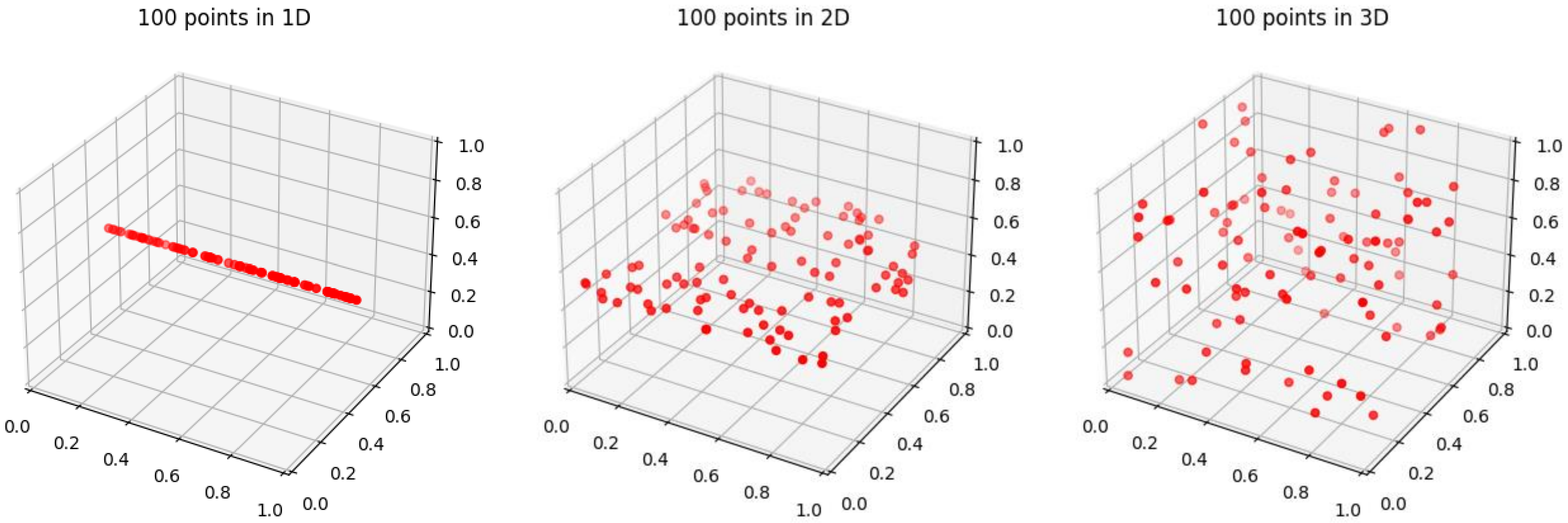
HIGH DIMENSIONS

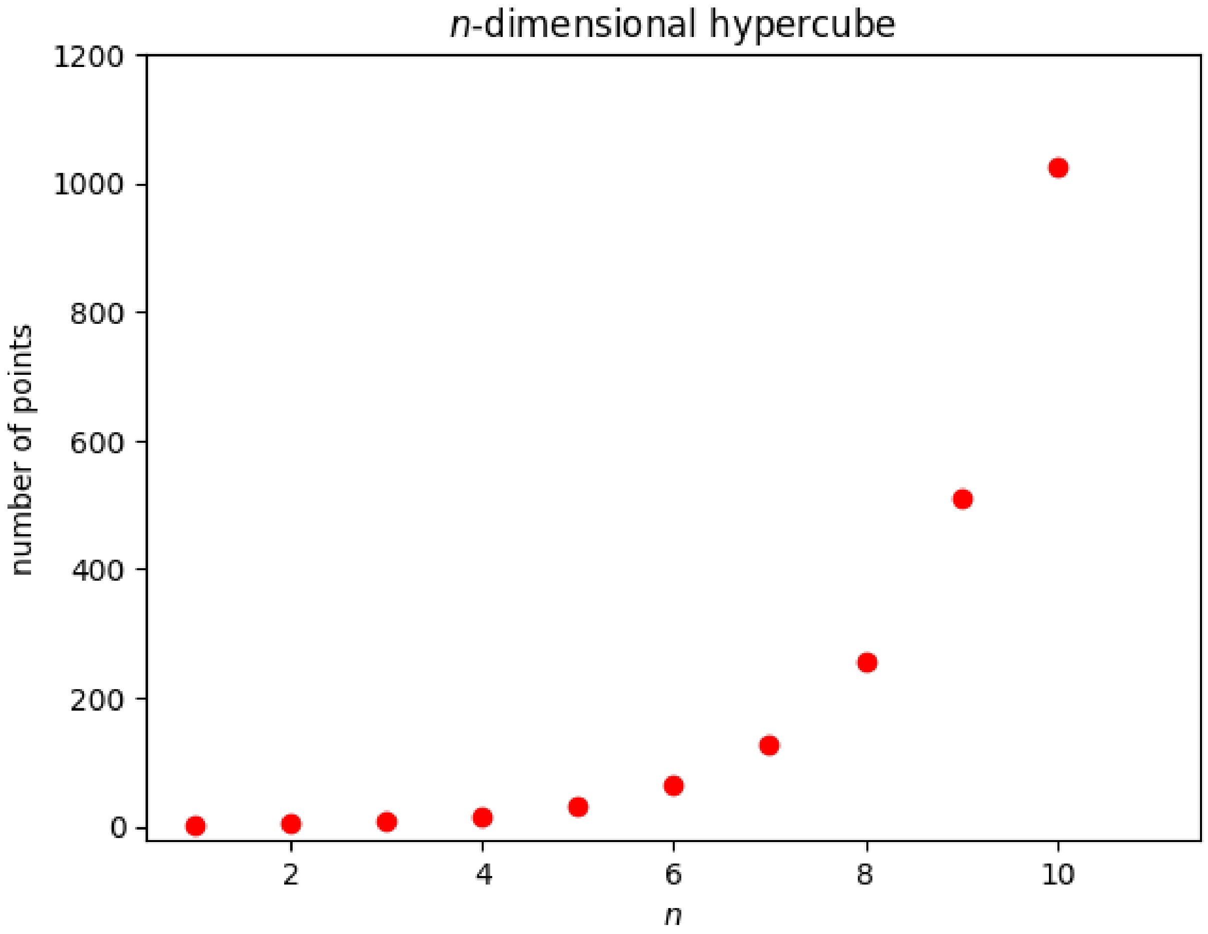
Do you understand where your models live?

