Problem 1

(a) We first consider the eigenvalue problem in the 3D momentum representation

$$\hat{\mathbf{p}} \left|\mathbf{p}\right\rangle = \mathbf{p} \left|\mathbf{p}\right\rangle \tag{1}$$

which is possible since $\hat{\mathbf{p}}$ is a vector of commuting, Hermitian operators. Each pair of operators in this vector then must possess a simultaneous eigenbasis. We can then represent the wavefunction of the state $|\psi\rangle$ in the momentum representation as $\phi(\mathbf{p}) \equiv \langle \mathbf{p}|\psi\rangle$. The question asks to represent $\hat{\mathbf{p}}\,|\psi\rangle$ on momentum basis. We do this by multiplying $\langle \mathbf{p}|$:

$$\langle \mathbf{p}|\hat{\mathbf{p}}|\psi\rangle = \mathbf{p}\,\langle \mathbf{p}|\psi\rangle \tag{2}$$

where we deliberately made the operator $\hat{\mathbf{p}}$ act on $\langle \mathbf{p}|$ by translating equation (1) into an eigenbra problem:

$$\langle \mathbf{p}|\,\hat{\mathbf{p}} = \mathbf{p}\,\langle \mathbf{p}| \tag{3}$$

which is equivalent to (1) because $\hat{\mathbf{p}}$ is a Hermitian and is not affected by conjugation. We then see that the action of $\hat{\mathbf{p}}$ on $\langle \mathbf{p}|\psi\rangle$ is to multiply it by $\mathbf{p}$, which we can rewrite as

$$\boxed{(\hat{\mathbf{p}}\phi)(\mathbf{p}) = \mathbf{p}\phi(\mathbf{p})} \tag{4}$$

(b) We now show the action of the same operator on $\langle \mathbf{x}|\psi\rangle$, which corresponds to the position representation of the state ψ. Recall in 1D that the momentum operator can be defined from the generator of infinitesimal translation operators T(a) by defining

$$\hat{p} \equiv i\hbar \frac{dT}{da}\bigg|_{a=0}. \tag{5}$$

Making it act on $\psi(x)$, we get

$$(\hat{p}\psi)(x) = i\hbar \left(\frac{dT(a)}{da}\psi\right)(x)\bigg|_{a=0} \tag{6}$$

$$= i\hbar \frac{d}{da}\psi(x-a)\bigg|_{a=0} \tag{7}$$

$$= -i\hbar \frac{d\psi}{dx} \tag{8}$$

where we used the action of a translation operator on a wave function, $(T(a)\psi)(x) = \psi(x-a)$ and the chain rule at $x+a=0$. This is true for all directions, and so for the 3D case,

$$\boxed{(\hat{\mathbf{p}}\psi)(\mathbf{x}) = -i\hbar\nabla\psi(\mathbf{x})}. \tag{9}$$

■

Problem 2

(a) In this problem, we explore how the translation operator changes the expectation values of both the position and momentum operators.

For $\langle \mathbf{x} \rangle_\psi = \langle \psi | \hat{\mathbf{x}} | \psi \rangle$, let $|\phi\rangle = T(\mathbf{a})\,|\psi\rangle$ be the translated state ket. Getting the expectation value of $|\phi\rangle$, we get

$$\langle \phi | \hat{\mathbf{x}} | \phi \rangle = \langle \psi | T(\mathbf{a})^\dagger \hat{\mathbf{x}} T(\mathbf{a}) | \psi \rangle . \tag{1}$$

We then use the commutation relations of the translation and position operators:

$$\begin{aligned} [\hat{\mathbf{x}}, T(\mathbf{a})] &= \hat{\mathbf{x}} T(\mathbf{a}) - T(\mathbf{a}) \hat{\mathbf{x}} & (2) \\ &= \mathbf{a} T(\mathbf{a}). & (3) \end{aligned}$$

