Section 6.4

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Green's fxns for the wave eqn

Recall: Lorenz gauge of the DEs for the potentials:

$$\Box^2 \Phi = -\frac{\rho}{\epsilon_0} \qquad (1)$$
$$\Box^2 \vec{A} = -\mu_0 \vec{J}$$

where $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t}$ is the D'Alembertian box operator. This is a wave equation with source terms: in general,

$$\Box^2 \psi(\vec{x},t) = -4\pi f(x,t) \qquad (2)$$

Here, $\Box^2$ can be thought of as a 4D Laplacian (where time is the 4th dimension) so that we can use Green's functions to solve them. We first recast ψ & f into the Fourier domain (i.e. we are specifying conditions for which ψ & f is able to have a Fourier transform):

$$\psi(\vec{x},\omega) = \int_{-\infty}^{\infty} \psi(\vec{x},t) e^{i\omega t}\, dt \qquad (3)$$

$$f(\vec{x},\omega) = \int_{-\infty}^{\infty} f(\vec{x},t)\, e^{i\omega t}\, dt \quad (3)$$

so that

$$\psi(\vec{x},t) = \int_{-\infty}^{\infty} \psi(\vec{x},\omega)\, e^{-i\omega t}\, d\omega \quad (4)$$
$$f(\vec{x},t) = \int_{-\infty}^{\infty} f(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

letting $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ act on (4a),

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\psi\, e^{-i\omega t}\, d\omega$$
$$= -\frac{1}{c^2}\int_{-\infty}^{\infty}\psi(\vec{x},\omega)\frac{\partial^2}{\partial t^2} e^{-i\omega t}\, d\omega$$
$$= \frac{\omega^2}{c^2}\int_{-\infty}^{\infty}\psi(\vec{x},\omega) e^{-i\omega t}\, d\omega$$
$$= \frac{\omega^2}{c^2}\psi(\vec{x},t) \quad (5)$$

Dispersion relation

So that our wave eqn becomes

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega) \quad (6)$$

We now insert a Green's fxn $G(\vec{x},\omega)$ to this PDE st.

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G(\vec{x},\omega) = -4\pi\,\delta(\vec{x}-\vec{x}') \quad (7)$$

Here, G becomes the instantaneous response for ψ. Without boundary conditions, G must emanate from

the origin in a spherically symmetric way
((cf. Laplace eqn's spherical decomposition) so that
$G = G(|\vec{x}-\vec{x}'|) = G(\mathscr{r})$ (Griffiths notation)
Then the Helmholtz eqn. (7) becomes

$$\frac{1}{\mathscr{r}}\frac{d^2}{d\mathscr{r}^2}(\mathscr{r}G) + k^2 G = -4\pi\delta(\vec{\mathscr{r}}) \quad (8)$$

Solving the homogeneous eqn., we find

$$\frac{d^2}{d\mathscr{r}^2}(\mathscr{r}G) + k^2(\mathscr{r}G) = 0$$

$$\mathscr{r}G(\mathscr{r}) = Ae^{ik\mathscr{r}} + Be^{-ik\mathscr{r}} \quad (9)$$

Recall from electrostatics that a
point charge's Green fxn will have
a $\frac{1}{\mathscr{r}}$ dependence, especially since $k \ll \frac{1}{\mathscr{r}}$
so that the homogenous eqn must be
satisfied; in this limit,

$$\lim_{k\mathscr{r}\to 0} G(\mathscr{r}) = \frac{1}{\mathscr{r}}$$

which allows us to write the general
solution (for a particular value of k)

$$G(\mathscr{r}) = \frac{A}{\mathscr{r}}e^{ik\mathscr{r}} + \frac{B}{\mathscr{r}}e^{-ik\mathscr{r}} \quad (10)$$

$$G_k(r) = \frac{A}{r} e^{ikr} + \frac{B}{r} e^{-ikr} \quad (10)$$

Diverging spherical wavefront   Converging spherical wavefront

Normalized such that $A+B=1$. To figure out the time dependence of (10), we insert another delta fxn in time:

$$\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(r,t,t') = -4\pi\delta(\vec{r})\,\delta(t-t')$$

cf. eq (6), as well as from the (inverse) Fourier transform of $f(\vec{x},t)$, we find the source term (RHS) to be $-4\pi\delta(\vec{r})e^{i\omega t'}$. Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\,d\omega$$

so the time-dependent solution becomes

$$G_k(r,\tau) = A\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikr}}{r}e^{-i\omega\tau}\,d\omega + B\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\tau}\frac{e^{-ikr}}{r}\,d\omega \quad (11)$$

where $\tau \equiv t-t'$ is the relative time bet observation & source. Simpli

$$G(r,\tau) = \frac{1}{r}\left(\frac{1}{2\pi}A\int_{-\infty}^{\infty} e^{i(kr-\omega\tau)}\,d\omega\right.$$

$$+\frac{1}{2\pi}B\int_{-\infty}^{\infty} e^{i(-kR-\omega t)}d\omega\Big)$$

$$=\frac{1}{R}\left(\frac{1}{2\pi}A\int_{-\infty}^{\infty} e^{i\omega(\frac{k}{\omega}R-\tau)}d\omega\right.$$
$$\left.+\frac{1}{2\pi}B\int_{-\infty}^{\infty} e^{-i\omega(\frac{k}{\omega}R-\tau)}d\omega\right)$$

For a simple relation $\omega = ck$, we get

$$G_k(R,\tau)=\frac{1}{R}\Big[\underbrace{A\delta\left(\tau-\frac{R}{c}\right)}_{\text{Retarded}}+\underbrace{B\delta\left(\tau+\frac{R}{c}\right)}_{\text{Advanced}}\Big] \quad (12)$$

We can now build the full solution Ψ by integrating Gf over spacetime

$$\Psi_k^{(\pm)}(\vec{x},t)=\iint G^{(\pm)}(\vec{x},t;\vec{x}',t')\,f(\vec{x}',t')\,d^3x'\,dt' \quad (13)$$

We then decompose this solution to accommodate limiting cases as well as to solve physical problems. For a localized $f(\vec{x}',t')$ in time & space. We further restrict its existence by saying that it is finite & nonzero for a finite interval around $t'=0$.

① For $t\to-\infty$, $f\to 0$, and the Poisson equation reverts to its Laplacian equivalent. Then that would also have its own homogenous solution $\psi_{in}(\vec{x},t)$ to be added to (13) for $G^{(+)}$ (the diverging wave).

② For $t \to \infty$, the same thing happens; this time some $\psi_{out}(\vec{x},t)$ will be added to (13) for $G^{(-)}$ (the converging wave).