Paul Dubois - MICS Christmas day 2023

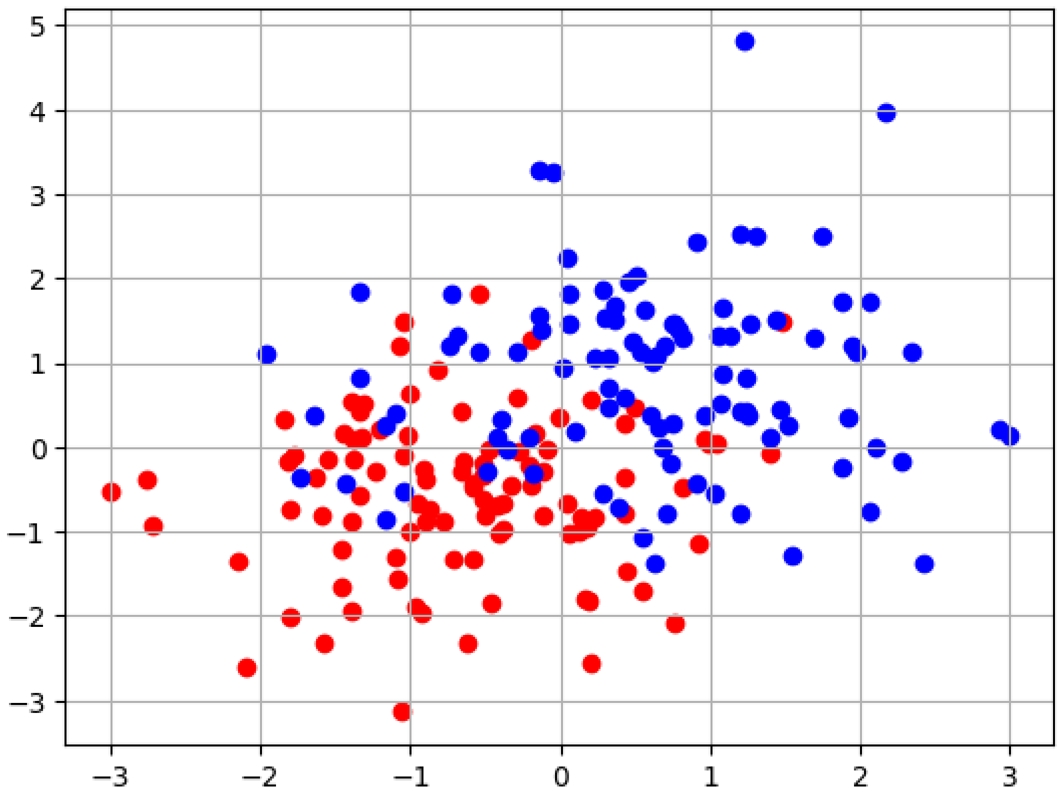
Curse of dimensionality

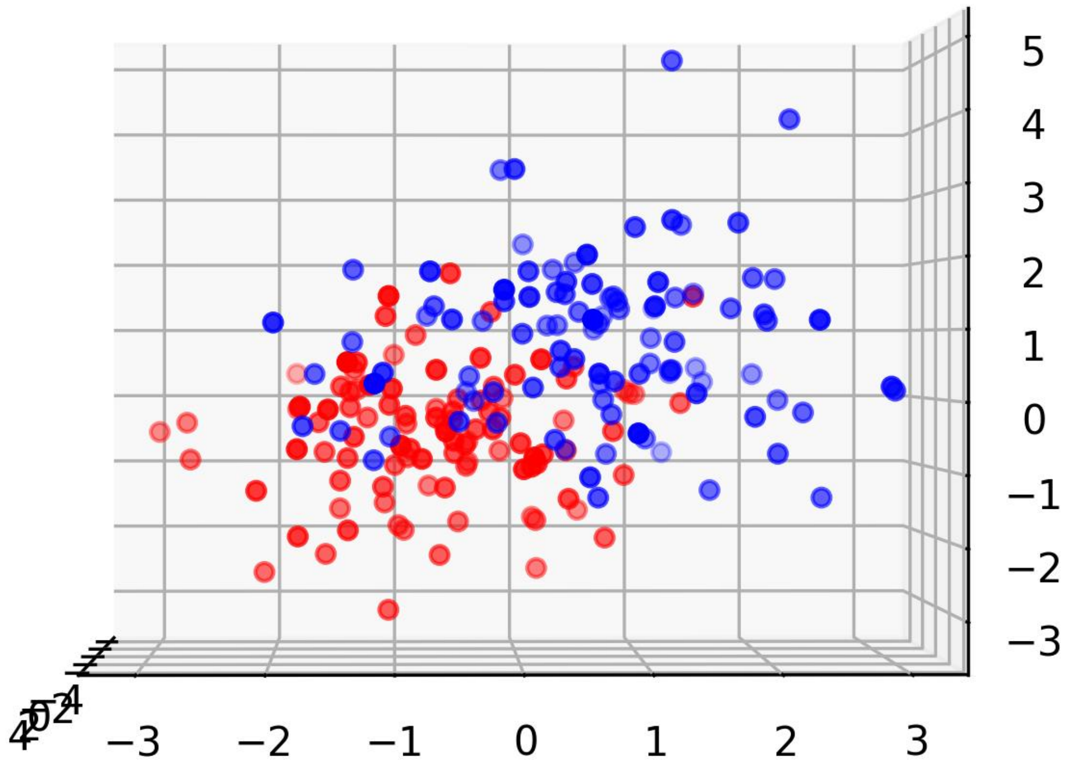


Curse of dimensionality



Blessing of dimensionality





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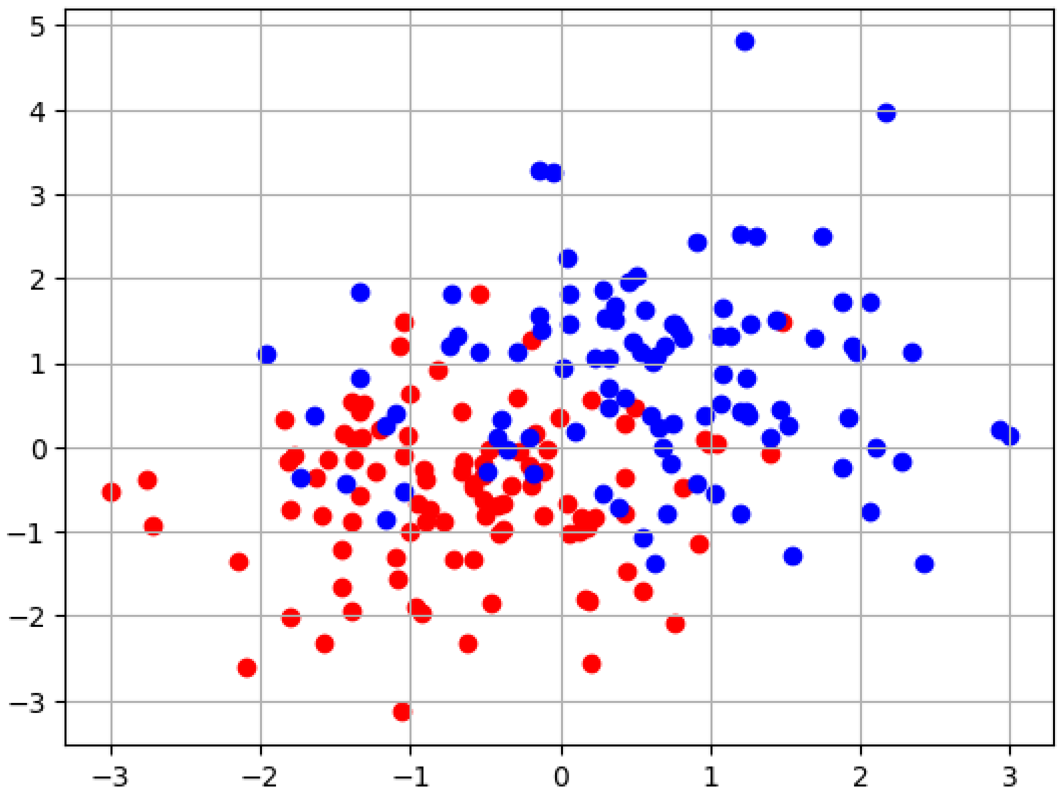

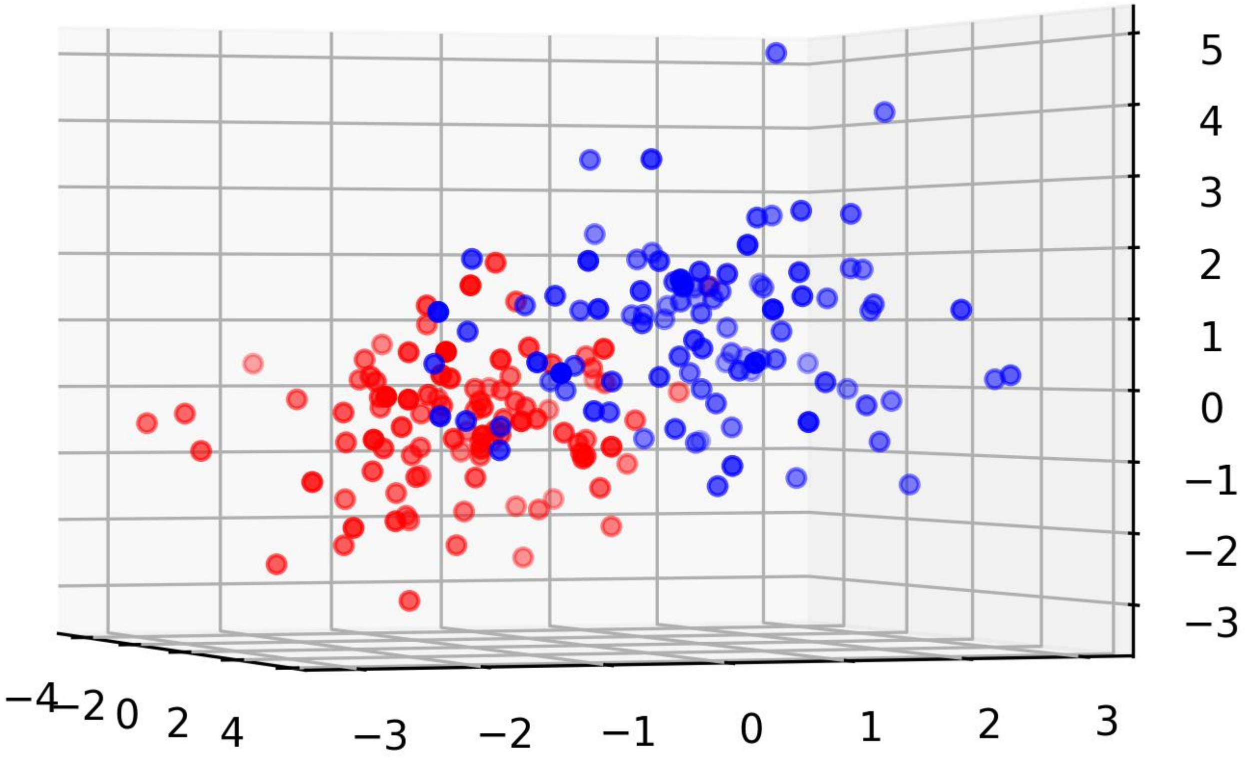

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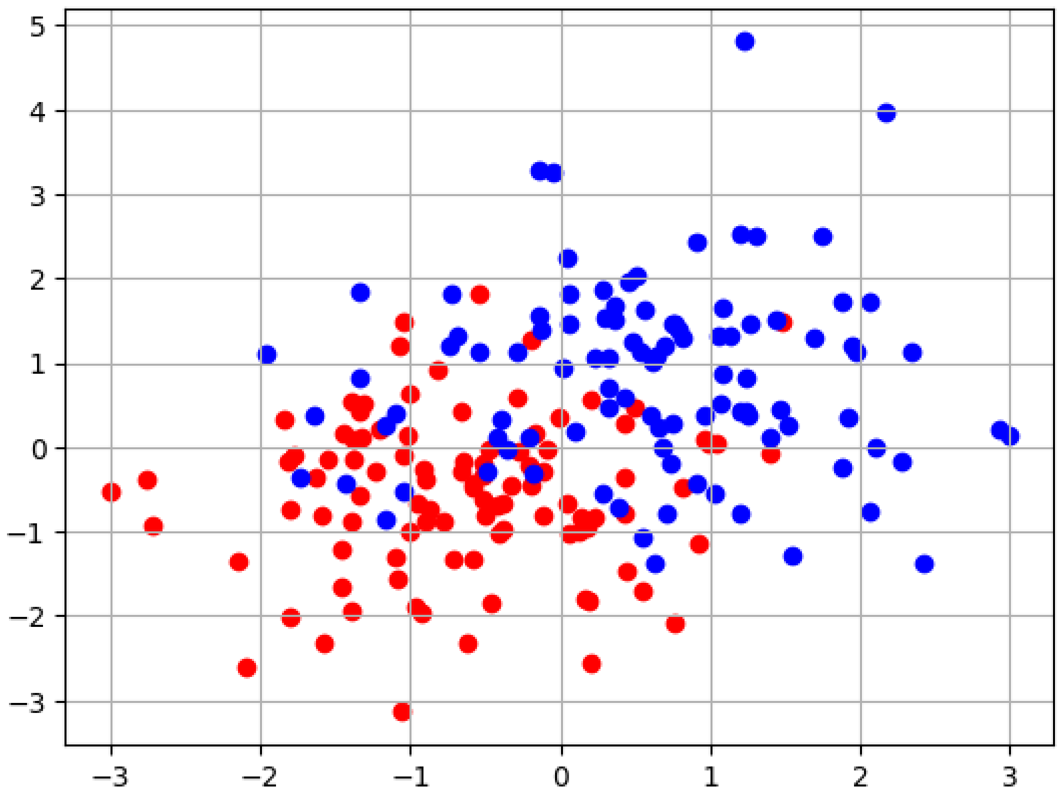

