Shato Robot Kinematic Model

Reference: *Kinematic Model of a Four Mecanum Wheeled Mobile Robot*

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1. Introduction

Omni-differential locomotion is being using in current mobile robots in order to obtain the additional **maneuverability** and **productivity**. These features are expanded at the expense of improved mechanical complication and increased complexity in control mechanism. Omni-differential systems work by applying rotating force of each individual wheel in one direction similar to regular wheels with a different in the fact that Omni-differential systems are able to slide freely in a different direction, in other word, they can slide frequently perpendicular to the torque vector. The main advantage of using Omni-drive systems is that translational and rotational motions are decoupled for simple motion although in making an allowance for the fastest possible motion this is not essentially the case.

1.1 Mecanum Wheels

In this design, there are a **series of free moving rollers attached to the hub but with an 45° of angle about the hub's circumference** but still the overall side profile of the wheel is circular.

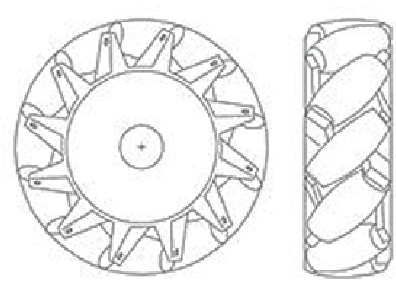

Fig 1: Mecanum wheel design

Omnidirectional motion can be reached by mounting four

Mecanum wheels on the corners of a four-sided base.

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2. Kinematic

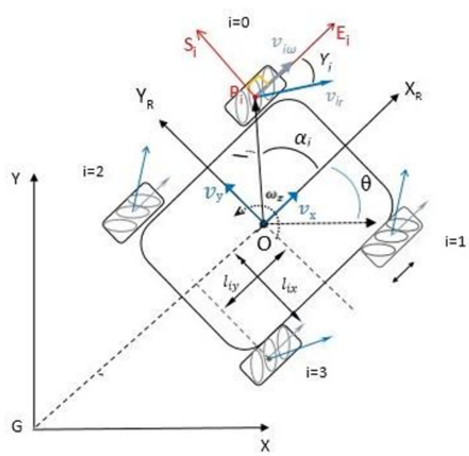

Fig 2: Wheels Configuration and Posture definition

- $S_i P_i E_i$ - coordinate system of i-th wheel in the wheel’s center point P_i.
- O, P_i - the inertial basis of the Robot in Robot’s frame.
- $\overrightarrow{OP_i}$ - is a vector that indicates the distance between Robot’s center and the center of the wheel i-th.
- l_{ix}, l_{iy} - l_{ix} half of the distance between front wheels and l_{iy} half of the

Figure 2 shows the configuration of a robot with four omnidirectional wheels.

The configuration parameters and system velocities are defined as follows:

- x, y, θ - robot’s position (x, y) and it orientation angle (the angle between X and X_R).
- XGY - inertial frame, x, y are the coordinates of the reference point O.
- $X_R O Y_R$ - robot’s base frame; Cartesian coordinate system associated with the movement of the body center.
- a_i - the angle between OP_i and X_R.
- β_i - the angle between S_i and X_R.
- γ_i - the angle between vir and Ei

distance between front wheel and the rear wheels.

- l_i - distance between wheels and the base (center of the robot O);
- r_i - denotes the radius of the wheel i (Distance of the wheel’s center to the roller center)
- r_r - denotes the radius of the rollers on the wheels.
- w_i [rad/s] - wheels angular velocity;
- v_{iw} $[m/s]$, i=0,1,2,3 $\in R$, is the velocity vector corresponding to wheel revolutions.
- v_{ir} - the velocity of the passive roller in the wheel i;
- $[W_{Si} W_{Ei} w_i]^T$ - Generalized velocity of point P_i in the frame $S_i P_i E_i$.
- $[v_{Si} v_{Ei} w_i]^T$ - Generalized velocity of point Pi in the frame $X_R O Y_R$.
- v_x, v_y [m/s] - Robot linear velocity.
- w_z [rad/s] - Robot angular velocity.

