Linear emulator approach for bound orbits under the influence of the Paczynski-Wiita potential

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Abstract

We studied orbits under the influence of the Paczynski-Wiita pseudo-Newtonian potential. This potential results in a modified Binet equation that is nonlinear. We can approximate this nonlinearity by using the least squares method to produce a linear emulator. We showed that this approximation produces orbits that precesses over time, outperforming the standard linear Taylor series approximation.

Keywords: pseudo-Newtonian potentials, linear emulator

1 Introduction

The Paczynski-Wiita potential [1] is a pseudo-Newtonian potential given by

$$V_{PW}(r) = -\frac{GM}{r - r_{\mathrm{G}}}, \tag{1}$$

where $r_{\mathrm{G}} \equiv 2GM/c^2$. It is called as such due to the fact that it's not a solution to the gravitational Poisson equation $\nabla^2\phi = 4\pi G\rho$. Modifications of the Newtonian potential such as Eq. (1) allow one to analyze low-order general relativistic effects [2].

Wang and Zuo [2] used several pseudo-Newtonian potentials to show the precession of orbits and proceeded to compare the Einsteinian precession rate with that of their calculations. In their paper, they considered retaining the lowest-order terms of the Taylor expansion of the pseudo-Newtonian potentials (in the inverse radial distance) to produce a linear Binet equation. For an arbitrary central potential, this equation can be written as

$$u''(\theta) + u(\theta) = -\frac{1}{\ell^2}\frac{\partial V}{\partial u}, \tag{2}$$

where $u \equiv 1/r$, ℓ is the angular momentum per unit mass, and that the double prime corresponds to the second derivative with respect to the azimuthal angle θ.

After transforming the potential Eq. (1) from a function of $r \to u$, we apply it to Eq. (2) and obtain a second order nonlinear differential equation, which we can further simplify with the substitution $\xi \equiv u/u_{\mathrm{G}}$, $u_{\mathrm{G}} = 1/r_{\mathrm{G}}$:

$$\xi''(\theta) + \xi(\theta) = 2\alpha\left(\frac{1}{(1-\xi(\theta))^2}\right); \qquad f(\xi) = \xi - \frac{2\alpha}{(1-\xi)^2}, \tag{3}$$

where $\alpha \equiv (GM/c\ell)^2$, a constant which will parametrize our bound system.

Since a Binet equation effectively mimics a one-dimensional harmonic oscillator, we can use approximation techniques such as linearization [3] and least squares emulation [4, 5] for the nonlinear terms of the radial component of motion.

In this paper, we use the least squares method to approximate the nonlinear pseudo-force $f(\xi)$ arising from the Paczynski-Wiita potential. The method produces expressions for the coefficients of a differential equation emulator with known analytical solution. We then use this emulator to describe the bound orbits under the influence of the Paczynski-Wiita potential. We also calculate the precession per revolution between two successive perihelia from the emulator solution and then compare this with numerical and Taylor-linearized solutions.

2 The Least Squares method and its application

In this section, we introduce the method of least squares, which can be used for solving matrix equations as well as approximating nonlinear functions by polynomials [4, 5]. We apply this method to Eq. (3) to find an approximate form which can be solved analytically. We define the least squares integral to be

$$\int_{\xi_{\mathrm{A}}}^{\xi_{\mathrm{P}}} \left(f(\xi) - f_{\mathrm{approx}}(\xi)\right)^2 \, d\xi, \tag{4}$$

where $f_{\mathrm{approx}}(\xi)$ is an approximating function from a differential equation with known analytical solution when placed in a form $\xi'' + f_{\mathrm{approx}}(\xi) = 0$. For this paper, we use the forced linear oscillator [3]

$$\begin{aligned} \xi''(\theta) + \lambda\xi(\theta) + \gamma = 0; \\ \xi(\theta = 0) = \xi_{\mathrm{P}}, \quad \xi'(\theta = 0) = 0, \end{aligned} \tag{5}$$

with a particular solution according to the given boundary conditions:

$$\xi(\theta) = \left(\xi_{\mathrm{P}} - \frac{\gamma}{\lambda}\right)\cos\left(\sqrt{\lambda}\theta\right) + \frac{\gamma}{\lambda}, \tag{6}$$

where $\omega_\theta = \sqrt{\lambda}$.

