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# Problem 1



**Jackson 4.2.** Recall the energy of a dipole immersed in an external field

$$W = -\mathbf{p} \cdot \mathbf{E} \tag{1}$$

while in general, the energy due to a charge distribution can be written as

$$W = \int \rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}. \tag{2}$$

We can equate these expressions for $W$ in order to see how the dipole moment can be related to an effective charge distribution:

$$-\mathbf{p} \cdot \mathbf{E} = -\mathbf{p} \cdot (\boldsymbol{\nabla}\Phi)_{\mathbf{x}=\mathbf{x}_0} \tag{3}$$

We can insert a Dirac delta integral to the gradient of the potential in order to compare it with equation (2):

$$-\mathbf{p} \cdot (\boldsymbol{\nabla}\Phi)_{\mathbf{x}=\mathbf{x}_0} = -\int \mathbf{p} \cdot (\boldsymbol{\nabla}\Phi)_{\mathbf{x}=\mathbf{x}_0}\delta(\mathbf{x}-\mathbf{x}_0)d\mathbf{x} = \int \rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}. \tag{4}$$

We need to match up $\Phi$ on both sides. Thus we can use integration by parts in order to simplify the first integral:

$$\begin{aligned} -\int \mathbf{p} \cdot (\boldsymbol{\nabla}\Phi)_{\mathbf{x}}\ d\mathbf{x} = \mathbf{x}_0\delta(\mathbf{x}-\mathbf{x}_0) &= -\left[\Phi(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)\right]_{\mathbf{x}=\infty} - \int \mathbf{p} \cdot \boldsymbol{\nabla}(\delta(\mathbf{x}-\mathbf{x}_0))\Phi(\mathbf{x})d\mathbf{x} \\ &= \int \rho(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x} \end{aligned} \tag{5}$$

The boundary term just vanishes since we set the potential to vanish at infinity. When we compare the integrands,

$$\boxed{\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p} \cdot \boldsymbol{\nabla}(\delta(\mathbf{x}-\mathbf{x}_0))} \tag{6}$$

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**Jackson 4.5.** In this problem, a localized charge density $\rho(x, y, z)$ is placed in an external electrostatic field described by a slowly-varying potential $\Phi^{(0)}(x, y, z)$ in regions where the charge density is nonzero.

(a) We first calculate the total force acting on the charge distribution as a multipole expansion (up to the quadrupole moment). We can find this by generalizing $\mathbf{F} = q\mathbf{E}$ to

$$\mathbf{F} = \int \rho(\mathbf{x})\mathbf{E}(\mathbf{x})d^3\mathbf{x} \tag{1}$$

$$= \sum_i \hat{\mathbf{x}}_i \int \rho(\mathbf{x})E_i^{(0)}(\mathbf{x})d^3\mathbf{x}, \tag{2}$$

where we expanded the components of $\mathbf{F}$ and $\mathbf{E}$. We also need to expand each $E_i^{(0)}$ via Taylor series and retain the lowest-order terms because the field is slowly varying:

$$E_i^{(0)}(\mathbf{x}) = \left[E_i^{(0)}(\mathbf{x}') + \sum_j x_j \frac{\partial}{\partial x_j'} E_i^{(0)}(\mathbf{x}') + \frac{1}{2}\sum_j \sum_k x_j x_k \frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_k'} E_i^{(0)}(\mathbf{x}') + \dots\right]_{\mathbf{x}'=0}, \tag{3}$$

where the primed variables allow us to separate them from the integration variables. It does not concern about the charge densities that create the field. We can then substitute this to equation (2):

$$\mathbf{F} = \sum_i \hat{\mathbf{x}}_i \int d^3\mathbf{x}\ \rho(\mathbf{x})\left[E_i^{(0)}(\mathbf{x}') + \sum_j x_j \frac{\partial}{\partial x_j'} E_i^{(0)}(\mathbf{x}') + \frac{1}{2}\sum_j\sum_k x_j x_k \frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_k'} E_i^{(0)}(\mathbf{x}') + \dots\right]_{\mathbf{x}'=0} \tag{4}$$

$$= \left[q\mathbf{E}^{(0)}(\mathbf{x}') + \sum_i \hat{\mathbf{x}}_i \frac{\partial}{\partial x_j'} E_i^{(0)}(\mathbf{x}') \int d^3\mathbf{x}\ \rho(\mathbf{x}) \sum_j x_j + \frac{1}{2}\sum_i\sum_j\sum_k \hat{\mathbf{x}}_i \frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_k'} E_i^{(0)}(\mathbf{x}') \int d^3\mathbf{x}\ \rho(\mathbf{x}) x_j x_k + \dots\right]_{\mathbf{x}'=0} \tag{5}$$

