Density-Induced Ordered Weighted Averaging Operators

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We provide a special type of induced ordered weighted averaging (OWA) operator called density-induced OWA (DIOWA) operator, which takes the density around the arguments as the inducing variables to reorder the arguments. The density around the argument, which can measure the degree of similarity between the argument and its nearest neighbors, is associated with both the number of its nearest neighbors and its weighted average distance to these neighbors. To determine the DIOWA weights, we redefine the orness measure, and propose a new maximum orness model under a dispersion constraint. The DIOWA weights generated by the traditional maximum orness model depend upon the order of the arguments and the dispersion degree. Differently, the DIOWA weights generated by the new maximum orness model also depend upon the specific values of the density around the arguments. Finally, we illustrate how the DIOWA operator is used in the decision making, and prove the effectiveness of the DIOWA operator through comparing the DIOWA operator with other operators, i.e., the centered OWA operator, the Olympic OWA operator, the majority additive-OWA (MA-OWA) operator, and the kNN-DOWA operator. © 2011 Wiley Periodicals, Inc.

1. INTRODUCTION

The ordered weighted averaging (OWA) operators introduced by Yager[1] provide a parameterized family of aggregation operators. Since the appearance of OWA operators, they have been used in many domains, such as multicriteria decision making, multiperson decision making, flexible database querying and information retrieval, learning and classification, etc.[2–16] An important feature of OWA operators is the reordering of the arguments based upon their values, that is, the weight in the OWA operator is associated with a particular position in the ordering, rather than being associated with a specific argument as in other weighted averaging operators. Then Yager[17,18] introduced a more general type of OWA operator called the induced ordered weighted averaging (IOWA) operator, in which the ordering of the arguments is induced by the order inducing variables, rather than the values of the arguments.

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Numerous methods have been proposed to determine the weighting vector associated with the order of the arguments in the OWA operator, including programming-based approaches,[19–29] quantifier guided approaches,[30–32] learning-based approaches,[5,17,33] and extreme point approach,[34] etc. However, few studies have been conducted on the determination of the order-inducing variable in the induced OWA operator. In this paper, we take the density around the argument as the order inducing variable and provide a special type of IOWA operator called density-induced ordered weighted averaging (DIOWA) operator.

There are two common methods to define the density. One is to define the density around an object as the reciprocal of its average distance to its k nearest neighbors. If this distance is small, the density is high, and vice versa.[35] It is similar to the approach to measuring the reliability of the argument.[36] However, the selection of the parameter k can be difficult.[35] On the one hand, empirical results have shown that given a small k, the proposed metric is robust to the setting of this parameter[36]; on the other hand, given a smaller k, less information can be utilized to estimate the reliability of each argument value. Hence a suitable parameter k, neither too large nor too small, should be chosen carefully to preserve the local constitution of data. In another approach, the density around an object is defined as the number of neighbors within a given radius r of the object.[35] However, this approach neglects the specific distance from the object to these nearest neighbors.

In fact, the density around the object is associated with both the number of its nearest neighbors and its weighted averaging distance to these neighbors. When the number of these nearest neighbors is constant, the density around the object depends upon its weighted averaging distance to these neighbors. When its weighted averaging distance to these neighbors is constant, the density around the object depends upon the number of its nearest neighbors. Synthesizing the two decisive factors mentioned above, we provide a new definition of the density around the argument from the viewpoint of combining multiple arguments.

The actual result of aggregation performed by the DIOWA operator depends upon not only the order inducing variables, i.e., density around the arguments, but also the weighting vector used. In this paper, we redefine the orness measure, and provide a new maximum orness model to determine the density-induced OWA (DIOWA) weights under a dispersion constraint. The DIOWA weights generated by the traditional maximum orness model depend upon the dispersion degree and the order of the arguments. Differently, the DIOWA weights generated by the new maximum orness model also depend upon the specific values of the density around the arguments.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries of OWA operators and IOWA operators. In Section 3, we define the density around the argument, then we introduce the DIOWA operator. To obtain the weights associated with the DIOWA operator, we provide a new maximum orness model. In Section 4, we illustrate how the DIOWA operator is used in the decision process and compare the DIOWA operator with other operators, i.e., the centered OWA operator, the Olympic OWA operator, the majority additive OWA (MA-OWA) operator, and the kNN-DOWA operator. Finally, the paper ends in Section 5 with concluding remarks.

2. PRELIMINARIES

In this section, we briefly describe the OWA operator and the IOWA operator.

2.1. OWA Operators

The OWA operators introduced by Yager[1] provide a parameterized family of mean type aggregation operators. The important feature of the OWA operator is the reordering of the arguments based upon their values.

DEFINITION 1. *An OWA operator*[1] *of dimension n is a mapping* $OWA: R^n \to R$ *defined by an associated weighting vector* $\boldsymbol{w}$ *of dimension n, such that* $\sum_{i=1}^{n} w_i = 1$ *and* $w_i \in [0, 1]$*, according to the following formula:*

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where $(b_1, b_2, \ldots, b_n)$ *is simply* $(a_1, a_2, \ldots, a_n)$ *reordered from largest to smallest.*

The OWA operator has the following properties[1]:

(1) Monotonicity: $OWA(a_1, a_2, \ldots, a_n) \geq OWA(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$ if $a_i \geq \hat{a}_i$ for all i.
(2) Commutativity: the initial indexing of the arguments doesn't matter.
(3) Bounded: $\min_i[a_i] \leq OWA(a_1, a_2, \ldots, a_n) \leq \max_i[a_i]$.
(4) Idempotent: $OWA(a_1, a_2, \ldots, a_n) = a$ if $a_i = a$ for all i.

2.2. IOWA Operators

The IOWA operator introduced by Yager and Filev[17] represents a more general type of OWA operator. The main feature of the IOWA operator is that the ordering of the arguments is induced by the order inducing variables, rather than the values of the arguments.

DEFINITION 2. *An IOWA operator*[17] *of dimension n is a mapping* $IOWA: R^n \to R$ *defined by a set of order inducing variables* u_i*, and an associated weighting vector* $\boldsymbol{w}$ *of dimension n such that* $\sum_{i=1}^{n} w_i = 1$ *and* $w_i \in [0, 1]$*, according to the following formula:*

$$IOWA(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

where $(b_1, b_2, \ldots, b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ reordered in decreasing order of the values of u_i, and a_i is the argument variable.

The IOWA operator is also commutative, monotonic, bounded, and idempotent.[17] Especially, if $\forall i\, w_i = 1/n$, the IOWA operator is reduced to the arithmetic averaging (AA) operator; if $\forall i\, u_i = a_i$, then the IOWA operator is reduced to OWA operator.

3. DENSITY-INDUCED OWA OPERATORS

In this section, we introduce a new local-distribution-based operator called DIOWA operator, whose distinctive characteristic is that the ordering of the arguments is induced by the density around the arguments, rather than the values of the arguments.

