Problem 1

This problem tackles the properties of a conductor, which are just listed down on Griffiths. Jackson wants us to prove, using the divergence and curl of **E**, the following:

(a) Any excess charge placed on a conductor must entirely lie on its surface.
PROOF: In electrostatics, we have charges that are **NOT** in motion. By definition, a conductor have free charges that CAN move around inside it through the action of an electric field. Thus, we should reject the condition that THERE IS an electric field inside, or $\mathbf{E} = 0$. By Gauss' law, creating a Gaussian surface S just inside the surface of a conductor with volume V gives

$$\oint_S \mathbf{E} \cdot da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x}) d^3x, \tag{1}$$

So that we have no charge $\rho(\mathbf{x})$ inside! Where did the "free" charges go? The only sensible answer is in the **surface**, since they cannot leave the conductor.

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.
PROOF: If there are charges outside the conductor, their sole purpose is to maintain the vanishing of the electric field inside. They do this by inducing charges on the outer surface of the conductor that will negate the free charges residing on the inner surface (as discussed in the first part). Thus, $\mathbf{E} = 0$ inside still. The converse is not true, since when we place charges inside the conductor, a similar effect will happen on the inner surface: charges will be induced in the inner surface to maintain the vanishing of the electric field inside. In turn, this induced charge INDUCES another set of charges, this time lying on the OUTSIDE surface of the conductor. It is this set of induced charges that produce $\mathbf{E} \neq 0$ outside via (1), thus the conductor cannot shield the exterior from fields emanating from the charges inside it.

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density per unit area on the surface.
PROOF: To prove that the electric field is normal to the surface is equivalent to saying the field has no tangential component. We use equation (1.21) from Jackson to prove this:

$$\oint \mathbf{E} \cdot \mathbf{dl} = 0 \tag{2}$$

We use the path depicted in figure 1 in calculating equation (2):

Recalling that $\mathbf{E} = 0$ inside, and that the integration paths on both sides cancel, we are left with the upper path consisting of the integrand $\mathbf{E} \cdot \vec{\ell}$ which vanishes according to (2). For this to happen, the electric field must be perpendicular to ℓ, which makes it

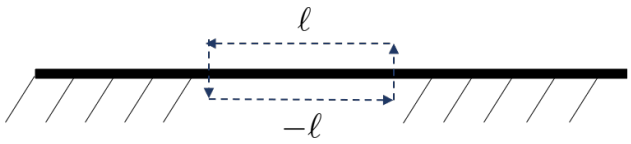

Figure 1: Loop used for taking the line integral of $\mathbf{E}$ around the surface of the conductor.

completely normal to the surface. For the magnitude of this electric field, we obviously need to use (1). If we use a Gaussian pillbox that consists of one face outside the conductor (in vacuum) and another inside, with the thickness of the pillbox approaching zero, we see that the only nonzero contribution would be from the normal electric field we solved earlier. This is due to $\mathbf{E} = 0$ inside as well as $da = 0$ on the sides. (A) Since we already showed that the electric field on the outside surface is normal to the surface, this field is parallel to the face of the Gaussian pillbox. (B) We can also shrink the pillbox so that effectively, the electric field is constant throughout, so

$$\begin{aligned} \int \mathbf{E} \cdot \mathbf{da} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ (A) \quad \int E \; da &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ (B) \quad E \int da &= \frac{Q_{\text{enc}}}{\epsilon_0} \end{aligned} \tag{3}$$

If we define the integral to be A_{out}, and that $\sigma \equiv Q/A_{\text{out}}$, then we have $\boxed{E = \sigma/\epsilon_0}$.

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Problem 2

In this problem, we are tasked to express localized charge densities $\rho(\mathbf{x})$ in different geometries. We do this by converting the Dirac delta function in different coordinate systems, since localized charge densities are proportional to $\delta(\mathbf{x})$, with the proportionality constant (which we call f) being a function of the charge distribution.

(a) In spherical coordinates (r, θ, φ), we have a spherical shell with radius R and uniformly distributed charge Q. Thus we infer that the localization of the charge is in terms of r only, since every angle θ, ϕ are completely covered by this shell. Thus the delta function can be written as $\delta(r - R)$. Subsequently, the function f must be a function only of $r = R$, since the delta function permits it to be localized at that region only:

$$\begin{aligned} \rho(\mathbf{x}) &= f(\mathbf{x})\delta(r - R) \\ &= f(R)\delta(r - R) \end{aligned} \tag{1}$$

If we integrate both sides over all space, we get the total charge Q on the left-hand side, while the right hand side becomes

$$\begin{aligned} Q &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(R)\delta(r - R)r^2 dr d(\cos\theta)d\varphi \\ Q &= 4\pi f(R)\int_0^{\infty} r^2\delta(r - R)dr \\ Q &= 4\pi R^2 f(R) \end{aligned} \tag{2}$$

or $f(R) = Q/4\pi R^2$. Thus $\boxed{\rho(\mathbf{x}) = \dfrac{Q}{4\pi R^2}\delta(r - R)}$.