The first integral appearing in equation (5) resembles the dipole moment formula. We can then manipulate the indices of $E_i$ and the derivative in the same term by using $\boldsymbol{\nabla} \times \mathbf{E} = 0$:

$$\nabla \times \mathbf{E} = 0 \rightarrow \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i} \tag{6}$$

This allows us to recognize that we're getting the dot product of $\mathbf{p}$ and $\mathbf{E}$ in the 2nd term. We can also do the same manipulation of indices for the last term. Just as we did in the lecture notes for §4.2, we add an extra term, $\frac{1}{6} r^2 \nabla \cdot \mathbf{E} = 0$ that doesn't affect anything in the summation. We can then rewrite the last term to have a quadrupole term and an extra integral:

$$\begin{aligned}\mathbf{F} = \Bigg[ & q\mathbf{E}^{(0)}(\mathbf{x}') + \sum_i \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i'} \mathbf{p} \cdot \mathbf{E}^{(0)} + \frac{1}{6} \sum_i \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i'} \Bigg( \sum_j \sum_k Q_{jk} \frac{\partial}{\partial x_k'} E_j^{(0)}(\mathbf{x}') \\ & + \sum_j \frac{\partial}{\partial x_j'} E_j^{(0)}(\mathbf{x}') \int d^3\mathbf{x}\; r^2 \rho(\mathbf{x}) \Bigg) + \dots \Bigg]_{\mathbf{x}'=0}\end{aligned} \tag{7}$$

We then recognize the derivative of $E$ on the last term of equation (7) as the divergence of $\mathbf{E}$. Since we know that there's no charges outside, this term must vanish ($\nabla \cdot \mathbf{E} = 0$. Thus we are left with the three dominant terms that can be simplified using vector notation:

$$\boxed{\mathbf{F} = q\mathbf{E}^{(0)}(0) + [\nabla(\mathbf{p} \cdot \mathbf{E}^{(0)})]_0 + \nabla \frac{1}{6} \left( \sum_j \sum_k Q_{jk} \frac{\partial}{\partial x_k'} E_j^{(0)}(\mathbf{x}') \right)_0 + \dots} \tag{8}$$

We can recover a familiar expression by factoring out a negative gradient from each term:

$$\mathbf{F} = -\nabla \left[ q\Phi^{(0)} - \mathbf{p} \cdot \mathbf{E}^{(0)} - \frac{1}{6} \sum_j \sum_k Q_{jk} \frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x_k'} \right]_0 \tag{9}$$

Recalling equation (4.24) from Jackson 3rd ed.,

$$W = q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{6} \sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial x_i}(0) + \dots \tag{4.24}$$

Then equation (9) becomes

$$\mathbf{F} = -\nabla W \tag{10}$$

$$W = -\int \mathbf{F} \cdot d\mathbf{x}_{(0)} \tag{11}$$

(b) We do a similar calculation for the torque produced by the external electric field acting on the charge distribution. In general,

$$\boldsymbol{\tau} = \int \mathbf{x} \times (\rho(\mathbf{x})\mathbf{E}^{(0)}(\mathbf{x}))d^3\mathbf{x} \tag{12}$$

We also solve for the torque component-wise then build the vector later on. For one component, say $\tau_1$,

$$\tau_1 = \int \rho(\mathbf{x})(x_2 E_3^{(0)} - x_3 E_2^{(0)})d^3\mathbf{x}. \tag{13}$$

Then we expand both $E_2$ and $E_3$ using Taylor series and retain the lowest-order terms as well. We can use equation (3) for this case, though now we can expand up to the linear term $\dfrac{\partial E_i^{(0)}(\mathbf{x}')}{\partial x_j'}$ only, as required by the problem.

