Problem 1

Jackson 5.4. In this problem, we are given the field components decomposed using cylindrical coordinates. We are also given the field component value B_z at the axis, or $\rho = 0$. We are also told that the magnitude of the same component varies slowly along the axial coordinate.

(a) Near the axis, we can expand both components in terms of a Taylor series expansion

$$B_\rho(\rho \approx 0, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} a_n(z) \tag{1}$$

$$B_z(\rho \approx 0, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} b_n(z) \tag{2}$$

We can then solve for the coefficients a_n and b_n by applying the magnetostatic equations for $\mathbf{B}$: since we're working on a current-free region,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{3}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = 0 \tag{4}$$

We first work with the divergence. Recall

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial}{\partial z} B_z = 0 \tag{5}$$

$$= \frac{1}{\rho}\sum_{n=0}^{\infty}(n+1)\frac{\rho^n}{n!} a_n(z) + \sum_{n=0}^{\infty}\frac{\rho^n}{n!} b_n'(z) \tag{6}$$

The first summation can then be realigned with the second one by making the substitution $n \to n+1$ so that

$$\sum_{n=0}^{\infty}(n+1)\frac{\rho^{n-1}}{n!} a_n(z) \to \sum_{n=-1}^{\infty}(n+2)\frac{\rho^n}{(n+1)!} a_{n+1}(z) = \frac{a_0(z)}{\rho} + \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\frac{n+2}{n+1} a_{n+1}(z) \tag{7}$$

Going back to the divergence equation, since all these terms equate to zero, each term should vanish. Then,

$$\frac{1}{\rho} a_0(z) + \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left(\frac{n+2}{n+1} a_{n+1}(z) + b_n'(z)\right) \tag{8}$$

and so for any ρ,

$$a_0(z) = 0 \tag{9}$$
$$a_{n+1}(z) = -\frac{n+1}{n+2} b_n'(z) \tag{10}$$

For the curl equation, we only consider one component since the third component of $\mathbf{B}$ vanishes, $B_\varphi = 0$. Then only $[\boldsymbol{\nabla} \times \mathbf{B}]_\varphi = 0$ is nontrivial:

$$[\boldsymbol{\nabla} \times \mathbf{B}]_\varphi = 0 = \frac{\partial}{\partial z} B_\rho - \frac{\partial}{\partial \rho} B_z \tag{11}$$
$$0 = \sum_{n=0}^{\infty} \left(\frac{\rho^n}{n!} a_n'(z) - \frac{\rho^{n-1}}{(n-1)!} b_n(z) \right) \tag{12}$$
$$0 = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} (a_n'(z) - b_{n+1}(z)) \tag{13}$$

where again, in the last line we used the substitution procedure to realign the summations. Combining equation (13) with equation (10), we get the recursion relation for b_n

$$b_{n+1}(z) = -\frac{n}{n+1} b_{n-1}''(z) \tag{14}$$

We can immediately deduce from here that $b_1 = 0$. Now we need to determine b_0. From the recursion relation (14),

$$\begin{aligned} b_{n+2}(z) &= -\frac{n+1}{n+2} b_n''(z) \\ b_2(z) &= -\frac{1}{2} b_0''(z) \\ b_4(z) &= -\frac{3}{4} b_2''(z) = \frac{3}{4}\frac{1}{2} b_0^{(4)}(z) \\ &\vdots \\ b_n(z) &= (-1)^{n/2} \frac{(n-1)(n-3)\ldots 3 \cdot 1}{n(n-2)\ldots 4 \cdot 2} b_0^{(n)}(z) \end{aligned} \tag{15}$$

so that we know the higher-order derivatives of b_0. Using (9) and (10) then

$$
\begin{aligned}
a_1(z) &= -\frac{1}{2}b_0'(z) \\
a_{a3}(z) &= -\frac{3}{4}b_2'(z) = \frac{3}{4}\cdot\frac{1}{2}b_0''(z) \\
&\vdots \\
a_{n+1}(z) &= \frac{(-1)^{n/2+1}}{2^n}\frac{(n+1)!}{(n+2)[(n/2)!]^2}b_0^{(n+1)}(z) \qquad (16)
\end{aligned}
$$

