Fourier Series and transforms

A series expansion of a func^n in terms of trigonometric functions $\cos(mx)$ and $\sin(nx)$ is called fourier series.

Many func^ns including some discontinuous periodic func^ns can be written in form a Fourier series.

Hence, it has wide application in solving ODE's and PDE's.

consider a set of func^ns,

$\{f_1, f_2, f_3 \cdots\}$ defined in $[a,b]$

these are said to be orthogonal if $\int_a^b f_i(x) f_j(x)\, dx = 0, \forall i \neq j$

∴ Now, we see $\left\{1, \cos\left(\frac{\pi x}{l}\right), \cos\left(\frac{2\pi x}{l}\right), \dots \sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right) \dots\right\}$ —①

is orthogonal in $[-l, l]$ since,

- $\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right) dx = 0$

- $\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \cos\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0, & m \neq n \\ l, & m = n \end{cases}$

- $\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \begin{cases} 0, & m \neq n \\ l, & m = n \end{cases}$

- $\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = \{0, \forall m, n$

Hence, if we take any 2 diff. func^ns from set ① and integrate their product from $[-l, l]$ then we get '0' which means given set of func^n is orthogonal.

Now consider a funcn $f(x)$ which is periodic with period $2l$ defined on $[-l, l]$.

Assuming $f(x)$ can be expressed as a linear combination of trig. funcns $\cos(nx)$, $\sin(nx)$ then

$$f(x) = \text{constant} \times \left[1 + \cos\frac{\pi x}{l} + \cos\frac{2\pi x}{l} + \dots + \sin\frac{\pi x}{l} + \sin\frac{2\pi x}{l} + \dots\right]$$

$$f(x) = \left[\frac{a_0}{2} + \frac{a_1'}{2}\cos\frac{\pi x}{l} + \frac{a_2'}{2}\cos\frac{2\pi x}{l} + \dots + \frac{b_1'}{2}\sin\frac{\pi x}{l} + \frac{b_2'}{2}\sin\frac{2\pi x}{l}\right]$$

$$= \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + \dots + b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right)}$$

Now to find 'a_0' we integrate both sides from $-l$ to l →

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{l}\frac{a_0}{2}\,dx + \int_{-l}^{l}\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right)dx$$

$$\int_{-l}^{l} f(x)\,dx = \frac{a_0}{2}\times 2l + 0$$

$$\boxed{a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx}$$

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right) = 0$$

$$\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right) = 0$$

for any 'n'

to find 'a_n' we multiply both sides by $\cos\left(\frac{n\pi x}{l}\right)$ and $\int_{-l}^{l}$.

$$\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{a_0}{2}\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)dx + a_n\int_{-l}^{l}\cos^2\left(\frac{n\pi x}{l}\right)dx + 0$$

$$\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = 0 + a_n \times l$$

$$\therefore \boxed{a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx}$$

$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right) = 0$ $\forall\ m \neq n$

$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right) = 0$ $\forall\ m, n$

$\int_{-l}^{l}\cos^2\left(\frac{n\pi x}{l}\right) = l$

Similarly, for 'b_n', we mult. both sides by $\sin\left(\frac{n\pi x}{l}\right)$ and $\int_{-l}^{l}$ and we get,

$$\boxed{b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

Q. $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ 0, & 0 \le x < \pi \end{cases}$ Find the fourier series for $f(x)$.

here $l = \pi$ $\therefore$ period is 2π i.e. $f(x+2\pi) = f(x)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\int_{-\pi}^{0}(\pi + x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx$$

$$a_0 = \frac{1}{\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} \Rightarrow \frac{\pi}{2}$$

now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx$$

$$a_n = \frac{1}{\pi}\left[(\pi+x)\frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{-\pi}^{0}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos(n\pi)}{n^2}\right]$$

$$a_n = \frac{1}{\pi n^2}(1-(-1)^n)$$

$\cos(n\pi) = (-1)^n$

$n = 1, 2, 3 \ldots$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\sin(n\pi)\,dx \Rightarrow \frac{1}{\pi}\left[(\pi+x)\left(-\frac{\cos(nx)}{n}\right) + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\right) \Rightarrow b_n = -\frac{1}{n}$$

$\therefore$ fourier series of $f(x)$ is,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}[1-(-1)^n]\cos\left(\frac{n\pi x}{l}\right) + \left(-\frac{1}{n}\right)\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}[1-(-1)^n]\cos(nx) - \frac{1}{n}\sin(nx)\right]$$

Convergence of Fourier series for continuous func^n.