Inserting equation (3) to (13) for $E_2$ and $E_3$,

$$\begin{aligned}
\tau_1 &= \int \rho(\mathbf{x})\Bigg[x_2\Bigg(E_3^{(0)}(\mathbf{x}') + \sum_j x_j \frac{\partial}{\partial x_j'} E_3^{(0)}(\mathbf{x}') + \dots\Bigg) \\
&\quad - x_3\Bigg(E_2^{(0)}(\mathbf{x}') + \sum_j x_j \frac{\partial}{\partial x_j'} E_2^{(0)}(\mathbf{x}') + \dots\Bigg)\Bigg]d^3\mathbf{x} \\
&= \Bigg[E_3^{(0)}(\mathbf{x}')\int \rho(\mathbf{x})x_2 d^3\mathbf{x} - E_2^{(0)}(\mathbf{x}')\int \rho(\mathbf{x})x_3 d^3\mathbf{x} \\
&\quad + \sum_j x_j \frac{\partial}{\partial x_j'} E_3^{(0)}(\mathbf{x}')\int \rho(\mathbf{x})x_2 x_j d^3\mathbf{x} - \sum_j x_j \frac{\partial}{\partial x_j'} E_2^{(0)}(\mathbf{x}')\int \rho(\mathbf{x})x_3 x_j d^3\mathbf{x}\Bigg]_{\mathbf{x}'=0} + \dots
\end{aligned} \tag{14}$$

We then see that the first two terms comprise of dipole components. Again, we invoke $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ to simplify the last term. We also add the extra term $\frac{1}{6}r^2\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ to the last two terms as we did from the calculation of the force to "force" out the quadrupole terms:

$$\tau_1 = \left[ p_2 E_3^{(0)}(\mathbf{x}') - p_3 E_2^{(0)}(\mathbf{x}') + \sum_j x_j \frac{\partial}{\partial x'_j} E_3^{(0)}(\mathbf{x}') \int \rho(\mathbf{x}) x_2 x_j d^3\mathbf{x} \right.$$
$$\left. + \frac{1}{6} \sum_i x_i^2 \frac{\partial E_i^{(0)}(\mathbf{x}')}{\partial x'_i} - \sum_j x_j \frac{\partial}{\partial x'_j} E_2^{(0)}(\mathbf{x}') \int \rho(\mathbf{x}) x_3 x_j d^3\mathbf{x} + \frac{1}{6} \sum_i x_i^2 \frac{\partial E_i^{(0)}(\mathbf{x}')}{\partial x'_i} \right]_{\mathbf{x}'=0} + \dots \tag{15}$$
$$= \left[ p_2 E_3^{(0)}(\mathbf{x}') - p_3 E_2^{(0)}(\mathbf{x}') + \frac{1}{3} \sum_j \frac{\partial}{\partial x'_j} E_3^{(0)}(\mathbf{x}') Q_{2j} - \frac{1}{3} \sum_j \frac{\partial}{\partial x'_j} E_2^{(0)}(\mathbf{x}') Q_{3j} \right]_{\mathbf{x}'=0} + \dots \tag{16}$$

We notice then that the first two terms, when simplified, look like the component of $\mathbf{p} \times \mathbf{E}^{(0)}(0)$ along index 1. Then we use the curl-less property of $\mathbf{E}$ to get out the partial differential operator, particularly $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$:

$$\tau_1 = [\mathbf{p} \times \mathbf{E}^{(0)}(0)]_1 + \left[ \frac{1}{3} \frac{\partial}{\partial x'_3} \sum_j E_j^{(0)}(\mathbf{x}') Q_{2j} - \frac{1}{3} \frac{\partial}{\partial x'_2} \sum_j E_j^{(0)}(\mathbf{x}') Q_{3j} \right]_{\mathbf{x}'=0} + \dots \tag{17}$$

so that

$$\boxed{\tau_1 = [\mathbf{p} \times \mathbf{E}^{(0)}(0)]_1 + \frac{1}{3} \left[ \frac{\partial}{\partial x_3} \sum_j E_j^{(0)} Q_{2j} - \frac{\partial}{\partial x_2} \sum_j E_j^{(0)} Q_{3j} \right]_0 + \dots} \tag{18}$$

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**Jackson 4.8** In this problem, we solve the potentials, fields, and lines of forces for the configuration shown below.

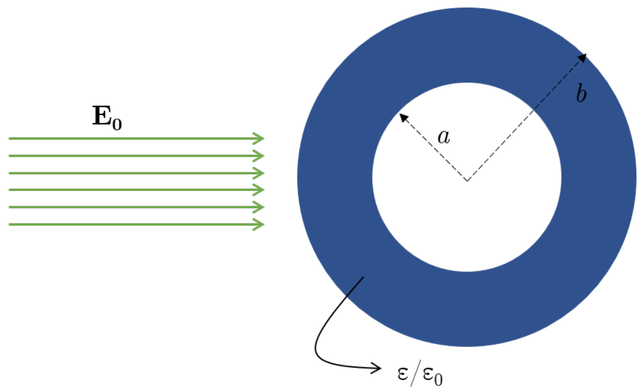


Figure 1: A cylindrical shell of inner radius $a$ and outer radius $b$ with dielectric constant $\epsilon/\epsilon_0$ immersed in a uniform electric field $E_0$ aligned perpendicular to the axis of the cylinder.

