Unit - 4

Beta and Gamma function

✓ ① Improper Integrals : first and second kind
✓ ② Beta function
✓ ③ Gamma function

✓ ④ Improper integrals involving a parameter (Leibnitz formula)

Definite Integral :- $\int_a^b f(x)\,dx$ where

(i) a, b are finite.
(ii) $f(x)$ is bounded $\forall\ x \in [a,b]$.

Improper Integral :- If any condition (i) and (ii) is not satisfied.

① first type :- a, b are infinite
i.e
(i) $a \to \infty$ or
(ii) $b \to -\infty$ or
(iii) $a \to \infty$ & $b \to -\infty$

② second type :- $f(x)$ has infinite discontinuity in $[a,b]$.

Improper Integral of first kind :-

① $\int_a^{\infty} f(x)\,dx = \lim_{p\to\infty} \int_a^{p} f(x)\,dx$

② $\int_{-\infty}^{b} f(x)\,dx = \lim_{p\to-\infty} \int_p^{b} f(x)\,dx$

③ $\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{c} f(x)\,dx}_{\text{by ②}} + \underbrace{\int_c^{\infty} f(x)\,dx}_{\text{by ①}}$

Example 1 :- find value of

(i) $\int_0^{\infty} \frac{1}{a^2+x^2}\,dx \quad , \quad a>0$

Sol^n)

$I = \lim_{p\to\infty} \int_0^{p} \frac{1}{a^2+x^2}\,dx = \lim_{p\to 0} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^{p}$

$I = \lim_{p\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{p}{a}\right] = \frac{\pi}{2a}$

Convergent

(ii) $\int_{-\infty}^{0} e^x dx$

Sol^n) $I = \lim_{p\to -\infty} \int_p^0 e^x dx = \lim_{p\to -\infty} [1-e^p] = 1$

Convergence of Improper Integral :- (for first type)

① **Comparison Test 1 :-**

If $0 \le f(x) \le g(x)$ $\forall x$, then

(i) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

(ii) $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges

② **Comparison Test 2 :-**

Suppose $f(x) > 0$ and $g(x) > 0$ then

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, $0 < L < \infty$

Then the improper integral $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$ converges / diverges together.

Example 1:- Test for Convergent

(i) $\int_1^\infty e^{-x^2}\,dx$

Sol^n) as $e^{-x^2} < e^{-x} \quad \forall\ x > 1$

and $\int_1^\infty e^{-x}\,dx = \lim_{p\to\infty}\left[-e^{-x}\right]_1^p$

$= \lim_{p\to\infty}\left(\frac{1}{e} - e^{p}\right) = \frac{1}{e}$

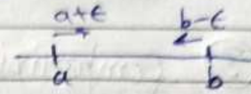
$\Rightarrow$ Convergent

hence $\int_1^\infty e^{-x^2}\,dx$ is also convergent by

Comparison Test.

Improper Integral of 2nd Type :-

$$I = \int_a^b f(x)\,dx$$



case (i) :- Infinite discontinuity at $x = a$

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)\,dx$$

case (ii) :- Infinite discontinuity at $x = b$

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)\,dx$$

case (iii) Infinite discontinuity at $x = a$ & $x = b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\epsilon \to 0} \int_{a+\epsilon}^{c} f(x)\,dx + \lim_{\epsilon \to 0} \int_c^{b-\epsilon} f(x)\,dx$$

case (iv) :- Infinite discontinuity at c, $a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon \to 0} \int_{c+\epsilon}^{b} f(x)\,dx$$

Convergence of Improper Integral :- (IInd type) :-

③ Comparision Test 3 :-

If $0 \leq f(x) \leq g(x) \quad \forall\, x \in [a,b]$ then

(i) $\int_a^b f(x)\,dx$ converges if $\int_a^b g(x)\,dx$ is convergent

(ii) $\int_a^b g(x)\,dx$ diverges if $\int_a^b f(x)\,dx$ is divergent.

