Chapter 4: Multipoles, electrostatics of macroscopic media, dielectrics

> Multipole expansion (§ 4.1)

Consider: localized ρ within some region R.

For distances $\gg R$, $\rho \sim$ sphere/pt. charge

Then, for distances $> R$, Expand using spherical harmonics & keep few terms only.

> Recall the full solution to Laplace's eqn in spherical coordinates for the full azimuthal range

$$\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (A_l r^l + B_l r^{-l-1}) Y_{lm}(\theta,\varphi)$$

> For distances $r > 0 \to \infty$, we want to prevent the potential from blowing up, so we set $A_l = 0$:

$$\Phi(r>0,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_l r^{-l-1} Y_{lm}(\theta,\varphi)$$

> We then set $B_l := \frac{1}{4\pi\epsilon_0}\frac{4\pi}{2l+1} q_{lm}$ to retrieve the multipole expansion

retrieve the multipole expansion

$$\Phi(r>0,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1} q_{lm}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}$$

since $l=0 \rightarrow$ monopole, $l=1 \rightarrow$ dipole, etc. (cf. Mass quadrupole fxn in GW)

> The problem then reduces to solving q_{lm} (see cf earlier)

> e.g. The Coulomb solution

$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')}{|\vec{x}-\vec{x}'|} d\vec{x}'$$

in which we expand $\frac{1}{|\vec{x}-\vec{x}'|}$ for $x > x'$:

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r'^l}{r^{l+1}}\frac{1}{2l+1} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)$$

→ Green's fxn expansion

Thus, inserting to the Coulomb soln

$$\Phi = \frac{1}{4\pi\epsilon_0}\int d\vec{x}'\rho(\vec{x}')\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r'^l}{r^{l+1}}\frac{4\pi}{2l+1}Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)\right]$$

$$\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\underbrace{\left[\int d\vec{x}'\rho(\vec{x}')Y^*_{lm}(\theta',\varphi') r'^l\right]}_{q_{lm}}$$

> Define q_{lm} := spherical multipole moments

Physical significance:

- $l=0$: monopole moment, no θ/φ dependence since $Y_{00} = \sqrt{\frac{1}{4\pi}}$ s.t

$$q_{00} = \sqrt{\frac{1}{4\pi}} \int \rho(\vec{x}')\, d\vec{x}' = \sqrt{\frac{1}{4\pi}}\, q \rightarrow \text{total charge}$$

- $\ell = 1$: dipole moment, $\propto \vec{p}$ (electric dipole moment)

$$q_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \int d\vec{x}'\, r'\sin\theta'\, e^{\mp i\varphi'}\, \rho(\vec{x}')$$

$$= \mp\sqrt{\frac{3}{8\pi}} \int d\vec{x}' \Big(\underbrace{r'\sin\theta'\cos\varphi'}_{x'} \mp i\, \underbrace{r'\sin\theta'\sin\varphi'}_{y'}\Big)\rho(\vec{x}')$$

$$= \mp\sqrt{\frac{3}{8\pi}}\,(p_x \mp i p_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int d\vec{x}'\, \rho(\vec{x}')\, \underbrace{r'\cos\theta'}_{z'} = \sqrt{\frac{3}{4\pi}}\, p_z$$

- $\ell = 2$: quadrupole moment (cf. Grav. wave)

$$q_{2,\pm2} = \pm\sqrt{\frac{15}{32\pi}} \int d\vec{x}'\, \rho(\vec{x}')\, \underbrace{r'^2\sin^2\theta'\, e^{\mp i2\varphi'}}_{(x' \mp i y')^2}$$

$$= \frac{1}{3}\sqrt{\frac{15}{32\pi}}\,(Q_{11} \mp 2i Q_{12} - Q_{22})$$

where $Q_{ij} := \int d\vec{x}'\, \rho(\vec{x}')\,(3x'_i x'_j - r'^2\delta_{ij})$

$$q_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}} \int d\vec{x}'\, \rho(\vec{x}')\, r'^2 \sin\theta'\cos\theta'\, e^{\mp i\varphi'}$$

$$= \mp\sqrt{\frac{15}{8\pi}} \int d\vec{x}'\, \rho(\vec{x}')\, z'(x' \mp i y')$$

$$= \mp\frac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} \mp i Q_{23})$$

$$q_{20} = \sqrt{\frac{5}{16\pi}} \int d\vec{x}'\, \rho(\vec{x}')\, r'^2(3\cos^2\theta' - 1)$$

$$= \sqrt{\frac{5}{16\pi}} \int d\vec{x}'\, \rho(\vec{x}')\,(3z'^2 - r'^2)$$

$$= \sqrt{\frac{5}{16\pi}}\, Q_{33}$$

> Thus we can rewrite the full potential as

$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, q_{\ell m}\, \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}$$

$$= \frac{1}{\epsilon_0} q_{00} \frac{Y_{00}}{r} + \frac{1}{3\epsilon_0 r^2}\left[q_{10} Y_{10} + q_{11} Y_{11} + q_{1,-1} Y_{1,-1}\right]$$

$$+ \frac{1}{5\epsilon_0 r^3}\left[q_{20} Y_{20} + q_{21} Y_{21} + q_{2,-1} Y_{2,-1} + q_{22} Y_{22} + \dots\right]$$

$$\boxed{= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}\frac{x_i x_j}{r^5} + \dots\right]}$$

ELECTRIC FIELD in MULTIPOLE EXPANSION (§4.1)

> Recall the potential in multipole expansion

$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}q_{\ell m}\frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}} \qquad (1)$$

> We use $\vec{E} = -\nabla\Phi$, $\nabla = \left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right]$

so that

$$\vec{E} = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{q_{\ell m}}{2\ell+1}\left[-\frac{\partial}{\partial r}\frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}\frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}\hat{\varphi}\right] \qquad (2)$$

$$\vec{E} = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{q_{\ell m}}{2\ell+1}\frac{1}{r^{\ell+2}}\left[Y_{\ell m}(\ell+1)\hat{r} - \frac{\partial Y_{\ell m}}{\partial\theta}\hat{\theta} - \frac{im}{\sin\theta}Y_{\ell m}\hat{\varphi}\right] \qquad (3)$$

> let's check the dominant contributions:

- monopole ($\ell=0$):

$$\vec{E}_{\ell=0} = \frac{1}{\epsilon_0}\frac{q_{00}}{r^2}Y_{00}\hat{r} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \qquad (4)$$

→ pt. charge at origin

- dipole ($\ell=1$)

$$\begin{aligned}\vec{E}_{\ell=1} &= \frac{1}{3\epsilon_0}\frac{q_{1,-1}}{r^3}\left[2Y_{1,-1}\hat{r} - \frac{\partial Y_{1,-1}}{\partial\theta}\hat{\theta} + i\frac{Y_{1,-1}}{\sin\theta}\hat{\varphi}\right] \\ &+ \frac{1}{3\epsilon_0}\frac{q_{10}}{r^3}\left[2Y_{10}\hat{r} - \frac{\partial Y_{10}}{\partial\theta}\hat{\theta}\right] \\ &+ \frac{1}{3\epsilon_0}\frac{q_{11}}{r^3}\left[2Y_{11}\hat{r} - \frac{\partial Y_{11}}{\partial\theta}\hat{\theta} - i\frac{Y_{11}}{\sin\theta}\hat{\varphi}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{3\hat{x}(\vec{p}\cdot\hat{x}) - \vec{p}}{|\vec{x}|^3}\ ; \quad \vec{p} = \int d\vec{x}'\,\rho(\vec{x}')\,\vec{x}'\end{aligned} \qquad (5)$$

> Generally, multipole moments depend on origin;

That's why for gravitationally interacting bodies, the dipole moment can vanish with a change in coordinates. The lowest nonvanishing moment is ALWAYS independent of the origin, but higher multipoles aren't.

> However, we see that the previous result (5) of $\vec{E}$ will be problematic once we integrate over an off-centered sphere (say at $\vec{x} = \vec{x}_0$). How do we account for this singularity?

- First consider a localized charge distribution $\rho(\vec{x})$ giving rise to an e-field $\vec{E}(\vec{x})$. To calculate the field's integral over a sphere of radius R, we consider two cases:
  - Case I: All charge is contained within the sphere

Then,
$$\int_{r<R} \vec{E}(\vec{x})\, d^3\vec{x} = -\int_{r<R} \vec{\nabla}\Phi\, d^3\vec{x} \quad \{\text{Stokes'}$$
$$= -\oint_{r=R} \Phi\, R^2\, \hat{n}\, d\Omega \qquad (6)$$

where $\hat{n} = \vec{x}/R$ (directed outside)

We do know the solution for the potential using the Green's fxn of the sphere,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (7)$$

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s.t.

$$\int_{r<R} \vec{E}(\vec{x})\, d^3\vec{x} = -\frac{R^2}{4\pi\epsilon_0}\int d^3x'\, \rho(\vec{x}') \oint \frac{\hat{n}}{|\vec{x}-\vec{x}'|} d\Omega \qquad (8)$$

we then rewrite $\hat{n}$ in terms of spherical harmonics instead of its usual Cartesian coordinate rep.:

$$\hat{n} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} \qquad (9)$$

We need to use combinations of Y_{1m} (or $l=1$) in order to substitute these terms. Specifically,

$$\left.\begin{aligned} \sin\theta\cos\varphi &= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\,(Y_{11} - Y_{1,-1}) \\ \sin\theta\sin\varphi &= \frac{i}{2}\sqrt{\frac{8\pi}{3}}\,(Y_{11} + Y_{1,-1}) \\ \cos\theta &= \sqrt{\frac{4\pi}{3}}\, Y_{10} \end{aligned}\right\} \qquad (10)$$

We also replace the spherical Green's fxn

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^* Y_{lm} \qquad (11)$$

As we multiply this term w/ $\hat{n}$, we see that only $l=1$ terms will survive, so that the last integral in ( ) becomes

$$\oint \frac{\hat{n}}{|\vec{x}-\vec{x}'|} d\Omega = \oint d\Omega\, \frac{4\pi}{3}\frac{r_<}{r_>^2} \underbrace{\sum_{m=-1}^{1} Y_{1m}^* Y_{1m}}_{\frac{3}{4\pi}\left[\sin\theta'\sin\theta\cos(\varphi-\varphi') + \cos\theta'\cos\theta\right]}\, \hat{n} \qquad (12)$$

$$\frac{3}{4\pi}\underbrace{\left[\sin\theta'\sin\theta\cos(\varphi-\varphi') + \cos\theta'\cos\theta\right]}_{\cos\gamma \rightarrow \text{ angle bet}}$$

$\cos\gamma \to$ angle bet source & observation pt.

- To further simplify, we use orthogonality

$$\oint d\Omega\, Y^*_{l'm'} Y_{lm} = \delta_{l'l}\,\delta_{m'm} \tag{13}$$

- We wish to integrate the spherical harmonics from $\frac{1}{|\vec{x}-\vec{x}'|}$ term per term, leading to the simplest form

$$\oint_{r=R} d\Omega \frac{\hat{n}}{|\vec{x}-\vec{x}'|} = \frac{4\pi}{3}\frac{r_<}{r_>^2}\hat{n}', \quad \hat{n}' = \frac{\vec{r}'}{r}. \tag{14}$$

- Thus we are left with

$$\int_{r<R} \vec{E}(\vec{x})\, d^3x = -\frac{R^2}{\epsilon_0}\int d^3x' \frac{r_<}{r_>^2}\hat{n}' \rho(\vec{x}') \tag{15}$$

- For a sphere completely enclosing the charge density, $r_< = r'$ and $r_> = R^2$, leading to

$$\int_{r<R} \vec{E}(\vec{x})\, d^3x = -\frac{\vec{p}}{3\epsilon_0} \tag{16}$$

- For case II: charges outside the sphere, we replace $r_< = R$ & $r_> = r'$ in (15). Thus

$$\int_{r<R} \vec{E}(\vec{x})\, d^3x = -\frac{R^3}{3\epsilon_0}\underbrace{\int d^3x \frac{\rho(\vec{x}')}{r'^2}\hat{n}'}_{-4\pi\epsilon_0 \vec{E}(\vec{0})}$$

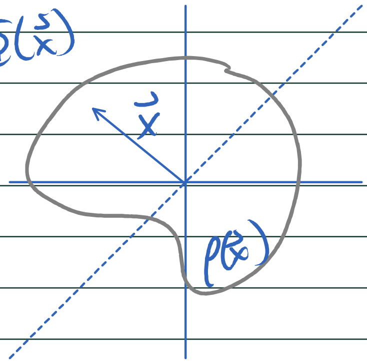
$$= \frac{4\pi R^3}{3}\vec{E}(\vec{0}) \tag{17}$$

> Thus, we need to modify (5) due to (16)'s nonzero contribution:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{\vec{p}}{3}4\pi\delta(\vec{x}-\vec{x}_0)\right] \tag{18}$$

(§4.2) Multipole expansion of the energy of a charge distribution in an external field.

> Consider the setup to the right, for which $\Phi(\vec{x})$ is slowly varying:



$$\Phi(\vec{x}) = \Phi(0) + \vec{x}\cdot\nabla\Phi(0) + \frac{1}{2}\sum_i\sum_j x_i x_j \frac{\partial^2\Phi}{\partial x_i \partial x_j}(0) + \ldots \quad (1)$$

> The second term can be replaced with $-\vec{E}(0)$ while the third term can be replaced by

$$\frac{\partial^2\Phi}{\partial x_i \partial x_j}(0) = \left[\frac{\partial}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\right](0) = \left[\frac{\partial}{\partial x_i}(-E_j)\right](0) = -\frac{\partial E_j}{\partial x_i}(0) \quad (2)$$

> Since the external potential (field) contains no charges, $\vec{\nabla}\cdot\vec{E} = 0$, and we can further subtract $\frac{1}{6}r^2\vec{\nabla}\cdot\vec{E} = 0$ to obtain

$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_i\sum_j (3x_i x_j - r^2\delta_{ij})\frac{\partial E_j}{\partial x_i}(0) + \ldots \quad (3)$$

(since $\frac{1}{6}r^2\vec{\nabla}\cdot\vec{E} = \frac{1}{6}r^2(\partial_i E_i) = \frac{1}{6}r^2(\partial_j E_i \delta_{ij})$ (Einstein s.c.)

