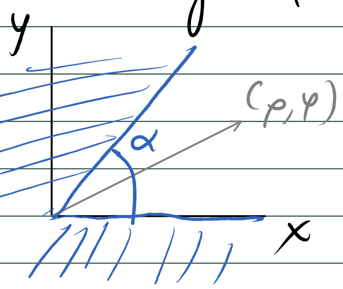
Laplace's eqn. for intersecting planes

- use polar coordinates
- Boundary conditions:



$$\Phi(\rho,\varphi=0)=\Phi(\rho,\varphi=\alpha)=V$$
$$\Phi(\rho=\rho_0,\varphi)=V_1(\varphi)$$

- in this problem, we include the origin in the region of interest, s.t. $B_\nu=0 \ \forall \nu$ in the gen. sol'n:

$$\Phi(\rho,\varphi)=(A_0+B_0\ln\rho)(C_0+D_0\varphi)$$
$$+\sum_{\nu\neq 0}(A_\nu\rho^\nu+B_\nu\rho^{-\nu})(C_\nu e^{i\nu\varphi}+D_\nu e^{-i\nu\varphi})$$

to prevent blowing up. We then apply the first BC:

$$\Phi(\rho,\varphi=0)=V=\underbrace{A_0}_{C_0 \text{ absorbed}}+\sum_{\nu\neq 0}\underbrace{A_\nu\rho^\nu}_{C_\nu \text{ absorbed}}+D_\nu\rho^\nu$$

so that $\boxed{V=A_0 \ \& \ A_\nu=-D_\nu}$ and we update:

$$\Phi(\rho,\varphi)=V+D_0\varphi+\sum_{\nu\neq 0}A_\nu\rho^\nu(e^{i\nu\varphi}-e^{-i\nu\varphi})$$
$$=V+D_0\varphi+\sum_{\nu\neq 0}\underbrace{A_\nu}_{2 \text{ absorbed}}\rho^\nu\sin\nu\varphi$$

Apply the next BC:

$$\Phi(\rho,\varphi=\alpha)=V=V+D_0\alpha+\sum_{\nu\neq 0}A_\nu\rho^\nu\sin\nu\alpha$$

to be valid $\forall \rho$, set $D_0=0$ and $\sin\nu\alpha=0$ or $\boxed{\nu=\frac{n\pi}{\alpha}}$, $n=0,1,2,\dots$ We then update:

$$\Phi(\rho,\varphi)=V+\sum_{n=1}^{\infty}A_n\rho^{\frac{n\pi}{\alpha}}\sin\left(\frac{n\pi}{\alpha}\varphi\right)$$

$$\Phi(\rho,\varphi) = V + \sum_{n=1} A_n \rho^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\varphi\right)$$

Lastly, the final BC is used:

$$\Phi(\rho=\rho_0, \varphi) = V_1(\varphi) = V + \sum_{n=1}^{\infty} A_n \rho_0^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\varphi\right)$$

We then multiply both sides by $\sin\frac{m\pi}{\alpha}\varphi$ to exploit orthogonality:

$$(V_1(\varphi) - V)\sin\frac{m\pi}{\alpha}\varphi = \sum_{n=0}^{\infty} A_n \rho_0^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\varphi\right)\sin\left(\frac{m\pi}{\alpha}\varphi\right)$$

Integrate both sides for $\varphi \in (0,\alpha)$:

$$\int_0^\alpha (V_1 - V)\sin\frac{m\pi}{\alpha}\,d\varphi = \sum_{n=0}^{\infty} A_n \rho_0^{\frac{n\pi}{\alpha}} \underbrace{\int_0^\alpha \sin\frac{n\pi}{\alpha}\varphi \sin\frac{m\pi}{\alpha}\varphi\,d\varphi}_{\frac{\alpha}{2}\delta_{mn}}.$$

$$\int_0^\alpha (V_1 - V)\sin\frac{m\pi}{\alpha}\,d\varphi = \frac{\alpha}{2}\sum_{n=0}^{\infty} A_n \rho_0^{\frac{n\pi}{\alpha}} \delta_{mn} = \frac{\alpha}{2} A_m \rho_0^{\frac{m\pi}{\alpha}}$$

