Motilal Nehru National Institute of Technology, Allahabad
NUMERICAl METHODS AND STATISTICAL TECHNIQUES
Unit: Cubic Spline
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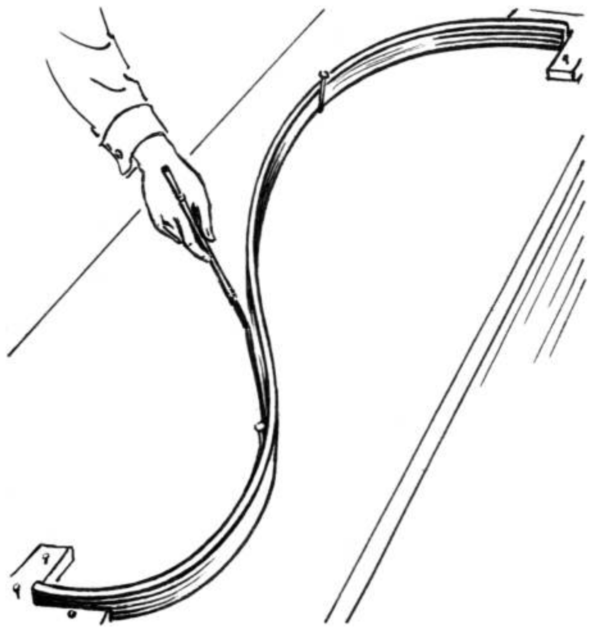
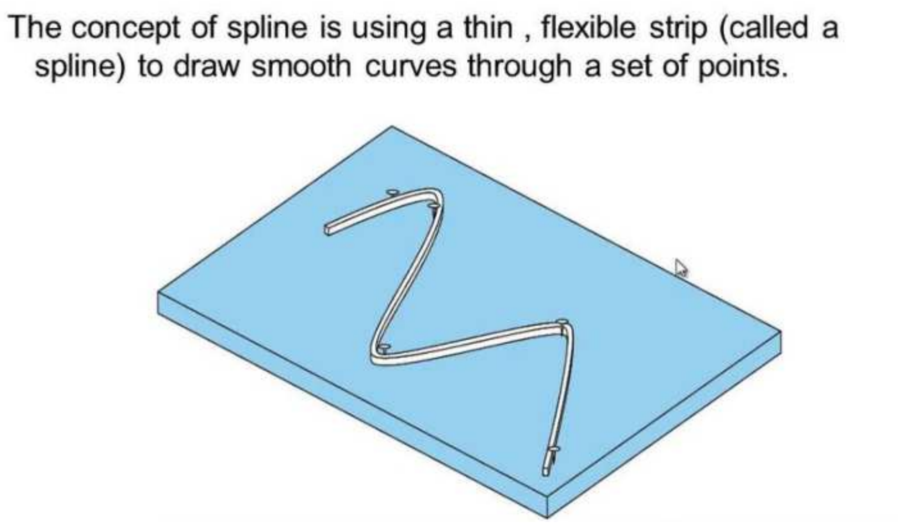
1 Introduction

In interpolation chapter we have discussed the method of finding nth order polynomial passing through $(n+1)$ given data points. In certain cases, this polynomials tend to give erroneous results due to round off and other errors. Further, it has been found that lower order polynomial approximation in each sub-interval provide a better approximation to the tabulated data than fitting a higher order polynomial in the entire interval. Such an interpolation is called piecewise polynomial interpolation and spline functions are such piecewise connecting polynomials.

The terms "spline" has been adopted following the draftsman's device of using a thin flexible strip (called a spline) to draw a smooth curve through given points. The points at which two connecting splines meet are called *knots*. Connecting polynomial may be of any degree. So we have different types of splines: linear, quadratic, cubic, quintic etc. Of these, the cubic spline has found to be most popular in engineering applications.





We first discuss little about the linear and quadratic spline and then will eventually justify the development of the cubic spline.

2 Linear Spline

Let the data points be $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$. Assume that $a = x_0 < x_1 < ... < x_n = b$ and let $h_i = x_i - x_{i-1}$, for $i = 1, 2, ..., n$.

