System of Linear equations

Methods to solve

Direct ~~Exact~~ Methods (Exact Methods)

1. Gauss's Elimination. Method
2. Gauss Jordan Method
3. Crout's Method (factorisation method) (LU Decomposition Method)

Indirect Methods (Iterative Methods)

1. Gauss-Jacobi Method
2. Gauss-seidel Method
3. Relaxation Method

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① Gauss's Elimination Method :-

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form, $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Argumented matric,

$$C = [A : B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Reduce augmented → echelon form

$$C = \left[\begin{array}{ccc|c} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{array}\right]$$

The corresponding system of equ^n are –

$$\boxed{\begin{aligned} c_{11}x + c_{12}y + c_{13}z &= d_1 \\ c_{22}y + c_{23}z &= d_2 \\ 0 \quad c_{33}z &= d_3 \end{aligned}}$$

→ solve the equ^n by back substitution.

eg)

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

sol^n) matrix form –

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 10 \end{bmatrix}$$

augmented matrix

$$C = [A \vdots B] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \end{array}\right]$$

echelon form –

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 2 & 1 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

Corresponding equⁿ are –

$$x + 3y + 2z = 5$$
$$-2y - 10z = -14$$
$$-9z = -9$$

$\boxed{z = 1}$

$$-2y - 10 = -14$$
$$2y + 10 = 14$$
$\boxed{y = 2}$

$$x + 6 + 2 = 5$$
$\boxed{x = -3}$

(2) Gauss-Jordan Method :

matrix form - $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Augmented matrix, $C = [A \mid B]$

$$= \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

by row transformations, convert augmented $\rightarrow$ unit matrix

Unit matrix :
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right]$$

Corresponding system of equ^n -

$$\boxed{\begin{array}{l} x = d_1 \\ y = d_2 \\ z = d_3 \end{array}}$$

Eg)
$$\begin{cases} x - 2y = -4 \\ -5y + z = -9 \\ 4x - 3z = -10 \end{cases}$$

Solⁿ) $AX = B$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -5 & 1 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \\ -10 \end{bmatrix}$$

Augmented matrix,

$$C = \left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 4 & 0 & -3 & -10 \end{array}\right]$$

<u>Unit matrix:</u>

$$C = \left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 0 & +8 & -3 & 6 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & -2/5 & -2/5 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 0 & -7/5 & -42/5 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & -2/5 & -2/5 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Corresponding equ^n

$$\boxed{\begin{array}{l} x = 2 \\ y = 3 \\ z = 6 \end{array}}$$

(3) LU Decomposition method :- (fails if any diagonal element of L or U = 0)

Principle :- Every square matrix A can be expressed as the product of a lower triangular matrix & an upper triangular matrix.

Matrix form— $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $A = L \cdot U$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} ; \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

then $LUX = B$

Let $UX = Y$

$$\boxed{LY = B} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

solve $LY = B$ get Y

solve $UX = Y$ get X Ans

Eg) $x+y+z=1$
$4x+3y-z=6$
$3x+5y+3z=4$

Sol^n) <u>matrix</u> :- $AX=B$

$$A=\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 3\end{bmatrix};\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix};\quad B=\begin{bmatrix}1\\ 6\\ 4\end{bmatrix}$$

Let $A=L\cdot U$

$$\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 3\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}\cdot u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

$u_{11}=1$, $u_{12}=1$, $u_{13}=1$
$l_{21}=4$ $l_{31}=3$ $l_{32}=-2$
$u_{22}=-1$, $u_{23}=-5$, $u_{33}=-10$

$A = LU$

$LUX = B$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Now $UX = y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

corresponding equ^n –

$$x + y + z = 1$$
$$-y - 5z = 2$$
$$-10z = 5$$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 1/2 \\ z &= -1/2 \end{aligned}}$$ Ans

(4) Gauss Jacobi Iteration Method -

principle: diagonal dominent property must be satisfied i.e.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Now Condition is -

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{32}| + |a_{31}|$$

if property is satisfied →
Rewriting the equ^n

$$x = \frac{1}{a_{11}}\left(b_1 - a_{12}y - a_{13}z\right)$$
$$y = \frac{1}{a_{22}}\left(b_2 - a_{21}x - a_{23}z\right)$$
$$z = \frac{1}{a_{33}}\left(b_3 - a_{31}x - a_{32}y\right)$$

perform iterations -

iteration 1: put $x = x_0$, $y = y_0$, $z = z_0$ (if initial value given in ques put that otherwise consider $x_0 = 0$, $y_0 = 0$, $z_0 = 0$)

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}y_0 - a_{13}z_0\right)$$
$$y_1 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_0 - a_{23}z_0\right)$$
$$z_1 = \frac{1}{a_{33}}\left(b_3 - a_{31}x_0 - a_{32}y_0\right)$$

iteration 2! put $x = x_1$, $y = y_1$, $z = z_1$ obtained from iteration 1

get (x_2, y_2, z_2) ~~and~~

repeat iteration ~~until~~ until 2 iterations value are not equal upto given decimal places.