which is only for odd n. We then have expressions for a_n and b_n in terms of b_0, which is technically $B_z(\rho = 0)$, already given for us. Then, we can plug these back to equations (1) and (2). Separating the index n into its odd $(2k+1)$ and even $(2k)$ terms,

$$
\begin{aligned}
B_\rho(\rho, z) &= \sum_{k=0}^{\infty}\frac{\rho^{(2k+1)}}{(2k+2)(k!)^2}\frac{(-1)^{k+1}}{4^k}\left[\frac{\partial^{2k+1}B_z(0,z)}{\partial z^{2k+1}}\right] \qquad (17) \\
&= -\frac{\rho}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \frac{\rho^3}{16}\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \cdots \qquad (18) \\
B_z(\rho, z) &= \sum_{k=0}^{\infty}\frac{\rho^{2k}}{(k!)^2}\frac{(-1)^k}{4^k}\left[\frac{\partial^{2k}B_z(0,k)}{\partial z^{2k}}\right] \qquad (19) \\
&= B_z(0,z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \cdots \qquad (20)
\end{aligned}
$$

(b) We then find the magnitude of the neglected terms to see how we can find a length scale for considering only the first-order terms. From equations (17) and (19), we have a scaling of the nth term in the series expansion,

$$
a_n \propto \frac{\rho^n}{2^n[(n/2)!]^2}\left[\frac{\partial^n B_z(0,z)}{\partial z^n}\right] \qquad (21)
$$

We then use the ratio test to make the series converge: the ratio of every adjacent terms should be less than 1:

$$
\begin{aligned}
\frac{a_{n+2}}{a_n} \propto \rho^2\left[\frac{\partial^{n+2}}{\partial z^{n+2}}B_z(0,z)\right]\cdot\left[\frac{\partial^n}{\partial z^n}B_z(0,z)\right]^{-1} << 1 \qquad (22) \\
\rho << \sqrt{\left[\frac{\partial^{n+2}}{\partial z^{n+2}}B_z(0,z)\right]^{-1}\cdot\left[\frac{\partial^n}{\partial z^n}B_z(0,z)\right]} \qquad (23)
\end{aligned}
$$

Problem 1

Recall that a derivative scales as $1/L$ for a smooth function, where L is a length scale signifying the variation of the field. Then the constraint demands $\rho << \sqrt{\frac{1/L^n}{1/L^{n+2}}} = L$ where in this case, L represents the length scale for the variation of B_z along the z-axis, which is given to us as a condition.

■

Jackson 5.7. In this problem, we try to express the magnetic field for a compact circular coil of radius a, carrying a current I (perhaps N turns, each with current I/N), lying in the x-y plane with its center at the origin, showed in figure 1:

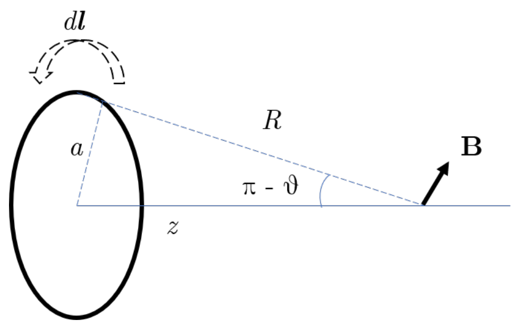

Figure 1: Geometry for finding the magnetic field due to a compact circular coil.

(a) To find the magnetic field along the z-axis, we use the Biot-Savart law

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I d\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2} \tag{1}$$

where due to symmetry of rotation around the z-axis, only the z-axis term of $\mathbf{B}$ survives:

$$\begin{aligned} B_z &= \frac{\mu_0 I}{4\pi} \int \frac{[d\boldsymbol{\ell} \times \hat{\mathbf{r}}]_z}{r^2} \\ &= \frac{\mu_0 I}{4\pi} \int \frac{\sin(\pi - \vartheta)}{R^2} d\ell \\ &= \frac{\mu_0 I}{4\pi} (2\pi a) \frac{a}{R^3} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \end{aligned} \tag{2}$$

where we used $R^2 = a^2 + z^2$ from the geometry of the problem.