> We can use this expansion (3) in solving the electrostatic energy of the system $W = \int \rho(\vec{x})\,\Phi(\vec{x})\,d\vec{x}$, integrating term per term:

total charge "q"

$$W = \Phi(0)\left(\int \rho(\vec{x})\,d^3x\right) - \left(\int \rho(\vec{x})\,\vec{E}(0)\right)d^3x - \frac{1}{6}\sum\sum \frac{\partial E_j}{\partial x_i}(0)$$

Term per term: $\longrightarrow$ total charge q

$$W = \Phi(0)\underbrace{\int \rho(\vec{x})\, d^3x}_{\text{total charge } q} - \int \underbrace{\rho(\vec{x})\,\vec{x}}_{\vec{p}(\vec{x})}\cdot\vec{E}(0)\, d^3x - \frac{1}{6}\sum_i \sum_j \frac{\partial E_j}{\partial x_i}(0) \underbrace{\int \rho\,(3x_i x_j - r^2\delta_{ij})\, d^3x}_{Q_{ij}} + \dots \qquad (4)$$

$$W = q\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_i \sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \dots \qquad (5)$$

> let's digest eq. (5) term per term.

$q\Phi(0) \rightarrow$ Energy contribution from the total charge; interaction w/ the potential gives the charge energy

$\vec{p}\cdot\vec{E}(0) \rightarrow$ Energy contribution from the electric dipole; interaction w/ the electric field (at the origin) gives the dipole energy

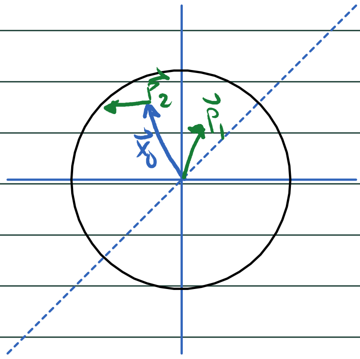
$\underbrace{\frac{1}{6}\sum_i \sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0)}_{}$ $\rightarrow$ Energy contribution from the electric quadrupole; interaction w/ the electric field gradient gives the quadrupole energy. Useful in nuclear physics

Is there a similar contribution for GW?

> The existence of electric quadrupole can be used as a probe for the properties of atomic nuclei, including the shape of nuclei & intranuclear forces.

> Consider now the dipole-dipole interaction (cf. Chem).

What is its energy contribution? Consider two dipoles $\vec{p}_1$ & $\vec{p}_2$. Then, the electric field due to $\vec{p}_2$ is



$$\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}_2\cdot\hat{n}) - \vec{p}_2}{|\vec{x}_0|^3}\right] \quad (6)$$

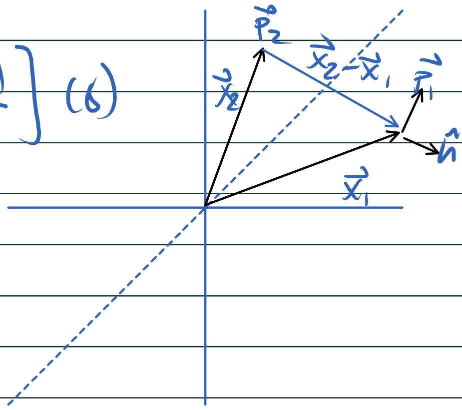
> Here, the Dirac delta vanishes due to off-location of the field point $\vec{x}=0$. Thus the energy of $\vec{p}_2$ becomes

$$-\vec{p}_1\cdot\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{n})(\vec{p}_2\cdot\hat{n})}{|\vec{x}_0|^3}\right] \quad (7)$$

> In general, $\vec{p}_1$ may also be off-centered w.r.t.

$$-\vec{p}_1\cdot\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_2\cdot\hat{n})(\vec{p}_1\cdot\hat{n})}{|\vec{x}_1-\vec{x}_2|^3}\right] \quad (8)$$

↳ invariant to $\hat{n}\to-\hat{n}$ or
$\vec{x}_1-\vec{x}_2 \to \vec{x}_2-\vec{x}_1$



(§4.3) Electrostatics in ponderable media

> Ponderable media/material: pertains to material that can accept applied EM fields; such fields cause the material to be polarized (E-field) or magnetized (B-field).

> Consider a material that has infinitely small charge distributions scattered; we can determine the electric potential due to the whole material as follows:

- Expand the infinitesimal potential in terms of the multipole expansion

$$d\Phi = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3} + \dots \quad (1)$$

- Multiply the RHS by $\frac{dV}{dV}$:

$$d\Phi = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\underbrace{\left(\frac{dq}{dV}\right)}_{\rho(\vec{x}') \text{ charge density}} dV + \frac{1}{4\pi\epsilon_0}dV\underbrace{\left(\frac{\vec{p}}{dV}\right)}_{\text{polarization definition } (\vec{P})}\cdot\frac{\vec{x}}{r^3} + \dots \quad (2)$$

- Integrate both sides to get the full potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}dV + \frac{1}{4\pi\epsilon_0}\int\frac{\vec{P}(\vec{x}')\cdot\vec{x}}{|\vec{x}-\vec{x}'|}dV + \dots \quad (3)$$

- First term pertains to the contribution of the free charges; 2nd term gives the contribution of the "bound" charges that were polarized.

- Typically, the applied field induces the polarization $\vec{P}$, then this polarization creates its own electric field.

> We can rewrite eqn. (3) to recover Coulomb's law as follows:

- Using the identity $\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$ (derivative wrt $\vec{x}'$), we can rewrite the 2nd term as

$$\frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}\cdot\vec{x}}{|\vec{x}-\vec{x}'|^3}\,d^3x = \frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)d^3x \qquad (4)$$

- Invoke IBP to rewrite (4) as

$$\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)d^3x = \underbrace{\left(\frac{\vec{P}\cdot\hat{n}}{|\vec{x}-\vec{x}'|}\right)_S}_{\text{boundary term evaluated at the surface}} - \int \frac{\vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x \quad (5)$$

- As we evaluate for all space, the boundary term vanishes; the potential now reads

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x - \frac{1}{4\pi\epsilon_0}\int \frac{\vec{\nabla}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|}\,d^3x \qquad (6)$$

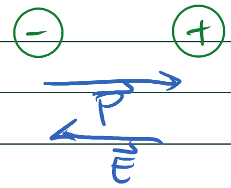
$$= \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}') - \vec{\nabla}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|}\,d^3x}_{\text{Coulomb's law for } \rho_{total} = \rho - \underbrace{\vec{\nabla}'\cdot\vec{P}}_{\text{polarized}}}$$

charges

- If we define $\rho_{bound} = -\vec{\nabla}' \cdot \vec{P}$, we can make it look like Gauss' law by dividing both sides by ϵ_0:

points opposite to $\vec{E}_{bound}$, recall

$$\vec{\nabla}' \cdot \underbrace{\left(\frac{-\vec{P}}{\epsilon_0}\right)}_{\vec{E}_{bound},\ \text{due to}} = \frac{\rho_{bound}}{\epsilon_0} \qquad (7)$$



> We can already rewrite Gauss' law to accommodate for bound charges; define $\vec{D} \rightarrow$ electric displacement field

$$\vec{\nabla} \cdot \frac{\vec{D}}{\epsilon_0} = \frac{\rho(\vec{x})}{\epsilon_0} \qquad (8)$$

> Thus, the full divergence of $\vec{E}$ becomes

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{tot}}{\epsilon_0} = \frac{1}{\epsilon_0}\left(\rho - \vec{\nabla}' \cdot \vec{P}\right) = \frac{1}{\epsilon_0}\left(\vec{\nabla} \cdot \vec{D} - \vec{\nabla}' \cdot \vec{P}\right) \qquad (9)$$

$$\boxed{\vec{E} = \frac{1}{\epsilon_0}\left(\vec{D} - \vec{P}\right)} \qquad (10)$$

> This applies to materials for which the dipole moment is the dominant contributor to the field.

The linear, isotropic dielectric

> has a property $\vec{P} = \left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\vec{D}$; $\epsilon_r = \frac{\epsilon}{\epsilon_0}$ (11)

$\epsilon_r \hookrightarrow$ relative electric permittivity OR "dielectric constant"

> Plugging to eq. (10), we get

$$\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \frac{D}{\epsilon_0}\left(\cancel{1} - \cancel{1} + \frac{1}{\epsilon_r}\right) = \frac{D}{\epsilon}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon} \longrightarrow \text{Linear, isotropic materials ONLY} \qquad (12)$$

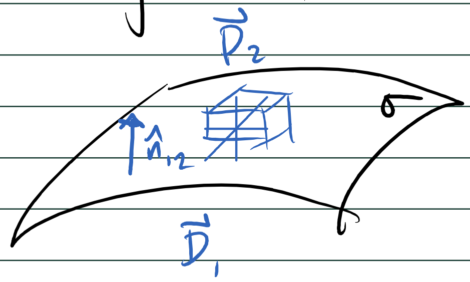
Thus, $\vec{\nabla} \cdot \vec{D} = \rho \longrightarrow \vec{\nabla} \cdot \epsilon \vec{E} = \rho$

(§4.4)

Boundary-value problems for linear, dielectric media

> Recall $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}$ $(\epsilon \neq \epsilon_0)$ (1)

- equation is only applicable for regions of unif. dielectric constant
- Use superposition to solve regions w/ diff materials
- First BC: discontinuity of electric displacement

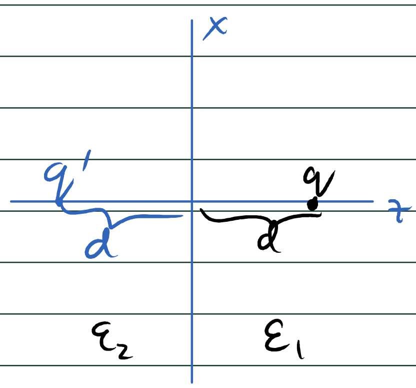


$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{12} = \sigma \quad (2)$$

- 2nd BC: discontinuity of Electric Field (around a loop)

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n}_{12} = \vec{0} \quad (3)$$

> Consider the configuration to the right, for which we want to find the field for $z > 0$.



- We use the method of images to solve this problem.
- First set up the equs governing the fields

$$\nabla \cdot \vec{E}_1 = \rho/\epsilon_1 \quad , \quad z > 0 \qquad (4)$$
$$\epsilon_2 (\vec{\nabla} \cdot \vec{E}_2) = 0 \quad , \quad z < 0$$

- Then set up the BCs at $z=0$ (using cyl. coord.)

$$\left.\begin{aligned} \epsilon_1 E_{1,z} &= \epsilon_2 E_{2,z} \\ E_{1,r} &= E_{2,r} \end{aligned}\right\} \forall \varphi \qquad (5)$$

- Using the method of images, we can easily describe the potential in region 1 as the sum of the point charges' contributions:

$$\Phi_1 = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{(r^2+(z-d)^2)^{1/2}} + \frac{q'}{(r^2+(z+d)^2)^{1/2}}\right) \qquad (6)$$

- But we still don't know q'. To solve it, we need another eqn: the potential in region 2. To do this, we need to place another image charge at $z=d$ (where the original pt. charge is located). This gives us

$$\Phi_2 = \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{(r^2+(z-d)^2)^{1/2}}\right) \qquad (6)$$

- Now we can freely apply the BC (5a):

$$\epsilon_1 \frac{\partial}{\partial z}\Phi_1 = \epsilon_2 \frac{\partial}{\partial z}\Phi_2 \quad , \quad z=0 \qquad (7)$$

$$\cancel{\epsilon_1}\left(\frac{1}{\cancel{4\pi\epsilon_1}}\right)\left[\cancel{-\tfrac{1}{2}}\frac{q}{\cancel{(r^2+d^2)^{3/2}}}(-\cancel{2}d) \cancel{-\tfrac{1}{2}}\frac{q'}{\cancel{(r^2+d^2)^{3/2}}}(\cancel{2}d)\right] = \cancel{\epsilon_2}\frac{1}{\cancel{4\pi\epsilon_2}}\frac{q''}{\cancel{(r^2+d^2)^{3/2}}}(-\cancel{2}d) \qquad (8)$$

$$q - q' = q'' \qquad (9)$$

- As for BC (5b),

$$\frac{\partial}{\partial r}\Phi_1 = \frac{\partial}{\partial r}\Phi_2 \;,\quad z=0 \qquad (10)$$

$$\frac{1}{\cancel{4\pi}\epsilon_1}\left[-\cancel{\tfrac{1}{2}}\frac{q}{\cancel{(r^2+d^2)^{3/2}}}\cancel{(2r)} - \cancel{\tfrac{1}{2}}\frac{q'}{\cancel{(r^2+d^2)^{3/2}}}\cancel{(2r)}\right] = \frac{1}{\cancel{4\pi}\epsilon_2}\left[-\cancel{\tfrac{1}{2}}\frac{q''}{\cancel{(r^2+d^2)^{3/2}}}\cancel{2r}\right] \qquad (11)$$

$$\frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2} \qquad (12)$$

- Combining eqs (9) & (12),

$$\boxed{\begin{aligned} q' &= \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\,q \\ q'' &= \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\,q \end{aligned}} \qquad (13)$$

