Laplace's eqn in polar coordinates

- Example: uniform conditions along the z-axis and a circular boundary along the xy-plane = Cylindrical Coordinates!
- In 2D polar coordinates, Laplace's eqn is given by

$$\nabla^2 V = 0 \longrightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} = 0$$

where $x = \rho\cos\varphi$ & $y = \rho\sin\varphi$

or $\rho = \sqrt{x^2+y^2}$ & $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

- Again, substitute the ansatz

$$V(\rho,\varphi) = R(\rho)\,\Phi(\varphi)$$

so that

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R\Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 R\Phi}{\partial \varphi^2} = 0$$

or

$$\frac{\Phi}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{R}{\rho^2}\frac{d^2\Phi}{d\varphi^2} = 0$$

divide both sides by $V = R\Phi$, and expand

$$\frac{1}{R\rho}\left(\frac{dR}{d\rho} + \rho\frac{d^2R}{d\rho^2}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\varphi^2} = 0$$

multiply by ρ^2 and expand further:

$$\underbrace{\frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2}}_{\text{entirely a fxn of } \rho} \quad \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\text{entirely } \varphi} = 0$$

- Again, for fxns of diff variables to add to 0, they must both be constants. Set that constant to be ν^2, st

$$\frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} = \nu^2$$

&

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\nu^2$$

$\Phi(\varphi)$ is easily solved as

$$\Phi(\varphi) = C_\nu e^{i\nu\varphi} + D_\nu e^{-i\nu\varphi}$$

Rewrite the DE in $R(\rho)$ as

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \nu^2$$

If $R(\rho) = A\rho^\nu + B\rho^{-\nu}$, then

$$\rho\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\left(A\rho^\nu + B\rho^{-\nu}\right)\right) = \nu^2\left(A\rho^\nu + B\rho^{-\nu}\right)$$

$$\rho\frac{d}{d\rho}\left(\rho\,\nu A\rho^{\nu-1} - \nu\rho B\rho^{-\nu-1}\right) = \boxed{\nu^2 A\rho^\nu + \nu^2 B\rho^{-\nu}}$$

$$\rho\left(\nu A\rho^{\nu-1} + \rho\nu(\nu-1)A\rho^{\nu-2} - \nu B\rho^{-\nu-1} - \rho\nu(-\nu-1)B\rho^{-\nu-2}\right)$$

$$= \nu\cancel{A}\rho^\nu + \rho\nu^2 A\rho^{\nu-1} - \cancel{\rho\nu A\rho^{\nu-1}}$$

$$= \cancel{v A \rho} + \rho v^2 A\rho - \cancel{\rho v A\rho}$$

$$- \cancel{v B\rho^{-v}} + \rho v^2 B\rho^{-v-1} + \cancel{\rho v B}e^{-v-1}$$

$$= v^2 A\rho^{v} + v^2 B\rho^{-v} \quad //$$

Thus $R(\rho) = A_v \rho^v + B_v \rho^{-v}$

The general solution is therefore

$$V(\rho, \varphi) = \sum_v (A_v \rho^v + B_v \rho^{-v})(C_v e^{iv\varphi} + D_v e^{-iv\varphi})$$

If $v = 0$, however, the DEs become

$$\underbrace{\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0} \quad \& \quad \underbrace{\frac{d^2\Phi}{d\varphi^2} = 0}$$

$$\rho \frac{dR}{d\rho} = B \qquad\qquad \Phi(\varphi) = C + D\varphi$$

$$\frac{dR}{d\rho} = \frac{B}{\rho}$$

$$dR = B\frac{d\rho}{\rho}$$

$$R(\rho) = A + B\ln\rho$$

so that

$$V(\rho, \varphi) = (A_0 + B_0 \ln\rho)(C_0 + D_0\varphi) + \sum_{v \neq 0} (A_v \rho^v + B_v \rho^{-v})(C_v e^{iv\varphi} + D_v e^{-iv\varphi})$$

- For a single-valued solution, we restrict φ s.t.

$$\varphi(0) = \varphi(2\pi) \quad \text{or} \quad V(\rho, \varphi) = V(\rho, \varphi + 2\pi)$$

