Laplace Transforms

Laplace transforms are very helpful in solving many linear and non-linear ODEs, PDEs, integral equations etc.
Solution of a differential equation can be directly found without finding general sol^n (for which we need to find CF and PI).

Laplace transform of any func^n $f(t)$ is defined as,

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt = F(p) \quad \text{where } t \geq 0,\ p \to \text{parameter.}$$

(Provided integral exists)

Similarly, if $L\{f(t)\} = F(p)$ then $L^{-1}\{F(p)\} = f(t)$

Laplace is a linear operator so, linearity property holds →

$$L\{af(t) + b\,g(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$$

$$L^{-1}\{aF(p) + b\,G(p)\} = aL^{-1}\{F(p)\} + b\,L^{-1}\{G(p)\}$$

Laplace of some func^{n}s -

- $L\{1\} = \frac{1}{p}$, $p > 0$

Proof: By definition, $L\{1\} = \int_0^\infty e^{-pt} \times 1 \, dt \Rightarrow \int_0^\infty e^{-pt} dt$

$$= -\frac{1}{p}\left(e^{-pt}\right)_0^\infty \Rightarrow -\frac{1}{p}\left[0-1\right] \Rightarrow \frac{1}{p}$$

Here, $p > 0$ so that as $t \to \infty$, $e^{-pt} \to 0$

Proof: $L\{e^{at}\} = \int_0^\infty e^{-pt} \cdot e^{at}\, dt \Rightarrow \int_0^\infty e^{-(p-a)t} \cdot dt$

$$= \left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_0^\infty \Rightarrow \frac{1}{p-a}$$

$p > a$ so that as $t \to \infty$, $e^{-(p-a)t} \to 0$

Q. Find the Laplace transform of $f(t) = (1 + 2e^{3t})$

By linearity property, $L\{1 + 2e^{3t}\} = L\{1\} + 2L\{e^{3t}\}$

$$= \frac{1}{p} + 2 \times \frac{1}{p-3} \Rightarrow \frac{p-3+2p}{p(p-3)} \Rightarrow \frac{3(p-1)}{p(p-3)}$$

Q. Find the Laplace transform of $f(t) = t$.

$$L\{t\} = \int_0^\infty e^{-pt}\, t\, dt \Rightarrow \left(\frac{te^{-pt}}{-p} - \int \frac{e^{-pt}}{-p}\, dt\right)_0^\infty$$

$$= \left(\frac{te^{-pt}}{-p} - \frac{1}{p^2} e^{-pt}\right)_0^\infty \Rightarrow \left(0 - 0 + 0 + \frac{1}{p^2}\right) \Rightarrow \frac{1}{p^2}$$

Here, $p > 0$ so that as $t \to \infty$, $e^{-pt} \to 0$

$\therefore \quad L^{-1}\left\{\frac{1}{p^2}\right\} = t$

Laplace transforms of some standard funcns.

(1) $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$, $p>0$, $n>-1$

Proof:

$L\{t^n\} = \int_0^\infty e^{-pt} t^n \, dt$

Put $pt = z$
$p\,dt = dz$

$= \int_0^\infty e^{-z} \left(\frac{z}{p}\right)^n \frac{dz}{p} \Rightarrow \frac{1}{p^{n+1}} \underbrace{\int_0^\infty z^n e^{-z} \, dz}_{\Gamma(n+1)}$

for $n \in N$
$\Gamma(n+1) = n!$

$L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$

$\therefore L\{t^n\} = \frac{n!}{p^{n+1}}$, $n \in N$

(2) $L\{\sin at\} = \frac{a}{a^2+p^2}$, $p>0$

Proof: we know,

$e^{iat} = \cos(at) + i\sin(at)$ —①

$e^{-iat} = \cos(at) - i\sin at$ —②

eqn ① - ②:

$e^{iat} - e^{-iat} = 2i\sin at \Rightarrow \sin at = \frac{e^{iat} - e^{-iat}}{2i}$

