Example 18·4. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement y (x, t).* **(Madras, 1995)**

Sol. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, \ y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \dfrac{\pi x}{l}$ ...(iv)

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $\quad y(0, t) = c_1 (c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore \quad y(x, t) = c_2 \sin px \, (c_3 \cos cpt + c_4 \sin cpt)$

Also $\quad y(l, t) = c_2 \sin pl \, (c_3 \cos cpt + c_4 \sin cpt) = 0 \quad$ for all t.

This gives $\quad pl = n\pi$ or $p = \dfrac{n\pi}{l}$, n being an integer.

Thus $\quad y(x, t) = c_2 \sin \dfrac{n\pi x}{l}\left(c_3 \cos \dfrac{cn\pi}{l} t + c_4 \sin \dfrac{cn\pi}{l} t\right)$

By (iii), $\quad 0 = c_2 c_3 \sin \dfrac{n\pi x}{l} \quad$ for all x i.e. $c_2 c_3 = 0$

$\therefore \quad y(x, t) = b_n \sin \dfrac{n\pi x}{l} \sin \dfrac{cn\pi t}{l} \quad$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \Sigma\, b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \qquad ...(v)$$

Now $\quad \dfrac{\partial y}{\partial t} = \Sigma\, b_n \sin \dfrac{n\pi x}{l} \cdot \dfrac{cn\pi}{l} \cos \dfrac{cn\pi t}{l}$

By (iv), $\quad v_0 \sin^3 \dfrac{\pi x}{l} = \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \Sigma \dfrac{cn\pi}{l} b_n \sin \dfrac{n\pi x}{l}$

or

$$\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \Sigma \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l} \qquad [\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \ldots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$\therefore \quad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right)$$

Example 18·5. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.*

Sol. Let B and C be the points of the trisection of the string $OA(= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...}(i)$$

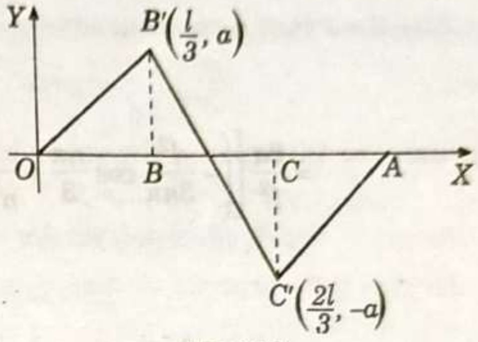

Fig. 18·2.

and the boundary conditions are

$$y(0, t) = 0 \qquad \text{...}(ii)$$

$$y(l, t) = 0 \qquad \text{...}(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{...}(iv)$$

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\, x$;

the equation of $B'C'$ is $y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, *i.e.* $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is
$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}\, x, \quad 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x), \quad \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l), \quad \frac{2l}{3} \le x \le l \end{aligned}\right\} \qquad \text{...}(v)$$

As in example 18·4, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad [\text{where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \text{...}(vi)$$

Putting $t = 0$, we have $y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$ ...(vii)

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ By (1) of § 10·7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6a}{l^2}\left[\left|x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l-2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left. + \left|(x-l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\cdot\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2}\cdot\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1 + (-1)^n\right] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n\sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad ...(vii)$$

Putting $x = l/2$ in (vii), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v. Then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$

and
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\partial y} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.* u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.* of the form

$$u = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t} \quad \text{...(8)}$$

is the only suitable solution of the heat equation.

Example 18·7. *Solve the differential equation* $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(i) u is not infinite for $t \to \infty$, *(ii)* $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(iii) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. *(Assam, 1999)*

Sol. Substituting $u = X(x)\, T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e.\ X''/X = \frac{T'}{\alpha^2 T} = -k^2 \quad \text{(say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + k^2 \alpha^2 T = 0 \quad \text{...(1)}$$

Their solutions are

$$X = c_1 \cos kx + c_2 \sin kx, \quad T = c_3\, e^{-k^2 \alpha^2 t} \quad \text{...(2)}$$

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4\, e^{kx} + c_5\, e^{-kx}, \quad T = c_6\, e^{k^2 \alpha^2 t} \quad \text{...(3)}$$

If $k^2 = 0$, the solutions are

$$X = c_7 x + c_8, \quad T = c_9 \quad \text{...(4)}$$

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.* the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \quad \text{...(5)}$$

From (2), $$\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, k c_3\, e^{-k^2 \alpha^2 t}$$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e. $$c_2 = 0 \quad \text{and} \quad kl = n\pi \quad (n \text{ an integer})$$

$$\therefore \quad u = c_1 \cos kx \cdot c_3\, e^{-k^2 \alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2 \pi^2 \alpha^2 t}}{l^2} \quad \text{...(6)}$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma\, a_n \cos(n\pi x/l)\, e^{-n^2 \pi^2 \alpha^2 t/l^2} \quad \text{...(7)}$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma\, a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\, dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and $$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\, dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18·8. *(a) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* *(Andhra, 1998)*

(b) Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C. *(Madras, 2000S)*

Sol. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...}(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$ ...(*ii*)

Its general solution is $u = ax + b$

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (2) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$u(0, t) = 0$ for all values of t ...(*iv*)

and $u(l, t) = 0$ for all values of t ...(*v*)

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1\cos px + C_2\sin px)\, e^{-c^2p^2t} \qquad \text{...}(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2\sin px \cdot e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2\sin pl \cdot e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $\therefore p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2}$, where $b_n = C_2$.

