Problem 1

TIME EVOLUTION OF A GAUSSIAN WAVEPACKET. In this problem, we are going to use propagators to determine how a wave function evolves in time.

We can describe the initial condition of the particle as a Gaussian wavepacket: a ground state wave function for a particle in the harmonic oscillator potential which can be expressed in the position representation as

$$\psi_0(x', t=0) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x'^2\right) \tag{1}$$

We then recall the free particle propagator

$$K(x, x', t) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{i}{\hbar}\frac{m(x-x')^2}{2t}\right) \tag{2}$$

We then evolve the wave function of the ground state quantum harmonic oscillator by using the relation of the initial state to the final state by

$$|\psi(x,t)\rangle = K(x, x', t)\,|\psi(x', t=0)\rangle \tag{3}$$

Thus, operating the propagator on the initial state,

$$\psi(x,t) = \int K(x, x', t) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x'^2\right) dx' \tag{4}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int \exp\left(\frac{i}{\hbar}\frac{m(x-x')^2}{2t}\right) \exp\left(-\frac{m\omega}{2\hbar}x'^2\right)\ dx' \tag{5}$$

We then see that we're adding Gaussians, so that

$$\exp\left(\frac{i}{\hbar}\frac{m(x-x')^2}{2t}\right) = \exp\left(\frac{imx^2}{2\hbar t}\right) \exp\left(-\frac{imx}{\hbar t}x'\right) \exp\left(\frac{imx'^2}{2\hbar t}\right) \tag{6}$$

and we can use the property of Gaussian integrals

$$\int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right)\ dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \tag{7}$$

to further simplify equation (5):

$$\psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{imx^2}{2\hbar t}\right) \sqrt{\frac{2\pi\hbar t}{m\omega t - im}} \exp\left(-\frac{m^2x^2}{2m\hbar\omega t^2 - 2im\hbar t}\right)$$

$$\boxed{\psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{1}{i\omega t+1}} \exp\left(\frac{mx^2}{2\hbar}\frac{i\omega}{\omega t - i}\right)} \tag{8}$$

From equation (8), we can recover equation (1) by setting $t = 0$.
We then determine the probability density $|\psi(x,t)|^2$:

$$\begin{aligned}
\psi^*(x,t)\psi(x,t) &= \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{1}{-i\omega t+1}} \exp\left(\frac{mx^2}{2\hbar}\frac{i\omega}{\omega t + i}\right)\right] \\
&\quad \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{1}{i\omega t+1}} \exp\left(\frac{mx^2}{2\hbar}\frac{-i\omega}{\omega t - i}\right)\right] \\
&= \left(\frac{m\omega}{\pi\hbar(\omega^2t^2+1)}\right)^{1/2} \exp\left(-\frac{m\omega x^2}{\hbar(\omega^2t^2+1)}\right) \qquad (9) \\
&= \frac{A(t)}{\sqrt{\pi}} e^{-A(t)^2x^2} \qquad (10)
\end{aligned}$$

where $A(t) = \sqrt{\dfrac{m\omega}{\hbar(\omega^2t^2+1)}}$.

■

AHARONOV-BOHM EFFECT. The Aharonov-Bohm effect provides an understanding of the significance of the vector and scalar potentials that appear as we rewrite the magnetic and electric fields of Maxwell's theory. Recall that for a charged particle of mass m and charge q in an electromagnetic field, the Hamiltonian (for a potential-less field) becomes

$$H = \frac{1}{2m}\left[\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x})\right]^2 + q\Phi(\mathbf{x}) \tag{1}$$

where $\mathbf{p} = \mathrm{m}\mathbf{v}$, and $\mathbf{A}$ & Φ are the magnetic vector and electric scalar potentials, respectively. They are defined from the magnetic field and electric field as follows: $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{E} = -\boldsymbol{\nabla}\Phi$. In the Schrodinger picture, we convert the Hamiltonian into its operator form, yielding

$$\left[\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right)^2\right]\Psi + q\Phi = i\hbar\frac{\partial\Psi}{\partial t} \tag{2}$$

