Problem 1

Jackson 2.1. For the trivial case of a point charge q placed at a distance $z = d$ against a conducting plane, we find the potential using the method of images (as in the class notes). We find that the image charge induced by this configuration has the same magnitude but opposite sign as the original charge, and is placed at $z = -d$, collinear with the original charge.

The total potential due to the new configuration is then

$$\Phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x} - \mathbf{x}'|} + \frac{-q}{|\mathbf{x} - \mathbf{x}''|}\right] \tag{1}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}}\right] \tag{2}$$

We are now asked to find the following important physical quantities:

(a) the surface charge density can be easily found using $\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial z}\right|_{z=0}$:

$$\sigma = -\epsilon_0 \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}}\right]\Bigg|_{z=0} \tag{3}$$

$$= \frac{-q}{4\pi}\left[\frac{-2(z - d)}{2\sqrt[3]{x^2 + y^2 + (z - d)^2}} - \frac{-2(z + d)}{2\sqrt[3]{x^2 + y^2 + (z + d)^2}}\right]\Bigg|_{z=0} \tag{4}$$

$$= -q\frac{1}{2\pi}\left[\frac{d}{\sqrt[3]{x^2 + y^2 + d^2}}\right] \tag{5}$$

The plot of the surface charge density at $z = 0$ is given in Figure 1.

(b) Since they are collinear, the force on the original charge due to the image charge is given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q(-q)}{|\mathbf{x} - \mathbf{x}'|^2}\hat{\mathbf{x}} \tag{6}$$

$$= \frac{-q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\hat{\mathbf{z}} \tag{7}$$

$$= \frac{-q^2}{16\pi\epsilon_0 d^2}\hat{\mathbf{z}} \tag{8}$$

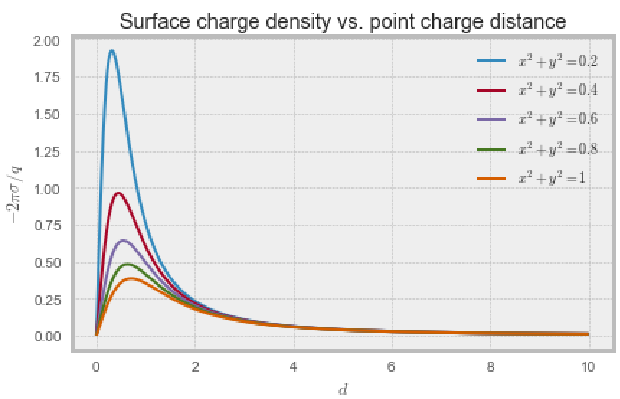

Figure 1: Surface charge density versus original charge distance.

(c) From $\mathbf{F} = \mathrm{q}\mathbf{E}$, we obtain the so-called electrostatic pressure by $\dfrac{d\mathbf{F}}{da} = \dfrac{dq}{da}\mathbf{E} = \sigma\mathbf{E} = \dfrac{(\sigma(x,y))^2}{2\epsilon_0}\hat{\mathbf{n}}$. We then integrate this over the whole xy-plane to obtain the force on the conducting plane (which we assume would be equal to the force on the original charge but with opposite sign):

$$\mathbf{F} = \hat{\mathbf{n}}\frac{1}{2\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q^2}{4\pi^2}\frac{d^2}{(x^2+y^2+d^2)^3}dx\ dy \tag{9}$$

$$= \hat{\mathbf{z}}\frac{q^2d^2}{8\pi^2\epsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{1}{(\rho^2+d^2)^3}\rho\ d\rho\ d\varphi \tag{10}$$

$$= \hat{\mathbf{z}}\frac{q^2d^2}{4\pi\epsilon_0}\left[-\frac{1}{4(\rho^2+d^2)^2}\right]\Bigg|_0^{\infty} = \hat{\mathbf{z}}\frac{q^2d^2}{16\pi\epsilon_0}\frac{1}{d^4} \tag{11}$$

$$= \frac{1}{16\pi\epsilon_0}\frac{q^2}{d^2}\hat{\mathbf{z}} \tag{12}$$

which is as expected.

