Problem 1

Jackson 3.3.

(a) We're going to exploit the azimuthal symmetry of the problem so we can write the potential as the general solution

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})P_\ell(\cos\theta) \tag{1}$$

Next, we divide the space $r > R$ into two regions: one for $\theta < \pi/2$ and another for $\theta > \pi/2$, which indicate the upper and lower parts of the disk (recall that $\theta = \pi/2$, $r \in [0, R]$ constitute the disk). We avoid blowing up as $r \to \infty$ so we set all $A_\ell = 0$. Now, there are two sets of coefficients we must solve, one for the upper and one for the lower, so let's indicate them with B_ℓ for the upper and $B_{\ell,\text{lower}}$ for the lower.

Next, we apply symmetry of the potential in the upper and lower regions in order to match the boundary conditions at $\theta = \pi/2$. This allows us to write

$$\Phi(z) = \Phi(-z) \to \Phi(\cos\theta) = \Phi_{\text{lower}}(\cos\theta) \tag{2}$$

$$\sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1}P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} B_{\ell,\text{lower}} r^{-\ell-1}P_\ell(-\cos\theta) \tag{3}$$

But then, P_ℓ is even for even ℓ and odd for odd ℓ, so we can write the relation between the lower and upper coefficients as $B_{\ell,\text{lower}} = B_\ell(-1)^\ell$.

At this point, we exhausted the conditions available for $r > R$. We really can't do much in answering B_ℓ unless we start answering $r < R$, so let's go ahead and do that.

(b) For the region $r < R$, we follow the same steps in (a). We divide the region into upper ($\theta < \pi/2$) and lower ($\theta > \pi/2$), prevent the potential from blowing up as $r \to 0$ (which leads us to $B_{\ell,\text{in}} = 0$), and apply symmetry to the upper and lower regions. Summarizing our findings, we get

$$\Phi(r,\theta) = \begin{cases} \sum\limits_{\ell=0}^{\infty} B_\ell r^{-\ell-1}P_\ell(\cos\theta) & r > R \;\&\; \theta < \pi/2 \\ \sum\limits_{\ell=0}^{\infty} B_\ell r^{-\ell-1}(-1)^\ell P_\ell(\cos\theta) & r > R \;\&\; \theta > \pi/2 \\ \sum\limits_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r < R \;\&\; \theta < \pi/2 \\ \sum\limits_{\ell=0}^{\infty} A_\ell r^\ell (-1)^\ell P_\ell(\cos\theta) & r < R \;\&\; \theta > \pi/2 \end{cases} \tag{4}$$

Another condition we can place is that for $r < R$ and $\theta = \pi/2$, the potential should be V:

$$V = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos \pi/2) \tag{5}$$

$$V = \sum_{\ell=0}^{\infty} A_\ell r^\ell (-1)^\ell P_\ell(\cos \pi/2) \tag{6}$$

Since there are no higher-order r terms present in the LHS, the constant term $A_0 = V$, which would cause the summation for the higher terms to vanish. We can then conclude that $A_\ell = 0$ for $\ell > 0$. We can relax this condition by noting that $P_\ell(0) = 0$ for all odd ℓ, so that we only collect the odd terms for which $A_\ell \neq 0$. We then update our potential for $r < R$:

$$\Phi(r < R, \theta) = \begin{cases} V + \sum\limits_{\ell=1,3,5,\ldots}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & \theta < \pi/2 \\ V - \sum\limits_{\ell=1,3,5,\ldots}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & \theta > \pi/2 \end{cases} \tag{7}$$

We then match the potentials at $r = R$ and invoke orthogonality in order to drop the higher even terms for the solution in $r > R$. We can then rewrite the solution to make it symmetric to the solution in $r < R$:

$$\Phi(r > R, \theta) = \begin{cases} \dfrac{B_0}{r} + \sum\limits_{\ell=1,3,5,\ldots}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) & \theta < \pi/2 \\ \dfrac{B_0}{r} - \sum\limits_{\ell=1,3,5,\ldots}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) & \theta > \pi/2 \end{cases} \tag{8}$$