To solve this, we recast Ψ in a more intuitive form,

$$\Psi(\mathbf{x}, t) = e^{iS(\mathbf{x})/\hbar}\Psi'(\mathbf{x}, t) \tag{3}$$

where S is the action integral

$$S(\mathbf{x}) \equiv \frac{q}{c}\int_{\mathcal{O}}^{\mathbf{x}} \mathbf{A}(\mathbf{x}') \cdot \mathbf{dx}' \tag{4}$$

where $\mathcal{O}$ is an arbitrary point, due to the gauge freedom for electromagnetic potentials. To make the action independent of the path, we recall that for irrotational vector fields, $\boldsymbol{\nabla} \times \mathbf{A} = 0$, but this precisely leads us to choosing $\mathbf{B} = 0$, or the field outside a very long solenoid. (Of course, this works by hindsight, since we are interested in how the field of a solenoid brings up the Aharonov-Bohm effect.)

We calculate the Laplacian of Ψ in terms of the "shifted" wavefunction Ψ': first, get the gradient of Ψ using the chain rule,

$$\begin{aligned}
\boldsymbol{\nabla}\Psi &= \left(\frac{i}{\hbar}\boldsymbol{\nabla}S\right)\exp\{iS(\mathbf{x})/\hbar\}\Psi'(\mathbf{x}) + \exp\{iS(\mathbf{x})/\hbar\}\boldsymbol{\nabla}\Psi'(\mathbf{x}) &(5)\\
&= \frac{iq}{\hbar c}\mathbf{A}\Psi + \exp\{iS/\hbar\}\boldsymbol{\nabla}\Psi' &(6)
\end{aligned}$$

where we used equation (4) and the 3D equivalent of the first fundamental theorem of calculus to simplify $\boldsymbol{\nabla}S$. We rearrange it a bit and then we square both sides to get

$$\left(-i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right)\Psi = -i\hbar\exp\{iS/\hbar\}\boldsymbol{\nabla}\Psi' \tag{7}$$

$$\left(-i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right)^2\Psi = -\hbar^2\exp\{iS/\hbar\}\boldsymbol{\nabla}^2\Psi' \tag{8}$$

which we can readily substitute to equation (2):

$$\frac{1}{2m}\left(-\hbar^2 e^{i\mathbf{S}/\hbar}\boldsymbol{\nabla}^2\Psi'\right) = i\hbar e^{i\mathbf{S}/\hbar}\frac{\partial\Psi'}{\partial t} \tag{9}$$

$$\frac{-\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi' = i\hbar\frac{\partial\Psi'}{\partial t}, \tag{10}$$

which looks like another Schrodinger equation, only this time the potential is turned off. We can then easily solve the original wavefunction by solving equation (10) and attaching an exponential term involving the action of $\mathbf{A}$.

We now setup the experiment. We make a beam of electrons flow past a solenoid in each of its sides as in figure 1:

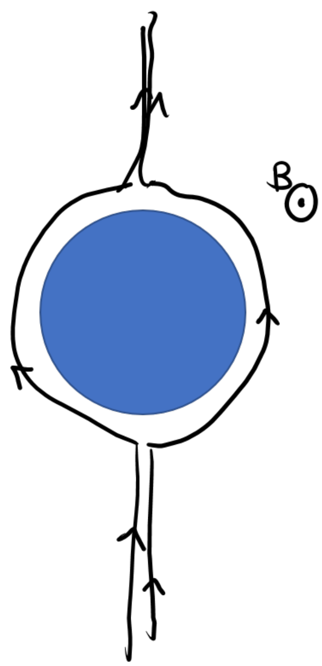

Figure 1: Cross-section view of a long solenoid with a magnetic field present inside due to its tightly wound coils. The field outside is zero.