$\therefore L\{\sin at\} = \frac{1}{2i}\{L(e^{iat}) - L(e^{-iat})\}$

$= \frac{1}{2i}\left[\frac{1}{p-ia} - \frac{1}{p+ia}\right]$

$= \frac{1}{2i} \times \frac{2ia}{p^2+a^2} \Rightarrow \frac{a}{p^2+a^2}$ //

$L\{e^{at}\} = \frac{1}{p-a}$
here replace 'a' ↔ ia
and '-a' with $-ia$

$\therefore L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$

(3) $L\{\cos(at)\} = \dfrac{p}{p^2+a^2}$

Proof: $\cos(at) = \dfrac{e^{iat}+e^{-iat}}{2}$

$\therefore L\{\cos(at)\} = \dfrac{1}{2} L\{e^{iat}+e^{-iat}\}$

$= \dfrac{1}{2}\left[\dfrac{1}{p-ia}+\dfrac{1}{p+ia}\right] \Rightarrow \dfrac{1}{\cancel{2}}\left[\dfrac{\cancel{2}p}{p^2+a^2}\right] \Rightarrow \dfrac{p}{p^2+a^2}$ //

$\therefore L^{-1}\left\{\dfrac{p}{p^2+a^2}\right\} = \cos(at)$

(4) $L\{\sin(hat)\} = \dfrac{a}{p^2-a^2}$, $p>|a|$

Proof: $\sin\{hat\} = \dfrac{e^{at}-e^{-at}}{2}$

$L\{\sin(hat)\} = \dfrac{1}{2} L\{e^{at}-e^{-at}\}$

$= \dfrac{1}{2}\left(\dfrac{1}{p-a}-\dfrac{1}{p+a}\right) \Rightarrow \dfrac{a}{p^2-a^2}$ //

$\therefore L^{-1}\left\{\dfrac{a}{p^2-a^2}\right\} = \dfrac{1}{a}\sin(hat)$

(5) $L\{\cos(hat)\} = \dfrac{p}{p^2-a^2}$, $p>|a|$

Proof: $\cos(hat) = \dfrac{e^{at}+e^{-at}}{2}$

$L\{\cos(hat)\} = \dfrac{1}{2} L\{e^{at}+e^{-at}\}$

$= \dfrac{1}{2}\left[\dfrac{1}{p-a}+\dfrac{1}{p+a}\right] \Rightarrow \dfrac{p}{p^2-a^2}$

$\therefore L^{-1}\left\{\dfrac{p}{p^2-a^2}\right\} = \cos(hat)$

Finding Laplace inverse transforms using partial fraction method.

Q. $F(p) = \frac{p+2}{(p-1)(p^2+4)}$, find $L^{-1}\{F(p)\} = ?$

First we reduce $F(p)$ in partial fraction then find L^{-1}

$\therefore$

$$\frac{p+2}{(p-1)(p^2+4)} = \frac{A}{(p-1)} + \frac{Bp+C}{p^2+4}$$

$$p+2 = A(p^2+4) + (Bp+C)(p-1)$$

comparing the coeff. from both sides $\rightarrow$

$A+B=0$ -(1), $-B+C=1$ -(2), $4A-C=2$ -(3)

using (1), (2), (3) $\rightarrow$

$A = \frac{3}{5}$, $B = -\frac{3}{5}$, $C = \frac{2}{5}$

$$F(p) = \frac{3}{5(p-1)} + \frac{2-3p}{5(p^2+4)}$$

$$L^{-1}\{F(p)\} = \frac{3}{5} L^{-1}\left\{\frac{1}{p-1}\right\} + \frac{2}{5} L^{-1}\left\{\frac{1}{p^2+4}\right\} - \frac{3}{5} L^{-1}\left\{\frac{p}{p^2+4}\right\}$$

$$= \frac{3}{5} e^{t} + \frac{\cancel{2}}{5} \frac{\sin(2t)}{\cancel{2}} - \frac{3}{5} \cos(2t)$$

$$= \frac{1}{5}\left[3e^{t} + \sin(2t) - 3\cos(2t)\right]$$

Properties of Laplace transforms

(1) Shifting property

- First translation or shifting property :