We start to match the solution at $r = R$:

$$\frac{B_0}{R} + \sum_{\ell=1,3,5,\ldots}^{\infty} B_\ell R^{-\ell-1} P_\ell(\cos\theta) = V + \sum_{\ell=1,3,5,\ldots}^{\infty} A_\ell R^{-\ell-1} P_\ell(\cos\theta) \tag{9}$$

$$\frac{B_0}{R} - \sum_{\ell=1,3,5,\ldots}^{\infty} B_\ell R^{-\ell-1} P_\ell(\cos\theta) = V - \sum_{\ell=1,3,5,\ldots}^{\infty} A_\ell R^{-\ell-1} P_\ell(\cos\theta) \tag{10}$$

which allows us to see the relation between B_ℓ and A_ℓ: $B_0 = VR$ and $B_\ell = A_\ell R^{2\ell+1}$, leaving us with determining A_ℓ. It turns out we can find these coefficients by using the discontinuity of the electric field across the plate:

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\boldsymbol{n}}_{12} = \frac{\sigma}{\epsilon_0} \tag{11}$$

$$(\mathbf{E}_{\text{in, lower}} - \mathbf{E}_{\text{in, upper}}) \cdot \hat{\boldsymbol{\theta}} = \frac{\sigma}{\epsilon_0} \tag{12}$$

We are then given the information that $\sigma \propto 1/\sqrt{R^2 - \rho^2}$. We just need to find the constant of proportionality. We can do this by using Coulomb's law at the origin and setting it equal to V:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{x}')}{\mathbf{x}'} da' \tag{13}$$

Set C to be the proportionality constant and expand the integral in 2D spherical coordinates:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{C}{r'\sqrt{R^2 - r'^2}} r'\ dr'\ d\varphi' && (14) \\ &= \frac{C}{2\epsilon_0} \int_0^R \frac{dr'}{\sqrt{R^2 - r'^2}} && (15) \\ &= \frac{C}{2\epsilon_0} \left[\arcsin \frac{r'}{R}\right]_0^R = \frac{C}{2\epsilon_0} \frac{\pi}{2} && (16) \end{aligned}$$

which makes $C = \dfrac{4\epsilon_0 V}{\pi}$. Going back to the discontinuity condition,

$$\begin{aligned} -\frac{1}{r}\frac{\partial \Phi_{\text{in, lower}}}{\partial\theta} + \frac{1}{r}\frac{\partial \Phi_{\text{in, upper}}}{\partial\theta} &= \frac{4V}{\pi}\frac{1}{\sqrt{R^2 - r^2}} && (17) \\ -\frac{\partial \Phi_{\text{in, lower}}}{\partial\theta} + \frac{\partial \Phi_{\text{in, upper}}}{\partial\theta} &= \frac{4V}{\pi}\frac{r}{R}\frac{1}{\sqrt{1 - \left(\frac{r}{R}\right)^2}} && (18) \end{aligned}$$

We then evaluate the derivatives using the following relations for P_ℓ:

$$\frac{\partial}{\partial\theta}P_\ell(\cos\theta)\bigg|_{\theta=\pi/2} = -\frac{\partial}{\partial x}P_\ell(x)\bigg|_{x=0} \tag{19}$$

$$= \frac{\ell x P_\ell(x) - \ell P_{\ell-1}(x)}{x^2-1}\bigg|_{x=0} \tag{20}$$

so that when we simplify it, the discontinuity equation becomes

$$-2\sum_{\ell=1,3,5,\ldots}^{\infty} A_\ell \left(\frac{r}{R}\right)^\ell (\ell P_{\ell-1}(0)) = \frac{4V}{\pi}\frac{r}{R}\frac{1}{\sqrt{1-\left(\frac{r}{R}\right)^2}} \tag{21}$$