(b) Now, we modify the situation by placing a similar coil with similar (directed) current a distance b from the original one, and then adjusting the coordinate system such that the original is now at $z = -b/2$, as in figure 2

Now we want to find the field near $z = 0$ as a power series in z, up to z^4:

$$B_z = \frac{\mu_0 I a^2}{d^3} \left[1 + \frac{3(b^2 - a^2)}{2d^4} z^2 + \frac{15(b^4 - 6b^2 a^2 + 2a^4)}{16 d^8} z^4 + \dots \right] \tag{3}$$

where $d^2 = a^2 + b^2/4$.

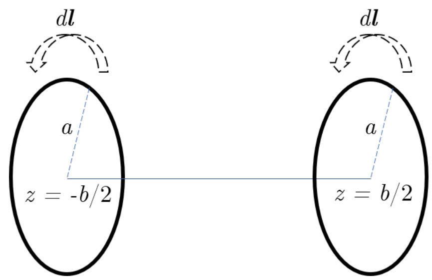

Figure 2: Adjusting the configuration to add a second similar current-carrying coil.

We can salvage our original answer by noting that since we're just moving the configuration along the z-axis, we can shift z:

$$B_z = B_{-b/2} + B_{b/2} = \frac{\mu_0 I a^2}{2}\left[\frac{1}{(a^2+(z-b/2)^2)^{3/2}} + \frac{1}{(a^2+(z+b/2)^2)^{3/2}}\right] \tag{4}$$

Now we need to expand in powers of z. Recalling the binomial expansion for negative powers and for $|z| < 1$,

$$(1+z)^{-n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}(-1)^k x^k = 1 - nx + \frac{(-n)(-n-1)}{2}x^2 + \ldots \tag{5}$$

Simplifying the terms earlier,

$$\begin{aligned} B_z &= \frac{\mu_0 I a^2}{2}\left[\frac{1}{(a^2+z^2-bz+b^2/4)^{3/2}} + \frac{1}{(a^2+z^2+bz+b^2/4)^{3/2}}\right] \\ &= \frac{\mu_0 I a^2}{2}\left[\frac{1}{d^3(1+z^2/d^2-bz/d^2)^{3/2}} + \frac{1}{d^3(1+z^2/d^2+bz/d^2)^{3/2}}\right] \end{aligned} \tag{6}$$

where we introduced $d^2 \equiv a^2 + b^2/4$. Then, using the series expansion (5),

$$B_z = \frac{\mu_0 I a^2}{2d^3}\left[1 + \frac{3}{2}(b^2-a^2)\frac{z^2}{d^4} + \frac{15}{16}(b^4 - 6b^2a^2 + 2a^4)\frac{z^4}{d^8} + \ldots\right] \tag{7}$$

(c) We then compute for the expansion of the field off-axis and derive an expression for B_ρ and B_z. We can then use Problem 1's result for near the axis. We use equation (7) as $B_z(0,z)$. Recall

$$B_z(\rho, z) = B_z(0,z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \dots \tag{8}$$
$$B_\rho(\rho, z) = -\frac{\rho}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \frac{\rho^3}{16}\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \dots \tag{9}$$

Differentiating equation (7) with respect to z,

$$\frac{\partial B_z(0,z)}{\partial z} = \frac{\mu_0 I a^2}{2d^3}\left[3(b^2-a^2)\frac{z}{d^4} + \frac{15}{4}(b^4 - 6b^2a^2 + 2a^4)\frac{z^3}{d^8} + \dots\right] \tag{10}$$
$$\frac{\partial^2 B_z(0,z)}{\partial z^2} = \frac{\mu_0 I a^2}{2d^3}\left[3(b^2-a^2)\frac{1}{d^4} + \frac{45}{4}(b^4 - 6b^2a^2 + 2a^4)\frac{z^2}{d^8} + \dots\right] \tag{11}$$
$$\frac{\partial^3 B_z(0,z)}{\partial z^3} = \frac{\mu_0 I a^2}{2d^3}\left[\frac{90}{4}(b^4 - 6b^2a^2 + 2a^4)\frac{z}{d^8} + \dots\right] \tag{12}$$

