Previous Exam solution

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Problem 1

Periodicity

i) $x(t) = 3\cos(3t + \pi/3)$ is periodic as it is a simple sinusoid with $T = \frac{2\pi}{3}$.

ii) $x(t) = 3\cos^2(3t+\pi/3)$ is also periodic. To see this note that $y(t) = 3\cos(3t+\pi/3)$ is periodic as before. Then that $y(t + \frac{\pi}{3}) = -y(t)$. but since we take the square $x(t) = y^2(t) = y^2(t + \frac{\pi}{3})$ it now is periodic with half the period of $y(t)$, namely $T = \frac{\pi}{3}$.

iii) $x(t) = \cos(\frac{\pi}{2}t) + \cos(\frac{1}{2}t)$ is a sum of sinusoids with periods of $T_1 = 4$ and $T_2 = 4\pi$. $x(t)$ is periodic if there exist two integers n, m such that $n \cdot T_1 = m \cdot T_2$. This is not possible as the fraction of $n/m \propto \pi$, which is irrational. Hence, no such integer numbers exist.

iv) $x(t) = e^{j\pi t}$ is trivially periodic, as it is a simple complex exponential and thus by Eulers rule, a sum of sinusoids of equal periods.

System properties

$$y(t) = x(t)x(t + 2). \tag{1}$$

memoryless? No, as the system depends on future values.

causal? No, as the system depends on future values.

stable? Yes. Let $x(t)$ be bounded by $|x(t)| < M < \infty$. Then $|y(t)| = |x(t) \cdot x(t+2)| \leq M^2 < \infty$. Hence, $y(t)$ is bounded by M^2 for M bounded input and thus stable.

TI Yes. To be time invarient we need

$$y(t + t_0) = x(t + t_0)x(t + 2 + t_0), \tag{2}$$

which can be seen by insertion.

linear No. Let $x(t) = a \cdot x_0(t) + b \cdot x_1(t)$. Then

$$\begin{aligned} x(t)x(t+2) &= (a \cdot x_0(t) + b \cdot x_1(t))(a \cdot x_0(t+2) + b \cdot x_1(t+2)) && (3)\\ &= a^2 \cdot x_0(t)x_0(t+2) + b^2 \cdot x_1(t)x_1(t+2) \\ &+ ab \cdot (x_0(t)x_1(t+2) + x_0(t+2)x_1(t)) && (4)\\ &\neq a \cdot \underbrace{x_0(t)x_0(t+2)}_{y(t)|_{x(t)=x_0(t)}} + b \cdot \underbrace{x_1(t)x_1(t+2)}_{y(t)|_{x(t)=x_1(t)}}. && (5) \end{aligned}$$

$$y[n] = max\{x[n], x[n+1]\}. \tag{6}$$

memoryless? No, as the system depends on future values.

causal? No, as the system depends on future values.

stable? Yes. Let $x(t)$ be bounded by M as before. Then by definition $y(t)$ is also bounded by M and thus it is stable.

TI? Yes. Any values we input will shift the output by the same. As we compare to the one-setp-ahead value, we will always cover the same pairings. Different integer jumps could produce different pairing depending on time delays of the imput and could be time variant.

Linear No. Generally[1]

$$\begin{aligned} &a \cdot max\{x_0[n], x_0[n+1]\} + b \cdot max\{x_1[n], x_1[n+1]\} \\ &\neq max\{ax_0[n] + bx_1[n], ax_0[n+1] + bx_1[n+1]\} && (7) \end{aligned}$$

Consider e.g.

$$\begin{aligned} x_0[n] &= [\ldots, 1, 2, 1, 2, \cdots] && (8)\\ x_1[n] &= [\ldots, 1, -1, 1, -1, \cdots] && (9)\\ \Rightarrow x_0[n] + x_1[n] &= [\ldots, 2, 1, 2, 1, \ldots] \text{ and} && (10)\\ a &= b = 1. && (11) \end{aligned}$$

Then

$$\begin{aligned} y_0[n] &= [\ldots, 2, 2, 2, \ldots] && (12)\\ y_1[n] &= [\ldots, 1, 1, 1, \ldots] && (13)\\ \Rightarrow y_0[n] + y_1[n] &= [\ldots, 3, 3, 3, \ldots], && (14) \end{aligned}$$

but

$$\begin{aligned} y_{x_0+x_1}[n] &= [\ldots, 2, 2, 2, \ldots] && (15)\\ &\neq y_0[n] + y_1[n]. && (16) \end{aligned}$$

[1] There are exceptions, but that's not the point.