(d) We can compute the work done in moving the original charge from $z = d$ to infinity by using $W = \int_d^{\infty}\mathbf{F}(\ell)\cdot d\boldsymbol{\ell}$:

$$W = \int_d^\infty \frac{q^2}{16\pi\epsilon_0 \ell^2} d\ell \tag{13}$$
$$= \frac{q^2}{16\pi\epsilon_0} \int_d^\infty \frac{1}{\ell^2} d\ell \tag{14}$$
$$= \frac{q^2}{16\pi\epsilon_0} \left[-\frac{1}{\ell} \right] \Bigg|_d^\infty \tag{15}$$

So that $\boxed{W = \frac{q^2}{16\pi\epsilon_0 d}}$

(e) We can also compute the potential energy between the original and image charge by using $W = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$. Substituting the distance of the original and image charge as well as their charge magnitudes, we get

$$W = -\frac{q^2}{8\pi\epsilon_0 d} \tag{16}$$

which is twice the work done on the particle to move it from its position at $z = d$ to infinity. It doesn't have the same value as the earlier work derived because we're looking at the potential energy between the original and image charge, whereas the work done derived earlier was in accordance to the original problem. By calculating the potential energy between the original and the image charge, we ignore the fact that there are no actual fields in the conductor itself. We count the energy of the fields inside, so that's why it's larger.

(f) We then subsitute to $W = \frac{q^2}{16\pi\epsilon_0 d}$ the following: a charge of 1 electron (denoted as 1 e), a distance of 1 angstrom from the surface (with magnitude 10^{-10} [m]), and a vacuum permittivity constant of 5.526×10^7 e/Vm, which lets us cancel e. Thus,

$$W = \frac{(1e)^2}{16\pi(5.526 \times 10^7 \text{e/Vm})(10^{-10}\text{m})} \tag{17}$$
$$W = 3.6 \text{ eV} \tag{18}$$

■

Jackson 2.5

(a) This is pretty similar to (1d), except we first need to find the force acting on a charge q near a grounded conducting sphere, which we can do by either substituting the charge and distance of both the original and image charge to Coulomb's law (cf. (1b)), or to integrate the electrostatic pressure over the whole surface area of the sphere (cf. (1c)). We resort to the former.

We know that if the original charge q is placed at a distance y near a conducting sphere of radius a, then the image charge q' has an opposite charge with magnitude $\frac{a}{y}q$, and it's positioned at $\frac{a^2}{y}$, along the line connecting the original charge to the origin. Since they're collinear, it's easy to find the distance between them, and subsequently, the force:

$$
\begin{aligned}
|\mathbf{F}| &= \frac{1}{4\pi\epsilon_0}\frac{q'q}{|\mathbf{x}'-\mathbf{y}'|^2} && (1)\\
&= \frac{1}{4\pi\epsilon_0}\frac{q(qa/y)}{\left(y-\frac{a^2}{y}\right)^2} = \frac{1}{4\pi\epsilon_0}\frac{a}{y}\frac{q^2}{y^2\left(1-\frac{a^2}{y^2}\right)^2} && (2)\\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{a}{y}\right)^3\frac{q^2}{a^2}\left(1-\frac{a^2}{y^2}\right)^{-2} && (3)
\end{aligned}
$$

Then, the work done to remove it from its distance $r > a$ to infinity is

$$
\begin{aligned}
W &= \int_r^\infty \frac{1}{4\pi\epsilon_0}\left(\frac{a}{y}\right)^3\frac{q^2}{a^2}\left(1-\frac{a^2}{y^2}\right)^{-2} dy && (4)\\
&= \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{y}{(y^2-a^2)^2}dy && (5)
\end{aligned}
$$

Letting $x = y^2 - a^2$, $dx = 2y\ dy$, and $y = r$ becomes $x = r^2 - a^2$. Then, W becomes

$$
\begin{aligned}
W &= \frac{q^2 a}{8\pi\epsilon_0}\int_{r^2-a^2}^\infty \frac{dx}{x^2} && (6)\\
&= \frac{q^2 a}{8\pi\epsilon_0}\left[-\frac{1}{x}\right]\Bigg|_{r^2-a^2}^\infty && (7)
\end{aligned}
$$

so that

$$\boxed{W = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)}} \tag{8}$$

We know that the work done for a conservative force is proportional to the potential difference; in this case, the proportionality constant is the charge. We recall the potential at the surface $x = a$ due to a point charge in the presence of a grounded conducting sphere to be equation (2.3) in Jackson:

