Orthogonal functions & expansions

I Recall from P113: a set of fxns $U_n(x)$ is considered orthogonal if:

$$\int_a^b U_n^*(x) U_m(x)\, dx = 0, \quad m \neq n \qquad (1)$$

in the interval (a, b).

- modified to orthonormality if we replace the RHS of (1) by δ_{mn}.
- Sweet properties of U_n:
  ① Complete† — rewrite any fxn as an expansion of U_n:

$$f(x) = \sum_i c_i U_i(x) \qquad (2)$$

to find c_i, use orthonormality (summation killer):

> multiply both sides of (2) by U_j^*:

$$f(x) U_j^*(x) = \sum_i c_i U_i(x) U_j^*(x) \qquad (3)$$

> integrate from a to b

$$\int_a^b f(x) U_j^*(x)\, dx = \int_a^b \sum_i c_i U_i(x) U_j^*(x)\, dx \qquad (4)$$

> use the trick that pisses mathematicians.

$$\int_a^b f(x) U_j^*(x)\, dx = \sum_i c_i \int_a^b U_i(x) U_j^*(x)\, dx \qquad (5)$$

> kill the summation via δ_{ij}:

$$\boxed{\int_a^b f(x) U_j^*(x)\, dx = c_j} \qquad (6)$$

† if U_n is NOT complete, the expansion is just an approximation! (cf. Taylor)

- in the continuous limit of n, U_n

becomes a continuum of fxns for which

$\delta_{mn} \rightarrow \boxed{\delta(k-k')} \rightarrow$ Dirac

↳ Kronecker

$$\int_a^b U_k(x)\, U^*_{k'}(x)\, dx = \delta(k-k')$$

II. - most common orthogonal fxns: sines & cosines

- Expanding fxns in terms of sines & cosines is called Fourier series expansion.

- Most general Fourier expansion over $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$:

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(k_n x) + B_n \sin(k_n x)$$

- conditions on using the Fourier series:

> function must be periodic outside the interval: $f(-a/2) = f(a/2)$:

$$f(a/2) - f(-a/2) = 0$$

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{a k_n}{2}\right) + B_n \sin\left(\frac{a}{2}k_n\right) - \sum_{n=0}^{\infty} A_n \cos\left(-\frac{a}{2}k_n\right) + B_n \sin\left(-\frac{a}{2}k_n\right) = 0$$

$$\sum_{n=0}^{\infty} A_n \cancelto{0}{\left(\cos \tfrac{a}{2}k_n - \cos \tfrac{a}{2}k_n\right)} + B_n \left(\sin \tfrac{a}{2}k_n + \sin \tfrac{a}{2}k_n\right) = 0$$

- thus, for all A_n, B_n,

$$2 \sin \frac{a}{2} k_n = 0$$

$$\frac{a}{2} k_n = n\pi$$

$$\boxed{k_n = \frac{2\pi n}{a}}$$

- Now, we solve for A_n & B_n by using the steps in extracting coeffs (3) - (6):

> multiply both sides by $\cos \frac{2\pi \ldots}{a}$:

$$f(x) \cos \frac{2\pi m x}{a} = \sum^{\infty} A_n \cos \frac{2\pi n x}{a} \cos \frac{2\pi m x}{a}$$

$$f(x)\cos\frac{2\pi m x}{a} = \sum_{n=0}^{\infty} A_n \cos\frac{2\pi n x}{a}\cos\frac{2\pi m x}{a} + B_n \sin\frac{2\pi n x}{a}\cos\frac{2\pi m x}{a}$$

> integrate over $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$:

$$\int_{-a/2}^{a/2} f(x)\cos\frac{2\pi m x}{a}\,dx = \sum_{n=0}^{\infty} A_n \int_{-a/2}^{a/2} \cos\frac{2\pi n x}{a}\cos\frac{2\pi m x}{a}\,dx + \cancel{B_n \int_{-a/2}^{a/2} \sin\frac{2\pi n x}{a}\cos\frac{2\pi m x}{a}\,dx} \to 0 \;\forall\, m,n$$

odd $\times$ even = odd, integrated over an even interval $\Rightarrow 0$

$$\int_{-a/2}^{a/2} f(x)\cos\frac{2\pi m x}{a}\,dx = \sum_{n=0}^{\infty} A_n \frac{a}{2}\delta_{mn} = A_m \frac{a}{2}$$

$$\boxed{A_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\frac{2\pi n x}{a}\,dx}$$

- same method for extracting B_n:

$$\int_{-a/2}^{a/2} f(x)\sin\frac{2\pi n x}{a}\,dx = B_n\frac{a}{2}$$

$$B_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\frac{2\pi n x}{a}\,dx$$

- we can rewrite this in terms of complex exponentials since $e^{i\theta} = \cos\theta + i\sin\theta$:

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{a}}\sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{2\pi n x}{a}\right)} \\ A_n &= \frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} f(x)\, e^{-i\left(\frac{2\pi n x}{a}\right)}\,dx \end{aligned}}$$