Arranging and simplifying,

$$\begin{aligned} B_z(\rho, z) &= \frac{\mu_0 I a^2}{2d^3}\left[1 - \frac{\rho^2}{4}\right]\left[3(b^2-a^2)\frac{z}{d^4} + \frac{15}{4}(b^4 - 6b^2a^2 + 2a^4)\frac{z^3}{d^8} + \dots\right] \\ &= \frac{\mu_0 I a^2}{d^3} + \frac{\mu_0 I a^2}{d^3}\left(\frac{3(b^2-a^2)}{2d^4}\right)\left(z^2 - \frac{\rho^2}{2}\right) + \dots \end{aligned} \tag{13}$$
$$\begin{aligned} B_\rho(\rho, z) &= -\frac{\rho}{2}\frac{\mu_0 I a^2}{2d^3}\left[3(b^2-a^2)\frac{z}{d^4} + \frac{15}{4}(b^4 - 6b^2a^2 + 2a^4)\frac{z^3}{d^8} + \dots\right] \\ &= -\frac{\mu_0 I a^2}{d^3}\left(\frac{3(b^2-a^2)}{2d^4}\right)\rho z + \dots \end{aligned} \tag{14}$$

(d) For large $|z|$, we can use the series expansion in (b) (equation (6)), where instead of taking $d^2 = a^2 + b^2/4$, we take $d = |z|$ so that equation (6) becomes

$$B_z(0,z) = \frac{\mu_0 I a^2}{2|z|^3}\left[\frac{1}{(1 - (a^2 + b^2/4)/z^2 - b/z)^{3/2}} + \frac{1}{(1 + (a^2 + b^2/4)/z^2 + b/z)^{3/2}}\right] \tag{15}$$

This can then be expanded in powers of $1/z$ for which we take an expansion around $1/z \to 0$ or $z \to \infty$.

(e) Lastly, we take a limiting case for which $b = a$ for the two coils, which simplifies to a Helmholtz coil setup. Substituting that to equation (7),

$$\begin{aligned} B_z(0,z) &= \frac{\mu_0 I a^2}{2(5/4)^{3/2} a^{3/2}} \left[1 - \frac{45}{16} \frac{a^4}{(5/4)^4 a^8} z^4 + \dots \right] \\ &= \frac{4\mu_0 I}{5^{3/2} a} \left[1 - \frac{144}{125} \frac{z^4}{a^4} + \dots \right] \end{aligned} \tag{16}$$

Then, taking the lowest term (z^4) as the small correction, we see that the maximum value for $|z|/a$ for B_z to be uniform should be dependent on that term:

$$\frac{\delta B_z}{B_z} \approx \frac{144}{125} \frac{z^4}{a^4} \tag{17}$$

To test some values, if we want to achieve uniformity to one part per 10^4 (0.01%), we need $|z|/a < \sqrt{\sqrt{\frac{125}{144}\frac{1}{10000}}} = 0.097$, while for one part per 10^2 (1%), we need $|z|/a < 0.305$.

■

Jackson 5.8. In this problem, we are given the current density, localized such that $\mathbf{J} = J(r,\theta)\hat{\varphi}$. We can easily adopt the derivations of Jackson's section 5.5 to this problem so that we can derive the (a) vector potential and its (b) multipole moments.

Recall the Coulomb gauge for the Laplacian of $\mathbf{A}$:

$$\boldsymbol{\nabla}^2\mathbf{A} = -\mu_0\mathbf{J} = -\mu_0 J(r,\theta)\hat{\varphi}. \tag{1}$$

Since the RHS only has a nonzero component in the azimuthal direction, we are expecting the solution for $\mathbf{A}$ to be the same: $\mathbf{A} = A_\varphi(r,\theta)$. Expanding the (vector) Laplacian in spherical coordinates yields

$$\left[\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\right]A_\varphi(r,\theta)\hat{\varphi} = -\mu_0 J_\varphi(r,\theta)\hat{\varphi} \tag{2}$$

Now the last term in the LHS would mean we would need to differentiate $\hat{\varphi}$: $\dfrac{\partial^2}{\partial\varphi^2}\varphi = -\varphi$. We can now simplify the equation into a scalar one, since we are working on the azimuthal coordinate:

$$\left[\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{1}{r^2\sin^2\theta}\right]A_\varphi(r,\theta) = -\mu_0 J_\varphi(r,\theta) \tag{3}$$

This is now of the form of a Poisson equation. The homogenous solution then follows the spherical harmonics. With the ansatz $A_\varphi(r,\theta) = \dfrac{U(r)}{r}\Theta(\theta)$, the angular equation follows