- Substituting back to Φ_1 & Φ_2,

$$\boxed{\begin{aligned} \Phi &= \frac{q}{4\pi\epsilon_1}\left[\frac{1}{\sqrt{r^2+(z-d)^2}} + \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)\frac{1}{\sqrt{r^2+(z+d)^2}}\right], \quad z>0 \\ &= \frac{q}{4\pi\epsilon_2}\left(\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right)\frac{1}{\sqrt{r^2+(z-d)^2}}, \quad z<0 \end{aligned}} \qquad (14)$$

Animation of Fig. 4.5 of Jackson https://twitter.com/j_bertolotti/status/1315975665385123842?s=20

- What is the bound charge density σ_{pol}? We can extract a third boundary condition by using the same pillbox setup but only enclosing the bound charge, s.t.

$$\sigma_{pol} = -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n}_{12} \;,\quad z=0 \qquad (15)$$

- Assuming we have linear dielectrics,

$$\sigma_{pol} = -\left(\frac{\epsilon_{r,2}-1}{\epsilon_{r,2}}\vec{D}_2 - \frac{\epsilon_{r,1}-1}{\epsilon_{r,1}}\vec{D}_1\right)\cdot\hat{n}_{12}$$

$$= -\left(\frac{\epsilon_{r,2}-1}{\epsilon_{r,2}}(\epsilon_2\vec{E}_2) - \frac{\epsilon_{r,1}-1}{\epsilon_{r,1}}(\epsilon_1\vec{E}_1)\right)\cdot\hat{n}_{12}$$

$$= -\left(\frac{\epsilon_{r,2}-1}{\epsilon_{r,2}} (\epsilon_2 \vec{E}_2) - \frac{\epsilon_{r,1}-1}{\epsilon_{r,1}} (\epsilon_1 \vec{E}_1) \right) \cdot \hat{n}_{12}$$

$$= -\left[(\epsilon_2 - \epsilon_0) \vec{E}_2 - (\epsilon_1 - \epsilon_0) \vec{E}_1 \right] \cdot \hat{n}_{12} \qquad (16)$$

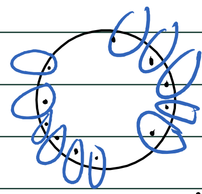
- Using eq. (7) for $\vec{E}_2$ & $\vec{E}_1$,

$$\sigma_{pol} = \left[(\epsilon_0 - \epsilon_2) \frac{q}{4\pi\epsilon_2} \left(\frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} \right) \left(-\frac{1}{2} \frac{(-2d)}{(r^2+d^2)^{3/2}} \right) \right.$$
$$\left. + (\epsilon_1 - \epsilon_0) \frac{-q}{8\pi\epsilon_1} \left(\frac{-2d}{(r^2+d^2)^{3/2}} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{2d}{(r^2+d^2)^{3/2}} \right) \right]$$

$$\boxed{\sigma_{pol} = -\frac{q}{2\pi} \frac{\epsilon_0 (\epsilon_2 - \epsilon_1)}{\epsilon_1 (\epsilon_2 + \epsilon_1)} \frac{d}{(r^2+d^2)^{3/2}}} \qquad (17)$$

The concept of Polarizability & Susceptibility (§4.3)

> Recall the modification of the divergence of $\vec{E}$ for ponderable media



$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left[\rho - \vec{\nabla}\cdot\vec{P}\right] \qquad (1)$$

where $\vec{P}(\vec{x}) = \sum_i N_i \langle \vec{p}_i \rangle$, $\vec{p}_i$ is the dipole moment of the ith type of molecule and the average is taken over a small volume centered at $\vec{x}$ and N_i is the ave. number per unit vol of the ith type of molecule at pt $\vec{x}$.

> We then define the electric displacement $\vec{D}$ as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad (2)$$

to reduce eq (1) to

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad (3)$$

> A constitutive relation bet. $\vec{D}$ & $\vec{E}$ is necessary before a solution for the electrostatic potential or fields can be obtained. (Ch. 1)

> Isotropic medium then allows for $\vec{P} \parallel \vec{E}$

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \qquad (4)$$

where χ_e is called electric susceptibility and is independent of the medium. Thus we can define the electric permittivity of the material ϵ as

$$\epsilon = \epsilon_0 (1+\chi_e) \qquad (5)$$

s.t.

$$\vec{D} = \epsilon \vec{E} \qquad (6)$$

and

$$\frac{\epsilon}{\epsilon_0} = (1+\chi_e) \qquad (7)$$

is known as the dielectric constant, or relative electric permittivity.

> Dense materials give rise to extra electric fields $\vec{E}_i = \vec{E}_{near} - \vec{E}_p$ so that the total field at the molecule is $\vec{E} + \vec{E}_i$. We can calculate $\vec{E}_p$ by redoing the procedure on lec 3 for calculating the electric field inside a spherical volume of radius R and volume V, st the total dipole moment inside is

$$\vec{p} = \frac{4}{3}\pi R^3 \vec{P} \qquad (8)$$

so that

$$\vec{E}_p = \frac{3}{4\pi R^3} \int_{r<R} \vec{E}\, d^3\vec{x} = -\frac{\vec{P}}{3\epsilon_0} \qquad (9)$$

and

$$\vec{E}_i = \frac{1}{3\epsilon_0} \vec{P} + \vec{E}_{near} \qquad (10)$$

> $\vec{E}_{near} = 0$ for highly symmetric systems such as a cubic lattice (Lorentz). Same is true for amorphous ones.

> Define molecular polarizability γ_{mol} as the ratio

$$\gamma_{mol} = \frac{\overbrace{\langle p_{mol} \rangle}^{\text{ave. dipole moment}}}{\epsilon_0 \underbrace{(\vec{E} + \vec{E}_i)}_{\text{applied field}}} \tag{11}$$

so that

$$\vec{P} = N \gamma_{mol} \left(\epsilon_0 \vec{E} + \frac{1}{3}\vec{P} \right)$$

$$\underbrace{\chi_e}_{\text{macroscopic}} = \frac{N \gamma_{mol} \rightarrow \text{microscopic}}{1 - \frac{1}{3} N \gamma_{mol}} \tag{12}$$

or, since $1 + \chi_e = \frac{\epsilon}{\epsilon_0}$,

$$\gamma_{mol} = \frac{3}{N} \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} \right) \tag{13}$$

↳ Clausius-Mossotti eqn.

> Electrostatic energy in dielectric media (§4.7)

- Recall electrostatic ~~PE~~ stored in A CHARGE dist.

$$W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d\vec{x} \qquad (1)$$

- We can still use eq. (1) for LINEAR dielectrics by building ρ piece by piece, letting each piece interact with the TOTAL potential Φ (including dielectric effects)
- Recall the eqn governing free charges

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad (2)$$

- Substituting to (1), we get

$$W = \frac{1}{2}\int (\vec{\nabla}\cdot\vec{D})\,\Phi(\vec{x})\,d\vec{x} \qquad (3)$$

$$= \frac{1}{2}\left[\cancel{\Phi\vec{D}\Big|_{\text{all space}}}^{\,0} - \int \vec{D}\cdot\vec{\nabla}\Phi\,d\vec{x}\right] \qquad (4)$$

$$= \boxed{\frac{1}{2}\int \vec{D}\cdot\vec{E}\,d\vec{x}} \qquad (5)$$

where we used IBP in (4).

- We can then substitute $\epsilon\vec{E} = \vec{D}$ to recover a familiar expression for W (works for linear dielectrics only)

$$\boxed{W = \frac{\epsilon}{2}\int |\vec{E}|^2\,d\vec{x}} \qquad (6)$$

Chapter 5: Magnetostatics (§5.1)

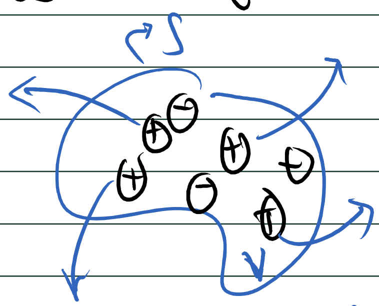
> Recall: Static charges → static electric fields
? → static magnetic fields

> Turns out: magnetostatics is a special case of electrodynamics → electric charges move

> Define $\vec{J}$: positive charge per unit area per unit time
: aka current density
: direction is given by direction of charge flow

> Relationship of $\vec{J}$ with the familiar I:
integrate $\vec{J}$ over the cross-sectional area of the wire in which $\vec{J}$ is contained.

> Using our knowledge of volumes and their bounding surfaces, we can relate Q to $\vec{J}$ as



$$\oint_S \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} Q_V \qquad (1)$$

Converting the integral via Gauss' Law,

$$\int_V (\vec{\nabla} \cdot \vec{J})\, dV = -\frac{\partial}{\partial t} Q_V \qquad (2)$$
$$= -\frac{\partial}{\partial t} \int_V \rho_v \, dV$$

OR, getting the integrands,

$$\boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}} \rightarrow \text{continuity eqn.} \qquad (3)$$

> Magnetostatics is then defined when we assume no build-up or depletion of charge

$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = 0} \qquad (4)$$

Thus, the ? earlier should refer to "steady currents".

- From (4), we can immediately infer properties of $\vec{J}$:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{J}) = \nabla^2 \vec{J} = 0 \qquad (5)$$

Divergence-less fields imply the vanishing of point sources.

Lec 08: 25 Mar 2021

Thursday, 25 March 2021 16:25

The Biot-Savart and Ampere Laws (§5.2)

> The experimental results (Oersted, Biot, Savart):

- magnetic field is directly proportional to length of wire

$$dB \propto dl \tag{1}$$

- magnetic field is directly proportional to current

$$dB \propto I\,dl \tag{2}$$

- Magnetic field is inversely proportional to the square of the distance from the wire

$$dB \propto \frac{I\,dl}{r^2} \tag{3}$$

- Magnetic field points perpendicular to the plane in which the wire and field point

$$d\vec{B} \propto \frac{I\,dl}{r^2}(\hat{l}\times\hat{r}) \tag{4}$$

$$d\vec{B} = K\frac{I\,dl}{r^2}(\hat{l}\times\hat{r}) \tag{5}$$

$$= K\frac{I}{r^2}\left(d\vec{l}\times\frac{\vec{r}}{r}\right) \tag{6}$$

$$d\vec{B} = K\frac{I}{\ }(d\vec{l}\times\vec{r}) \tag{7}$$

$$\vec{dB} = K \frac{I}{r^3}(\vec{dl} \times \vec{r}) \quad (7)$$

- Combining units,

$$[\vec{dB}] = [K]\frac{[A]}{[m^3]}[\cancel{m^2}] = [G] \quad (8)$$

Gauss

$$[K] = \frac{[G][m]}{[A]} \rightarrow \mu_0 \quad (9)$$

permeability!

- In SI,

$$\boxed{\vec{dB} = \frac{\mu_0}{4\pi}\frac{I}{r^3}(\vec{dl} \times \vec{r})} \quad (10)$$

Biot-Savart law

- Integrating both sides,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\vec{dl} \times \vec{r}}{r^3} \quad (11)$$

For more complex materials, we expand I:

$$\vec{B} = \frac{\mu_0}{4\pi}\iint \frac{(\vec{J}\cdot\hat{l})\vec{dl} \times \vec{r}}{r^3}\,da \quad (12)$$

or

$$\vec{B} = \frac{\mu_0}{4\pi}\iint \frac{\vec{J} \times \vec{r}}{r^3}\,dl\,da \quad (13)$$

In a cleaner form, we rewrite (13) as

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}\,d^3\vec{x}'} \quad (14)$$

where as always, primed coordinates refer to locations of source (currents).

> Eq. (14) is the equivalent of Coulomb's Law

for determining the magnetic field (density). To use it, we must integrate over all space. However, as is the trouble of using Coulomb's law, most sources are localized. (§ 5.3)

> To find a localized version of eq. (14), we first convert it to a differential equation.

- Recall: $-\frac{\hat{r}}{r^2} = -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \nabla\left(\frac{1}{r}\right)$ (15)

so that eq (14) becomes

$$\vec{B} = -\frac{\mu_0}{4\pi}\int \vec{J} \times \vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3\vec{x}' \quad (16)$$

- Using IBP for curls,

$$\vec{J} \times \vec{\nabla} f = f(\vec{\nabla}\times\vec{J}) - \vec{\nabla}\times(f\vec{J}) \quad (17)$$

or

$$\vec{B} = \frac{\mu_0}{4\pi}\int\left(\vec{\nabla}\times\frac{\vec{J}}{|\vec{x}-\vec{x}'|}\right)d^3\vec{x}' - \int\frac{d^3x'}{|\vec{x}-\vec{x}'|}\,\cancel{\vec{\nabla}\times\vec{J}}^{\,0} \quad (18)$$

↳ vanishes since $\vec{J}(\vec{x}')$, not $\vec{J}(\vec{x})$

- Moving the curl out of the integral since it is wrt unprimed coordinates,

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\vec{\nabla}\times\int\frac{\vec{J}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right) \quad (19)$$

so that by taking the divergence of both sides, (since $\nabla\cdot\nabla\times\vec{A}=0$)

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\left(\cancel{\vec{\nabla}\cdot\vec{\nabla}\times\int}\right)^{0} \quad (20)$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

- Recall: $\vec{\nabla} \cdot \vec{E} = 0$ only when $\rho = 0$

so $\rho_{mag} = 0 \rightarrow$ NO MAGNETIC CHARGE!

