Motilal Nehru National Institute of Technology, Allahabad
NUMERICAl METHODS AND STATISTICAL TECHNIQUES
Unit: Interpolation
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1 Introduction

Consider the following table:

$x:$	x_0	x_1	x_2	x_3	$\cdots$	x_n
$y:$	y_0	y_1	y_2	y_3	$\cdots$	y_n

In this table, $y_k,\ (k = 0, 1, \cdots, n)$ are assumed to be the values of a certain function $y = f(x)$, evaluated at $x = x_k,\ (k = 0, 1, \cdots, n)$ in an interval containing these points. **Note that only the functional values are known, not the function $f(x)$ itself**. The problem is to find y_p corresponding to a non-tabulated intermediate value x_p $(x_0 < x_p < x_n)$. Such a problem is called an Interpolation Problem. The numbers $x_0, x_1, x_2, \cdots, x_n$ are called the nodes. The interpolation problem is a classical problem and dates back to the time of Newton and Kepler, who needed to solve such a problem in analyzing data on the positions of stars and planets. It is also of interest in numerous other practical applications.

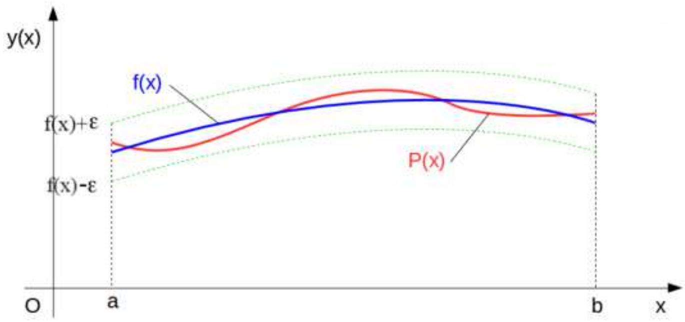
Given a set of tabular values $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$, satisfying the relation $y = f(x)$ where explicit nature of $f(x)$ is not known, it is required to find a simple function, say $\phi(x)$, such that $f(x)$ and $\phi(x)$ agree at the tabulated points, that means $f(x) = \phi(x),\ i = 0, 1, 2, \cdots n$. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. Different types of interpolation arise depending on whether $\phi(x)$ is finite trigonometric series, series of Bessel functions, etc. In this unit we shall be concerned with polynomial interpolation only.

When approximating a function $f(x)$ by means of polynomial $\phi(x)$, one may be tempted to ask : (i) How should the closeness of the approximation be measured? (ii) What is the criterion to decide the best polynomial approximation to the function? Answer to these question, important though, are outside the scope of the current syllabus and will not be attempted here. We will, however, derive in the next section a formula for finding the error associated with the approximation of a tabulated function by means of a polynomial.

As a justification for the approximation of a unknown function by meas of polynomial, we state here the following result.

Existence Results: It is well-known that a continuous function $f(x)$ on $[a, b]$ can be approximated as close as possible by means of a polynomial. Specifically, for each $\epsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$ for all x in $[a, b]$. This is a classical result, known as **Weierstrass Approximation Theorem (1885)**. This means that it is possible to find a polynomial $P(x)$ whose graph remain within the region bounded by $y = f(x) - \epsilon$ and

$y = f(x) + \epsilon$ for all $x \in [a, b]$ and for all ϵ however small may be.



Remark: Through two distinct points, we can construct a unique polynomial of degree 1 (straight line). Through three distinct points, we can construct a unique polynomial of degree 2 (parabola) or a unique polynomial of degree 1 (straight line). That is, through three distinct points, we can construct a unique polynomial of degree ≤ 2. In general, through $n+1$ distinct points, we can construct a unique polynomial of degree $\leq n$. We may express it in various forms but are otherwise the same polynomial. For example, $f(x) = x^2 - 2x - 1$ can be written as $x^2 - 2x - 1 = -2 + (x - 1) + (x - 1)(x - 2)$.

The uniqueness of the interpolating polynomial of degree $\leq n$ for $n + 1$ data points can also be proved by the method of contradiction. If possible suppose there are more than one interpolating polynomial of degree $\leq n$ for the given $n + 1$ data points

$x:$	x_0	x_1	x_2	x_3	$\cdots$	x_n
$y:$	y_0	y_1	y_2	y_3	$\cdots$	y_n

Let two such polynomial be $p(x)$ and $q(x)$ interpolating the given data points. Let $r(x) = p(x) - q(x)$. Then $r(x)$ is a polynomial of degree $\leq n$. Here, $r(x_i) = p(x_i) - q(x_i) = y_i - y_i = 0$ for $i = 0, 1, 2, ..., n$. Thus the polynomial equation $r(x) = 0$ has $n + 1$ roots. But we know that any polynomial equation of degree n has exactly n roots. Since $r(x)$ is a polynomial of degree $\leq n$, number of roots of $r(x) = 0$ should be $\leq n$. But we have shown that polynomial equation $r(x) = 0$ has $n+1$ roots, which is a contradiction. Only way it is possible if $r(x)$ is a zero polynomial, i.e., $r(x) \equiv 0$. That means $p(x) = q(x)$. Thus the interpolating polynomial of degree $\leq n$ for $n+1$ data points is unique (The interpolating polynomial of degree $> n$ for $n + 1$ data points is not unique).

The interpolating polynomial $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ interpolating the points (x_0, y_0), (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), can be found by solving for the coefficients $a_0, a_1, ..., a_n$

from the following $n+1$ equations:

$$
\begin{aligned}
&a_0 + a_1 x_0 + a_2 {x_0}^2 + ... + a_n {x_0}^n = y_0 \\
&a_0 + a_1 x_1 + a_2 {x_1}^2 + ... + a_n {x_1}^n = y_1 \\
&\vdots \\
&a_0 + a_1 x_n + a_2 {x_n}^2 + ... + a_n {x_n}^n = y_n
\end{aligned}
$$

Although, finding solution for large system of equations is not an easy job. Moreover, if the number of data points are increased, then the degree of the interpolating polynomial may be increased. To find the polynomial, we need to solve a new system of equations. There is no way to use the previous results. So, our aim is to study some sophisticated method to find the polynomial which interpolates the given tabulated points.