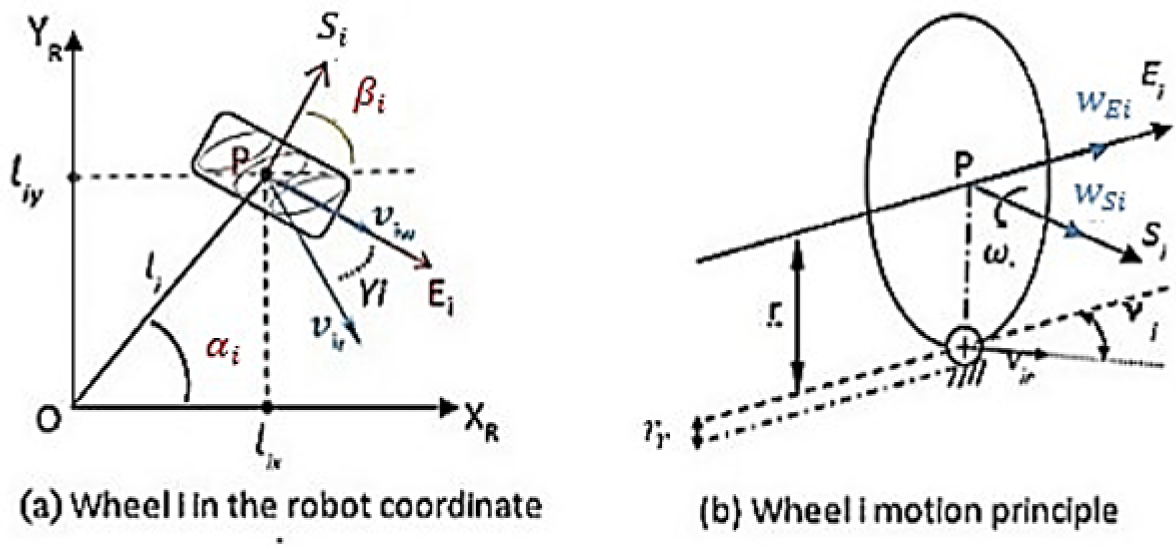

Fig 3: Parameters of ith wheel

Equation 1 describes the velocity of the passive rollers attached to the Mecanum wheels. It expresses the relationship between the rotational speed of the wheel (angular velocity denoted as w_i) and the tangential velocity of the roller (denoted as v_{ir}).

$$v_{ir} = \frac{1}{cos45} r_r w_i,$$

$$w_{Ei} = r_i w_i$$

Explanation:

- The rollers on a Mecanum wheel are mounted at a 45° angle to the wheel’s circumference, allowing the wheel to generate forces in both the longitudinal and lateral directions.
- The velocity v_{ir} represents the linear speed of the roller as it rolls on the ground.
- Since the roller is angled at 45°, the tangential velocity along the roller's contact point is affected by this angle,

hence the $cos(45^\circ)$ factor is used to project the velocity onto the relevant axis.

Equation 2 describes how the velocity components of the **Mecanum wheel** are determined based on the configuration of the roller system. It gives the velocity of the wheel at the center of the roller where it contacts the ground.

$$v_{Si} = v_{ir} sin(\gamma_i)$$

$$v_{Ei} = w_i r_i + v_{ir} cos(\gamma_i)$$

$$\begin{bmatrix} v_{Si} \\ v_{Ei} \end{bmatrix} = \begin{bmatrix} 0 & sin(\gamma_i) \\ r_i & cos(\gamma_i) \end{bmatrix} \begin{bmatrix} w_i \\ v_{ir} \end{bmatrix} = {}^{w_i}T_{Pi} \begin{bmatrix} w_i \\ v_{ir} \end{bmatrix}$$

Explanation:

This equation breaks down the wheel's motion into two components:

1. Sideways Velocity (v_{Si}):
    - This equation captures the lateral velocity of the wheel due to the roller's rotation. The sine term appears because the lateral movement depends on the angle between the roller’s velocity and the direction of the wheel.
    - Since the rollers are angled, they contribute to the wheel’s sideways motion, and this component is influenced by the sine of the angle γ_i.
2. Longitudinal Velocity (v_{Ei}):
    - This part represents the forward velocity of the wheel, which is determined by the wheel's own rotational speed ($r_i w_1$) plus the effect of the roller's tangential velocity in the longitudinal direction.