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} + \ell(\ell+1)\right]\Theta(\theta) = 0 \tag{4}$$

having a regular solution $\Theta(\theta) = P_\ell^1(\cos\theta)$. The radial equation

$$\frac{d^2}{dr^2}U(r) - \frac{\ell(\ell+1)}{r^2}U(r) = 0 \tag{5}$$

having a solution $U(r) = C_\ell r^{\ell+1} + D_\ell r^{-\ell}$. Now we can admit the solution following a multipole expansion for both the interior and exterior domains: picking out the part of $U(r)$ that would make the potential finite inside, we require $D_\ell = 0\forall\ell$:

$$A_\varphi(r<r_0,\theta) = -\frac{\mu_0}{4\pi}\sum_{\ell^1}^{\infty} m_\ell r^\ell P_\ell^1(\cos\theta) \tag{6}$$

where we recast C_ℓ to get the constants out of the summation. We also defined an arbitrary r_0 signifying the radius of the cylindrical current. The same goes for the exterior solution, only this time, C_ℓ vanishes:

$$A_\varphi(r > r_0, \theta) = -\frac{\mu_0}{4\pi}\sum_{\ell=1}^{\infty}\mu_\ell r^{-\ell-1}P_\ell^1(\cos\theta) \tag{7}$$

where again, we recast D_ℓ. We are now ready to solve for the multipole moments, μ_ℓ and m_ℓ. Recall that we can write the magnetic vector potential in terms of the current density as an integral:

$$A_\varphi(r, \theta) = \frac{\mu_0}{4\pi}\int\frac{J_\varphi(r,\theta)\varphi'\cdot\varphi}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{8}$$

We can then expand the term $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$ in terms of spherical harmonics. Following the solution of section 5.5, we place the observation point at $\varphi = 0$, yielding

$$\begin{aligned}A_\varphi(r,\theta) &= \frac{\mu_0}{4\pi}\int\frac{J_\varphi(r',\theta')\cos\varphi'}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{\mu_0}{4pi}\int\sum_{\ell,m}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell,m}(\theta,\varphi=0)Y^*_{\ell m}(\theta',\varphi)\left[\frac{\exp(i\varphi')+\exp(-i\varphi')}{2}\right]J_\varphi(r',\theta')d^3x'\end{aligned} \tag{9}$$

Now our goal is to make this equivalent to equations (7) and (8). It is then obvious to pick out $m = 1$ (which is equivalently the only nonzero contribution as we integrate out φ' together with $m = -1$, which has the same nonzero contribution). We can then leave the $m = 1$ contribution alone together with all ℓ:

$$\begin{aligned}A_\varphi(r,\theta) &= 2\Bigg(\frac{\mu_0}{4\pi}\sum_{\ell,m=1}\frac{4\pi}{2\ell+1}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-1)!}{(\ell+1)!}}P_\ell^1(\cos\theta)\times \\ &\qquad\int\frac{r_<^\ell}{r_>^{\ell+1}}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-1)!}{(\ell+1)!}}P_\ell^1(\cos\theta')\frac{J_\varphi(r',\theta')}{2}d^3x'\Bigg) \\ &= \frac{\mu_0}{4\pi}\sum_{\ell,m=1}\frac{\ell(\ell-1)!}{\ell(\ell+1)(\ell!)}P_\ell^1(\cos\theta)\int\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell^1(\cos\theta')J_\varphi(r',\theta')d^3x'\end{aligned} \tag{10}$$

We can now fully simplify this to obtain the moments: for the interior region, the observation point's radial distance is less than the source point's, so $r_< = r$ and $r_> = r'$:

Problem 3

$$A_\varphi(r<r_0,\theta) = -\frac{\mu_0}{4\pi}\sum_\ell \left[-\frac{1}{\ell(\ell+1)}\int \frac{1}{r'^{\ell+1}}P_\ell^1(\cos\theta')J_\varphi(r',\theta')d^3x'\right] r^\ell P_\ell^1(\cos\theta) \tag{11}$$

$$m_\ell = \left[-\frac{1}{\ell(\ell+1)}\int \frac{1}{r'^{\ell+1}}P_\ell^1(\cos\theta')J_\varphi(r',\theta')d^3x'\right] \tag{12}$$