- All magnetic fields are created only by moving electric charges
- Magnetic field lines have no beginning or end.
- Taking the curl instead of both sides of eq (19),

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\vec{\nabla} \times \int \right) \qquad (21)$$

Use $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \left[\vec{\nabla} \left(\vec{\nabla} \cdot \int \right) - \vec{\nabla}^2 \int \right] \qquad (22)$$

$$\frac{}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \Big] \qquad (23)$$

$$= \frac{\mu_0}{4\pi} \left[\vec{\nabla} \left(\int \vec{J}(\vec{x}') \cdot \underbrace{\nabla \left(\frac{1}{|\vec{x} - \vec{x}'|} \right)}_{\rightarrow\, -\nabla' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right)} d^3\vec{x}' \right) - \int \vec{J}(\vec{x}') \underbrace{\vec{\nabla}^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right)}_{\rightarrow\, -4\pi \delta(\vec{x} - \vec{x}')} d^3\vec{x}' \right] \qquad (24)$$

$$= \frac{\mu_0}{4\pi} \left[-\vec{\nabla} \left(\int \vec{J} \cdot \nabla' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3\vec{x}' \right) + 4\pi \vec{J}(\vec{x}) \right] \qquad (25)$$

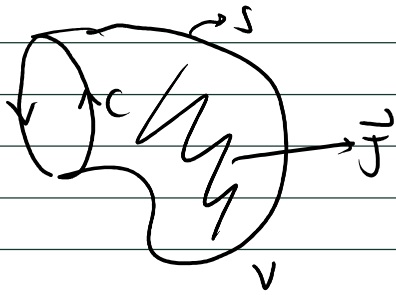
magnetostatics

$$+ 4\pi \vec{J}(\vec{x}) \Big] \qquad (25)$$

$$= \frac{\mu_0}{4\pi}\left[-\vec{\nabla}\left(\cancelto{0}{\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\Big|_{\text{all space}}} - \int \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\cancelto{0}{(\vec{\nabla}'\cdot\vec{J})}}_{\text{magnetostatics}} d^3\vec{x}' \right) + 4\pi \vec{J}(\vec{x}) \right] \qquad (26)$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}(\vec{x})} \longrightarrow \text{Ampere's Law} \qquad (27)$$

- Converting to integral form, (27) becomes



$$\int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

Stokes'

$$\boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}}$$

7 The Magnetic Force

- physically measurable
- EMstatics: fields are just explanations for identifying the effects of charges

Electrodynamics: fields are independent entities apart from sources

- Recall: expt. results

① Magnetic force

$$\vec{F} = q\vec{v} \times \vec{B} \tag{1}$$

② Magnetostatics approximation

$$d\vec{F} = I\, d\vec{\ell} \times \vec{B} \tag{2}$$

③ Force on a wire: integrate (2)

$$\vec{F} = \int I\, d\vec{\ell} \times \vec{B} \tag{3}$$

④ Force on general current density

$$\vec{F} = \int \vec{J} \times \vec{B}\; d^3\vec{x} \tag{4}$$

⑤ Torque

$$\vec{\tau} = \int \vec{x} \times \vec{F}\, d\vec{x} \tag{5}$$

> The vector potential, $\vec{A}$ (§5.4)

- Recall: since $\vec{\nabla}\times\vec{E}=0$, $\vec{E}$ can be written as a gradient of some scalar potential since $\vec{\nabla}\times\vec{\nabla}f=0$.

- Is there a magnetic field analog?

Recall:

$$\vec{\nabla}\cdot\vec{B}=0 \tag{6}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \tag{7}$$

- We can write $\vec{B}=\vec{\nabla}\times\vec{A}$ since $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}=0$ so that $\vec{A}$ is our vector potential analogous to Φ.

- Recall Biot-Savart law in curl form

$$\vec{B}=\frac{\mu_0}{4\pi}\left(\vec{\nabla}\times\int\frac{\vec{J}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right) \tag{8}$$

since $\vec{B}=\vec{\nabla}\times\vec{A}$

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'+\nabla\Psi \tag{9}$$

since $\vec{\nabla}\times\nabla f=0$, there should be a gradient "constant" that vanishes as we take the curl: "gauge".

- Choosing $\Psi=$ const. as gauge allows us to write

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}}{|\vec{x}-\vec{x}'|}d\vec{x}' \tag{10}$$

- We can then rewrite Ampere's law in terms of $\vec{A}$:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (11)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \quad (12)$$

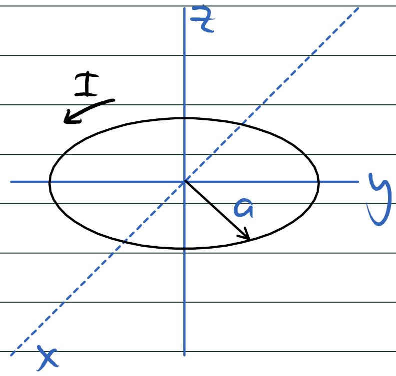
$$\vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{A}})^{0} - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J} \quad (13)$$

$\vec{A} \rightarrow \vec{A} + \nabla \Psi$
(Coulomb gauge)

$$\boxed{\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}} \quad (14)$$

$\vec{A}$ & $\vec{B}$ for a circular loop of $\vec{J}$ (§5.5)

Given: Circular loop, radius a, center (0,0,0), carrying I, current density is given by



$$J_\varphi = I \sin\theta' \, \delta(\cos\theta') \frac{\delta(r'-a)}{a} \quad (1)$$

$(J_r = J_\theta = 0)$

Decompose $\vec{J}$ into its cartesian components,

$$\vec{J} = -J_\varphi \sin\varphi' \hat{i} + J_\varphi \cos\varphi' \hat{j} \quad (2)$$

Choose $\varphi = 0$ to coincide w/ x-z plane and that we set this as the field point direction (i.e. $\vec{x}(\varphi=0)$). Then, $\vec{A}$ becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \quad (3)$$

$$= \frac{\mu_0}{4\pi} \hat{i} \int \frac{(-J_\varphi \sin\varphi') \, r'^2 dr' d\Omega'}{\sqrt{r^2+r'^2-2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi')}}$$

$$+ \frac{\mu_0}{4\pi} \hat{j} \int \frac{(J_\varphi \cos\varphi') \, r'^2 dr' d\Omega'}{\sqrt{r^2+r'^2-2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi')}} \quad (4)$$

But $A_y = A_\varphi$ due to our placing of field point, so

$$\vec{A} = A_\varphi \hat{\varphi} \quad (5)$$

$$A_\varphi(r,\theta) = \frac{\mu_0 I}{4\pi a} \int r'^2 dr' d\Omega' \frac{\sin\theta' \cos\varphi' \, \delta(\cos\theta') \, \delta(r'-a)}{\sqrt{r^2+r'^2-2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi')}}$$

Expanding and evaluating the Dirac deltas,

$$\int \frac{r'^2 \, \delta(r'-a)}{\sqrt{\cdots}} dr' = \frac{a^2}{\sqrt{a^2+r^2-2ar(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi')}} \quad (6)$$

$$\int \frac{r'\,\delta(r'-a)}{\sqrt{\cdots}}\,dr' = \frac{a}{\sqrt{a^2+r^2-2ar(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi')}} \quad (6)$$

$$\int \frac{\sin^2\theta'\,\delta(\cos\theta')}{\sqrt{\cdots}}\,d\theta' = \frac{1}{\sqrt{a^2+r^2-2ar\sin\theta\cos\varphi'}} \quad (7)$$

so that (5) becomes

$$A_\varphi(r,\theta) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{\cos\varphi'\,d\varphi'}{(a^2+r^2-2ar\sin\theta\cos\varphi')^{1/2}} \quad (8)$$

We can immediately simplify the integral by using elliptic integrals. Recall the complete elliptic integral of the first kind,

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}} \quad (9)$$

and the incomplete elliptic integral of the second kind,

$$E(m) = \int_0^{\pi/2} \sqrt{1-m^2\sin^2\theta}\,d\theta \quad (10)$$

Then, since the integrand in (8) is even, we could divide it by 4 and divide the limits by 4 as well. Thus

$$\int_0^{\pi/2} \frac{\cos\varphi'\,d\varphi'}{(a^2+r^2-2ar\sin\theta\cos\varphi')^{1/2}} = \int_0^{\pi/2} \frac{\cos\varphi'\,d\varphi'}{a\left(1+\frac{r^2}{a^2}-2\frac{r}{a}\sin\theta\cos\varphi'\right)^{1/2}}$$

$$= \int_0^{\pi/2} \frac{\cos\varphi'\,d\varphi'}{ar\left(\frac{1}{r^2}+\frac{1}{a^2}-\frac{2}{ar}\sin\theta\cos\varphi'\right)^{1/2}} = \int_0^{\pi/2} \frac{\cos\varphi'\,d\varphi'}{ar\sin\theta\left(\frac{1}{r^2\sin^2\theta}+\frac{1}{a^2\sin^2\theta}-\frac{2\cos\varphi'}{ar\sin\theta}\right)^{1/2}}$$

Letting $m^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}$, we get

$$A_\varphi(r,\theta) = \frac{\mu_0}{4\pi}4Ia\,\frac{(a^2+r^2+2ar\sin\theta)^{1/2}}{4ar\sin\theta}\left[(2-m^2)K(m)-E(m)\right]$$

verifying,

$$A_\varphi(r,\theta) = \frac{\mu_0 I a}{\ldots}\,4\,\frac{(a^2+r^2+2ar\sin\theta)^{1/2}}{\ldots}\Big[\ldots\ \frac{4ar\sin\theta}{\ldots}\ \Big)$$

verifying,

$$A_\varphi(r,\theta) = \frac{\mu_0 I a}{4\pi}\, 4\, \frac{(a^2+r^2+2ar\sin\theta)^{1/2}}{4ar\sin\theta}\Bigg[\left(2 - \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}\right)\int_0^{\pi/2}\frac{d\varphi'}{\sqrt{1-\frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}\sin^2\varphi'}}$$

$$-2\int_0^{\pi/2}\sqrt{1-\frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}\sin^2\varphi'}\,d\varphi'\Bigg]$$

$$= \frac{\mu_0 I a}{4\pi}\, 4\, \frac{(a^2+r^2+2ar\sin\theta)^{1/2}}{4ar\sin\theta}\Bigg[2\left(\int_0^{\pi/2}\frac{1-\sqrt{1-m\sin^2\varphi' - m^2\sin^4\varphi' + m^3\sin^4\varphi'}}{\sqrt{1-m\sin^2\varphi'}}\,d\varphi'\right)$$

$$-m^2\int_0^{\pi/2}\frac{d\varphi'}{\sqrt{1-m\sin^2\varphi'}}\Bigg]$$

so that

$$\boxed{A_\varphi(r,\theta) = \frac{\mu_0}{\pi}\,\frac{Ia}{\sqrt{a^2+r^2+2ar\sin\theta}}\left[\frac{(2-m^2)K(m)-2E(m)}{m^2}\right]} \qquad (11)$$

and, using $\vec{B} = \vec{\nabla}\times\vec{A}$,

$$\boxed{\begin{aligned} B_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) \\ B_\theta &= -\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) \\ B_\varphi &= 0 \end{aligned}} \qquad (12)$$

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(§5.6)

> Magnetic fields of a localized current distribution

- Recall: Use of multipole expansion in finding $\vec{E}$ for locations far away from source charges
- We will now develop its $\vec{B}$ equivalent. Recall

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3\vec{x}' \qquad (1)$$

- Expand $\frac{1}{|\vec{x}-\vec{x}'|}$ in terms of $\vec{x}'$ (source locations)

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \dots \qquad (2)$$

such that (1) becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\Big[\underbrace{\cancel{\frac{1}{|\vec{x}|}\int \overbrace{\vec{J}(\vec{x}')}^{I}\,d^3\vec{x}}^{0}}_{\substack{\text{magnetic}\\ \text{monopole}\\ \text{source} = \text{DNE}}} + \underbrace{\overbrace{\frac{\vec{x}}{|\vec{x}|}}^{\hat{x}}\cdot\int \vec{x}'\vec{J}(\vec{x}')\,d^3\vec{x}'}_{\substack{\text{magnetic dipole}\\ \text{source}}} + \dots\Big] \qquad (3)$$

- To prove the non-existence of the magnetic monopole, we prove that the first term in (3) vanishes.

Consider scalar fxns $f(\vec{x}')$ & $g(\vec{x}')$ as well as a vector fxn $\vec{J}(\vec{x}')$. Then, by using IBP,

$$\int_V g(\vec{J}\cdot\vec{\nabla}' f)\,d^3\vec{x}' = \underbrace{\cancel{\int_S fg\,\vec{J}\cdot\hat{x}'\,da}^{0}}_{\substack{\text{localized current,}\\ \text{no } \vec{J} \text{ on } S}} - \int_V f\,\vec{\nabla}'\cdot(g\vec{J})\,d^3\vec{x}' \qquad (4)$$

$$= -\left[\int_V f(\vec{J}\cdot\vec{\nabla}' g)\,d^3\vec{x}' + \int fg(\vec{\nabla}'\cdot\vec{J})\,d^3\vec{x}'\right] \qquad (5)$$

Rearranging both sides,

$$\int\left[g(\vec{J}\cdot\vec{\nabla}' f) + f(\vec{J}\cdot\vec{\nabla}' g) + fg\cancel{(\vec{\nabla}'\cdot\vec{J})}^{0}\right]d^3\vec{x}' = 0 \qquad (6)$$

$$\int_V \left[g(\vec{J}\cdot\vec{\nabla}'f) + f(\vec{J}\cdot\vec{\nabla}'g) + fg(\underbrace{\vec{\nabla}\cdot\vec{J}'}_{0})\right] d^3\vec{x}' = 0 \quad (6)$$

magnetostatics defn.

If we set $f=1$ & $g=x_i'$, the first term of (6) vanishes as well, and

$$\int_V (\vec{J}\cdot\vec{\nabla}' x_i')\, d^3\vec{x}' = \int_V J_i\, d^3\vec{x} = 0 \quad (7)$$

Generalizing eq. (7), $\int \vec{J}(\vec{x}')\, d^3\vec{x}' = 0$, which is the magnetic monopole integral.