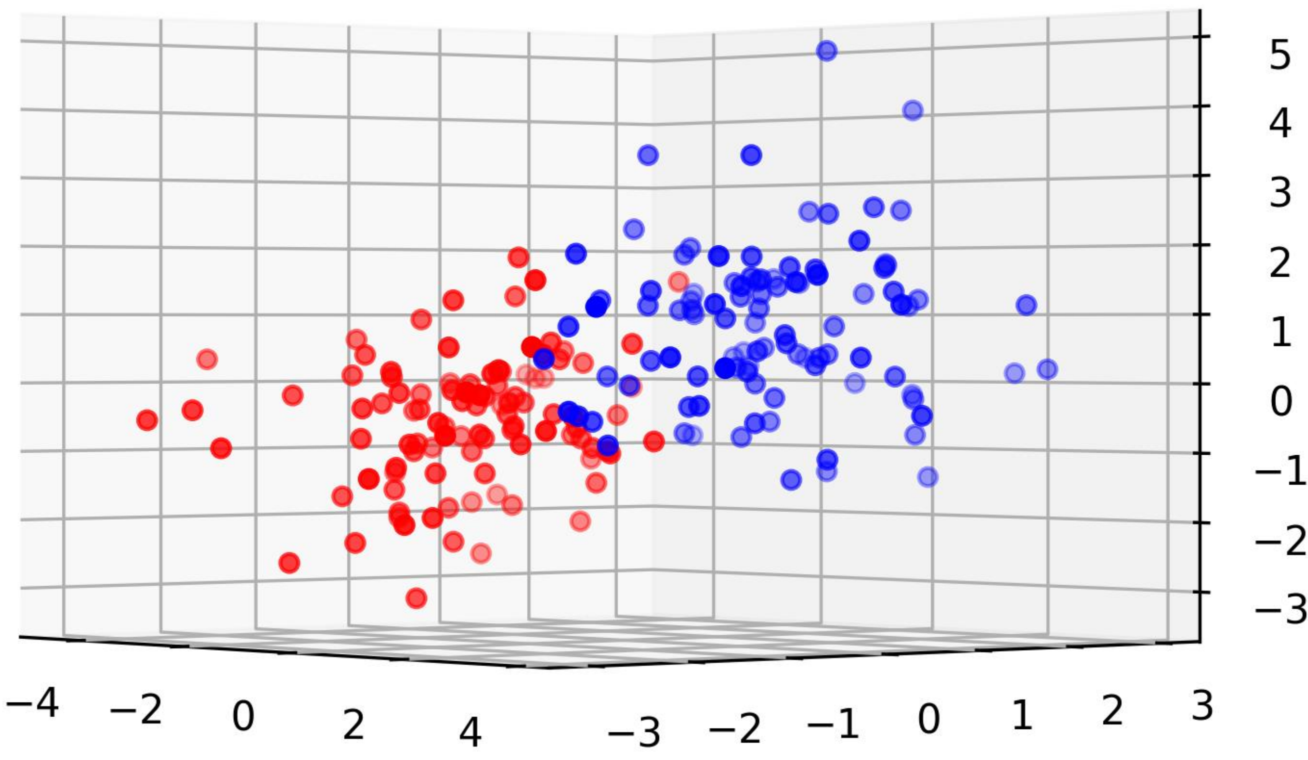

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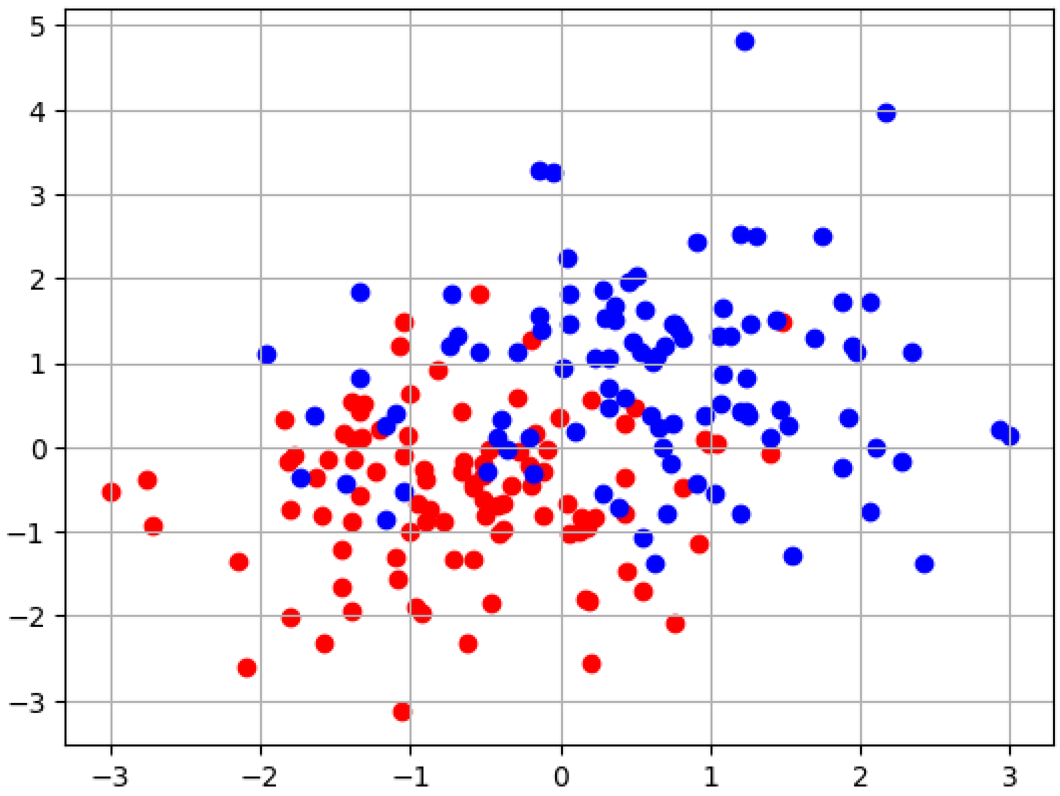