And for the exterior region, $r_> = r$ and $r_< = r'$:

$$A_\varphi(r>r_0,\theta) = -\frac{\mu_0}{4\pi}\sum_\ell \left[-\frac{1}{\ell(\ell+1)}\int r'^\ell P_\ell^1(\cos\theta')J_\varphi(r',\theta')d^3x'\right] \frac{1}{r^{\ell+1}} P_\ell^1(\cos\theta) \tag{13}$$

$$\mu_\ell = \left[-\frac{1}{\ell(\ell+1)}\int r'^\ell P_\ell^1(\cos\theta')J_\varphi(r',\theta')d^3x'\right] \tag{14}$$

$\blacksquare$

Jackson 5.18 In this problem, we first orient our given to make the calculations easier. First, we place the surface of the semi-infinite slab at $z = 0$ and cover the whole $z < 0$ area. Then, for parts (a) and (b) we place the center of the current loop on the z-axis at $z = d$, albeit different orientations. To maintain the boundary conditions, we can think of the problem as a method of images, just as in Problem 5.17. Then,

(a) For this item, we consider the geometry shown in figure 1

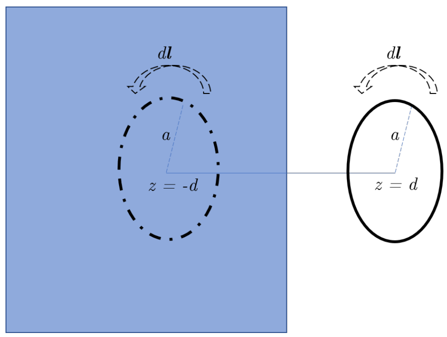

Figure 1: Geometry for part (a), in which the plane of the circular loop is parallel to the surface of the slab.

We can then calculate the magnetic field at the loop by replacing the semi-infinite slab with a mirror current loop of magnitude I' located at $z = -d$. This will lead to the vector potential along the azimuthal coordinate for a circular loop. Recall that when the loop is at $z = 0$, the expression for $\mathbf{A}$ is

$$A_\varphi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{\cos\varphi'}{\sqrt{a^2 + r^2 - 2ar\sin\theta\cos\varphi'}} d\varphi' \tag{1}$$

Since we're using the method of images, the potential must be due to these two loops only. Shifting the coordinates of the loops from $z = 0$ to $z = -d$ and $z = d$, we find

$$\begin{aligned} A_\varphi(z>0) =& \frac{\mu_0 a}{4\pi}\int_0^{2\pi} d\varphi' \ \cos\varphi' \Bigg[\frac{I}{\sqrt{a^2 + d^2 + r^2 - 2dr\cos\theta - 2ar\sin\theta\cos\varphi'}} \\ &+ \frac{I'}{\sqrt{a^2 + d^2 + r^2 - +2dr\cos\theta - 2ar\sin\theta\cos\varphi'}}\Bigg] \end{aligned} \tag{2}$$

Next we solve the potential for $z < 0$. Since there's no current there, only the mirror current image at $z > 0$ due to the original mirror image. We can then use equation (1), albeit the source is at $z = d$ again and the permeability to be used is μ since we're working inside the semi-infinite slab (cf. Eq. (5.48) and (5.49) of Jackson):

$$A_\varphi(z<0) = \frac{\mu I\text{"} a}{4\pi}\int_0^{2\pi}\frac{\cos\varphi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\varphi'}}d\varphi' \tag{3}$$

Using the boundary conditions for the magnetic field, we find that there must be a continuity for the field due to $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, or translating,

$$\mathbf{B}(z>0)\cdot\hat{\mathbf{n}} = \mathbf{B}(z<0)\cdot\hat{\mathbf{n}} \tag{4}$$