The most common solution is then given by setting $\psi_{in} = 0$:

$$\psi(\vec{x},t) = \iint \frac{1}{R}\,\delta\left(t' - \left(t + \frac{R}{c}\right)\right) f(\vec{x}',t')\, d^3x'\, dt'$$

$$\boxed{\psi(\vec{x},t) = \int \frac{1}{R} f\left(\vec{x}', t - \frac{R}{c}\right) d^3x'} \quad (14)$$

e.g. if $f = \rho$ (charge density)

$$\psi(\vec{x},t) = \int \frac{\rho\left(\vec{x}', t - \frac{R}{c}\right)}{R}\, d^3x'$$

Coulomb solution

retarded time

Section 6.7

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Poynting's Theorem & Conservation of $\vec{p}$ & E for a system of charged particles & EM fields

> Recall: Power for a single charge:

$$\vec{F}\cdot\vec{v} = q\vec{E}\cdot\vec{v} + \cancel{q\,\vec{v}\times\vec{B}\cdot\vec{v}} \qquad (1)$$

$$= \int \vec{E}\cdot\vec{J}\, d^3x \qquad (2)$$

↳ continuous charge/current distribution

Power - rate of converting energy stored to useful forms

∴ Conservation of energy dictates that the nonzero positive power must be balanced by a decrease in some energy → energy stored in the EM field. Recasting $\vec{J}$ from (2) using Maxwell's eqns,

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = \int_V \left(\nabla\times\vec{H} - \frac{\partial}{\partial t}\vec{D}\right)\cdot\vec{E}\, d^3x \qquad (3)$$

$$= \int_V \left(\vec{E}\cdot\vec{\nabla}\times\vec{H} - \vec{E}\cdot\frac{\partial}{\partial t}\vec{D}\right) d^3x \qquad (4)$$

Using $\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}) \qquad (5)$

$$= -\int \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H}) \qquad (7)$$

$$= -\int \vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \nabla\cdot(\vec{E}\times\vec{H}) \quad (7)$$

Recalling that if $W_{electrostatics} = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x$ and $W_{magnetostatics} = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x$, then the total energy density stored in the fields is $W_{electrostatics} + W_{magnetostatics} = \int u_{EM}\, d^3x$ where $u_{EM} = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})$. Differentiating wrt t,

$$\frac{\quad}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial t}\vec{E}\cdot\vec{D} + \frac{\partial}{\partial t}\vec{H}\cdot\vec{B}\right)$$

for linear materials w/ negligible loss/dispersion, $\frac{\partial}{\partial t}\vec{H}\cdot\vec{B} = \vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \frac{\partial \vec{H}}{\partial t}\cdot\vec{B} = 2\vec{H}\cdot\frac{\partial \vec{B}}{\partial t}$ (same for $\frac{\partial}{\partial t}\vec{E}\cdot\vec{D}$) so

$$\frac{\partial u_{EM}}{\partial t} = \frac{1}{2}\left(2\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + 2\vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right.$$
$$= \vec{E}\cdot\frac{\partial \vec{D}}{\quad} \quad (8)$$

cf. eq. (7),

$$\int \vec{J}\cdot\vec{E}\, d^3x = -\int_V \left(\frac{\partial u_{EM}}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right) d^3x \quad (9)$$

since this is true for any volume V, we can get both integrands:

$$\vec{J}\cdot\vec{E} = -\frac{\partial u_{EM}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{H}) \quad (10)$$

Recalling the form of the full time derivative of a vector field $\vec{F}(\vec{x}(t), \hat{t})$, of a vector field $\vec{F}(\vec{x}(t), t)$,

$$\frac{\partial}{\partial t}\vec{F}(\vec{x}(t), t) = \frac{\partial \vec{F}}{\partial t} + \frac{\partial \vec{F}}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial t}$$
$$= \frac{\partial \vec{F}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{F}$$

so, from eq.(10), u_{EM} is the vector field in question: the flow of energy density in time is given by the 2nd term, $\vec{\nabla} \cdot (\vec{E} \times \vec{H})$ so it must be special. Denoting

$$\vec{S} \equiv \vec{E} \times \vec{H}$$

we have the Poynting vector.

From eq (9), we can interpret the conservation of energy as follows:

$$\underbrace{\int \vec{J} \cdot \vec{E}\, d^3x}_{\text{nonzero power (positive)}} = -\int \left(\frac{\partial u_{EM}}{\partial t} + \nabla \cdot \vec{S} \right) d^3x \tag{11}$$
$$= -\left[\frac{\partial}{\partial t} \underbrace{\int u_{EM}\, d^3x}_{\text{temporal flow of energy}} + \underbrace{\oint_{\delta V} \vec{S} \cdot d\vec{a}}_{\text{spatial flow of energy}} \right]$$

$-$ → decrease in power

$\frac{\text{energy}}{\text{time}}$ $\qquad$ $\frac{\text{energy}}{\text{time}}$ $\qquad$ $\frac{\text{energy}}{\text{time}}$

Thus the units of $\vec{S}$ is $\frac{[\text{energy}]}{[\text{area}][\text{time}]}$.

We can also interpret $\vec{J} \cdot \vec{E}$ as the rate of increase of energy of charged particles PER unit volume.

PER unit volume:

$$\frac{dE_{mech}}{dt\,dV} = \vec{J}\cdot\vec{E}$$

$$\frac{dE_{mech}}{dt} = \int \vec{J}\cdot\vec{E}\, d^3x \qquad (12)$$

From Poynting's Theorem, the total power is

$$\frac{dE}{dt} = \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt}$$
$$= \int \vec{J}\cdot\vec{E}\, d^3x + \int \frac{\partial u_{EM}}{\partial t} d^3x \qquad (13)$$
$$= -\oint \vec{S}\cdot d\vec{a}$$

where $\frac{\partial u_{EM}}{\partial t} = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$

$$= \frac{1}{2}\left(\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right)$$
$$= \frac{\epsilon_0}{2}\left(|\vec{E}|^2 + c^2|\vec{B}|^2\right)$$

so equation (13) dictates the conservation of energy. Let's now check the conservation of linear momentum. From the Lorentz force, $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$, we can use Newton's 2nd law to write (in general)

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \int_V (\rho\vec{E} + \vec{J}\times\vec{B})\, d^3x \qquad (14)$$

To derive $\frac{dP_{field}}{dt}$, we recast the source $\vec{J}$

$$\frac{\quad}{dt} = \int_V \left(\rho\vec{E} + \left(\vec{\nabla}\times H - \frac{\partial\vec{D}}{\partial t}\right)\times\vec{B}\right) d^3x \quad (15)$$

We also replace ρ by Gauss' law:

$$\frac{dP_{mech}}{dt} = \int_V \left[(\vec\nabla\cdot\vec D)\vec E + (\vec\nabla\times\vec H)\times\vec B - \frac{\partial\vec D}{\partial t}\times\vec B \right] d^3x$$

$$\vec D\times\frac{\partial\vec B}{\partial t} - \frac{\partial}{\partial t}(\vec D\times\vec B)$$

$$= \int_V \left[(\vec\nabla\cdot\vec D)\vec E - \vec B\times(\vec\nabla\times\vec H) + \vec D\times\frac{\partial\vec B}{\partial t} - \frac{\partial}{\partial t}(\vec D\times\vec B) \right] d^3x \qquad (16)$$

$$\frac{\partial\vec B}{\partial t} \to -\nabla\times\vec E$$

Let's add the magnetic Gauss's law to the fray: $\vec\nabla\cdot\vec B = 0 \to \vec H(\vec\nabla\cdot\vec B) = 0$

$$\frac{dP_{mech}}{dt} = \int_V \left[(\vec\nabla\cdot\vec D)\vec E + (\vec\nabla\cdot\vec B)\vec H \right] d^3x - \int_V \left[\vec B\times(\vec\nabla\times\vec H) + \vec D\times(\vec\nabla\times\vec E) \right] d^3x - \int \frac{\partial}{\partial t}\vec D\times\vec B\, d^3x \qquad (17)$$

Recasting the last integral in terms of $\vec S$ and replacing the $\vec D$'s by $\epsilon_0\vec E$'s & the $\vec H$'s by $\frac{1}{\mu_0}\vec B$'s (assuming linearity),

$$\frac{dP_{mech}}{dt} = \int_V \epsilon_0\left[(\vec\nabla\cdot\vec E)\vec E + c^2(\vec\nabla\cdot\vec B)\vec B \right] - \int_V \epsilon_0\left[\vec B\times(\vec\nabla\times c^2\vec B) + \vec E\times(\vec\nabla\times\vec E) \right] - \int_V \frac{1}{c^2}\frac{\partial}{\partial t}\vec S\, d^3x \qquad (18)$$

Since the last integral has a time derivative, it may seem that $\int \frac{1}{c^2}\vec S\, d^3x$ is the