- Dropping h.o.t., we get

$$\vec{A}(\vec{x} >> \vec{x}') = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int (\vec{x}\cdot\vec{x}')\,\vec{J}(\vec{x}')\, d^3\vec{x}' \quad (8)$$

- We can develop a better formula than (8) to mimic the electric dipole formula by rearranging terms. We first want to reach eq. (6). To do this, we list the components of the integral

$$A(\vec{x} >> \vec{x}') = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int \sum_i x_i x_i' \sum_j \hat{x}_j J_j\, d^3\vec{x}' \quad (9)$$

$$= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int \sum_j \hat{x}_j \underbrace{\sum_i x_i (x_i' J_j)}_{\frac{1}{2}\left(\sum_i x_i (x_i' J_j + x_i' J_j)\right)}\, d^3\vec{x}' \quad (10)$$

$$= -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int \sum_j \hat{x}_j \sum_i x_i (-x_i' J_j - x_i' J_j)\, d^3\vec{x}' \quad (11)$$

Now, if we set $f = x_i'$ & $g = x_j'$ in (6),

$$\int_V \left[x_j' (\vec{J}\cdot\nabla' x_i') + x_i' (\vec{J}\cdot\vec{\nabla}' x_j')\right] d^3\vec{x}' = 0 \quad (12)$$

or

$$\int x_j' (\vec{J}\cdot\vec{\nabla}' x_i')\, d^3\vec{x}' = -\int x_i' (\vec{J}\cdot\vec{\nabla}' x_j')\, d^3\vec{x}' \quad (13)$$

or

$$\int_V x_i' (\vec{J}\cdot\vec{\nabla}' x_i') \, d^3\vec{x}' = -\int_V x_i' (\vec{J}\cdot\vec{\nabla}' x_j') d^3\vec{x}' \quad (13)$$

$$\int_V x_j' J_i \, d^3\vec{x}' = -\int x_i' J_j \, d^3\vec{x}' \quad (14)$$

which we can use to simplify (11):

$$\vec{A}(\vec{x} >> \vec{x}') = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int \sum_j \hat{x}_j \underbrace{\sum_i x_i (x_j' J_i - x_i' J_j)}_{\sum_i x_i \left(\underbrace{x_i' J_j \varepsilon_{ijk}}_{(\vec{x}' \times \vec{J})_k}\right)} d^3\vec{x}' \quad (15)$$

$$= -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int \sum_j \hat{x}_j' \sum_i x_i (\vec{x}' \times \vec{J})_k \, d^3\vec{x}' \quad (16)$$

$$= -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int \vec{x} \times (\vec{x}' \times \vec{J}) \, d^3\vec{x}' \quad (17)$$

or $$\boxed{\vec{A}(\vec{x} >> \vec{x}') = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\vec{x} \times \int \vec{x}' \times \vec{J} \, d^3\vec{x}'}$$

> Potential of a magnetic dipole (§5.6)

Recall

$$\vec{A}(\vec{x} >> \vec{x}') = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\,\vec{x} \times \int \vec{x}' \times \vec{J}\, d^3\vec{x}' \qquad (1)$$

Define
$$\vec{m} = \frac{1}{2}\int \vec{x}' \times \vec{J}\, d^3\vec{x}' \qquad (2)$$
(magnetic dipole moment)

Then $\vec{A}$ becomes
$$\vec{A}(\vec{x} >> \vec{x}') = -\frac{\mu_0}{4\pi}\frac{\vec{x} \times \vec{m}}{|\vec{x}|^3} = \boxed{\frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{x}}{|\vec{x}|^2}} \qquad (3)$$

so that when we calculate $\vec{B}$,
$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi}\vec{\nabla} \times \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} = \boxed{\frac{\mu_0}{4\pi}\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}} \qquad (4)$$

> Behavior of localized $\vec{J}$ in external $\vec{B}$ (§5.7)

- As with any test charges/currents, $\vec{J}$ will feel a force; only this time, there's a corresponding torque.
- We also use Taylor expansion for a slowly-varying magnetic field $\vec{B}^{(0)}$ (cf. PS1) to arrive at
$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}^{(0)}) \qquad (5)$$
- The gradient tells us that constant fields exert no force on $\vec{m}$ (or $\vec{J}$).
- Cf. $\vec{F} = q\vec{v} \times \vec{B}$ for a charged (free)

particle. This particle spirals around the field lines, creating an effective current loop. This current loop produces a magnetic moment $\vec{m} \parallel \vec{B}$, so that when $\vec{B}$ changes, $\vec{F} = \vec{\nabla}(mB)$ is nonzero.

- The torque can then be calculated as

$$\vec{\tau} = \vec{m} \times \vec{B}^{(0)} \tag{6}$$

- e.g. setting a uniform magnetic field $\vec{B}^{(0)} = B_0 \hat{i}$ and setting the magnetic moment on the xy-plane,

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}^{(0)}) = 0} \tag{7}$$

$$\vec{\tau} = \vec{m} \times \vec{B}^{(0)} \tag{8}$$

$$= m(\cos\theta_m \hat{i} + \sin\theta \hat{j}) \times B_0 \hat{i} \tag{9}$$

$$\boxed{\vec{\tau} = -m B_0 \sin\theta \hat{k}} \tag{10}$$

Magnetostatics in ponderable media (§5.8)

- Similar to §4.3 – 4.7, we analyze the behavior of $\vec{B}$ in matter that can be magnetized.
- We can derive a similar quantity to $\vec{P}$ and $\vec{D}$. Recall

$$\vec{P} = \frac{\vec{p}}{V} \quad (1)$$

→ electric dipole moment

then

$$\vec{M} = \frac{\vec{m}}{V} \quad (2)$$

→ magnetic dipole moment

↳ magnetization

- If we let currents pass through a material, the material's response can be recorded through $\vec{M}$, adding magnetic effects to the vacuum equations.
- However, as opposed to electric polarization which is purely induced, magnetization can exist on its own, as in the case of permanent magnets.
- In a magnetic material, the macroscopic description of magnetization can be attributed

ultimately to the orbits of electrons. These small regions of nonzero magnetic moment are called "domains".

- Thus, a non-permanent magnet still has domains, but the individual magnetic moments point randomly and cancel on average.
- The overall magnetization appears when an external magnetic field is present or a current density is applied. The domains feel a torque until they are all aligned.
- IMPORTANT: NOTATION

$\mu_0 \vec{H}$: Applied magnetic field due to free currents + magnetic field due to effective magnetic charges (see: $\vec{H}$ is called "auxiliary field")

$\mu_0 \vec{M}$: induced or permanent magnetic field due to "bound currents" (cf. bound charges)

$\vec{B}$: TOTAL magnetic field including the applied field & material's response

$\vec{J}$: FREE current density which results in $\vec{\nabla} \times \mu_0 \vec{H} = \vec{J}$ (if $\vec{J} = 0$, $\vec{H}$ can still exist, albeit non-curling)

$\vec{J}_M$: Bound current density, either induced or permanent, giving rise to $\vec{M}$

$\vec{J}_{tot}$: TOTAL current density, $\vec{J} + \vec{J}_M$, giving rise to $\vec{B}$.

- Recall : for a single dipole, the far-field approximation for $\vec{A}$ reads

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} \qquad (3)$$

$$\vec{m} = \frac{1}{2} \int \vec{x}' \times \vec{J} \, d\vec{x}' \qquad (4)$$

But in general,

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{x}' \qquad (5)$$

We now add the contributions from $\vec{J}_M$, s.t.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{J(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{\vec{M} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d\vec{x}' \qquad (6)$$

Recall

$$\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) \qquad (7)$$

Then, shifting the derivative to $\vec{x}'$ & using IBP,

$\left(\nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right)\right)$

$$-\int \vec{M} \times \nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3\vec{x}' = \frac{-M}{|\vec{x}-\vec{x}'|} + \int \frac{\vec{\nabla}' \times \vec{M}}{|\vec{x}-\vec{x}'|} \qquad (8)$$

so that

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} + \vec{\nabla}' \times \vec{M}}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \qquad (9)$$

Define $\vec{J}_{tot} = \vec{J} + \vec{J}_M = \vec{J} + \vec{\nabla}' \times \vec{M}$, thus

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_{tot}}{|\vec{x}-\vec{x}'|} d^3\vec{x}'} \qquad (10)$$

and Ampere's law becomes

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{\nabla} \times \vec{M})} \qquad (11)$$

> Recall

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{\nabla} \times \vec{M}) \quad (1)$$

The free current $\vec{J}$ gives rise to $\vec{H}$ via

$$\vec{J} = \vec{\nabla} \times \vec{H} \quad (2)$$

so that, from eq. (1),

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \quad (3)$$

> We can then use all vacuum eqns by changing the defn. of $\vec{J}$ to $\vec{J}_{tot}$, and that the effects of both free and bound currents give rise to the total field.

> We can still simplify the equations for certain types of material, analogous to the linear, isotropic dielectric.

> The linear, isotropic, uniform dia/paramagnetic

Linearity response: $\vec{M} = (\mu_r - 1) \vec{H}$ (4)

$$\mu_r = \frac{\mu}{\mu_0}$$

(Free space: $\mu_r = 1$, so $\vec{M} = 0$ & $\vec{B} = \mu_0 \vec{H}$)

Paramagnetics: $\mu_r > 1$, $\vec{M} = +|\mu_r - 1| \vec{H}$ (5)

Diamagnetics: $\mu_r < 1$, $\vec{M} = -|\mu_r - 1| \vec{H}$ (6)

- consequences:

$$\vec{B} = \mu \vec{H}$$

$$\vec{A} = \frac{\mu}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}$$

$$\vec{\nabla}^2 \vec{A} = -\mu \vec{J}(\vec{x})$$

(7)

- ex. steel/iron → paramagnetic materials, attracted to high-flux regions such as magnet poles; applied field $\mu_0 \vec{H}$ is enhanced
- Higher values of μ make a material more responsive to a magnetic field
- Recall: valid for linear, uniform materials only! Change $\mu_0 \to \mu$

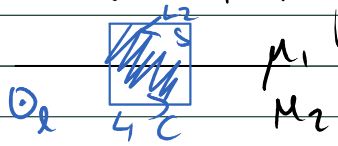
> Boundary conditions for different materials
- Analogous to dielectrics consisting of multiple materials, we break down the dielectric and solve for the equations in each region. We then match BCs for the complete soln.
- Start with $\vec{\nabla} \cdot \vec{B} = 0$, then $\vec{B}$ is continuous along the normal:

$$\oint \vec{B} \cdot \hat{n}\, da = 0$$

$$\oint \vec{B}\cdot\hat{n}\,da = 0$$

$$\boxed{(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0} \qquad (8)$$

- Next, use $\vec{\nabla}\times\vec{H} = \vec{J}$. We integrate over some square surface S



(Assume 3D)

Then
$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot\hat{t}\,da \qquad (9)$$

where $\hat{t}$ is the tangential vector to the boundary.
Define $\hat{l} = \hat{t}\times\hat{n}$ as the loop vector, thus
$$\vec{H}_2\cdot L_1\hat{l} - \vec{H}_1\cdot L_1\hat{l} = \vec{J}\cdot\hat{t}\,L_1L_2 \qquad (10)$$

Define $\vec{K} = L_2\vec{J}$ as the surface current density.
Thus,
$$(\vec{H}_2 - \vec{H}_1)\cdot(\hat{t}\times\hat{n}) = \vec{K}\cdot\hat{t} \qquad (11)$$
$$\hat{n}\times(\vec{H}_2 - \vec{H}_1)\cdot\hat{t} = \vec{K}\cdot\hat{t} \qquad (12)$$
$$\boxed{\hat{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{K}} \qquad (13)$$

The free surface current then creates a discontinuity bet. the tangential components of $\vec{H}$.

> For linear media, we can use alternatives for eq. (8) & (13) as
$$\vec{H}_2\cdot\hat{n} = \frac{\mu_1}{\mu_2}\vec{H}_1\cdot\hat{n} \qquad (14)$$
$$\vec{B}_2\times\hat{n} = \frac{\mu_2}{\mu_1}\vec{B}_1\times\hat{n} \qquad (15)$$

(§5.9)

Review of magnetostatics in magnetic materials

> Recall the ff:

- Currents give rise to fields w/ nonzero curl:

$$\left.\begin{aligned} \vec{\nabla}\times\vec{B} &= \mu_0 \vec{J}_{tot}, \\ \vec{\nabla}\times\vec{H} &= \vec{J} \\ \vec{\nabla}\times\vec{M} &= \vec{J}_M \end{aligned}\right\} \vec{J}_{tot} = \vec{J} + \vec{J}_M \qquad (1)$$

- There are no magnetic monopoles:

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \qquad (2)$$

- Defining $\vec{A}$ from $\vec{\nabla}\times\vec{A} = \vec{B}$ yields

$$\vec{\nabla}^2\vec{A} = -\mu_0 \vec{J}_{tot}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_{tot}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\vec{x}' \qquad (3)$$

- For materials w/ uniform, linear, & isotropic magnetization ($\vec{B} = \mu\vec{H}$), replace $\mu_0 \rightarrow \mu$ & $\vec{J}_{tot} \rightarrow \vec{J}$ in eqs. (2) & (3).

- Boundary conditions for previous point:

$$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0$$

$$\hat{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{K} \text{ (current surface density)} \qquad (4)$$

> Recall: special cases

- linear materials w/ no current density in

- linear materials w/ no current density in the region where we want to measure $\vec{B}$:

$$\vec{\nabla}^2 \vec{A} = 0 \qquad (5)$$

(we can also define $\vec{B} = -\vec{\nabla}\Phi_M$ to get $\vec{\nabla}^2\Phi_M = 0$)

- If the material is nonlinear & $\vec{J} = 0$ with a given $\vec{M}$, then

$$\left.\begin{aligned}\vec{\nabla}^2\vec{A} &= -\mu_0 \vec{J}_M \\ \vec{A} &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}_M}{|\vec{x}-\vec{x}'|}\,d\vec{x}'\end{aligned}\right\} \vec{J}_M = \vec{\nabla}\times\vec{M} \qquad (6)$$

• We can also let $\vec{H} = -\nabla\Phi_M$ s.t.