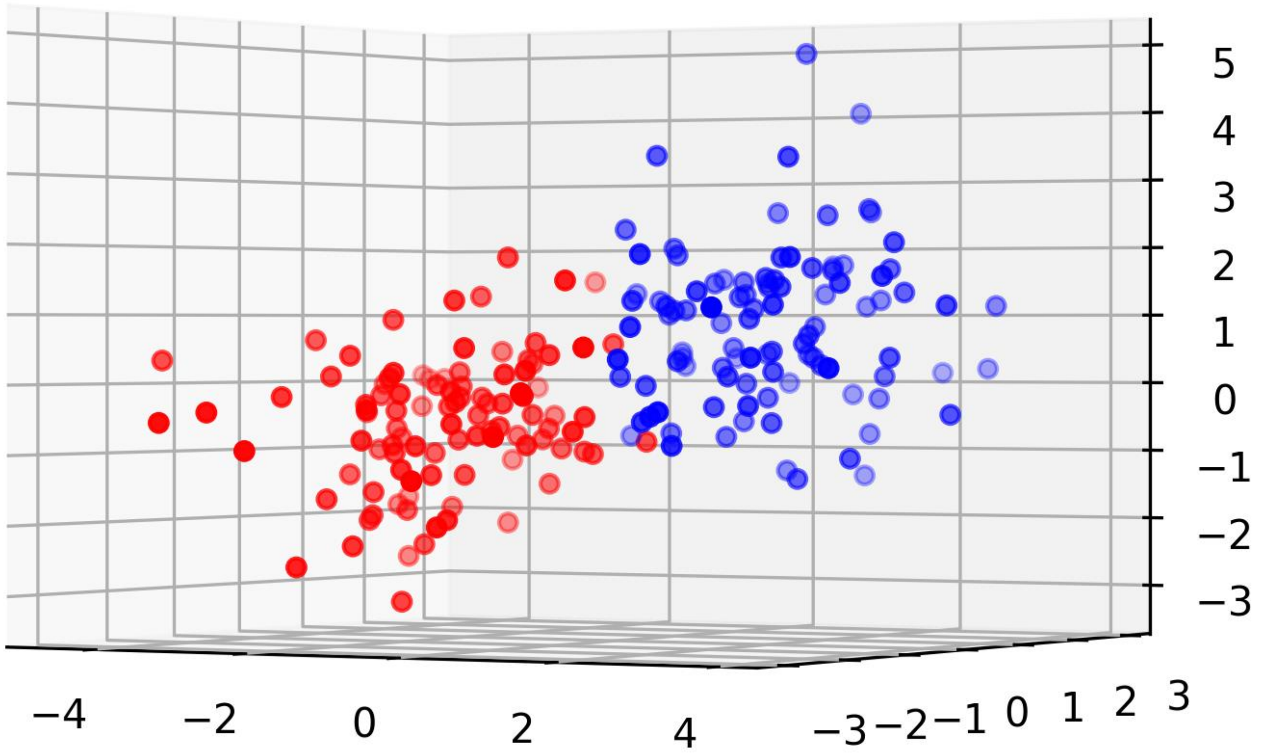

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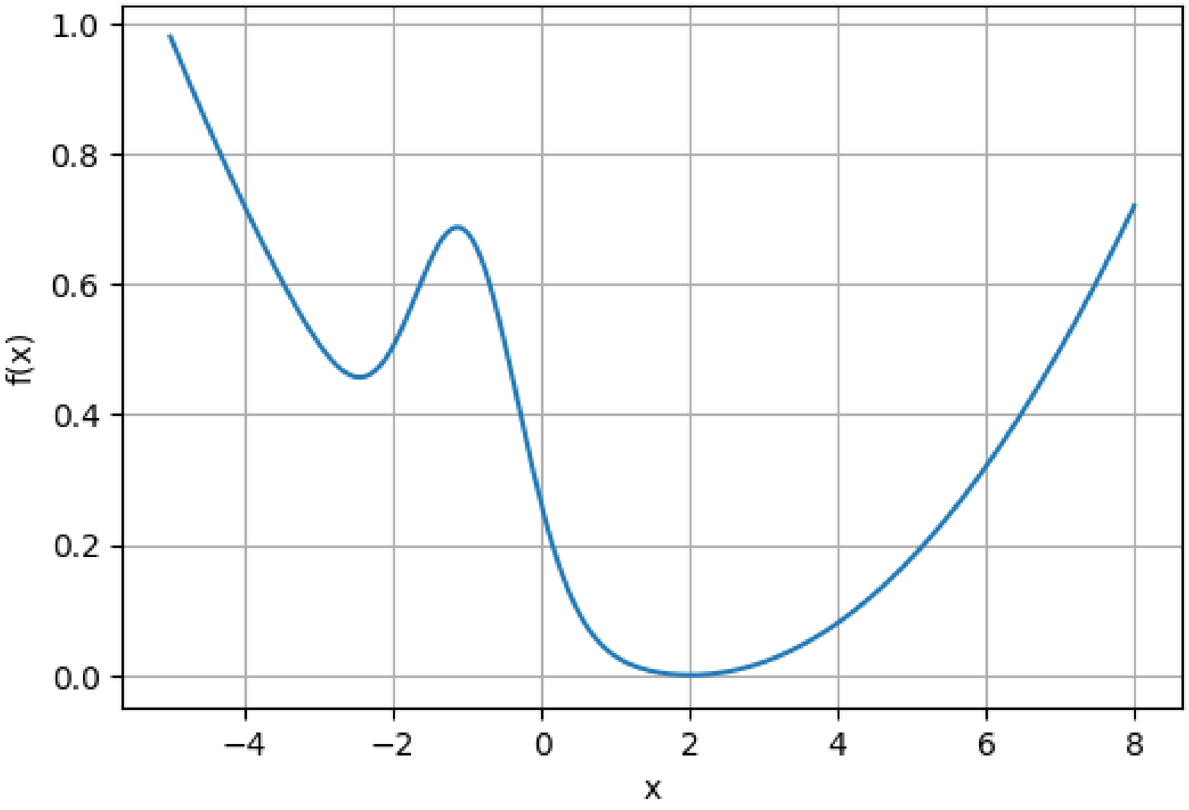
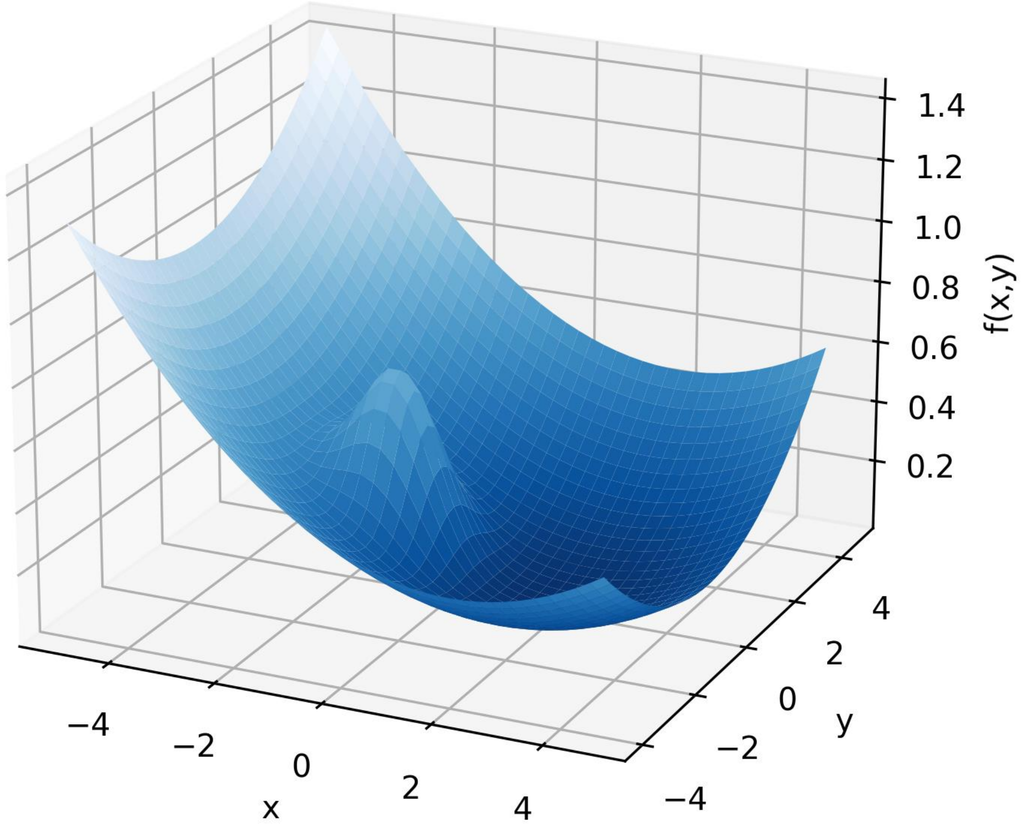




Higher dimensions of π

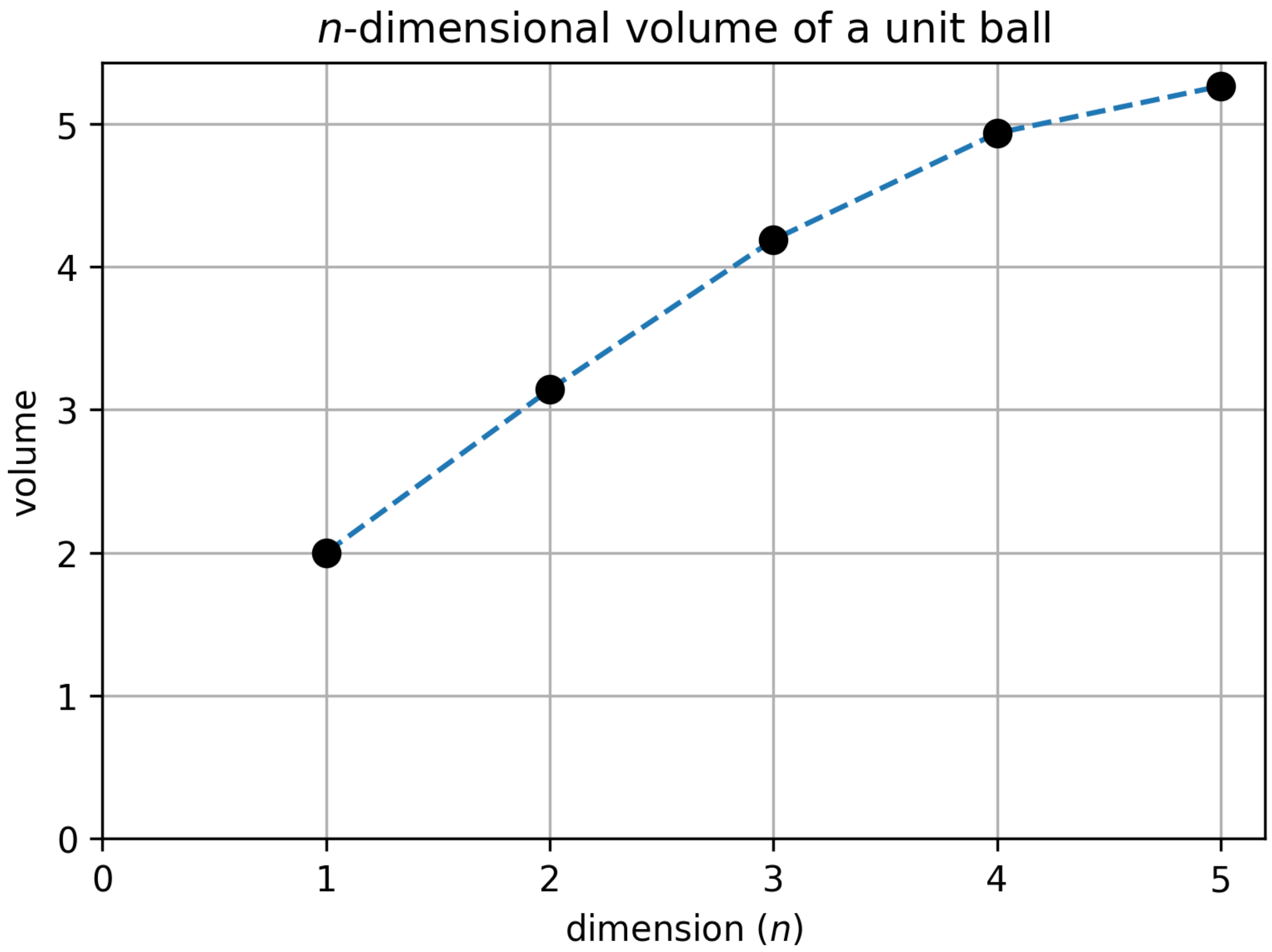
$$\boldsymbol{\mathcal{V}_n(\mathcal{B}_n) = \pi_n r^n}$$

(length)

(area)

- A $1D$-ball of radius 1 has $1D$-volume of 2
- A $2D$-ball of radius 1 has $2D$-volume of $\pi \approx 3.14159$
- A $3D$-ball of radius 1 has $3D$-volume of $\frac{4\,\pi}{3} \approx 4.18879$
- A $4D$-ball of radius 1 has $4D$-volume of $\frac{\pi^2}{2} \approx 4.93480$
- A $5D$-ball of radius 1 has $5D$-volume of $\frac{8\pi^2}{15} \approx 5.26379$