In order to match coefficients, we need to expand the RHS using the generating function for P_ℓ: $g(t,x) = (1-2xt+t^2)^{1/2} = 1 + \sum_{\ell=1}^{\infty}\frac{(2\ell-1)!!}{(2n)!!}(2xt-t^2)^n = \sum_{\ell=0}^{\infty}P_\ell(x)t^n$ (cf. Arfken 6th, equation (12.4-5)). Rewriting, we get

$$-2\sum_{\ell=1,3,5,\ldots}^{\infty} A_\ell \left(\frac{r}{R}\right)^\ell (\ell P_{\ell-1}(0)) = \frac{4V}{\pi}\sum_{\ell=1,3,5,\ldots}^{\infty}(-1)^{(\ell-1)/2}\left(\frac{r}{R}\right)^\ell (\ell P_{\ell-1}(0)) \tag{22}$$

which tells us that $A_\ell = \dfrac{2V}{\pi\ell}(-1)^{(\ell-1)/2}$. Thus the particular solution for both (a) and (b) is

$$\boxed{\Phi(r,\theta) = \begin{cases} \dfrac{VR}{r} + \dfrac{2V}{\pi}\displaystyle\sum_{\ell=1,3,5,\ldots}^{\infty}\frac{(-1)^{(\ell-1)/2}}{\ell}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\theta) & r>R \;\&\; \theta<\pi/2 \\ \dfrac{VR}{r} - \dfrac{2V}{\pi}\displaystyle\sum_{\ell=1,3,5,\ldots}^{\infty}\frac{(-1)^{(\ell-1)/2}}{\ell}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\theta) & r>R \;\&\; \theta>\pi/2 \\ V + \dfrac{2V}{\pi}\displaystyle\sum_{\ell=1,3,5,\ldots}^{\infty}\frac{(-1)^{(\ell-1)/2}}{\ell}\left(\frac{r}{R}\right)^{\ell+1}P_\ell(\cos\theta) & r<R \;\&\; \theta<\pi/2 \\ V - \dfrac{2V}{\pi}\displaystyle\sum_{\ell=1,3,5,\ldots}^{\infty}\frac{(-1)^{(\ell-1)/2}}{\ell}\left(\frac{r}{R}\right)^{\ell+1}P_\ell(\cos\theta) & r>R \;\&\; \theta>\pi/2 \end{cases}} \tag{23}$$

(c) To obtain the capacitance, we first need the total charge which can be obtained by integrating $\sigma = \dfrac{4\epsilon_0 V}{\pi\sqrt{R^2-r^2}}$ over all space and dividing this by the constant potential V:

Problem 1

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\begin{aligned}
Q &= \frac{4\epsilon_0 V}{\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{1}{\sqrt{R^2 - r'^2}} \frac{\delta(\theta' - \pi/2)}{r'} \; r'^2 \sin\theta' \; dr' \; d\theta' d\varphi' && (24) \\
&= 8\epsilon_0 V \int_0^R \frac{r' dr'}{\sqrt{R^2 - r'^2}} && (25)
\end{aligned}
$$

which gives us $Q = 8\epsilon_0 VR$, and thus, $\boxed{C = Q/V = 8\epsilon_0 R}$.

■

Jackson 3.5 The equivalence of the two solutions can be proven indirectly by solving the same problem and invoking the uniqueness of solutions.

(a) This solution comes from using the Green function method. Recall the general solution to the Green function method:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\ d^3\mathbf{x}' - \frac{1}{4\pi}\oint\left(\Phi(\mathbf{x}')\frac{dG}{dn'}\right)da' \tag{1}$$

There's no charge/charge density present, so the first term vanishes. For a hollow sphere of radius a and a boundary condition $\Phi(r=a,\theta,\varphi) = V(\theta,\varphi)$, we can immediately deduce the Green's function using the method of images:

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\mathbf{x}-\mathbf{x}'} - \frac{1}{\left|\frac{x'}{a}\mathbf{x} - \frac{a}{x'}\mathbf{x}'\right|} \tag{2}$$

Converting into spherical coordinates,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x'^2}{a^2}x^2+a^2-2xx'\cos\gamma}} \tag{3}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$. To complete our solution, we also need the normal derivative of the (spherical) Green's function evaluated at the sphere's surface. Since we're looking at the potential inside, we need a negative (towards the origin) normal:

$$\begin{aligned}
\left.\frac{dG}{dn'}\right|_{x'=a} &= -\frac{\partial}{\partial x'}\left[\frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x'^2}{a^2}x^2+a^2-2xx'\cos\gamma}}\right]_{x'=a} && (4)\\
&= -\left[\frac{(2x'-2x\cos\gamma)}{2(x^2+x'^2-2xx'\cos\gamma)^{3/2}} - \frac{2\frac{x^2}{a^2}x'-2x\cos\gamma}{2\left(\frac{x^2}{a^2}x'^2+a^2-2xx'\cos\gamma\right)^{3/2}}\right]_{x'=a} && (5)\\
&= \frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}} && (6)
\end{aligned}$$