if $L\{f(t)\} = F(p)$ then $L\{e^{at} f(t)\} = F(p-a)$

Proof: $L\{e^{at} f(t)\} = \int_0^\infty e^{-pt} e^{at} f(t)\, dt$

$= \int_0^\infty e^{-(p-a)t} f(t)\, dt$

let $\phi = p-a$
then $F(\phi) = F(p-a)$

$= \int_0^\infty e^{-\phi t} f(t) dt = F(\phi) = F(p-a)$ //

- Second translation or shifting property :

if $L\{f(t)\} = F(p)$ & $G(t) = \begin{cases} f(t-a) & , t>a \\ 0 & , t<a \end{cases}$

then $L\{G(t)\} = e^{-ap} F(p)$

Proof: $L\{G(t)\} = \int_0^\infty e^{-pt} G(t)\, dt$

$= \int_0^a e^{-pt} G(t)\, dt + \int_a^\infty e^{-pt} G(t) dt$

let $t-a = m$
$dt = dm$

$= 0 + \int_a^\infty e^{-pt} f(t-a)\, dt$

$= \int_0^\infty e^{-p(a+m)} f(m)\, dm \Rightarrow \int_0^\infty e^{-ap} (e^{-pm} f(m)\, dm)$

$= e^{-ap} F(p)$

ex. Find $L\{e^{-4t}\cosh 2t\}$

$\rightarrow$ $L\{\cos(h\,2t)\} = \dfrac{p}{p^2-4} = F(p)$

$\therefore$ $L\{e^{-4t}\cos(h\,2t)\} = \dfrac{p+4}{(p+4)^2-4}$ $\left[\begin{array}{c}\text{Replacing } p \text{ by } p-a \\ \text{here } a=-4\end{array}\right]$

ex. Find $L\{e^{-t}\sin^2 t\}$

$L\{\sin^2 t\} = L\left\{\dfrac{1-\cos 2t}{2}\right\} \Rightarrow \dfrac{1}{2}L\{1-\cos 2t\}$

$= \dfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{p}{p^2+4}\right) = F(p)$

$\therefore$ Replacing p by $p-a$, here $a=-1$

$L\{e^{-t}\sin^2 t\} = \dfrac{1}{2}\left(\dfrac{1}{p+1} - \dfrac{(p+1)}{(p+1)^2+4}\right)$

ex. Find $L\{e^{-2t}\sin\sqrt{t}\}$

we know, $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots$

$\therefore$ $\sin\sqrt{t} = \sqrt{t} - \dfrac{(\sqrt{t})^3}{3!} + \dfrac{(\sqrt{t})^5}{5!} - \dots$

$\therefore$ $L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \dfrac{1}{3!}L\{t^{3/2}\} + \dfrac{1}{5!}L\{t^{5/2}\} - \dots$

$= \dfrac{\Gamma 3/2}{p^{3/2}} - \dfrac{1}{3!}\dfrac{\Gamma 5/2}{p^{5/2}} + \dfrac{1}{5!}\dfrac{\Gamma 7/2}{p^{7/2}} - \dots$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{3!\,p^{5/2}} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{5!\,p^{7/2}} \cdots\cdots$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{3}{3!\,2p^{2/2}} + \frac{5\cdot 3}{5!\,2\cdot 2\cdot p^{4/2}}\cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{\left(\frac{1}{2^2 p}\right)}{1!} + \frac{\left(\frac{1}{2^2 p}\right)^2}{2!}\cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}}\, e^{-1/4p} = F(p)$$

$$\therefore\ L\{e^{-2t}\sin\sqrt{t}\} = \frac{\sqrt{\pi}\, e^{-1/4(p+2)}}{2(p+2)^{3/2}} \text{ //}$$

ex: $L\{G(t)\} = ?$ if $G(t) = \begin{cases} (t-1)^3, & t>1 \\ 0, & t<1 \end{cases}$

using second trans. property, if $G(t) = \begin{cases} (t-a1)^3, & t>1 \\ 0, & t<1 \end{cases}$

then $f(t) = t^3$

$$\therefore\ L\{f(t)\} = L(t^{3/1}) = \frac{\lfloor 3}{p^4} \Rightarrow \frac{6}{p^4}$$

$$\therefore\ L\{G(t)\} = e^{-p}\,\frac{6}{p^4} \text{ //}$$

(2) change of scale property :

if $L\{f(t)\} = F(p)$ then, $L\{f(at)\} = \frac{1}{a} F\left(\frac{p}{a}\right)$, $a \neq 0$