If a periodic func^n $f(x)$ with period $2l$ is conti. in $[-l, l]$ and has continuous 1^{st} and 2^{nd} order derivatives at each point in that interval, then the Fourier series of $f(x)$ is convergent

Proof: $$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\left(\frac{f(x)\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}\right)_{-l}^{l} - \int_{-l}^{l} f'(x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}dx\right]$$

$$= \frac{l}{n\pi l}\left[\left(f(x)\sin\left(\frac{n\pi x}{l}\right)\right)_{-l}^{l} - \int_{-l}^{l} f'(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{-1}{n\pi}\left[-\left(\frac{f'(x)\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right)_{-l}^{l} + \int_{-l}^{l}\frac{f''(x)\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}dx\right]$$

$$= \frac{-1}{n\pi}\left[-\frac{l}{n\pi}\left(f'(l)\cos n\pi - f'(-l)\cos n\pi\right) + \frac{l}{n\pi}\int_{-l}^{l} f''(x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{-l}{n^2\pi^2}\int_{-l}^{l} f''(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Since, $f(x)$ has $T = 2l$
i.e. $f(x+2l) = f(x)$
$\because$ 1^{st} derivative is conti.
$\therefore$ $f'(x+2l) = f'(x)$
at $x = -l$
$f'(l) = f'(-l)$

Since, $f''(x)$ is also conti. in $[-l, l]$ so, for some 'M'

$$|f''(x)| < M \quad \text{also} \quad \left|\cos\left(\frac{n\pi x}{l}\right)\right| \leq 1$$

$$\therefore |a_n| = \frac{l}{n^2\pi^2}\left|\int_{-l}^{l} f''(x)\cos\left(\frac{n\pi x}{l}\right)dx\right| < \frac{l}{n^2\pi^2}\int_{-l}^{l}(M\times 1)dx = \frac{2Ml}{n^2\pi^2}$$

Similarly, $|b_n| < \frac{2Ml}{n^2\pi^2}$

$$\therefore \quad |f(x)| = \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \right|$$

$$< \frac{|a_0|}{2} + \sum_{n=1}^{\infty}\left(\frac{2Ml}{n^2\pi^2} + \frac{2Ml}{n^2\pi^2}\right)$$

$$< \frac{|a_0|}{2} + \frac{4Ml}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

By P-series test $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) \rightarrow$ convergent as $P>1$

$\therefore$ $|f(x)|$ is convergent

and $f(x) \leq |f(x)|$

$\therefore$ $f(x) \rightarrow$ also convergent

convergence of fourier series for piecewise conti. func^n.

A function f is said to be piecewise conti. in $[-l, l]$ if,

- $f(x)$ is defined and conti. in $\forall\, x \in [-l, l]$ except at finite number of points in $[-l, l]$.
- At a point $x_0 \in (-l, l)$, if func^n is not continuous, then $\lim_{x \to x_0} f(x)$ exist and are finite.
- At the end point of interval, $\lim_{x \to -l^-} f(x)$ and $\lim_{x \to -l^+} f(x)$ exist and are finite

now, if

(1) $f(x) \to$ periodic with period $2l$, single valued & finite.
(2) $f(x) \to$ piecewise conti. in $[-l, l]$
(3) $f(x) \to$ has left hand derivative & RHD at each point in $[-l, l]$

then, for convergence of piecewise cont. func^n we have a

<u>theorem:</u> Let $f(x)$ and $f'(x)$ be piecewise cont. func^n on the interval $[-l, l]$ then the fourier series of $f(x)$ converges to $f(x)$ at the point of continuity

At the point of discontinuity, say $x_0 \in (-l, l)$ fourier series converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$ where $f(x_0^+)$ & $f(x_0^-)$ are RHL, LHL of $f(x)$ at x_0.