OR $\quad A_m = \frac{2}{\alpha}\rho_0^{-\frac{m\pi}{\alpha}} \int_0^\alpha (V_1(\varphi) - V)\sin\frac{m\pi}{\alpha}\,d\varphi$

Thus, the final soln is

$$\boxed{\Phi(\rho,\varphi) = V + \sum_{m=0}^{\infty} A_m \rho^{\frac{m\pi}{\alpha}} \sin\left(\frac{m\pi}{\alpha}\varphi\right), \quad A_m = \frac{2}{\alpha}\rho_0^{-\frac{m\pi}{\alpha}} \int_0^\alpha (V_1 - V)\sin\frac{m\pi}{\alpha}\,d\varphi}$$

Let us try to get the electric field & surface charge density for this case. Recall $\vec{E} = -\vec{\nabla}\Phi$. Near the origin, we only keep the first term of the series, s.t.

$$\Phi(\rho \ll 1, \varphi) = V + A_1 \rho^{\frac{\pi}{\alpha}} \sin\frac{\pi}{\alpha}\varphi$$

$$\& \quad \vec{E} = -\vec{\nabla}\Phi = -\left(\frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\hat{\varphi}\right)\Phi$$

$$= -\left[A_1 \sin\frac{\pi}{\alpha}\varphi\left(\frac{\pi}{\alpha}\rho^{\frac{\pi}{\alpha}-1}\right)\hat{\rho} + A_1 \rho^{\frac{\pi}{\alpha}-1}\left(\frac{\pi}{\alpha}\cos\frac{\pi}{\alpha}\varphi\right)\hat{\varphi}\right]$$

$$= -\frac{A_1\pi}{\alpha}\rho^{\frac{\pi}{\alpha}-1}\left(\sin\frac{\pi}{\alpha}\varphi\,\hat{\rho} + \cos\frac{\pi}{\alpha}\varphi\,\hat{\varphi}\right)$$

Thus, the surface charge densities at $\varphi = 0$ & $\varphi = \alpha$ are

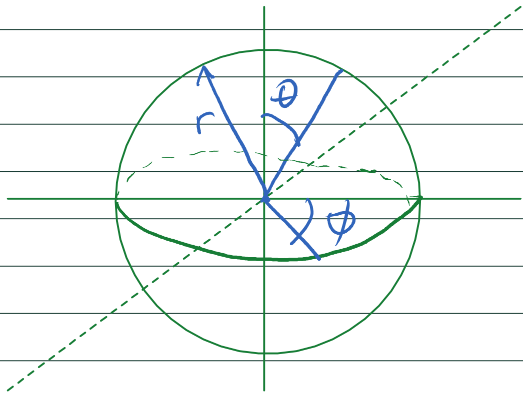
$$\sigma(\rho) = \epsilon_0[\vec{E}\cdot\hat{n}] = \epsilon_0\vec{E}(\rho,\varphi=0)\cdot\hat{\varphi} = -\frac{\epsilon_0 A_1\pi}{\alpha}\rho^{\frac{\pi}{\alpha}-1}$$

$$\epsilon_0\vec{E}(\rho,\varphi=\alpha)\cdot\hat{\varphi} = \frac{\epsilon_0 A_1\pi}{\alpha}\rho^{\frac{\pi}{\alpha}-1}$$

Laplace's eqn. in spherical coordinates

$$\nabla^2\Phi = 0$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$



Again, use sepvar, now w/ ansatz

$$\Phi(r,\theta,\phi) = \frac{R(r)}{r}P(\theta)Q(\phi)$$

$\hookrightarrow$ keeps Φ unitless & R a fxn of r.

