Lec 10: 15 Oct 2020

Tuesday, 20 October 2020 18:01

III. Separation of variables

Directly attacking Laplace's eqn

$$\nabla^2 \Phi = 0$$

in a coordinate system that takes into account the symmetries of the geometry.

① In rectangular coordinates

- Laplacian operator reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0$$

- use the ansatz $\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\frac{\partial^2 XYZ}{\partial x^2} + \frac{\partial^2 XYZ}{\partial y^2} + \frac{\partial^2 XYZ}{\partial z^2} = 0$$

- we can now separate the variables:

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

- dividing both sides by XYZ,

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{fxn of X completely}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{fxn of Y entirely}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{fxn of Z ONLY}} = 0$$

- for 3 very diff. terms to add to zero, each of them MUST be constants that add to zero!

LET

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= k^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= -l^2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= m^2 \end{aligned}\right\} k^2 - l^2 + m^2 = 0$$

- thus, the general solutions are

$$\left.\begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= Ce^{ily} + De^{-ily} \\ Z(z) &= Ee^{mz} + Fe^{-mz} \end{aligned}\right\} k\neq 0,\ l\neq 0,\ m\neq 0$$

- in the case of $k=l=m=0$, we have

$$X(x) = A + Bx$$
$$Y(y) = C + Dy$$
$$Z(z) = E + Fz$$

- therefore, the particular soln for nonzero k, l, m is

$$\Phi(x,y,z) = (Ae^{ikx} + Be^{-ikx})(Ce^{ily} + De^{-ily})(Ee^{-mz} + Fe^{-mz})$$

- for $k=l=m=0$,

$$\Phi(x,y,z) = (A + Bx)(C + Dy)(E + Fz)$$

- if not all constants vanish, we can constrain the solns s.t.

$$\Phi(x,y,z) = (A_{0l} + B_{0l}x)(C_{0l}e^{ily} + D_{0l}e^{-ily})(E_{0l}e^{lz} + F_{0l}e^{-lz}) \text{ for } k=0, l \neq 0$$

$$\Phi(x,y,z) = (A_{k0}e^{ikx} + B_{k0}e^{-ikx})(C_{k0} + D_{k0}y)(E_{k0}e^{kz} + D_{k0}e^{-kz}) \text{ for } k \neq 0, l=0$$

- thus, the most general soln must be the sum of ALL particular solutions

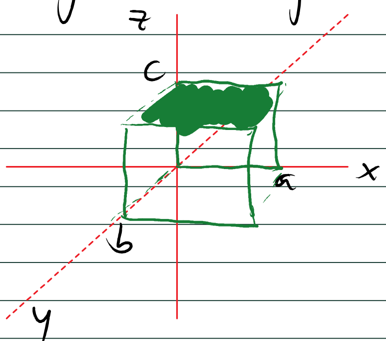
$$\Phi(x,y,z) = \sum_{k\neq 0}\sum_{l \neq 0} (A_{kl}e^{ikx} + B_{kl}e^{-ikx})(C_{kl}e^{ily} + D_{kl}e^{-iky})(E_{kl}e^{mz} + F_{kl}e^{-imz})$$
$$+ \sum_{l\neq 0} (A_{0l} + B_{0l}x)(C_{0l}e^{ily} + D_{0l}e^{-ily})(E_{0}e^{lz} + F_{0l}e^{-lz})$$
$$+ \sum_{k \neq 0} (A_{k0}e^{ikx} + B_{k0}e^{-ikx})(C_{k0} + D_{k0}y)(E_{k0}e^{kz} + D_{k0}e^{-kz})$$
$$+ (A_{00} + B_{00}x)(C_{00} + D_{00}y)(E_{00} + F_{00}z)$$

- the MOST general solution
- in 3D, the Laplace eqn has 3 2nd-order derivatives, meaning 6 BOUNDARY conditions
- to tackle: we write down ALL unknown

- to tackle, we write down ALL unknown constants & ALL BCs.