2 Error in polynomial interpolation

Let the function $y(x)$ be defined at $(n+1)$ points $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$. Also assume that $y(x)$ be continuous and $(n+1)$ times differentiable. Also, let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that $\phi_n(x_i) = y_i$ for $i = 0, 1, 2, ..., n$. If we now use $\phi_n(x)$ to obtain a approximate value of $y(x)$ at some points other than the tabulated values, then what would be the accuracy of the approximation.

Since the expression $y(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, ..., x_n$, we put

$$y(x) - \phi_n(x) = L\Pi_{n+1}(x), \tag{1}$$

where $\Pi_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_n)$ and L is to be determined such that (1) holds for any intermediate value of x say x', where $x_0 < x' < x_n$. Clearly,

$$L = \frac{y(x') - \phi_n(x')}{\Pi_{n+1}(x')}. \tag{2}$$

Let us consider

$$F(x) = y(x) - \phi_n(x) - L\Pi_{n+1}(x), \tag{3}$$

where L is given by above relation. It is clear that

$$F(x_0) = F(x_1) = ... = F(x_n) = 0 = F(x'), \tag{4}$$

that is, $F(x)$ vanishes $n+2$ times in the interval $x_0 \le x \le x_n$. So, by **Generalized Rolle's**

> **Generalized Rolle's Theorem:** Suppose that $f(x)$ is continuous on $[a, b]$ and n times differentiable on (a, b). If $f(x)$ has n distinct zeros in $[a, b]$, then $f^{(n+1)}(c) = 0$ where $a < c < b$.

Theorem, $F^{(n+1)}(x)$ must vanish once in the interval $x_0 \le x \le x_n$. Let this point be given by η, where $x_0 \leq \eta \leq x_n$. Now,

$$F^{(n+1)}(\eta) = 0 = y^{(n+1)}(\eta) - 0 - L * (n+1)!$$
$$\big(since\ \Pi_{n+1}(x)\ is\ a\ polynomial\ of\ degree\ (n+1)\big)$$
$$\Rightarrow L = \frac{y^{(n+1)}(\eta)}{(n+1)!}.$$

Hence, (2) gives

$$y(x') - \phi_n(x') = \frac{y^{(n+1)}(\eta)}{(n+1)!}\Pi_{n+1}(x').$$

Dropping x' to x, we get

$$y(x) - \phi_n(x) = \frac{y^{(n+1)}(\eta)}{(n+1)!}\Pi_{n+1}(x), \tag{5}$$

which is the required expression for errors.

Since the explicit form of the function $y(x)$ is not known in general, hence we do not have any information concerning $y^{(n+1)}(x)$. Thus this form of the error formula is useless in practical calculation. Though, for theoretical numerical study, it has huge importance. The error formula will be useful if somehow we can replace $y^{(n+1)}(x)$ in terms of the given data points. We will discuss this later at some point.

Tabulated points or data points are classified in two categories: (i) equi-spaced points/evenly spaced points and (ii) unevenly spaced points. If the difference between two consecutive x-values are constants, i.e., $|x_{i+1} - x_i| = k$ for $i = 0, 1, 2, ..., n$ then the points are evenly spaced, otherwise the points are unevenly spaced.

For evenly spaced points we are going to discuss the following interpolation methods:
(i) Newton's forward difference formula
(ii) Newton's backward difference formula
(iii) Gauss forward formula
(iv) Gauss backward formula
(v) Stirling's interpolation formula
(vi) Bessel's formula
(vii) Everett's formula.

For arbitrary data points (specially unevenly data points), we are going to discuss the following interpolation methods:
(i) Lagrange's interpolation method
(ii) Newton's divided difference formula/ Newton's general interpolation formula.

Note: The methods for unevenly spaced points data points can also be used for evenly spaced points. But the converse is not true.

3 Methods for Evenly-spaced Points

To derive the interpolative methods, we have to define some symbolic operators.

3.1 Finite differences

Suppose the following data points are given. We define the following difference operators.

$x:$	x_0	$x_1 = x_0 + h$	$x_2 = x_0 + 2h$	$x_3 = x_0 + 3h$	$\cdots$	$x_n = x_0 + nh$
$y:$	y_0	y_1	y_2	y_3	$\cdots$	y_n

(i) **Shift Operator E:** When the operator E is applies on y_i, we have
$Ey_k = Ef(x_k) = f(x_k + h) = f(x_{k+1}) = y_{k+1}$.
That is, $Ey_0 = y_1$, $Ey_1 = y_2$,..., $Ey_{n-1} = y_n$ etc.
Therefore, the operator E when applied on $f(x)$ shifts it to the value at the next nodal point.
We have $E^2 y_k = E^2 f(x_k) = E(E(f(x_k))) = E(f(x_k + h)) = f(x_k + 2h) = y_{k+2}$.
In general, $E^r y_k = y_{k+r}$, where r is any real number. For example, we define $E^{\frac{1}{2}} y_k = y_{k+\frac{1}{2}}$.

(ii) **Forward Difference Operator:** Forward difference operator is denoted by Δ and when Δ is applied on y_k, we have
$\Delta y_k = y_{k+1} - y_k$. That is,
$\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, ..., $\Delta y_{n-1} = y_n - y_{n-1}$.
$\Delta y_0, \Delta y_1, ..., \Delta y_{n-1}$ are called first forward differences. Differences of the first forward differences are called second forward differences.
Thus, $\Delta^2 y_0 = \Delta(\Delta(y_0)) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$.
Similarly,
$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$,
$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$, etc.

Relation between shift operator and forward difference operator:
$\Delta y_k = y_{k+1} - y_k = Ey_k - y_k = (E-1)y_k$, implies that $\Delta = (E-1)$ or $E = 1 + \Delta$. Using this relation, we can write the n-th order forward difference as $\Delta^n y_k = (E-1)^n y_k$.

The forward differences can be written in a tabular form as in Table 6. The diagonal containing the terms y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ is called the forward diagonal in the forward difference table. These values will be used in Newton's forward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
x_4	y_4				

Table 1: Forward Difference Table

For example, the forward difference table for the data

$x:$	1	2	3	4	5
$y:$	2	4	9	7	8

can be created as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	2				
		2			
2	4		3		
		5		-10	
3	9		-7		14
		-2		4	
4	7		-3		
		1			
5	8				

Table 2: Forward Difference Table

(ii) **Backward Difference Operator:** Backward difference operator is denoted by ∇ and when ∇ is applied on y_k, we have
$\nabla y_k = y_k - y_{k-1}$. That is,
$\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, ..., $\nabla y_n = y_n - y_{n-1}$.
$\nabla y_1, \nabla y_2, ..., \nabla y_n$ are called first beckward differences. Differences of the first backward differences are called second backward differences.
Thus, $\nabla^2 y_2 = \nabla(\nabla(y_2)) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$.
Similarly, $\nabla^3 y_3 = y_3 - 3y_2 + 3y_1 - y_0$,
$\nabla^4 y_4 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$, etc.