Multiply both sides by $T(\mathbf{a})^{-1}$ from the left:

$$\begin{aligned} T(\mathbf{a})^{-1} \hat{\mathbf{x}} T(\mathbf{a}) - T(\mathbf{a})^{-1} T(\mathbf{a}) \hat{\mathbf{x}} &= T(\mathbf{a})^{-1} \mathbf{a} T(\mathbf{a}) & (4) \\ T(\mathbf{a})^{-1} \hat{\mathbf{x}} T(\mathbf{a}) &= \hat{\mathbf{x}} + \mathbf{a}\mathbb{I}, & (5) \end{aligned}$$

where $\mathbb{I}$ is the identity operator. We then exploit the fact that the translation operator is unitary so we can change $T(\mathbf{a})^{-1}$ to $T(\mathbf{a})^\dagger$, which we can replace (1) with to make

$$\begin{aligned} \langle \phi | \hat{\mathbf{x}} | \phi \rangle &= \langle \psi | \hat{\mathbf{x}} + \mathbf{a}\mathbb{I} | \psi \rangle & (6) \\ \langle \phi | \hat{\mathbf{x}} | \phi \rangle &= \langle \psi | \hat{\mathbf{x}} | \psi \rangle + \langle \psi | \mathbf{a}\mathbb{I} | \psi \rangle & (7) \\ \langle \phi | \hat{\mathbf{x}} | \phi \rangle &= \langle \psi | \hat{\mathbf{x}} | \psi \rangle + \mathbf{a}\mathbb{I} & (8) \end{aligned}$$

where we used the normalization condition for $|\psi\rangle$. We see that the translation operator just did as it was defined: it translates the expectation value by some vector $\mathbf{a}$.

(b) For the momentum expectation value, however, we repeat the procedure above, except using the commutation relation

$$\begin{aligned} [\hat{\mathbf{p}}, T(\mathbf{a})] &= 0 \\ \hat{\mathbf{p}} T(\mathbf{a}) &= T(\mathbf{a}) \hat{\mathbf{p}} \\ T(\mathbf{a})^{-1} \hat{\mathbf{p}} T(\mathbf{a}) &= T(\mathbf{a})^{-1} T(\mathbf{a}) \hat{\mathbf{p}} = \hat{\mathbf{p}} \end{aligned} \tag{9}$$

Then, we can rewrite the momentum expectation value of $|\phi\rangle$ as

$$\begin{aligned} \langle \phi | \hat{\mathbf{p}} | \phi \rangle &= \langle \psi | T(\mathbf{a})^\dagger \hat{\mathbf{p}} T(\mathbf{a}) | \psi \rangle \\ \langle \phi | \hat{\mathbf{p}} | \phi \rangle &= \langle \psi | \hat{\mathbf{p}} | \psi \rangle \end{aligned} \tag{10}$$

which works as expected: the momentum expectation value does not change.

■

(a) A spin 1/2 system interacts with an external magnetic field explicitly through the spin S. This gives us a Hamiltonian of the form

$$H = -\frac{e}{mc}\mathbf{B}\cdot\mathbf{S} = -\frac{eB}{mc}S_z \tag{1}$$

where we set the constant magnetic field to be pointing along the z-axis $\mathbf{B} = B\hat{\mathbf{z}}$ so that only one S component would survive. Now we want to solve for $\mathbf{S}(t)$ by evolving the operator through the Heisenberg picture.

Recall the Heisenberg equation of motion for a general operator A as

$$\frac{dA_H}{dt} = -\frac{i}{\hbar}\left[A_H, H\right] \tag{2}$$

where A_H is the operator in the Heisenberg picture. We separate this equation for the different $\mathbf{S}$-components as follows:

$$\frac{dS_x}{dt} = -\frac{i}{\hbar}\left[S_x, H\right] \tag{3}$$
$$\frac{dS_y}{dt} = -\frac{i}{\hbar}\left[S_y, H\right] \tag{4}$$
$$\frac{dS_z}{dt} = -\frac{i}{\hbar}\left[S_z, H\right] \tag{5}$$

where H is given by equation (1). In order to solve this system of differential equations, we need to recall the commutation relations between the $\mathbf{S}$-components:

$$[S_i, S_i] = 0 \tag{6}$$
$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k \tag{7}$$