At both end points of $[-l, l]$, fourier series converges to

$$\frac{f(-l^+) + f(l^-)}{2}.$$

Q. Find the fourier series expansion of the following func^n

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

Hence, deduce :

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

we have our func^n in $[-\pi, \pi]$

$\therefore l = \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$= \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right] \Rightarrow -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos(nx)\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\sin nx}{n}\right)_0^{-\pi} + \left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{n^2\pi}\cdot\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left[x\left(-\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)_0^{\pi}\right]\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin nx$$

we need sq. terms only in denomi. so $\sin nx = 0$

$\therefore$ put $x = 0$

$$f(0) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$f(x)$ is discont. at $x=0$ so, $f(0)$ converges to

$\dfrac{f(0^+) + f(0^-)}{2}$ ie. $f(0) = \dfrac{x - \pi}{2} \Rightarrow f(0) = -\dfrac{\pi}{2}$

$\therefore$

$$\Rightarrow -\frac{\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}[(-1)^n - 1]$$

$$\Rightarrow -\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}[(-1)^n - 1]$$

$$\Rightarrow -\frac{\pi^2}{4} = -\frac{2}{1^2} - \frac{2}{3^2} - \frac{2}{5^2} \cdots$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad //$$

Fourier cosine transform-

Let $f(x)$ be defined $\forall\ x \geq 0$ then the fourier cosine transform of $f(x)$ is given by

$$F(\lambda) = \int_0^{\infty} f(x) \cos(\lambda x)\, dx \quad , \quad \lambda \to \text{parameter} \geq 0$$

Similarly, Fourier inverse cosine transform of $F(\lambda)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(\lambda) \cos(\lambda x)\, d\lambda$$

Q: $f(x) = \begin{cases} \cos x & , \ 0 < x < a \\ 0 & , \ x > a \end{cases}$ Find fourier cosine transform of $f(x)$

$$F(\lambda) = \int_0^a \cos x \cos \lambda x\, dx + \int_a^{\infty} 0 \cdot \cos \lambda x\, dx$$

$$= \frac{1}{2} \int_0^a 2 \cos x \cos \lambda x\, dx$$

$$= \frac{1}{2}\left[\int_0^a \cos(1-\lambda)x\, dx + \int_0^a \cos(1+\lambda)x\, dx \right.$$

$$= \frac{1}{2}\left[\frac{\sin(1-\lambda)x}{1-\lambda} + \frac{\sin(1+\lambda)x}{1+\lambda}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(1-\lambda)a}{1-\lambda} + \frac{\sin(1+\lambda)a}{1+\lambda}\right]$$

Q: Find Fourier cosine transform of $f_1(x) = \frac{1}{1+x^2}$ then find F. sine transform of $f_2(x) = \frac{x}{1+x^2}$

$$F_1(\lambda) = \int_0^\infty f_1(x)\cos(\lambda x)\,dx$$

$$= \int_0^\infty \frac{\cos(\lambda x)\,dx}{1+x^2} = I(\lambda) \quad -(1)$$

$$\frac{dI}{d\lambda} = -\int_0^\infty \frac{x\sin\lambda x}{(1+x^2)}\,dx = -\int_0^\infty \frac{x^2\sin\lambda x}{(1+x^2)\,x}\,dx$$

$$= -\int_0^\infty \frac{(x^2+1-1)\sin\lambda x}{(1+x^2)\,x}\,dx$$

$$= -\int_0^\infty \frac{\sin\lambda x}{x} + \int_0^\infty \frac{\sin\lambda x}{(1+x^2)x}\,dx$$

$$= -\frac{\pi}{2} + \int_0^\infty \frac{\sin\lambda x}{x(1+x^2)}\,dx \quad -(2)$$

$$\frac{d^2I}{d\lambda^2} = \int_0^\infty \frac{\cos\lambda x}{(1+x^2)}\,dx = I$$

$$\therefore \frac{d^2I}{d\lambda^2} - I = 0 \Rightarrow \frac{d}{d\lambda}\left(\frac{dI}{d\lambda}\right) = I$$

$$\int d\left(\frac{dI}{d\lambda}\right) = \int I\,d\lambda \Rightarrow \cancel{\frac{dI}{d\lambda}}$$ can't be evaluated becoz $I(\lambda)$ is a general funcn.