3.1. The Definition of Density

The density around the argument is associated with both the number of its nearest neighbors and its weighted averaging distance to these neighbors. Synthesizing the two decisive factors mentioned above, we provide a new definition of the density around the argument from the viewpoint of combining multiple arguments.

Definition 3. *Let $A = \{a_1, a_2, \ldots, a_n\}$ be a collection of data arguments. The density around a_i in the set A is defined as*

$$density(a_i, r) = \begin{cases} \left(\dfrac{1 + \sum_{a_j \in A_i} w_{ij} d(a_i, a_j)}{|A_i|} \right)^{-1}, & if\ A_i \neq \phi, \\ 0, & if\ A_i = \phi, \end{cases} \tag{3}$$

where $d(a_i, a_j)$ denotes the distance (Euclidean distance if no further specification) between two arguments a_i and a_j, $A_i = \{a_j | d(a_i, a_j) \le r,\ j \neq i,\ and a_i, a_j \in A\}$, $|A_i|$ is the size of the set A_i, and w_{ij} is the weight associated with $d(a_i, a_j)$.

The distance $d(a_i, a_j)$ as a proximity measure can be used to refer to either the similarity or dissimilarity between a_i and a_j. The only parameter r is a user specified similarity threshold or allowable inaccuracy threshold. If $d(a_i, a_j) \le r$, a_j can be thought to be similar to a_i. Hence, A_i is a set whose members are similar to a_i.

Intuitively, different neighbors of the argument have different contributions to the whole neighborhood. Taking this difference into account, we consider the weighted average instead of arithmetic average in Definition 3. In the following, we just set

$$w_{ij} = e^{-d(a_i, a_j)} \Big/ \sum_{a_j \in A_i} e^{-d(a_i, a_j)}. \tag{4}$$

From Equation 3, we can conclude

(1) The $density(a_i, r)$ is associated with both the cardinality of the set A_i whose members are similar to a_i and the weighted average distance from a_i to the arguments $a_j \in A_i$;
(2) If the cardinality of the set A_i is constant, then the $density(a_i, r)$ depends upon the weighted average distance from a_i to the arguments $a_j \in A_i$;
(3) If the weighted average distance from a_i to the arguments $a_j \in A_i$ is constant, then the $density(a_i, r)$ depends upon the cardinality of the set A_i;
(4) If $A_i = \phi$, i.e., there is no argument that is similar to a_i, then the value of $density(a_i, r)$ is zero;
(5) If $A_i \neq \phi$ and the weighted average distance from a_i to the arguments $a_j \in A_i$ is zero, i.e., the value of the argument $a_j \in A_i$ is equal to a_i, then $density(a_i, r)$ is equal to the cardinality of the set A_i.

From what is mentioned above, we can see that $density(a_i, r)$ can be used to measure the degree of similarity between a_i and its neighbors. If the value of $density(a_i, r)$ is higher, the degree of similarity between a_i and its neighbors is higher and the degree of divergence between a_i and its neighbors is lower, and vice versa.

Example. Ten users marked a product's quality and after service. The preference data is shown in Table I.

Let $A = \{\mathbf{a}_1, \boldsymbol{a}_2, \ldots, \mathbf{a}_{10}\}$. $\mathbf{a}_i$ represents the information given by user i. Let $\mathbf{a}_1 = (3, 3)$, $\mathbf{a}_2 = (4, 7)$, $\mathbf{a}_3 = (4, 9)$, $\mathbf{a}_4 = (5, 8)$, $\mathbf{a}_5 = (5.5, 9)$, $\mathbf{a}_6 = (5.5, 6.5)$, $\mathbf{a}_7 = (6.5, 7.5)$, $\mathbf{a}_8 = (5, 6)$, $\mathbf{a}_9 = (9.5, 8)$, and $\mathbf{a}_{10} = (9, 9.5)$. Suppose that the user specified similarity threshold or allowable inaccuracy threshold $r = 0.5$. The $density(\mathbf{a}_i, r)$, which can be calculated from Equations 3 and 4, are shown in Table II and Figure 1.

3.2. The Definition of DIOWA Operators

Yager[17,18] introduced a general type of OWA operator called the IOWA operator, in which the ordering of the arguments is induced by the order inducing variables, rather than the values of the arguments. But quite few studies have been conducted on the determination of the order inducing variables. We take the density

Table I. Scores of a product's quality and after service marked by 10 users.

	User 1	User 2	User 3	User 4	User 5	User 6	User 7	User 8	User 9	User 10
Quality	3	4	4	5	5.5	5.5	6.5	5	9.5	9
After service	3	7	9	8	9	6.5	7.5	6	8	9.5

Table II. The density around $\mathbf{a}_i$.

$density(\mathbf{a}_i, r)$ $(r = 2)$ its true									
$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$
0	1.5677	1.1638	2.4472	1.2525	1.8431	1.1653	1.4251	0.3874	0.3874

Note: r is a user specified similarity threshold or allowable inaccuracy threshold.

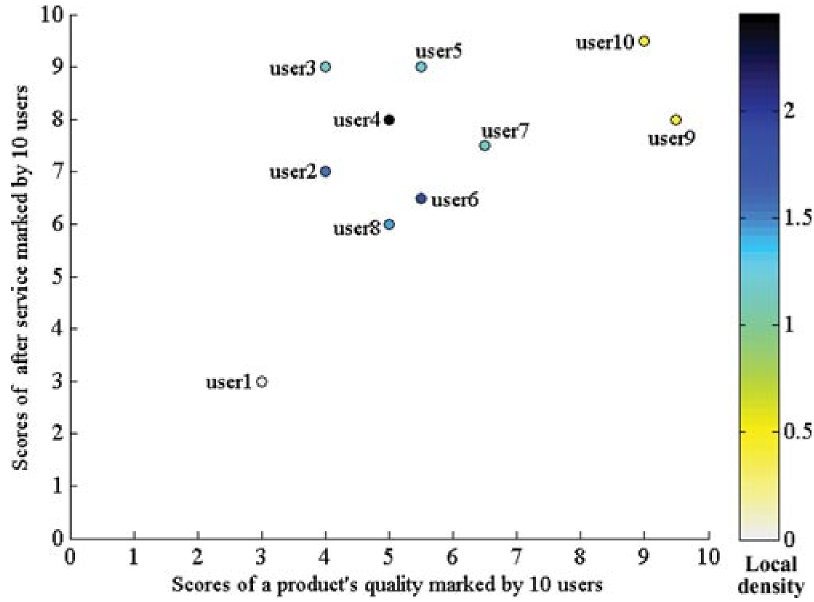

Figure 1. The density around $\boldsymbol{a}_i$ given by user i.

around the arguments as the order inducing variables and provide a special type of IOWA operator called DIOWA operator.