EM field's momentum. Thus, everything that remains in the RHS of (18) is the total momentum's total time derivative. To make this more obvious, we recast the RHS as a surface integral (as we did in the energy flow calculations). We do this by recasting the integrand as follows:

$$\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})\right]_i = E_i \partial_j E_j - \epsilon_{ijk} E_j (\epsilon_{klm} \partial_l E_m)$$

$$= E_i \partial_j E_j + \epsilon_{ikj} (\epsilon_{klm} \partial_l E_m) E_j$$

$$= E_i \partial_j E_j - \epsilon_{kij} \epsilon_{klm} \partial_l E_m E_j$$

$$= E_i \partial_j E_j - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_l E_m E_j$$

$$= E_i \partial_j E_j - (\partial_i E_j^2 - E_i \partial_j E_j)$$

$$= \tfrac{1}{2}\partial_j (E_i E_j) + \tfrac{1}{2}\partial_j (E_i E_j) - \delta_{ij} \partial_j E_j^2$$

$$= \partial_j (E_i E_j) - \tfrac{1}{2} \partial_j \delta_{ij} E_j^2$$

$$= \partial_j \underbrace{\left[E_i E_j - \tfrac{1}{2}\delta_{ij} E_j^2\right]}_{T_{ij}} \qquad (19)$$

Thus, we treat the ith component of the total momentum as the total momentum as

$$\frac{d}{dt}(\vec{P}_{mech} + \vec{P}_{field})_i = \int_V \partial_j T_{ij}\, d^3x$$

$$= \oint \partial_j \underbrace{T_{ij}\, n_j}_{\substack{\text{ith component} \\ \text{of the momentum} \\ \text{flow in space}}}\, da$$

Similar arguments can be used in describing the angular momentum conservation, for which $\vec{l}_{field} = \vec{x} \times \frac{1}{c^2}\vec{S}$, $\vec{l}_{mech} = \vec{x} \times \vec{p}_{mech}$ & $\epsilon_{ijk} T_{jk} x_l$ is the ith component of the angular momentum flow

Poynting's theorem in linear dispersive media with losses

> In this section, we get rid of the assumptions that simplify Poynting's theorem to

$$\frac{\partial u_{EM}}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E} \qquad (1)$$

i.e. we now allow linear & isotropic dispersion so that $\vec{D}(\omega) = \epsilon(\omega)\,\vec{E}(\vec{x},\omega)$, where $\vec{D}(\vec{x},\omega)$ & $\vec{E}(\vec{x},\omega)$ are the Fourier transforms of $\vec{D}(\vec{x},t)$ & $\vec{E}(\vec{x},t)$, respectively. Same goes for $\vec{B}$: $\vec{B}(\omega) = \mu(\omega)\,\vec{H}(\vec{x},\omega)$. In these two statements, we prevent the simplification

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \neq \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) \qquad (2)$$

since $\frac{\partial}{\partial t}\vec{D} = -i\omega\vec{D}$; Recall,

$$\mathcal{F}\left[\frac{\partial f}{\partial t}\right] = \int_{-\infty}^{\infty}\frac{\partial f}{\partial t}e^{i\omega t}\,dt$$

$$= \cancel{f e^{i\omega t}\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} i\omega e^{i\omega t} f\,dt$$

$$= -i\omega\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt$$

$$= -i\omega\, \mathcal{F}[f(t)] \qquad (3)$$

Since $\vec{E}$, $\vec{D}$, & ϵ are real, then

$$\vec{E}(\vec{x},\omega) = \vec{E}^*(\vec{x},\omega)$$
$$\vec{D}(\vec{x},\omega) = \vec{D}^*(\vec{x},\omega) \qquad (4)$$
$$\epsilon(\omega) = \epsilon^*(\omega)$$

We now rewrite $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$ using the Fourier integrals for both $\vec{E}$ & $\vec{D}$:

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \iint \vec{E}^*(\omega')\, e^{i\omega' t} \cdot [-i\omega]\epsilon(\omega)\vec{E}(\omega)\, e^{i\omega t}\, d\omega\, d\omega' \qquad (5)$$

We split the RHS integrand into two:

$$\vec{E}^*(\omega')[-i\omega\,\epsilon(\omega)]\cdot\vec{E}(\omega)\, e^{-i(\omega'-\omega)t} =$$
$$\frac{1}{2}\vec{E}^*(\omega')[-i\omega\,\epsilon(\omega) - i\omega\epsilon(\omega)]\cdot\vec{E}(\omega)\, e^{-i(\omega'-\omega)t}$$

sub. $-\omega'$ ϵ^* $-\omega'$

Now we need to expand $i\omega'\epsilon^*(\omega')$ around $\omega'=\omega$ and retain the low-order terms iff we assume a stationary phase for $\vec{E}$, i.e. $\vec{E}(\omega') \lesssim \frac{d}{d\omega}\epsilon(\omega)$:

$$\epsilon^*(\omega') = \epsilon^*(\omega) + \frac{d}{d\omega}(\omega\,\epsilon^*(\omega)) + \ldots$$

$$-i\omega\,\epsilon(\omega) + i\omega'\epsilon^*(\omega') = -i\omega\,\epsilon(\omega) + i\omega\epsilon^* + i(\omega'-\omega)\frac{d}{d\omega}(\omega\epsilon^*) + \ldots = -i\omega(\underbrace{\epsilon-\epsilon^*}_{2\,\mathrm{Im}\,\epsilon}) + i(\omega'-\omega')\frac{d}{d\omega}(\omega\epsilon^*) + \ldots$$

so that, since $-i\omega = \frac{\partial}{\partial t}$,

$$\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = \frac{1}{2}\int d\omega \int d\omega' \, \vec{E}^*(\omega')\cdot\vec{E}(\omega) e^{-i(\omega'-\omega)t}$$
$$\left[2\omega \, \mathrm{Im}\, \epsilon(\omega) + \frac{\partial}{\partial t}\frac{d}{d\omega}(\omega \epsilon^*(\omega))\right]$$
$$= \int d\omega \int d\omega \, \vec{E}^*(\omega')\cdot\vec{E}(\omega)\, \omega \, \mathrm{Im}\, \epsilon(\omega) e^{-i(\omega'-\omega)t} \quad (6)$$

To better explain this approximation, we recall that for linear media, different ω-components would not interfare. Thus our assumption that $\vec{E}(\omega)$ & $\vec{H}(\omega)$ is dominant over a relatively narrow range of ω' can be readily shown by supposing

$$\vec{E}(\vec{x},t) = \mathrm{Re}\, \vec{E}(\vec{x}) e^{-i\omega t} = \vec{E}(\vec{x})\cos(\omega t) \quad (7)$$

called the harmonic t-dependence. Now, we calculate for $\vec{J}\cdot\vec{E}$ using this formulation:

$$\vec{J}\cdot\vec{E} = \mathrm{Re}\, \vec{J}(\vec{x}) e^{-i\omega t} \cdot \mathrm{Re}\, \vec{E}(\vec{x}) e^{-i\omega t}$$
$$= \frac{1}{2}\left(\vec{J}(\vec{x})e^{-i\omega t} + \vec{J}^*(\vec{x})e^{i\omega t}\right)\cdot\frac{1}{2}\left(\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}^*(\vec{x})e^{i\omega t}\right)$$
$$= \frac{1}{4}\Big[\underbrace{\vec{J}\cdot\vec{E}e^{-2i\omega t} + \vec{J}^*\cdot\vec{E}^* e^{2i\omega t}}_{\text{oscillatory}} + \underbrace{\vec{J}^*\cdot\vec{E} + \vec{J}\cdot\vec{E}^*}_{\text{no t-dependence}}\Big]$$

oscillatory

so that, getting the time average over one period, we integrate out the t-dependence:

$$\langle \vec{J}\cdot\vec{E}\rangle = \frac{1}{4}(\vec{J}^*\cdot\vec{E} + \vec{J}\cdot\vec{E}^*) = \frac{1}{2}\mathrm{Re}[\vec{J}^*(\vec{x},t)\cdot\vec{E}(\vec{x},t)]$$

