① $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$

② $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$

③ $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

④ $\sin x = \frac{e^{ix} - e^{-ix}}{2}$

⑤ $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ (gamma function)

⑥ $e^{isx} = (\cos sx - i \sin sx)$

# "Fourier series"

# Periodic function :- A $f^n$ repeat itself after a certain period.

# Properties of definite Integral :

(1) $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even}$$

$$= 0 \quad \text{if } f(x) \text{ is odd}$$

★ (2) Chain rule of integration :-

$$\int x^2 \sin 2x\,dx = x^2\left(\frac{-\cos 2x}{2}\right) - 2x\left(\frac{-\sin 2x}{4}\right) + 2\left(\frac{\cos 2x}{8}\right) - 0$$

diff diff

Int Int

only for 3 $f^n$

$$\boxed{\int (\text{algebric } f^n)(\sin/\cos/\text{expo})\,dx}$$



# Fourier series :- Let function $f(x)$ is defined and periodic in the interval $(a,b)$, Then its fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right)\right]$$

where

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2n\pi x}{b-a}\right)dx$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

Ques-1) $f(x) = x^2$ , $(-\pi, \pi)$ $(a = -\pi,\ b = \pi)$

$(b - a = 2\pi)$

Sol$^n$)

~~Order~~ fourier series will be -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{2\pi}\right) + b_n \sin\left(\frac{2n\pi x}{2\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad \text{---- (i)}$$

Now

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2x(\cos nx)}{(n^2)} - 2\left(\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$0 \qquad\qquad 0$$

$\left[\begin{array}{l}\text{as} \\ \sin n\pi = 0 \\ \& \\ \cos n\pi = (-1)^n\end{array}\right]$

$$a_n = \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx = \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{\sin nx}{n^2}\right) - 2\left(\frac{-\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{\sin nx}{n^2}\right) + \left(\frac{2\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$\Downarrow 0$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} + \frac{1}{n} - \frac{1}{n^3}\right] = 0$$

$x^2\sin(nx) =$ odd $f^n$

$$\boxed{b_n = \frac{1}{\pi}(0) = 0}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

So the required fourier series is

$$F(x) = \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos(nx) + 0\right]$$

$$F(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

Ans

# Fourier series for discontinuous function :-

Eg:

$$F(x) \rightarrow \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Sol^n) The function $f(x)$ is discont. at point $x=0$ in interval $(-\pi, \pi)$

$\boxed{\text{Period} = 2\pi}$

The fourier series in the general interval $(a,b)$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right)\right] \quad \text{---(i)}$$

here

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right] \quad \text{---(ii)}$$

where

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} F(x)\cos(nx)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos(nx)\,dx + \int_0^{\pi} x\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\sin(nx)}{n}\right)_{-\pi}^{0} + \left[x\,\frac{\sin(nx)}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cdot 0 + \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\boxed{a_n = \frac{(-1)^n - 1}{\pi n^2}}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin(nx)\,dx + \int_{0}^{\pi} x \sin nx \,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\pi\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \cancel{\frac{1}{n}\left((-1)^n - 1\right)} + \frac{1}{\pi}\left[\cancel{\pi \frac{(-1)^n}{n} - 0}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n}\left(1 - (-1)^n\right) - \frac{\pi}{n}(-1)^n\right]$$

$$= \frac{1}{n}\left[1 - (-1)^n - (-1)^n\right]$$

$$\boxed{b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - \pi^2 + \frac{\pi^2}{2} - 0\right]$$

$$\cancel{a_0 = \quad = \quad} \frac{1}{\pi}\left[-\frac{\pi^2}{2}\right]$$

$$\boxed{a_0 = -\frac{\pi}{2}}$$

putting values we get,

$$F(x) = -\frac{\pi}{2}\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{1}{n}\left[1 - 2(-1)^n\right]\sin(nx)\right]$$

$$F(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2 \pi}\cos(nx) + \frac{1}{n}\left(1 - 2(-1)^n\right)\sin(nx)\right]$$

Remark :- As function is discontinuous at $x=0$.
so we find its value at $x=0$ by

$$\boxed{F(a) = \frac{F(a+0)+f(a-0)}{2}}$$

$$\boxed{\begin{aligned} F(a+0) &= \lim_{x\to a^+} F(x) \\ F(a-0) &= \lim_{x\to a^-} F(x) \end{aligned}}$$

and

$$F(0) = \frac{F(0+0)+F(0-0)}{2} = \frac{-\pi+0}{2} = -\pi/2$$

Also we get →

$$F(0) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\right] = \frac{-\pi}{2}$$