Then we are left with the following: let $\omega = \dfrac{eB}{mc}$, so $H = -\omega S_z$, and

$$\frac{dS_x}{dt} = -\frac{i}{\hbar}(-\omega\left[S_x, S_z\right]) = \frac{i\omega}{\hbar}(i\hbar\varepsilon_{xzy}S_y) \tag{8}$$
$$= -\omega S_y \tag{9}$$
$$\frac{dS_y}{dt} = -\frac{i}{\hbar}(-\omega\left[S_y, S_z\right] = \frac{i\omega}{\hbar}(i\hbar\varepsilon_{yzx}S_x) \tag{10}$$
$$= \omega S_x \tag{11}$$
$$\frac{dS_z}{dt} = -\frac{i}{\hbar}(0) = 0 \tag{12}$$

where we see that we are left with coupled first-order differential equations in S_x and S_y. This could be resolved by differentiating both sides:

$$\frac{d^2 S_x}{dt^2} = -\omega \frac{dS_y}{dt} = -\omega(\omega S_x) \tag{13}$$
$$\frac{d^2 S_y}{dt^2} = \omega \frac{dS_x}{dt} = \omega(-\omega S_y) \tag{14}$$

Now we have isolated the components to ordinary differential equations (including S_z in equation (12)), it's time we gather the general solutions of all three:

$$S_x(t) = A\exp(-i\omega t) \tag{15}$$
$$S_y(t) = B\exp(-i\omega t) \tag{16}$$
$$S_z(t) = C \tag{17}$$

where $A,\ B,\ C$ are undetermined coefficients that could be solved according to initial conditions. If we set $S_x(t=0) = \hbar/2$, we can then write equation (15) as $S_x(t) = \dfrac{\hbar}{2}\exp(-i\omega t)$. We then use the commutation relation to find out B and C.

(b) We know that getting the expectation value through the Schrodinger and Heisenberg pictures are basically the same process. We then get the expectation value by the use of the Pauli form of the spin operator $\mathbf{S} = \dfrac{\hbar}{2}\boldsymbol{\sigma}$, where the Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{18}$$

For an initial state with spin along the x-axis, we know that equation (1.4.17a) of Sakurai gives us $(1/\sqrt{2})\,|+\rangle + (1/\sqrt{2})\,|-\rangle$. We can solve the expectation value of $\langle \mathbf{S}(t) \rangle$ for each component as follows.

$$\langle S_x \rangle = \frac{\hbar}{2} \langle \chi | \sigma_x | \chi \rangle = \frac{\hbar}{4} \begin{pmatrix} e^{-\frac{i\omega t}{2}} & e^{\frac{i\omega t}{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ e^{\frac{i\omega t}{2}} \end{pmatrix} \tag{19}$$

$$= \frac{\hbar}{4}(e^{-i\omega t} + e^{i\omega t}) = \frac{\hbar}{2}\cos(\omega t) \tag{20}$$

$$\langle S_y \rangle = \frac{\hbar}{2} \langle \chi | \sigma_y | \chi \rangle = \frac{\hbar}{4} \begin{pmatrix} e^{-\frac{i\omega t}{2}} & e^{\frac{i\omega t}{2}} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ e^{\frac{i\omega t}{2}} \end{pmatrix} \tag{21}$$

$$= \frac{\hbar}{4}(ie^{-i\omega t} - ie^{i\omega t}) = \frac{\hbar}{2}\sin(\omega t) \tag{22}$$

$$\langle S_z \rangle = \frac{\hbar}{2} \langle \chi | \sigma_z | \chi \rangle = \frac{\hbar}{4} \begin{pmatrix} e^{-\frac{i\omega t}{2}} & e^{\frac{i\omega t}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ e^{\frac{i\omega t}{2}} \end{pmatrix} \tag{23}$$

$$= \frac{\hbar}{4}(1-1) = 0 \tag{24}$$

$$\tag{25}$$

Thus, as is classically expected, the expectation value of $\mathbf{S}(t)$ precesses about the z-axis.

■

(a) Our knowledge of the raising ($a^\dagger$) and lowering (a) operators allows us to construct coherent states. For example, we can write a coherent state $|\lambda\rangle$ in terms of the basis of harmonic oscillator energy eigenkets $|n\rangle$ by calculating $\langle n|\lambda\rangle$. We obtain the form of $|n\rangle$ and $\langle n|$ from the repeated application of $a^\dagger$, until we obtain $|0\rangle$:

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{1}$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{2}$$

Getting the adjoint of both sides of equation (2) yields

$$\langle n| = \langle 0|\,\frac{a^n}{\sqrt{n!}} \tag{3}$$

Now we are ready to calculate $\langle n|\lambda\rangle$:

$$\langle n|\lambda\rangle = \langle 0|\frac{a^n}{\sqrt{n!}}|\lambda\rangle \tag{4}$$