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a|\mathbf{n} - \frac{y}{a}\mathbf{n}'|} + \frac{q'}{y'|\mathbf{n} - \frac{a}{y'}\mathbf{n}'|}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a|\mathbf{n} - \frac{y}{a}\mathbf{n}'|} - \frac{1}{a|\mathbf{n} - \frac{y}{a}\mathbf{n}'|}\right]. \tag{9}$$

Since the one that's creating the potential (the original charge) is the same one we're doing work on, $\mathbf{n}$' = $\mathbf{n}$, and $y = r$. We then drop the first term which corresponds to the infinite self-energy (a nonphysical term), recovering an expression for W that is pretty much similar to equation (8):

$$\boxed{W = q\Phi = -\frac{q^2 a}{4\pi\epsilon_0 (r^2 - a^2)}} \tag{10}$$

We note the differences: in equation (10), the sign is negative due to removing the particle **against** the force. It's also up by a factor of 2 since we double counted the work in considering the charge that created the potential being the same charge that works against that potential.

(b) We then do the same calculations for a point charge in the field of an insulated conducting sphere with total charge Q. This time, we build the work done from the force acting on charge q (2.9):

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2 - a^2)}{y(y^2 - a^2)^2}\right]\hat{\mathbf{y}} \tag{11}$$

We calculate the work done as the line integral of the force as we bring the charge from r to infinity:

$$W = -\int_r^\infty \mathbf{F}\cdot\mathbf{dr} \tag{12}$$

$$= -\frac{q}{4\pi\epsilon_0}\int_r^\infty \frac{1}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]dy \tag{13}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[Q\left(-\frac{1}{r}\bigg|_r^\infty\right) - qa^3\int_r^\infty \frac{(2y^2-a^2)}{y^3(y^2-a^2)^2}dy\right] \tag{14}$$

We then let $x = y^2 - a^2$ so that $y^2 = x + a^2$ and $dx = 2y\ dy$ to get

$$W = -\frac{q}{4\pi\epsilon_0}\left[\frac{Q}{r} - qa^3\int_{r^2-a^2}^\infty \frac{(2x+a^2)}{2x^2(x+a^2)^2}dx\right] \tag{15}$$

We then use partial fraction expansion to solve the remaining integral:

$$\int_{r^2-a^2}^\infty \frac{(2x+a^2)}{x^2(x+a^2)^2}dx = \frac{A}{x^2} + \frac{B}{(x+a^2)^2} \tag{16}$$

$$= \frac{A(x+a^2)^2 + Bx^2}{x^2(x+a^2)^2} \tag{17}$$

$$2x + a^2 = A(x^2 + 2a^2x + a^4) + Bx^2 \tag{18}$$

$$x: \qquad 2Aa^2 = 2 \rightarrow A = \frac{1}{a^2} \tag{19}$$

$$x^2: \qquad A + B = 0 \rightarrow B = -\frac{1}{a^2} \tag{20}$$

Thus,

$$W = -\frac{q}{4\pi\epsilon_0}\left[\frac{Q}{r} - \frac{qa}{2}\left(\int_{r^2+a^2}^\infty \frac{dx}{x^2} - \int_{r^2+a^2}^\infty \frac{dx}{(x+a^2)^2}\right)\right] \tag{21}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{Q}{r} - \frac{qa}{2}\left(\frac{-1}{x}\bigg|_{r^2+a^2}^\infty\right) - \left(\frac{-1}{x+a^2}\bigg|_{r^2+a^2}^\infty\right)\right] \tag{22}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{Q}{r} - \frac{qa}{2(r^2+a^2)} + \frac{qa}{2r^2}\right] \tag{23}$$

or

$$\boxed{W = -\frac{1}{4\pi\epsilon_0}\left[\frac{Qq}{r} - \frac{q^2a}{2(r^2+a^2)} + \frac{q^2a}{2r^2}\right]} \tag{24}$$

Again, we relate this calculation to the product of the charge and the potential, equation (2.8) in the text (where we set y=r and **n**'=**n**):

$$W = q\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{Qq}{r} - \frac{q^2 a}{(r^2+a^2)} + \frac{q^2 a}{r^2}\right] \tag{25}$$

where again, we notice two differences: the overall change of sign (due to the change in perspective in removing the charge) as well as the factor of 2 in the charge and its image's potential contributions, due to double counting.

■

Jackson 2.7 We consider a potential problem for the half space $z \geqslant 0$ with Dirichlet boundary conditions on the plane $z = 0$ and at infinity.