Now applying FTs to the Maxwell eqns, and assuming harmonic time dependence (defined for a single frequency ω),

$$\vec{\nabla}\cdot\vec{D} = \rho$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = i\omega\vec{B}$$
$$\vec{\nabla}\times\vec{H} = \vec{J} - i\omega\vec{D}$$

Then we recalculate Poynting's Theorem, this time starting from $\frac{1}{2}\mathrm{Re}(\vec{J}^*\cdot\vec{E})$:

$$\int\langle\vec{J}\cdot\vec{E}\rangle d^3x = \frac{1}{2}\mathrm{Re}\int\vec{J}^*\cdot\vec{E}\, d^3x$$
$$= \frac{1}{2}\mathrm{Re}\int(\vec{\nabla}\times\vec{H}^* - i\omega\vec{D}^*)\cdot\vec{E}\, d^3x$$
$$= \frac{1}{2}\mathrm{Re}\int\left[\vec{H}^*\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{H}^*) - i\omega(\vec{E}\cdot\vec{D}^*)\right] d^3x$$
$$= \frac{1}{2}\mathrm{Re}\int\left[\vec{H}^*\cdot i\omega\vec{B} - i\omega(\vec{E}\cdot\vec{D}^*) - \vec{\nabla}\cdot(\vec{E}\times\vec{H}^*)\right] d^3x$$

$-\vec{\nabla}\cdot(\vec{E}\times\vec{H}^*)]\, d^3x$

Thus, defining $\vec{S} \equiv \frac{1}{2}\vec{E}\times\vec{H}^*$ as the harmonic Poynting vector

$$= \frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\, d^3x = \frac{1}{2}\int [-i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*) - \vec{\nabla}\cdot 2\vec{S}]\, d^3x$$

$$= \int [-2i\omega(w_e - w_m) - \vec{\nabla}\cdot\vec{S}]\, d^3x$$

where $w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^*$ is the electric energy density & $w_m = \frac{1}{4}\vec{B}\cdot\vec{H}^*$ is the magnetic energy density. We see from the form of w_e & w_m that their time-averaged value over one period (related to the characteristic frequency ω) is

$$-i\omega\langle\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*\rangle = \langle\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\rangle - \frac{\partial u_{eff}}{\partial t}$$

$$= \omega\, \mathrm{Im}\, \epsilon(\omega)\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\rangle$$

$$+ \omega\, \mathrm{Im}\, \mu(\omega)\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\rangle$$

so that the effective EM energy density

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$$u_{eff} = \frac{1}{2} \mathrm{Re}\left[\frac{d(\omega\epsilon)}{d\omega} \epsilon(\omega)\right] \langle \vec{E}\cdot\vec{E} \rangle + \frac{1}{2} \mathrm{Re}\left[\frac{d(\omega\mu)}{d\omega} \epsilon(\omega)\right] \langle \vec{H}\cdot\vec{H} \rangle$$

using the approximation earlier.

Plane waves in a nonconducting medium

> Simplest waves: single frequency (defined constant ω), transverse (direction of propagation $\perp$ direction of oscillation), plane wave (can write in the form $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$)

> Simplest materials: linear ($\vec{D}=\epsilon\vec{E}$, $\vec{B}=\mu\vec{H}$), uniform & isotropic ($\epsilon,\mu \neq \epsilon(r,\varphi,\theta), \mu(r,\theta,\varphi)$), frequency-dependent ($\epsilon,\mu = \epsilon(\omega), \mu(\omega)$), no charges & currents present ($\rho=\vec{J}=0$)

> Start from Maxwell's eqns w/ no sources

$$\begin{aligned}
\vec{\nabla}\cdot\vec{D} &= 0 \\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\
\vec{\nabla}\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{H} &= \frac{\partial\vec{D}}{\partial t}
\end{aligned} \qquad (1)$$

Apply linearity & uniformity:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 0 \\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\
\vec{\nabla}\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{B} &= \mu(\omega)\epsilon(\omega)\frac{\partial\vec{E}}{\partial t}
\end{aligned} \qquad (2)$$

$$\vec{\nabla}\times\vec{B} = \mu(\omega)\epsilon(\omega)\frac{\partial \vec{E}}{\partial t}$$

Take the curl of the curl eqns & use the identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \vec{\nabla}^2\vec{F}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial \vec{B}}{\partial t}\right)$$
$$\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}})^{0} - \vec{\nabla}^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

no singularities for $\vec{B}(\vec{x},t)$

$$-\vec{\nabla}^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu(\omega)\epsilon(\omega)\frac{\partial \vec{E}}{\partial t}\right)$$

Apply FT to $\frac{\partial}{\partial t}$: $\frac{\partial}{\partial t} \to -i\omega$

$$\nabla^2\vec{E} + \mu(\omega)\epsilon(\omega)\omega^2\vec{E} = 0 \qquad (3)$$

Same goes for $\vec{B}$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu(\omega)\epsilon(\omega)\frac{\partial \vec{E}}{\partial t}\right)$$
$$\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{B}})^{0} - \vec{\nabla}^2\vec{B} = \mu(\omega)\epsilon(\omega)\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$
$$-\vec{\nabla}^2\vec{B} = \mu(\omega)\epsilon(\omega)\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$
$$\vec{\nabla}^2\vec{B} + \mu(\omega)\epsilon(\omega)\omega^2\vec{B} = 0 \qquad (4)$$

We then encounter two Helmholtz PDEs with the ansatz (for 1D)

$$\vec{B},\vec{E}(x,t) \propto e^{i(kx-\omega t)} \qquad (5)$$

Then, applying to (3) & (4),

$$\left(\frac{\partial^2}{\partial x^2} + \mu\epsilon\omega^2\right)e^{i(kx-\omega t)} = 0$$
$$\left(-k^2 + \mu\epsilon\omega^2\right)e^{i(kx-\omega t)} = 0 \qquad (\ldots)$$

$\underbrace{\qquad}_{0}$ $\underbrace{\qquad}_{\text{Cannot vanish}}$ $\neq 0$

Thus, k is related to ω by $k = \pm\sqrt{\mu\epsilon}\,\omega$ (dispersion relation); from the form of the ansatz, the phase velocity $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$, where c is the speed of light & $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$ is the index of refraction of the material. Thus, in one dimension, the general solution is

$$u(x,t) = a\underbrace{e^{i(kx-\omega t)}}_{\text{traveling to } +x\text{-direction}} + b\underbrace{e^{-i(kx+\omega t)}}_{\text{traveling to } -x\text{-direction}} \qquad (7)$$

If we use the basic dispersion relation $v = \frac{\omega}{k}$, we get

$$u(x,t) = ae^{i(x-vt)} + be^{-i(x+vt)} \qquad (8)$$

If $v \to$ frequency-independent, then the gen. soln. involves waves of speed v traveling to both + & − x directions. For not-so-basic dispersion relations, eq. (8) would not hold. This then portrays a wave that changes in shape as time progresses.