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\cdot(\mu_0\vec{H} + \mu_0\vec{M}) = 0$$

$$-\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\vec{M}$$

$$\vec{\nabla}^2\Phi_M = \vec{\nabla}\cdot\vec{M} \rightarrow -\rho_M \qquad (7)$$

$\rightarrow$ effective magnetic charge density

> We then list the diff. situations for which we can use diff. techniques for solving BVPs in magnetostatics.

A. Generally applicable method for $\vec{A}$

By having a constitutive relation $\vec{H} = \vec{H}(\vec{B})$, we can write (1a) as we can write (1a) as

$$\vec{\nabla}\times\vec{H}(\vec{\nabla}\times\vec{A})=\vec{J} \qquad (8)$$

which is difficult to solve, unless $\vec{H}(\vec{B})$ is simple, e.g. linear media. Then, using $\vec{B}=\mu\vec{H}$,

$$\vec{\nabla}\times\left(\frac{1}{\mu}\vec{\nabla}\times\vec{A}\right)=\vec{J} \qquad (9)$$

If μ is constant over a finite region, it can get out of the derivatives; using the double curl identity,

$$\vec{\nabla}^2\vec{A}-\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{A}})=-\mu\vec{J} \qquad (10)$$

Coulomb gauge

$$\vec{\nabla}^2\vec{A}=-\mu\vec{J} \qquad (11)$$

$\rightarrow$ modified $\vec{J}$: $\frac{\mu}{\mu_0}\vec{J}$

B. $\vec{J}=0$; Φ_M

Recall that for $\vec{J}=0$, $\vec{\nabla}\times\vec{H}=0$. Then

$$\vec{H}=-\nabla\Phi_M \qquad (12)$$

$(\vec{B}=\vec{B}[\vec{H}])$ $\quad \vec{\nabla}\cdot\vec{B}[-\vec{\nabla}\Phi_M]=0 \qquad (13)$

(linear) $\quad \vec{\nabla}\cdot(\mu\vec{\nabla}\Phi_M)=0 \qquad (14)$

(const. μ) $\quad \vec{\nabla}^2\Phi_M=0 \rightarrow$ Laplace eqn for Φ_M

we can then connect diff. regions via BCs (Lec 14). Also, $\Phi_M \propto \Omega$ subtended by a closed current loop's boundary.

C. Hard Ferromagnets ($\vec{M}$ given, $\vec{J}=0$)

$\vec{M}$ is independent of applied fields, so that $\vec{M}(\vec{x})$ is fixed. We then use eq. (7) since $\vec{J}=0$, so that for boundless surfaces,

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (15)$$

We can then use IBP if $\vec{M}$ is well-behaved and localized; here, $u = \frac{1}{|\vec{x}-\vec{x}'|}$, $dv = \vec{\nabla}'\cdot\vec{M}\, d^3x'$.

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\int \vec{M}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' \qquad (16)$$

$$= -\frac{1}{4\pi}\int \vec{M}\cdot\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' \qquad (17)$$

$$= -\frac{1}{4\pi}\nabla\cdot\left(\int \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right) \qquad (18)$$

For $r >> |\vec{x}-\vec{x}'|$,

$$\Phi_M(\vec{x}) \approx -\frac{1}{4\pi}\nabla\left(\frac{1}{r}\right)\cdot\int \vec{M}(\vec{x}')\, d^3x' \qquad (19)$$

$$= \frac{\vec{m}\cdot\vec{x}}{4\pi r^3} \rightarrow \text{dipole potential} \qquad (20)$$

We can also use Green's first identity for discontinuous $\vec{M}$, which should NOT be used in conjunction w/ eq (18). We can also use $\vec{A}$ to develop the same situation.

Lec 16: 06 May 2021

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E.g. Uniformly magnetized sphere (§5.10)

Consider: sphere of radius a, uniform permanent magnetization $\vec{M} = M_0 \hat{z}$, embedded in a non permeable medium. Since it vanishes at the surface discontinuously, we use

$$\Phi_M = -\frac{1}{4\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \frac{1}{4\pi}\oint \frac{\sigma_M da'}{|\vec{x}-\vec{x}'|} \qquad (1)$$

where $\sigma_M = \hat{n}'\cdot\vec{M}(\vec{x}') = M_0\cos\theta'$. Thus,

$$\Phi_M = \frac{1}{4\pi}\oint \frac{M_0\cos\theta'(a^2 d\Omega')}{|\vec{x}-\vec{x}'|} = \frac{M_0 a^2}{4\pi}\int \frac{\cos\theta' d\Omega'}{|\vec{x}-\vec{x}'|} \qquad (2)$$

Recall that since we are solving Laplace's eqn for ρ_M, we can use the Legendre polynomial expansion for $\frac{1}{|\vec{x}-\vec{x}'|}$:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \qquad (3)$$

where $r_<$ is $\min(r,a)$ & $r_> = \max(r,a)$. so that for $r<a$ (inside the sphere),

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r^l}{a^{l+1}}P_l(\cos\gamma) \qquad (4)$$

and

$$\Phi_M(r,\theta) = \frac{M_0 a^2}{4\pi}\int d\Omega' \cos\theta'\left(\frac{r}{a^2}\cos\gamma\right) \qquad (8)$$

$$= \frac{M_0}{4\pi}\left(\frac{4\pi}{3}\cos\theta\right) = \frac{1}{3}M_0 r\cos\theta$$

z

& for $r>a$,

$$\Phi_M(r,\theta) = \frac{1}{3}M_0\frac{a^3}{r^2}\cos\theta \qquad (9)$$

$$\Phi_M(r,\theta) = \frac{1}{3} M_0 \frac{a^3}{r^2}\cos\theta \qquad (9)$$

Inside the sphere then,

$$\vec{H}(r<a) = -\vec{\nabla}\Phi_M = -\frac{1}{3}\vec{M}$$
$$\vec{B}(r<a) = \frac{2}{3}\mu_0\vec{M} \qquad (10)$$

Outside, we see that (9) is the potential for a dipole moment $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$. No higher multipole terms appear. We can also do this calculation using $\vec{A}$. ▲

E.g. unmagnetized sphere in an external field

> Again, consider a sphere of radius a made w/ uniform, linear, magnetic material w/ permeability μ, placed in an external magnetic field $\vec{B} = B_0\hat{z}$.

> Since $\vec{J} = 0$ and $\vec{B} = \mu\vec{H}$, we can solve

$$\nabla^2\Psi_M = 0 \qquad (11)$$
$$\vec{B} = -\nabla\Psi_M \qquad (12)$$

&

$$\Psi_M = -B_0 r\cos\theta \quad \text{for } r \gg a \qquad (13)$$

> Since we're effectively solving Laplace's eqn,

$$\Psi_M(r,\theta,\varphi) = \sum_\ell \left(F_\ell r^\ell + G_\ell r^{-\ell-1}\right) P_\ell(\cos\theta) \qquad (14)$$

↳ Azimuthally symmetric

> Apply BCs:

- At $r \gg a$,

$$-B_0 r\cos\theta = \sum_\ell F_\ell r^\ell P_\ell(\cos\theta)$$
$$-B_0 = F_1 \qquad (F_\ell = 0,\ \ell \neq 1) \quad (15)$$

so that outside the sphere,

$$\Psi_M(r>a,\theta) = -B_0 r\cos\theta + \sum_{\ell=0}^{\infty} G_\ell^{out} r^{-\ell-1} P_\ell$$

- At $r<a$, $G_\ell = 0$ to keep Ψ_M from blowing up:

$$\Psi_M(r<a,\theta) = \sum_{\ell=0}^{\infty} F_\ell^{in} r^\ell P_\ell(\cos\theta) \quad (16)$$

- Use the BCs for $\vec{J}=0$ & $\vec{B}=\mu\vec{H}$:

$$\left.\begin{aligned}(\vec{B}^{out}-\vec{B}^{in})\cdot\hat{r} &= 0\\ \left(\frac{1}{\mu}\vec{B}^{out}-\frac{1}{\mu}\vec{B}^{in}\right)\cdot\hat{\theta} &= 0\end{aligned}\right\} r=a \quad (17)$$

so that,

$$\left.\frac{\partial\Psi^{out}}{\partial r}\right|_{r=a} = \left.\frac{\partial\Psi^{in}}{\partial r}\right|_{r=a} \quad (18)$$

$$-B_0\cos\theta + \sum_{\ell=0}^{\infty} G_\ell^{out}(-\ell-1)a^{-\ell-2}P_\ell(\cos\theta) = \sum_{\ell=1}^{\infty} F_\ell^{in}\,\ell a^{\ell-1}P_\ell(\cos\theta)$$

$G_\ell = 0$ for $\ell = 0$

$F_1^{in} = -B_0 - \frac{2}{a^3}G_1^{in}$

so that for all ℓ, $F_\ell^{in} = G_\ell^{out}\dfrac{(-\ell-1)}{\ell}a^{-2\ell-1} \quad (19)$

and for the second BC (17b),

$$\frac{1}{\mu_0}\frac{\partial\Psi_M^{out}}{\partial\theta} = \frac{1}{\mu}\frac{\partial\Psi_M^{in}}{\partial\theta} \quad (20)$$

$$\frac{1}{\mu_0}\left[B_0 a\sin\theta + \sum_{\ell=1}^{\infty} G_\ell^{out} a^{-\ell-1} P_\ell'(\cos\theta)\right]$$
$$= \frac{1}{\mu}\left[\qquad \sum^{\infty} F_\ell^{in} a^\ell P_\ell'(\cos\theta)\right]$$

$$= \frac{1}{\mu}\Big[\qquad \sum_{\ell=1}^{\infty} F_\ell^{in} a^\ell P_\ell'(\cos\theta) \Big]$$

$$F_1^{in} = \frac{\mu}{\mu_0}\left[-B_0 + G_1^{out} a^{-3}\right]$$

so that for all ℓ $\qquad F_\ell^{in} = \frac{\mu}{\mu_0} G_\ell^{out} a^{-2\ell-1} \qquad (21)$

Thus, from (21) & (19), every coefficient (F^{in} & G^{out}) vanishes for $\ell > 1$ in order to keep the BCs.

For $\ell = 1$, however, we can solve the linear eqns for F_1^{in} & G_1^{out}:

$$F_1^{in} = B_0\left(\frac{-3\mu}{\mu + 2\mu_0}\right) \qquad (22)$$

$$G_1^{out} = B_0\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right) a^3 \qquad (23)$$

> Unmagnetized sphere in external $\vec{B}$ (cont'd)

Recall the coefficients solved last lec:

$$F_1^{in} = B_0\left(\frac{-3\mu}{\mu+2\mu_0}\right) \qquad (22)$$

$$G_1^{out} = B_0\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)a^3 \qquad (23)$$

From our solutions

$$\Psi_M(r>a,\theta) = -B_0 r\cos\theta + \sum_{l=0}^{\infty} G_l^{out} r^{-l-1}P_l(\cos\theta)$$

$$\Psi_M(r<a,\theta) = \sum_{l=0}^{\infty} F_l^{in} r^l P_l(\cos\theta)$$

We can now simplify them, leaving

$$\boxed{\begin{aligned}\Psi_M(r>a,\theta) &= -B_0 r\cos\theta + B_0\frac{a^3}{r^2}\cos\theta\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\\ \Psi_M(r<a,\theta) &= B_0 r\cos\theta\left(\frac{-3\mu}{\mu+2\mu_0}\right)\end{aligned}}$$

Using $B = -\nabla\Psi_M$,

$$\vec{B}(r>a,\theta) = B_0\hat{z} + B_0\frac{\mu-\mu_0}{\mu+2\mu_0}\left(\frac{a}{r}\right)^3\left[3\hat{r}\cos\theta - \hat{z}\right]$$

$$\vec{B}(r<a,\theta) = B_0\frac{3\mu}{\mu+2\mu_0}\hat{z}$$

We can then use $\vec{B} = \mu\vec{H}$ inside & $\vec{B} = \mu_0\vec{H}$ outside the sphere, as well as

$$\vec{M}(r<a) = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\vec{B}(r<a)$$

$$\vec{M}(r>a) = \left(\frac{1}{\mu_0} - \frac{1}{\mu_0}\right)\vec{B}(r>a) = 0$$

so that for a sphere with $\mu = \mu_0$, it's like there's no sphere there & the external field $B_0\hat{z}$ permeates all space. If $\mu \to \infty$ (perfectly paramagnetic), the fields reduce to

$$\vec{H}(r<a) = 0$$
$$\vec{M}(r<a) = \frac{1}{\mu_0}\vec{B}(r<a) = 3B_0\hat{z}$$
$$\vec{H}(r>a) = \frac{1}{\mu_0}\vec{B}(r>a)$$
$$= B_0\hat{z} + B_0\left(\frac{a}{r}\right)^3[3\hat{r}\cos\theta - \hat{z}]$$
$$\vec{M}(r>a) = 0$$

(§ 5.11) Existence of ferromagnets & permanent magnets

> Recall the same conditions from the sphere in §5.10 giving rise to the relations

$$\vec{B}(r<a) = \vec{B}_0 + \frac{2\mu_0}{3}\vec{M} \qquad (1)$$
$$\vec{H}(r<a) = \frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}$$

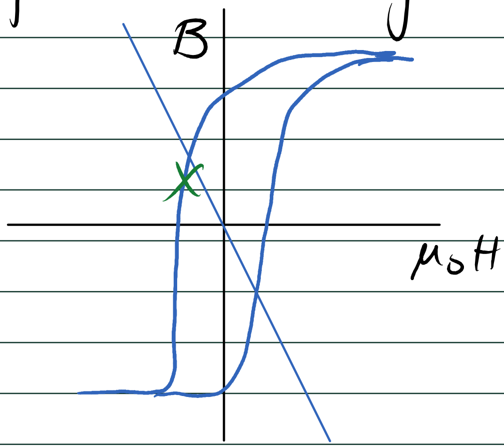
Now we assert that the sphere is para/diamagnetic w/ permeability μ (cf. example). Using $\vec{B}(r<a) = \mu\vec{H}(r<a)$, we simplify (1) as

$$\vec{M} = \frac{3}{\mu_0}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\vec{B}_0 \qquad (2)$$

which shows that for $\vec{B}_0 = 0$ (no external field), the magnetization vanishes. However, permanent magnets exist, which contradicts the limiting case. Thus, a nonlinear functional relationship $\vec{B} = \vec{F}(\vec{H})$ must be employed, together with the existence of hysteresis. Eliminating $\vec{M}$ from eq. (1),

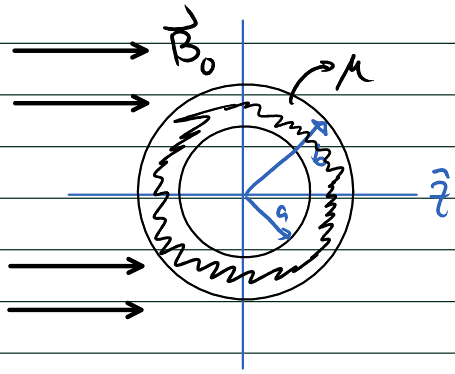
$$\vec{B}(r<a) + 2\mu_0 \vec{H}(r<a) = 3\vec{B}_0 \qquad (3)$$

Using the hysteresis diagram for relation (3),



§5.12 Magnetic shielding, spherical shell of permeable material in a uniform field.