(a) We write the Green function in general as $G(\mathbf{x}, \mathbf{x}') = \dfrac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}')$, where an unknown function F is to be solved using the given boundary conditions. In a Dirichlet problem, we are assured that at the bounding surface (in this case, the xy-plane at $z = 0$), the value of Green's function is 0, i.e. $G_D = 0$ at $z = 0$. But this is exactly similar to the problem in the book where we had a unit charge placed at $\mathbf{x}'$ in the presence of a perfectly conducting flat surface at $z = 0$ (since this also makes the Green function vanish at $z = 0$). We can easily solve this using the method of images.

As outlined in the notes, we can show that the potential at the positive half-space is equivalent to a potential due to two charges: one being the original charge q at $z = z'$, and the other at the exact opposite side with the opposite charge sign $-q$ (since they follow the same conditions for Φ: zero at $z = 0$, vanishes at infinity). Again, this is possible due to the uniqueness theorem for Laplace's equation. We can easily write the potential as a sum of the potentials due to each charge:

$$\Phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{q}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}\right] \tag{1}$$

Thus, in order to extract G_D, we need to recast the method images but this time, a unit charge $q = 4\pi\epsilon_0$ must be placed in $z = z'$:

$$\boxed{G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}} \tag{2}$$

(b) We then proceed to modify the potential at the $z = 0$ plane to contain a nonzero V inside a circular region of radius a centered at the origin. To obtain an integral expression for the potential at point P, we use the Green function solution for Dirichlet boundary conditions,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x}, \mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}da' \tag{3}$$

where one implicit assumption in the problem is that there are no charges supplying the nonzero potential, so that the first term must vanish. The surface integral can then be expanded to be the surface of a very large box with all surfaces but $z = 0$ surface at infinity. We then want $\Phi \to 0$ as the surfaces approach infinity, so these sides do not contribute to the surface integral. We are then left with the surface at $z = 0$, for which the only nonzero contribution is inside the circular region specified above. Thus,

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\int_{\text{circ}}\left(\Phi\frac{\partial G_D}{\partial n'}\right) dS && (4)\\
&= -\frac{V}{4\pi}\int_0^{2\pi}\int_0^{a}\frac{\partial G_D}{\partial n'}\rho'\ d\rho'\ d\varphi' && (5)
\end{aligned}$$

For n', we determine the normal to the volume enclosing this box. For the surface at $z = 0$, this corresponds to $-z'$, so that $\dfrac{\partial G_D}{\partial n'} = -\dfrac{\partial G_D}{\partial z'}$. Thus,

$$\frac{\partial G_D}{\partial z'} = \frac{\partial}{\partial z'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right] \tag{6}$$

$$= \frac{(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} + \frac{(z+z')}{((x-x')^2+(y-y')^2+(z+z')^2)^{3/2}} \tag{7}$$

We can then substitute this to equation (5) and replace x and y by their cylindrical coordinate counterparts: $(x-x')^2 + (y-y')^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi')$, so that Φ becomes

$$\begin{aligned}
\Phi(\mathbf{x}) = -\frac{V}{4\pi}\int_0^{2\pi}\int_0^{a} &\frac{(z-z')}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')^2)^{3/2}} && (8)\\
&+ \frac{(z+z')}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z+z')^2)^{3/2}}\rho'\ d\rho'\ d\varphi'
\end{aligned}$$

we then evaluate this at $z' = 0$ since that's where the nonzero $\Phi(\mathbf{x}')$ is located:

$$\begin{aligned}
\Phi(\mathbf{x}) = \frac{V}{4\pi}\int_0^{2\pi}\int_0^{a} &\Bigg[\frac{(z)}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z)^2)^{3/2}} && (9)\\
&+ \frac{(z)}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z)^2)^{3/2}}\rho'\ d\rho'\ d\varphi'
\end{aligned}$$

$$\boxed{\Phi(\mathbf{x}) = V\frac{z}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{1}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}}\rho'\ d\rho'\ d\varphi'} \tag{10}$$