Plugging this back to the general solution, we get

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint V(\theta',\varphi')\frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}\ da' \tag{7}$$

We can convert the differential area da' into its equivalent differential surface solid angle $a^2\ d\Omega'$. Simplifying, we get

$$\Phi(\mathbf{x}) = \frac{a(a^2-r^2)}{4\pi}\int\frac{V(\theta',\varphi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\ d\Omega' \tag{8}$$

(b) We then use an alternate method to solve the same problem. Using Laplace's equation (no charges present), we start with the general solution (exploiting azimuthal symmetry):

$$\Phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell m}r^{\ell}+B_{\ell m}r^{-\ell-1})Y_{\ell m}(\theta,\varphi) \tag{9}$$

Since we're solving the potential inside the sphere, we want the potential to be finite even as $r\to 0$, so all $B_{\ell m}=0$. Applying the boundary condition $\Phi(r=a,\theta,\varphi)=V(\theta,\varphi)$, we get

$$V(\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}a^{\ell}Y_{\ell m}(\theta,\varphi) \tag{10}$$

We then kill the summations by using the orthogonality relations for the spherical harmonics:

$$\int_0^{2\pi}\int_0^{\pi}Y^*_{\ell' m'}(\theta,\varphi)Y_{\ell m}(\theta,\varphi)\sin\theta\ d\theta\ d\varphi = \delta_{\ell'\ell}\delta_{m'm} \tag{11}$$

Thus, we multiply both sides by $Y^*_{\ell' m'}\sin\theta$ and integrate in $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi]$ to get

$$\int_0^{2\pi}\int_0^{\pi}Y^*_{\ell m}(\theta,\varphi)V(\theta,\varphi)\sin\theta\ d\theta\ d\varphi = A_{\ell' m'}a^{\ell'} \tag{12}$$

So that our full solution becomes

$$\Phi(\mathbf{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}\left(\frac{r}{a}\right)^{\ell}Y_{\ell m}(\theta,\varphi), \qquad A_{\ell m} = \int_0^{2\pi}\int_0^{\pi}Y^*_{\ell m}(\theta',\varphi')V(\theta',\varphi')\sin\theta'\ d\theta'\ d\varphi' \tag{13}$$

$$= \int d\Omega'\ Y^*_{\ell m}(\theta,\varphi)V(\theta,\varphi) \tag{14}$$

which is equivalent to the Green's function solution since we solved the same problem.

■

Jackson 3.12 In this problem, we are free to exploit the azimuthal symmetry again, so we tweak the 3D Laplace equation in cylindrical coordinates to yield no φ term. Recall the ansatz $\Phi(\rho,\varphi,z) = R(\rho)\phi(\varphi)Z(z)$. Then the Laplace equation becomes

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{R\rho}\frac{\partial R}{\partial \rho} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0. \tag{1}$$

(a) Because we have no φ terms in the equation, the two former terms must be equal to the negative of the third term. This implies that their sum must be a constant. Let that constant be $-k^2$, $k > 0$. Then

$$\frac{1}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{R\rho}\frac{\partial R}{\partial \rho} = -k^2 \tag{2}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 \tag{3}$$

We can easily solve the equation for $Z(z)$:

$$Z(z) = \sum_k A_k \exp(kz) + B_k \exp(-kz) \tag{4}$$

We then see that we can easily eliminate one of the integration constants by letting the potential be finite as $z \to \infty$, so that $A_k = 0$. We next solve $R(\rho)$ with the remaining equation from earlier:

$$\frac{1}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{R\rho}\frac{\partial R}{\partial \rho} + k^2 = 0 \tag{5}$$