Then, since only the φ term of $\mathbf{A}$ exists, we find that

$$\begin{aligned}\frac{\partial}{\partial r}(rA_\varphi(z>0))\bigg|_{\theta=\pi/2} &= \frac{\partial}{\partial r}(rA_\varphi(z<0))\bigg|_{\theta=\pi/2} \qquad (5)\\ \mu_0\frac{\partial}{\partial r}\int_0^{2\pi}d\varphi'\ \cos\varphi'\Bigg[&\frac{Ir}{\sqrt{a^2+d^2+r^2-2dr\cos\theta-2ar\sin\theta\cos\varphi'}}\\ &+\frac{I'r}{\sqrt{a^2+d^2+r^2-+2dr\cos\theta-2ar\sin\theta\cos\varphi'}}\Bigg]\\ &=\mu I\text{"}\int_0^{2\pi}\frac{r\cos\varphi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\varphi'}}d\varphi'\\ \mu_0(I+I') &= \mu I\text{"} \qquad (6)\end{aligned}$$

Then we use the next boundary condition for the partial field $\mathbf{H}$. Since there's no free current at the boundary surface, all $\mathbf{K}$ terms will vanish:

$$\hat{\mathbf{n}}\times\mathbf{H}(z>0)\big|_{\theta=\pi/2} = \hat{\mathbf{n}}\times\mathbf{H}(z<0)\big|_{\theta=\pi/2} \tag{7}$$

$$\frac{1}{\mu_0}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi(z>0)\bigg|_{\theta=\pi/2} = \frac{1}{\mu}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi(z<0)\bigg|_{\theta=\pi/2} \quad I-I' \quad = I\text{"} \tag{8}$$

Solving the two equations yields I' and I":

$$I' = \frac{\mu-\mu_0}{\mu+\mu_0}I \tag{9}$$

$$I\text{"} = \frac{2\mu_0}{\mu+\mu_0}I \tag{10}$$

Now we can plug this back to $\mathbf{A}_\varphi$. To calculate the force acting on the loop due to the slab, we use

$$\mathbf{F} = \oint I d\boldsymbol{\ell} \times \mathbf{B} \tag{11}$$

Since the field is constant around the loop, this simplifies to getting the circumference of the loop:

$$\mathbf{F} = 2\pi a I(\hat{\varphi} \times (\boldsymbol{\nabla} \times \mathbf{A}(z > 0))) \tag{12}$$

Again, since A_φ is the only existing term, we decompose the vector equation into its terms and retain only the z-direction term

$$\mathbf{F} = 2\pi a I \hat{z} \frac{\partial}{\partial z} A_\varphi(\rho, z > 0)\bigg|_{\rho=a, z=d} \tag{13}$$

We then find that there are two terms: the force of the mirror loop and the "self-force" of the loop on itself. The latter, having two terms on A_φ, tends to cancel its contributions on one another due to symmetry of the force vectors on the loop. Thus, the only nonzero contribution would be from the mirror loop I". Since $z = r\cos\theta$, we can immediately get the derivative as

$$\boxed{\mathbf{F} = -\mu_0 I^2 a^2 d \frac{\mu - \mu_0}{\mu + \mu_0} \int_0^{2\pi} d\varphi' \ \frac{\cos\varphi'}{(4d^2 + 2a^2(1 - \cos\varphi'))^{3/2}} \hat{z}} \tag{14}$$

(b) We then reorient the loop such that its plane is perpendicular to the slab's face, as in figure 2

Reusing the answers from part (a) and recasting them to Cartesian coordinates, we find that the potential due to the original loop is

$$\mathbf{A}(x < 0, y, z) = \frac{\mu_0 I a}{4\pi} \frac{(-y\hat{x} + x\hat{y})}{\sqrt{x^2 + y^2}} int_0^{2\pi} \frac{\cos\varphi'}{\sqrt{a^2 + x^2 + y^2 + z^2 - 2a(\sqrt{x^2 + y^2})\cos\varphi'}} d\varphi' \tag{15}$$

Now the potential for the mirror current will mimic this form except that the x-coordinate would be shifted to $x - 2d$. This would lead to the same conditions on I' so that it would have the same form as in part (a), or $I' = I\dfrac{\mu - \mu_0}{\mu + \mu_0}$.

Problem 4

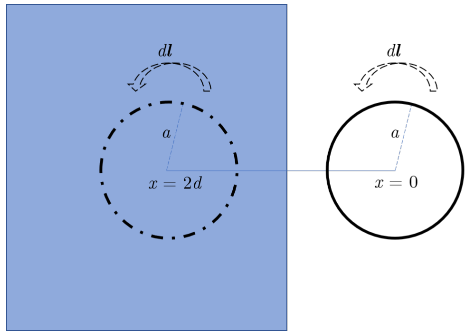

Figure 2: Reorienting the loop so that its plane is perpendicular to the interface at $x = d$, effectively shifting the coordinates.