(c) We can further simplify this expression by considering the potential along the axis of the circle ($\rho' = 0$, $\varphi' = [0, 2\pi)$). Substituting, we get

$$\Phi(\mathbf{x}) = V\frac{z}{2\pi}\int_0^{2\pi}\int_0^a \frac{1}{(\rho'^2 + z^2)^{3/2}}\rho'\ d\rho'\ d\varphi' \tag{11}$$

$$= Vz\int_0^a \frac{\rho'\ d\rho'}{(\rho'^2 + z^2)^{3/2}} \tag{12}$$

Letting $u = \rho'^2 + z^2$, $du = 2\rho' d\rho'$, $\rho' = 0 \to u = z^2$ and $\rho' = a \to u = a^2 + z^2$ so that

$$\Phi(\mathbf{x}) = \frac{Vz}{2}\int_{z^2}^{a^2+z^2} \frac{du}{u^{3/2}} \tag{13}$$

$$= \frac{Vz}{2}\left(\frac{-2}{\sqrt{u}}\right)\Bigg|_{z^2}^{a^2+z^2} \tag{14}$$

$$= -Vz\left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{z}\right) \tag{15}$$

Simplifying,

$$\boxed{\Phi(\mathbf{x}) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)} \tag{16}$$

(d) Going back to equation (10), we can expand the solution in terms of large distances for which $x^2 + y^2 + z^2 = \rho^2 + z^2 >> a^2$, where a as we recall is the radius of the circular region in the $z = 0$ plane for which the potential is a nonzero constant V.

We do this by multiplying equation (12) with $\dfrac{(\rho^2 + z^2)^{3/2}}{(\rho^2 + z^2)^{3/2}}$ in order to expand the integrand:

$$\begin{aligned}\Phi(\mathbf{x}) &= V\frac{z}{2\pi}\int_0^{2\pi}\int_0^a \frac{(\rho^2+z^2)^{3/2}}{(\rho^2+z^2)^{3/2}}\frac{\rho'\ d\rho'\ d\varphi'}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}} \\ &= \frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}\int_0^a \left[\frac{\rho'\ d\rho'\ d\varphi'}{(\rho^2+z^2)^{-3/2}\left((\rho^2+z^2)+\rho'^2-2\rho\rho'\cos\varphi'\right)^{3/2}}\right] \\ &= \frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}\int_0^a \left(1+\frac{\rho'^2-2\rho\rho'\cos\varphi'}{\rho^2+z^2}\right)^{-3/2}\rho'\ d\rho'\ d\varphi' \end{aligned} \tag{17}$$

Although not looking too good, the integrand can certainly be expanded using the Binomial series for a general $n \in \mathbb{R}$: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2}x^2 + \ldots$, so that

$$\left(1 + \frac{\rho'^2 - 2\rho\rho'\cos\varphi'}{\rho^2+z^2}\right)^{-3/2} = 1 + (-3/2)\frac{\rho'^2 - 2\rho\rho'\cos\varphi'}{\rho^2+z^2} + (15/8)\left(\frac{\rho'^2 - 2\rho\rho'\cos\varphi'}{\rho^2+z^2}\right)^2 + \ldots \tag{18}$$

and that the potential can be integrated term by term in this series (we only show the first few terms for lack of space and time):

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\left(\int_0^{2\pi}\int_0^a \rho'\ d\rho'\ d\varphi'\right) - \frac{3}{2}\left(\int_0^{2\pi}\int_0^a \rho'\ d\rho'\ d\varphi' \frac{\rho'^2 - 2\rho\rho'\cos\varphi'}{\rho^2+z^2}\right) \\ &\quad + \frac{15}{8}\left(\int_0^{2\pi}\int_0^a \rho'\ d\rho'\ d\varphi' \left(\frac{\rho'^2 - 2\rho\rho'\cos\varphi'}{\rho^2+z^2}\right)^2\right) + \ldots \\ \Phi(\mathbf{x}) &= \frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\Bigg[\pi a^2 - \frac{3}{2(\rho^2+z^2)}\left(2\pi\int_0^a \rho'^3 d\rho' - 2\rho\cancel{\int_0^{2\pi}\cos\varphi'\ d\varphi'}\int_0^a \rho'^2\ d\rho'\right) \\ &\quad + \frac{15}{8(\rho^2+z^2)^2}\Bigg(2\pi\int_0^a \rho'^5 d\rho' - 4\rho\cancel{\int_0^{2\pi}\cos\varphi'\ d\varphi'}\int_0^a \rho'^4\ d\rho' \\ &\quad + 4\rho^2\int_0^{2\pi}\cos^2\varphi'\ d\varphi'\int_0^a \rho'^3\ d\rho'\Bigg) + \ldots\Bigg]\end{aligned} \tag{19}$$