# Half Range Cosine series :-

Let $F(x)$ is defined in the interval $(-l, l)$. The half range fourier Cosine series in interval $(0,l)$ is defined as –

$$\boxed{F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}$$

where,

$$a_n = \frac{2}{l}\int_0^l F(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_0 = \frac{2}{l}\int_0^l F(x)\,dx$$

# Half Range sine series :- The half range fourier sine series in interval $(0,l)$ is defined as –

$$\boxed{F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}, \quad \text{where } b_n = \frac{2}{l}\int_0^l F(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Q) Find half range sine and cosine series for $F(x) = x$ in interval $(0, \pi)$

Sol$^n$) Half range sine series —

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{--- (i)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \sin(nx)\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \frac{\sin(nx)}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} - \frac{\sin(n\pi)}{n^2}\right) - (-0 - 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n\right] = \frac{-2}{n}(-1)^n$$

Putting in equ$^n$ (i)

$$\boxed{F(x) = x = \sum_{n=1}^{\infty} \frac{(-2)}{n}(-1)^n \sin(nx)}$$

half range Cosine series —

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{--- (ii)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \cos(nx)\,dx = \frac{2}{\pi}\left[x \cdot \frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{\pi}{n}\sin(n\pi) + \frac{\cos(n\pi)}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left[0 + \frac{(-1)^n - 1}{n^2}\right] = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

putting in equ$^n$ (ii)

$$\boxed{F(x) = \frac{\pi}{2} + \frac{2}{n^2\pi}((-1)^n - 1)\cos(nx)}$$

**Fourier Transform :-** Let function $f(x)$ is given. The fourier transform of $f(x)$ is denoted by $F\{f(x)\}$ or $F(s)$ and given by-

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx} \quad \text{----(i)}$$

where $i = \sqrt{-1}$
$s$ = parameter

The Inverse fourier Transform of $F(s)$ is $f(x)$ and it is given by-

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds}$$

**Fourier sine Transform :-** The fourier sine transform of function $f(x)$ is denoted by $F_s(x)$ or $F_s\{f(x)\}$ and given by

$$\boxed{F_s(s) = \int_{0}^{\infty} f(x) \sin(sx)\, dx}$$



**Inverse fourier sine transform :-**

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_s(s) \sin(sx)\, ds$$

**Fourier Cosine Transform :-** " " Cosine " " $f(x)$ is " by $F_c(x)$ or $F_c\{f(x)\}$ and given by-

$$\boxed{F_c(s) = \int_{0}^{\infty} f(x) \cos(sx)\, dx}$$

**Inverse fourier Cosine transform :-**

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_c(s) \cos(sx)\, ds$$

# Parseval Identity for fourier series –

The fourier series of $f(x)$ in interval $(0,2l)$ is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

If fourier series of $F(x)$ converge uniformly in $(0,2l)$, then

$$\int_0^{2l} [F(x)]^2 dx = l \cdot \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

**Proof!** $$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Multiply both sides by $F(x)$ & integrate from $x=0$ to $2l$

$$\int_0^{2l} [F(x)]^2 dx = \frac{a_0}{2}\int_0^{2l} F(x)\,dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2l} F(x) \cos\left(\frac{n\pi x}{l}\right) dx + b_n \int_0^{2l} F(x) \sin\left(\frac{n\pi x}{l}\right) dx\right]$$

$$= \frac{a_0}{2} \cdot a_0 l + \sum_{n=1}^{\infty} \left[a_n . a_n . l + b_n . b_n . l\right]$$

$$\boxed{\int_0^{2l} [F(x)]^2 dx = l \cdot \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]}$$

① **Parseval for half range cosine series in $(0,l)$ :-**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$\int_0^{l} [F(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

② **Parseval for half range sine series in $(0,l)$ :-**

$$\int_0^{l} [F(x)]^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^2\right]$$

## Complex fourier Transform :-

Let $f(x)$ defined for all real value then complex fourier Transform of $f(x)$ is denoted by $F(s)$ or $F\{f(x)\}$ defined as