Relation between shift operator and backward difference operator:

$\nabla y_k = y_k - y_{k-1} = y_k - E^{-1}y_k = (1-E^{-1})y_k$, implies that $\nabla = (1-E^{-1})$. Using this relation, we can write the n-th order backward difference as $\nabla^n y_k = (1-E^{-1})^n y_k$.

The backward differences can be written in a tabular form as in Table 3.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
		$\nabla y_4 = y_4 - y_3$			
x_4	y_4				

Table 3: Backward Difference Table

The diagonal containing the terms y_4, ∇y_4, $\nabla^2 y_4$, $\nabla^3 y_4$, $\nabla^4 y_4$ is called the backward diagonal in the backward difference table. These values will be used in Newton's backward difference formula.

3.2 Forward Difference of Polynomial

Let $y(x)$ be a polynomial of n-th degree so that

$$y(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n.$$

Then, we have

$$\begin{aligned}
&\triangle y(x) = y(x+h) - y(x)\\
&= a_0(x+h)^n + a_1(x+h)^{n-1} + \ldots + a_n - (a_0x^n + a_1x^{n-1} + \ldots + a_n)\\
&= a_0(nh)x^{n-1} + {a_1}' x^{n-2} + \ldots + {a_n}',
\end{aligned}$$

where ${a_1}', {a_2}', ..., {a_n}'$ are the new constants. This relation shows that the first difference of a polynomial of n-th degree is a polynomial of degree $(n-1)$. Similarly, the second difference of a polynomial of n-th degree is a polynomial of degree $(n-2)$ and the coefficient of x^{n-2} is $a_0 n(n-1)h^2$. Thus the n-th difference of n-th degree is a polynomial is $a_0 n! h^n$, which is a constant. Hence, the $(n+1)$ and higher differences of polynomial of degree n will be zero.

Conversely, if the n-th order differences of a tabulated function are constant and $(n+1)$-th, $(n+2)$-th,... differences all vanish, then tabulated function represent a polynomial of degree n. It should be noted that these results holds good if the values of x are equally spaced. The converse is important in numerical analysis since it enables us to approximate a function by a polynomial if its differences of some order become nearly constant.

3.3 Missing Data Technique

Suppose n values of out of $(n+1)$ values of $y = f(x)$ are given. The values of x being equidistant. Let the unknown value be N. We construct the forward difference table. Since only n values of y are known, we can assume $y = f(x)$ to be a polynomial of degree $(n-1)$ in x. Equating the n-th difference to zero, we can gen the value of N.

Similarly, Suppose $(n+1)-r$ values of out of $(n+1)$ values of $y = f(x)$ are given. The values of x being equidistant. Let the unknown value be $N_1, N_2, ..., N_r$. We construct the forward difference table. Since only $(n+1)-r$ values of y are known, we can assume $y = f(x)$ to be a polynomial of degree $(n-r)$ in x. Equating the $((n+1)-r)$-th differences to zero, we can gen the values of $N_1, N_2, ..., N_r$.

Example 1 Find the missing values in the table

x :	45	50	55	60	65
y :	3	?	2	?	-2.4

Solution: The difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$y_1 = 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	2		$y_1 + y_2 - 4$	
		$y_3 - 2$		$-y_1 - 3y_3 + 3.6$
60	y_2		$-0.4 - 2y_3$	
		$-2.4 - y_3$		
65	-2.4			

As only three entries y_0, y_2, y_4 are given, the function can be represented by a second degree polynomial.

$$\therefore \Delta^3 y_0 = 0 \text{ and } \Delta^3 y_1 = 0$$
$$\Rightarrow 3y_1 + y_3 = 9 \text{ and } y_1 + 3y_3 = 3.6$$

Solving these equations, we get

$$y_1 = 2.925 \text{ and } y_3 = 0.225.$$

3.4 Newton's Forward Difference Formula

Let $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$ be the given data points such that $x_i = x_0 + ih$ for $i = 1, 2, 3, ..., n$. We need to find a polynomial of degree $\leq n$ such that the polynomial will

agree with the given data points.
Let the polynomial be

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \quad (6)$$

Now, putting $x = x_0$ we get

$$P_n(x_0) = y_0 = a_0 \Rightarrow a_0 = y_0.$$

Putting $x = x_1$ we get

$$P_n(x_1) = y_1 = a_0 + a_1(x_1 - x_0) \Rightarrow a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}.$$

Putting $x = x_2$ we get

$$\begin{aligned}
&P_n(x_2) = y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)\\
&\Rightarrow a_2 * 2h * h = y_2 - y_0 - 2h * \frac{\Delta y_0}{h}\\
&\Rightarrow a_2 * 2h^2 = y_2 - y_0 - 2(y_1 - y_0)\\
&\Rightarrow a_2 * 2h^2 = y_2 - 2y_1 + y_0\\
&\Rightarrow a_2 * 2h^2 = \Delta^2 y_0\\
&\Rightarrow a_2 = \frac{\Delta^2 y_0}{2!h^2}.
\end{aligned}$$

Similarly, putting $x = x_n$, we get

$$a_n = \frac{\Delta^n y_0}{n!h^n}.$$

Substituting these values in (6), we get

$$P_n(x) = y_0 + (x - x_0)\frac{\Delta y_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_0}{2!h^2} + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})\frac{\Delta^n y_0}{n!h^n}, \quad (7)$$

which is the required Newton's forward difference formula.

Let $x = x_0 + ph$, then

$$\begin{aligned}
&x - x_0 = ph\\
&x - x_1 = x - x_0 + (x_0 - x_1) = ph - h = (p - 1)h\\
&\vdots\\
&x - x_{n-1} = x - x_0 + (x_0 - x_{n-1}) = ph - (n - 1)h = (p - n + 1)h
\end{aligned}$$

Substituting these values in (7), we get

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n y_0, \tag{8}$$

which is another form of Newton's forward difference formula.