(a) As we see in the configuration, solving the potential amounts to solving Laplace's equation in 2D polar form for three regions: (1) $\rho < a$, (2) $a < \rho < b$, and (3) $\rho > b$. This came from the assumptions that the field outside is uniform, the cylindrical shell is significantly long along its axis, and that there are no free charges anywhere.

For $\rho < a$, we recall the solution of Laplace's equation that includes the origin (so that the $1/\rho^\nu$ term vanishes):

$$\Phi(\rho < a) = \sum_{\nu=0}^{\infty} a_\nu \rho^\nu (A_\nu e^{i\nu\varphi} + B_\nu e^{-i\nu\varphi}) \tag{1}$$

where we can further absorb $a_\nu$ into $A_\nu$ and $B_\nu$.

We first solve the potential outside so that we can let the solution inside the cylindrical shell connect the outside and inside potentials for continuity. For $\rho > b$, we immediately have a solution for $\Phi$ coming from the external uniform field: calculating $\Phi = -\int \mathbf{E}\cdot d\boldsymbol{\ell}$, we get $\Phi(\rho > b) = -E_0\rho\cos\varphi$. We can immediately equate this to the general solution of Laplace's equation valid for $\rho > b$:

$$\Phi(\rho > b) = -E_0\rho\cos\varphi = a_0 + b_0 \ln\rho + \sum_{\nu=1}^{\infty} (a_\nu \rho^\nu + b_\nu \rho^{-\nu})(A_\nu e^{i\nu\varphi} + B_\nu e^{-i\nu\varphi}) \tag{2}$$

Comparing the coefficients, we immediately see that $a_0 = b_0 = 0$ and $a_\nu = b_\nu = 0$ for all $\nu \neq 1$. To extract $\cos\varphi$, we need to set $A_1 = B_1$. Thus we get $-E_0 = 2a_1B_1$. We still lack $b_1$ so we simply put it in the final solution,

$$\Phi(\rho > b) = \left(-E_0\rho + \frac{b_1}{\rho}\right)\cos\varphi \tag{3}$$

To connect the outer and inner solutions, we use the general solution of the 2D polar Laplace equation with a different set of coefficients:

$$\Phi(a < \rho < b) = m_0 + n_0 \ln\rho + \sum_{\nu=1}^{\infty}(m_\nu \rho^\nu + n_\nu \rho^{-\nu})(M_\nu e^{i\nu\varphi} + N_\nu e^{-i\nu\varphi}) \tag{4}$$

We then solve this middle solution by applying boundary conditions. At the outer boundary $\rho = b$, $(\epsilon_2 \mathbf{E}_2 - \epsilon_1 \mathbf{E}_1) \cdot \hat{\mathbf{n}} = \sigma$. Here, the normal is in the radial direction, and that there are no free charges, so the derivative of the potential is related by

$$\epsilon_2 \mathbf{E}_2 \cdot \hat{\boldsymbol{\rho}} = \epsilon_1 \mathbf{E}_1 \cdot \hat{\boldsymbol{\rho}} \tag{5}$$

$$\epsilon_2 \frac{\partial \Phi_2}{\partial \rho}\bigg|_{\rho=b} = \epsilon_1 \frac{\partial \Phi_1}{\partial \rho}\bigg|_{\rho=b} \tag{6}$$

Differentiating equations (3) and (4) at $\rho = b$ and applying to equation (6), we get

$$\epsilon_0 \left(-E_0 - \frac{b_1}{b^2}\right)\cos\varphi = \epsilon\left[\frac{n_0}{b} + \sum_{\nu=1}^{\infty}(\nu m_\nu b^{\nu-1} - \nu n_\nu b^{-\nu-1})(M_\nu e^{i\nu\varphi} + N_\nu e^{-i\nu\varphi})\right] \tag{7}$$

Again, we compare coefficients for each power of $b$. We can immediately see that $n_0 = 0$, $M_1 = N_1$ to extract $\cos\varphi$ and that $m_\nu = n_\nu = 0$ for $\nu \neq 1$. Simplifying,

$$\epsilon_0 \left(-E_0 - \frac{b_1}{b^2}\right) = 2\epsilon M_1 \left(m_1 - \frac{n_1}{b^2}\right) \tag{8}$$

$$M_1 = \frac{-\epsilon_0\left(E_0 + \frac{b_1}{b^2}\right)}{2\epsilon\left(1 - \frac{n_1}{b^2}\right)} \tag{9}$$

where we absorbed $2\epsilon m_1$ onto $M_1$ for further simplification. We can then simplify the middle solution as

$$\Phi(a < \rho < b) = -\left(\rho + \frac{n_1}{\rho}\right)\frac{\epsilon_0\left(E_0 + \frac{b_1}{b^2}\right)}{2\epsilon\left(1 - \frac{n_1}{b^2}\right)}\cos\varphi \tag{10}$$