(b) In cylindrical coordinates (ρ, ϕ, z), we have a cylindrical surface of radius b and uniformly distributed charge per unit length λ. We assume that the cylinder's axis is aligned with the z-axis of the coordinate system, and that the length of the cylinder along this axis is L. Thus, We localize the charge distribution to $\rho = b$ only. We do the same thing with (a):

$$\rho(\mathbf{x}) = f(\mathbf{x})\delta(\rho - b) \tag{3}$$

$$= f(b)\delta(\rho - b) \tag{4}$$

Integrating both sides over all space, we get

$$\lambda L = f(b)\int_0^{L}\int_0^{2\pi}\int_0^{\infty} \delta(\rho - b)\rho d\rho d\phi dz \tag{5}$$

$$\lambda L = 2\pi L f(b)b \tag{6}$$

$$f(b) = \frac{\lambda}{2\pi b} \tag{7}$$

Thus $\boxed{\rho(\mathbf{x}) = \frac{\lambda}{2\pi b}\delta(\rho - b)}$.

(c) Again, in cylindrical coordinates defined earlier, we have a flat circular disk of negliglible thickness (along the z-axis) and radius R. Compared to (b), the negligible thickness of this disk prompts us to use a delta function in the z-component. To ease our calculations, we position this disk at $z = 0$. For the radial ρ component though, we need a function that has produces unity at $\rho < R$ and vanishes completely for $\rho > R$. We find that this function is the (Heaviside) step function, denoted by $\Theta(R - \rho)$ (instead of the usual $(\rho - R)$, since we need to reverse the function according to the given). The proportionality factor f will now depend on the radial distance R, since we will distribute the charge evenly from $\rho = 0$ to $\rho = R$. Thus $\rho(\mathbf{x}) = f(R)\delta(z - 0)\theta(R - \rho)$. Integrating both sides, we get

$$Q = f(R)\int_0^{\infty}\int_0^{2\pi}\int_0^{R}\rho\Theta(R-\rho)\delta(z)d\rho d\phi dz \tag{8}$$

$$Q = 2\pi f(R)\left.\frac{\rho^2}{2}\right|_0^R = \pi R^2 f(R) \tag{9}$$

$$f(R) = \frac{Q}{\pi R^2} \tag{10}$$

where we used the fact that the Heaviside function is defined as the derivative of a ramp function, that outputs the input. Thus, $\boxed{\rho(\mathbf{x}) = \frac{Q}{\pi R^2}\delta(z)\Theta(R - \rho)}$.

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Problem 3

To calculate the capacitance between two conductors, we first calculate the potential difference Φ between them, since

$$C = \frac{q}{\Phi} \tag{1}$$

where q is the charge per unit length on one conductor. First, we find the electric field due to one cylindrical conductor (the one with radius a_1). We create a Gaussian cylinder with radius $r > a_1$ that is coaxial with this cylinder. The enclosed charge is then proportional to the surface area $2\pi a_1 L$ where L is an arbitrary length along the axis of the cylinder. The circular ends of the cylinder do not contribute to the problem, as they cancel in $\vec{da}$. Thus Gauss' law becomes (in integral form)

$$\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{2\pi a_1 L \sigma}{\epsilon_0} \tag{2}$$

where σ is the charge density per area. We see that throughout this Gaussian surface, $\mathbf{E}$ is parallel to $\mathbf{da}$ and that it is constant in both ϕ and r (in the standard cylindrical coordinate system r, ϕ, z positioned at the axis of the cylinder). Thus we can bring it out of the integral:

$$E \oint da = \frac{2\pi a_1 L \sigma}{\epsilon_0} \tag{3}$$

$$E(2\pi r L) = \frac{2\pi a_1 L \sigma}{\epsilon_0} \tag{4}$$

$$E(r) = \frac{a_1 \sigma}{\epsilon_0} \frac{1}{r} = \frac{q}{2\pi \epsilon_0 r} \tag{5}$$

where we reverted to the definition of the charge per unit length q in terms of the charge density. Getting the potential difference from the field is then straightforward: we set the points at which we calculate the difference in potentials. To support our next point, we introduce the coordinate system in figure 1:

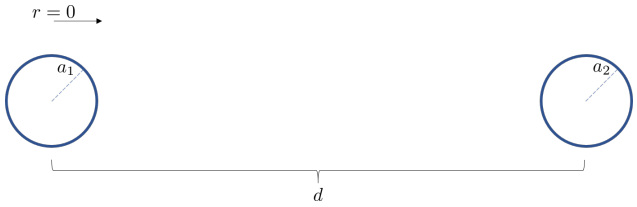

Figure 1: Coordinate system used in Problem 3, with $r = 0$ from the center of one cylinder.