DEFINITION 4. *A DIOWA operator of dimension n is a mapping DIOWA:$R^n \to R$ defined by a set of order inducing variables $u(a_i) = density(a_i, r)$, and an associated weighting vector $\boldsymbol{w}$ of dimension n such that $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$, according to the following formula:*

$$u(a_i) = density(a_i, r), \tag{5}$$

$$DIOWA(\langle u(a_1), a_1\rangle, \langle u(a_2), a_2\rangle, \ldots, \langle u(a_n), a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \tag{6}$$

where a_i is the argument variable, $(b_1, b_2, \ldots, b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ re-ordered in decreasing order of the values of $u(a_i)$, and

$$u(b_i) = density(b_i, r), \tag{7}$$

$$u(b_1) \geq u(b_2) \geq \cdots \geq u(b_n). \tag{8}$$

As a special type of IOWA operator, the DIOWA operator has the following basic properties:

(1) Commutativity: the initial indexing of the arguments doesn't matter.
(2) Bounded: $\min_i[a_i] \le DIOWA(a_1, a_2, \ldots, a_n) \le \max_i[a_i]$.
(3) Idempotent: $DIOWA(a_1, a_2, \ldots, a_n) = a$ if $a_i = a$ for all i.

Example. Ten users marked a product's quality and after service. The preference data is shown in Table I, and the density around $\boldsymbol{a}_i$, denoted by *density* $(\boldsymbol{a}_i, r)$, is shown in Table II. Then $\boldsymbol{u}_a = (u(a_1), u(a_2), \ldots, u(a_n)) = (0, 1.5677, 1.1638, 2.4472,$ $1.2525, 1.8431, 1.1653, 1.4251, 0.3874, 0.3874)$, $\boldsymbol{u}_b = (u(b_1), u(b_2), \ldots, u(b_n)) =$ $(2.4472, 1.8431, 1.5677, 1.4251, 1.2525, 1.1653, 1.1638, 0.3874, 0.3874, 0)$, $\boldsymbol{b}_1 =$ $(5, 8)$, $\boldsymbol{b}_2 = (5.5, 6.5)$, $\boldsymbol{b}_3 = (4, 7)$, $\boldsymbol{b}_4 = (5, 6)$, $\boldsymbol{b}_5 = (5.5, 9)$, $\boldsymbol{b}_6 = (6.5, 7.5)$, $\boldsymbol{b}_7 =$ $(4, 9)$, $\boldsymbol{b}_8 = (9.5, 8)$, $\boldsymbol{b}_9 = (9, 9.5)$, and $\boldsymbol{b}_{10} = (3, 3)$.

The weights can be obtained using the method proposed in the next subsection. When the weights are determined, we can gain the comprehensive evaluation of a product's quality and after service according to Equation 6.

3.3. The New Maximum Orness Approach for Generating DIOWA Weights

The actual result of aggregation performed by the DIOWA operator depends upon not only the order inducing variables, i.e., density around the arguments, but also the weighting vector used. In this section, we redefine the orness measure and provide a new maximum orness method to determine the DIOWA weights with a fixed dispersion.

The new orness measure of the aggregation is defined as

$$orness(\mathbf{w}) = \sum_{j=1}^{n} \frac{u(b_j) - \min_j[u(b_j)]}{\max_j[u(b_j)] - \min_j[u(b_j)]} w_j \ , \tag{9}$$

where $u(b_j)$ is the order inducing variable associated with b_j. According to Equation 8, we have $\min_j[u(b_j)] = u(b_n)$ and $\max_j[u(b_j)] = u(b_1)$, thus Equation 9 can be written as

$$orness(\mathbf{w}) = \sum_{j=1}^{n} \frac{u(b_j) - u(b_n)}{u(b_1) - u(b_n)} w_j. \tag{10}$$

This measure can reflect the degree to which $\mathbf{w}$ gives preference to the higher or lower values of $u(b_j)$.

The dispersion measure introduced by Yager[1] is written as

$$Disp(\mathbf{w}) = \frac{1}{n} \sum_{j=1}^{n} \left(w_j - \frac{1}{n}\right)^2 = \frac{1}{n} \sum_{j=1}^{n} w_j^2 - \frac{1}{n^2}. \tag{11}$$

This measure can reflect the degree to which $\boldsymbol{w}$ takes into account all information in the aggregation.

Based on the two characterizing measures mentioned above, we propose a new maximum orness method to determine the DIOWA weights with a fixed dispersion:

$$\max \sum_{j=1}^{n} \frac{u(b_j) - u(b_n)}{u(b_1) - u(b_n)} w_j$$

$$\text{s.t. } Disp(\mathbf{w}) = \frac{1}{n} \sum_{j=1}^{n} \left(w_j - \frac{1}{n} \right)^2 = \frac{1}{n} \sum_{j=1}^{n} w_j^2 - \frac{1}{n^2} = \beta, \tag{12}$$

$$\sum_{j=1}^{n} w_j = 1, w_j \in [0, 1], \ j = 1, \ldots, n,$$

where $u(b_j) = density\ (b_j, r)$, $\beta = \lambda \cdot \max\_disp\ (0 \leq \lambda \leq 1)$, $\max\_disp = (n|K|)^{-1} - n^{-2}$.

β represents the dispersion of $\boldsymbol{w}$, and $\max\_disp$ represents the allowable upper bound of β. λ is specified by the user, which is the adjustment coefficient of β. If λ is higher, β is higher. When $\lambda = 1$, β reaches its allowable upper bound and $\beta = \max\_disp$. When $\lambda = 0$, β reaches its lower bound and $\beta = 0$.

$|K|$ represents the size of the set K, and $K = \{u(b_j) | u(b_j) = u(b_1),\ j \in N\}$. If

$$w_j = \begin{cases} 1/|K|, & u(b_j) = u(b_1) \\ 0, & u(b_j) \neq u(b_1) \end{cases} \quad (j \in N)$$

then $Disp(\boldsymbol{w}) = \max\_disp = (n|K|)^{-1} - n^{-2}$.

For example, if $\boldsymbol{u}_b = (u(b_1), u(b_2), \ldots, u(b_5)) = (2.5,\ 2.5,\ 2,\ 1,\ 0)$, then $K = \{u(b_1), u(b_2)\}$, $|K| = 2$, and $\max\_disp = (n|K|)^{-1} - n^{-2} = 0.06$. When $\boldsymbol{w} = (w_1, \ldots, w_5) = (1/2, 1/2, 0, 0, 0)$, $Disp(\boldsymbol{w}) = 0.06 = \max\_disp$.

To analyze the properties of the proposed maximum orness model 12, the following theorems are stated and proved.