We still need to solve for $n_1$ so that the middle solution would be complete. We can do this by using another boundary condition from $\nabla \times \mathbf{E} = 0$ which allows for continuity of the tangential component of $\mathbf{E}$ at the boundary:

$$\left.\frac{\partial \Phi(\rho > b)}{\partial \varphi}\right|_{\rho=b} = \left.\frac{\partial \Phi(a < \rho < b)}{\partial \varphi}\right|_{\rho=b} \tag{11}$$

Again, applying equations (3) and (10) to equation (11), we get

$$\begin{aligned} -\left(-E_0 b - \frac{b_1}{b}\right)\sin\varphi &= -\left(b + \frac{n_1}{b}\right)\frac{\epsilon_0\left(E_0 + \dfrac{b_1}{b^2}\right)}{2\epsilon\left(1 - \dfrac{n_1}{b^2}\right)}\sin\varphi \\ \epsilon\left(E_0 - \frac{b_1}{b^2}\right)\left(1 - \frac{n_1}{b^2}\right) &= \epsilon_0\left(E_0 + \frac{b_1}{b^2}\right)\left(1 + \frac{n_1}{b^2}\right) \\ n_1 &= \frac{b_1(\epsilon + \epsilon_0) - b^2 E_0(\epsilon - \epsilon_0)}{b_1(\epsilon - \epsilon_0) - b^2 E_0(\epsilon + \epsilon_0)} \end{aligned} \tag{12}$$

So that in terms of $b_1$ (the coefficient of $1/\rho$ outside), $b$ (the outer radius of the cylindrical shell), and $\kappa \equiv \epsilon/\epsilon_0$ (the dielectric constant), the middle solution becomes

$$\boxed{\Phi(a < \rho < b, \varphi) = \frac{1}{2\kappa b^2}\left[\rho\left(b_1(\kappa - 1) - b^2 E_0(\kappa + 1)\right) + \frac{b^2}{\rho}\left(b_1(\kappa + 1) - b^2 E_0(\kappa - 1)\right)\right]\cos\varphi} \tag{13}$$

We then apply the same normal and tangential boundary conditions to the boundary at $\rho = a$ to solve for the inner solution. Here, we are at an advantage since we know the complete middle solution in terms of the coefficient of the outer solution.

For the normal BC, we use equation (6) to relate (1) and (13) to each other; thus

$$\epsilon_2 \left.\frac{\partial \Phi(a < \rho < b)}{\partial \rho}\right|_{\rho=a} = \epsilon_1 \left.\frac{\partial \Phi(\rho < a)}{\partial \rho}\right|_{\rho=a} \tag{14}$$

$$\epsilon \frac{1}{2\kappa b^2}\left[a\left(b_1(\kappa - 1) - b^2 E_0(\kappa + 1)\right) + \frac{b^2}{a}\left(b_1(\kappa + 1) - b^2 E_0(\kappa - 1)\right)\right] = \epsilon_0 a_1 a(2A_1)$$

$$\frac{1}{2b^2}\left[a\left(b_1(\kappa - 1) - b^2 E_0(\kappa + 1)\right) + \frac{b^2}{a}\left(b_1(\kappa + 1) - b^2 E_0(\kappa - 1)\right)\right] = A_1 \tag{15}$$

where we immediately deduced that only $\nu = 1$ will survive the summation of equation (1) by comparing the coefficients. Here, we have two unknowns, $A_1$ and $b_1$. We can add another equation from the tangential BC and solve these system of linear equations.

$$\left.\frac{\partial \Phi(a<\rho<b)}{\partial \varphi}\right|_{\rho=a} = \left.\frac{\partial \Phi(\rho<a)}{\partial \varphi}\right|_{\rho=a} \tag{16}$$

$$\frac{1}{2\kappa a b^2}\left[a\left(b_1(\kappa-1) - b^2 E_0(\kappa+1)\right) + \frac{b^2}{a}\left(b_1(\kappa+1) - b^2 E_0(\kappa-1)\right)\right] = A_1 \tag{17}$$