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + k^2 R = 0 \tag{6}$$

We rescale the problem to solve $k\rho$ that enables us to show that the solution is (cf. Arfken 6th equation 11.22a):

$$R(\rho) = J_0(k\rho) \tag{7}$$

We now expand the possibilities of k, from being a discrete spectrum to a continuous one, and write the whole solution as

$$\Phi(\rho, z) = \int_0^\infty B(k)\exp(-kz)J_0(k\rho)\ dk \tag{8}$$

for which we limit the values of k in order to prevent writing $\exp(-kz)$. Now we apply the boundary condition at $z = 0$ and $\rho < a$:

$$\Phi(\rho < a, z = 0) = V(\rho < a) = \int_0^\infty B(k) J_0(k\rho)\ dk \tag{9}$$

We then use the integral killer: a Dirac delta from the orthogonality relations of the Bessel function of the first kind (eq. 3.108 of Jackson 3rd or eq. 11.59 of Arfken 6th):

$$\int_0^\infty x J_0(kx) J_0(k'x)\ dx = \frac{1}{k}\delta(k' - k) \tag{3.108}$$

Thus, we multiply both sides by $\rho J_0(k'\rho)$ and integrate in ρ from 0 to a:

$$\int_0^a \rho J_0(k'\rho) V\ d\rho = \frac{1}{k'} B(k') \tag{10}$$

so that our potential becomes

$$\Phi(\rho, z) = V \int_0^\infty \left(\int_0^a k\rho' J_0(k\rho')\ d\rho' \right) e^{-kz} J_0(k\rho)\ dk \tag{11}$$

which can be simplified further by recalling the Bessel identity $k \int_0^a \rho J_0(k\rho)\, d\rho = a J_1(ka)$ so that

$$\Phi(\rho, z) = Va \int_0^\infty e^{-kz} J_1(ka) J_0(k\rho)\ dk \tag{12}$$

(b) At $\rho = 0$, equation (11) reduces to the following form:

$$\Phi(\rho = 0, z) = V \int_0^\infty \int_0^a k\rho' J_0(k\rho') e^{-kz}\ dk\ d\rho' \tag{13}$$

We can regroup terms in order to obtain a familiar expression involving J_0:

$$\begin{aligned}
\Phi(\rho = 0, z) &= V J_0(0) \int_0^a \rho' \left[\int_0^\infty k e^{-kz} J_0(k\rho')\ dk \right] d\rho' && (14)\\
&= V \int_0^a \rho' \left[-\frac{\partial}{\partial z} \int_0^\infty e^{-kz} J_0(k\rho')\ dk \right] d\rho' && (15)\\
&= V \int_0^a \left[-\frac{\partial}{\partial z} \left(\frac{1}{\sqrt{\rho'^2 + z^2}} \right) \right] d\rho' && (16)\\
&= Vz \int_0^a \frac{\rho'\ d\rho'}{(\rho'^2 + z^2)^{3/2}} && (17)
\end{aligned}$$

where we used Feynman's trick of differentiating under the integral sign as well as the form of the generating function used for J_0. Then, we let $r = \rho'^2 + z^2$, $dr = 2\rho' d\rho'$ $\rho' = 0 \to r = z^2$ and $\rho' = a \to r = a^2 + z^2$, and the above integral simplifies to

$$\Phi(\rho = 0, z) = \frac{Vz}{2} \int_{z^2}^{a^2+z^2} r^{-3/2} dr \tag{18}$$

$$= -Vz \left(\frac{1}{\sqrt{r}} \right) \Bigg|_{z^2}^{a^2+z^2} \tag{19}$$

Which, when simplified, gives us $\boxed{\Phi(\rho = 0, z) = V \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right)}$.

(c) When we transfer our point of inquiry to the edge of the disk, we let $\rho = a$ in equation (12):

$$\Phi(\rho = a, z) = Va \int_0^\infty e^{-kz} J_1(ak) J_0(ak)\ dk \tag{20}$$

Using tables of integrals involving Bessel functions (Okui, 1974)[1], we get to simplify this integral directly using the complete elliptic integral:

$$\int_0^\infty e^{-px} J_1(ax) J_0(ax)\ dx = -\frac{pk}{2\pi a^2} K(k) + \frac{1}{2a} \tag{LU 317}$$