Given: $\vec{B}_0 = \mu_0 \vec{H}_0$, spherical shell, permeability μ. Find: $\vec{B}(r<a)$. Cf. $\vec{B}_0$.



› In this problem, $\vec{J} = 0$. Then we can solve $\vec{H}$ from a scalar potential, $\vec{H} = -\nabla \Phi_M$. Since $\vec{H}$ is related to $\vec{B}$ linearly, $\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{\nabla} \cdot \vec{H} = 0$, & $\nabla^2 \Phi_M = 0$. Thus we immediately use the solution to Laplace's eqn. We then find the proper BCs at $r=a$ & $r=b$. For $r>b$,

$$\Phi_M(r>b, \theta) = -H_0 r \cos\theta + \sum_{\ell=0}^{\infty} \frac{\alpha_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad (1)$$

↳ Far-field uniform H_0

For $a<r<b$,

$$\Phi_M(a<r<b, \theta) = \sum_{\ell=0}^{\infty} \left(\beta_\ell r^\ell + \frac{\gamma_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \qquad (2)$$

and for $r<a$,

$$\Phi_M(r<a, \theta) = \sum_{\ell=0}^{\infty} \eta_\ell r^\ell P_\ell(\cos\theta) \qquad (3)$$

(BC1): $B_r(r=a) = B_r(r<a) = B_r(a<r)$, same for H_θ & $r=b$

Using Φ_M to follow these BCs.

Using Φ_M to follow these BCs,

i $\mu_0 \frac{\partial \Phi_M}{\partial r}\Big|_{r=a^-} = \mu \frac{\partial \Phi_M}{\partial r}\Big|_{r=a^+}$, ii $\mu_0 \frac{\partial \Phi_M}{\partial r}\Big|_{r=b^+} = \mu \frac{\partial \Phi_M}{\partial r}\Big|_{r=b^-}$

iii $\frac{\partial \Phi_M}{\partial \theta}\Big|_{r=a^+} = \frac{\partial \Phi_M}{\partial \theta}\Big|_{r=a^-}$, $\frac{\partial \Phi_M}{\partial \theta}\Big|_{r=b^+} = \frac{\partial \Phi_M}{\partial \theta}\Big|_{r=b^-}$ iv

true $\forall\ \theta$

Condition (iv) immediately allows us to drop $l \neq 1$ terms (because of the existence of $\cos\theta$).

We then expand on all (4) for $l=1$:

i $\mu_0 \eta_1 \cancel{\cos\theta} = \mu\left(\beta_1 - \frac{2\gamma_1}{a^3}\right)\cancel{\cos\theta}$

ii $\mu_0\left(-H_0 \cancel{\cos\theta} - \frac{2\alpha_1}{b^3}\cancel{\cos\theta}\right) = \mu\left(\beta_1 - \frac{2\gamma_1}{b^3}\right)\cancel{\cos\theta}$

iii $-\eta_1 a \cancel{\sin\theta} = -\left(\beta_1 a + \frac{\gamma_1}{a^2}\right)\cancel{\sin\theta}$

iv $-\left(\beta_1 b + \frac{\gamma_1}{b^2}\right)\cancel{\sin\theta} = H_0 b \cancel{\sin\theta} - \frac{\alpha_1}{b^2}\cancel{\sin\theta}$

Simplifying using $K = \frac{\mu}{\mu_0}$,

i $\eta_1 = K\left(\beta_1 - \frac{2\gamma_1}{a^3}\right)$

ii $\left(H_0 + \frac{2\alpha_1}{b^3}\right) = K\left(\frac{2\gamma_1}{b^3} - \beta_1\right)$

iii $\eta_1 = \beta_1 + \frac{\gamma_1}{a^3}$

iv $\beta_1 b + \frac{\gamma_1}{b^2} = \frac{\alpha_1}{b^2} - H_0 b$

i $a^3\eta_1 + 2K\gamma_1 - Ka^3\beta_1 = 0$

iii $a^3\eta_1 - \gamma_1 - a^3\beta_1 = 0$

ii $H_0 b^3 + 2\alpha_1 - 2K\gamma_1 + Kb^3\beta_1 = 0$

iv $H_0 b^3 - \alpha_1 + \gamma_1 + b^3\beta_1 = 0$

i - iii $\quad \gamma_1(2K+1) + \beta_1 a^3(1-K) = 0$

ii + 2iv $\quad \gamma_1(2-2K) + \beta_1 b^3(2+K) = -3H_0 b^3$

$$\begin{pmatrix} 2K+1 & a^3(1-K) \\ 2-2K & b^3(2+K) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3H_0 b^3 \end{pmatrix}$$

$$\det() = b^3(2+K)(2K+1) - a^3(1-K)(2-2K)$$

$$\text{inv}() = \det() \begin{pmatrix} b^3(2+K) & a^3(K-1) \\ 2K-2 & 2K+1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \gamma_1 \\ \beta_1 \end{pmatrix} = \text{inv}() \begin{pmatrix} 0 \\ -3H_0 b^3 \end{pmatrix}$$

↳ substitute to (iii) & (iv) to get

dipole moment $\leftarrow \alpha_1 = H_0 \dfrac{(2K+1)(K-1)}{(2K+1)(K+2) - 2\frac{a^3}{b^3}(K-1)^2}(b^3 - a^3)$

uniform H $\leftarrow \eta_1 = -H_0 \dfrac{9K}{(2K+1)(K+2) - 2\frac{a^3}{b^3}(K-1)^2}$

For $\mu >> \mu_0$, $\frac{\mu}{\mu_0} = K >> 1$, so

$$\alpha_1 \to b^3 H_0$$

$$-\eta_1 \to \frac{9}{2K\left(1-\frac{a^3}{b^3}\right)} H_0$$

↳ High K, low η_1!

§ 5.15 Faraday's Law

> First exposure to magnetodynamics

> Consider the effect of changing $\vec{B}(t)$

> Expt. observations of Faraday:

- moving magnets
- moving wires w/ currents

changing the current inside the wire

All created currents in nearby wires

> Changing B-fields → creation of E-fields

$$\mathcal{E}_{\delta S} = - \frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \qquad (1)$$

opposes the original flux

$$\oint_{\delta S} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \qquad (2)$$

The RHS can be expanded as

$$\frac{d}{dt}\vec{B}(\vec{r},t) = \frac{\partial \vec{B}}{\partial t} + (\vec{\nabla}\cdot\vec{B})\frac{d\vec{r}}{dt} \qquad (3)$$

So that

$\oint_{\delta S}(\vec{B}\times\vec{v})\cdot d\vec{l}$

$$\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} = \int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} + \int_S (\vec{\nabla}\cdot\vec{B})\vec{v}\cdot d\vec{a} \qquad (4)$$

$$\oint_{\delta S}\left[\vec{E}' - (\vec{v}\times\vec{B})\right]\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} \qquad (5)$$

Using Stokes' theorem on the LHS,

$$\int (\vec{\nabla}\times\vec{E}')\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

Galilean transformation

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5}$$

With $\vec{B}$ = constant, $\vec{\nabla}\times\vec{E} = 0 \rightarrow$ electrostatics

Note that this only holds true for nonrelativistic speeds, as is the case for when we shifted our frame from the inertial lab frame to the rest frame of the moving charges. //

§5.16 Energy in $\vec{B}$-fields

Recall: The work done by an emf $\mathcal{E}$ on a current loop I is related by

$$\frac{dW}{dt} = -I\mathcal{E} \tag{6}$$

Then, using Faraday's law,

$$\frac{dW}{dt} = I\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \tag{7}$$

or

$$dW = I\, d\int \vec{B}\cdot d\vec{a} \tag{8}$$

In terms of $\vec{A}$,

$$dW = I\, d\int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} \tag{9}$$

In terms of J,

In terms of J,

$$\Delta(dW) = J\,\Delta\sigma \int (\vec{\nabla}\times d\vec{A})\cdot d\vec{a} \qquad (10)$$

and since $J\,\Delta\sigma\, d\vec{l} = \vec{J}\, d^3x$

$$dW = \int d\vec{A}\cdot\vec{J}\, d^3x \qquad (11)$$

Recall $\vec{\nabla}\times\vec{H} = \vec{J}$,

$$dW = \int d\vec{A}\cdot(\vec{\nabla}\times\vec{H})\, d^3x \qquad (12)$$

and $\vec{\nabla}\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\vec{\nabla}\times\vec{P}) - \vec{P}\cdot(\vec{\nabla}\times\vec{Q})$,

$$dW = \int \left[\vec{\nabla}\cdot(d\vec{A}\times\vec{H}) + \vec{H}\cdot(\vec{\nabla}\times d\vec{A})\right] d^3x$$

localized $\vec{H}$

$$dW = \int \vec{H}\cdot d\vec{B}\, d^3x \qquad (13)$$

$$\left(\text{cf. } dW = \int \vec{E}\cdot d\vec{D}\, d^3x\right)$$

But $\vec{H}\cdot d\vec{B} = \frac{1}{2} d(\vec{H}\cdot\vec{B})$ for linear para/dia magnetics, so

$$\boxed{W = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x} \qquad (14)$$

$$\text{cf. } W = \frac{1}{2\mu}\int |\vec{B}|^2\, d^3x$$

§5.16 (contd)

If we also assume a linear relationship bet $\vec{A}$ & $\vec{J}$, then

$$dW = \int d\vec{A}\cdot\vec{J}\, d^3x$$

$$\hookrightarrow W = \frac{1}{2}\int \vec{A}\cdot\vec{J}\, d^3x \qquad (1)$$

Jackson's challenge: Find the change in energy when an object of permeability μ_1 is placed in a magnetic field $\vec{B}_0$ with permeability μ_0. Answer:

$$W = \frac{1}{2}\int_{V_1} (\vec{B}\cdot\vec{H}_0 - \vec{H}\cdot\vec{B}_0)\, d^3x \qquad (2)$$

§5.17 Energy & self/mutual inductance

Recall: capacitance → electrostatics

inductance → magnetodynamics

From eq. (1), we can rewrite the total energy as

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij} I_i I_j \qquad (3)$$

self-inductance   mutual inductance

Converting $\vec{A}$ to its integral form, (1) becomes

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (4)$$

$$W = \frac{\mu_0}{8\pi}\int d^3x \int d^3x' \frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \qquad (5)$$

$$W = \frac{\mu_0}{8\pi} \int d^3x \int d^3x' \frac{\vec{J}(\vec{x}) \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad (5)$$

Now for each circuit element i & j,

$$W = \frac{\mu_0}{8\pi} \sum_{i=1}^{N} \int d^3x_i \sum_{j=1}^{N} \int d^3x_j' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$$

For $i=j$, we get L_i

$$L_i = \frac{\mu_0}{4\pi I_i^2} \int_{C_i} d^3x_i \int_{C_i} d^3x_i' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_i')}{|\vec{x}_i - \vec{x}_i'|} \quad (6)$$

and for $i \neq j$, we get M_{ij}

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{C_i} d^3x_i \int_{C_j} d^3x_j' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|} \quad (7)$$

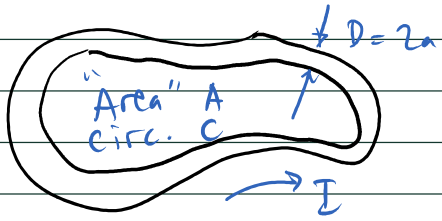
Recall in elementary EM classes that inductance (whether self or mutual) is defined in terms of flux linkage. To relate the above definitions, consider eq. (7).