- Then, V becomes

$$(A_0 + B_0 \ln\rho)(C_0 + D_0\varphi) + \sum_{v \neq 0} (A_v \rho^v + B_v \rho^{-v})(C_v e^{iv\varphi} + D_v e^{-iv\varphi})$$

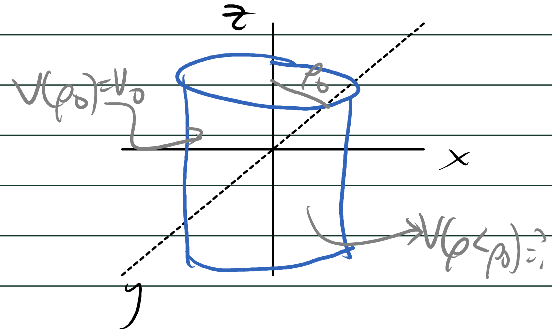
$$= (A_0 + B_0 \ln\rho)(C_0 + D_0(\varphi + 2\pi)) + \sum_{\nu \neq 0} (A_\nu \rho^\nu + B_\nu \rho^{-\nu})(C_\nu e^{i\nu(\varphi+2\pi)} + D_\nu e^{-i\nu(\varphi+2\pi)})$$

- Since $D_0 2\pi = 0$, $D_0 = 0$
and since $e^{i\nu 2\pi} = 1$, ν must be an integer
because $e^{2\pi i} = 1$, $e^{4\pi i} = 1$, ...

Update:

$$V(\rho,\varphi) = (A_0 + B_0 \ln\rho)(C_0) + \sum_{\nu=1}^{\infty} (A_\nu \rho^\nu + B_\nu \rho^{-\nu})(C_\nu e^{i\nu\varphi} + D_\nu e^{-i\nu\varphi})$$

- If we want to solve for the potential in the region $\rho < \rho_0$, for which $V(\rho_0) = V_0$ is a boundary condition, then we need to include $\rho = 0$ in our calculations;



- For this reason, $B_\nu = 0$ to prevent blowing up of V due to $\ln 0$ and $\frac{1}{0}$.

Update:

$$V(\rho,\varphi) = A_0 + \sum_{\nu=1}^{\infty} A_\nu \rho^\nu (C_\nu e^{i\nu\varphi} + D_\nu e^{-i\nu\varphi})$$

to compensate, make ν run from $-\infty$ to $+\infty$:

$$V(\rho,\varphi) = \sum_{\nu=-\infty}^{\infty} A_\nu \rho^{|\nu|} e^{i\nu\varphi}$$, A_ν absorbs A_0, C_ν, & D_ν.

We then apply the boundary condition $V(\rho_0) = V_0$:

$$V(\rho_0,\varphi) = V_0 = \sum_{\nu=-\infty}^{\infty} A_\nu \rho_0^{|\nu|} e^{i\nu\varphi}$$

- using the orthogonality of exponentials,

$$\int_{-\pi}^{\pi} e^{i(k-k')x} dx = 2\pi \delta(k-k')$$

we multiply both sides with $e^{-i\nu\varphi'}$ to get we multiply both sides with $e^{-i\nu\varphi'}$ to get

$$\int_{-\pi}^{\pi} V_0 e^{-i\nu\varphi'} d\varphi' = \sum_{\nu=-\infty}^{\infty} A_\nu \rho_0^{|\nu|} \int_{-\pi}^{\pi} e^{i\nu(\varphi-\varphi')} d\varphi'$$

or $$A_n = \frac{1}{2\pi} \frac{1}{\rho^{|n|}} \int_{-\pi}^{\pi} V_0 e^{-in\varphi'} d\varphi'$$

or in general, if $V(\rho_0) = V(\varphi)$ fxn of azim. angle.

- Thus, our potential inside the cylinder assumes the form

$$V(\rho,\varphi) = \sum_{m=-\infty}^{\infty} A_m \left(\frac{\rho}{\rho_0}\right)^{|m|} e^{im\varphi},$$

$$A_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\varphi') e^{-im\varphi'} d\varphi'$$