Thus, we have the solutions for $A_1$ and $b_1$:

$$b_1 = \frac{(b^2-a^2)(\kappa^2-1)}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2} E_0 b^2 \tag{18}$$

$$A_1 = \frac{-4b^2\kappa}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2} E_0 \tag{19}$$

So we can summarize the three solutions as follows:

$$\Phi(\rho<a) = -E_0 \rho \cos\varphi \left[\frac{4b^2\kappa}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right] \tag{20}$$

$$\Phi(a<\rho<b) = -E_0 a \cos\varphi \left[\frac{2b^2\kappa}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right]\left[\frac{\rho}{a}\left(1+\frac{1}{\kappa}\right) + \frac{a}{\rho}\left(1-\frac{1}{\kappa}\right)\right] \tag{21}$$

$$\Phi(\rho>b) = -E_0 \rho \cos\varphi \left[1 - \frac{b^2}{\rho^2}\frac{(b^2-a^2)(\kappa^2-1)}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right] \tag{22}$$

To determine the electric field, we just use the polar gradient $\mathbf{E} = -\boldsymbol{\nabla}\Phi = -\frac{\partial \Phi}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi}\hat{\varphi}$ on equations (20)-(22):

$$\begin{aligned}
\mathbf{E}(\rho < a) &= E_0(\cos\varphi\hat{\rho} - \sin\varphi\hat{\varphi})\left[\frac{4b^2\kappa}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right] \\
&= E_0\left[\frac{4b^2\kappa}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right]\hat{i} \qquad (23) \\
\mathbf{E}(a < \rho < b) &= E_0\left[\frac{2b^2}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right]\Bigg[(\kappa+1)\hat{i} - \frac{a^2}{\rho^2}(\kappa-1)\hat{i} \\
&\quad - 2\frac{a^2}{\rho^2}(\kappa-1)\sin\varphi\hat{\varphi}\Bigg] \qquad (24) \\
\mathbf{E}(\rho > b) &= E_0\Bigg[\left(1 + \frac{b^2}{\rho^2}\frac{(b^2-a^2)(\kappa^2-1)}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right)\hat{i} \\
&\quad + 2\left(\frac{b^2}{\rho^2}\frac{(b^2-a^2)(\kappa^2-1)}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\right)\sin\varphi\hat{\varphi}\Bigg] \qquad (25)
\end{aligned}$$

(b) We then sketch lines of force for $b \approx 2a$. Simplifying equations (23)-(25),

$$\begin{aligned}
\mathbf{E}(\rho < a) &= E_0\left[\frac{16\kappa}{4(\kappa+1)^2 - (\kappa-1)^2}\right](\cos\varphi\hat{\rho} - \sin\varphi\hat{\varphi}) \qquad (26) \\
\mathbf{E}(a < \rho < 2a) &= E_0\left[\frac{8}{4(\kappa+1)^2 - (\kappa-1)^2}\right]\Bigg[(\kappa+1)(\cos\varphi\hat{\rho} - \sin\varphi\hat{\varphi}) \\
&\quad - \frac{a^2}{\rho^2}(\kappa-1)\cos\varphi\hat{\rho} - \frac{a^2}{\rho^2}(\kappa-1)\sin\varphi\hat{\varphi}\Bigg] \qquad (27) \\
\mathbf{E}(\rho > 2a) &= E_0\Bigg[\left(1 + \frac{a^2}{\rho^2}\frac{12(\kappa^2-1)}{4(\kappa+1)^2 - (\kappa-1)^2}\right)(\cos\varphi\hat{\rho} - \sin\varphi\hat{\varphi}) \\
&\quad + \frac{a^2}{\rho^2}\left(\frac{24(\kappa^2-1)}{4(\kappa+1)^2 - (\kappa-1)^2}\right)\sin\varphi\hat{\varphi}\Bigg] \qquad (28)
\end{aligned}$$

We plot the fields in a 2D polar coordinate system for three regions: (1) $\rho/a < 1$, (2) $1 < \rho/a < 2$, and (3) $\rho/a > 2$, and a full sweep for $\varphi$ for all three regions. We then see in Figure 2 the lines of force emanating from the three electric fields (26)-(28).