Simplifying this big mess, we get

$$\Phi(\mathbf{x}) = V\frac{z}{(\rho^2+z^2)^{3/2}}\left[\frac{a^2}{2} - \frac{3a^4}{8(\rho^2+z^2)} + \frac{15}{8(\rho^2+z^2)^2}\left(\frac{a^6}{6} + 2\rho^2\frac{a^4}{4}\right) + \ldots\right]$$

$$\boxed{\Phi(\mathbf{x}) = V\frac{a^2 z}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{15}{8(\rho^2+z^2)^2}\left(\frac{a^4}{3} + \rho^2 a^2\right) + \ldots\right]} \tag{20}$$

To recover our answer in part (c), we set $\rho = 0$ (along the axis) in the previous equation and simplify:

$$\begin{aligned}
\Phi(\mathbf{x}) &= V\frac{a^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4} + \dots\right] \\
&= V\left[\frac{a^2}{2z^2} - \frac{3a^4}{8z^4} + \frac{5a^6}{16z^6} + \dots\right] \\
&= V\left[1 - \left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} - \frac{5a^6}{16z^6} - \dots\right)\right] \qquad (21)
\end{aligned}$$

We then recognize the series inside the parenthesis as the binomial expansion for $\left(1 + \dfrac{a^2}{z^2}\right)^{-1/2}$, so that we could further simplify the expression:

$$\begin{aligned}
\Phi(\mathbf{x}) &= V\left[1 - \left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right] \\
&= V\left[1 - \left(\frac{z}{\sqrt{z^2 + a^2}}\right)\right] \qquad (22)
\end{aligned}$$

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Problem 4

Jackson 2.15

(a) To obtain the appropriate Green's function for the specified geometry, we start by recalling the Poisson equation for the Green's function in two dimensions

$$\begin{aligned} \nabla'^2 G(x,y;x',y') &= -4\pi\delta(x'-x)\delta(y'-y) \\ \left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)G(x,y;x',y') &= -4\pi\delta(x'-x)\delta(y'-y) \end{aligned} \tag{1}$$

The Dirichlet boundary conditions assert that the Green's function must vanish at the endpoints (for the surface charge in x'). We can then conclude that the Green's function can be expanded in x' as a Fourier sine series (we cannot use the cosine series because these have nonzero values in the bounds)

$$G(x,y;x',y') = \sum_{n=1}^{\infty} f_n(x,y;y')\sin(n\pi x') \tag{2}$$

Substituting to equation (1), we get

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sum_{n=1}^{\infty} f_n(x,y;y')\sin(n\pi x') &= -4\pi\delta(x'-x)\delta(y'-y) \\ \sum_{n=1}^{\infty}\left(\frac{\partial^2}{\partial y'^2}-n^2\pi^2\right)f_n(x,y;y')\sin(n\pi x') &= -4\pi\delta(x'-x)\delta(y'-y) \end{aligned} \tag{3}$$

where we used $\frac{\partial}{\partial x'}\sin(n\pi x') = n\pi\cos(n\pi x')$ twice. To proceed, we make the deltas in the right-hand side Fourier sine series as well, where we use the completeness relation for sines

$$\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x') = \frac{1}{2}\delta(x-x') \tag{4}$$

We then note the symmetry of Dirac delta on its two arguments so we can sucessfully substitute this to equation (3):

$$\sum_{n=1}^{\infty}\left(\frac{\partial^2}{\partial y'^2}-n^2\pi^2\right)f_n(x,y;y')\sin(n\pi x') = -8\pi\delta(y'-y)\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x') \tag{5}$$

How do we relate f_n to the g_n in the final expression? Looking at the previous equation, we see that f_n is proportional to $\sin(n\pi x)$; the proportionality factor must then be g_n:

$$f_n(x, y; y') = g_n(y, y') \sin(n\pi x) \tag{6}$$

Building G, we get

$$G(x, y; x', y') = \sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x') \tag{7}$$

We then insert a factor of 2 in order for g_n to satisfy the remaining pieces of equation (5):

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = -8\pi\delta(y' - y) \tag{8}$$

so that when $g_n \to 2g_n$,

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = -4\pi\delta(y' - y) \tag{9}$$

and

$$\boxed{G(x, y; x', y') = 2\sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')} \tag{10}$$