The integral for d^3x_j' looks like the integral definition for $\vec{A}(\vec{x}_i)$. With some approximations, we can write $\vec{J}(\vec{x}_i)\, d^3x = J_{\parallel}\, da\, d\vec{l}$, where da is a differential cross-section & $d\vec{l}$ is a directed longitudinal differential following the current flow. Thus, eq. (7) becomes

$$M_{ij} = \frac{1}{I_i I_j} I_i \oint \underbrace{\vec{A}_{ij}}_{\text{const. throughout small cross-sectional area}} \cdot d\vec{l} \xrightarrow{\text{Stokes}} \frac{1}{I_j} \int_{S_i} (\nabla \times \vec{A}_{ij}) \cdot d\vec{a} \quad (8)$$

or, $$M_{ij} = \frac{F_{ij}}{I_j} \quad \xrightarrow{} \text{magnetic Flux} \tag{9}$$

How then do we estimate self-inductance for simple circuits? E.g.



Then the self-inductance is

$$L = \frac{1}{I^2} \int \frac{\vec{B} \cdot \vec{B}}{\mu} \, d^3x \tag{10}$$

Where $B_\phi = \frac{\mu_0 I}{2\pi a} \frac{\rho_<}{\rho_>}$ ($\rho_< \to \min(\rho, a)$, $\rho_> \to \max(\rho, a)$) for nonpermeable wires & surroundings. Then the inductance per unit length are

$$\left.\frac{dL}{dl}\right|_{\rho<a} = \frac{\mu_0}{8\pi} \tag{11}$$

&

$$\left.\frac{dL}{dl}\right|_{\rho>a} = \frac{\mu_0}{4\pi} \ln\left(\frac{\rho_{max}^2}{a^2}\right) \tag{12}$$

and

$$L \approx \frac{\mu_0}{4\pi} C \left[\ln\left(\frac{4A}{a^2} + \frac{1}{2} \right) \right] \tag{13}$$

$4A \to \rho_{max}^2$

§5.18 Quasi-static Magnetic Fields in Conductors; Eddy currents; Magnetic Diffusion

"Quasi-static" → magnetic fields dominate before creation of $\vec{E}$ by Faraday's law; $c \to \infty$, propagation of fields is instantaneous. Relevant eqs:

$$\vec{\nabla} \times \vec{H} = \vec{J}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (1)$$
$$\vec{J} = \sigma \vec{E}$$

Since $\vec{B} = \vec{\nabla} \times \vec{A}$,

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$
$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Then
$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi \qquad (2)$$

Assuming no electric (free) charge, $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$ then $\vec{\nabla} \cdot \vec{E} = 0$ & $\vec{\nabla} \cdot \vec{A} = 0$ (Coulomb gauge?)

If μ is freq.-independent & uniform,

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} = \mu \sigma \vec{E} \qquad (3)$$
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu \sigma \vec{E}$$

$$\cancel{\vec\nabla(\vec\nabla\cdot\vec A)}^{\,0} - \vec\nabla^2\vec A = \mu\sigma\vec E \to -\frac{\partial A}{\partial t}$$

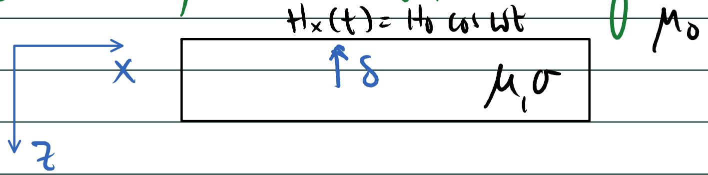
or the diffusion eqn,

$$\vec\nabla^2\vec A = \mu\sigma\frac{\partial\vec A}{\partial t} \qquad (4)$$

also valid for $\vec E$ when the conductivity σ is spatially-varying but freq.-independent. (If σ = const. then $\vec B$ & $\vec J$ also satisfy (4)).

A. Skin depth/Eddy currents/Induction heating

Consider:



We try to solve (4) for $z>0$. BCs:

① Continuity at $z=0$

$$H(z=0^+,t)=H_0\cos\omega t \qquad (5)$$

② Linearity of (4) $\to$ only H_x exists for $z>0$.

To get the steady-state solution then,

$$H_x(z,t)=h(z)e^{-i\omega t} \qquad (6)$$

satisfying

$$\left(\frac{d^2}{dz^2}+i\mu\sigma\omega\right)h(z)=0$$

with trial solution $h(z)=e^{ikz}$ for $k^2=i\mu\sigma\omega$

or $k=\pm(1+i)\sqrt{\frac{\mu\sigma\omega}{2}}$ $\to \frac{1}{\delta}$, skin depth!

Thus,

$$H_x(z,t) = H_0\, e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right) \qquad (7)$$

Using Ampere's & Ohm's laws, we can solve for $\vec{E}$. Since $H_x(z,t)$ is the only existing magnetic field, then only E_y exists and

$$E_y = \frac{1}{\sigma}\frac{dH_x}{dz} \qquad (8)$$

$$= -\frac{1+i}{\sigma\delta} H_0 e^{-z/\delta} e^{iz/\delta - i\omega t}$$

$$\mathrm{Re}(E_y) = \frac{\mu\omega\delta}{\sqrt{2}} H_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right) \qquad (9)$$

Comparing magnitudes,

$$\frac{E_y}{c\mu H_x} = \mathcal{O}\left(\frac{\omega\delta}{c}\right) \ll 1 \qquad \text{Quasi-static assumption}$$

And the associated $J_y(z>0,t)$ is

$$\mathrm{Re}(J_y) = \sigma\,\mathrm{Re}(E_y)$$

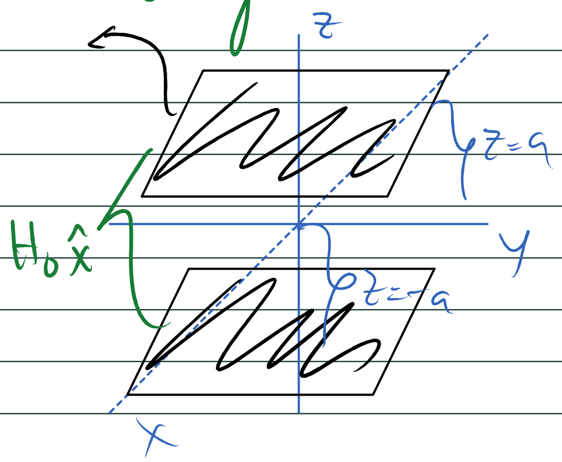
$$= \frac{\sqrt{2}}{\delta} H_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right)$$

B. Diffusion of $\vec{B}$ in conducting media

Consider: permeability μ & conductivity σ

so that

$$J_y = H_0\left[\delta(z+a) - \delta(z-a)\right]$$



At $t=0$, current $J_y = 0$ suddenly. Then, by using Laplace transforms,

$$H_x(z,t) = \int_0^\infty h(k) \cos(kz) e^{-k^2 t/\mu\sigma} \, dk$$

We use BCs to solve for $h(k)$:

At $t=0$,

$$H_x(z, 0^+) = \int_0^\infty h(k) \cos(kz) \, dk$$

const. H_0 bet $-a$ to a $\longrightarrow$ $= H_0 [\Theta(z+a) - \Theta(z-a)]$

Using Fourier inversion,

$$\boxed{h(k) = \frac{2}{\pi} H_0 \frac{\sin(ka)}{k}}$$

Ch. 6 Maxwell's Equations, Macroscopic EM, Conservation Laws

§6.1 Maxwell's Displacement Current; Maxwell Equations

> Recall: The static eqns

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \rightarrow \text{Gauss's law}$$ → creates diverging $\vec{E}$

$$\vec{\nabla}\times\vec{E} = 0 \rightarrow \text{Irrotational } \vec{E} \qquad (1)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \text{no magnetic monopole charge}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{tot} \rightarrow \text{steady moving charges} \qquad (2)$$

> Recall: Faraday's Law: considering time-varying $\vec{B}$ so that (1b) becomes

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (3)$$

> taking the divergence of (2b),

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J}_{tot})$$

$$0 = \mu_0(\vec{\nabla}\cdot\vec{J}_{tot}) \qquad (4)$$

↳ okay for electrostatics, BUT NOT for dynamics!

Recall: The continuity eqn.

$$\vec{\nabla}\cdot\vec{J}_{tot} = -\frac{\partial\rho_{tot}}{\partial t} \rightarrow \text{prevents buildup of charges} \qquad (5)$$

Since $\frac{\partial\rho}{\partial t} \neq 0$ for electrodynamics, (4) must be wrong!

(or is it?)

, With Maxwell's genius in Math (GG Stokes approved), the contradiction was resolved: from eq. (5),

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D})$$

or

$$\vec{\nabla} \cdot \underbrace{\left(\vec{J} + \frac{\partial}{\partial t}\vec{D}\right)}_{\text{some } ``\vec{J}"} = 0 \qquad (6)$$

so that eq. (4) becomes

$$\cancel{\vec{\nabla}} \cdot (\vec{\nabla} \times \vec{B}) = 0 = \mu_0 \cancel{\vec{\nabla}} \cdot \left(\vec{J} + \frac{\partial}{\partial t}\vec{D}\right)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial}{\partial t}\vec{D} \qquad (7)$$

↳ Ampere-Maxwell law

Which automatically satisfies eq (5). For partial fields,

$$\left(\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J} + \frac{\partial}{\partial t}\vec{D} \end{aligned}\right) \qquad (8)$$

Then, with the eqns that govern the creation of $\vec{E}$ & $\vec{B}$, we now have a complete theory for charges:

$$\vec{F} = \int [\rho(\vec{x})\vec{E}(\vec{x}) + \vec{J}(\vec{x}) \times \vec{B}(\vec{x})]\, d\vec{x} = m\frac{d^2\vec{x}}{dt^2} \qquad (9)$$

↳ motion of charged masses wrt fields

> Boundary conditions

Since the divergence eqns are the same for electrostatics & electrodynamics, we can still use the standard Gaussian pillbox treatment and get the fields normal to the surfaces:

$$(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = \sigma$$
$$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0 \qquad (10)$$

For the electrodynamic curl eqns, we employ again the standard Amperian loop procedure to the ff integral forms:

$$\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$
$$\oint \vec{B}\cdot d\vec{l} = \int \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{a} \qquad (11)$$

so that

$$\hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0$$
$$\hat{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{K} \qquad (12)$$

$\vec{K}$ → surface current density

> EM WAVES

Taking the curl of (8d)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = \vec{\nabla}\times\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \qquad (13)$$
$$\vec{\nabla}^2\vec{H} - \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{H})} = -\left(\vec{\nabla}\times\vec{J} + \frac{\partial}{\partial t}\vec{\nabla}\times\vec{D}\right)$$
$$\vec{\nabla}^2\vec{H} = -\left(\vec{\nabla}\times\vec{J} + \frac{\partial}{\partial t}\left(-\epsilon_0\frac{\partial}{\partial t}\vec{B}\right)\right)$$

$$\vec{\nabla}^2\vec{H} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{H} = -\vec{\nabla}\times\vec{J}$$

or in terms of $\vec{J}_{tot}$ & $\vec{B}$,

$$\boxed{\Box^2\vec{B} = -\mu_0\,\vec{\nabla}\times\vec{J}} \qquad (14)$$

We do a similar process for $\vec{E}$:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla}^2\vec{E} - \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) = \frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}^2\vec{E} - \vec{\nabla}\left(\frac{\rho}{\epsilon_0}\right) = \frac{\partial}{\partial t}\left(\mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}^2\vec{E} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = \vec{\nabla}\left(\frac{\rho}{\epsilon_0}\right) + \mu_0\frac{\partial\vec{J}}{\partial t}$$

§6.2 Vector & Scalar potentials

> Goal: to obtain electrodynamic potentials that reduce to the familiar ones when the fields are not time-varying.

> Since $\vec{\nabla}\cdot\vec{B}=0$ still holds, we can still use $\vec{A}$:

$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0 \quad \forall \vec{A} \qquad (1)$$

Plugging to Faraday's law,

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$
$$=\vec{\nabla}\times\left(-\frac{\partial\vec{A}}{\partial t}\right)$$

assume smoothness of $\vec{A}$ in x & t

$$\vec{\nabla}\times\underbrace{\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)}_{\text{equal to some gradient:}}=0 \qquad (2)$$

$$\vec{\nabla}\times(-\vec{\nabla}\Phi)=0$$

so that $\vec{\nabla}\Phi=-\vec{E}-\frac{\partial\vec{A}}{\partial t}$ or $\vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$

Inserting into Gauss' & Ampere's laws

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\epsilon_0}$$

$$\rightarrow \nabla^2\Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \qquad (3)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}_{tot} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} + \mu_0\epsilon_0\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}_{tot} \quad (4)$$

§6.3

For a unique solution (2 unknowns, 2 eqns), we need to set a relation bet. $\vec{A}$ & Φ (called "gauge"). From (4),

$$\vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right)}_{\text{must vanish to obtain a simple uncoupled PDE}} = \mu_0\vec{J} + \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$= \mu_0\vec{J} + \Box^2\vec{A}$$

so that our condition must be $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t} = 0$ (due to Lorenz). This is a valid excuse since from the definitions of $\vec{A}$ & Φ, additional freedom is given by

$\vec{A}$: $\vec{A} \rightarrow \vec{A}' + \nabla f$, since $\vec{\nabla}\times(\vec{A}' + \nabla f) = \vec{\nabla}\times\vec{A}'$

Φ: $\Phi \rightarrow \Phi' + \frac{\partial f}{\partial t}$, since $\nabla(\Phi' + \frac{\partial f}{\partial t}) = \nabla\Phi'$

> Then, the Maxwell eqns in the Lorenz gauge simplify to two from four:

$$\vec{\nabla}^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_{tot}}{\epsilon_0}$$

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}_{tot} \qquad (5)$$

→ potentials obey wave eqns just like fields

> A second gauge consists of letting $\vec{\nabla}\cdot\vec{A} = 0$; This is called the Coulomb gauge. Then, Maxwell's eqns reduce to

$$\vec{\nabla}^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J}_{tot} + \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial \Phi}{\partial t}\right) \qquad (6)$$

They look like nonhomogeneous wave eqns.