Higher dimensions of π



Higher dimensions of π

$$\boldsymbol{\pi_n = \frac{2\pi}{n} \cdot \pi_{n-2}}$$

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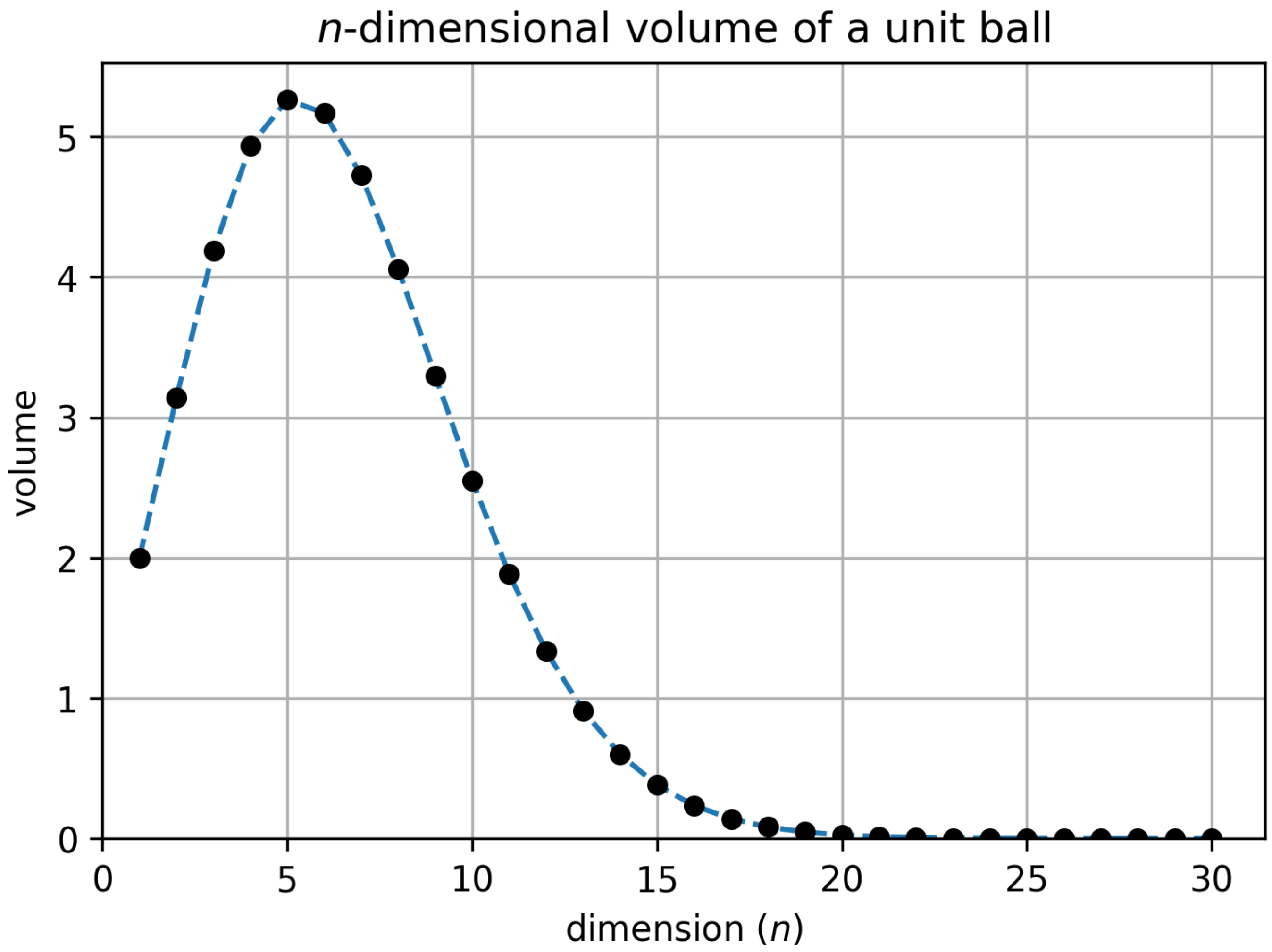
- A $6D$-ball of radius 1 has $6D$-volume of $\frac{\pi^3}{6} \approx 5.16771$
- A $7D$-ball of radius 1 has $7D$-volume of $\frac{16\pi^3}{105} \approx 4.72477$
- A $8D$-ball of radius 1 has $8D$-volume of $\frac{\pi^4}{24} \approx 4.05871$

Higher dimensions of π



ε-width sphere vs ball

$$\boldsymbol{\mathcal{V}_n(\mathcal{B}_n) = \pi_n r^n}$$

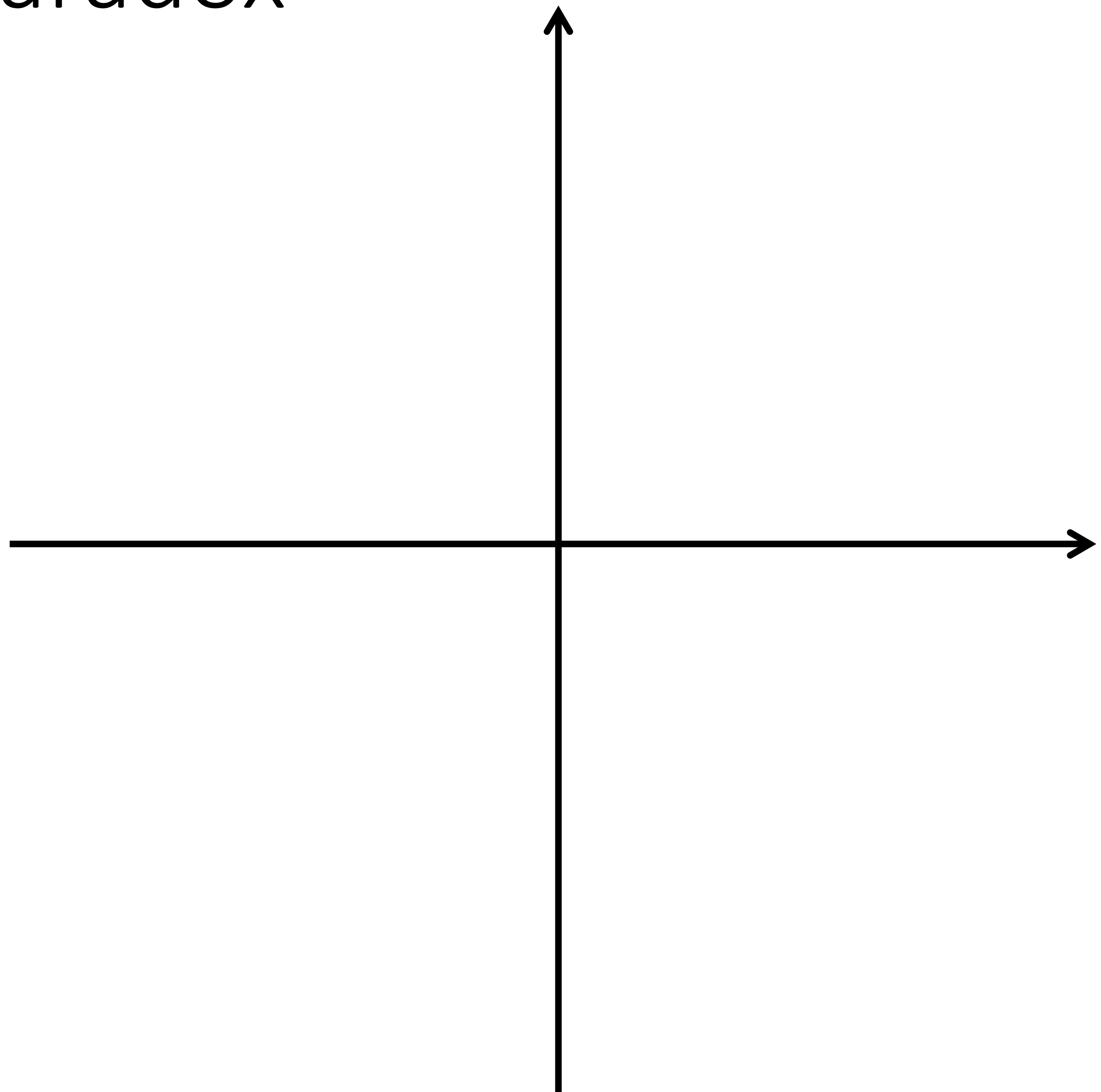
$$\boldsymbol{\mathcal{V}_n(\varepsilon\mathcal{S}_n) = \pi_n r^n - \pi_n r^n(1-\varepsilon)^n}$$

$$\boldsymbol{\frac{\mathcal{V}_n(\varepsilon\mathcal{S}_n)}{\mathcal{V}_n(\mathcal{B}_n)} = 1 - (1-\varepsilon)^n \to 0} \qquad \text{(as } n \to +\infty\text{)}$$