Further, let $s_i(x)$ be the spline of degree one in the interval $[x_{i-1}, x_i]$. Obviously, $s_i(x)$ represents a straight line joining the points (x_{i-1}, y_{i-1}) and (x_i, y_i). Hence, we write

$$s_i(x) = y_{i-1} + m_i(x - x_{i-1}), \tag{1}$$

where

$$m_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}. \tag{2}$$

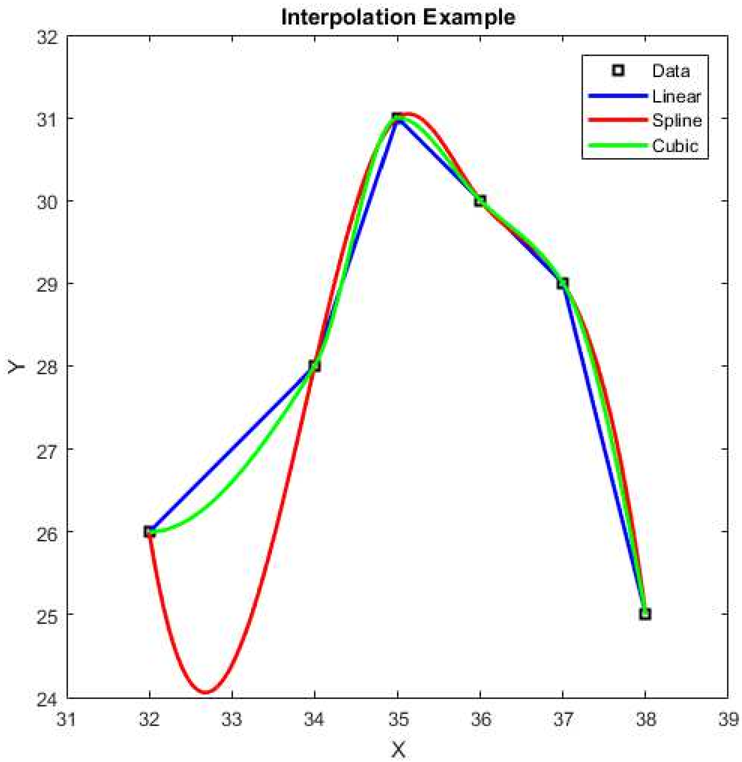
Setting $i = 1, 2, ..., n$ successively in equation (1), we obtain different splines of degree one valid in the subintervals 1 to n, respectively. It is easily seen that $s_i(x)$ is continuous at both the end points.

3 Quadratic Spline

In linear spline, the curve is continuous but slope is not continuous at every points. In quadratic spline, we will make sure that the spline curve as well as the slope of the spline curve is continuous. For this, in each subinterval the data points will be approximated by a second degree polynomial or a parabola.

4 Cubic Spline

In quadratic spline, the curve and the slope of the curve is continuous but the curvature is not continuous. In cubic spline the curve will be created in such a way that the curve, slope of the curve and the curvature of the curve is continuous.



A cubic spline $S(x)$ is defined by the following properties:
(i) $S(x_i) = y_i,\ i = 1, 2, ..., n$ (interpolation property)
(ii) $S(x)$, $S^{'}(x)$ and $S^{''}(x)$ are continuous on $[a, b]$.
(iii) $S(x)$ is a cubic polynomial say $s_i(x)$ in each sub-interval $[x_{i-1}, x_i]$ for $i = 1, 2, ..., n$.

From (i), (ii) and (iii), $S(x)$ is written as

$$S(x) = \begin{cases} s_1(x), & x_0 \leq x \leq x_1, \\ s_2(x), & x_1 \leq x \leq x_2, \\ \vdots & \vdots \\ s_n(x), & x_{n-1} \leq x \leq x_n. \end{cases}$$

How to find these $s_i(x)$: Since $s_i(x)$ is a cubic polynomial in $[x_{i-1}, x_i]$ for $i = 1, 2, ..., n$,

${s_i}''(x)$ is a linear polynomial. Let ${s_i}''(x) = ax + b$.