This flow of electrons suspiciously looks like a beam that is being split by a double slit setup; hence we expect interference as they regroup in the other side of the solenoid. The solution for the wave function of electrons traveling through each side then follows a plane wave form,

$$\Psi_l = Ae^{ikx_l}; \quad \Psi_r = Ae^{ikx_r}, \tag{11}$$

where k is the wave vector of the wavefunction and $x_{l,\ r}$ are the distances traveled for beams in the left and right of the solenoid, respectively. If there's no magnetic field inside the solenoid, then our choice of $\mathbf{A}$ would have to be zero outside, since by Stokes' theorem,

$$\oint_C \mathbf{A} \cdot \mathbf{dr} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{dS} \tag{12}$$

$$= \int_S \mathbf{B} \cdot \mathbf{dS} \equiv \Phi_m \tag{13}$$

where Φ_m is the magnetic field. In cylindrical coordinates $(\rho,\ \phi,\ z)$, this leads to a choice of $\mathbf{A} = \dfrac{\Phi_m}{2\pi\rho}\hat{\phi}$. (In our setup, z is parallel to the axis of the solenoid.)

For a zero magnetic field inside, the phase difference for the two wavefunctions will only depend on the distance between the traveled paths; if $\Delta\varphi$ is our phase difference, then $\Delta\varphi = k(x_r - x_l)$. By turning on the field inside (and consequently, the vector potential outside), we will induce additional phase difference between the wavefunctions given by $\Delta\varphi = (S_r - S_l)/\hbar$, where $S_{l,\ r}$ are the actions for the electrons passing through the left and right sides of the solenoid, respectively. We then expand this phase difference using equation (4):

$$\Delta\varphi = \frac{q}{\hbar c}\left[\int_{C_l} \mathbf{A}(\mathbf{x}') \cdot \mathbf{dx}' - \int_{C_r} \mathbf{A}(\mathbf{x}') \cdot \mathbf{dx}'\right] \tag{14}$$

$$= \frac{q}{\hbar c}\oint_C \mathbf{A}(\mathbf{x}') \cdot \mathbf{dx}' = \frac{q}{\hbar c}\Phi_m, \tag{15}$$

where we used equation (13) and that C is the total path encircling the solenoid.

Thus, the changes in the magnetic field of a solenoid would induce phase differences in the wavefunction, and in turn will shift the interference pattern of the beams. We can then measure the magnetic flux by measuring this additional phase difference due to the Aharonov-Bohm effect.

■

DENSITY OPERATOR.

(a) All wave functions describe pure states since these wave functions can be written as state kets $|\psi\rangle$. Then, we can write the density operator ρ for this state as

$$\rho = |\psi\rangle\langle\psi| \,. \tag{1}$$

State kets can be treated as vectors, and any values the wave function can take on can well be written as a superposition of observable eigenvectors. It is obvious that a matrix can hold more information than a vector; in this case, the density operator can be of more use than the wave function in describing mixed states.

(b) For some eigenbasis $|n\rangle$, we can write down the matrix elements of the density operator ρ as

$$\rho = \sum_n a_n \, |n\rangle\langle n| \tag{2}$$

Since we wrote ρ in terms of its eigenbasis, we can interpret a_n as the eigenvalues of ρ, satisfying the conditions

$$0 \leqslant a_n \leqslant 1 \tag{3}$$
$$\sum_{n=1}^{M} a_n = 1 \tag{4}$$

where the normalization of a_n comes from the fact that the trace $\mathrm{Tr}(\rho) = 1$, which we prove here:

$$\begin{aligned} \mathrm{Tr}(\rho) &= \sum_m \langle m|\rho|m\rangle && (5)\\ &= \sum_n \sum_m \langle m|n\rangle \langle n|m\rangle && (6)\\ &= \sum_m \langle n|m\rangle \langle m|n\rangle = \langle n|\hat{I}|n\rangle = \langle n|n\rangle = 1 && (7) \end{aligned}$$

due to the normalization of our eigenbasis and the definition for the identity operator.

We then see that the eigenvalue spectrum a_n always corresponds to some probability distribution, given at they follow the conditions for probability as well as the interpretation that we can always prepare a mixed state with density operator given by equation (2) with probability a_n.

■