To apply the least squares approximation, we emulate $f(\xi)$ by a linear emulator in an appropriate range of emulation, ξ_{A} to ξ_{P}. Here, ξ_{A} is the inverse of the apocenter, or the farthest point in the orbit, and ξ_{P} is the inverse of the pericenter, or the nearest point in the orbit. The values of ξ_{A} and ξ_{P} come from the energy integral

$$\frac{\dot{r}^2}{2} + \frac{\ell^2}{2r^2} - \frac{GM}{r - r_{\mathrm{G}}} = E \tag{7}$$

for r. We solve for the bounds of the orbit by setting $\dot{r} = 0$ and nondimensionalizing Eq. (7):

$$\frac{1}{8\alpha\chi^2} - \frac{1}{2(\chi - 1)} = \frac{E}{c^2} := \eta. \tag{8}$$

Here, $\chi \equiv r/r_{\mathrm{G}} = 1/\xi$. By specifying α, we get to solve this equation which yields $\chi_{\min} = 1/\xi_{\mathrm{P}}$ and $\chi_{\max} = 1/\xi_{\mathrm{A}}$. With the appropriate choice for $\eta < 0$, we can get bound orbits for this quasi-potential.

For convenience, we make another change of variable $\xi \to \xi - \xi_{\mathrm{SCO}}$, so that the emulator runs from $\xi_{\mathrm{A}} - \xi_{\mathrm{SCO}}$ to $\xi_{\mathrm{P}} - \xi_{\mathrm{SCO}}$. Here, ξ_{SCO} is the inverse of the radius of the stable circular orbit r_{SCO}, obtained by solving the radial equation with $\ddot{r} = 0$:

$$\frac{\ell^2}{(r_{\mathrm{SCO}})^3} = \frac{GM}{(r_{\mathrm{SCO}} - r_{\mathrm{G}})^2}. \tag{9}$$

We nondimensionalize and solve for $\chi_{\mathrm{SCO}} \equiv r_{\mathrm{SCO}}/r_{\mathrm{G}}$ for an appropriate choice of α:

$$\frac{1}{2\alpha\chi_{\mathrm{SCO}}^3} = \frac{1}{(\chi_{\mathrm{SCO}} - 1)^2}. \tag{10}$$

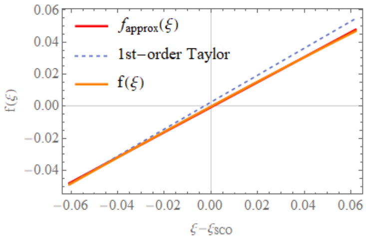
Using the designated changes on Eq. (3), the nonlinear differential equation to be emulated becomes

$$\xi''(\theta) + \xi(\theta) + \xi_{\mathrm{SCO}} - \left(\frac{2\alpha}{(1 - \xi(\theta) - \xi_{\mathrm{SCO}})^2}\right) = 0. \tag{11}$$

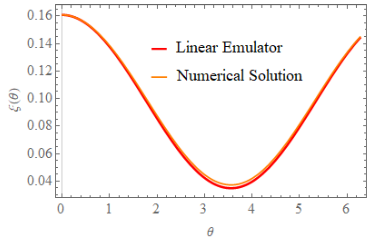
The least squares integral is therefore

$$I = \int_{\xi_{\mathrm{A}} - \xi_{\mathrm{SCO}}}^{\xi_{\mathrm{P}} - \xi_{\mathrm{SCO}}} \left(\xi - 2\alpha\left(\frac{1}{(1 - \xi_{\mathrm{SCO}} - \xi)^2}\right) - \lambda\xi - \gamma + \xi_{\mathrm{SCO}}\right)^2 d\xi. \tag{12}$$

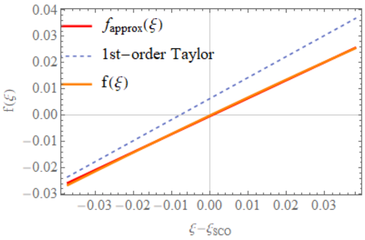
We now minimize the integral by differentiating with respect to the coefficients of the emulator, then setting these derivatives to zero. This will result in systems of equations in which we aim to solve λ and γ, the square of the angular frequency and the forcing term respectively.