Proof: $L\{f(at)\} = \int_0^\infty e^{-pt} f(at)\, dt$

Put $at = z$

$a\,dt = dz$

$\frac{p}{a} = \phi$

$= \int_0^\infty e^{-p\left(\frac{z}{a}\right)} f(z) \frac{dz}{a} \Rightarrow \frac{1}{a}\int_0^\infty e^{-p\left(\frac{z}{a}\right)} f(z)\, dz$

$= \frac{1}{a}\int_0^\infty e^{-\phi z} f(z)\, dz$

$= \frac{1}{a} F(\phi) \Rightarrow \frac{1}{a} F\left(\frac{p}{a}\right)$

ex. 2 if $L\{f(t)\} = \frac{1}{p(p^2+1)}$, then $L\{e^{-2t} f(3t)\}$

→ using change of scale property,

$L\{f(3t)\} = \frac{1}{3} f\left(\frac{p}{3}\right) \Rightarrow \frac{1}{3} \cdot \frac{1}{\frac{p}{3}\left(\frac{p^2}{9}+1\right)}$

$= \frac{1}{\cancel{3}} \times \frac{\cancel{3} \times 9}{p(p^2+9)} \Rightarrow \frac{9}{p(p^2+9)}$

Now, using first shifting property,

$L\{e^{-2t} f(3t)\} = \frac{9}{(p+2)((p+2)^2+9)}$

Laplace of error function.

Error func is defined as, $\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du$.

we know, $e^{-u^2} = 1 - \frac{u^2}{1!} + \frac{(u^2)^2}{2!} - \frac{(u^2)^3}{3!} + \cdots$

$\therefore$

$$L\{\operatorname{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} L\left\{ \int_0^{\sqrt{t}} \left(1 - \frac{u^2}{1!} + \frac{(u^2)^2}{2!} - \frac{(u^2)^3}{3!} + \cdots\right) du \right.$$

$$= \frac{2}{\sqrt{\pi}} L\left\{ \sqrt{t} - \frac{t^{3/2}}{3 \cdot 1!} + \frac{t^{5/2}}{5 \cdot 2!} - \frac{t^{7/2}}{7 \cdot 3!} \cdots \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma 3/2}{p^{3/2}} - \frac{\Gamma 5/2}{3 \cdot p^{5/2}} + \frac{\Gamma 7/2}{5 \cdot 2 \cdot p^{7/2}} \cdots \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{3 \cdot p^{5/2}} + \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{5 \cdot 2 \cdot p^{7/2}} \cdots \right]$$

$$= \frac{1}{p^{3/2}} \left[1 - \frac{1}{2p} + \frac{1}{2} \cdot \frac{3}{4} \left(\frac{1}{p}\right)^2 \cdots \right]$$

$$= \frac{1}{p^{3/2}} \left[1 - \frac{1}{2p} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!} \left(\frac{1}{p^2}\right) \cdots \right]$$

$$= \frac{1}{p^{3/2}} \left[1 + \frac{1}{p} \right]^{-1/2}$$

$$= \frac{1}{p^{3/2}} \cdot \frac{p^{1/2}}{\sqrt{p+1}} \Rightarrow \frac{1}{p\sqrt{p+1}} = F(p)$$

(3) Laplace transform of derivatives

Let $f(t)$ is continuous $\forall\ t \geq 0$ and is of exponential order α. Let $f'(t)$ be piecewise continuous on $[0, \infty)$. Then, $L[f'(t)]$ exists for $p > \alpha$ and

$$\boxed{L\{f'(t)\} = pF(p) - f(0)}$$

and if $f(t)$, $f'(t)$ are continuous and are of exponential order $\forall\ t \geqslant 0$ and $f''(t)$ is piecewise continuous on $[0, \infty)$ then,

$$\boxed{L\{f''(t)\} = p^2 L\{f(t)\} - pf(0) - f'(0)}$$

Proof:

① $$L\{f'(t)\} = \int_0^\infty \underset{\mathrm{I}}{e^{-pt}} \underset{\mathrm{II}}{f'(t)}\, dt$$

$$= \left[e^{-pt} f(t)\right]_0^\infty + \int_0^\infty p e^{-pt} f(t)\, dt$$

$$= -f(0) + pL\{f(t)\}$$

$$= pF(p) - f(0)$$

② $$L\{f''(t)\} = \int_0^\infty \underset{\mathrm{I}}{e^{-pt}} \underset{\mathrm{II}}{f''(t)}\, dt$$

$$= \left(e^{-pt} f'(t)\right)_0^\infty + \int_0^\infty p e^{-pt} f'(t)\, dt$$

$$= -f'(0) + p\left(pF(p) - f(0)\right)$$

$$= p^2 F(p) - pf(0) - f'(0)$$

∴ In general, if $f(t), f'(t), \dots f^{(n-1)}(t)$ be continuous on $[0,\infty)$ and be of exponential order.
Suppose $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$ then

$$L\{f^{n}(t)\} = p^{n}F(p) - p^{(n-1)}f(0) - p^{(n-2)}f'(0) - \dots f^{(n-1)}(0).$$

ex. $L\{te^{2t}\} = ?$

→ So, $f(t) = te^{2t}$ ∴ $f'(t) = 2te^{2t} + e^{2t}$

using $L\{f'(t)\} = pF(p) - f(0)$

$$L\{2te^{2t} + e^{2t}\} = pL\{te^{2t}\} - 0$$

$$2L\{te^{2t}\} + L\{e^{2t}\} = pL\{te^{2t}\}$$

$$(2-p)L\{te^{2t}\} = -L\{e^{2t}\}$$

$$L\{te^{2t}\} = -\frac{1}{(p-2)(2-p)} \Rightarrow \frac{1}{(p-2)^2}$$

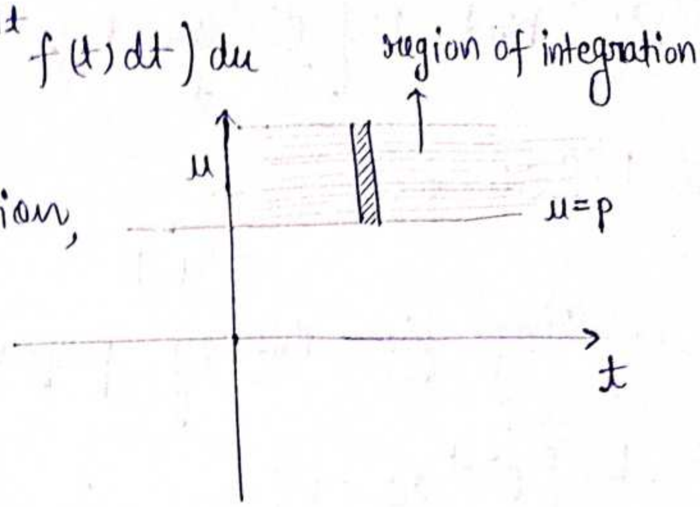
(4) Laplace transform of functions divided by 't'.

If $L\{f(t)\} = F(p)$ then

$$L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(u)\,du \quad \text{provided } \lim_{t\to 0^+} \frac{f(t)}{t} \text{ exists.}$$

Here $F(u)$ also $L\{f(t)\}$ only in terms of 'u'.

Proof: $\int_p^{\infty} F(u)\,du = \int_p^{\infty}\left(\int_0^{\infty} e^{-ut} f(t)\,dt\right) du$



changing the order of integration,

$$= \int_{t=0}^{\infty}\int_{u=p}^{\infty} e^{-ut} f(t)\,du\,dt$$

$$= \int_0^{\infty}\left(\frac{e^{-ut}}{-t}\right)_p^{\infty} f(t)\,dt \Rightarrow \int_0^{\infty} \frac{e^{-pt}}{t} f(t)\,dt = L\left\{\frac{f(t)}{t}\right\}$$

ex. Find $L\left\{\frac{\sinh t}{t}\right\} = ?$

$$L\{\sinh t\} = \frac{1}{p^2-1} = F(p)$$

$$\therefore L\left\{\frac{\sinh t}{t}\right\} = \int_p^{\infty} F(u)\,du$$

since, $F(p)$ and $F(u)$ are both $L\{f(t)\}$

$\therefore F(u) = \frac{1}{u^2-1}$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