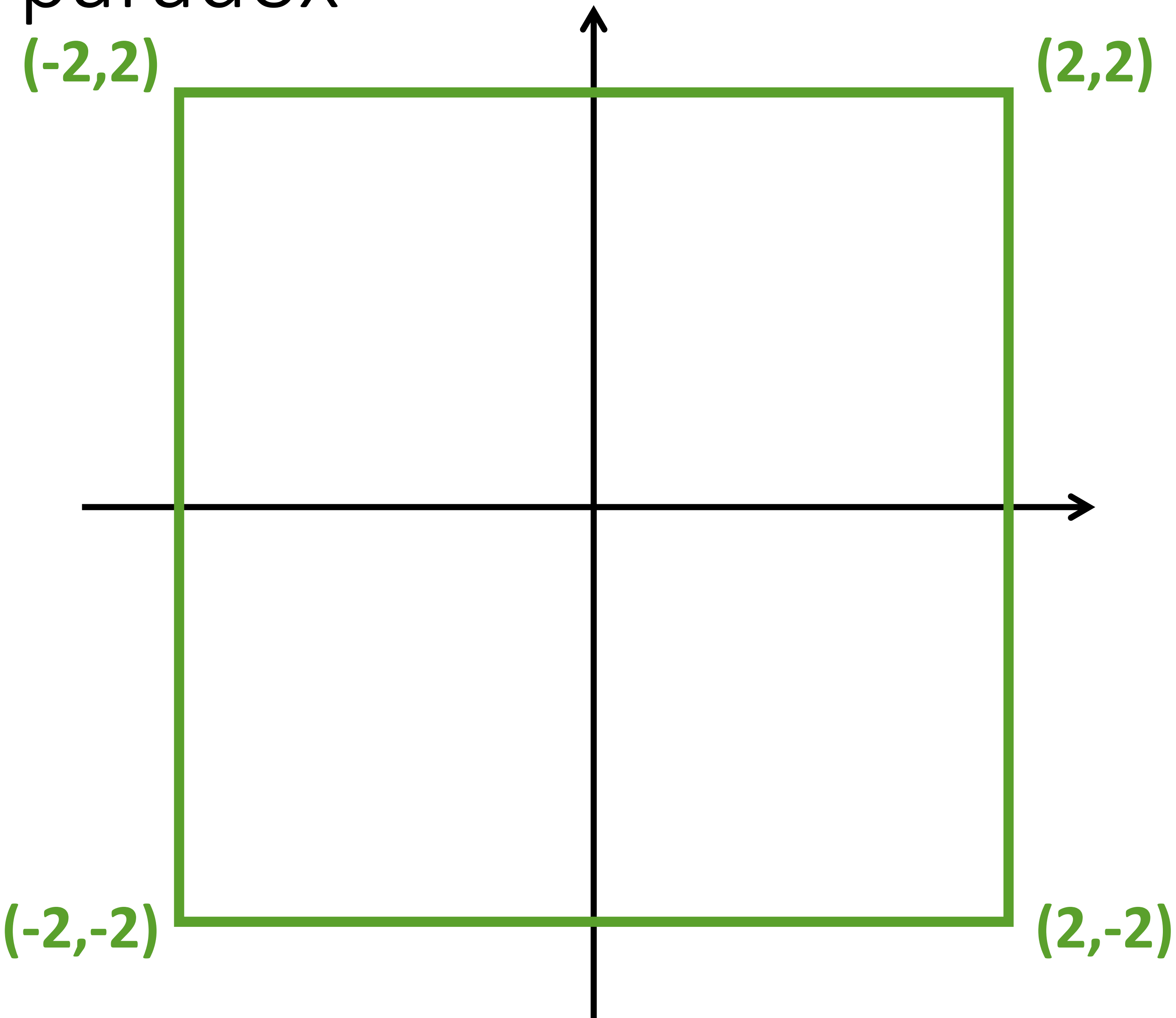
$$\boldsymbol{n = 3, \quad \varepsilon = 0.01:} \qquad \boldsymbol{\frac{\mathcal{V}_3(0.01\,\mathcal{S}_3)}{\mathcal{V}_3(\mathcal{B}_3)} < 3\,\%}$$

$$\boldsymbol{n = 300, \varepsilon = 0.01:} \qquad \boldsymbol{\frac{\mathcal{V}_{300}(0.01\,\mathcal{S}_{300})}{\mathcal{V}_{300}(\mathcal{B}_{300})} > 95\%}$$

4 circles paradox



4 circles paradox



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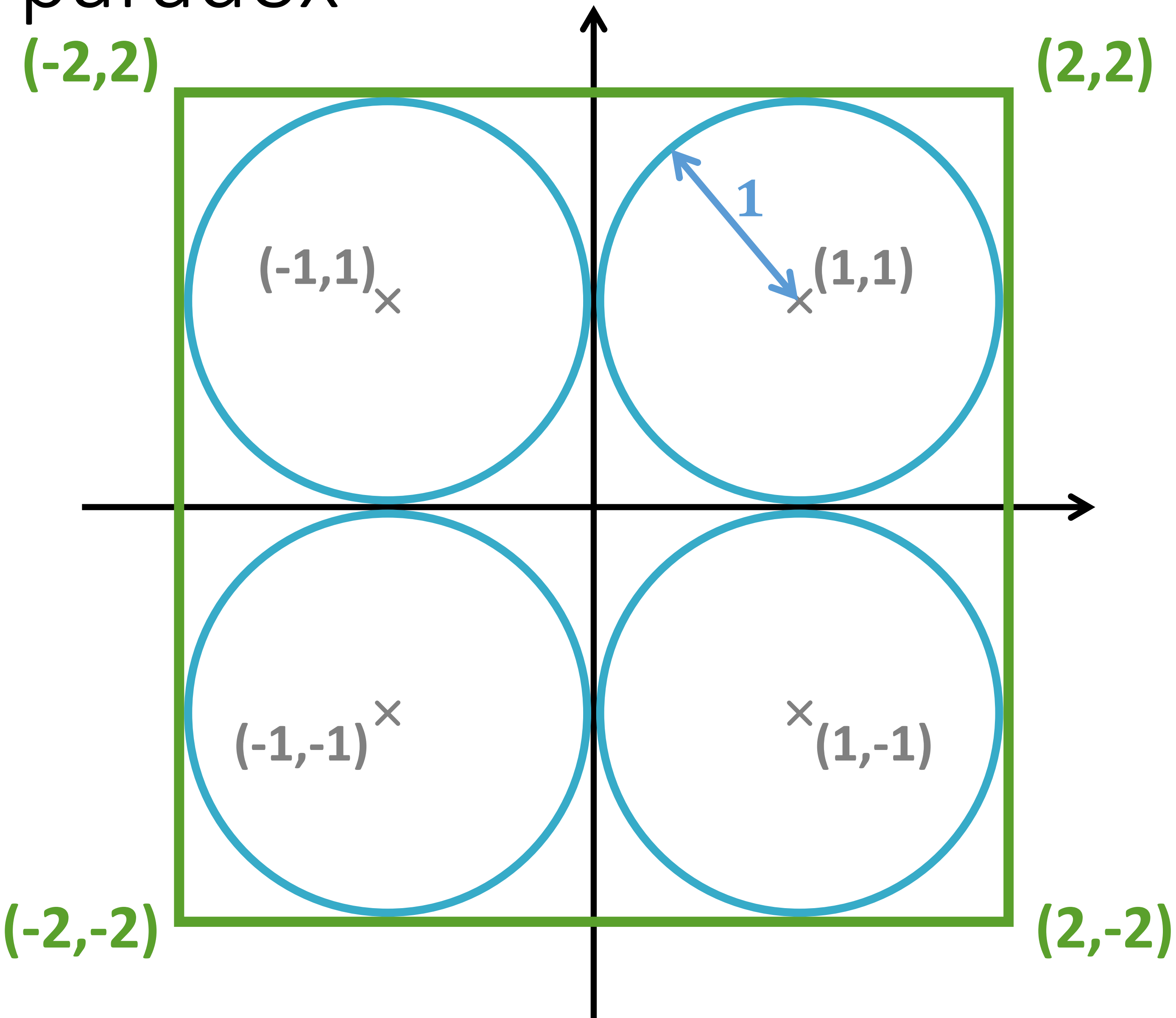