Example 2 Find a cubic polynomial by Newton's forward difference formula satisfying the following data.

x	0	1	2	3
y	1	0	1	10

Solution: The forward difference table is defined bellow.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here $h = 1$ and $y_0 = 1$, $\Delta y_0 = -1$, $\Delta^2 y_0 = 2$ and $\Delta^3 y_0 = 6$. Hence, by Newton's forward difference formula, the polynomial is

$$\begin{aligned}
&P_3(x) = y_0 + (x-x_0)\frac{\Delta y_0}{h} + (x-x_0)(x-x_1)\frac{\Delta^2 y_0}{2!h^2} + (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 y_0}{3!h^3}, \\
&\Rightarrow P_3(x) = 1 + (x-0)(-1) + (x-0)(x-1)\frac{2}{2!} + (x-0)(x-1)(x-2)\frac{6}{3!}, \\
&\Rightarrow P_3(x) = x^3 - 2x^2 + 1.
\end{aligned}$$

3.5 Newton's Backward Difference Formula

Let (x_0, y_0), (x_1, y_1), (x_2, y_2), $\ldots, (x_n, y_n)$ be the given data points such that $x_i = x_0 + ih$ for $i = 1, 2, 3, ..., n$. We need to find a polynomial of degree $\leq n$ such that the polynomial will agree with the given data points.
Let the polynomial be

$$P_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \cdots + a_n(x-x_n)(x-x_{n-1})\cdots(x-x_1). \tag{9}$$

Now, putting $x = x_n$ we get

$$P_n(x_n) = y_n = a_0 \Rightarrow a_0 = y_n.$$

Putting $x = x_{n-1}$ we get

$$P_n(x_{n-1}) = y_{n-1} = a_0 + a_1(x_{n-1} - x_n) \Rightarrow a_1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{\nabla y_n}{h}.$$

Putting $x = x_{n-2}$ we get

$$\begin{aligned}
&P_n(x_{n-2}) = y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1}) \\
&\Rightarrow a_2 * (-2h) * (-h) = y_{n-2} - y_n + 2h * \frac{\nabla y_n}{h} \\
&\Rightarrow a_2 * 2h^2 = y_{n-2} - y_n + 2(y_n - y_{n-1}) \\
&\Rightarrow a_2 * 2h^2 = y_n - 2y_{n-1} + y_{n-2} \\
&\Rightarrow a_2 * 2h^2 = \nabla^2 y_n \\
&\Rightarrow a_2 = \frac{\nabla^2 y_n}{2!h^2}.
\end{aligned}$$

Similarly, putting $x = x_1$, we get

$$a_n = \frac{\nabla^n y_n}{n!h^n}.$$

Substituting these values in (9), we get

$$P_n(x) = y_n + (x - x_n)\frac{\nabla y_n}{h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{2!h^2} + \cdots + (x - x_n)(x - x_{n-1}) \cdots (x - x_1)\frac{\nabla^n y_n}{n!h^n}, \tag{10}$$

which is the required Newton's backward difference formula.

Let $x = x_n + sh$, then

$$\begin{aligned}
&x - x_n = sh \\
&x - x_{n-1} = x - x_n + (x_n - x_{n-1}) = sh + h = (s + 1)h \\
&\vdots \\
&x - x_1 = x - x_n + (x_n - x_1) = sh + (n - 1)h = (s + n - 1)h
\end{aligned}$$

Substituting these values in (10), we get

$$P_n(x) = y_n + s\nabla y_n + \frac{s(s + 1)}{2!}\nabla^2 y_n + \cdots + \frac{s(s + 1) \cdots (s + n - 1)}{n!}\nabla^n y_n, \tag{11}$$

which is another form of Newton's backward difference formula.

Example 3 Find a cubic polynomial by Newton's backward difference formula satisfying the following data.

x	0	2	4	6
y	1	23	141	451

Solution: The backward difference table is defined bellow.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		22		
2	23		96	
		118		96
4	141		192	
		310		
6	451			

Here $h = 2$ and $y_3 = 451$, $\nabla y_3 = 310$, $\nabla^2 y_3 = 192$ and $\nabla^3 y_3 = 96$. Hence, by Newton backward difference formula, the polynomial is

$$P_3(x) = y_3 + (x - x_3)\frac{\nabla y_3}{h} + (x - x_3)(x - x_3)\frac{\nabla^2 y_3}{2!h^2} + (x - x_3)(x - x_2)(x - x_1)\frac{\nabla^3 y_3}{3!h^3},$$
$$\Rightarrow P_3(x) = 451 + (x - 6)(\frac{310}{2}) + (x - 6)(x - 4)\frac{192}{2! * 2^2} + (x - 6)(x - 4)(x - 2)\frac{96}{3! * 2^3},$$
$$\Rightarrow P_3(x) = 451 + 155(x - 6) + 24(x - 6)(x - 4) + 2(x - 6)(x - 4)(x - 2).$$

3.6 Central Difference Formula For Interpolation

In the preceding section, we derived Newton's forward and backward interpolation formulas which are applicable for the interpolation near the beginning and the end, respectively, of the tabulated data points. In this section we shall derive some central difference formulas which are most suited for interpolation near the middle of the tabulated data points.

3.6.1 Gauss Forward Formula

We consider the following difference table in which the central ordinate is taken for convenience as y_0 corresponding to $x = x_0$.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

The differences used in this formula lie on the forward line shown in the table. The formula is, therefore, of the form

$$y_p = y_0 + G_1\Delta y_0 + G_2\Delta^2 y_{-1} + G_3\Delta^3 y_{-1} + G_4\Delta^4 y_{-2} + \cdots , \tag{12}$$

where G_1, G_2,... have to be determined. The y_p on the left side can be expressed in terms of y_0, Δy_0, and higher order differences of y_0, as follows:

$$\begin{aligned} y_p &= E^p y_0 = (1+\Delta)^p y_0 \\ &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots . \end{aligned}$$

Similarly, the right hand side of the equation (12) an also be expressed in terms of y_0, Δy_0, and higher order differences of y_0. we have

$$\begin{aligned} &\Delta^2 y_{-1} \\ &= \Delta^2 E^{-1} y_0 \\ &= \Delta^2 (1+\Delta)^{-1} y_0 \\ &= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0 \\ &= \Delta^2 (y_0 - \Delta y_0 + \Delta^2 y_0 - \Delta^3 y_0 + \cdots) \\ &= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots \end{aligned}$$

$$\Delta^3 y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \cdots$$

$$\begin{aligned} &\Delta^4 y_{-2} \\ &= \Delta^4 E^{-2} y_0 \\ &= \Delta^4 (1+\Delta)^{-2} y_0 \\ &= \Delta^4 (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \cdots) y_0 \\ &= \Delta^4 (y_0 - 2\Delta y_0 + 3\Delta^2 y_0 - 4\Delta^3 y_0 + \cdots) \\ &= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots \end{aligned}$$