$$= \int_p^{\infty}\left(\frac{1}{u^2-1}\right) du$$

$$= \frac{1}{2}\left[\ln\left|\frac{u-1}{u+1}\right|\right]_p^{\infty}$$

$$= \frac{1}{2}\left[0 - \ln\left|\frac{p-1}{p+1}\right|\right]$$

$$= \frac{1}{2}\ln\left|\frac{p+1}{p-1}\right|$$

ex: Find $L\left\{\frac{e^{-2t}\sin 3t}{t}\right\}$

$L\{\sin 3t\} = \frac{3}{p^2+9}$

using first shifting property →

$L\{e^{-2t}\sin 3t\} = \frac{3}{(p+2)^2+9}$

$L\{e^{at}f(t)\} = F(p-a)$

here $a = -2$

Now,

$L\left\{\frac{e^{-2t}\sin 3t}{t}\right\} = \int_p^{\infty} F(u)\,du$

$= \int_p^{\infty} \frac{3}{(u+2)^2+9}\,du = \int_{2+p}^{\infty} \frac{3\,dm}{m^2+9}$

Put $2+u = m$

$du = dm$

$= \cancel{3} \times \frac{1}{\cancel{3}}\left[\tan^{-1}\left(\frac{m}{3}\right)\right]_{2+p}^{\infty} \Rightarrow \frac{\pi}{2} - \tan^{-1}\left(\frac{p+2}{3}\right)$

(5) Laplace transform of func^n mult. by 't^n'.

$L\{f(t)\} = F(p)$. then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{dp^n}(F(p))$

ex: Find $L\{te^{-2t}\sin 4t\} = ?$

Here $f(t) = e^{-2t}\sin 4t$

$\therefore L\{f(t)\} = \frac{4}{(p+2)^2+16} = F(p)$

now, $F'(P) = \dfrac{[(P+2)^2+16]\times 0 - 4[2P+4]}{[(P+2)^2+16]^2}$

$= \dfrac{-8(P+2)}{[(P+2)^2+16]^2}$

$\therefore L\{t e^{-2t} \sin 4t\} = (-1)^1 F'(P) \Rightarrow \dfrac{8(P+2)}{[(P+2)^2+16]^2}$

ex: Find $L^{-1}\{\tan^{-1}(\frac{2}{P})\}$

here, $F(P) = \tan^{-1}(\frac{2}{P})$

$\therefore F'(P) = \dfrac{1}{1+(\frac{2}{P})^2} \times \left(-\dfrac{2}{P^2}\right) \Rightarrow \dfrac{-2}{P^2+4}$

now, $L(t f(t)) = (-1)^1 F'(P)$

$\therefore L^{-1}\{F'(P)\} = -t f(t)$

$L^{-1}\left\{\dfrac{\cancel{-}2}{P^2+4}\right\} = \cancel{-} t f(t)$

$\dfrac{\sin 2t}{t} = f(t)$ //

(6) Laplace transforms of integrals

If $L\{F(t)\} = F(P)$ then $L\left[\int_0^t f(u)\,du\right] = \frac{1}{P}F(P)$

Ex $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\} = ?$

here $f(t) = t\int_0^{\infty} \frac{e^{-t}\sin t}{t}$

So,

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_p^{\infty} \frac{1}{(R+1)^2+1}\,dR$$

$\therefore$

$$\int_{p+1}^{\infty} \frac{dm}{m^2+1} \Rightarrow \tan^{-1}(m)\Big|_{p+1}^{\infty}$$

put, $R+1 = m$
$dR = dm$

$$= \frac{\pi}{2} - \tan^{-1}(p+1) = F(p)$$

Now, using transform of integrals,

$$L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\} = \frac{F(p)}{p} \Rightarrow \frac{\frac{\pi}{2} - \tan^{-1}(p+1)}{p}$$

$$\therefore L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\} = (-1)^1 \frac{d}{dp}\left(\frac{\frac{\pi}{2} - \tan^{-1}(p+1)}{p}\right)$$

$$= \frac{1}{p\left((p+1)^2+1\right)} + \frac{1}{p^2}\left(\frac{\pi}{2} - \tan^{-1}(p+1)\right)$$