THEOREM 1. *The DIOWA weights generated by the new maximum orness model 12 depend upon the fixed dispersion degree β and $u(b_j)$.*

Proof. Using the Lagrange method, the DIOWA weights generated by the new maximum orness model 12 must make zero the partial derivatives of Lagrange function:

$$L = \sum_{j=1}^{n} \frac{u(b_j) - u(b_n)}{u(b_1) - u(b_n)} w_j + \eta_1 \left(\frac{1}{n} \sum_{j=1}^{n} w_j^2 - \frac{1}{n^2} - \beta \right) + \eta_2 \left(\sum_{j=1}^{n} w_j - 1 \right).$$

Determining the partial derivatives of L with respect to parameters w_j and Lagrange multipliers η_1 and η_2, and setting them equal to zero, we obtain

$$\frac{\partial L}{\partial w_j} = \frac{u(b_j) - u(b_n)}{u(b_1) - u(b_n)} + \frac{2\eta_1 w_j}{n} + \eta_2 = 0, \quad j = 1, 2, \ldots n, \tag{13}$$

$$\frac{\partial L}{\partial \eta_1} = \frac{1}{n}\sum_{j=1}^{n} w_j^2 - \frac{1}{n^2} - \beta = 0, \tag{14}$$

$$\frac{\partial L}{\partial \eta_2} = \sum_{j=1}^{n} w_j - 1 = 0. \tag{15}$$

From Equations 13–15, we obtain

$$w_j = -f(u(b_j))\frac{n}{2\eta_1} + \frac{\sum_{j=1}^{n} f(u(b_j))}{2\eta_1} + \frac{1}{n}, \tag{16}$$

where

$$\eta_1 = -\left(\frac{n\sum_{j=1}^{n} f^2(u(b_j)) - \left(\sum_{j=1}^{n} f(u(b_j))\right)^2}{4\beta}\right)^{1/2}, \tag{17}$$

$$f(u(b_j)) = \frac{u(b_j) - u(b_n)}{u(b_1) - u(b_n)}. \tag{18}$$

From Equations 16–18, we can see that the DIOWA weights generated by the new maximum orness model 12 depend upon the fixed dispersion degree β and $u(b_j)$. ■

THEOREM 2. *When $u(b_i) = u(b_j)$ $(i \neq j)$, the weights w_i and w_j generated by the new maximum orness model 12 are equal.*

Proof. From Equations 16–18, we can see that when $u(b_i) = u(b_j)(i \neq j)$, the weights w_i and w_j generated by Model 12 are equal. ■

THEOREM 3. *When the sequence of $u(b_j)$ $(j \in N)$ is an arithmetic progression, i.e., $u(b_j) = u(b_i) - (j - i)d(d > 0)$ for all $j > i$, the DIOWA weights generated by the new maximum orness model 12 are equal to the weights generated by the traditional maximum orness model.*

Proof. The traditional maximum orness model introduced by Marchant[37] is written as

$$\max \sum_{j=1}^{n} \frac{n-i}{n-1} w_j$$

$$\text{s.t. } D^2(\mathbf{w}) = \frac{1}{n}\sum_{j=1}^{n} w_j^2 - \frac{1}{n^2} = \beta,\ 0 \le \beta \le \frac{1}{n} - \frac{1}{n^2} \tag{19}$$

$$\sum_{j=1}^{n} w_j = 1, w_j \in [0, 1],\ j = 1, \ldots, n.$$

When $u(b_j) = u(b_i) - (j - i)d(d > 0)$ for all $j > i$, the objective function in Model (12) can be written as

$$\sum_{j=1}^{n} \frac{u(b_j) - u(b_n)}{u(b_1) - u(b_n)} w_j = \sum_{j=1}^{n} \frac{(n-j)d}{(n-1)d} w_j = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j. \tag{20}$$

And the upper bound of β in Model 12 can be written as

$$(n\,|K|)^{-1} - n^{-2} = 1\big/n - 1\big/n^2, \tag{21}$$

which are the same as the objective function and the upper bound of β in the traditional maximum orness model 19.

From Models 12 and 19 and Equations 20 and 21, we can see that when the sequence of $u(b_j)$ $(j \in N)$ is an arithmetic progression, i.e., $u(b_j) = u(b_i) - (j - i)d(d > 0)$ for all $j > i$, the DIOWA weights generated by the new maximum orness model 12 are equal to the weights generated by the traditional maximum orness model 19. ■

Example. Ten users marked a product's quality and after service. The preference data is shown in Table I, and $\boldsymbol{u}_b = (u(b_1), u(b_2), \ldots, u(b_{10})) = (2.4472,$ $1.8431, 1.5677, 1.4251, 1.2525, 1.1653, 1.1638, 0.3874, 0.3874, 0)$. Then $|K| = 1$, $\max\_disp = 0.09$. The DIOWA weights generated by the new maximum orness model 12 are shown in Table III and Figure 2a. And the DIOWA weights generated by traditional maximum orness model 19 are shown in Table IV and Figure 2b.

Note: $u_i = u(b_j)\big/\sum_{i=1}^{n} u(b_j)$ $(n = 10)$; λ is the adjustment coefficient of the dispersion degree β.

From Tables III and IV and Figure 2, we can see that the DIOWA weights generated by the traditional maximum orness approach 19 depend upon the dispersion degree β and the order of the arguments, $\beta = \lambda \cdot (n^{-1} - n^{-2})$.

Different from the DIOWA weights generated by the traditional approach, the DIOWA weights generated by the new maximum orness approach 12 depend upon the dispersion degree β and the values of $u(b_j)$. Especially, when $u(b_8) = u(b_9)$, the weights w_8 and w_9 generated by Model 12 are equal.

Table III. The DIOWA weights generated by the new maximum orness approach 12.

λ	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.05	0.2263	0.1660	0.1385	0.1243	0.1070	0.0983	0.0982	0.0207	0.0207	0
0.1	0.3138	0.1984	0.1458	0.1186	0.0856	0.0690	0.0687	0	0	0
0.2	0.4344	0.2377	0.1479	0.1015	0.0453	0.0169	0.0164	0	0	0
0.3	0.5270	0.2599	0.1381	0.0750	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0

Table IV. The DIOWA weights generated by the traditional maximum orness approach 19.