> Now we want to determine how k relates to both $\vec{B}$ & $\vec{E}$, for which we lost the relation due to our transformation to Helmholtz. We can plug back our ansatz (5) back to Maxwell's eqns. but first, we go 3D. Let

$$\vec{E}(\vec{x},t) = \vec{\xi}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (9)$$
$$\vec{B}(\vec{x},t) = \vec{\beta}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

where $\mathrm{Re}\,\vec{E}$ & $\mathrm{Re}\,\vec{B}$ are the physical electric & magnetic fields. Now we know that (9) satisfies the Helmholtz PDEs provided $\vec{k}\cdot\vec{k} = \mu\epsilon\omega^2$. Plugging back to the divergence eqns (2a) & (2c),

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial}{\partial x_i}\hat{e}_i E_i = 0 \qquad (10)$$
$$= i(\vec{k}\cdot\vec{E}) = 0$$

Thus, $\vec{k}\perp\vec{E}$ and $\vec{k}\perp\vec{B}$. If $\vec{k}$ is the wave vector, it must describe the direction where the wave goes. Thus direction where the wave goes. Thus the oscillations of $\vec{E}$ & $\vec{B}$ are perpendicular to the propagation $\longrightarrow$ transverse wave.

Applying the ansatz to the curl eqns (2b) & (2d),

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\epsilon_{ijk}\,\hat{e}_i \frac{\partial}{\partial x_j}(\vec{E})_k = \omega \vec{B}$$

$$(\vec{k} \times \vec{E}) = \omega \vec{B} \longrightarrow \quad (\vec{k} \times \vec{E}) = \omega \vec{B} \qquad (11)$$

setting $\vec{k} = k\hat{n}$, where k is the wavenumber, and using the simple dispersion relation, we get $(\hat{n} \times \vec{E}) = \frac{\omega}{k}\vec{B} = \frac{1}{\sqrt{\mu\epsilon}}\vec{B}$,

Thus $\vec{E}$ & $\frac{1}{\sqrt{\mu\epsilon}}\vec{B}$ have the same dimensions.

For the next curl eqn,

$$\vec{\nabla} \times \vec{B} = \mu(\omega)\,\epsilon(\omega)\frac{\partial \vec{E}}{\partial t}$$

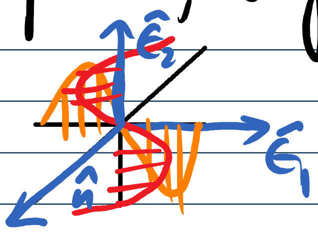
$$k\hat{n} \times \vec{B} = \mu(\omega)\,\epsilon(\omega)\,\omega \vec{E}$$

$$\hat{n} \times \vec{B} = \frac{1}{\sqrt{\mu\epsilon}}\vec{E} \qquad (12)$$

so that $\vec{E} \perp \vec{B}$ and $\vec{E}, \vec{B}, \hat{n}$ form a right-handed triplet, as we see for $\vec{E} \times \vec{B} = \sqrt{\mu\epsilon}\,|\vec{E}|^2\,\hat{n}$.

> For real $\hat{n}$, $\vec{E}$ & $\vec{B}$ are in phase. Then we can solve for $\vec{\mathcal{E}}$ & $\vec{\beta}$ (field strengths) by designating the unit vectors for the right-handed system, say $(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n})$.

Then $\hat{\epsilon}_1$ is the direction for $\vec{\mathcal{E}}$, or



$\vec{\mathcal{E}} = \hat{\epsilon}_1 E_0$.

For $\vec{\beta}$, we get $\vec{\beta} = \hat{\epsilon}_2 \sqrt{\mu\epsilon}\, E_0$.

> We can find the power density $\vec{S}$ by recalling its time-averaged value from §6.8:

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^* \quad (13)$$

$$= \frac{1}{2\mu} \vec{E} \times \vec{B}^*$$

$$= \frac{1}{2\mu} \sqrt{\mu\epsilon}\, |\vec{E}|^2 \hat{n} \quad (14)$$

We also recall the time-averaged energy density (over volume) as

$$\langle u \rangle = \frac{1}{4} (\vec{E} \cdot \vec{D}^* + \vec{B} \cdot \vec{H}^*) \quad (15)$$

$$= \frac{1}{4} \left(\epsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^*\right)$$

$$= \frac{1}{4} \left(\frac{\epsilon}{\mu\epsilon} \vec{B} \cdot \vec{B}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^*\right)$$

$$= \frac{1}{2\mu} |\vec{B}|^2 \quad (16)$$

$$= \frac{1}{2\mu}|\vec{B}|^2 \qquad (16)$$

$$= \frac{1}{4}\left(\epsilon \vec{E}\cdot\vec{E}^* + \frac{\mu\epsilon}{\mu}\vec{E}\cdot\vec{E}^*\right)$$

$$= \frac{1}{2}\epsilon|\vec{E}|^2 \qquad (17)$$

$\left(\frac{\langle S \rangle}{\langle u \rangle} = \frac{1}{\sqrt{\mu\epsilon}} = v\right) \to$ Poynting vector moves at phase speed

> We can further make $\hat{n}$ complex to get the most general solution. Let $\hat{n} = \hat{n}_R + i\hat{n}_I$. Then

$$e^{i(\vec{k}\cdot\vec{x}-\omega t)} = e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$

$$= e^{i(k(\hat{n}_R + i\hat{n}_I)\cdot\vec{x}-\omega t)}$$

$$= e^{i(k\hat{n}_R\cdot\vec{x} + ik\hat{n}_I\cdot\vec{x}-\omega t)}$$

$$= e^{-k\hat{n}_I\cdot\vec{x} + i(k\hat{n}_R\cdot\vec{x}-\omega t)}$$

$\hookrightarrow$ exponential growth/decay $\qquad$ $\hookrightarrow$ oscillatory terms

This wave is called an inhomogeneous plane wave. Since it's a unit vector, we see that

$$\hat{n}\cdot\hat{n} = 1$$

$$(\hat{n}_R + i\hat{n}_I)\cdot(\hat{n}_R + i\hat{n}_I) = 1$$

$$\hat{n}_R^2 + 2i(\hat{n}_R\cdot\hat{n}_I) - \hat{n}_I^2 = 1 \qquad (18)$$

Re: $\hat{n}_R^2 - \hat{n}_I^2 = 1$ } hyperbolic magnitudes

Im: $\underbrace{\hat{n}_R\cdot\hat{n}_I}_{\hat{n}_R \perp \hat{n}_I} = 0$

$\hat{n}_R \perp \hat{n}_I$

Thus, if we write $\hat{n} = \hat{n}_1 \cosh\theta + i\hat{n}_2 \sinh\theta$, we satisfy both Re(18) & Im(18).

If we are to satisfy $\hat{n}\cdot\vec{\varepsilon} = 0$, $\vec{\varepsilon}$ becomes

$$\vec{\varepsilon} = \hat{n}_1 (i\varepsilon \sinh\theta) + \hat{n}_2 (-\varepsilon \cosh\theta) + \hat{n}_3 \varepsilon'$$

where we see that for $\theta = 0$, $\vec{\varepsilon} = -\varepsilon \hat{n}_2$ which corresponds to the real $\hat{n}$ case. //

Linear & Circular Polarization; Stokes Parameters

> Polarization: "restricting the vibrations of a transverse wave to one direction"

> Recall the right-handed system $(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n})$ previously established. Then, the EM wave is polarized in two ways:

① Restricting $\hat{n}$ to be real & $\vec{E}$ to be in the $\hat{\epsilon}_1$ direction only

$$\vec{E}(\vec{x},t) = \hat{\epsilon}_1 E_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = \hat{\epsilon}_2 \sqrt{\mu\epsilon}\, E_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (1)$$

② Restricting $\hat{n}$ to be real & $\vec{E}$ to be in the $\hat{\epsilon}_2$ direction only

$$\vec{E}(\vec{x},t) = \hat{\epsilon}_2 E_0' e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = -\hat{\epsilon}_1 \sqrt{\mu\epsilon}\, E_0' e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (2)$$

In general, when $\vec{E}$ is restricted to the plane of $\hat{\epsilon}_1$ & $\hat{\epsilon}_2$,

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_0 + \hat{\epsilon}_2 E_0') e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = \sqrt{\mu\epsilon}\,(-\hat{\epsilon}_1 E_0' + \hat{\epsilon}_2 E_0) e^{i(k\hat{n}\cdot\vec{x}-\omega t)} \quad (3)$$

where E_0 & E_0' are generally complex.