(c) There are two limiting cases appropriate for this problem: If we want a solid cylinder immersed in a uniform electric field, we can then let the inner radius $a \to 0$; the inner solution vanishes, and we are left with

# Problem 3



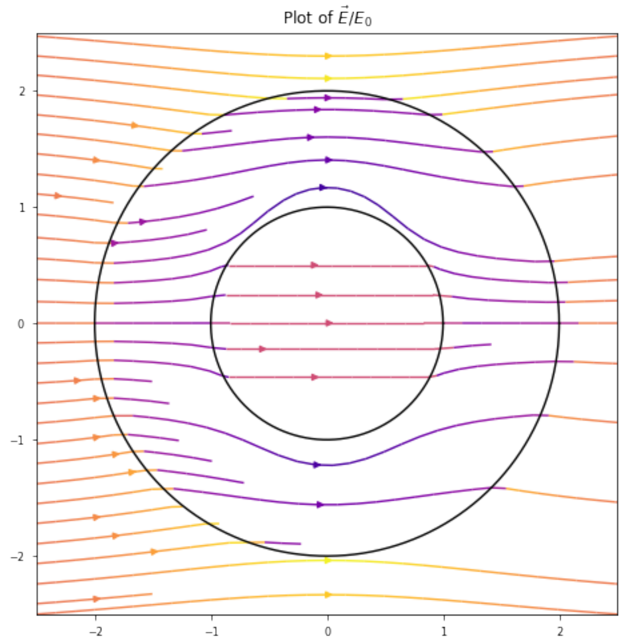


Figure 2: Electric field lines for $a = 1$, $b = 2$ cylindrical shell with dielectric constant $\kappa = 2$.

$$\mathbf{E}(\rho < b, \varphi) = E_0 \frac{2}{(\kappa+1)} \hat{i} \tag{29}$$

$$\mathbf{E}(\rho > b, \varphi) = E_0 \left[ \left( 1 + \frac{b^2}{\rho^2} \frac{\kappa - 1}{\kappa + 1} \right) \hat{i} + 2 \sin\varphi \frac{b^2}{\rho^2} \frac{\kappa - 1}{\kappa + 1} \hat{\varphi} \right] \tag{30}$$

We can also reverse the situation, for which a cylindrical cavity exists in a uniform dielectric $\epsilon/\epsilon_0 = \kappa$. For this case, we let $b \to \infty$, and $a$ the radius of the cavity. The outer solution vanishes, and we are left with

$$\mathbf{E}(\rho < a) = E_0 \frac{4\kappa}{(\kappa+1)^2} \hat{i} \tag{31}$$

$$\mathbf{E}(\rho > a) = E_0 \frac{2}{(\kappa+1)^2} \left[ (\kappa+1)\hat{i} - \frac{a^2}{\rho^2}(\kappa - 1)\hat{i} - 2\frac{a^2}{\rho^2}(\kappa - 1) \sin\varphi \hat{\varphi} \right] \tag{32}$$

■

**Jackson 4.13** In this problem, we wish to find an expression for the susceptibility of the liquid dielectric $\chi_e$, as shown in Figure 1

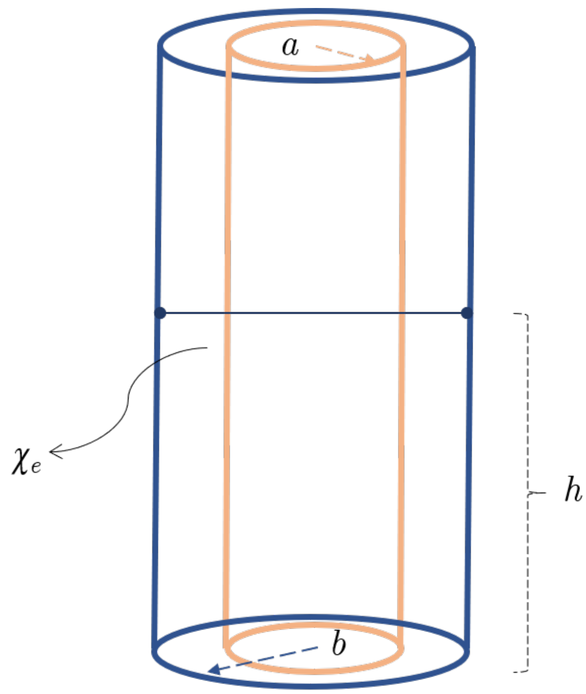


Figure 1: Two long coaxial cylindrical surfaces of radii $a$ and $b$ immersed in a liquid dielectric with electric susceptibility $\chi_e$.

To determine its expression, we can extract $\chi_e$ from the change in potential energy $\Delta W$ coming from two different scenarios of the same situation: (1) the change in energy can be stored in the fields in the fluid for which we use the formula $\frac{1}{2}\int \mathbf{E} \cdot \mathbf{D}\, d^3\mathbf{x}$, or (2) the change in energy can be stored in the particles while they're being raised, for which we employ a change in gravitational potential energy $\Delta W = mgh$.