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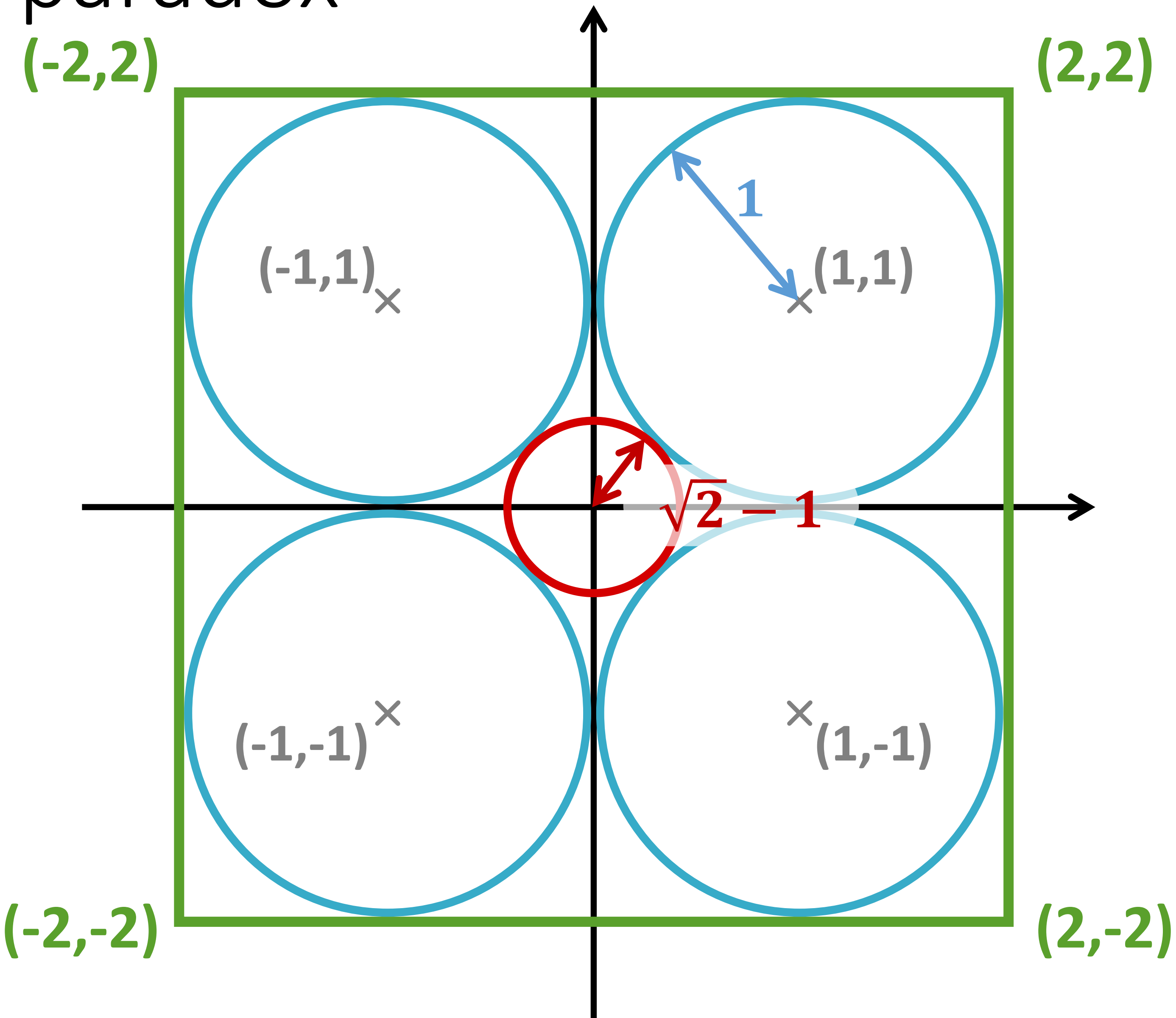
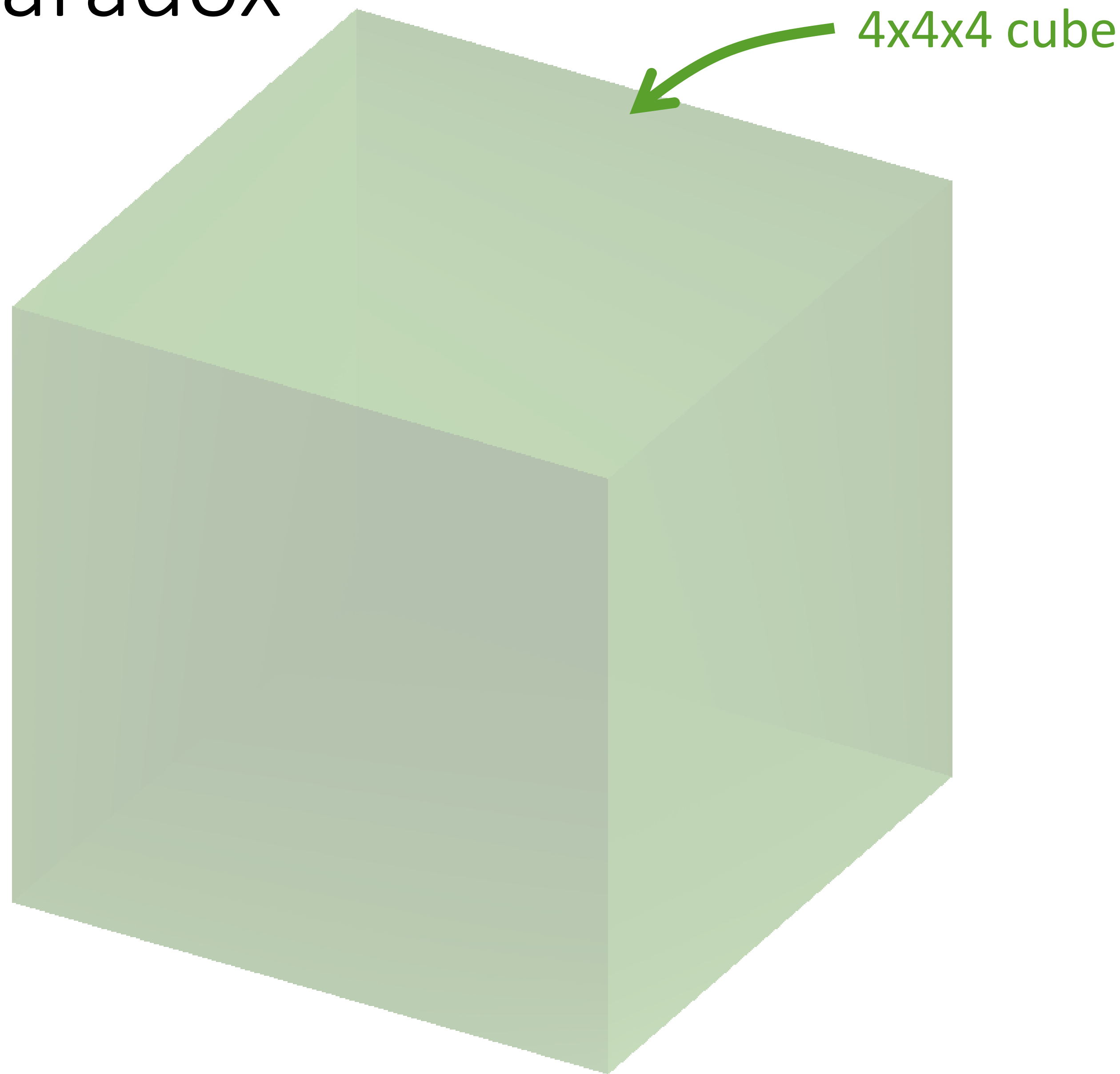


8 spheres paradox



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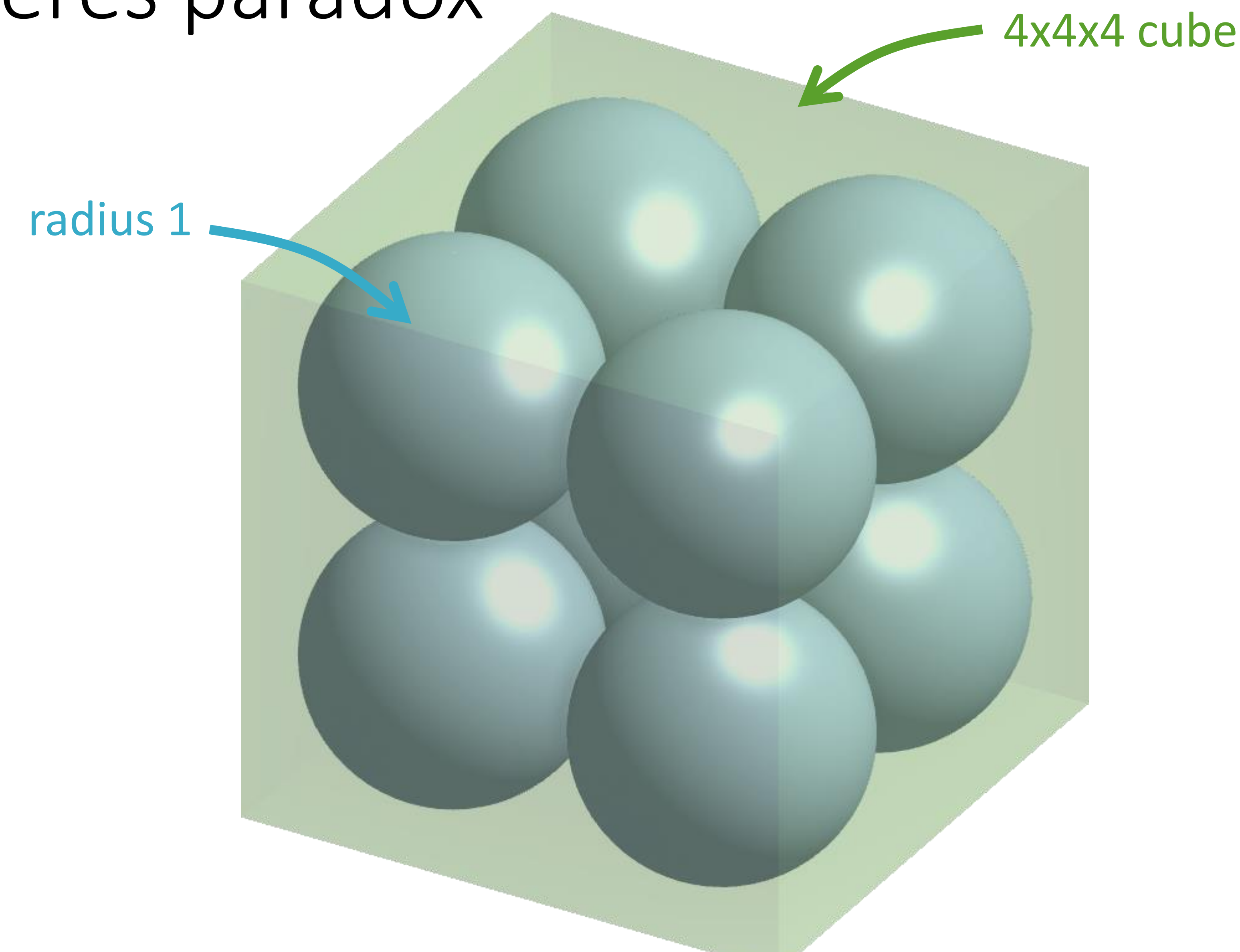