> For E_0 & E_0' having the same phase, eq. (3) is said to be linearly polarized, making an angle $\tan^{-1}(E_0'/E_0)$ wrt $\hat{\epsilon}_1$ and magnitude $\sqrt{E_0^2 + E_0'^2}$.

> For E_0 & E_0' having diff. phases, eq. (3) is said to be elliptically polarized.
A special case is when $E_0' = \pm i E_0$, or that they have the same magnitude but out-of-phase by $90°$. Then, eq. (3) is said to be circularly polarized. Then eq. (3) becomes

$$\vec{E}(\vec{x},t) = E_0 (\hat{\epsilon}_1 \pm i \hat{\epsilon}_2) e^{i(\vec{k}\cdot\vec{x} - \omega t)} \quad (4)$$

If we take $\hat{z}$ to be $\hat{n}$, and $\hat{\epsilon}_1 \to \hat{x}$ & $\hat{\epsilon}_2 \to \hat{y}$, we see that the physical electric field is given by

$$\begin{aligned} \mathrm{Re}\, \vec{E}(\vec{x},t) &= E_0 \,\mathrm{Re}\, (\hat{x} \pm i\hat{y}) e^{i(\vec{k}\cdot\vec{x} - \omega t)} \\ &= E_0 \,\mathrm{Re}\, (\hat{x} \pm i\hat{y}) [\cos(\vec{k}\cdot\vec{x} - \omega t) \\ &\quad + i \sin(\vec{k}\cdot\vec{x} - \omega t)] \end{aligned}$$

right circularly
left circularly polarized

$= E_0 \left[\hat{x} \cos(\vec{k}\cdot\vec{x} - \omega t) \mp \hat{y} \sin(\vec{k}\cdot\vec{x} - \omega t)\right]$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{circular motion of frequency } \omega}$

Nb. right circularly polarized = negative helicity, right-handed wave

left " " = positive helicity left-handed wave

> We can create another set of basis vectors from eq. (4): define $\hat{\epsilon}_\pm$:

$$\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2) \quad (5)$$

~~Having~~ properties

$$\hat{\epsilon}^*_\pm \cdot \hat{\epsilon}_\mp = \frac{1}{2}(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2)\cdot(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2)$$

$$= \frac{1}{2}(\hat{\epsilon}_1^2 \pm \hat{\epsilon}_2^2 \mp 2i\hat{\epsilon}_1\cdot\hat{\epsilon}_2) = 0$$

$\hat{n} \longrightarrow$

$$\hat{\epsilon}^*_\pm \cdot \hat{\epsilon}_3 = 0$$

$$\hat{\epsilon}^*_\pm \cdot \hat{\epsilon}_\pm = \frac{1}{2}(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2)\cdot(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)$$

$$= \frac{1}{2}(\hat{\epsilon}_1 + \hat{\epsilon}_2) = 1$$

We can then recast (4) in terms of $\hat{\epsilon}_\pm$ as

$$\vec{E}(\vec{x},t) = (E_+\hat{\epsilon}_+ + E_-\hat{\epsilon}_-)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \quad (6)$$

where $E_\pm$ are generally complex.

> We can get different polarizations for $\vec{E}$

by determining the relationship bet. E_+ & E_-.

e.g. If $|E_+| \neq |E_-|$, but their phases are equal, then,

$$\vec{E}(\vec{x},t) = (|E_+|\,\hat{\epsilon}_+ + |E_-|\,\hat{\epsilon}_-)\,e^{i(\vec{k}\cdot\vec{x}-\omega t+\theta)}$$

↳ traces an ellipse with semi-minor axis

$$\left|\frac{E_- + E_+}{E_- - E_+}\right| = \left|\frac{1-r}{1+r}\right|, \quad r = \frac{E_-}{E_+}$$

e.g. If they have diff phases,

$$\vec{E}(\vec{x},t) = (|E_+|\,\hat{\epsilon}_+ + |E_-|\,\hat{\epsilon}_-)\,e^{i(\vec{k}\cdot\vec{x}-\omega t+\theta_+ + \theta_-)}$$

or, if $\frac{E_-}{E_+} = re^{i\alpha}$, $\vec{E}(\vec{x},t) = (|E_+|\hat{\epsilon}_+ + |E_-|\hat{\epsilon}_-)\,e^{i(\vec{k}\cdot\vec{x}-\omega t+\theta+\frac{\alpha}{2})}$

and if $r = \pm 1$, we get back (5).

> We consider the converse problem: Given

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

(& its corresponding $\vec{B}$), how do we determine (E_1, E_2) or $(E_\pm)$? This can be solved by using the Stokes parameters

$$\vec{S} \equiv \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} |\vec{E}|^2 \\ |E_1|^2 - |E_2|^2 \\ 2\,\mathrm{Re}(E_1^* E_2) \\ 2\,\mathrm{Im}(E_1^* E_2) \end{bmatrix} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} \qquad (7)$$

> We motivate (7) by taking the scalar products of the basis vectors $(\hat{\epsilon}_{1,2,\pm})$

with the field:

$\hat{\epsilon}_1 \cdot \vec{E} = |\vec{E}|$ of linearly polarized wave in x

$\hat{\epsilon}_2 \cdot \vec{E} = |\vec{E}|$ " " " " " y

$\hat{\epsilon}_+^* \cdot \vec{E} = |\vec{E}|$ " polarized wave w/ positive helicity

$\hat{\epsilon}_-^* \cdot \vec{E} = |\vec{E}|$ " " " " negative "

Also, Intensity $\propto |\vec{E}|^2$, so these values can be infered. The phase can then be infered from the cross product.

By letting $E_{1,2} = a_{1,2} e^{i\delta_{1,2}}$, & $E_\pm = a_\pm e^{i\delta_\pm}$, we can get the Stokes parameters $s_{0,1,2,3}$ in 2 ways:

① in the linear polarization basis $(\hat{\epsilon}_{1,2})$

$$s_0 = |\hat{\epsilon}_1 \cdot \vec{E}|^2 + |\hat{\epsilon}_2 \cdot \vec{E}|^2 = |\hat{\epsilon}_1 \cdot (\hat{\epsilon}_1 a_1 e^{i\delta_1} + \cancel{\hat{\epsilon}_2 a_2 e^{i\delta_2}}) e^{i(\vec{k}\cdot\vec{x}-\omega t)}|^2$$
$$+ |\hat{\epsilon}_2 \cdot (\cancel{\hat{\epsilon}_1 a_1 e^{i\delta_1}} + \hat{\epsilon}_2 a_2 e^{i\delta_2}) e^{i(\vec{k}\cdot\vec{x}-\omega t)}|^2$$
$$= |a_1 e^{i(\vec{k}\cdot\vec{x}-\omega t+\delta_1)}|^2 + |a_2 e^{i(\vec{k}\cdot\vec{x}-\omega t+\delta_2)}|^2$$
$$= |a_1|^2 + |a_2|^2$$

$$s_1 = |\hat{\epsilon}_1 \cdot \vec{E}|^2 - |\hat{\epsilon}_2 \cdot \vec{E}|^2 = |a_1|^2 - |a_2|^2$$

$$s_2 = 2\mathrm{Re}[(\hat{\epsilon}_1 \cdot \vec{E})^* (\hat{\epsilon}_2 \cdot \vec{E})] = 2\mathrm{Re}[(a_1 e^{i(\vec{k}\cdot\vec{x}-\omega t+\delta_1)})^*$$
$$(a_2 e^{i(\delta_2+\vec{k}\cdot\vec{x}-\omega t)})] = 2\mathrm{Re}[a_1 a_2 e^{i(\delta_2-\delta_1)}]$$
$$= 2a_1 a_2 \cos(\delta_2 - \delta_1)$$

