boundary component arising from the first gluing, and this has winding number $-2\frac{q-1}{\ell}$. From the second gluing, we have that $\sigma_2^{-1}\tau_{\ell_1}\sigma_2\tau_{r_2}$ is given by

$$c \mapsto c - q,$$

and so there are $\gcd(q, \frac{p+q-1}{\ell})$ boundary components, each with winding number $-2\frac{(p-1)(q-1)}{\gcd(\ell q, p+q-1)}$.

Putting this all together, we have constructed a surface, call it $\Sigma_{\mathbf{w}_{\text{chain}},\Gamma}$, which has $1+\gcd(q, \frac{p+q-1}{\ell})$ components, Euler characteristic

$$-\chi(\Sigma_{\mathbf{w}_{\text{chain}},\Gamma}) = \frac{p(q-1)}{\ell},$$

and genus

$$g_{\text{chain}} = \frac{1}{2\ell}(pq - p + \ell - \gcd(\ell q, p+q-1)).$$

Applying Theorem 1 yields a quasi equivalence

$$D^b(\mathcal{A}_{Z_{\mathbf{w}_{\text{chain}},\Gamma}} - \text{mod}) \simeq \mathcal{W}\bigg(\Sigma_{\mathbf{w}_{\text{chain}},\Gamma}; 2\frac{q-1}{\ell}, \Big(2\frac{(p-1)(q-1)}{\gcd(\ell q, p+q-1)}\Big)^{\gcd(q,\frac{p+q-1}{\ell})}\bigg).$$

Applying Theorem 2 yields

$$D^b\,\mathrm{Coh}(Z_{\mathbf{w}_{\text{chain}},\Gamma}) \simeq \mathcal{W}(\Sigma_{\mathbf{w}_{\text{chain}},\Gamma}),$$
$$\mathrm{perf}\, Z_{\mathbf{w}_{\text{chain}},\Gamma} \simeq \mathcal{F}(\Sigma_{\mathbf{w}_{\text{chain}},\Gamma}).$$

In the case of maximally graded chain polynomials, observe that the above description differs from that of [25, Section 3.2.2] only be a rotation of the identification of the left boundary of the first annulus in the first column. Therefore, the surface constructed in the maximally graded case is graded symplectomorphic to the Milnor fibre of $\check{\mathbf{w}}$ in the maximally graded case. In the case of $\ell > 1$, we follow the same strategy as in Section 7.1.1 to deduce that $\check{V}/\check{\Gamma}$ is graded symplectomorphic to $\Sigma_{\text{chain},\Gamma}$, and this establishes Theorem 3 in the case of chain polynomials.

7.1.3. *Brieskorn–Pham polynomials.* The case of Brieskorn–Pham polynomials is covered in [35], although we include it here for completeness. For each Brieskorn–Pham polynomial $\mathbf{w} = x^p + y^q$, we consider $\mathbf{W} = x^p + y^q + xyz$, and $\Gamma \subseteq \Gamma_{\mathbf{w}}$ a subgroup of index ℓ containing the group generated by the grading element with identification $\Gamma \simeq \mathbb{C}^* \times \mu_{\frac{d}{\ell}}$. As in the previous cases, we define

$$Z_{\mathbf{w}_{\text{BP}},\Gamma} = [(\mathbf{W}^{-1}(0) \setminus \{\mathbf{0}\})/\Gamma],$$

where Γ acts by its inclusion into $\Gamma_{\mathbf{w}}$. This stack has one irreducible component, whose coarse moduli space is a nodal rational curve, and the normalisation is given by $\widetilde{\mathcal{C}} \simeq \mathbb{P}_{\frac{(p-1)(q-1)-1}{\ell},\frac{(p-1)(q-1)-1}{\ell}}$. We identify the coordinates in the patch of $\widetilde{\mathcal{C}}$ containing $q_+ = \infty$ as x, and in the patch containing $q_- = 0$ as y. Therefore, the presentation of $\mathcal{C}$ around the node q is given by the quotient of $xy = 0$ by $H = \mu_{\frac{(p-1)(q-1)-1}{\ell}}$, where the action is given by

$$t \cdot (x, y) = (t^{q-1}x, ty).$$

Correspondingly, H acts on the fibre of $\mathcal{O}(-q_-)$ at q_- with weight 1, and with weight $q-1$ on the fibre $\mathcal{O}(-q_+)$ at q_+.