Substituting,

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(RPQ) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{R}{r}PQ\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left(\frac{R}{r}PQ\right) = 0 \qquad (1)$$

$$\frac{PQ}{r}\frac{d^2R}{dr^2} + \frac{RQ}{r^3\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{RP}{r^3\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0$$

Divide both sides by Φ,

Multiply both sides by $r^2\sin^2\theta$,

$$\frac{r^2\sin^2\theta}{R}\frac{dR}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{} = 0$$

constant $(-m^2)$

We can easily solve $Q(\phi)$:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \longrightarrow \frac{d^2Q}{d\phi^2} = -m^2 Q$$

or

$$\boxed{Q(\phi) = \begin{cases} A_m e^{im\phi} + B_m e^{-im\phi}, & m \neq 0 \\ A_0 + B_0\phi, & m = 0 \end{cases}}$$

for a full sweep of ϕ, we need to keep Φ single-valued, so that m must be an integer & $Q(\phi) = A_0$, $m=0$.

We then go back to solving R & P:

$$\frac{r^2\sin^2\theta}{R}\frac{d^2R}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - m^2 = 0$$

$$\underbrace{\frac{r^2}{R}\frac{d^2R}{dr^2}}_{\text{fxn of } R(r)\ \&\ r} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\underbrace{\text{no } R(r) \text{ or } r \text{ terms}}_{\text{let } -\ell(\ell+1)}} = 0$$

so that
$$\frac{r^2}{R}\frac{d^2R}{dr^2} = \ell(\ell+1)$$

Let $R(r) = r^a$, α a constant. Then $\frac{dR}{dr} = ar^{a-1}$, $\frac{d^2R}{dr^2} = \alpha(\alpha-1)r^{a-2}$. Thus

$$\frac{d^2R}{dr^2} = \frac{R}{r^2}\ell(\ell+1) = \alpha(\alpha-1)r^{a-2}$$

$$= \frac{r^\alpha}{r^2}\ell(\ell+1) = \alpha(\alpha-1)\cancel{r^{\alpha-2}}$$

$$\alpha(\alpha-1) - \ell(\ell+1) = 0$$

$$\alpha^2 - \alpha - \ell(\ell+1) = 0$$

Solving for a in terms of ℓ,

$$\alpha = \frac{1 \pm \sqrt{1+4\ell(\ell+1)}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{4\ell^2+4\ell+1}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{(2\ell+1)^2}$$

$$= \frac{1}{2} \pm \frac{1}{2}(2\ell+1)$$

$$= \frac{1}{2} + \frac{1}{2} + \ell = \ell+1$$

$\alpha = \ell+1$

or $\alpha = -\ell$

or $\alpha = -\ell$

$$= \frac{1}{2} \pm \frac{1}{2}(2\ell+1)$$
$$= \frac{1}{2} + \frac{1}{2} + \ell = \ell+1$$
or $= -\ell$

Thus $\boxed{R(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}}$ $\forall \ell$

Now, solving for P is trickier & involves special fxns.

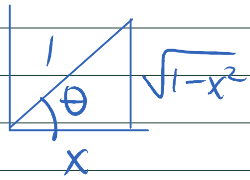
Recall

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1)$$

let $x = \cos\theta$, $dx = -\sin\theta\, d\theta$,

$$d\theta = \frac{-1}{\sqrt{1-x^2}}dx$$

$$\frac{d}{d\theta} = -\sqrt{1-x^2}\frac{d}{dx}$$



$$\frac{1}{P(x)\cancel{\sqrt{1-x^2}}}\left(-\cancel{\sqrt{1-x^2}}\frac{d}{dx}\left(\sqrt{1-x^2}\left(-\sqrt{1-x^2}\right)\frac{d}{dx}P(x)\right)\right) - \frac{m^2}{1-x^2} = -\ell(\ell+1)$$

$$\frac{1}{P(x)}\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) - \frac{m^2}{1-x^2} = -\ell(\ell+1)$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] = \left[\frac{m^2}{1-x^2} - \ell(\ell+1)\right]P(x)$$

We treat this equation in two separate cases:

① When $m=0$,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] = -\ell(\ell+1)P(x)$$

$$-2x\frac{dP}{dx} + (1-x^2)\frac{d^2P}{dx^2} = -\ell(\ell+1)P(x)$$

LET $P(x) = \sum_{j=0}^{\infty} a_j x^{j+\alpha} = x^\alpha \sum_{j=0}^{\infty} a_j x^j$