(b) We then proceed to solve g_n in equation (9). We first summarize the different conditions we impose on g_n:

- Boundary conditions: $g_n(y, y') = 0$ for $y' = 0, 1$
- Continuity conditions: $g_< = g_>$ for $y' = y$
- Discontinuity in the derivative (aka "jump") conditions: $\dfrac{\partial g_<}{\partial y'} - \dfrac{\partial g_>}{\partial y'} = 4\pi$

We first solve the homogeneous equation for which the right hand side of equation (9) vanishes. This gives us exponential solutions, which can then be converted into combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$. The solutions can then be separated depending on the value of y with respect to y':

$$g_n(y, y') = \begin{cases} g_< \equiv A_< \sinh(n\pi y') + B_< \cosh(n\pi y'), & y' < y \\ g_> \equiv A_> \sinh(n\pi y') + B_> \cosh(n\pi y'), & y' > y \end{cases} \tag{11}$$

We see that the two pairs of constants of integration should be related to each other since we're only solving a 2nd-order DE, and that these constants are functions of y. We now use the various conditions listed earlier: for the boundary conditions, substituting $y' = 0$ yields $B_< = 0$ since we use the first solution for $y' < y$ while $0 \leqslant y \leqslant 1$. Then, for $y' = 1$, we use the second solution:

$$A_> \sinh(n\pi) + B_> \cosh(n\pi) = 0 \longrightarrow B_> = -A_> \tanh(n\pi) \tag{12}$$

Thus we're left with two coefficients, $A_<$ and $A_>$. We then proceed to use the continuity and jump conditions:

$$\begin{aligned} A_<(y = y')\sinh(n\pi y') &= A_>(y = y')[\sinh(n\pi y') \\ &\quad - \tanh(n\pi)\cosh(n\pi y')] \end{aligned} \tag{13}$$

$$\begin{aligned} n\pi A_< \cosh(n\pi y') &= n\pi A_>[\cosh(n\pi y') \\ &\quad - \tanh(n\pi)\sinh(n\pi y')] + 4\pi \end{aligned} \tag{14}$$

$$\begin{pmatrix} \sinh(n\pi y') & \tanh(n\pi)\cosh(n\pi y') - \sinh(n\pi y') \\ \cosh(n\pi y') & \tanh(n\pi)\sinh(n\pi y') - \cosh(n\pi y') \end{pmatrix} \begin{pmatrix} A_< \\ A_> \end{pmatrix} = \begin{pmatrix} 0 \\ 4/n \end{pmatrix} \tag{15}$$

Inverting the matrix gives us the solutions for $A_<$ and $A_>$:

$$\begin{pmatrix} A_< \\ A_> \end{pmatrix} = -\frac{4}{n}\frac{1}{\sinh(n\pi)} \begin{pmatrix} \cosh(n\pi)\sinh(n\pi y) - \sinh(n\pi)\cosh(n\pi y) \\ \cosh(n\pi)\sinh(n\pi y) \end{pmatrix} \tag{16}$$

We substitute this back to $g_n(y, y')$ to obtain

$$\begin{aligned} g_n(y, y') &= \frac{4}{n\sinh(n\pi)} \\ &\quad \times \begin{cases} \sinh(n\pi y')[\sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)], & y' < y \\ \sinh(n\pi y)[\sinh(n\pi)\cosh(n\pi y') - \cosh(n\pi)\sinh(n\pi y')], & y' > y \end{cases} \end{aligned} \tag{17}$$

which we could further simplify by using the identity $\sinh(A - B) = \sinh A \cosh B - \cosh A \sinh B$ for $A = n\pi$ and $B = n\pi y$ or y'. To facilitate this "or" condition, we make use of the max and min function to determine which is bigger (or smaller) between y and y'. Defining $y_< \equiv \min(y, y')$ and $y_> \equiv \max(y, y')$, we get

$$g_n(y, y') = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y_<)\sinh(n\pi - n\pi y_>). \tag{18}$$

Substituting back to equation (10) yields the full expression for the Green's function

$$\boxed{G(x,y;x',y') = \sum_{n=1}^{\infty} \frac{8}{n\sinh(n\pi)} \sinh(n\pi y_<)\sinh[n\pi(1-y_>)]\sin(n\pi x)\sin(n\pi x')} \tag{19}$$

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