[These are the solutions of (*i*) satisfying the boundary conditions (*iv*) and (*v*). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (*i*) satisfying the boundary conditions (*iv*) and (*v*) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2} \qquad \text{...(viii)}$$

Putting $t = 0$, $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(*ix*)

In order that the condition (*iii*) may be satisfied, (*iii*) and (*ix*) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence (*viii*) gives $u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}\, e^{-(cn\pi/l)^2 t}$

(*b*) Here the initial condition remains the same as (*iii*) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad \text{...(x)}$$

$$u(l, t) = 80 \text{ for all values of } t \qquad \text{...(xi)}$$

In part (*a*), the boundary values (*i.e.* the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad \text{...(xii)}$$

where $u_s(x)$ is a solution of (*i*) involving x only and satisfying the boundary conditions (*x*) and (*xi*) ; $u_t(x, t)$ is then a function defined by (*xii*). Thus $u_s(x)$ is a steady state solution of the form (*ii*) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (*ii*) we get

$$u_s(x) = 20 + (60/l)x \qquad \text{...(xiii)}$$

Putting $x = 0$ in (*xii*), we have by (*x*),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad \text{...(xiv)}$$

Putting $x = l$ in (*xii*), we have by (*xi*),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad \text{...(xv)}$$

Also
$$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right)$$
$$= \frac{40x}{l} - 20 \qquad \text{[by (iii) and (xiii)} \qquad ...(xvi)$$

Hence (*xiv*) and (*xv*) give the boundary conditions and (*xvi*) gives the initial condition relative to the transient solution. Since the boundary values given by (*xiv*) and (*xv*) are both zero, therefore, as in part (*a*), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t}$

By (*xiv*), $\qquad u_t(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and $\qquad u_t(x, t) = C_2 \sin px \cdot e^{-c^2p^2t}$ $\qquad ...(xvii)$

Applying (*xv*), it gives $\qquad u_t(l, t) = C_2 \sin pl e^{-c^2p^2t} = 0$ for all values of t.

This requires $\qquad \sin pl = 0$, *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $\quad p = n\pi/l$, when n is any integer.

Hence (*xvii*) reduces to $u_t(x, t) = b_n \sin \frac{n\pi x}{l}\, e^{-c^2n^2\pi^2t/l^2}$ $\qquad$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is $u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-c^2n^2\pi^2t/l^2}$ $\qquad ...(xviii)$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ $\qquad ...(xix)$

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\,x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{where } b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin \frac{n\pi x}{l}\, dx = -\frac{40}{n\pi}(1 + \cos n\pi)$$

i.e. $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4...}^{\infty} \left(\frac{-80}{n\pi}\right) \sin \frac{n\pi x}{l} \cdot e^{-c^2n^2\pi^2t/l^2}$ $\qquad$ [Take $n = 2m$]

$$= -\frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2t/l^2} \qquad ...(xx)$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2t/l^2}$$

Example 18·9. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Sol. The temperature $u\,(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18·4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0,t)}{\partial x}=0,\ \frac{\partial u(l,t)}{\partial x}=0 \text{ for all } t \quad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2}=0$. Its solution is $u=ax+b$.

Since $u=0$ for $x=0$ and $u=100$ for $x=l$ $\therefore b=0$ and $a=1$.

Thus the initial condition is $u(x,0)=x \quad 0<x<l$. ...(iii)

Now the solution of (i) is of the form $u(x,t)=(c_1\cos px+c_2\sin px)\,e^{-c^2p^2t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x}=(-c_1 p\sin px+c_2 p\cos px)\,e^{-c^2p^2t} \quad ...(v)$$

Putting $x=0$, $\left(\frac{\partial u}{\partial x}\right)_0=c_2 p\,e^{-c^2p^2t}=0$ for all t. [by (ii)]

$\therefore \quad c_2=0$

Putting $x=l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l=-c_1 p\sin pl\,e^{-c^2p^2t}$ for all t. [by (ii)

$\therefore \quad c_1 p\sin pl=0$ *i.e.*, p being $\neq 0$, either $c_1=0$ or $\sin pl=0$.

When $c_1=0$, (iv) gives $u(x,t)=0$ which is a trivial solution, therefore $\sin pl=0$.

or $\quad pl=n\pi \quad$ or $\quad p=n\pi/l, \quad n=0,1,2,\ldots$

Hence (iv) becomes $u(x,t)=c_1\cos\frac{n\pi x}{l}\,e^{-c^2n^2\pi^2t/l^2}$.

$\therefore$ The most general solution of (i) satisfying the boundary conditions (ii) is

$$u(x,t)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{l}\,e^{-c^2n^2\pi^2t/l^2}=A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}\,e^{-c^2n^2\pi^2t/l^2} \quad (\text{where } A_n=c_1)\ ...(vi)$$

Putting $t=0$, $u(x,0)=A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}=x$ [by (iii)]

This requires the expansion of x into a half range cosine series in $(0,l)$.

Thus $\quad x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x/l \quad$ where $a_0=\frac{2}{l}\int_0^l x\,dx=l$

and $\quad a_n=\frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx=\frac{2l}{n^2\pi^2}(\cos n\pi-1)$

$=0$, where n is even ; $=-4l/n^2\pi^2$, when n is odd.

$\therefore \quad A_0=\frac{a_0}{2}=l/2$, and $A_n=a_n=0$ for n even ; $=-4l/n^2\pi^2$ for n odd.

Hence (vi) takes the form

$$u(x,t)=\frac{l}{2}+\sum_{n=1,3,\ldots}^{\infty}\frac{4l}{n^2\pi^2}\cos\frac{n\pi x}{l}\,e^{-c^2n^2\pi^2t/l^2}$$

$$=\frac{l}{2}-\frac{4l}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}\,e^{-c^2(2n-1)^2\pi^2t/l^2} \quad ...(vii)$$

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