Here, we see that the space in between the cylinders can have superimposing electric fields from both. Thus, we can easily rewrite equation (5) for the electric field due to the cylinder with radius a_2:

$$E(r) = \frac{-a_2\sigma}{\epsilon_0}\frac{1}{r} = \frac{-q}{2\pi\epsilon_0(d-r)} \tag{6}$$

Here, we set the charge in cylinder 2 to be $-q$ to compensate for the charge q in 1. Thus, we just subtract the two fields to get the field in between the cylinders. For the potential difference between the cylinders, we calculate at the points $r = a_1$ and $r = d - a_2$, since these points are collinear with the centers of the cylinders and with d. Therefore,

$$\Phi(r) = \int_a^b \mathbf{E}\cdot\mathbf{d}\ell \tag{7}$$

$$= \int_{a_1}^{d-a_2} \frac{q}{2\pi\epsilon_0 r} + \frac{q}{2\pi\epsilon_0(d-r)}dr \tag{8}$$

$$= \frac{q}{2\pi\epsilon_0}\left[\ln\left(\frac{d-a_2}{a_1}\right) + \ln\left(\frac{d-a_1}{a_2}\right)\right] \tag{9}$$

Here, the product $(d-a_2)(d-a_1) = d^2 - d(a_1+a_2) + a_1a_2$ can be approximated due to $d >> a_1$ and $d >> a_2$. We only retain the first term d^2, thus, $\Phi \approx \frac{q}{2\pi\epsilon_0}\ln\frac{d^2}{a_1a_2}$. Plugging back to (1),

$$C \approx 2\pi\epsilon_0\left(\ln\frac{d^2}{a_1a_2}\right)^{-1} \tag{10}$$

$$C \approx \pi\epsilon_0\left(\ln\left(\frac{d^2}{a_1a_2}\right)^{1/2}\right)^{-1} \tag{11}$$

$$C \approx \pi\epsilon_0\left(\ln\left(\frac{d}{a}\right)\right)^{-1} \tag{12}$$

where $a = \sqrt{a_1a_2}$ is the geometric mean between a_1 and a_2.

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Problem 4

For this problem, we use the configuration noted in problem 1.17: A volume V in vacuum is bounded by a surface S consisting of several separate conducting surfaces S_i. Except now, every conductor is held at zero potential except for S_1. In general, when we specify the potential Φ at a surface, we dictate the *Dirichlet boundary conditions* on the problem, which leads to equation (1.44) in the text:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')\ d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}\ da' \tag{1}$$

The problem specifies that all but S_1 are held at zero potential. Taking S_1 out of the problem first causes the second term to vanish.

We can then write the configuration of charges on S_1 as a sum of Dirac deltas per charge. This prompts us to write the potential due to that single unit charge as

$$\Phi_1(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V 4\pi\epsilon_0\delta(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\ d^3x' \tag{2}$$

which, when simplified, reverts back to $G(\mathbf{x},\mathbf{x'})$, as mentioned in the problem. If we plug this in to (1), together with the fact that the second term vanishes as well as the fact that the surface charges of S_1 constitute the totality of its charge, we rewrite (1) as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\ da' \tag{3}$$

where σ_1 is the surface charge density on S_1.

We then derive the electrostatic energy by plugging (3) into equation (1.53) in the text:

$$W = \frac{1}{2}\int_V \rho(\mathbf{x})\Phi(\mathbf{x})\ d^3x \tag{4}$$

$$= \frac{1}{2}\int_V \rho(\mathbf{x})\left[\oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\ da'\right]\ d^3x \tag{5}$$

$$\tag{6}$$

The volume integral over the charge density can then be converted into a surface integral over the surface charge density at S_1, since they're the only existing charges:

Problem 4

$$W = \frac{1}{8\pi\epsilon_0} \oint_{S_1} \sigma_1(\mathbf{x})\ da \oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\ da' \tag{7}$$

$$W = \frac{1}{8\pi\epsilon_0} \oint_{S_1}\ da \oint_{S_1} \sigma_1(\mathbf{x})G(\mathbf{x},\mathbf{x}')\sigma_1(\mathbf{x}')\ da' \tag{8}$$

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