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx}$$

## Inverse fourier Transform :-

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds}$$

**E.g.:** find fourier transform of $F(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$

**Sol$^n$)** $f(x) = \begin{cases} x, & -a \le x \le a \\ 0, & -a > x > a \end{cases}$

by defination,

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = \int_{-\infty}^{-a} f(x) e^{isx}\, dx + \int_{-a}^{a} f(x) e^{isx}\, dx + \int_{a}^{\infty} f(x) e^{isx}\, dx$$

$$F(s) = \int_{-a}^{a} x e^{isx}\, dx = \left[ x\frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2} \right]_{-a}^{a}$$

$$F(s) = \left[ \frac{a e^{isa}}{is} + \frac{e^{isa}}{s^2} - \left( \frac{-a e^{-isa}}{is} + \frac{e^{-isa}}{s^2} \right) \right]$$

$$F(s) = \frac{a e^{isa}}{is} + \frac{e^{isa}}{s^2} + \frac{a e^{-isa}}{is} - \frac{e^{-isa}}{s^2}$$

$$= \frac{a}{is}\left(e^{isa} + e^{-isa}\right) + \frac{1}{s^2}\left(e^{isa} - e^{-isa}\right)$$

$$= \frac{2a}{si}(\cos sa) + \frac{1}{s^2}\, 2i \sin(sa)$$

$$\boxed{F(s) = \frac{2a}{is}\cos(sa) + \frac{2i}{s^2}\sin(sa)}$$

Ans

Eg: Find fourier transform of -

$f(x) = e^{-ax^2} \quad (a>0)$

Sol$^n$) by definition

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = \int_{-\infty}^{\infty} e^{-ax^2} \cdot e^{isx} dx = \int_{-\infty}^{\infty} e^{-ax^2+isx} dx$$

$$F(s) = \int_{-\infty}^{\infty} e^{-a\left(x^2 - \frac{isx}{a}\right)} dx$$

$$= \int_{-\infty}^{\infty} e^{-a\left(x-\frac{is}{2a}\right)^2 - \frac{s^2}{4a}} dx = e^{-\frac{s^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(x-\frac{is}{2a}\right)^2} dx$$

Let $a\left(x-\frac{is}{2a}\right)^2 = y^2$

$\sqrt{a}\left(x-\frac{is}{2a}\right) = y$

$\sqrt{a}\, dx = dy$

$dx = \frac{dy}{\sqrt{a}}$

$$= \frac{e^{-s^2/4a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{e^{-s^2/4a}}{\sqrt{a}} \sqrt{\pi}$$ Ans

**Fourier Integral:-** Let $F(x)$ satisfy the following cond$^n$ –

① $F(x)$ satisfy the Dirchlet cond$^n$ in every finite interval $-l \le x \le l$

② $\int_{-\infty}^{\infty} |F(x)|\,dx$ converges i.e $F(x)$ absolutely converges in $-l \le x \le l$

then fourier Int. theorem state that $F(x)$ is defined as –

$$F(x) = \int_0^{\infty} \left[ A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \right] d\lambda$$

where
$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(x)\cos(\lambda x)\,dx$$
$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(x)\sin(\lambda x)\,dx$$

(P. T. O for sine & cosine Integral)

① **Sine Transform of F.I:-** interval $(0 < x < \infty)$

$$f_s(\lambda) = \int_0^{\infty} F(u)\sin(\lambda u)\,du$$

**inverse:**
$$F(x) = \frac{2}{\pi} \int_0^{\infty} f_s(\lambda)\sin(\lambda u)\,du$$

② **Cosine Transform of F.I:-**

$$F_c(\lambda) = \int_0^{\infty} F(u)\cos(\lambda u)\,du$$

**inverse:**
$$F(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\lambda)\cos(\lambda u)\,du$$

XXX

## Complex form of fourier integral:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\, d\lambda \int_{-\infty}^{\infty} F(u)\, e^{-i\lambda u}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u)\, e^{-i\lambda(x-u)}\, du\, d\lambda$$

**Fourier Transform of $F(\lambda)$ →**

$$\boxed{f(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda u} F(u)\, du}$$

**Inverse fourier Transform of:-**

$$\boxed{F(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda u} f(\lambda)\, d\lambda}$$

---

☆ ① **Cosine Transform of F.I:**

$$F(x) = \frac{1}{\pi}\int_{0}^{\infty} \cos\lambda x \underbrace{\left[\int_{-\infty}^{\infty} F(x)\cos(\lambda x)\, dx\right]}_{A(\lambda)} d\lambda$$

② **sine Transform of FI:**

$$F(x) = \frac{1}{\pi}\int_{0}^{\infty} \sin\lambda x \underbrace{\left[\int_{-\infty}^{\infty} F(x)\sin(\lambda x)\, dx\right]}_{B(\lambda)} d\lambda$$