Hence, equation (12) give the identity

$$\begin{aligned} & y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots . \\ &= y_0 + G_1\Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots) + G_3(\Delta^3 y_0 - \Delta^4 y_0 + \\ &\quad \Delta^5 y_0 - \Delta^6 y_0 + \cdots) + G_4(\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots) + \cdots \end{aligned}$$

Equating the coefficients of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, etc., on the both side of the above equation, we obtain

$$\begin{aligned} G_1 &= p, \\ G_2 &= \frac{p(p-1)}{2!}, \\ G_3 &= \frac{(p+1)p(p-1)}{3!}, \\ G_4 &= \frac{(p+1)p(p-1)(p-2)}{4!}, \ \ etc. \end{aligned}$$

Hence, the Gauss forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \tag{13}$$

Example 4 Apply Gauss forward formula to find $y(9)$ from the following data.

x	0	4	8	12	16
y	14	24	32	35	40

Solution: The difference table taking origin at $x = 8$, is created below:

Index	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(u)$	$\Delta^4 f(x)$
– 2	0	14				
			10			
– 1	4	24		– 2		
			8		– 3	
0	8	$\boxed{32}$		$\boxed{-5}$		$\boxed{10}$
			$\boxed{3}$		$\boxed{7}$	
1	12	35		2		
			5			
2	16	40				

Here, $x_0 = 8$, $h = 4$. Hence, from $x = x_0 + ph$, we get $9 = 8 + 4p \Rightarrow p = 0.25$.
Gauss forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}.$$

Hence,

$$\begin{aligned} y(9) = 32 + (.25)(3) + \frac{.25(.25-1)}{2!}(-5) + \frac{(.25+1)(.25)(.25-1)}{3!}(7) \\ + \frac{(.25+1)(.25)(.25-1)(.25-2)}{4!}(10) \\ = 33.1162 \end{aligned}$$

3.6.2 Gauss Backward Formula

The differences used in this formula lie on the backward line shown in the table.

Table 3.7 Gauss' Backward Formula

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
$\vdots$	$\vdots$						
x_{-1}	y_{-1}						
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1						
$\vdots$	$\vdots$						

The formula is, therefore, of the form

$$y_p = y_0 + B_1 \Delta y_{-1} + B_2 \Delta^2 y_{-1} + B_3 \Delta^3 y_{-2} + B_4 \Delta^4 y_{-2} + \cdots \,, \tag{14}$$

where B_1, B_2,... have to be determined. The y_p on the left side can be expressed in terms of y_0, Δy_0, and higher order differences of y_0, as follows:

$$\begin{aligned} y_p &= E^p y_0 = (1+\Delta)^p y_0 \\ &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots . \end{aligned}$$

Similarly, the right hand side of the equation (14) an also be expressed in terms of y_0, Δy_0, and higher order differences of y_0. we have

$$\begin{aligned} &\Delta y_{-1} \\ &= \Delta E^{-1} y_0 \\ &= \Delta(1+\Delta)^{-1} y_0 \\ &= \Delta(1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0 \\ &= \Delta(y_0 - \Delta y_0 + \Delta^2 y_0 - \Delta^3 y_0 + \cdots) \\ &= \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \cdots \end{aligned}$$

$$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots$$

$$\begin{aligned}
&\Delta^3 y_{-2}\\
&= \Delta^3 E^{-2} y_0\\
&= \Delta^3 (1+\Delta)^{-2} y_0\\
&= \Delta^3 (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \cdots) y_0\\
&= \Delta^3 (y_0 - 2\Delta y_0 + 3\Delta^2 y_0 - 4\Delta^3 y_0 + \cdots)\\
&= \Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - 4\Delta^6 y_0 + \cdots
\end{aligned}$$

and so on. Hence, equation (14) give the identity

$$\begin{aligned}
&y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots .\\
&= y_0 + B_1(\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \Delta^4 y_0 + \cdots) + B_2(\Delta^2 y_0 - \Delta^3 y_0 +\\
&\quad \Delta^4 y_0 - \Delta^5 y_0 + \cdots) + B_3(\Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 - 4\Delta^6 y_0 + \cdots) + \cdots
\end{aligned}$$

Equating the coefficients of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, etc., on the both side of the above equation, we obtain

$$\begin{aligned}
B_1 &= p,\\
B_2 &= \frac{p(p+1)}{2!},\\
B_3 &= \frac{(p+1)p(p-1)}{3!},\\
B_4 &= \frac{(p+2)(p+1)p(p-1)}{4!}, \ \textit{etc.}
\end{aligned}$$

Hence, the Gauss backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{15}$$

Example 5 Apply Gauss backward formula, find the population in the year 1936 given that

Year	1901	1911	1921	1931	1941	1951
Population (in thousands)	12	15	20	27	39	52

Solution: The difference table taking origin at $x = 1941$ is created below:

Index	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
– 4	12					
		3				
– 3	15		2			
		5		0		
– 2	20		2		3	
		7		3		– 10
– 1	27		5		– 7	
		$\boxed{12}$		$\boxed{-4}$		
0	$\boxed{39}$		$\boxed{1}$			
		13				
1	52					

Here, $x_0 = 1941$, $h = 10$. Hence, from $x = x_0 + ph$, we get $1936 = 1941 + 10p \Rightarrow p = -0.5$. Gauss backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Hence,

$$y(1936) = 39 + (-.5)(12) + \frac{(.5)(.5+1)}{2!}(1) + \frac{(.5+1)(.5)(.5-1)}{3!}(-4) = 32.625 \textit{ thousands}.$$

Note: $x = 1931$ can also be chosen as origin. In this case, the backward line will be the line containing $27,\ 7,\ 5,\ 3,\ -7,\ -10$. Accordingly, we have to use the Gauss backward formula.