Then we use the fact that coherent states satisfy $a\,|\lambda\rangle = \lambda\,|\lambda\rangle$ to get rid of the operator:

$$\langle n|\lambda\rangle = \frac{\lambda^n}{\sqrt{n!}}\,\langle 0|\lambda\rangle \tag{5}$$

How do we solve for $\langle 0|\lambda\rangle$? We use the completeness relation for $|n\rangle$ (since the set of $|n\rangle$ is a complete set of states) to extract $|\lambda\rangle$ from (5):

$$|\lambda\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\lambda\rangle = \langle 0|\lambda\rangle \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}\,|n\rangle \tag{6}$$

We normalize $|\lambda\rangle$ such that $\langle\lambda|\lambda\rangle = 1$ and $\langle n|n\rangle = 1$, so that

$$\langle z|z\rangle = |\langle 0|\lambda\rangle|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} = 1 \tag{7}$$

The summation is equivalent to $e^{|\lambda|^2}$, so that

$$\langle 0|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right) \tag{8}$$

Thus, (5) becomes

$$\boxed{\langle n|\lambda\rangle = \frac{\lambda^n}{\sqrt{n!}} \exp\left(-\frac{|\lambda|^2}{2}\right)} \tag{9}$$

(b) Next, we can calculate the expectation values of $\hat{x}$ and $\hat{p}$ by rewriting the definitions of the raising and lowering operators as follows: take the raising and lowering operator definitions

$$a = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} + i\hat{p}) \tag{10}$$

$$a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} - i\hat{p}) \tag{11}$$

Reverting these equations to solve for $\hat{x}$ and $\hat{p}$, we get

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left(a + a^\dagger\right) \tag{12}$$

$$\hat{p} = i\left(\frac{m\hbar\omega}{2}\right)^{1/2} \left(a^\dagger - a\right) \tag{13}$$

to get the expectation values of these operators, we let them act on $|\lambda\rangle$:

$$\langle\hat{x}\rangle = \langle\lambda|\hat{x}|\lambda\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \langle\lambda|\left(a + a^\dagger\right)|\lambda\rangle \tag{14}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\lambda + \lambda^*) \tag{15}$$

and

$$\langle\hat{p}\rangle = \langle\lambda|\hat{p}|\lambda\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2} \langle\lambda|\left(a^\dagger - a\right)|\lambda\rangle \tag{16}$$

$$= i\left(\frac{m\hbar\omega}{2}\right)^{1/2} (\lambda^* - \lambda) \tag{17}$$

where we used the eigenket (and eigenbra) relationship between a and λ in trying to expand (14) and (16),

$$a\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle \tag{18}$$
$$\left\langle\lambda\right| a^\dagger = \left\langle\lambda\right| \lambda^*. \tag{19}$$

To get the time dependence of these expectation values, we first evolve equation (6) in time by adding a factor of $\exp(-iE_n t/\hbar)$ to the summation in n:

$$\left|\lambda, t\right\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(\frac{-iE_n t}{\hbar}\right)\left|n\right\rangle \tag{20}$$

For the energy eigenvalues $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$, we expand and simplify $\left|\lambda, t\right\rangle$:

$$\left|\lambda, t\right\rangle = e^{-|\lambda|^2/2}e^{-i\omega t/2}\sum_{n=0}^{\infty}\frac{\left(\lambda e^{-i\omega t}\right)^n}{\sqrt{n!}}\left|n\right\rangle = \exp\left(\frac{-i\omega t}{2}\right)\left|\lambda e^{-i\omega t}\right\rangle \tag{21}$$

where we used equation (6). We can then recast the calculated expectation values as follows:

$$\left\langle\hat{x}(t)\right\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\lambda e^{-i\omega t}+\lambda^* e^{i\omega t}\right) = \left(\frac{2\hbar}{m\omega}\right)^{1/2}\mathrm{Re}\left(\lambda e^{-i\omega t}\right) \tag{22}$$
$$\left\langle\hat{p}(t)\right\rangle = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}\left(\lambda^* e^{i\omega t}-\lambda e^{-i\omega t}\right) = \left(2m\hbar\omega\right)^{1/2}\mathrm{Im}\left(\lambda e^{-i\omega t}\right). \tag{23}$$