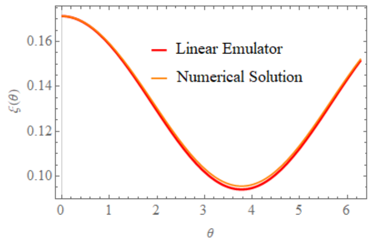
(a) $\alpha = 0.04$ and $\eta = -0.015$



(b) $\alpha = 0.04$ and $\eta = -0.015$



(c) $\alpha = 0.05$ and $\eta = -0.03$



(d) $\alpha = 0.05$ and $\eta = -0.03$

Figure 1: Plots of the nonlinear pseudo-force against the linear emulator pseudo-force and the Taylor expanded linear pseudo-force as well as their corresponding solutions in $\xi(\theta)$. Different values of α correspond to different pseudo-potential forms and admits only particular values of $\eta < 0$.

3 Results and discussion

We now present the form of the coefficients:

$$\lambda = \frac{1}{(\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^4}\left[12(\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^2\left(-(\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^2 - \frac{12\alpha}{-1+\xi_{\mathrm{A}}} - \frac{12\alpha}{-1+\xi_{\mathrm{P}}}\right)\right.$$
$$\left. + 24(\xi_{\mathrm{P}} - \xi_{\mathrm{A}})\alpha \ln\left(\frac{-1+\xi_{\mathrm{P}}}{-1+\xi_{\mathrm{A}}}\right)\right];$$
$$\gamma = \frac{1}{[(\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^3(-1+\xi_{\mathrm{A}})(-1+z_{\mathrm{P}})]}\left[4\left((3 - 2\xi_{\mathrm{A}} + 2\xi_{\mathrm{SCO}})(\xi_{\mathrm{A}} - \xi_{\mathrm{SCO}})^2 + (\xi_{\mathrm{P}} - \xi_{\mathrm{SCO}})^2(-3 + 2\xi_{\mathrm{P}} - 2\xi_{\mathrm{SCO}})\right)\alpha\right.$$
$$+ (\xi_{\mathrm{A}} - \xi_{\mathrm{P}})((-1 + \xi_{\mathrm{A}} - \xi_{\mathrm{SCO}})(\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^2(-1 + \xi_{\mathrm{P}} - \xi_{\mathrm{SCO}}) - 12(\xi_{\mathrm{A}} + \xi_{\mathrm{P}} - 2\xi_{\mathrm{SCO}})\alpha)\xi_{\mathrm{SCO}}$$
$$+ 24(\xi_{\mathrm{P}} - \xi_{\mathrm{A}})\alpha \ln\left(\frac{-1+\xi_{\mathrm{P}}}{-1+\xi_{\mathrm{A}}}\right)$$
$$+ (\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^3(-2 + \xi_{\mathrm{A}} + \xi_{\mathrm{P}} - 2\xi_{\mathrm{SCO}})\xi_{\mathrm{SCO}}^2 + (\xi_{\mathrm{A}} - \xi_{\mathrm{P}})^3\xi_{\mathrm{SCO}}^3$$
$$\left. - 12(\xi_{\mathrm{A}} + \xi_{\mathrm{P}} - 2\xi_{\mathrm{SCO}})\alpha(-1+\xi_{\mathrm{A}})(-1+\xi_{\mathrm{P}}) \ln\left(\frac{-1+\xi_{\mathrm{P}}}{-1+\xi_{\mathrm{A}}}\right)\right]$$