3.6.3 Stirling's Formula

Taking the mean of Gauss forward and Gauss backward formula, we obtain Stirling's formula, which is

$$y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \quad (16)$$

It is useful when $-\frac{1}{2} < p < \frac{1}{2}$ and gives best estimate when $-\frac{1}{4} < p < \frac{1}{4}$.

Example 6 ApplyStirling's formula to find $y(35)$ from the following data.

x	20	30	40	50
y	512	439	346	243

Solution: The difference table, taking origin at $x = 30$, is created below:

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 1	20	512			
			$\boxed{-73}$		
0	30	$\boxed{439}$		$\boxed{-20}$	
			$\boxed{-93}$		10
1	40	346		– 10	
			– 103		
2	50	243			

Here, $x_0 = 30$, $h = 10$. Hence, from $x = x_0 + ph$, we get $35 = 30 + 10p \Rightarrow p = 0.5$. Stirling's formula is

$$y_p = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Hence,

$$\begin{aligned} y(35) &= 439 + (.5)\frac{-93-73}{2} + \frac{(.5)^2}{2!}(-20) + \frac{(.5)((.5)^2-1)}{3!}(\frac{10}{2}) \\ &= 394.6875 \end{aligned}$$

Note: In the given problem, $x = 40$ can also be chosen as origin. Accordingly, we have to use the Stirling's formula. In this case, $p = -0.5$. Hence,

$$\begin{aligned} y(35) &= 346 + (-.5)\frac{-103-93}{2} + \frac{(-.5)^2}{2!}(-10) + \frac{(-.5)((-.5)^2-1)}{3!}(\frac{10}{2}) \\ &= 394.0625 \end{aligned}$$

3.6.4 Bessel's Formula

See the following table, where bracket means average value should be taken.

$$\begin{array}{cccccccc} \vdots & \vdots & & & & & & \\ x_{-1} & y_{-1} & & & & & & \\ \begin{matrix} x_0 \\ x_1 \end{matrix} & \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} & \Delta y_0 & \begin{pmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{pmatrix} & \Delta^3 y_{-1} & \begin{pmatrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{pmatrix} & \Delta^5 y_{-2} & \begin{pmatrix} \Delta^6 y_{-3} \\ \Delta^6 y_{-2} \end{pmatrix} \\ \vdots & \vdots & & & & & & \end{array}$$

Bessel's formula is assumed in the form

$$y_p = \frac{y_0 + y_1}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots \quad (17)$$

Using the method outlined in Gauss forward formula, we get

$$
\begin{aligned}
B_1 &= p - \frac{1}{2}, \\
B_2 &= \frac{p(p-1)}{2!}, \\
B_3 &= \frac{p(p-1)(p-\frac{1}{2})}{3!}, \\
B_4 &= \frac{(p+1)p(p-1)(p-2)}{4!}, \ \textit{etc.}
\end{aligned}
$$

Hence, the Bessel's interpolation formula is

$$
\begin{aligned}
y_p = & \frac{y_0 + y_1}{2} + (p - \frac{1}{2})\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} \\
& + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots
\end{aligned}
$$

Example 7 Find the value of y_{15}, using Bessel's formula, if $y_{10} = 2854$, $y_{14} = 3162$, $y_{18} = 3544$, and $y_{22} = 3992$.

Solution: The difference table, taking origin at $x = 14$, is created below:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
−1	10	2854			
			308		
0	14	$\boxed{3162}$		$\boxed{74}$	
			$\boxed{382}$		$\boxed{-8}$
1	18	$\boxed{3544}$		$\boxed{66}$	
			448		
2	22	3992			

Here, $x_0 = 14$, $h = 4$. Hence, from $x = x_0 + ph$, we get $14 = 14 + 4p \Rightarrow p = 0.25$.
Bessel's formula is

$$
\begin{aligned}
y_p = & \frac{y_0 + y_1}{2} + (p - \frac{1}{2})\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} \\
& + \frac{p(p-1)(p-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y(15) = y_{.25} &= \frac{(3162 + 3544)}{2} + (.25 - \frac{1}{2})(382) + \frac{(.25)(.25-1)}{2!}\frac{(74+66)}{2} + \frac{(.25)(.25-1)(.25-\frac{1}{2})}{3!}(-8) \\
&= 3250.875
\end{aligned}
$$

3.6.5 Everett's Formula

It usages even order differences as shown in the following table.

x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	
		$-$		$-$		$-$		$-$
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$	

Everett's formula assumed in the form

$$y_p = E_0 y_0 + E_2 \Delta^2 y_{-1} + E_4 \Delta^4 y_{-2} + \cdots + F_0 y_1 + F_2 \Delta^2 y_0 + F_4 \Delta^4 y_{-1} + \cdots$$

Using the method outlined in Gauss forward formula, we get

$$\begin{aligned} E_0 &= 1 - p = q \ (say), & F_0 &= p, \\ E_2 &= \frac{q(q^2 - 1^2)}{3!}, & F_2 &= \frac{p(p^2 - 1^2)}{3!}, \\ E_4 &= \frac{q(q^2 - 1^2)(q^2 - 2^2)}{5!}, & F_4 &= \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!}, \ \textit{and so on.} \end{aligned}$$

Hence, Everett's formula is

$$\begin{aligned} y_p = q y_0 &+ \frac{q(q^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)(q^2 - 2^2)}{5!} \Delta^4 y_{-2} + \cdots \\ &+ p y_1 + \frac{p(p^2 - 1^2)}{3!} \Delta^2 y_0 + \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!} \Delta^4 y_{-1} + \cdots \end{aligned}$$

Example 8 *Apply Everett's formula to find $y(30)$ from the following data.*

x	20	28	36	44
y	2854	3162	7088	7984

Solution: *The difference table taking origin at $x = 28$, is created below:*

u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$
-1	2854			
		308		
0	$\boxed{3162}$		$\boxed{3618}$	
		3926		-6648
1	$\boxed{7088}$		$\boxed{-3030}$	
		896		
2	7984			

Here, $x_0 = 28$, $h = 8$. Hence, from $x = x_0 + ph$, we get $30 = 28 + 8p \Rightarrow p = 0.25$. Hence $q = 1 - p = .75$

Everett's formula is

$$
\begin{aligned}
y_p &= qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots \\
&\quad + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y(30) = y_{.25} &= (.75)(3162) + \frac{(.75)((.75)^2-1^2)}{3!}(3618) + \\
&\quad + (.25)(7088) + \frac{(.25)((.25)^2-1^2)}{3!}(-3030) \\
&= 4064
\end{aligned}
$$

4 Interpolation Methods for Arbitrary Data Points (Specially for Unevenly Spaced Data Points)

The interpolation formulas derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is therefore desirable to develop some interpolation formulas for unevenly spaced values of x. We shall study two such formulas:
(i) Lagrange's interpolation formula
(ii) Newton's general interpolation formula with divided differences.