Furthermore, if we substitute $\lambda = |\lambda|e^{i\delta}$, we can further recast the above equations as

$$\left\langle\hat{x}(t)\right\rangle = \left(\frac{2\hbar}{m\omega}\right)^{1/2}|\lambda|\cos(\omega t-\delta) \tag{24}$$
$$\left\langle\hat{p}(t)\right\rangle = -\left(2m\hbar\omega\right)^{1/2}|\lambda|\sin(\omega t-\delta) \tag{25}$$

How does this relate to the classical time development of the harmonic oscillator? We plug these expectation values to the classical harmonic oscillator Hamiltonian

$$\left\langle\hat{H}\right\rangle = \frac{\left\langle\hat{p}\right\rangle^2}{2m}+\frac{1}{2}m\omega^2\left\langle\hat{x}\right\rangle^2. \tag{26}$$

We first let $A \equiv \left(\frac{2\hbar}{m\omega}\right)^{1/2}|\lambda|$ so that $\left\langle\hat{x}(t)\right\rangle = A\cos(\omega t-\delta)$ and $\left\langle\hat{p}(t)\right\rangle = -m\omega A\sin(\omega t-\delta)$. We can readily plug this to equation (26) to obtain

$$\left\langle \hat{H} \right\rangle = \frac{1}{2}m\omega^2 A^2 \left[\sin^2(\omega t - \delta) + \cos^2(\omega t - \delta)\right] = \frac{1}{2}m\omega^2 A^2. \tag{27}$$

Comparing with the quantum Hamiltonian, which can be rewritten in terms of the raising and lowering operators as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{-\hbar\omega}{4}(a^\dagger - a)^2 + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)\left(a + a^\dagger\right)^2 \tag{28}$$

$$= \frac{-\hbar\omega}{4}\left[(a^\dagger)^2 - 2aa^\dagger + a^2 - \mathbb{I}\right] + \frac{\hbar\omega}{4}\left[(a^2 + 2aa^\dagger + \left(a^\dagger\right)^2 + \mathbb{I}\right] \tag{29}$$

$$= \frac{\hbar\omega}{2}\left(aa^\dagger + \frac{1}{2}\mathbb{I}\right) \tag{30}$$

Taking the expectation value of Hamiltonian above and taking the limit as $|\lambda| >> 1$, we get $\left\langle \hat{H} \right\rangle \approx \hbar\omega|\lambda|^2 = \frac{1}{2}m\omega^2 A^2$, which is the same as the classical expectation value.

(c) Now we try to calculate the position-momentum uncertainty by using the calculated expectation values for $\hat{x}$ and $\hat{p}$ earlier. Now we solve for the expectation value for $\hat{x^2}$ and $\hat{p^2}$:

$$\left\langle \hat{x^2} \right\rangle = \langle\lambda|\hat{x^2}|\lambda\rangle = \frac{\hbar}{2m\omega}\left[(\lambda + \lambda^*)^2 + 1\right] \tag{31}$$

$$\left\langle \hat{p^2} \right\rangle = \langle\lambda|\hat{p^2}|\lambda\rangle = -\frac{1}{2}m\hbar\omega\left[(\lambda^* - \lambda)^2 - 1\right] \tag{32}$$

Together with equations (15) and (17), the variance can now be calculated for both position and momentum operators:

$$(\Delta x)^2 = \left\langle \hat{x^2} \right\rangle - (\langle\hat{x}\rangle)^2 = \frac{\hbar}{2m\omega} \tag{33}$$

$$(\Delta p)^2 = \left\langle \hat{p^2} \right\rangle - (\langle\hat{p}\rangle)^2 = \frac{m\hbar\omega}{2} \tag{34}$$

Multiplying and getting the positive square root, we get

$$\boxed{\Delta x \Delta p = \frac{\hbar}{2}} \tag{35}$$

which corresponds to the minimum uncertainty: a property of coherent states.