We see that these coefficients are functions of α and ξ_{P} which are related to the angular momentum and standard gravitational parameter, as well as the boundary amplitude, respectively. We then insert the particular values of ξ_{SCO}, ξ_{A}, and ξ_{P} according to a specific value of α and η. When the coefficients are evaluated, we can then construct the particular solution given by Eq. (6). Then, the orbit expression $r(\theta)$ becomes

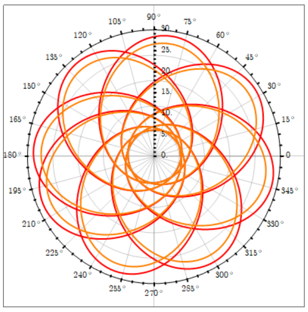
$$r(\theta) = \frac{r_{\mathrm{G}}}{\xi(\theta) + \xi_{\mathrm{SCO}}}. \tag{13}$$

We then compare our method with that of Wang and Zuo [2]. We note that they used a Taylor expansion, keeping only up to the linear pseudo-force term in u. In our notation, Eq. (8) from [2] becomes

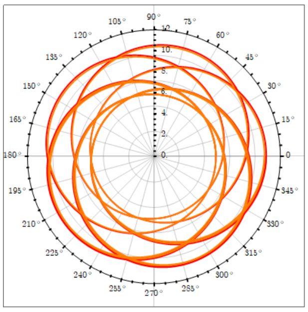
$$\xi''(\theta) + (1 - 4\alpha)\xi(\theta) - 2\alpha = 0 \tag{14}$$

We compare the produced linear emulator with their linear pseudo-force, $f_{\mathrm{Wang}}(\xi) = (1 - 4\alpha)\xi - 2\alpha$. Figure 1 shows this comparison, where the dashed line corresponds to f_{Wang}. As ξ increases, the least-squares linear emulator is closer to the actual pseudo-force than the Taylor pseudo-force.

Lastly, we calculate the values of precession. From the constructed emulator solution (Eq. 6), we find that the angular displacement between two succesive perihelia per revolution $\dfrac{\Delta\theta}{2\pi} - 1$ is equivalent to $\dfrac{1}{\sqrt{\lambda}} - 1$. We



(a) $\alpha = 0.04$ and $\eta = -0.015$



(b) $\alpha = 0.05$ and $\eta = -0.03$

Figure 2: Polar plot comparison of the least squares (red) and numerical solution (orange) for Eq. (13) for two distinct values of α.

compare this with $\dfrac{\Delta\theta}{2\pi} - 1 = \dfrac{1}{\sqrt{1-4\alpha}} - 1$ which is the precession value from the Taylor expanded solution. We also compare the precession for the numerical solution, which can be solved by computing for the root of the first derivative of the numerical solution near 2π (which is the period of radial oscillation, $\Delta\theta$), then applying to $\dfrac{\Delta\theta}{2\pi} - 1$. We summarize the results for two values of α in Tables 1 and 2.

An example of this precession can be seen in the polar plot of Eq. (13) for two values of α in figure 2.

4 Conclusions

We studied the Paczynski-Wiita pseudo-Newtonian potential, particularly the bound orbits it produces. We reproduce these orbits by considering a linear emulator to the modified Binet equation which gives $u(\theta) \equiv 1/r(\theta)$. By using the least squares method, the linear pseudo-force emulated the nonlinear pseudo-force up to a good approximation. We calculate the pericenter precession per revolution for the three solutions and find that the linear emulator is a better approximation to the pseudo-potential. Further, we can utilize other emulators (e.g. a quadratic or cubic pseudo-force emulator) that might fit the nonlinear pseudo-force better than the linear ones.

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Table 1: Pericenter precession per revolution for $\alpha = 0.04$

η	Numerical	Linear emulator	Taylor linear
-0.005	0.13426	0.13361	0.09109
-0.01	0.13342	0.13294	0.09109
-0.015	0.13258	0.13228	0.09109

Table 2: Pericenter precession per revolution for $\alpha = 0.05$

η	Numerical	Linear emulator	Taylor linear
-0.02	0.20738	0.20612	0.11803
-0.025	0.20484	0.20410	0.11803
-0.03	0.20238	0.21033	0.11803