In $\widehat{H} = \mathbb{Z}/(\frac{(p-1)(q-1)-1}{\ell})$, we label the characters such that $\chi_c = -c(q-1)$. Then, for each $c \in \mathbb{Z}/(\frac{(p-1)(q-1)-1}{\ell})$, we have the following morphisms in the exceptional collection:

$$\operatorname{Ext}^1(\mathcal{S}_q\{\chi_c\}, \mathcal{P}(0, c-1)) = \mathbb{C}\cdot a(c)$$
$$\operatorname{Ext}^1(\mathcal{S}_q\{\chi_c\}, \mathcal{P}((c(q-1)-1) \bmod \frac{(p-1)(q-1)-1}{\ell}, -1)) = \mathbb{C}\cdot b(c(q-1))$$

Correspondingly, the mirror surface is given by gluing the annulus $A(\frac{(p-1)(q-1)-1}{\ell}, \frac{(p-1)(q-1)-1}{\ell}; 1)$ to itself via the permutation $\sigma \in \mathfrak{S}_{\frac{(p-1)(q-1)-1}{\ell}}$ given by

$$c \mapsto -c(q-1).$$

The commutator $[\sigma, \tau] \in \mathfrak{S}_{\frac{(p-1)(q-1)-1}{\ell}}$, where τ is the permutation $c \mapsto c-1$, is given by

$$c \mapsto c - p.$$

Correspondingly, the constructed surface, call it $\Sigma_{\mathbf{w}_{\mathrm{BP}},\Gamma}$, has $\gcd(q, \frac{p+q}{\ell}) = \gcd(p, \frac{p+q}{\ell})$ boundary components, each of winding number $-2\frac{(p-1)(q-1)-1}{\gcd(\ell q, p+q)}$. Therefore, the Euler characteristic is

$$-\chi(\Sigma_{\mathbf{w}_{\mathrm{BP}},\Gamma}) = \frac{(p-1)(q-1)-1}{\ell},$$

and the genus is

$$g_{\mathrm{BP}} = \frac{1}{2\ell}(2\ell - 1 + (p-1)(q-1) - \gcd(\ell q, p+q)).$$

Applying Theorem 1 yields

$$D^b(\mathcal{A}_{Z_{\mathbf{w}_{\mathrm{BP}},\Gamma}} - \mathrm{mod}) \simeq \mathcal{W}\left(\Sigma_{\mathbf{w}_{\mathrm{BP}},\Gamma}; \left(2\frac{(p-1)(q-1)-1}{\gcd(\ell q, p+q)}\right)^{\gcd(q, \frac{p+q}{\ell})}\right),$$

and applying Theorem 2 yields

$$D^b \operatorname{Coh}(Z_{\mathbf{w}_{\mathrm{BP}},\Gamma}) \simeq \mathcal{W}(\Sigma_{\mathbf{w}_{\mathrm{BP}},\Gamma}),$$
$$\operatorname{perf} Z_{\mathbf{w}_{\mathrm{BP}},\Gamma} \simeq \mathcal{F}(\Sigma_{\mathbf{w}_{\mathrm{BP}},\Gamma}).$$

In the maximally graded case, the description of the mirror surface matches that of [25, Section 3.2.3] on-the-nose, and so is graded symplectomorphic to the Milnor fibre of $\check{\mathbf{w}}$. The proof that $\check{V}/\check{\Gamma}$ is graded symplectomorphic to $\Sigma_{\mathbf{w}_{\mathrm{BP}},\Gamma}$ follows as in the loop and chain cases, and this completes the proof of Theorem 3.

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