E.g. rectangular BC in charge-free region

- $\Phi = 0$ for all faces EXCEPT the top face
- $\Phi(x, y, z=c) = \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$



- for this example, we see that due to the vanishing of the potential at the sides and the bottom, the general sol'n reduces to

$$\Phi(x,y,z) = \sum_{k\neq 0}\sum_{l\neq 0} \left(A_{kl} e^{ikx} + B_{kl} e^{-ikx}\right)\left(C_{kl} e^{ily} + D_{kl} e^{-ily}\right)\left(E_{kl} e^{mz} + F_{kl} e^{-mz}\right)$$

- We now solve for A_{kl}, B_{kl}, C_{kl}, D_{kl}, E_{kl}, F_{kl}, k & l by applying the BCs one by one:

$$\Phi(x=0, y, z) = 0 = \sum_{k\neq 0}\sum_{l\neq 0} \left(A_{kl} \cancel{e^{ik0}}^{1} + B_{kl} \cancel{e^{-ik0}}^{1}\right)\left(C_{kl} e^{ily} + D_{kl} e^{-ily}\right)\left(E_{kl} e^{mz} + F_{kl} e^{-mz}\right)$$

$$0 = A_{kl} + B_{kl} \rightarrow B_{kl} = -A_{kl}$$

Update: $$\Phi(x,y,z) = \sum_{k\neq 0}\sum_{l\neq 0} A_{kl} \sin kx \left(C_{kl} e^{ily} + D_{kl} e^{-ily}\right)\left(E_{kl} e^{mz} + F_{kl} e^{-mz}\right)$$

Next: $$\Phi(x=a, y, z) = 0 = \sum_{k\neq 0}\sum_{k\neq 0} A_{kl} \sin ka \left(C_{kl} e^{ily} + D_{kl} e^{-ily}\right)\left(E_{kl} e^{mz} + F_{kl} e^{-mz}\right)$$

$0 = \sin ka \longrightarrow ka = n\pi$

$k = \frac{n\pi}{a}$

Update: $\Phi(x,y,z) = \sum_{k\neq 0} \sum_{l \neq 0} A_{kl} \sin\frac{n\pi x}{a} \left(C_{kl} e^{ily} + D_{kl} e^{-ily} \right) \left(E_{kl} e^{mz} + F_{kl} e^{-mz} \right)$

We do the same steps for the y-dimension to arrive at

$$\Phi(x,y,z) = \sum_{k,l\neq 0} A_{kl} \sin\frac{u\pi x}{a} C_{kl} \sin\frac{v\pi y}{b} \left(E_{kl} e^{mz} + F_{kl} e^{-mz} \right)$$

We also have $m^2 = k^2 + l^2 = \frac{u^2\pi^2}{a^2} + \frac{v^2\pi^2}{b^2}$

or $m = \pi\sqrt{\frac{u^2}{a^2} + \frac{v^2}{b^2}}$

Next, $\Phi(x,y,z=0) = 0 = \sum_{u,v\neq 0} A_{uv} \sin\frac{u\pi x}{a} C_{uv} \sin\frac{v\pi y}{b} \left(E_{uv} \cancel{e^{m0}}^{1} + F_{uv} \cancel{e^{-m0}}^{1} \right)$

$0 = E_{uv} + F_{uv} \longrightarrow F_{uv} = -E_{uv}$

Update: $\Phi(x,y,z) = \sum_{u,v\neq 0} A_{uv} C_{uv} \sin\frac{u\pi x}{a} \sin\frac{v\pi y}{b} E_{uv} \sinh mz$

And for the last BC,

$$\Phi(x,y,z=c) = \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} = \sum_{u,v\neq 0} A_{uv} C_{uv} E_{uv} \sin\frac{u\pi x}{a} \sin\frac{v\pi y}{b} \sinh\left(\pi\sqrt{\frac{u^2}{a^2} + \frac{v^2}{b^2}}\, c\right)$$

Absorb into A_{uv}

$\therefore \cancel{\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}} = A_{11} \cancel{\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}} \sinh\left(\pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}\right)$

or $$\Lambda_{11} = \frac{1}{\sinh\left(\pi c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\right)}$$

Which completes our solution:

$$\Phi(x,y,z) = \frac{1}{\sinh\left(\pi c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\right)} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sinh\left(\pi z\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\right)$$