Equation 3 provides the **transformation matrix** that relates the generalized velocity of the Mecanum wheel’s roller to the velocity of the wheel's center. This matrix is crucial for converting the velocity at the contact point of the roller (on the ground) into the wheel's velocity in the robot's frame of reference.

$$^{w_i}T_{Pi} = \begin{bmatrix} 0 & sin(\gamma_i) \\ r_i & cos(\gamma_i) \end{bmatrix}$$

Explanation:

- **Matrix on the left side**: This contains the two components of velocity for the wheel: lateral (v_{Si}) and longitudinal (v_{Ei}).
- **Matrix on the right side**: This is a transformation matrix that maps the roller velocity (v_{ir}) and the wheel's angular velocity (w_i) to the velocity components of the wheel.
  - The first row shows that the lateral velocity (v_{Si}) is purely a function of the roller velocity (v_{ir}) and the angle γ_i between the velocity vector and the wheel direction. Hence, the term $sin(\gamma_i)$.
  - The second row contains two parts: the wheel's own angular velocity component ($r_i w_i$) and the roller velocity projected onto the wheel's longitudinal direction ($v_{ir}.cos(\gamma_i)$).

Equation 4 provides a **transformation** from the velocity of the wheel’s center in the wheel’s local coordinate frame to the robot's coordinate frame $X_R O Y_R$. This is crucial because the wheels are mounted on a robot in different orientations, and their individual movements must be transformed into a common frame to determine the overall motion of the robot.

$$\begin{bmatrix} v_{iX_R} \\ v_{iY_R} \end{bmatrix} = \begin{bmatrix} cos\beta_i & -sin\beta_i \\ sin\beta_i & cos\beta_i \end{bmatrix} \begin{bmatrix} v_{S_i} \\ v_{E_i} \end{bmatrix} = {}^{w_i}T_{P_i} {}^{P_i}T_R \begin{bmatrix} w_i \\ v_{ir} \end{bmatrix}$$

Explanation:

1. Coordinate Transformation:
   - The equation uses a rotation matrix to transform the velocities of the wheel from its local frame to the robot’s global frame. This allows for understanding how each wheel's motion affects the robot's overall movement.
2. **Relating Wheel Motion to Robot Motion**:
   - By transforming each wheel’s velocity into the global frame, this equation allows the overall motion of the robot to be understood based on the velocities of its individual wheels.

Equation 5 refers to the transformation matrix that converts the velocity components of the wheel in its local frame of reference to the robot's global coordinate system. This transformation is necessary for understanding the overall kinematics of the robot, as it translates the local wheel velocities into a form that reflects their contribution to the robot’s global movement.

$${}^{P_i}T_R = \begin{bmatrix} cos\beta_i & -sin\beta_i \\ sin\beta_i & cos\beta_i \end{bmatrix}$$

Equation 6 expresses how the velocities at the robot’s center of mass (both translational and rotational) contribute to the velocities of each wheel, accounting for the geometry of the robot (i.e., the distances between the robot's center and the wheels).

$$\begin{bmatrix} v_{iX_R} \\ v_{iY_R} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -l_{iy} \\ 0 & 1 & l_i x \end{bmatrix} \begin{bmatrix} v_X \\ v_Y \\ w \end{bmatrix} = T' \begin{bmatrix} v_{X_R} \\ v_{Y_R} \\ w_R \end{bmatrix}$$

Where $T' = \begin{bmatrix} 1 & 0 & -l_{iy} \\ 0 & 1 & l_i x \end{bmatrix}$

To derive **Equation 6**, which relates the robot's translational and angular velocities to the velocity at each wheel's center, we need to understand the motion of the robot in both translation and rotation.

- Each wheel is positioned at a certain distance from the robot’s center of mass.
- If the robot rotates with angular velocity w, each wheel will have an additional velocity component due to the rotation.
- For any point on a rigid body (like a robot), the velocity due to rotation is: $v_{rotation} = w.r$; where w is the angular velocity and r is the vector from the center of mass to the point in question (the wheel in this case).
- For the i-th wheel, the velocity due to rotation is perpendicular to the vector $r = (l_{ix}, l_{iy})$ from the center of mass to the wheel. The velocities at each wheel due to rotation are:
    - In the x-direction: $-w.l_{iy}$
    - In the y-direction: $w.l_{ix}$
- The total velocity at the i-th wheel is the sum of the translational velocity of the robot and the rotational velocity caused by the robot’s angular velocity. Thus, for the i-th wheel, the velocity components are:
    - $v_{iX_R} = v_X - w.l_{iy}$
    - $v_{iY_R} = v_Y + w.l_{ix}$

- These equations describe how the robot's translational velocity and angular velocity (v_X, v_Y) combine to produce the velocity at the i-th wheel and can be represented in matrix form (Equation 7).