The second method's a lot easier to extract. The mass of the fluid can be expressed in terms of the density $\rho$ which is trapped in between the cylinders times the volume of that space, $\pi(b^2 - a^2)h$. Thus,

$$\Delta W_2 = \rho\pi(b^2 - a^2)gh^2 \tag{1}$$

For the first method, we first extract the electric fields and displacement fields before and after the raising of fluid. The electric displacements would be the same since we're answering the same problem $\boldsymbol{\nabla} \cdot D = \rho$, so we only solve for the differing electric fields, then apply $\mathbf{D} = \epsilon\mathbf{E}$.

To solve for the electric fields, we first solve the potential given by the solution to Laplace's equation. Here we exploit every available symmetry to simplify the problem: the long length would provide the same electric field throughout the $z$-axis which is placed coaxial with the cylinders, so we can use the same 2D polar Laplace solution in problem 3, equation (2):

$$\Phi(s,\varphi) = a_0 + b_0 \ln\rho + \sum_{\nu=1}^{\infty}(a_\nu \rho^\nu + b_\nu \rho^{-\nu})(A_\nu e^{i\nu\varphi} + B_\nu e^{-i\nu\varphi}) \tag{2}$$

where we set $s$ to be the radial variable to prevent confusion with $\rho$, the mass density. We can then further simplify this solution by exploting the azimuthal symmetry so that $a_\nu = b_\nu = 0 \forall \nu > 0$. Now we apply the boundary conditions. Since there's a constant potential $V$ between the cylinders, we can set $\Phi(s=a) = 0$ and $\Phi(s=b) = V$:

$$\Phi(s=a) = 0 = a_0 + b_0 \ln a \to a_0 = -b_0 \ln a \tag{3}$$

$$\Phi = b_0 \ln \frac{s}{a} \tag{4}$$

$$\Phi(s=b) = V = b_0 \ln\frac{b}{a} \to b_0 = \frac{V}{\ln\frac{b}{a}} \tag{5}$$

$$\Phi(s) = V\frac{\ln\frac{\rho}{a}}{\ln\frac{b}{a}} \tag{6}$$

so that using $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, we get

$$\mathbf{E}(s) = -\hat{s}\frac{V}{\ln\frac{b}{a}}\frac{1}{\rho} \tag{7}$$

Now we need to consider how this energy stored in this field changed. Before filling it with water, there is air in that region. To maintain the constant potential after changing the dielectric, we need to supply more energy. Thus

$$\Delta W_1 = \frac{1}{2}\int \mathbf{E}_{\text{final}}\cdot\mathbf{D}_{\text{final}} d^3\mathbf{x} - \frac{1}{2}\int \mathbf{E}_{\text{initial}}\cdot\mathbf{D}_{\text{initial}} d^3\mathbf{x} \tag{8}$$

$$= \left[\frac{1}{2}(L-h)\epsilon_0 \int E_{\text{air}}^2 d^2\mathbf{x} + \frac{1}{2}\epsilon \int E_{\text{liquid}}^2 d^2\mathbf{x}\right] - \frac{1}{2}L\epsilon_0 \int E_{\text{air}}^2 d^2\mathbf{x} \tag{9}$$

where we set the total length of the cylinders as $L$. Simplifying and substituting $\epsilon = \epsilon_0(1+\chi_e)$ as well as the same field (7), we get

$$\Delta W = \frac{1}{2} h \epsilon_0 \chi_e \int \left( \frac{V^2}{\ln^2 \dfrac{b}{a}} \right) \frac{1}{s^2} d^2\mathbf{x} \tag{10}$$

$$= \frac{1}{2} h \epsilon_0 \chi_e \left( \frac{V^2}{\ln^2 \dfrac{b}{a}} \right) \int \frac{1}{s^2} (2\pi s \ ds)$$

$$= \frac{\pi h \epsilon_0 \chi_e V^2}{\ln^2 \dfrac{b}{a}} \ln \frac{b}{a} = \frac{\pi h \epsilon_0 \chi_e V^2}{\ln \dfrac{b}{a}} \tag{11}$$

which we can readily equate with equation (1) to extract $\chi_e$:

$$\frac{\pi h \epsilon_0 \chi_e V^2}{\ln \dfrac{b}{a}} = \rho \pi (b^2 - a^2) g h^2 \tag{12}$$

$$\chi_e = \frac{(b^2 - a^2) \rho g h \ln \dfrac{b}{a}}{\epsilon_0 V^2} \tag{13}$$

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