■

Problem 5

(a) Given the wave function $\Psi(\mathbf{x}, t)$ and the probability current $\mathbf{j} = \left(\frac{\hbar}{m}\right) \mathrm{Im}\left[\Psi^* \boldsymbol{\nabla} \Psi\right]$, we can derive the continuity equation involving the probability density $\rho(t) = |\Psi|^2$ by invoking the representation of the Schrodinger equation in position space:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{x}, t) + V(\mathbf{x}, t)\Psi(\mathbf{x}, t) = i\hbar\frac{\partial\Psi(\mathbf{x}, t)}{\partial t} \tag{1}$$

as well as its conjugate:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi^*(\mathbf{x}, t) + V(\mathbf{x}, t)\Psi^*(\mathbf{x}, t) = -i\hbar\frac{\partial\Psi^*(\mathbf{x}, t)}{\partial t}. \tag{2}$$

We multiply both sides of (1) by Ψ^* and both sides of (2) by Ψ to arrive at

$$-\frac{\hbar^2}{2m}\Psi^*\nabla^2\Psi + \Psi^* V\Psi = i\hbar\Psi^*\frac{\partial\Psi}{\partial t} \tag{3}$$

and

$$-\frac{\hbar^2}{2m}\Psi\nabla^2\Psi^* + \Psi V\Psi^* = -i\hbar\Psi\frac{\partial\Psi^*}{\partial t}. \tag{4}$$

Dividing both sides by $i\hbar$ for (3) and $-i\hbar$ for (4), we then add both sides, yielding

$$\Psi^*\frac{\partial\Psi}{\partial t} + \psi\frac{\partial\Psi^*}{\partial t} = \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\Psi^*\nabla^2\Psi + \Psi^* V\Psi\right] - \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\Psi\nabla^2\Psi^* + \Psi V\Psi^*\right] \tag{5}$$

The 2nd and 4th terms of equation (5) cancels (since V is technically not an operator, we can switch positions with Ψ or Ψ^*). We can also recognize the left hand side as the time derivative of ρ, since

$$\frac{\partial\rho}{\partial t} = \frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial t}(\Psi^*\Psi) \tag{6}$$

$$= \Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}. \tag{7}$$

Thus we are left with

$$\frac{\partial\rho}{\partial t} = \frac{i\hbar}{2m}\left[\Psi^*(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Psi) - \Psi(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Psi^*)\right] \tag{8}$$

where we recalled the definition of a Laplacian operator as a divergence of a gradient. Next, we recast a part of the definition of $\mathbf{j}$ as $\text{Im}\left(\Psi^*\boldsymbol{\nabla}\Psi\right) = \frac{1}{2i}\left(\Psi\boldsymbol{\nabla}\Psi^* - \Psi^*\boldsymbol{\nabla}\Psi\right)$. Getting the divergence of $\mathbf{j}$,

$$\boldsymbol{\nabla}\cdot\mathbf{j} = \frac{\hbar}{2im}\nabla\left[\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Psi^* + \Psi\nabla^2\Psi^* - \boldsymbol{\nabla}\Psi^*\cdot\boldsymbol{\nabla}\Psi - \Psi^*\nabla^2\Psi\right]. \tag{9}$$

We see that the first and third terms cancel since they commute. Thus we are left with the negative of the right hand side of equation (8), or $\boxed{\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{j}}$.

(b) Next, if we use plane wave solution $\psi(\mathbf{x},t) = A(\mathbf{x},t)\exp\left[\frac{i}{\hbar}S(\mathbf{x},t)\right]$ on the definition of $\mathbf{j}$, and relating the probability density to the amplitude $\rho = A^2$, we get

$$\mathbf{j} = \frac{\hbar}{2im}\left[\sqrt{\rho}\ \exp\left(-\frac{i}{\hbar}S\right)\left(\boldsymbol{\nabla}\sqrt{\rho}\exp\left(\frac{i}{\hbar}S\right)\right) - \sqrt{\rho}\exp\left(\frac{i}{\hbar}S\right)\boldsymbol{\nabla}\sqrt{\rho}\exp\left(-\frac{i}{\hbar}S\right)\right] \tag{10}$$

Letting the del operator act on both the amplitude $\sqrt{\rho}$ and the phase S, we use the product rule to expand:

$$\begin{aligned}\mathbf{j} =& \frac{\hbar}{2im}\Bigg[\sqrt{\rho}\exp\left(-\frac{i}{\hbar}S\right)\left(\exp\left(\frac{i}{\hbar}S\right)\boldsymbol{\nabla}\sqrt{\rho} + \sqrt{\rho}\nabla\exp\left(\frac{i}{\hbar}S\right)\right)\\ &- \sqrt{\rho}\exp\left(\frac{i}{\hbar}S\right)\left(\exp\left(-\frac{i}{\hbar}S\right)\boldsymbol{\nabla}\sqrt{\rho} + \sqrt{\rho}\boldsymbol{\nabla}\exp\left(-\frac{i}{\hbar}S\right)\right)\Bigg]\end{aligned} \tag{11}$$