Equation 8 in the document provides the inverse kinematic model for the Mecanum-wheeled robot. It relates the robot's linear and angular velocities to the wheel velocities. The equation is used to calculate the angular velocities of the robot's wheels given the robot's desired translational and rotational velocities.

From (eq.3) and (eq.5), the inverse kinematic model can be obtained:

$$^{w_i}T_{P_i}{}^{P_i}T_R \begin{bmatrix} w_i \\ v_{ir} \end{bmatrix} = T' \begin{bmatrix} v_{X_R} \\ v_{Y_R} \\ w_R \end{bmatrix}, i = 0, 1, 2, 3$$

To derive **Equation 8**, we combine the concepts of **wheel kinematics** and **robot kinematics**, essentially transforming the robot's velocity into wheel velocities, accounting for the geometry and placement of each wheel.

1. **Describing Robot Motion**: The robot's motion is defined by its linear velocities (v_X, v_Y) and angular velocity (w_z).
2. **Velocity at Each Wheel**: Each wheel experiences a combination of the robot’s translational and rotational motion, which affects its velocity.
3. **Transformation Matrices**: Matrices $^{w_i}T_{P_i}$ and $^{P_i}T_R$ transform the wheel's local velocity components (angular velocity w_i and roller velocity v_{ir}) to the robot's global frame.
4. **Inverse Kinematics**: The equation expresses the wheel velocities needed to produce the desired robot motion, using the transformation matrix T'.

Equation 9 provides the relationship between the robot's base velocity (at the robot's center of mass) and the rotational velocities of the Mecanum wheels. This is done by combining the inverse of the transformation matrices from the wheel’s frame to the robot's frame.

$$\begin{bmatrix} w_i \\ v_{ir} \end{bmatrix} = {}^{w_i}T_{Pi}{}^{-1}\,{}^{P_i}T_R{}^{-1} \begin{bmatrix} v_{X_R} \\ v_{Y_R} \\ w_Z \end{bmatrix}, i = 0, 1, 2, 3$$

- **Inverse Kinematics**:
  - This equation is part of the **inverse kinematics** of the Mecanum wheel system. In inverse kinematics, the robot’s overall motion (in terms of velocity and rotation) is used to calculate the individual velocities of each wheel.
  - The transformation matrices combine the geometry and configuration of the wheels with their velocities to compute the motion of the robot.

According to eq.3 and eq.4 there is a relationship between variables in each robot’s wheels frames and its center. And with the inverse kinematic, the velocity of the system can be obtained by implementing v_{ir} the linear velocity and w_i the rotational speed of wheel i-th in Equation 10 and the contrary in

Equation 11.

$$\begin{bmatrix} v_{X_R} \\ v_{Y_R} \\ w_Z \end{bmatrix} = T^{+} \begin{bmatrix} w_i \\ v_{ir} \end{bmatrix}$$

$$\begin{bmatrix} w_i \\ v_{ir} \end{bmatrix} = T \begin{bmatrix} v_{X_R} \\ v_{Y_R} \\ w_Z \end{bmatrix}$$

Where $T = {}^{w_i}T_{Pi}{}^{-1} . {}^{P_i}T_R{}^{-1} . T'$, $T^{+} = (T^T T)^{-1} T^T$

Considering the fact that $l_{ix} = l_i cosa_i$ and $l_{iy} = l_i sina_i$, and assuming that the wheels are in a same size, the transformation matrix is:

$$T := \frac{1}{-r} \begin{bmatrix} \frac{\cos(\beta_i - y_i)}{\sin(y_i)} & \frac{\sin(\beta_i - y_i)}{\sin(y_i)} & \frac{li \sin(-\alpha_i + \beta_i - y_i)}{\sin(y_i)} \\ -\frac{r\cos(\beta_i)}{\sin(y_i)} & -\frac{r\sin(\beta_i)}{\sin(y_i)} & -\frac{li\sin(-\alpha_i + \beta_i)\, r}{\sin(y_i)} \end{bmatrix};$$

