Potential of a unit point charge in terms of Legendre polynomials

> Consider: unit pt. charge in z-axis. For $z>0$,

$$\Phi(r>0,\theta)=\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}=\frac{1}{\sqrt{r^2+r'^2+2rr'\cos\theta}}$$

$q \rightarrow 4\pi\epsilon_0$

> Since this is azimuthally symmetric, we can expand in terms of Legendre:

$$\frac{1}{\sqrt{r^2+r'^2+2rr'\cos\theta}}=\sum_\ell \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

> In this problem, we don't care about the location of the point charge in θ. So we choose $\theta=0$ to ease our calculations.

$$\frac{1}{r-r'}=\frac{1}{\sqrt{r^2+r'^2+2rr'}}=\sum_\ell A_\ell r^\ell + B_\ell r^{-\ell-1}$$

↳ separate the solutions for $r<r'$ & $r>r'$ to get A_ℓ & B_ℓ, respectively:

$$(r<r')\quad \frac{1}{r'-r}=\frac{1}{r'\left(1-\frac{r}{r'}\right)}=\sum_\ell A_\ell r^\ell + B_\ell r^{-\ell-1}$$

$$\sum_\ell\left(\frac{r}{r'}\right)^\ell, \longleftarrow \frac{1}{1-\frac{r}{r'}}=r'\left(\sum_\ell A_\ell r^\ell + B_\ell r^{-\ell-1}\right)$$

$\frac{r}{r'}<1$ ✓

> Thus, for $r<r'$, $B_\ell=0$, and $A_\ell=r'^{-\ell-1}$.

> For $r > r'$, we do the same thing and end up with $A_\ell = 0$ and $B_\ell = r'^\ell$.

> This solution is ONLY valid for when the charge is at $z > 0$:

$$\Phi(r,\theta) = \begin{cases} \sum_\ell \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\theta), & r < r' \\ \sum_\ell \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\theta), & r > r' \end{cases}$$

Fields in a conical hole

> conical holes destroy the symmetry in $x = \cos\theta$, which allows us to use series expansions in solving the DE for Legendre. Recall

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + \ell(\ell+1)P(x) = 0$$

> Consider a conical hole in some conductor with angle β relative to the axis. Instead of expanding at $x=0$, we expand at $x=1$ or $\theta = 0$:

Let $x' = \frac{1}{2}(1-x)$, $\frac{d}{dx'} = -2\frac{d}{dx}$. Then

$$\frac{d}{dx'}\left[(x'-x'^2)\frac{dP(x')}{dx'}\right] + \nu(\nu+1)P(x') = 0$$

where $P_\nu(x)$ is our new polynomial.

Let $P_\nu(x') = \sum_{j=0} a_j x^{\nu+\alpha}$

We then substitute this to the above DE to obtain

$$\underbrace{a_0 \alpha^2 x'^{\alpha-1}}_{\text{must vanish}} + \underbrace{\sum_{j=0}^{\infty}\left[a_{j+1}(j+1+\alpha)^2 + a_j\left[-(j+\alpha)(j+\alpha+1)+\nu(\nu+1)\right]\right] x^{j+\alpha}}_{\text{must vanish}} = 0$$

$\therefore$ $a_0 \alpha^2 = 0$ (α^2 must be zero; a_0 cannot be 0 because no series left)

OR $a_{j+1}(j+1+\alpha)^2 + a_j\left[-(j+\alpha)(j+\alpha+1) + \nu(\nu+1)\right] = 0$

$$\boxed{a_{j+1} = \frac{j(j+1) - \nu(\nu+1)}{(j+1)^2} a_j}$$

Choose $a_0 = 1$ for normalization so that

$$\boxed{P_\nu(x) = 1 + (-\nu(\nu+1))\left[\tfrac{1}{2}(1-x)\right] + \frac{-\nu(\nu+1)(2-\nu(\nu+1))}{4}\left[\tfrac{1}{2}(1-x)\right]^2 + \ldots}$$