Problem 2 - Fourier series

This exercise would usually require integration by parts, which we havn't used thorughout the course and will not appear in the final exam. I have sligtly changed the signal to make this problem more representative of the coming exam. Let

$$x(t) = \begin{cases} -1, & \text{if } 0 \leq t < 1 \\ 1, & \text{if } 1 \leq t < 2 \\ 0, & \text{if } 2 \leq t < 4 \end{cases} \tag{17}$$

and $T = 4$. We know

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T dt \quad x(t)e^{-jk\frac{2\pi}{T}t} && (18)\\
&= \frac{1}{4}\left(\int_0^1 dt \quad -e^{-jk\frac{\pi}{2}t} + \int_1^2 dt \quad e^{-jk\frac{\pi}{2}t}\right) && (19)\\
&= \frac{1}{4}\left(\left[\frac{2}{jk\pi}e^{-jk\frac{\pi}{2}t}\right]_1^2 - \left[\frac{2}{jk\pi}e^{-jk\frac{\pi}{2}t}\right]_0^1\right) && (20)\\
&= \frac{1}{jk2\pi}\left(e^{-jk\pi} - e^{-jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}} + 1\right) && (21)\\
&= \frac{1}{jk2\pi}\left(e^{-jk\pi} + 1 - 2\cos(k\frac{\pi}{2})\right) && (22)\\
&= \frac{1}{jk2\pi}\left(e^{-jk\frac{\pi}{2}}\left(e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}}\right) - 2\cos(k\frac{\pi}{2})\right) && (23)\\
&= \frac{\cos(k\frac{\pi}{2})}{jk\pi}\left(e^{-jk\frac{\pi}{2}} - 1\right) && (24)
\end{aligned}$$

Problem 3 LTI

Frequency response

Let $\dot{y}(t) + 3y(t) = x(t)$ be some LTI system.

We can solve the difference equation by using the Fourier transform.

$$\dot{y}(t) + 3y(t) = x(t) \tag{25}$$
$$\Rightarrow j\omega Y(j\omega) + 3Y(j\omega) = X(j\omega). \tag{26}$$

We now have eliminated the differential and can simply rearange as such

$$Y(j\omega)(j\omega + 3) = X(j\omega) \tag{27}$$
$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} \equiv H(j\omega) = \frac{1}{j\omega + 3} \tag{28}$$

Bode plot

You could just compute a few values and then sketch it by hand. Otherwise, here is some code that does it.

```
# imports
import numpy as np
import matplotlib.pyplot as plt


# frequencies
ws = np.linspace(0, np.pi, 100)

# analytical frequency response
def f_rsp(w):
    return 1 / (1j * w + 3)

# convert to dB
def dB(s):
    return 20*np.log10(s)

# plot
fig, ax = plt.subplots()
ax.plot(ws, np.abs(f_rsp(ws)))
ax.set_title("Bode plot")
ax.set_xlabel(r"$\omega$")

plt.savefig("figures/p3_bode.png")
plt.show()
```

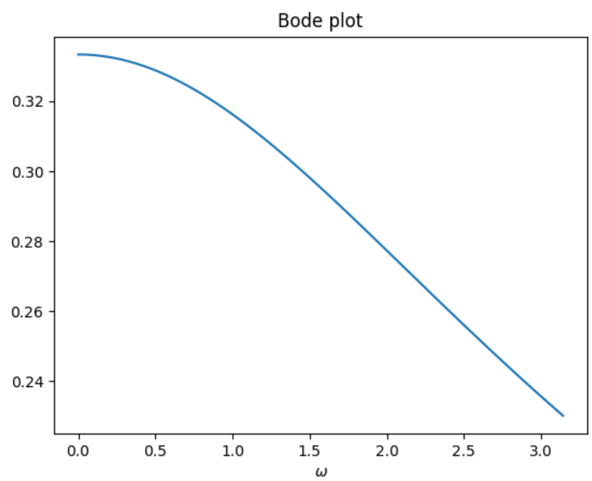

Figure 1: Bode plot of system.