$$T^{+} = \frac{1}{li^2 + 1} \begin{bmatrix} -\frac{1}{2}\left(li^2 \sin(\beta_i) - li^2 \sin(-\beta_i + 2\alpha_i) + 2\sin(Bi)\right) r & \frac{1}{2} li^2 \sin(y_i - \beta_i + 2\alpha_i) - \frac{1}{2}\sin(-y_i + \beta_i)\, li^2 - \sin(-y_i + \beta_i) \\ \frac{1}{2} r\left(li^2 \cos(\beta_i) - li^2 \cos(-\beta_i + 2\alpha_i) + 2\cos(\beta_i)\right) & -\frac{1}{2} li^2 \cos(y_i - \beta_i + 2\alpha_i) + \frac{1}{2}\cos(-y_i + \beta_i)\, li^2 + \cos(-y_i + \beta_i) \\ \cos(\alpha_i - \beta_i)\, li\, r & \cos(\alpha_i - \beta_i + y_i)\, li \end{bmatrix}$$

Equation 14 provides the **inverse kinematic solution**, which calculates the angular velocities (w_0, w_1, etc.) that the four wheels must rotate at in order to achieve the desired motion of the robot as specified by

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \frac{-1}{r}\begin{bmatrix}\frac{\cos(\beta_1-\gamma_1)}{\sin\gamma_1} & \frac{\sin(\beta_1-\gamma_1)}{\sin\gamma_1} & \frac{l_1\sin(\beta_1-\gamma_1-\alpha_1)}{\sin\gamma_1}\\ \frac{\cos(\beta_2-\gamma_2)}{\sin\gamma_2} & \frac{\sin(\beta_2-\gamma_2)}{\sin\gamma_2} & \frac{l_2\sin(\beta_2-\gamma_2-\alpha_2)}{\sin\gamma_2}\\ \frac{\cos(\beta_3-\gamma_3)}{\sin\gamma_3} & \frac{\sin(\beta_3-\gamma_3)}{\sin\gamma_3} & \frac{l_3\sin(\beta_3-\gamma_3-\alpha_3)}{\sin\gamma_3}\\ \frac{\cos(\beta_4-\gamma_4)}{\sin\gamma_4} & \frac{\sin(\beta_4-\gamma_4)}{\sin\gamma_4} & \frac{l_4\sin(\beta_4-\gamma_4-\alpha_4)}{\sin\gamma_4}\end{bmatrix}\begin{bmatrix}v_X\\ v_Y\\ \omega_z\end{bmatrix}$$

Equation 15 shows the Jacobian matrix for the system's inverse kinematic:

$$T = \frac{-1}{r}\begin{bmatrix}\frac{\cos(\beta_1-\gamma_1)}{\sin\gamma_1} & \frac{\sin(\beta_1-\gamma_1)}{\sin\gamma_1} & \frac{l_1\sin(\beta_1-\gamma_1-\alpha_1)}{\sin\gamma_1}\\ \frac{\cos(\beta_2-\gamma_2)}{\sin\gamma_2} & \frac{\sin(\beta_2-\gamma_2)}{\sin\gamma_2} & \frac{l_2\sin(\beta_2-\gamma_2-\alpha_2)}{\sin\gamma_2}\\ \frac{\cos(\beta_3-\gamma_3)}{\sin\gamma_3} & \frac{\sin(\beta_3-\gamma_3)}{\sin\gamma_3} & \frac{l_3\sin(\beta_3-\gamma_3-\alpha_3)}{\sin\gamma_3}\\ \frac{\cos(\beta_4-\gamma_4)}{\sin\gamma_4} & \frac{\sin(\beta_4-\gamma_4)}{\sin\gamma_4} & \frac{l_4\sin(\beta_4-\gamma_4-\alpha_4)}{\sin\gamma_4}\end{bmatrix}$$

Equation 16 establishes the relationship between the robot's overall velocity (both translational and rotational) and the angular velocities of its four wheels according to equation 10.

$$\begin{bmatrix}v_X\\ v_Y\\ w_Z\end{bmatrix} = T^{+}\begin{bmatrix}w_1\\ w_2\\ w3\\ w_4\end{bmatrix}$$

The matrix T^{+} accounts for the orientation and configuration of the wheels on the robot, allowing the angular velocities of the wheels to be converted into linear and angular velocities for the robot itself.

Table 1. Robot Parameters

i	Wheels	α_i	β_i	γ_i	l_i	l_{ix}	l_{iy}
0	1sw	$\pi/4$	$\pi/2$	$-\pi/4$	l	l_x	l_y
1	2sw	$-\pi/4$	$-\pi/2$	$\pi/4$	l	l_x	l_y
2	3sw	$3\pi/4$	$\pi/2$	$\pi/4$	l	l_x	l_y
3	4sw	$-3\pi/4$	$-\pi/2$	$-\pi/4$	l	l_x	l_y

Typical Mecanum four system shown in Figure 2; the parameters of this configuration are shown in table 1. In this configuration wheels sizes are the same.