For the del operator acting on the exponential, we have $\nabla e^{-iS/\hbar} = -i/\hbar e^{-iS/\hbar}\nabla S$ as well as $\nabla e^{iS/\hbar} = i/\hbar e^{iS/\hbar}\nabla S$. Then we can expand further as

$$\begin{aligned}\mathbf{j} =& \frac{\hbar}{2im}\Bigg[\sqrt{\rho}\exp\left(-\frac{i}{\hbar}S\right)\left(\exp\left(\frac{i}{\hbar}S\right)\boldsymbol{\nabla}\sqrt{\rho} + \sqrt{\rho}\frac{i}{\hbar}\exp\left(\frac{i}{\hbar}S\right)\boldsymbol{\nabla}S\right)\\ &- \sqrt{\rho}\exp\left(\frac{i}{\hbar}S\right)\left(\exp\left(-\frac{i}{\hbar}S\right)\boldsymbol{\nabla}\sqrt{\rho} + \sqrt{\rho}\frac{-i}{\hbar}\exp\left(-\frac{i}{\hbar}S\right)\boldsymbol{\nabla}S\right)\Bigg]\end{aligned} \tag{12}$$

Factoring out the extra exponentials allows us to cancel all exponential terms. This also allows us to cancel the first and third terms. For the remaining terms, we cancel $\hbar/i$

with the $i/\hbar$ inside. We multiply $\sqrt{\rho}$ to itself, and then we add the second and fourth terms to end up with $2\nabla S$, which ends up canceling with the 2 in the denominator. Ultimately we are left with $\boxed{\mathbf{j} = (\rho/m)\nabla S}$.

■

Problem 6

Recall that the basic idea of the WKB approximation to expand the solution of the wave equation in the ratio of the wavelength to the scale length of the environment in which the wave propagates, when that ratio is small. From here, we can deduce the limits of the validity of the approximation in terms of the spatial variation of the potential.

In a constant potential $V = V_0$, we are inclined to write the wavefunction solution to the Schrodinger equation as plane waves, i.e. $\psi(\mathbf{x}) = A \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}\right]$. The de Broglie relation then dictates the wavelength of this wavefunction, that is $\lambda = \frac{2\pi\hbar}{|\mathbf{p}|}$. In the event that the potential V is slowly varying, this plane wave solution might still be the case, but for it to work, we must constrain $\lambda << L$, where L is some characteristic scale length of the potential. This would change the assumed solution into

$$\psi(\mathbf{x}) = A(\mathbf{x}) \exp\left[\frac{i}{\hbar}S(\mathbf{x})\right] \tag{1}$$

where $S(\mathbf{x}) = p(\mathbf{x}) \cdot \mathbf{x}$ and $A(\mathbf{x})$ are to be determined. If we try to expand ψ around some fixed point $\mathbf{x}_0$ by a small amount $\boldsymbol{\xi}$, such that $\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}$, we evaluate the amplitude at that fixed point, but include the first-order correction term to the phase:

$$\psi(\mathbf{x}_0 + \boldsymbol{\xi}) \approx A(\mathbf{x}_0) \exp\left[\frac{i}{\hbar}\left(S(\mathbf{x}_0) + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}S\right)\right] \tag{2}$$

from which we deduce $p(\mathbf{x}) = \nabla S$. So we want to use the WKB approximation for potentials that are **slowly varying** (in space).

As for the spatial variation of the wavefunction, the WKB approximation can be used for approximating ψ far from any turning points, i.e. points wherein $E = V(x)$. The WKB solutions generally blow up near this regions, so they aren't normalizable in general. In these regions near the turning points, we are forced to solve the Schrodinger equation analytically, arriving at the Airy functions as (rescaled) solutions. Then we end up with connection formulas relating the wave functions between the classically allowed and forbidden regions. Thus the WKB solutions are valid for wavefunctions with **slowly varying wavelengths** as well.

■