λ	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.05	0.2062	0.1824	0.1586	0.1349	0.1111	0.0873	0.0636	0.0398	0.0160	0
0.1	0.2660	0.2249	0.1839	0.1429	0.1018	0.0608	0.0198	0	0	0
0.2	0.3789	0.2894	0.2000	0.1106	0.0211	0	0	0	0	0
0.3	0.4824	0.3275	0.1725	0.0176	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0

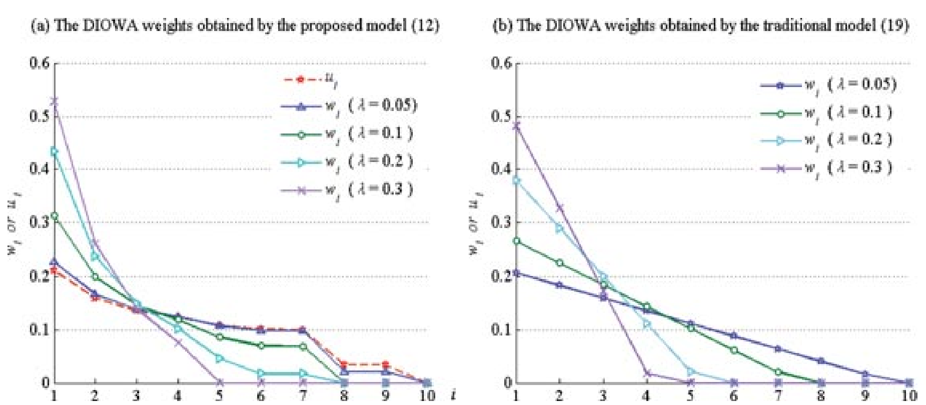

Figure 2. The DIOWA weights generated by the new maximum orness model and the traditional one.

4. APPLICATION OF THE DIOWA OPERATOR IN THE DECISION MAKING

In this section, we illustrate how the DIOWA operator is used in the decision process, and compare the DIOWA operator with other operators, i.e., the centered OWA operator, the Olympic OWA operator, the majority additive OWA (MA-OWA) operator, and the kNN-DOWA operator.

4.1. An Illustrative Example

In gymnastics, the Execution (E) Score is determined by a six-person E Panel. Each judge independently determines his/her score. And the gymnast's E-Score is calculated by aggregating the six judges' scores. Here's an example on how the DIOWA operator is used to aggregate the six judges' scores for each athlete in a competition.

Let $A_k = \{a_{k1}, \ldots, a_{k6}\}$ be the set of six judges' scores for the athlete k, $k = k_1, k_2, k_3, k_4$. $A_{k_1} = \{a_{k_1 1}, \ldots, a_{k_1 6}\} = \{8, 9.3, 9.1, 8.5, 9.3, 9.4\}$, $A_{k_2} = \{a_{k_2 1}, \ldots, a_{k_2 6}\} = \{8.4, 8.4, 9, 9, 9, 9\}$, $A_{k_3} = \{a_{k_3 1}, \ldots, a_{k_3 6}\} = \{8.7, 9.3, 8.8, 8.8, 8.7, 9.9\}$, $A_{k_4} = \{a_{k_4 1}, \ldots, a_{k_4 6}\} = \{9.3, 8.1, 8.6, 9.2, 9.2, 8\}$.

Using the DIOWA operator to aggregate the scores for each athlete, we take the following steps:

Step 1: Determine the density around a_{ki} in the set A_k.

According to Equations 3 and 4, the density around a_{ki} in the set A_k, denoted by $density(a_{ki}, r)$ $(a_{ki} \in A_k)$, can be written as

$$density(a_{ki}, r) = \begin{cases} \left(\dfrac{1 + \sum_{a_{kj} \in A_{ki}} w_{kij} d(a_{ki}, a_{kj})}{|A_{ki}|} \right)^{-1} & \text{if } A_{ki} \neq \phi, \\ 0 & \text{if } A_{ki} = \phi, \end{cases} \tag{22}$$

$$w_{kij} = e^{-d(a_{ki}, a_{kj})} \Big/ \sum_{a_{kj} \in A_{ki}} e^{-d(a_{ki}, a_{kj})}, \tag{23}$$

where $a_{ki} \in A_k$, $A_{ki} = \{a_{kj} | d(a_{ki}, a_{kj}) \leq r,\ j \neq i,\ \text{and } a_{ki}, a_{kj} \in A_k\}$, $k = k_1, k_2, k_3, k_4$.

According to Equation 5, the order inducing variables $u(a_{ki})$ in the DIOWA operator can be written as

$$u(a_{ki}) = density(a_{ki}, r) \quad a_{ki} \in A_k,\ k = k_1,\ k_2,\ k_3, k_4. \tag{24}$$

Suppose that the user specified similarity threshold or allowable inaccuracy threshold $r = 0.5$. The order inducing variables $u(a_{ki})$ $(a_{ki} \in A_k)$, which can be calculated from Equations 22, 23, and 24, are shown in Table V and Figure 3.

Table V. The density around a_{ki} in the set A_k.

	$u(a_{ki}) = density(a_{ki}, r)$ $(a_{ki} \in A_k$ and $r = 0.5)$					
Athlete k	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$k = k_1$	0.6667	2.7439	2.4367	0.6667	2.7439	2.5905
$k = k_2$	2	2	4	4	4	4
$k = k_3$	2.8185	1.3333	3.5034	3.5034	2.8185	0
$k = k_4$	1.8182	1.5866	0.6667	1.9093	1.9093	0.9091

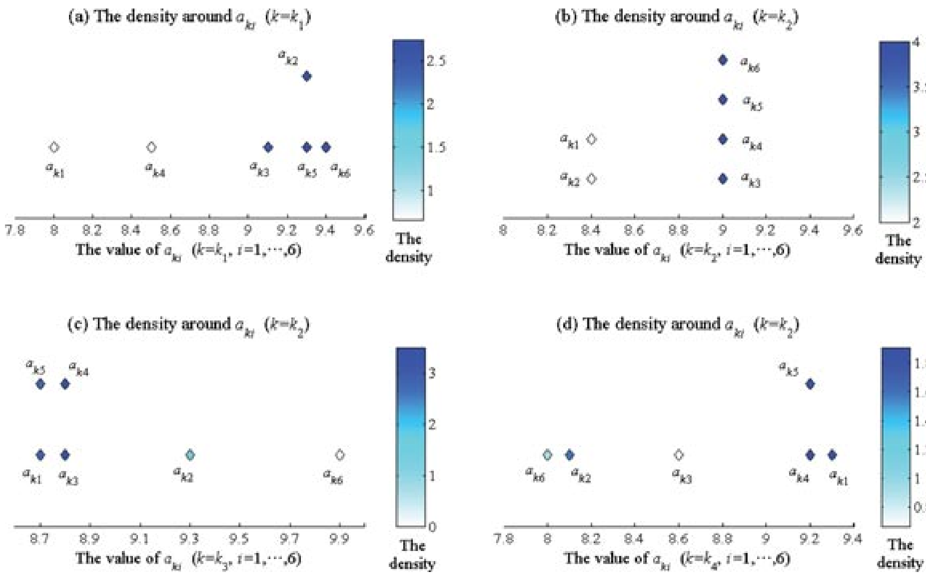

Figure 3. The density around the six judges' scores for each athlete.

Note: a_{ki} denotes the judge i's scores for athlete k, $k = k_1, k_2, k_3, k_4$.

Step 2: Determine the DIOWA weights.