so, $\frac{d^2 I}{d\lambda^2} - I = 0 \Rightarrow (D^2-1)I = 0$ | $\frac{d}{d\lambda} = D$

aux. eqn: $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore \quad I = CF + PI$ (But RHS $= 0 \Rightarrow PI = 0$)

$\therefore \quad I = CF = c_1 e^{\lambda} + c_2 e^{-\lambda}$ — (3)

Now, we need to find $c_1, c_2 \rightarrow$

$\therefore$ Put $\lambda = 0$ ie. $I(0) = c_1 + c_2$ — (4)

& $\left.\frac{dI}{d\lambda}\right|_{\lambda=0} = c_1 e^{\lambda} - c_2 e^{-\lambda} \Rightarrow c_1 - c_2$ — (5)

put $\lambda = 0$ in (2) so that $\frac{dI}{d\lambda} = -\frac{\pi}{2}$

now $c_1 - c_2 = -\frac{\pi}{2}$ — (6)

Put $\lambda = 0$ in (1)

$I(0) = \int_0^\infty \frac{1}{1+x^2} dx = \tan^{-1}(x)\Big|_0^\infty$

ie. $c_1 + c_2 = \frac{\pi}{2}$ — (7) $= \frac{\pi}{2}$

(6) + (7) $\rightarrow$ $c_1 = 0$ $\therefore$ $c_2 = \frac{\pi}{2}$

$\therefore \quad I = \frac{\pi}{2} e^{-\lambda}$ //

Now, fourier sine transform of $f_2(x) = \frac{x}{1+x^2}$

ie. $F_2(\lambda) = \int_0^\infty \frac{x}{1+x^2} \sin(\lambda x)\, dx = D(\lambda)$ (say)

we already evaluated, $\int_0^\infty \frac{\cos(\lambda x)}{1+x^2} dx = \frac{\pi}{2} e^{-\lambda}$ — (8)

differentiating (ii) w.r.t λ →

$$-\int_0^\infty \frac{x \sin(\lambda x)}{1+x^2} = -\frac{\pi}{2} e^{-\lambda}$$

$$\therefore \quad F_2(\lambda) = \frac{\pi}{2} e^{-\lambda}$$

Fourier Sine transform:

If $f(x)$ is defined $\forall$ $x > 0$, fourier sine transform of $f(x)$ is given by

$$F(\lambda) = \int_0^\infty f(x) \sin(\lambda x)\, dx \quad , \quad \lambda > 0$$

&

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \sin(\lambda x)\, d\lambda$$ (Inverse Fourier sine transform of $F(\lambda)$)

Q. $f(x) = e^{-|x|}$ evaluate fourier sine transform of $f(x)$ hence evaluate $\int_0^\infty \left(\frac{x \sin mx}{1+x^2}\right) dx$

→ $F(\lambda) = \int_0^\infty e^{-x} \sin \lambda x\, dx = I(\lambda)$ | $\because x > 0$ $\therefore |x| = x$

$$I = -\left(\frac{e^{-x}\cos \lambda x}{\lambda}\right)_0^\infty - \int_0^\infty \frac{e^{-x}\cos \lambda x}{\lambda}\, dx$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda}\left[\frac{e^{-x}\sin \lambda x}{\lambda} + \int \frac{e^{-x}\sin \lambda x}{\lambda}\, dx\right]$$

$$I = \frac{1}{\lambda} - \frac{I}{\lambda^2} \Rightarrow I\left(1 + \frac{1}{\lambda^2}\right) = \frac{1}{\lambda}$$

$$I = \frac{\lambda}{\lambda^2+1} \quad //$$

now, $\int_0^\infty e^{-x} \sin \lambda x \, dx = \frac{\lambda}{\lambda^2+1} = F(\lambda)$