④ Comparision Test 4 :-

If $f(x) \geq 0$, $g(x) > 0$ and if

(i) a is point of infinite discontinuity such that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = l_1$$

$$(0 < l_1 < \infty)$$

(ii) b is point of infinite discontinuity s.t

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(b-h)}{g(b-h)} = l_2$$

$$(0 < l_2 < \infty)$$

⇒ Then both integration $\int_a^b f(x)\,dx$ & $\int_a^b g(x)\,dx$

are Converges / diverges together.

Improper Integral involving a parameter :-

Consider,

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx \quad \text{---(i)}$$

where α is a parameter and integrand f is such that the integral can not be evaluated by standard method.

Leibnitz formula :-

If $a(\alpha)$, $b(\alpha)$, $f(x,\alpha)$ and $\frac{\partial f}{\partial \alpha}$ are

continuous function of α, then

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \left(\frac{\partial f}{\partial \alpha} + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} \right) dx \quad \text{----(2)}$$

if $\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx \quad \text{---(1)}$

Proof :- in Pdf (Math1_Unit4_L1.Pdf)

Example 1:- Evaluate the integral

$$\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx \quad , \; a>0$$

and hence deduce that

(i) $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ (ii) $\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$, $a>0$

Solⁿ)

let $\phi(a) = \int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$ ---(i)

$$\frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[e^{-ax}\frac{\sin x}{x}\right]dx = \lim_{p\to\infty}\int_0^p \text{''}\,dx$$

$$= -\int_0^\infty e^{-ax}\sin x\,dx$$

on integrating

$$\frac{d\phi}{da} = \lim_{p\to\infty}\left[\frac{e^{-ax}}{1+a^2}\left[\cos x + a\sin x\right]\right]_0^p$$

$$\boxed{\frac{d\phi}{da} = -\frac{1}{1+a^2}}$$

now

$$\phi = \int -\frac{1}{1+a^2}\,da = -\tan^{-1}a + c \quad ---(2)$$

where c is constant of integration

by using equ^n (i)

$$\lim_{a\to\infty} \phi(a) = 0$$

$$\Rightarrow \lim_{a\to\infty} \phi = \lim_{a\to\infty} -\tan^{-1} a + c = 0$$

$$\Rightarrow 0 = -\frac{\pi}{2} + c$$

$$c = +\frac{\pi}{2}$$

putting in (2)

$$\boxed{\phi = -\tan^{-1} a + \frac{\pi}{2}} \text{ or } \boxed{\phi = \cot^{-1} a}$$

Deduction:- of (i) & (ii)

(1) $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ (by putting $a=0$)

(2) at $x = ay$

$$\int_0^\infty \frac{\sin ax}{x}\, dx = \int_0^\infty \frac{a \sin y}{ay}\, dy = \frac{\pi}{2}$$

Beta function :-

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad , \ m, n > 0$$

① If $0 < m < 1$, $0 < n < 1$ $\Rightarrow$ improper integral (Convergent)

② If $m \geq 1$, $n \geq 1$ $\Rightarrow$ proper integral

Results :-

① $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

ⓐ If m is positive integer

$$\boxed{\beta(m,n) = \frac{(m-1)!}{n \cdot n+1 \cdots n+(m-1)}}$$

ⓑ If n is positive integer

$$\boxed{\beta(m,n) = \frac{(n-1)!}{m \cdot m+1 \cdot m+2 \cdots (m+n-1)}}$$

ⓒ If m, n are positive integers —

★★ $$\boxed{\beta(m,n) = \frac{(n-1)!\,(m-1)!}{(m+n-1)!}} = \left(\frac{\Gamma n \,\Gamma m}{\Gamma m+n}\right)$$