As $(b_{k1}, b_{k2}, \ldots, b_{kn})$ is simply $(a_{k1}, a_{k2}, \ldots, a_{kn})$ reordered in decreasing order of the values of $u(a_{ki})$, we can obtain the vector $\mathbf{b}_k = (b_{k1}, b_{k2}, \ldots, b_{k6})$ and $\mathbf{u}(b_k) = (u(b_{k1}), \ldots, u(b_{k6}))$, $k = k_1, k_2, k_3, k_4$. $\boldsymbol{b}_{k_1} = (9.3, 9.3, 9.4, 9.1, 8, 8.5)$, $\boldsymbol{b}_{k_2} = (9, 9, 9, 9, 8.4, 8.4)$, $\boldsymbol{b}_{k_3} = (8.8, 8.8, 8.7, 8.7, 9.3, 9.9)$, $\boldsymbol{b}_{k_4} = (9.2, 9.2, 9.3, 8.1, 8, 8.6)$, $\boldsymbol{u}(b_{k_1}) = (2.7439, 2.7439, 2.5905, 2.4367, 0.6667, 0.6667)$, $\boldsymbol{u}(b_{k_2}) = (4, 4, 4, 4, 2, 2)$, $\boldsymbol{u}(b_{k_3}) = (3.5034, 3.5034, 2.8185, 2.8185, 1.3333, 0)$, $\boldsymbol{u}(b_{k_4}) = (1.9093, 1.9093, 1.8182, 1.5866, 0.9091, 0.6667)$. Taking $\lambda = 0.2$ as an example, we determine the DIOWA weights w_{ki} associated with b_{ki} by the new maximum orness model 12, $k = k_1, k_2, k_3, k_4$, $i = 1, \ldots, 6$. The results are shown in Table VI and Figure 4.

Table VI. The DIOWA weights w_{ki} obtained by the new maximum orness model 12.

	w_{ki} ($\lambda = 0.2$)					
Athlete k	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$k = k_1$	0.25	0.25	0.24	0.22	0.02	0.02
$k = k_2$	0.20	0.20	0.20	0.20	0.09	0.09
$k = k_3$	0.27	0.27	0.20	0.20	0.05	0.00
$k = k_4$	0.26	0.26	0.24	0.19	0.04	0.00

Note: λ is the adjustment coefficient of the dispersion of $\mathbf{w}$.

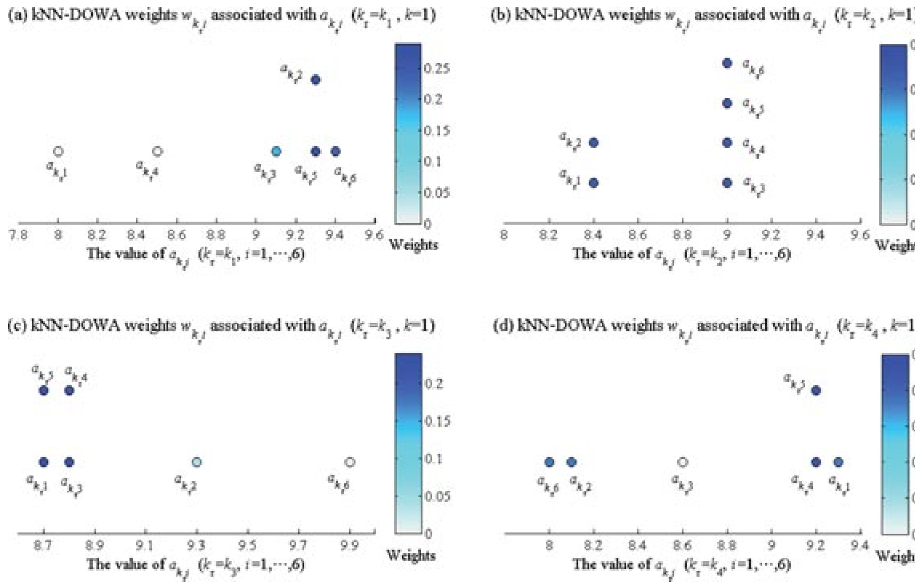

Figure 4. The DIOWA weights obtained by the new maximum orness model (12).

Table VII. The Gaussian OWA weights w_{ki} associated with b_{ki}.

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
w_{ki}	0.09	0.17	0.24	0.24	0.17	0.09

Note: $k = k_1, k_2, k_3, k_4$.

Step 3: Aggregate the six judges' scores for each athlete using the DIOWA operator.

According to Equation 6, the DIOWA operator can be written as

$$DIOWA(\langle u(a_{k1}),a_{k1}\rangle, \langle u(a_{k2}), a_{k2}\rangle, \ldots, \langle u(a_{kn}), a_{kn}\rangle) = \sum\nolimits_{j=1}^{n} w_{kj} b_{kj}. \quad (25)$$

From Equation 25, Table VI and the value of b_{kj}, we can obtain the final Execution (E) Scores for each athlete, which are 9.24, 8.89, 8.78, 8.96.

4.2. Comparison with Other Operators

4.2.1. Comparison with Gaussian OWA Operators and Olympic OWA Operators

4.2.1.1. Gaussian OWA Operators. The b_{ki} and OWA weights w_{ki} associated with b_{ki} in the Gaussian OWA operator are shown in Table VII and Figure 5. The final Execution (E) Scores for each athlete are 9.01, 8.85, 8.96, 8.77, respectively.

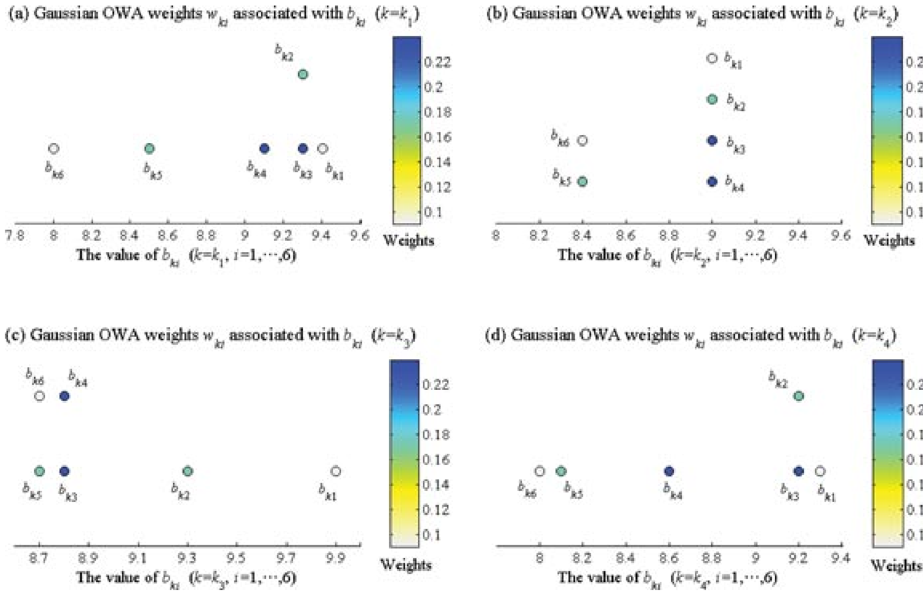

Figure 5. The Gaussian OWA weights w_{ki} associated with b_{ki}.