4.1 Lagrange's Interpolation Formula

Let $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$ be the given data points. The data point are not necessarily evenly-spaced. We need to find a polynomial of degree $\le n$ such that the polynomial will agree with the given data points.
The polynomial $P(x)$ may be written as

$$
\begin{aligned}
P(x) &= A_0(x-x_1)(x-x_2)\cdots(x-x_n) + A_1(x-x_0)(x-x_2)\cdots(x-x_n) \\
&\quad + \cdots + A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}),
\end{aligned} \tag{18}
$$

where $A_0, A_1, \cdots, A_n$ are constants to be determined.
Putting $x = x_0$ in (18), we obtain

$$
\begin{aligned}
P(x_0) = y_0 &= A_0(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n), \\
\Rightarrow A_0 &= \frac{y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}.
\end{aligned}
$$

Putting $x = x_1$ in (18), we obtain

$$P(x_1) = y_1 = A_1(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n),$$
$$\Rightarrow A_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}.$$

Similarly, putting $x = x_n$ in (18), we obtain

$$P(x_n) = y_n = A_n(x_n - x_0)(x_n - x_2)\cdots(x_n - x_{n-1}),$$
$$\Rightarrow A_n = \frac{y_n}{(x_n - x_0)(x_n - x_2)\cdots(x_n - x_{n-1})}.$$

Substituting these values, we get

$$P(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\cdots(x_n-x_{n-1})}y_n, \tag{19}$$

which is the required Lagrange's interpolation formula.

Another form of Lagrange's formula: The formula (19) can be written as

$$P(x_n) = \sum_{r=0}^{n} \frac{\phi(x)}{\phi'(x)(x-x_r)} y_r,$$
$$or,\ P(x_n) = \sum_{r=0}^{n} l_r(x) y_r,\ where\ l_r(x) = \frac{\phi(x)}{\phi'(x)(x-x_r)}.$$

Here, $\phi(x) = (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$, and $l_r(x) = \frac{\phi(x)}{\phi'(x)(x-x_r)}$ is call the **Lagrange's interpolation coefficients**.

Advantage:
(i) A major advantage of this method is that the coefficients are easily calculated.
(ii) It is applicable to both evenly spaced and unevenly spaced data points.
(iii)Abscissae or x-values need not be in increasing or decreasing order. In other method it is necessary to write them in certain order.

Disadvantage:
If one more point (x_{n+1}, y_{n+1}) is included with the given $(n+1)$ data points $(x_0, y_0), \ldots, (x_n, y_n)$, then we will have a polynomial of degree $(n+1)$ for $(n+2)$ points $(x_0, y_0), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})$. In this case, once again, we have to start the calculation from the beginning. The initial calculation of n degree polynomial for the $(n+1)$ data points $(x_0, y_0),\ (x_1, y_1),\ \ldots, (x_n, y_n)$ will be of no work.
While other methods, for example, Newton's forward or backward differences formulas, are recursive.

Example 9 Using the Lagrange's interpolation formula, find $y(10)$ from the following data:

x	5	6	9	11
y	12	13	14	16

Solution: Let $x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11$
and $\quad y_0 = 12,\ y_1 = 12,\ y_2 = 14,\ y_3 = 16.$
Then by Lagrange's interpolation formula, the polynomial is

$$\begin{aligned}
P_3(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \\
& \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3. \\
= & \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}13 \\
& + \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}16 \\
= & -\frac{1}{2}(x-6)(x-9)(x-11) + \frac{13}{15}(x-5)(x-9)(x-11) + \\
& -\frac{7}{12}(x-5)(x-6)(x-11) + \frac{4}{15}(x-5)(x-6)(x-9).
\end{aligned}$$

Putting $x = 10$, we get

$$\begin{aligned}
P_3(10) = & -\frac{1}{2}(10-6)(10-9)(10-11) + \frac{13}{15}(10-5)(10-9)(10-11) + \\
& -\frac{7}{12}(10-5)(10-6)(10-11) + \frac{4}{15}(10-5)(10-6)(10-9). \\
= & \ 14.666667
\end{aligned}$$

Hence, $y(10) = 14.666667$.

4.2 Newton's Divided Difference Formula

Lagranges interpolation formula has the disadvantage that if another point were added, all the interpolation coefficients for the new polynomial will have to be recomputed.

We therefore seek an interpolation polynomial which has the property that a polynomial of higher degree may be derived from it by simply adding new terms. That means, we are looking for a formula which will work recursively. If another interpolation point is added to the original data points, then we only have add one more term with the polynomial interpolating the original given points. In other word, we want to find a formula which makes use of $P_{k-1}(x)$ in computing $P_k(x)$.

Newtons general interpolation formula is one such formula and it employs divided differences.

4.2.1 Divided Difference

Let $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$ be the given data points. The data point are not necessarily evenly-spaced. Then first order divided difference for the points x_0 and x_1 is defined by

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}.$$

Similarly,

$$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}.$$

Second order divided difference for the points x_0, x_1 and x_2 is defined by

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}.$$

Third order divided difference for the points x_0, x_1, x_2 and x_3 is defined by

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

4.2.2 Newton's Divided Difference Formula or Newton's General Interpolation Formula*

(* This is sometimes call as Newton's General Interpolation Formula because Newton's forward and backward difference interpolation formula can be derived from this)

Let $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$ be the given data points. The data point are not necessarily evenly-spaced. Then from the definition of divided difference, we have

$$[x, x_0] = \frac{y_0 - y(x)}{x_0 - x}$$

$$\Rightarrow y(x) = y_0 + (x - x_0)[x, x_0]. \tag{20}$$

Again,

$$[x, x_0, x_1] = \frac{[x_0, x_1] - [x, x_0]}{x_1 - x}.$$

$$\Rightarrow [x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]. \tag{21}$$

From (20) and (21), we have

$$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]. \tag{22}$$

Again,

$$[x, x_0, x_1, x_2] = \frac{[x_0, x_1, x_2] - [x, x_0, x_1]}{x_2 - x}.$$
$$\Rightarrow [x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]. \tag{23}$$