(5) Gauss - seidel Method :-

Principle: diagonal dominance property must satisfied -

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewrite equ^n

$$x = \frac{1}{a_{11}} \left(b_1 - a_{12} y - a_{13} z \right)$$

$$y = \frac{1}{a_{22}} \left(b_2 - a_{21} x - a_{23} z \right)$$

$$z = \frac{1}{a_{33}} \left(b_3 - a_{31} x - a_{32} y \right)$$

Iteration 1:-

$$x_1 = \frac{1}{a_{11}} \left(b_1 - a_{12} y_0 - a_{13} z_0 \right)$$

$$y_1 = \frac{1}{a_{22}} \left(b_2 - a_{21} x_1 - a_{23} z_0 \right)$$

$$z_1 = \frac{1}{a_{33}} \left(b_3 - a_{31} x_1 - a_{32} y_1 \right)$$

using values in itself iteration Converges faster

Iteration 2:

$$x_2 = \frac{1}{a_{11}} \left(b_1 - a_{12} y_1 - a_{13} z_1 \right)$$

$$y_2 = \frac{1}{a_{22}} \left(b_2 - a_{21} x_2 - a_{23} z_1 \right)$$

$$z_2 = \frac{1}{a_{33}} \left(b_3 - a_{31} x_2 - a_{32} y_2 \right)$$

Continue iteration until 2 successive repeatation

Eg) $27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$

Soln) Rewriting equn -

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

Let
$(y_0 = 0, z_0 = 0)$

iteration 1:-

$$x_1 = \frac{85}{27} = 3.1481$$

$$y_1 = \frac{1}{15}\left(72 - 6 \times \frac{85}{27}\right) = 3.5408$$

$$z_1 = \frac{1}{54}\left(110 - \frac{85}{27} - 3.5408\right) = 1.9132$$

iteration 2:-

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.4322$$

$$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.5720$$

$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.9258$$

& so on - Ans in 4th iteration
$\boxed{x = 2.426, \quad y = 3.573, \quad z = 1.926}$

Basic

1. Positive Definite of matrix = all eigen value are greater than 0 (zero)
2. -ve " " " = " " less than zero
3. Indefinite = some +ve & some -ve

4. Necessary and sufficient condition for Convergence of Jacobian & Gauss Jordon -

Let
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_3 + a_{33}x_3 = b_3$$

$$Ax = b$$

$$x^{k+1} = mx^k + c$$

$$\begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \end{pmatrix} \pm \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$\downarrow$

M is called iteration matrix

condition

$\rho(m) < 1$

spectral Radius = M

eigen value of M is strictly less than n

(5) Gauss seidel Method is twice faster than Gauss Jacobian method

(6) sufficient condition for LU decomposition of matrix A = A = positive definite

Power Method - (used to find largest eigen value)

eg) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ find largest eigen value & Corresponding eigen vector using power method

Sol^n) Let

$$Y_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Y_1 = A\,Y_0 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

Let $K_1 = 2$

$$Z_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$Y_2 = AZ_1 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix} = 2.5 \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

$K_2 \; Z_2$

$$Y_3 = AZ_2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix}$$

$K_3 \; Z_3$

$$Y_4 = AZ_3 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix} = \underset{K_4}{2.93} \underset{Z_4}{\begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}}$$

$$Y_5 = AZ_4 = 2.98 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = K_5 Z_5$$

$$Y_6 = AZ_5 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = K_6 Z_6$$

here $\boxed{K_6 = \lambda = 3}$ largest eigen vector

& $\boxed{Z_6 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \text{eigen vector}}$

LL^T Decomposition / Choleski method –

if $Ax = b$

and A is a symmetric matrix & positive definite

then $\boxed{A = LL^T}$ $\quad L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$

$Ax = b$

$LL^T x = b$.

$\boxed{LY = b}$ get Y then $\boxed{L^T x = Y}$ get $\underline{\underline{x}}$