From Table VII and Figure 5, we can see that as a type of centered OWA operator, the Gaussian OWA operator gives preference to argument values that lie in the middle between the largest and the smallest. One potential application of this is to aggregate ratings of the members of the group so that eliminate the extreme values in group decision making.[38]

4.2.1.2. Olympic OWA operators. The b_{ki} and OWA weights w_{ki} associated with b_{ki} in the Olympic OWA operator are shown in Table VIII and Figure 6. The final Execution (E) Scores for each athlete are 9.05, 8.85, 8.9, 8.78, respectively.

From Table VIII and Figure 6, we can see that the highest and lowest scores are dropped, and the gymnast's Execution Score is the average of the remaining four judges' scores.

4.2.1.3. Comparison with Gaussian OWA operators and Olympic OWA operators. An extreme value is an observation that lies far away from the mean of a given distribution.[39] If most arguments are large, the small one is the extreme value. If most arguments are small, the large one is the extreme value.

Table VIII. The Olympic OWA weights w_{ki} associated with b_{ki}.

	$I=1$	$I=2$	$I=3$	$I=4$	$I=5$	$i=6$
w_{ki}	0	0.25	0.25	0.25	0.25	0

Note: $k = k_1, k_2, k_3, k_4$.

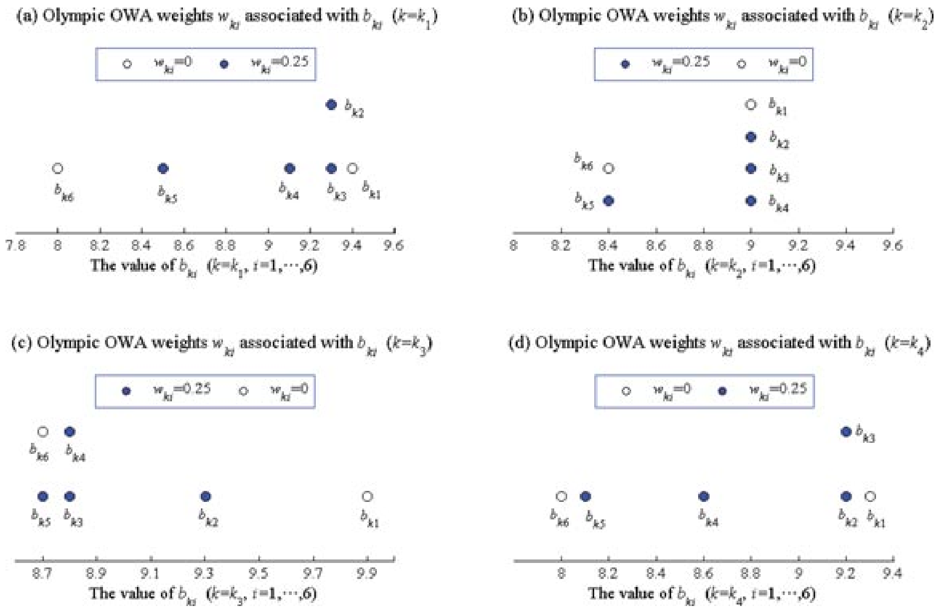

Figure 6. The Olympic OWA weights w_{ki} associated with b_{ki}.

Table IX. The MA-OWA weights w_{ki} associated with a_{ki}.

	w_{ki}					
Athlete k	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$k=k_1$	0.1	0.5	0.1	0.1	0.1	0.1
$k=k_2$	0.04	0.08	0.04	0.08	0.25	0.50
$k=k_3$	0.08	0.08	0.08	0.33	0.33	0.08
$k=k_4$	0.1	0.1	0.1	0.1	0.5	0.1

From Figure 3 to Figure 6, we can see that $b_{k_1 1}$ in Figures 5a and 6a, $b_{k_2 1}$ in Figures 5b and 6b, $b_{k_3 6}$ in Figures 5c and 6c, and $b_{k_4 1}$ in Figures 5d and 6d are not the ones far from others, although they are the largest or the smallest ones among the arguments.

In comparison with the Gaussian OWA operator and Olympic OWA operator, the DIOWA operator can identify the extreme values more precisely, because the DIOWA operator is a type of local-distribution-based operator.

4.2.2. Comparison with Majority Additive OWA (MA-OWA) Operators

The MA-OWA weights w_{ki} associated with a_{ki} are shown in Table IX and Figure 7. The final Execution (E) Scores for each athlete are 9.08, 8.88, 8.89, 8.92, respectively.

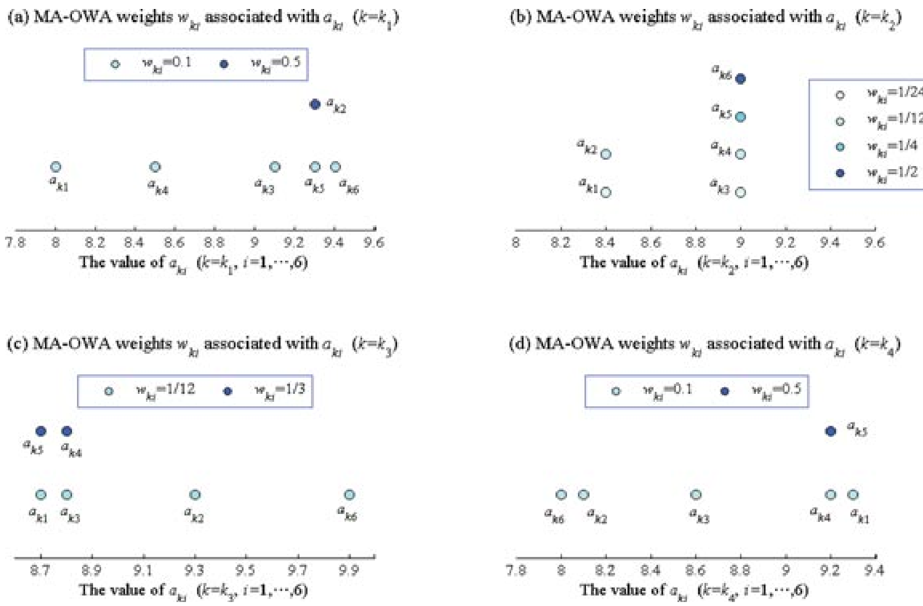

Figure 7. The MA-OWA weights w_{ki} associated with a_{ki}.

From Table IX and Figure 7, we can see that the Majority Additive OWA (MA-OWA) operator gives preference to the argument values that have cardinality greater than one.