Here, $k^2 = \dfrac{4a^2}{p^2 + a^2}$ and $K(k)$ is the complete elliptic integral of the first kind. Thus, applying back the equation earlier, we get

$$\boxed{\Phi(\rho = a, z) = Va \left(-\frac{kz}{2\pi a^2} K(k) + \frac{1}{2a} \right) = \frac{V}{2} \left(1 - \frac{kz}{\pi a} K(k) \right)} \tag{21}$$

where $k = \dfrac{4a^2}{z^2 + 4a^2}$ and $K(k)$ is the complete elliptic integral of the first kind, obtained by solving Jacobi's elliptic equation: $K(k^2) \equiv \int_0^{\pi/2} \dfrac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}$.

■

[1] https://nvlpubs.nist.gov/nistpubs/jres/78B/jresv78Bn3p113_A1b.pdf

Jackson 3.19 We are invited to compare this problem with the configurations in Problem 3 (3.12) as well as Problem 3.18 (not included in this problem set). Recall that in 3.12 we considered an infinite plane in $z = 0$ that has a circular conducting disk of radius a centered on the origin, with an insulating ring that keeps it from redistributing its constant potential V to the grounded plane. For 3.18, we modify this configuration by placing a similar conducting disk-conducting plane configuration at $z = L$ and leaving the plane at $z = 0$ fully grounded. In this problem, we modify the system yet again by removing the disk and replacing the plane with a full conducting one, both at $z = 0$ and $z = L$. We then place a point charge q at the $z = z_0$ for which $0 < z_0 < L$. The potential and induced charges can then be solved by exploiting Green's reciprocation theorem.

(a) Recall Green's reciprocation theorem (Problem 1.12) which allows us to determine charge densities and potentials of one volume with the knowledge from another volume:

$$\int_V \rho'\Phi \; dV + \int_S \sigma'\Phi \; dA = \int_V \rho\Phi' \; dV + \int_S \sigma\Phi' \; dA \tag{1}$$

where ρ here refers to the volume charge densities and not the radial cylindrical coordinate (which we assign to r). We then get all the unprimed variables and assign to them the configuration presented in Problem 3.18, while 3.19 gets assigned to the primed ones. For problem 3.18, $\rho(r, z) = 0$, $\Phi(r, z = 0) = 0$, $\Phi(r > a, z = L) = 0$, $\Phi(r < a, z = L) = 0$, and $\Phi(r, 0 < z < L) = V\int_0^\infty dk \; aJ_1(ak)J_0(rk)\dfrac{\sinh(kz)}{\sinh(kL)}$. For problem 3.19, $\rho(r, z) = q\delta(r - 0)\delta(z - z_0)$ and $\Phi(r, z = 0, L) = 0$. To get the induced charge density, we plug the relevant values to Green's reciprocation theorem to solve σ' which, when integrated, will yield the induced charge.

$$\int_V \rho'\Phi \; dV + \int_S \sigma'\Phi \; dA = \cancel{\int_V \rho\Phi' \; dV} + \cancel{\int_S \sigma\Phi' \; dA} \tag{2}$$

$$q\left[\int_0^\infty dk\left(\int_0^\infty J_0(rk)\delta(r)\; dr\int_0^L \frac{\sinh(kz)}{\sinh(kL)}\delta(z - z_0)\; dz\right) aJ_1(ak)\right] + V\int_{z=L,\; r<a} \sigma'(r, z)\; dA = 0 \tag{3}$$

Integrating the surface charge density, we get the induced charge on the circular disk at $z = L$:

$$\int_{z=L,\; r<a} \sigma'(r, z)\; dA = -q\int_0^\infty dk\; aJ_1(ak)\frac{\sinh(kz_0)}{\sinh(kL)}\frac{V}{V} = -\frac{q}{V}\Phi(z_0, 0) \tag{4}$$

(b) We then try to extract σ' from the integral. We can do this by expanding the surface integral along r and φ:

$$\int_{z=L,\ r<a} \sigma'(r,z)\ dA = \int_0^{2\pi}\int_0^a \sigma(r,L)\ r\ dr\ d\varphi = 2\pi\int_0^a \sigma'(r,L)\ r\ dr \tag{5}$$