② $\beta(m,n) = \beta(n,m)$

③ Some forms of beta function-

ⓐ $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

ⓑ $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

ⓒ $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Proof:-

ⓐ As $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

putting $x = \frac{1}{1+y}$ $\Rightarrow$ $x=0 \Rightarrow y=\infty$

$x=1 \Rightarrow y=0$

$dx = \frac{-1}{(1+y)^2}\,dy$

$\beta(m,n) = -\int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1} \cdot \frac{1}{(1+y)^2}\,dy$

$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$\because (y=x)$

(b) Using $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

(c) $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

taking

$$x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$$

$$\& \quad x = 0 \Rightarrow \theta = 0$$

$$x = 1 \Rightarrow \theta = \pi/2$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta\,d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta}$$

Gamma function :-

$$\boxed{\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx} \quad , \quad n > 0$$

☆ (if $n > 0$, Γn is convergent)

Properties of gamma function :-

① $\Gamma 1 = 1$

② $\Gamma n+1 = n \Gamma n$

③ $\Gamma n+1 = n!$ if n is integer and $n > 0$

④ $\Gamma \frac{1}{2} = \sqrt{\pi}$

⑤ $\Gamma \frac{-1}{2} = -2\sqrt{\pi}$

as $\Gamma n+1 = n \Gamma n$

$\Rightarrow \Gamma \frac{1}{2} = \Gamma \frac{-1}{2}+1 = \frac{-1}{2} \Gamma \frac{-1}{2}$

$\sqrt{\pi} = \frac{-1}{2} \Gamma \frac{-1}{2}$

$\Rightarrow \boxed{\Gamma \frac{-1}{2} = -2\sqrt{\pi}}$

Relation b/w^n beta and gamma function –

** $$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

Proof:–

we have $\Gamma m = \int_0^\infty e^{-t} t^{m-1} dt$

putting $t = x^2$

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1} dx$$

similarly, $\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

$$\Gamma m \, \Gamma n = 4 \int_0^\infty e^{-x^2} x^{2m-1} dx \int_0^\infty e^{-y^2} y^{2n-1} dx$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\,dy$$

changing to polar form

$$= 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \, dr$$

$$= \Gamma m+n \;\; \beta(m,n)$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

Note:-

(1) $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

(2) If $p=0$, $q=n$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

(3) If $p=n$, $q=0$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

Transformation of Gamma Function –

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

① $$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$$

② $$\Gamma(n+1) = \int_0^\infty e^{-y^{1/n}} dy$$

③ $$\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

Proofs :-

① $$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

putting $x = zy$, z is constant
$dx = z\,dy$, $y = 0$ to ∞

$$\Gamma n = \int_0^\infty e^{-zy} z^{n-1} y^{n-1} z\,dy$$

$$= z^n \int_0^\infty e^{-zy} y^{n-1} dy$$

$$\boxed{\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zy} x^{n-1} dx}$$

by $x = y$
$dx = dy$

② $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Taking $x = y^{1/n} \Rightarrow x^n = y$

$n x^{n-1} dx = dy$

$\Gamma n = \int_0^\infty e^{-y^{1/n}} \frac{1}{n} dy$

$n\Gamma n = \int_0^\infty e^{-y^{1/n}} dy$

$$\boxed{\Gamma n+1 = \int_0^\infty e^{-y^{1/n}} dy}$$

③ $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$x = \log \frac{1}{y} \Rightarrow y = e^{-x}$

$dy = -e^{-x} dx$

$dx = -\frac{1}{y} dy$

$$\boxed{\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy}$$

Duplication formula :-

$$\boxed{\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m} \quad , m>0$$

Proof :- we know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{---(i)}$$

putting $n = \frac{1}{2}$

$$\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \quad \text{---(ii)}$$

putting $m = n$ in (i)

$$\beta(m,m) = \frac{2}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1} \, d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