For example, both values of $a_{k_1 2}$ and $a_{k_1 5}$ in Figure 7a are 9.3, that is, the cardinality of the argument value 9.3 is two, which is greater than one. Hence $a_{k_1 2}$ gains more weight than others.

In comparison with the MA-OWA weight associated with the argument a_{ki}, the DIOWA weight associated with a_{ki} depends upon *density* (a_{ki}, r). And *density* (a_{ki}, r) depends upon both the cardinality of the set A_{ki} and the weighted averaging distance from a_{ki} to those arguments $a_{kj} \in A_{ki}$. The members of the set A_{ki} are the ones which are similar to a_{ki}. The parameter r is a user specified similarity threshold or allowable inaccuracy threshold. When $r = 0$, the DIOWA weight associated with a_{ki} depends upon the cardinality of the set A_{ki} whose members are equal to a_{ki}, that is, if the number of the arguments which are equal to a_{ki} is larger, the DIOWA weight associated with a_{ki} is greater.

In brief, when $r = 0$, both the DIOWA operator and the MA-OWA operator give preference to the argument values that have cardinality greater than one.

4.2.3. Comparison with kNN-DOWA Operators

In the k nearest-neighbor-based dependent OWA (kNN-DOWA) operator, k represents the number of the nearest neighbors of the argument. Here we use $k_\tau = k_1, k_2, k_3, k_4$ to represent the four athletes. When $k = 1$, the kNN-DOWA weights $w_{k_\tau i}$ associated with $a_{k_\tau i}$ are shown in Figure 8 and Table X.

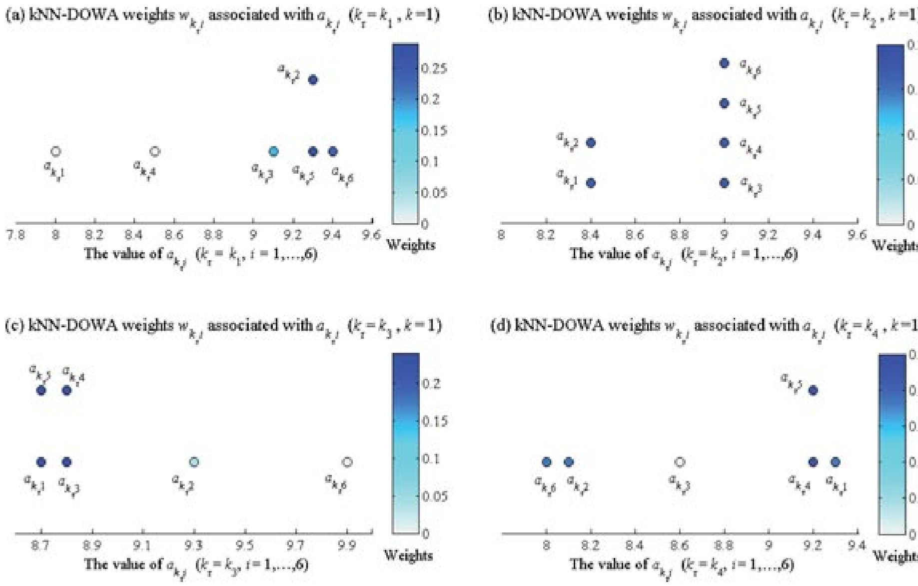

Figure 8. The kNN-DOWA weights $w_{k_\tau i}$ associated with $a_{k_\tau i}$.

From Table X and Figure 8, we can see that as a kind of local-distribution-based operator, the kNN-DOWA operator gives preference to the argument whose average distance to its k nearest neighbors is smaller.

However, the selection of the parameter k can be difficult. On the one hand, empirical results have shown that given a small k, the proposed metric is robust to the setting of this parameter[36]; on the other hand, given a smaller k, less information can be utilized to estimate the reliability of each argument value.

For example, when $k = 1$, each argument in Figure 8b has equal average distance to its k nearest neighbor. Hence the weights associated with the arguments are equal. And from Figure 8b we can see that the cardinality of the argument value 9 is greater than the cardinality of the argument value 8.4. However, as k is too small, the average distance from the argument to its k nearest neighbor cannot reflect the difference in the local data structure.

Table X. The kNN-DOWA weights $w_{k_\tau i}$ associated with $a_{k_\tau i}$ (k =1).

	$w_{k_\tau i}$					
Athlete k_τ	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$k_\tau = k_1$	0	0.29	0.18	0.00	0.29	0.24
$k_\tau = k_2$	0.17	0.17	0.17	0.17	0.17	0.17
$k_\tau = k_3$	0.24	0.04	0.24	0.24	0.24	0
$k_\tau = k_4$	0.18	0.18	0.00	0.23	0.23	0.18

Note: k represents the number of the nearest neighbors; $k_\tau = k_1, k_2, k_3, k_4$ represent the four athletes.

In comparison with the kNN-DOWA operator, the DIOWA operator gives preference to the argument with more nearest neighbors and smaller weighted average distance to its nearest neighbors. The nearest neighbors of the argument are the ones that are within the distance r of the argument. The parameter r is a user specified similarity threshold or allowable inaccuracy threshold. If it is more convenient to provide the similarity threshold or allowable inaccuracy threshold than to determine the number k of the nearest neighbors of arguments for some users, the DIOWA operator is more suitable for them than the kNN-DOWA operator.

5. CONCLUSIONS

In this paper, we introduced a new local-distribution-based operator called DIOWA operator, whose characteristic is that the ordering of the arguments is induced by the density around the arguments, rather than the values of the arguments. The density around the argument is associated with both the number of its nearest neighbors and its weighted average distance to these neighbors. The nearest neighbors of an argument are the ones which are within a similarity threshold or an allowable inaccuracy threshold r of the argument. The DIOWA operator gives preference to the arguments with higher similarity to their nearest neighbors. Especially, when $r = 0$, the DIOWA operator gives preference to the argument values that have cardinality greater than one.

To determine the DIOWA weights, we redefined the orness measure, and proposed a new maximum orness model under a dispersion constraint. The weights generated by the traditional maximum orness model depend upon the dispersion degree and the order of the arguments. Differently, the DIOWA weights generated by the new maximum orness model also depend upon the specific values of the density around the arguments. When the values of the density around the arguments are equal, i.e., $u(b_i) = u(b_j)(i \neq j)$, we can directly obtain that $w_i = w_j$. And when the sequence of the density around the arguments is an arithmetic progression, the weights generated by the new maximum orness model are the same as those generated by the traditional one introduced by Marchant.[37]

Finally, to prove the effectiveness of the DIOWA operator, we illustrated how the DIOWA operator is used in the decision process and compared the DIOWA operator with other operators, i.e., the centered OWA operator, the Olympic OWA operator, the majority additive OWA (MA-OWA) operator, and the kNN-DOWA operator.

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