# Problems w/ azimuthal symmetry

> Conditions for azimuthal symmetry:

Recall the ~~most~~ general solution for $\Phi(r,\theta,\varphi)$ including the whole azimuthal sweep:

$$\Phi(r,\theta,\varphi) = \sum_m \sum_\ell (A_\ell r^\ell + B_\ell r^{-\ell-1})(A_m e^{im\varphi} + B_m e^{-im\varphi}) \quad (1)$$

Associated Legendre $\rightarrow P_\ell^m(\cos\theta)$

$$\boxed{\text{Azimuthal symmetry is then achieved when } m=0 \text{ AND the region of interest includes } \varphi \in [0, 2\pi].}$$

> with this condition, $P_\ell^{m=0}$ reduces to the Ordinary Legendre polynomials (Lec 12-13)

E.g. A sphere of radius $a$ with BC $\Phi(r=a,\theta) = V(\theta)$ and we want to find $\Phi(r<a)$. Since this includes $r=0$, we immediately conclude

$B_\ell = 0$, so that

$$\Phi(r<a,\theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

At $r=a$,

$$\Phi(r=a,\theta) = V(\theta) = \sum_\ell A_\ell a^\ell P_\ell(\cos\theta)$$

Use the summation killer aka $\int^{\ell'}$ low

Use the summation killer, aka $\delta_{\ell}^{\ell'}$ by multiplying both sides with $P_{\ell'}(\cos\theta)\sin\theta$ and using the orthogonality condition

$$\int_0^{\pi} P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$\Rightarrow \int_0^{\pi} V(\theta)\, P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \sum_\ell A_\ell a^\ell \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$= A_{\ell'} a^{\ell'} \frac{2}{2\ell'+1}$$

so that our expression for $A_\ell$ becomes

$$A_\ell = \frac{2\ell+1}{2a^\ell}\int_0^{\pi} V(\theta)\, P_\ell(\cos\theta)\sin\theta\, d\theta$$

that completes the solution. ■

Eg. Two hemispherical shells with radii $a$, & BCs

$\Phi_{bottom}(r=a) = 0$ & $\Phi_{top}(r=a) = V$

We can directly exploit the solution obtained earlier; we just make use of two diff. BCs. To further ease our calculations, we can exploit the recursion relations for $P_\ell$ (and subsequently, for $A_\ell$).

We start with $A_0$ ($P_0 = 1$). Recall

(whole sphere) $$A_\ell = \frac{2\ell+1}{2}\int_0^{\pi} V(\theta)\, P_\ell(\cos\theta)\sin\theta\, d\theta.$$

But $V(\theta)$ is const., so $A_0$ becomes

$$A_0 = \frac{1}{2}V\int_0^{\pi/2} P_0(\cos\theta)\sin\theta\, d\theta$$ (top hemisphere)

$$= \frac{1}{2}V\int_0^{\pi/2}\sin\theta\, d\theta$$

$$= \frac{1}{2}V$$

Then we can use $P_\ell(x) = \frac{1}{2\ell+1}\frac{d}{dx}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right]$ for $\ell > 0$ to get

$$A_{\ell>0} = \frac{2\ell+1}{2}V\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta\, d\theta$$

(let $x = \cos\theta$, $dx = -\sin\theta\, d\theta$, $\theta = 0 \to x = 1$, $\theta = \pi/2 \to x = 0$)

$$A_{\ell>0} = \frac{2\ell+1}{2}V\int_0^1 P_\ell(x)\, dx$$

$$= \frac{\cancel{2\ell+1}}{2}V\int_0^1 \frac{1}{\cancel{2\ell+1}}\frac{d}{\cancel{dx}}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right]\cancel{dx}$$

$$= \frac{V}{2}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right]\Big|_0^1$$

$$= \frac{V}{2}\left[\cancel{P_{\ell+1}(1)} - P_{\ell+1}(0) - \cancel{P_{\ell-1}(1)} + P_{\ell-1}(0)\right]$$

$$= \frac{V}{2}\left[P_{\ell-1}(0) - P_{\ell+1}(0)\right]$$

so that

$$\Phi(r<a,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{\ell=1}^{\infty}\left[P_{\ell-1}(0) - P_{\ell+1}(0)\right]\left(\frac{r}{a}\right)^\ell P_\ell(\cos\theta)$$