We can then assemble the vector potential due to this mirror image as

$$\begin{aligned}\mathbf{A}(x > 0, y, z) =& \frac{\mu_0 I a}{4\pi} \frac{(-y\hat{x} + (x - 2d)\hat{y})}{\sqrt{(x-2d)^2 + y^2}} \times \\ & \int_0^{2\pi} \frac{\cos\varphi'}{\sqrt{a^2 + (x-2d)^2 + y^2 + z^2 - 2a(\sqrt{(x-2d)^2 + y^2})\cos\varphi'}} d\varphi'\end{aligned} \tag{16}$$

To successfully plug this to the force equation (11), we can use cylindrical coordinates. However, the magnetic field in this case would not be constant around the loop (cf. the nonzero x,y, and z terms in (15)). We then expand the force equation as follows:

$$\mathbf{F} = \int_0^{2\pi} I(a\ d\varphi\ \hat{\varphi}) \times \mathbf{B} \tag{17}$$

Decomposing $\mathbf{B}$ into its cylindrical components, putting everything in terms of $\mathbf{A}$, and expanding the unit vectors in Cartesian coordinates, we find

$$\begin{aligned}F =& Ia \int_0^{2\pi} d\varphi \left[\frac{1}{\rho} \frac{\partial}{\partial\rho}(\rho A_\varphi(x > 0)) - \frac{1}{\rho}\frac{\partial}{\partial\varphi} A_\rho(x > 0) \right] (\cos\varphi\hat{x} + \sin\varphi\hat{y}) \\ & - \left[-\frac{\partial}{\partial z} A_\varphi(x > 0) \right]\end{aligned} \tag{18}$$

We see that only the A_ρ and A_φ survive this force equation, so we can arbitrarily set $z = 0$ for the integration. Then, integrating at $\rho = a$ and recalling the symmetry argument made earlier for retaining only the z-component of the force (which now becomes the x-component),

$$\boxed{\mathbf{F} = I\int_0^{2\pi} d\varphi \cos\varphi \left[\frac{\partial}{\partial\rho}(\rho A_\varphi(x>0)) - \frac{\partial}{\partial\varphi}A_\rho(x>0)\right]_{\rho=a}\hat{x}} \tag{19}$$

where $\mathbf{A}$ is given by equation (16).

(c) Now we proceed to getting a limiting case: for $d >> a$ or $a/d << 1$, we can extract $4d^2$ from the denominator of equation (14) and simplify:

$$\mathbf{F} = -\mu_0 I^2 \frac{a^2}{8d^2}\frac{\mu-\mu_0}{\mu+\mu_0}\int_0^{2\pi} d\varphi' \; \frac{\cos\varphi'}{(1+1/2(a^2/d^2)(1-\cos\varphi'))^{3/2}}\hat{z} \tag{20}$$

We can further simplify the denominator by Taylor expanding: $(1+x)^{-3/2} \approx 1 - \cdot\frac{3}{2}x$, so that the force would be

$$\mathbf{F} = -\mu_0 I^2 \frac{a^2}{8d^2}\frac{\mu-\mu_0}{\mu+\mu_0}\int_0^{2\pi} d\varphi' \;\; \cos\varphi'\left[1 - \frac{3}{4}\frac{a^2}{d^2}(1-\cos\varphi')\right]\hat{z} \tag{21}$$

which leads to three integrals. Recalling $\int_0^{2\pi}\cos\theta \; d\theta = \pi$,

$$\mathbf{F} = -\mu_0 I^2 \frac{a^2}{8d^2}\frac{\mu-\mu_0}{\mu+\mu_0}\left[0 - \frac{3}{2}\pi\frac{a^2}{d^2} + \frac{3}{4}\pi\frac{a^2}{d^2}\right] \tag{22}$$

$$\boxed{\mathbf{F} = -\frac{3\pi}{32}\frac{\mu-\mu_0}{\mu+\mu_0}\frac{a^4}{d^4}\mu_0 I^2\hat{z}} \tag{23}$$

which is attractive and very small in magnitude.

■