Let $2\theta = \phi$

$d\theta = \frac{d\phi}{2}$

$$\beta(m,m) = \frac{1}{2^{2m-2}} \times 2 \int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi \quad \text{-(iii)}$$

by (ii) & (iii)

$$\beta(m,m) = \frac{1}{2^{2m-1}} \beta\left(m, \frac{1}{2}\right)$$

$$\boxed{\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{1}{2^{2m-1}} \sqrt{\pi} \, \Gamma 2m}$$

Example 1 :- Express in term of Gamma function-

(i) $\int_0^1 x^5 \left[\log \frac{1}{x}\right]^3 dx$

Sol^n)

$$I = \int_0^1 x^5 \left(\log \frac{1}{x}\right)^3 dx$$

$x^5 = e^{-t}$

put $x = e^{-t/5}$

$$I = \int_\infty^0 e^{-t}\left(\frac{t}{5}\right)^3 \cdot \frac{-1}{5} e^{-t/5}\, dt$$

$$I = \frac{1}{625}\int_0^\infty e^{-(1+\frac{1}{5})t} \cdot t^3\, dt$$

$$= \frac{1}{625}\int_0^\infty e^{-\frac{6}{5}t} \cdot t^3\, dt$$

Let $\frac{6}{5}t = x$

$t = \frac{5}{6}x$

$$= \frac{1}{625}\int_0^\infty e^{-x}\, \frac{5^4}{6^4} \cdot x^3\, dx$$

$$\boxed{I = \frac{1}{6^4}\int_0^\infty e^{-x} x^{(4-1)}\, dx}$$

$$I = \frac{1}{6^4}\Gamma 4 = \frac{1}{6^3}$$

(ii) $\int_0^\infty \frac{x^c}{c^x}\,dx$

Sol^n) $I = \int_0^\infty \frac{x^c}{c^x}\,dx$

$\cancel{\text{Let} \quad x\log c = t}$

$\cancel{dx = \frac{1}{\log c}\,dt}$

$$I = \int_0^\infty \frac{x^c}{e^{x\log c}}\,dx$$

$$I = \int_0^\infty e^{-x\log c} \cdot x^c\,dx$$

Let $\quad x\log c = t$

$dx = \frac{1}{\log c}\,dt$

$$I = \int_0^\infty e^{-t}\left(\frac{t}{\log c}\right)^c \frac{1}{\log c}\,dt$$

$$\boxed{I = \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t}\, t^c\,dt}$$

$$I = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

(iii) $\int_0^\infty a^{-bx^2}\,dx$

solⁿ) $I = \int_0^\infty e^{-bx^2 \log a}\,dx$

Let $x = \sqrt{\frac{1}{b\log a}}\sqrt{t}$

$bx^2 \log a = t$

$2bx \log a \; dx = dt$

$dx = \frac{1}{2\sqrt{tb\log a}}\,dt$

$$\boxed{I = \frac{1}{2\sqrt{b\log a}}\int_0^\infty e^{-t}\,t^{-1/2}\,dt}$$

$$I = \frac{1}{2\sqrt{b\log a}}\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{\frac{\pi}{b\log a}}$$

(iv) $\int_0^\infty e^{-x^3}\sqrt{x}\,dx$

solⁿ) Let $x^3 = t$

$x = t^{1/3}$

$dx = \frac{1}{3}t^{-2/3}\,dt$

$I = \int_0^\infty e^{-t}\cdot t^{1/6}\cdot \frac{1}{3}t^{-2/3}\,dt$

$$\boxed{I = \frac{1}{3}\int_0^\infty e^{-t}\, t^{-1/2}\, dt}$$

$$I = \frac{1}{3}\Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{3}$$

$$I = \cos(2ax) \int_0^\infty \frac{\sqrt{\pi}}{2}$$

$$x^2 = t$$

$$2x\,dx = dt$$

$$dx = \frac{1}{2\sqrt{t}}\,dt$$

$$\int_0^\infty e^{-t}\,\frac{\cos(2a\sqrt{t})}{2\sqrt{t}}\,dt$$

(v)