$$ {s_i}''(x) = ax + b. \tag{3} $$

Let us assume ${s_i}''(x_{i-1}) = M_{i-1}$ and ${s_i}''(x_i) = M_i$. So, from (3), we get

$$ \begin{aligned} ax_i + b &= M_i \\ ax_{i-1} + b &= M_{i-1}. \end{aligned} $$

Solving the above equations, we get

$$ \begin{aligned} a &= \frac{M_i - M_{i-1}}{h_i} \\ b &= \frac{M_{i-1}x_i - M_i x_{i-1}}{h_i}, \end{aligned} $$

where $h_i = x_i - x_{i-1}$. Substituting these values of a and b in (3), we get

$$ {s_i}''(x) = \frac{1}{h_i}[(x_i - x)M_{i-1} + (x - x_{i-1})M_i]. \tag{4} $$

Integrating (4) twice, we get

$$ s_i(x) = \frac{1}{h_i}[\frac{(x_i - x)^3}{6}M_{i-1} + \frac{(x - x_{i-1})^3}{6}M_i] + C_1 x + C_2, \tag{5} $$

where C_1 and C_2 are integrating constants. Since $s_i(x)$ passing through the points (x_{i-1}, y_{i-1}) and (x_i, y_i), we have $s_i(x_{i-1}) = y_{i-1}$ and $s_i(x_i) = y_i$. Hence,

$$ \begin{aligned} y_{i-1} &= \frac{1}{h_i}\frac{(x_i - x_{i-1})^3}{6}M_{i-1} + C_1 x_{i-1} + C_2, \\ y_i &= \frac{1}{h_i}\frac{(x_i - x_{i-1})^3}{6}M_i + C_1 x_i + C_2. \end{aligned} $$

Solving the above two equations, we get

$$ \begin{aligned} C_1 &= \frac{1}{h_i}(y_i - y_{i-1}) - \frac{h_i}{6}(M_i - M_{i-1}), \\ C_2 &= \frac{1}{h_i}(x_i y_{i-1} - x_{i-1} y_i) - \frac{h_i}{6}(x_i M_{i-1} - x_{i-1} M_i). \end{aligned} $$

Substituting C_1 and C_2 in (5), we get

$$ s_i(x) = \frac{1}{6h_i}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i + (x_i - x)(6y_{i-1} - M_{i-1}h_i^2) + (x - x_{i-1})(6y_i - M_i h_i^2)] \tag{6} $$

In (6), M_i are still unknown. To determine these, we use the condition of continuity of $s_i'(x)$.

Differentiating (6), we get

$$s_i^{'}(x) = \frac{1}{6h_i}[-3(x_i - x)^2 M_{i-1} + 3(x - x_{i-1})^2 M_i] + \frac{1}{h_i}(y_i - y_{i-1}) - \frac{h_i}{6}(M_i - M_{i-1}), \quad (7)$$

which is defined in $[x_{i-1}, x_i]$. Setting $i = i + 1$, we get

$$s_{i+1}^{'}(x) = \frac{1}{6h_{i+1}}[-3(x_{i+1} - x)^2 M_i + 3(x - x_i)^2 M_{i+1}] + \frac{1}{h_{i+1}}(y_{i+1} - y_i) - \frac{h_{i+1}}{6}(M_{i+1} - M_i), \quad (8)$$

which is defined in $[x_i, x_{i+1}]$.
Letting $s_i^{'}(x_i-) = s_{i+1}^{'}(x_i+)$, after simplification, we get

$$\frac{h_i}{6}M_{i-1} + \frac{h_i + h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{1}{h_{i+1}}((y_{i+1} - y_i)) - \frac{1}{h_i}(y_i - y_{i-1}), \quad (9)$$

for $i = 1, 2, ..., n-1$. The above system gives $n-1$ equation with $n+1$ unknowns $M_0, M_1, M_2, ..., M_n$. For natural spline, two additional conditions are taken in the following form: $M_0 = 0 = M_n$. Geometric meaning of these two conditions are that the spline curves are straight line at the beginning and the end.

4.1 Step To Obtain Cubic Spline For Given Data

• Hence, for the data points be $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_n, y_n)$, the cubic spline $S(x)$ is written as

$$S(x) = \begin{cases} s_1(x), & x_0 \le x \le x_1, \\ s_2(x), & x_1 \le x \le x_2, \\ \vdots & \vdots \\ s_n(x), & x_{n-1} \le x \le x_n. \end{cases}$$

where

$$s_i(x) = \frac{1}{6h_i}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i + (x_i - x)(6y_{i-1} - M_{i-1}h_i^2) + (x - x_{i-1})(6y_i - M_i h_i^2)] \quad (10)$$

and M_i are given by

$$\frac{h_i}{6}M_{i-1} + \frac{h_i + h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{1}{h_{i+1}}((y_{i+1} - y_i)) - \frac{1}{h_i}(y_i - y_{i-1}), \quad (11)$$

for $i = 1, 2, ..., n - 1$. Also, for natural spline: $M_0 = 0 = M_n$.