We can extract σ' by differentiating with respect to a and using the first fundamental theorem of calculus to get $2\pi a\sigma(a,L)$. We then do the same to the second integral in equation (4):

$$2\pi a\sigma'(a,L) = -q\int_0^\infty dk\left(\frac{\partial}{\partial a}\ aJ_1(ak)\right)\frac{\sinh(kz_0)}{\sinh(kL)} \tag{6}$$

We then use the recursion relation for Bessel functions of the first kind: $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ so we can simplify the product rule in the right-hand side:

$$\frac{\partial}{\partial a}(aJ_1(ak)) = \cancel{J_1(ak)} + akJ_0(ak) = akJ_0(ak) \tag{7}$$

Thus, we get

$$\sigma'(a,L) = -\frac{q}{2\pi}\int_0^\infty dk\ kJ_0(ak)\frac{\sinh(kz_0)}{\sinh(kL)} \tag{8}$$

and in general r,

$$\boxed{\sigma'(r,L) = -\frac{q}{2\pi}\int_0^\infty dk\ kJ_0(kr)\frac{\sinh(kz_0)}{\sinh(kL)}} \tag{9}$$

(c) Lastly, when we calculate the charge density at $r=0$,

$$\sigma'(r,L) = -\frac{q}{2\pi}\int_0^\infty dk\ k\frac{\sinh(kz_0)}{\sinh(kL)} \tag{10}$$

We can solve this via contour integral methods. Consider the integral of the fraction first before differentiating under the integral sign to include the remaining k factor. We first integrate $\frac{\sinh(kz_0)}{\sinh(kL)}$ over $k \in [0,\infty]$. We do this by considering that the integrand is even, so we solve $\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sinh(kz_0)}{\sinh(kL)}$. We then shift to the complex plane and consider the contour integral $\oint_C \frac{\sinh(zz_0)}{\sinh(zL)}dz$, where C is the rectangular indented contour with length $2R$ from $-R$ to R and width $L/2$ from 0 to $iL/2$. The indent is a semicircle of radius ε centered on $z = 0$ that will allow us to bypass the singularity there. Then, by Cauchy's theorem,

$$\oint_C \frac{\sinh(zz_0)}{\sinh(zL)} dz = 0 \tag{11}$$

which can be expanded into six contour paths, 3 of which will eventually vanish as we let $R \to \infty$:

$$\int_{-R}^{-\varepsilon} \frac{e^{z_0 x} - e^{-z_0 x}}{e^{Lx} - e^{-Lx}} dx + \int_{\varepsilon}^{R} \frac{e^{z_0 x} - e^{-z_0 x}}{e^{Lx} - e^{-Lx}} dx = \int_0^{\pi} \frac{e^{z_0 \varepsilon e^{i\theta}} - e^{-z_0 \varepsilon e^{i\theta}}}{e^{L\varepsilon e^{i\theta}} - e^{-L\varepsilon e^{i\theta}}} i\varepsilon e^{i\theta}\, d\theta \tag{12}$$

Using L'Hopital's rule and simplifying the RHS, we let the limits $R \to \infty$ and $\varepsilon \to 0$ to get the desired result:

$$\int_0^{\infty} \frac{\sinh(xz_0)}{\sinh(Lx)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sinh(xz_0)}{\sinh(Lx)} dx = \frac{1}{2} \frac{\pi}{L} \tan\left(\frac{\pi z_0}{2L}\right) \tag{13}$$

We then differentiate equation (13) with respect to z_0 by 2ℓ times to arrive at the desired integral:

$$\int_0^{\infty} \frac{\sinh(xz_0)}{\sinh(Lx)} x^{2\ell} dx = \frac{\partial^{2\ell}}{\partial z_0^{2\ell}} \frac{\pi}{2L} \tan\left(\frac{\pi z_0}{2L}\right) \tag{14}$$

Letting $\ell = 1/2$, we arrive at the answer:

$$\boxed{\sigma'(0, L) = -\frac{q}{2\pi} \left[\left(\frac{\pi}{2L}\right)^2 \sec^2\left(\frac{\pi z_0}{2L}\right) \right] = -\frac{q\pi}{8L^2} \sec^2\left(\frac{\pi z_0}{2L}\right)} \tag{15}$$

■