From (22) and (23), we have

$$\begin{aligned} y(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &\quad + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]. \end{aligned} \tag{24}$$

Proceeding in similar way, we obtain

$$\begin{aligned} y(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &\quad + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})[x_0, x_1, \cdots, x_n] \\ &\quad + (x - x_0)(x - x_1) \cdots (x - x_{n-1})(x - x_n)[x, x_0, x_1, \cdots, x_n], \end{aligned} \tag{25}$$

which is called Newton's divided difference interpolation formula. The last term is called the reminder or the error term. Hence, the interpolating polynomial $P_n(x)$ in terms of the divided differences is written as

$$\begin{aligned} y(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &\quad + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})[x_0, x_1, \cdots, x_n]. \end{aligned} \tag{26}$$

4.2.3 Table of Divided Difference

Let (x_0, y_0), (x_1, y_1), $\ldots, (x_4, y_4)$ be the given data points. The divided difference table is created as follows:

x	y	1st Divided Difference	2nd Divided Difference	3rd Divided Difference	4th Divided Difference
x_0	y_0				
		$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$			
x_1	y_1		$[x_0, x_1, x_2]$		
		$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$		$[x_0, x_1, x_2, x_3]$	
x_2	y_2		$[x_1, x_2, x_3]$		$[x_1, x_2, x_3, x_4, x_5]$
		$[x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}$		$[x_1, x_2, x_3, x_4]$	
x_3	y_3		$[x_2, x_3, x_4]$		
		$[x_3, x_4] = \frac{y_4 - y_3}{x_4 - x_3}$			
x_4	y_4				

Table 4: Divided Difference Table for n=4 (i.e., For Five Points)

Note:
(i) Each divided difference can be obtained from two previous ones of lower orders. For example, $[x_0, x_1, x_2]$ can be computed from $[x_0, x_1]$ and $[x_1, x_2]$ by $[x_0, x_1, x_2] = \frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$ and so on.
(ii) In computing $P_n(x)$ by Newton's divided difference, we need only the first diagonal entries of the above table; that is, we need only y_0, $[x_0, x_1]$, $[x_0, x_1, x_2]$, $[x_0, x_1, x_2, x_3], \cdots,$ $[x_0, x_1, \cdots, x_n]$.
(iii) Since the divided differences are generated recursively, the interpolating polynomials of successively higher degrees can also be generated recursively. Thus the work done previously can be used gainfully. For example,

$$P_1(x) = y_0 + (x - x_0)[x_0, x_1].$$

$$\begin{aligned} P_2(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &= P_1(x) + (x - x_0)(x - x_1)[x_0, x_1, x_2]. \end{aligned}$$

$$\begin{aligned} P_3(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] \\ &= P_2(x) + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3], \end{aligned}$$

and so on.

Example 10 Construct the divided difference table for the following data:

x	1	2	4	7	12
y	22	30	82	106	216

Solution: The divided difference table is created as follows:

x	y	1st Div. Dif.	2nd Div. Dif.	3rd Div. Dif.	4th Div. Dif.
1	22				
		$\frac{30-22}{2-1} = 8$			
2	30		$\frac{26-8}{4-1} = 6$		
		$\frac{82-30}{4-2} = 26$		$\frac{-3.6-6}{7-1} = -1.6$	
4	82		$\frac{8-26}{7-2} = -3.6$		$\frac{0.535-(-1.6)}{12-1} = 0.194$
		$\frac{106-82}{7-4} = 8$		$\frac{1.75-(-3.6)}{12-2} = 0.535$	
7	106		$\frac{22-8}{12-4} = 1.75$		
		$\frac{216-106}{12-7} = 22$			
12	216				

Table 5: Divided Difference Table

Example 11 Using Newton's divided difference formula, find a polynomial function of degree 4 satisfying the data

x	-4	-1	0	2	5
y	1245	33	5	9	1335

Solution: First we construct the divided difference table as follows:

x	y	1st Divided Difference	2nd Divided Difference	3rd Divided Difference	4th Divided Difference
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Table 6: Divided Difference Table

Applying Newton's divided difference formula, we have

$$
\begin{aligned}
P_4(x) &= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\
&\quad + \cdots + (x - x_0)(x - x_1)\cdots(x - x_3)[x_0, x_1, \cdots, x_4] \\
&= 1245 + (x + 4) \times (-404) + (x + 4)(x + 1) \times 94 \\
&\quad\; + (x + 4)(x + 1)(x - 0) \times (-14) + (x + 4)(x + 1)(x - 0)(x - 2) \times (3) \\
&= 3x^4 - 5x^3 + 6x^2 - 14x + 5.
\end{aligned}
$$

4.2.4 Newton's Divided Difference Formula for Equally Spaced Nodes

Suppose again, $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$ are equally space points with spacing h, that is $x_{i+1} - x_i = h$ for $i = 0, 1, 2, ..., n - 1$. Let $x = x_0 + ph$, then

$$
\begin{aligned}
&x - x_0 = ph \\
&x - x_1 = x - x_0 + (x_0 - x_1) = ph - h = (p - 1)h \\
&\vdots \\
&x - x_{n-1} = x - x_0 + (x_0 - x_{n-1}) = ph - (n - 1)h = (p - n + 1)h
\end{aligned}
$$

Substituting these values in Newton's divided difference formula (26), we get

$$\begin{aligned} y(x) &= y_0 + ph[x_0, x_1] + p(p-1)h^2[x_0, x_1, x_2] \\ &\quad + \cdots + p(p-1)(p-2)\cdots(p-n+1)h^n[x_0, x_1, \cdots, x_n]. \end{aligned} \tag{27}$$

4.2.5 Deduction of Newton's forward Difference Formula from Newton's Divided Difference Formula

We have,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}.$$

$$[x_0, x_1, x_2] = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}.$$

In general.

$$[x_0, x_1, \cdots, x_k] = \frac{\Delta^k y_0}{k!h^k} \text{ (\textbf{proof is left as exercise})}.$$

Hence, equation (27) can be written as

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n y_0, \tag{28}$$

Similarly, it is also possible to derive Newton's backward difference formula from Newton's divided difference formula sing the relation

$$[x_{n-k}, \cdots, x_{n-1}, x_n] = \frac{\nabla^k y_n}{k!h^k} \text{ (\textbf{proof is left as exercise})}.$$