• If the data points are evenly spaced then $h_i = h_{i+1} = h$ (say). Then the above formulae can be written as

$$s_i(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i + (x_i - x)(6y_{i-1} - M_{i-1}h^2) + (x - x_{i-1})(6y_i - M_i h^2)] \tag{12}$$

and M_i are given by

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \tag{13}$$

for $i = 1, 2, ..., n-1$. Also, for natural spline: $M_0 = 0 = M_n$.

Example 1 Obtain the cubic spline for the following data points:

$x:$	0	1	2	3
$y:$	2	-6	-8	2

Solution Since the data points are evenly spaced,

$$S(x) = \begin{cases} s_1(x), & 0 \le x \le 1, \\ s_2(x), & 1 \le x \le 2, \\ s_3(x), & 2 \le x \le 3. \end{cases}$$

where,

$$s_i(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i + (x_i - x)(6y_{i-1} - M_{i-1}h^2) + (x - x_{i-1})(6y_i - M_i h^2)] \tag{14}$$

for $i = 1, 2$ and M_i are given by

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \tag{15}$$

for $i = 1, 2$. Also, for natural spline: $M_0 = 0 = M_3$ (for natural spline). Here $h = 1$. From (15), we get

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2),$$
$$M_1 + 4M_2 + M_3 = \frac{6}{h^2}(y_1 - 2y_2 + y_3).$$

Here,

$$M_0 + 4M_1 + M_2 = 6[2 - 2(-6) - 8] = 36$$
$$M_1 + 4M_2 + M_3 = 6[-6 - 2(-8) + 2] = 72.$$

Putting $M_0 = 0 = M_3$, we get

$$4M_1 + M_2 = 36$$
$$M_1 + 4M_2 = 72.$$

Solving we get $M_1 = 4.8$ and $M_2 = 16.8$.
Hence, for $0 \le x \le 1$, we get where,

$$\begin{aligned} s_1(x) &= \frac{1}{6h}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1 + (x_1 - x)(6y_0 - M_0 h^2) + (x - x_0)(6y_1 - M_1 h^2)] \\ &= \frac{1}{6}[(1 - x)^3 \times 0 + (x - 0)^3 \times 4.8 + (1 - x)(6 \times 2 - 0 \times 1^2) + (x - 0)(6 \times (-6) - 4.8 \times 1^2)] \\ &= \frac{1}{6}[4.8x^3 + 12(1 - x) + (-36 - 4.8)x] \\ &= 0.8x^3 - 8.8x + 2 \end{aligned}$$

For $1 \le x \le 2$, we get where,

$$\begin{aligned} s_2(x) &= \frac{1}{6h}[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2 + (x_2 - x)(6y_1 - M_1 h^2) + (x - x_1)(6y_2 - M_2 h^2)] \\ &= \frac{1}{6}[(2 - x)^3 \times 4.8 + (x - 1)^3 \times 16.8 + (2 - x)(6 \times (-6) - 4.8 \times 1^2) + (x - 1)(6 \times (-8) - 16.8 \times 1^2)] \\ &= 2x^3 - 3.6x^2 - 5.2x + 0.8 \end{aligned}$$

And, for $2 \le x \le 3$, we get where,

$$\begin{aligned} s_3(x) &= \frac{1}{6h}[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3 + (x_3 - x)(6y_2 - M_2 h^2) + (x - x_2)(6y_3 - M_3 h^2)] \\ &= \frac{1}{6}[(3 - x)^3 \times 16.8 + (x - 2)^3 \times 0 + (3 - x)(6 \times (-8) - 16.8 \times 1^2) + (x - 2)(6 \times 2 - 0 \times 1^2)] \\ &= -2.8x^3 + 25.2x^2 - 62.8x + 39.2 \end{aligned}$$

Hence, the cubic spline passing through the given data points is given by

$$S(x) = \begin{cases} 0.8x^3 - 8.8x + 2, & 0 \le x \le 1, \\ 2x^3 - 3.6x^2 - 5.2x + 0.8, & 1 \le x \le 2, \\ -2.8x^3 + 25.2x^2 - 62.8x + 39.2, & 2 \le x \le 3. \end{cases}$$