$= 2a_1 a_2 \cos(\delta_2 - \delta_1)$

$S_3 \equiv 2\,\mathrm{Im}[(\hat{\epsilon}_1 \cdot \vec{E})^*(\hat{\epsilon}_2 \cdot \vec{E})] = 2\,\mathrm{Im}[a_1 a_2 e^{i(\delta_2 - \delta_1)}]$

$= 2a_1 a_2 \sin(\delta_2 - \delta_1)$

(2) In terms of the circular basis vectors $\hat{\epsilon}_\pm$,

$S_0 = |\hat{\epsilon}_+^* \cdot \vec{E}|^2 + |\hat{\epsilon}_-^* \cdot \vec{E}|^2 = |\hat{\epsilon}_+^* \cdot (\hat{\epsilon}_+ a_+ e^{i\delta_+} + \hat{\epsilon}_- a_- e^{i\delta_-})|^2$

$+ |\hat{\epsilon}_-^* \cdot (\hat{\epsilon}_+ a_+ e^{i\delta_+} + \hat{\epsilon}_- a_- e^{i\delta_-})|^2$

$= a_+^2 + a_-^2$

$S_1 = 2\,\mathrm{Re}[(\hat{\epsilon}_+^* \cdot \vec{E})^*(\hat{\epsilon}_-^* \cdot \vec{E})] = 2\,\mathrm{Re}[a_+ a_- e^{i(\delta_- - \delta_+)}]$

$= 2a_+ a_- \cos(\delta_- - \delta_+)$

$S_2 = 2\,\mathrm{Im}[(\hat{\epsilon}_+^* \cdot \vec{E})(\hat{\epsilon}_- \cdot \vec{E})] = 2\,\mathrm{Im}[a_+ a_- e^{i(\delta_- - \delta_+)}]$

$= 2a_+ a_- \sin(\delta_- - \delta_+)$

$S_3 = |\hat{\epsilon}_+^* \cdot \vec{E}|^2 - |\hat{\epsilon}_-^* \cdot \vec{E}|^2 = a_+^2 - a_-^2$

From either basis, we see the relation

$$S_0^2 = (a_1^2 + a_2^2)^2 = (a_1^2)^2 + 2a_1^2 a_2^2 + a_2^2$$

$$S_1^2 = (a_1^2 - a_2^2)^2 = (a_1^2)^2 - 2a_1^2 a_2^2 + a_2^2$$

$$S_2^2 = 4a_1^2 a_2^2 \cos^2(\delta_2 - \delta_1)$$

$$S_3^2 = 4a_1^2 a_2^2 \sin^2(\delta_2 - \delta_1)$$

so that $S_0^2 = S_1^2 + S_2^2 + S_3^2$.

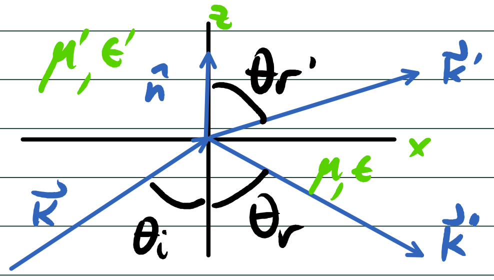
Reflection & Refraction of EM waves at a plane interface bet. dielectrics

> Due to the wave nature of light, there are diff. properties we see when light hits a boundary. Some are due to its kinematic properties ($\theta_r = \theta_i$, $n_i \sin\theta_i = n_r \sin\theta_r$), and some are due to the dynamic properties (Intensities, phase changes, polarization).

> We first clarify the notation before we solve anything:

$n(z<0) = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$

$n'(z>0) = \sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}$



Then, the three waves that meet at $x=z=0$ are, according to (7.18),

① Incident

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad (1)$$
$$\vec{B} = \sqrt{\mu\epsilon}\,\frac{\vec{k}\times\vec{E}}{k}$$

② Refracted

$$\vec{E}' = \vec{E}_0' e^{i(\vec{k}'\cdot\vec{x} - \omega t)} \qquad (2)$$

$$\vec{E}' = \vec{E}_0' e^{i(\vec{k}'\cdot\vec{x} - \omega t)}$$
$$\vec{B}' = \sqrt{\mu'\epsilon'} \frac{\vec{k}' \times \vec{E}'}{k'} \quad (2)$$

③ Reflected

$$\vec{E}'' = \vec{E}_0'' e^{i(\vec{k}''\cdot\vec{x} - \omega t)}$$
$$\vec{B}' = \sqrt{\mu\epsilon} \frac{\vec{k}'' \times \vec{E}''}{k} \quad (3)$$

for $k = |\vec{k}| = \omega\sqrt{\mu\epsilon} = |\vec{k}''|$ and
$k' = |\vec{k}'| = \omega\sqrt{\mu'\epsilon'}$.

> Since we need to get BCs at $z=0$, the plane wave factors $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ must be satisfied for all t & all $\vec{x}$ at $z=0$, which means

$$\omega = \omega' = \omega''$$

$$e^{i(\vec{k}\cdot\vec{x} - \omega t)} = e^{i(\vec{k}'\cdot\vec{x} - \omega t)} = e^{i(\vec{k}''\cdot\vec{x} - \omega'' t)}\Big|_{z=0}$$

$$(k_x x + k_z z)_{z=0} = (k_x' x + k_z' z)_{z=0} = (k_x'' x + k_z'' z)_{z=0} \quad (4)$$

projection onto $\vec{x}$

$$k_x = k_x' = k_x''$$

all equal → all in a plane!

This condition already contains the kinematic aspects of reflection & refraction:

$$k_x = k \underbrace{\sin\theta_i = k_x'' = k \sin\theta_r}_{\theta_i = \theta_r} \quad (5)$$

$$k_x = k_x'$$

$$k \sin\theta_i = k' \sin\theta_{r'} \rightarrow k\sin\theta_i = k'\sin\theta_{r'} \quad (6)$$

Since $k = \omega\sqrt{\mu\epsilon}$, $k' = \omega\sqrt{\mu'\epsilon'}$,

$$(6) = \sqrt{\mu\epsilon}\sin\theta_i = \sqrt{\mu'\epsilon'}\sin\theta_{r'} \quad (7)$$

$$n\sin\theta_i = n'\sin\theta_{r'} \quad (8)$$

> The dynamic properties depend more on the BCs themselves, including the type of EM wave present. For fields w/ no sources,

① $$(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = 0$$

$$[\epsilon(\vec{E}_0 + \vec{E}_0'') - \epsilon'\vec{E}_0']\cdot\hat{n} = 0 \quad (10)$$

② $$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0$$

$$[(\vec{k}\times\vec{E}_0 + \vec{k}\times\vec{E}_0'') - (\vec{k}'\times\vec{E}_0')]\cdot\hat{n} = 0 \quad (11)$$

③ $$(\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0$$

$$[(\vec{E}_0 + \vec{E}_0'') - \vec{E}_0']\times\hat{n} = 0 \quad (12)$$

④ $$(\vec{H}_2 - \vec{H}_1)\times\hat{n} = 0$$

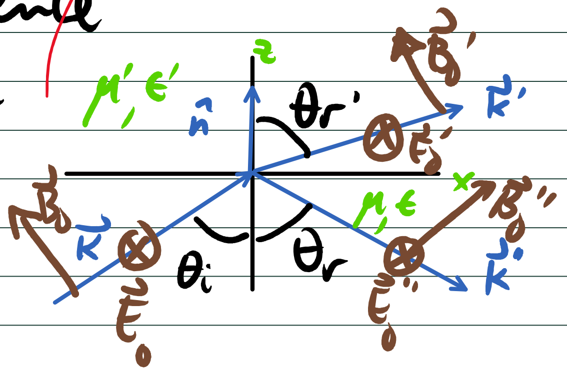
$$[\tfrac{1}{\mu}(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0'') - \tfrac{1}{\mu'}(\vec{k}'\times\vec{E}_0')]\times\hat{n} = 0 \quad (13)$$

> To get the most general solution of elliptical polarization, we first separate the parallel & perpendicular polarizations (wrt the plane of incidence defined by $\vec{k}$'s and $\hat{n}$).