Q. $\int_0^\infty t e^{-t} \sin^3 t \, dt = ?$

let us solve using Laplace transform →

if we say $f(t) = t \sin^3 t$ then $L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$

i.e. $L\{t \sin^3 t\} = \int_0^\infty e^{-pt}\, t \sin^3 t \, dt$

clearly, RHS reduces to given integral at $p=1$

$\therefore$ value of given integral $= L\{t \sin^3 t\}$ at $p=1$.

$\therefore L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$

$= \frac{3}{4} L\{\sin t\} - \frac{1}{4} L\{\sin 3t\}$

$= \frac{3}{4} \cdot \frac{1}{(p^2+1)} - \frac{1}{4} \cdot \frac{3}{(p^2+9)} = F(p)$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

$\therefore L\{t \sin^3 t\} = (-1) F'(p) = -\left[\frac{-3}{4} \times \frac{2p}{(p^2+1)^2} + \frac{3}{4} \cdot \times \frac{2p}{(p^2+9)^2}\right]$

$= \frac{3p}{2}\left[\frac{1}{(p^2+1)^2} - \frac{1}{(p^2+9)^2}\right]$

at $p=1$

$L\{t \sin^3 t\} = \frac{3}{2}\left[\frac{1}{4} - \frac{1}{100}\right] \Rightarrow \frac{96 \times 3}{400 \times 2} \Rightarrow \frac{9}{25}$

* Convalution Theorem for Laplace transforms

Let $f(t)$ and $g(t)$ be defined for $t \geq 0$ then the convolution of $f(t)$ and $g(t)$, denoted by $(f*g)t$ is defined as,

$$(f*g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

• convalution of two functions obeys commutative property

Proof: By the definition, $(g*f)(t) = \int_0^t g(u) f(t-u)\, du$

now put $t-u = z \Rightarrow -du = dz$

$$(g*f)(t) = -\int_t^0 g(t-z) f(z)\, dz \Rightarrow \int_0^t f(z)\, g(t-z)\, dz = (f*g)t$$

i.e. $(f*g) = (g*f)$

Similarly, convalution of f and g is associative & distributive

ie. $f*(g*h) = (f*g)*h$ and $f*(g+h) = f*g + f*h$

$\therefore$ ① $f*0 = 0*f = 0$ but ② $f*1 = 1*f = \begin{cases} \text{may be 'f'} \\ \text{may not be 'f'} \end{cases}$

Now, convolution theorem - If $L\{f(t)\} = F(p)$ and $L\{g(t)\} = G(p)$ then,

$$L\{(f*g)(t)\} = L\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = F(p)\, G(p)$$

ex. $L^{-1}\left\{\frac{P}{(P^2+4)^2}\right\}$

$$L^{-1}\left\{\frac{P}{(P^2+4)}\cdot\frac{1}{(P^2+4)}\right\}$$

$$\underset{F(P)}{\downarrow} \qquad \underset{G(P)}{\downarrow}$$

$$L^{-1}\{F(P)\} = L^{-1}\left\{\frac{P}{P^2+4}\right\} = \cos 2t = f(t)$$

$$L^{-1}\{G(P)\} = L^{-1}\left\{\frac{1}{P^2+4}\right\} = \frac{1}{2}\sin 2t = g(t)$$

using convolution theorem, $L^{-1}[F(P)\,G(P)] = (f*g)(t)$

$$(f*g)(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \cos 2u \; \frac{1}{2}\sin 2(t-u)\,du \Rightarrow \frac{1}{4}\int_0^t 2\sin 2(t-u)\cdot\cos 2u\,du$$

$$= \frac{1}{4}\int_0^t [\sin 2t + \sin(2t-4u)]\,du$$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$$= \frac{1}{4}\left[\sin 2t\,(u)_0^t + \left(\frac{\cos(2t-4u)}{4}\right)_0^t\right]$$