Then $\frac{dP}{dx} = \frac{d}{dx}\left[x^\alpha(a_0 + a_1 x + a_2 x^2 + \dots)\right]$

$$= \alpha x^{\alpha-1}(a_0 + a_1 x + \dots) + x^\alpha(a_1 + 2a_2 x + 3a_3 x^2 + \dots)$$

$$= x^\alpha\left[\frac{\alpha}{x}(a_0 + a_1 x + \dots) + (a_1 + 2a_2 x + 3a_3 x^2 + \dots)\right]$$

$$= x^\alpha\left[\frac{\alpha a_0}{x} + a_1(\alpha+1) + a_2 x(\alpha+2) + a_3 x^2(\alpha+3) + \dots\right]$$

$$= x^{\alpha-1}\left[\alpha a_0 + a_1 x(\alpha+1) + a_2 x^2(\alpha+2) + \dots\right]$$

$$= x^{\alpha-1}\sum_{j=0}^{\infty}(\alpha+j)\,a_j x^j$$

$$\frac{dp}{dx} = \sum_{j=0}^{\infty}(\alpha+j)\,a_j\,x^{j+\alpha-1}$$

& $$\frac{d^2p}{dx^2} = \frac{d}{dx}(\ \ ) = \frac{d}{dx}\left[x^{\alpha-1}\left(\alpha a_0 + a_1 x(\alpha+1) + a_2 x^2(\alpha+2) + \dots\right)\right]$$

$$= (\alpha-1)x^{\alpha-2}\left(\alpha a_0 + a_1 x(\alpha+1) + \dots\right) + x^{\alpha-1}\left(a_1(\alpha+1) + 2a_2(\alpha+2)x + \dots\right)$$

$$= x^{\alpha-2}\Big[\left(a_0\alpha(\alpha-1) + a_1 x(\alpha-1)(\alpha+1) + a_2 x^2(\alpha-1)(\alpha+2)\right. + \left(a_1 x(\alpha+1) + 2a_2 x^2(\alpha+2) + \dots\right)\Big]$$

$$= x^{\alpha-2}\left[a_0(\alpha+0)(\alpha-1) + a_1 x\alpha(\alpha+1) + a_2 x^2(\alpha+1)(\alpha+2) + \dots\right]$$

$$\frac{d^2p}{dx^2} = x^{\alpha-2}\left[\sum_{j=0}^{\infty} a_j x^j(\alpha+j-1)(\alpha+j)\right] = \sum_{j=0}^{\infty}(\alpha+j)(\alpha+j-1)a_j x^{j+\alpha-2}$$

Thus,

$$-2x\sum_{j=0}^{\infty}(j+\alpha)\cdot a_j x^{j+\alpha-1} + (1-x^2)\sum_{j=0}^{\infty}(j+\alpha)(j+\alpha-1)a_j x^{j+\alpha-2} + \ell(\ell+1)\sum_{j=0}^{\infty}a_j x^{j+\alpha} = 0$$

This must be equal for ALL powers of x, s.t the coefficients for each power of x MUST vanish.

$$-2\sum_{j=0}^{\infty}(j+\alpha)a_j x^{j+\alpha} + \sum_{j=0}^{\infty}(j+\alpha)(j+\alpha-1)a_j x^{j+\alpha-2} + \ell(\ell+1)\sum_{j=0}^{\infty}a_j x^{j+\alpha} - \sum_{j=0}^{\infty}(j+\alpha)(j+\alpha-1)a_j x^{j+\alpha} = 0$$

$$\alpha(\alpha-1)a_0 x^{\alpha-2} + \alpha(\alpha+1)a_1 x^{\alpha-1} + \sum_{j=0}^{\infty}(j+2+\alpha)(j+1+\alpha)a_{j+2}x^{j+\alpha} + \sum_{j=0}^{\infty}\left[-2(j+\alpha) + \ell(\ell+1) - (j+\alpha)(j+\alpha-1)\right]a_j x^{j+\alpha} = 0$$

$$\alpha(\alpha-1)a_0 = 0, \quad \alpha(\alpha+1)a_1 = 0,$$

$$(j+2+\alpha)(j+1+\alpha)a_{j+2} = 2(j+\alpha) + (j+\alpha)(j+\alpha-1) - \ell(\ell+1)a_j$$