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \sin(\lambda x) \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2+1} \cdot \sin \lambda x \, d\lambda$$

$$\frac{\pi}{2} e^{-x} = \int_0^\infty \frac{\lambda}{\lambda^2+1} \sin(\lambda x) \, d\lambda$$

Put $x = m$ both sides →

$$\frac{\pi}{2} e^{-m} = \int_0^\infty \frac{\lambda}{\lambda^2+1} \cdot \sin(\lambda m) \, d\lambda$$

using property $\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$

$$\frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x}{x^2+1} \cdot \sin(mx) \, dx \quad //$$

Half range Fourier sine and cosine serieses

Till now we defined fourier series in complete interval $(-l, l)$ to get $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$

but now, we are defining fourier series in half range i.e. $(0, l)$ so, depending on the case whether $f(x)$ is even or odd in $(-l, l)$ we get Half range fourier cosine series & sine series.

So, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

where $a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

{Fourier half range cosine series}

Similarly, $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

{ fourier half range sine series }

&

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ when $f(x) \to$ even

$a_0 = 0$ when $f(x) \to$ odd

$$a_n = \begin{cases} \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, & f(x) \to \text{even} \\ 0, & f(x) \to \text{odd} \end{cases}$$

$$b_n = \begin{cases} \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, & f(x) \to \text{odd} \\ 0, & f(x) \to \text{even} \end{cases}$$

Note: ① odd func^n x odd func^n → even func^n
even " " even " → " "
even " x odd " → odd "

② depending upon $f(x) \to$ odd or even diff. values of a_0, a_n, b_n are coming which ultimately giving half range sine and cosine series.

This is sometimes also known as fourier series for even and odd func^{n}s.

Fourier transforms and Integrals

Fourier transform of f(x) multi. by x^n.

$$\mathcal{F}[x^n f(x)] = i^n F^n(\lambda).$$

Finite Fourier sine and cosine transform.

for $f(x)$, in the interval $[0, l]$

F. Fourier cosine transform, $F_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

F. " sine " , $F_s(n) = \int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Similarly, their inverse are defined, as

$$f(x) = \frac{1}{l}\left[F_c(0) + 2\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\left(\frac{n\pi x}{l}\right)$$

Convolution Theorem-

Let $f(x), g(x)$ be piecewise conti. on every interval $[-l, l]$ & $\int_{-\infty}^{\infty}|f(x)|dx$, $\int_{-\infty}^{\infty}|g(x)|dx$ converge. $\mathcal{F}[f(x)] = F(\lambda)$ and $\mathcal{F}[g(x)] = G(\lambda)$ then,

$$(f*g)(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du = (g*f)(x)$$

is known as convolution of 'f' & 'g' w.r.t 'x'.

Now,

$$\mathcal{F}[\{f*g\}(x)] = F(\lambda)\, G(\lambda) \quad \text{(convolution Theorem)}$$

Fourier cosine and sine transform of derivatives :

$$\mathcal{F}_c[f''(x)] = \int_0^\infty f''(x)\cos(\lambda x)\,dx = -\lambda^2 F_c(\lambda) - f'(0)$$

$$\mathcal{F}_s[f''(x)] = \int_0^\infty f''(x)\sin(\lambda x)\,dx = -\lambda^2 F_s(\lambda) + \lambda f(0)$$

Fourier integrals -

If $f(x)$ satisfies the fourier condⁿ →

① $f(x)$ is piecewise conti. in every interval $[-l, l]$.

② $f(x)$ is absolutely integrable on x-axis ie. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

③ At every 'x' on real line $f(x)$ has LHD & RHD.

then fourier integral representation of $f(x)$ is,

$$f(x) = \int_0^{\infty} [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda$$

where, $A(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\, dx$

and $B(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\, dx$

Note: fourier integral converges to $f(x)$ at a point of continuity but if x is a point of discontinuity then fourier integral converges to $\dfrac{f(x^+) + f(x^-)}{2}$.