① $\vec{E}_0 \perp$ plane of incidence

① $\vec{E}_0 \perp$ plane of incidence

$= \vec{E}_0$ only has a component parallel to the interface ($z=0$ plane)



Since $\vec{E}_0 \parallel z=0$ plane, no $\vec{E}_0 \perp z=0$ is present, so normal BC (10) is irrelevant: from (12),

$$\left[\left(\vec{E}_0 + \vec{E}_0''\right) \times \hat{n}\right]_z = \left[\vec{E}_0' \times \hat{n}\right]_z$$

$$E_0 + E_0'' = E_0' \qquad (14)$$

& from (13),

$$\frac{1}{\mu}\left(\vec{k}\times\vec{E} + \vec{k}''\times\vec{E}''\right)\times\hat{n} = \frac{1}{\mu'}\left(\vec{k}'\times\vec{E}'\right)\times\hat{n}$$

$$\underbrace{\left(\vec{E}\times\left(\vec{k}''\times\hat{n}\right)\right)_y}_{-\omega\sqrt{\mu\epsilon}\,E_0'\cos\theta_r}$$

$$\underbrace{\sqrt{\frac{\epsilon}{\mu}}}_{Z} E_0 \cos\theta_i = \sqrt{\frac{\epsilon}{\mu}} E_0'' \cos\theta_i + \underbrace{\sqrt{\frac{\epsilon'}{\mu'}}}_{Z'} E_0' \cos\theta_{r'} \qquad (15)$$

Now we need to solve for E_0' & E_0'' in terms of E_0:

$$Z E_0 \cos\theta_i = Z E_0'' \cos\theta_i + Z'(E_0'' + E_0)\cos\theta_{r'}$$

$$(Z\cos\theta_i - Z'\cos\theta_{r'})E_0 = (Z\cos\theta_i + Z'\cos\theta_{r'})E_0''$$

$$E_0'' = \frac{Z\cos\theta_i - Z'\cos\theta_{r'}}{\quad}$$

$$E_0'' = \frac{Z\cos\theta_i - Z'\cos\theta_r'}{Z\cos\theta_i + Z'\cos\theta_r'} E_0$$

$$= \frac{\mu' n\cos\theta_i - \mu n'\sqrt{1-\frac{n^2}{n'^2}\sin^2\theta_i}}{\mu' n\cos\theta_i + \mu n'\sqrt{1-\frac{n^2}{n'^2}\sin^2\theta_i}} E_0$$

$$= \frac{n\cos\theta_i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2-n^2\sin^2\theta_i}} E_0 \quad (16)$$

$$Z E_0 \cos\theta_i = Z(E_0' - E_0)\cos\theta_i + Z' E_0' \cos\theta_r'$$

$$2 Z E_0 \cos\theta_i = (Z\cos\theta_i + Z'\cos\theta_r') E_0'$$

$$E_0' = \frac{2Z\cos\theta_i}{Z\cos\theta_i + Z'\cos\theta_r'} E_0$$

$$= \frac{2\mu' n\cos\theta_i}{\mu n\cos\theta_i + \mu n'\sqrt{1-\frac{n^2}{n'^2}\sin^2\theta_i}} E_0$$

$$= \frac{2n\cos\theta_i}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}} E_0 \quad (17)$$

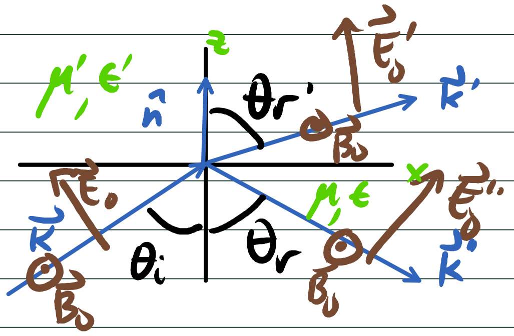
Here, (16) & (17) are called the Fresnel formulas.

(We note that the 2nd BC (11) yields $\sqrt{\mu'\epsilon'}E_0'\sin\theta_r' = \sqrt{\mu\epsilon}\sin\theta_i (E_0 + E_0'')$, similar to (15)).

> For E ∥ plane of incidence, everything

is normal to the interface, and so eq. (11) is irrelevant.

Using the tangential conditions (12)-(13),



$$\left[\left(\vec{E}_0+\vec{E}_0''\right)\times\hat{n} = \vec{E}_0'\times\hat{n}\right]_z$$

same procedure from (14),

$$(E_0-E_0'')\cos\theta_i = E_0'\cos\theta_{r'} \qquad (18)$$

and from BC @ (13),

$$\left(\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''\right)-\frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n}=0\right)_y$$

$$\frac{\sqrt{\mu\epsilon}}{\mu}(E_0+E_0'') - \frac{\sqrt{\mu'\epsilon'}}{\mu'}E_0' = 0 \qquad (19)$$

Again, solving for E_0' & E_0'' in terms of E_0,

$$\frac{\cos\theta_i}{\cos\theta_{r'}}(E_0-E_0'') = \frac{\mu'}{\mu}\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}(E_0+E_0'')$$

$$\left(\frac{\cos\theta_i}{\cos\theta_{r'}} - \frac{\mu'}{\mu}\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\right)E_0 = \left(\frac{\mu'}{\mu}\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}+\frac{\cos\theta_i}{\cos\theta_{r'}}\right)E_0''$$

$$E_0'' = \frac{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i - \sqrt{\frac{\epsilon}{\mu}}\cos\theta_{r'}}{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i + \sqrt{\frac{\epsilon}{\mu}}\cos\theta_{r'}}E_0 \qquad (20)$$

$$\frac{1}{\cos\theta_i}\left(E_0\cos\theta_i - E_0'\cos\theta_{r'}\right) = E_0''$$

$$\sqrt{\frac{\epsilon}{\mu}}\left[E_0 + \frac{1}{\cos\theta_i}\left(E_0\cos\theta_i - E_0'\cos\theta_{r'}\right)\right] = \sqrt{\frac{\epsilon'}{\mu'}}E_0'$$

$$2 z E_0 = z' E_0' \left(1 + \frac{\cos\theta_r'}{\cos\theta_i'}\right)$$

$$E_0' = \frac{2z/z'}{1 + \frac{\cos\theta_r'}{\cos\theta_i'}} E_0$$

$$E_0' = \frac{2nn'\cos\theta_i}{\frac{\mu}{\mu'} n'^2 \cos\theta_i + n\sqrt{n'^2 - n^2 \sin\theta_i}} E_0 \quad (21)$$

For non-magnetic materials $\frac{\mu}{\mu'} = 1$.
For normal incidence ($\theta_i' = 0$), (16)-(17) or (20)-(21) reduce to

$$\frac{E_0'}{E_0} = \frac{2nn'}{n'^2 + nn'} = \frac{2}{\frac{n'}{n} + 1} = \frac{2n}{n' + n}$$

$$\frac{E_0''}{E_0} = \frac{\sqrt{\epsilon'} - \sqrt{\epsilon}}{\sqrt{\epsilon'} + \sqrt{\epsilon}} = \frac{n' - n}{n' + n}$$

→ phase shift when $n' > n$