By replacing the parameters of Table 1 in matrix Equation 15 and considering Equation 14, we come up with:

$$T = \frac{1}{r}\begin{bmatrix} 1 & -1 & -(l_x + l_y) \\ 1 & 1 & (l_x + l_y) \\ 1 & 1 & -(l_x + l_y) \\ 1 & -1 & (l_x + l_y) \end{bmatrix}$$

$$T^{+} = \frac{r}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -\frac{1}{(l_x+l_y)} & \frac{1}{(l_x+l_y)} & -\frac{1}{(l_x+l_y)} & \frac{1}{(l_x+l_y)} \end{bmatrix}$$

According to equations (10) and (11) for Forward and Inverse kinematics there is:

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \frac{1}{r}\begin{bmatrix}1 & -1 & -(l_x + l_y)\\ 1 & 1 & (l_x + l_y)\\ 1 & 1 & -(l_x + l_y)\\ 1 & -1 & (l_x + l_y)\end{bmatrix}\begin{bmatrix}v_x\\ v_y\\ \omega_z\end{bmatrix}.$$

Equation 19 calculates the angular velocity of each wheel necessary to achieve the robot's desired linear velocity (v_X, v_Y) and rotational velocity (w_Z).

$$\begin{cases}\omega_1 = \frac{1}{r}\left(v_x - v_y - (l_x + l_y)\omega\right),\\ \omega_2 = \frac{1}{r}\left(v_x + v_y + (l_x + l_y)\omega\right),\\ \omega_3 = \frac{1}{r}\left(v_x + v_y - (l_x + l_y)\omega\right),\\ \omega_4 = \frac{1}{r}\left(v_x - v_y + (l_x + l_y)\omega\right).\end{cases}$$

Equation 20 relates the individual wheel angular velocities to the robot's overall linear velocities in the X and Y directions and its angular velocity around the Z-axis. This is a specific representation of how each Mecanum wheel's angular velocity contributes to the robot's overall movement.

$$\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} = \frac{r}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -\frac{1}{(l_x+l_y)} & \frac{1}{(l_x+l_y)} & -\frac{1}{(l_x+l_y)} & \frac{1}{(l_x+l_y)} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

Equation 21 describes the **forward kinematics** of the Mecanum-wheeled robot, which calculates the robot's overall linear and angular velocities based on the angular velocities of the four wheels.

1.

Linear Velocity v_X: The sum of the angular velocities of all four wheels influences the robot’s motion in the X direction.

2. **Linear Velocity** v_Y: The difference between the wheels on opposite sides contributes to the sideways motion of the robot.
3. **Angular Velocity** w_Z: The robot’s ability to rotate depends on the difference in the speeds of the wheels and the robot’s geometry, with $l_x + l_y$ affecting how much rotation is generated by wheel speed differences.

Explanation:

- **Forward Kinematics**: This equation is part of the **forward kinematics**, which means it calculates the robot’s overall movement based on the angular velocities of the wheels.

- **Matrix Structure**:
  - The first row adds up the wheel velocities to determine the forward/backward velocity (v_X).
  - The second row calculates the sideways motion (v_Y) by subtracting wheel velocities on opposite sides of the robot.

- The third row determines the angular velocity based on the difference in wheel velocities, accounting for the geometry of the robot (i.e., the distances $l_x + l_y$)

Transformation to Global Frame:

If you have velocity (v_X, v_Y, w_Z) the robot's local frame, you can find its coordinates in the global frame by multiplying the 3x3 rotation matrix with the local coordinates:

$$\begin{bmatrix} v_{gX} \\ v_{gY} \\ w_{gZ} \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_X \\ v_Y \\ w_Z \end{bmatrix}$$

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3. Demo

Welcome to this demonstration of the kinematic modeling of a four-wheeled Mecanum robot. Mecanum wheels allow omnidirectional movement, meaning the robot can move in any direction without changing its orientation. This is achieved through the unique design of Mecanum wheels, which have rollers set at an angle, enabling complex motion control.

FourMecanumWheeledRobotDemo.ipynb - Colab (google.com)