Fourier cosine and sine Integrals -

$$f(x) = \int_0^{\infty} A(\lambda)\cos(\lambda x)\, d\lambda$$

$$f(x) = \int_0^{\infty} B(\lambda)\sin(\lambda x)\, d\lambda$$

Complex form of fourier integral-

we have $f(x) = \int_0^{\infty} [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda$ -①

and $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\, dx$ -②

$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\, dx$ -③

using ② & ③ in ①

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos(\lambda(x-u))\, du\, d\lambda \quad -④$$

Since, $i\sin(\lambda(u-x)) \to$ odd funch $\therefore \int_{-\infty}^{\infty}\sin(\lambda(u-x))\, du = 0$

$\therefore$ we add $i\sin(\lambda(u-x))$ in ④ becoz it doesn't affect ④.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)[\cos(\lambda(u-x)) + i\sin(\lambda(u-x))]\, du\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\lambda(u-x)}\, du\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{i\lambda u}\, du\right] e^{-i\lambda x}\, d\lambda$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\lambda)e^{-i\lambda x}\, d\lambda}$$ where, $C(\lambda) = \int_{-\infty}^{\infty} f(u)e^{i\lambda u}\, du$

Complex form of fourier series

we know $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$

is fourier series' real form, in interval $[-l, l]$

$\because$ $\cos\left(\frac{n\pi x}{l}\right) = \left[\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right]$

$\sin\left(\frac{n\pi x}{l}\right) = \left[\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right]$

$\therefore$ complex form of fourier series is given by,

$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{(in\pi x/l)}$ where $C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{(in\pi x/l)}\, dx$

$l \rightarrow$ half length of the interval in which fourier series is formed $(n = 0, \pm 1, \pm 2 \dots)$

[For ex. in $[-l, l]$, interval length $= 2l$ $\therefore$ we take 'l']

* Some useful results:

(1) $\int e^{a\lambda} \sin(\lambda x)\, d\lambda = \frac{e^{a\lambda}}{a^2+x^2}\left[a \sin(\lambda x) + x \cos(\lambda x)\right]$

(2) $\int e^{a\lambda} \cos(\lambda x)\, d\lambda = \frac{e^{a\lambda}}{a^2+x^2}\left[a \cos(\lambda x) + x \sin(\lambda x)\right]$

(3) $\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

(4) $\int e^{ax} \sin(\lambda x)\, dx = \frac{e^{ax}}{a^2+\lambda^2}\left[a \sin(\lambda x) - \lambda \cos(\lambda x)\right]$

Fourier transforms -

An integral transform similar to Laplace transform,
$f(x)$ be piecewise conti. in $(-\infty, \infty)$ and assuming $f(x)$ is absolutely convergent i.e. $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges then,
Fourier transform of $f(x)$ is defined as,

$$\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\,dx = F(\lambda)$$

then $$\mathcal{F}^{-1}(F(\lambda)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x}\,d\lambda = f(x)$$ (Inverse fourier transform).

Properties →

1. $\mathcal{F}[f(x-a)] = e^{-ia\lambda} F(\lambda)$ Shifting on x-axis
2. $\mathcal{F}[e^{ixa} f(x)] = F(\lambda - a)$ Frequency shifting
3. $\mathcal{F}[f(x)\cos(ax)] = \frac{1}{2}[F(\lambda+a) + F(\lambda-a)]$
4. $\mathcal{F}[f(x)\sin(ax)] = \frac{1}{2}[F(\lambda+a) - F(\lambda-a)]$
   {Modulation theorem}
5. $\mathcal{F}[f(x)] = 2\pi f(-\lambda)$ Symmetry property
6. $\mathcal{F}[a f(x) + b g(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$ Linearity property

Note:
① Here $\mathcal{F}[f(x)] = F(\lambda)$ and $a \in R$
② Most of these properties are similar to Laplace properties.

Fourier transform of derivatives →

$\mathcal{F}[f(x)] = F(\lambda)$ then, $\mathcal{F}[f^n(x)] = (i\lambda)^n F(\lambda)$

Fourier transform of integral →

If above defi^n. of fourier transform holds under stated cond^n. and $F(0) = 0$ then $\mathcal{F}\left[\int_{-\infty}^{x} f(u)\,du\right] = \frac{1}{i\lambda} F(\lambda)$