Lec 13: 10 Nov 2020

Tuesday, 10 November 2020 16:05

Cont'd: Laplace's eqn in spherical coordinates

Recall: for $m=0$,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P(x) = 0$$

When $P(x) = \sum_{j=0}^{\infty} a_j x^{j+\alpha}$, $\alpha \in \mathbb{R}$, we have the ff. recurrence relation:

$$a_{j+2} = \left[\frac{(j+\alpha)(j+\alpha+1) - \ell(\ell+1)}{(j+\alpha+2)(j+\alpha+1)}\right] a_j$$

& $\alpha(\alpha-1)a_0 = 0$, $\alpha(\alpha+1)a_1 = 0 \rightarrow$ REDUNDANT

so choose $a_1 = 0$. By choosing this, we set all odd coeffs to zero!

$$P(x) = \sum_{j=0}^{\infty} a_{2j} x^{2j+\alpha} = a_0 + a_2 x^{2+\alpha} + a_4 x^{4+\alpha} + \ldots$$

- From $\alpha(\alpha-1)a_0 = 0$, $a_0 \neq 0$, so $\alpha = 0, 1$

so $P(x) = \sum_{j=0}^{\infty} a_{2j} x^{2j}$ or $\sum_{j=0}^{\infty} a_{2j} x^{2j+1}$

$\hookrightarrow$ BOTH converge for $x=1$ ONLY when the series is finite (HOW?) To specify the finiteness of the series, set the recurrence relation to zero at some j, call this j_{max}:

$$\frac{(j_{max}+\alpha)(j_{max}+\alpha+1) - \ell(\ell+1)}{(j_{max}+\alpha+2)(j_{max}+\alpha+1)} = 0$$

So, for $\alpha = 0$,

$$j_{max}(j_{max}+1) - l(l+1) = 0$$

$$j_{max} = l \longrightarrow P_l(x) = \sum_{j=0}^{l} a_j x^j,$$

$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+2)(j+1)} a_j$$

for $\alpha = 1$,

$$(j_{max}+1)(j_{max}+2) - l(l+1) = 0$$

$$j_{max} = l-1 \longrightarrow P_l(x) = \sum_{j=0}^{l-1} a_j x^{j+1}$$

Recall a_j for odd j is zero, so we can recast the series as

$$\alpha = 0: \quad P_l(x) = \sum_{j=0}^{l/2} a_{2j} x^{2j}, \quad a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+2)(j+1)} a_j$$

$$\alpha = 1: \quad P_l(x) = \sum_{j=0}^{(l-1)/2} a_{2j} x^{2j+1}, \quad a_{j+2} = \frac{(j+1)(j+2) - l(l+1)}{(j+3)(j+2)} a_j$$

EVEN l ODD l

Thus,

$$P_0(x) = a_0$$

$$P_1(x) = a_0 x$$

$$P_2(x) = a_0 x + a_2 x^2, \quad a_2 = -3a_0$$

$$P_3(x) = a_0 x + a_2 x^3, \quad a_2 = -\frac{5}{3} a_0$$

LEGENDRE POLYNOMIALS

(set $a_0 = 1$ for $P_l(1) = 1$)

Thus, the general soln for the Laplace's eqn in spherical coordinates is (for $m = 0$)

$$\boxed{\Phi(r, \theta, \varphi) = \sum A_l r^l + B_l r^{-l} P_l(\cos\theta)}$$

$$\boxed{\Phi(r,\theta,\varphi) = \sum_{\ell} A_\ell r^\ell + B_\ell r^{-\ell}\, P_\ell(\cos\theta)}$$