$$= \frac{1}{4}\left[t\sin 2t + \frac{1}{4}(\cos 2t - \cos 2t)\right]$$

$$= \frac{1}{4}(t\sin 2t)$$

//

* Laplace transform of periodic functions

suppose $f(t)$ has a period $T > 0$ then,

$$L[f(t)] = \frac{\int_0^T e^{-Pt} f(t)\,dt}{1-e^{-PT}}$$

$f(t)$ is periodic with Period 'T'

i.e. $f(t+T) = f(t)$ -①

where 'T' is smallest R^+ satisfying eqn ① is the period of $f(t)$

ex. Find the Laplace transform of

$$f(t) = \begin{cases} K & , 0 \le t < a \\ -K & , a \le t < 2a \end{cases}$$

Period $= 2a$

$$L\{f(t)\} = \frac{\int_0^T e^{-Pt} f(t)\,dt}{1-e^{-PT}} = \frac{\int_0^{2a} e^{-Pt} f(t)\,dt}{1-e^{-2Pa}}$$

$$= \frac{1}{1-e^{-2Pa}}\left[\int_0^a e^{-Pt} K\,dt + \int_a^{2a} e^{-Pt}(-K)\,dt\right]$$

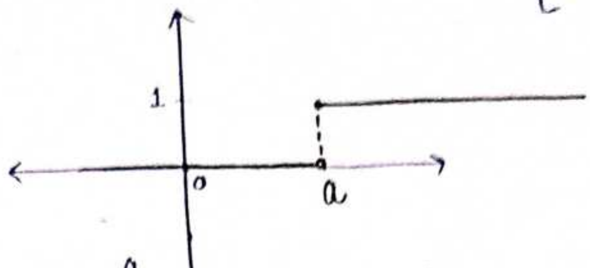
$$= \frac{K}{1-e^{-2Pa}}\left(\frac{(e^{-Pt})_0^a}{-P} + \frac{(e^{-Pt})_0^{2a}}{P}\right)$$

$$= \frac{K}{P(1-e^{-2aP})}\left[e^{-2aP} - \cancel{1} - e^{-aP} + \cancel{1}\right]$$

$$= \frac{K}{P}\left[\frac{e^{-2aP} - e^{-aP}}{1-e^{-2aP}}\right]$$

* Laplace transform of unit step function

Unit step function is defined as $u(t-a) = u_a(t) = \begin{cases} 0 , & t<a \\ 1 , & t \geq a \end{cases}$ at a point $t=a$.



Now,

$$L\{u_a(t)\} = \int_0^\infty e^{-pt} u_a(t)\,dt = \int_0^a e^{-pt} \times 0 \times dt + \int_a^\infty e^{-pt} \times 1 \times dt$$

$$= \int_a^\infty e^{-pt} \times 1 \times dt \Rightarrow \left(\frac{e^{-pt}}{-p}\right)_a^\infty \Rightarrow \frac{e^{-ap}}{p} \ //$$

- $L\{f(t-a)\, u_a(t)\} = e^{-ap} F(p)$ (Second shifting property)

here $F(p) = L\{f(t)\}$

ex: Find $L\{(t^2+1)\, u_1(t)\} = ?$

here $a=-1$, $a=1$ to apply second shifting property a must be same. so we express t^2+1 in terms of $t-1$

$\therefore$ $t^2+1 = (t-1+1)^2+1 \Rightarrow (t-1)^2+1+2(t-1)+1$

$t^2+1 = (t-1)^2 + 2(t-1) + 2$

now, $L\{((t-1)^2 + 2(t-1)+2)\, u_1(t)\}$

$= L\{(t-1)^2 u_1(t)\} + 2L\{(t-1)\, u_1(t)\} + 2L\{u_1(t)\}$

$= e^{-p}\dfrac{\lceil 2}{p^3} + 2e^{-p}\dfrac{\lceil 1}{p^2} + 2\dfrac{e^{-p}}{p}$

$= \dfrac{e^{-p}}{p}\left(\dfrac{1}{p^2} + \dfrac{2}{p} + 2\right)$

for $(t-1)^2$

$f(t) = t^2$

$\therefore L(t^2) = \dfrac{\lceil 2}{p^3}$

for $(t-1)$

$f(t) = t$

$L(t) = \dfrac{1}{p^2}$