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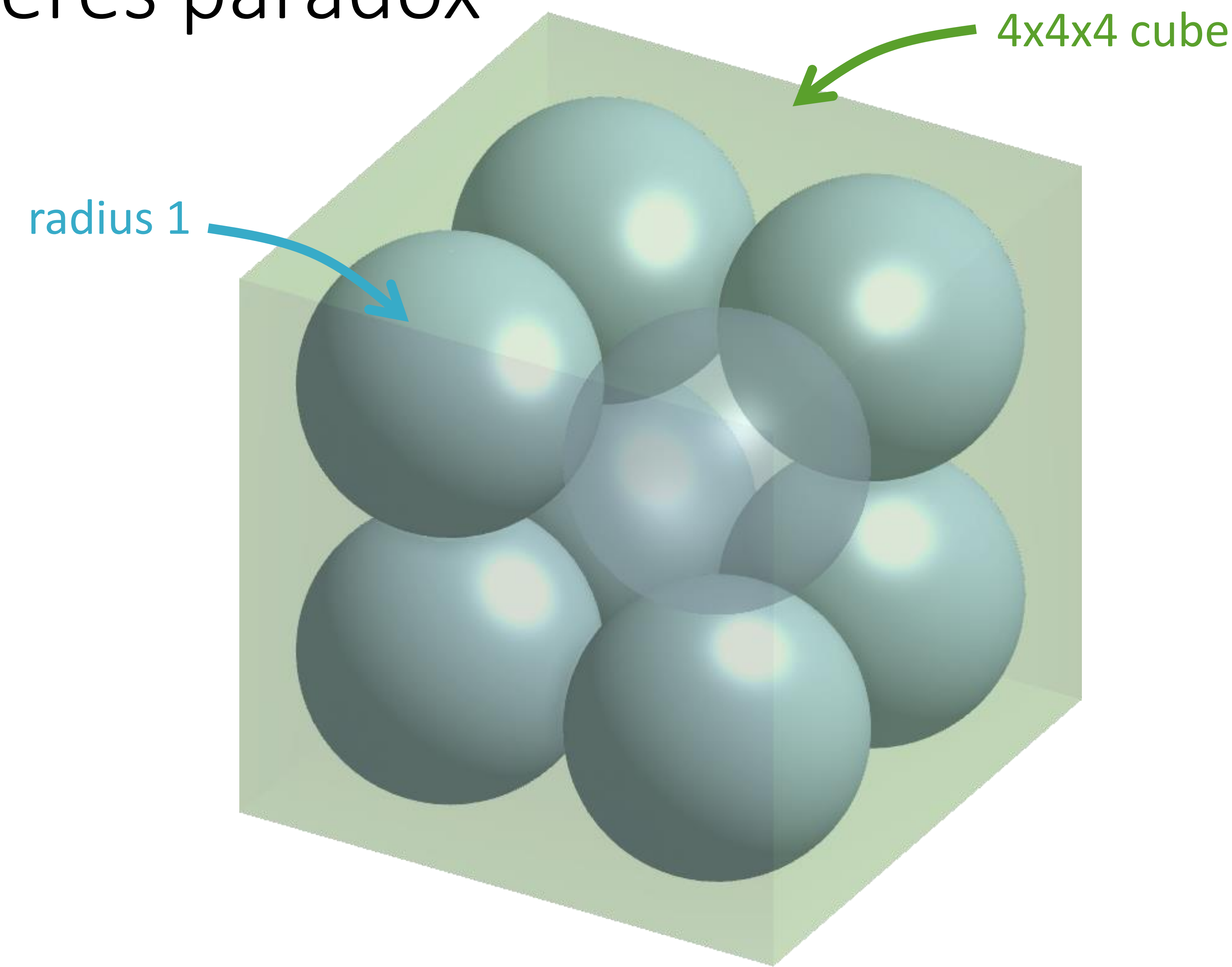

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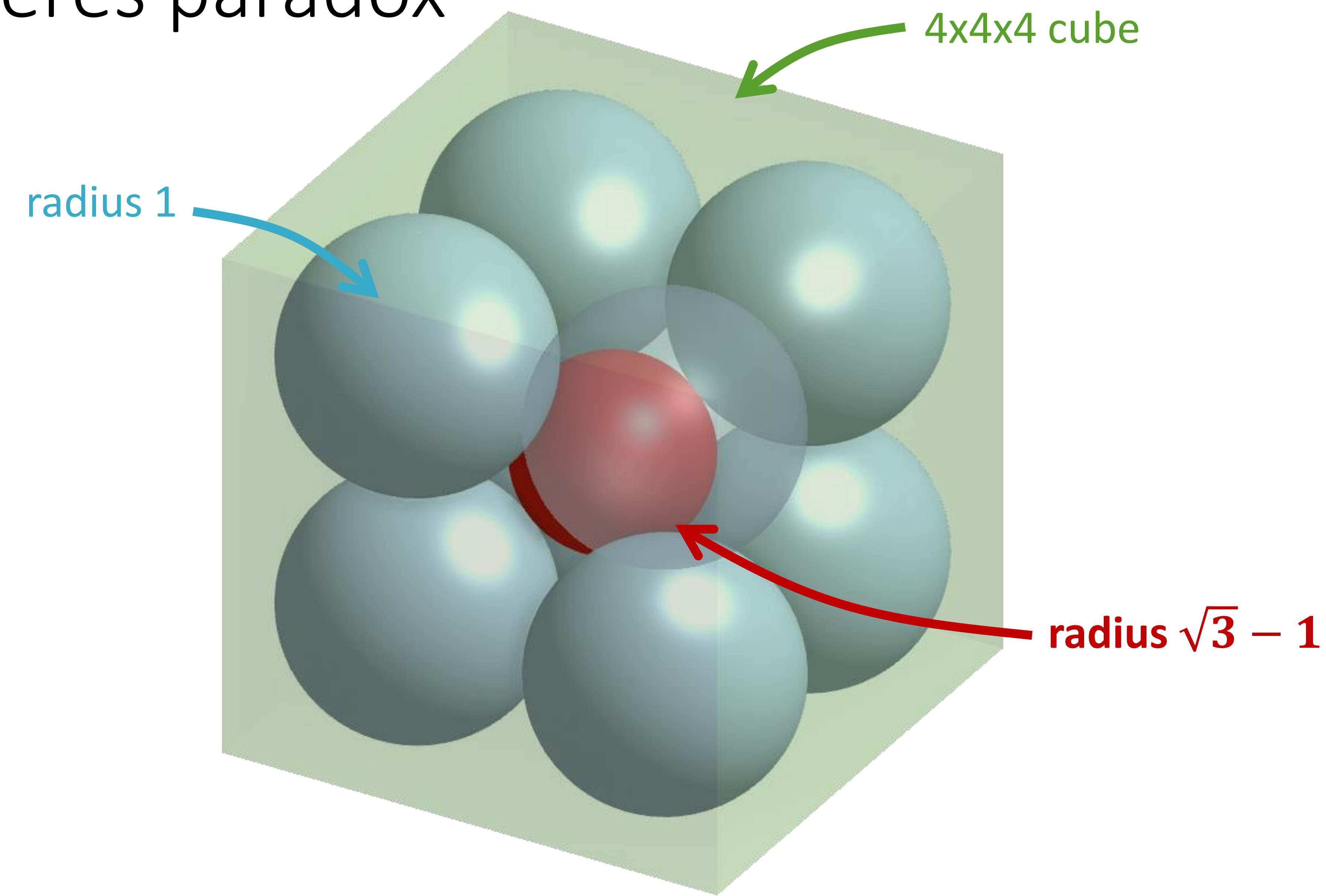

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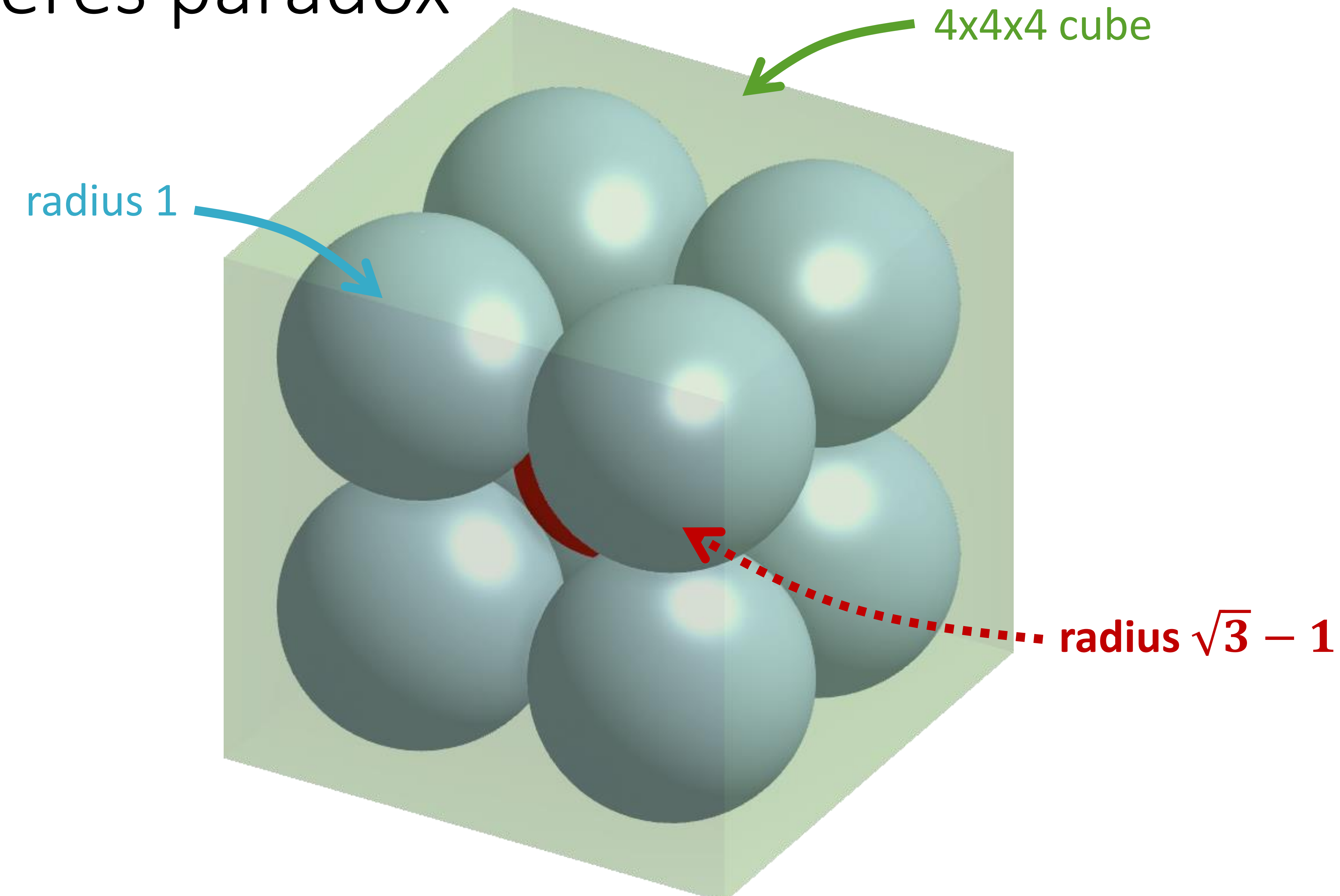


2^n n-spheres paradox

- 1 *dimension:* 2 *"blue"* spheres, *"red"* spheres *radius:* 0 $\left(= \sqrt{1} - 1\right)$
- 2 dimensions: 4 "blue" spheres, "red" spheres radius: $\sqrt{2} - 1$
- 3 dimensions: 8 "blue" spheres, "red" spheres radius: $\sqrt{3} - 1$
- 4 dimensions: 16 "blue" spheres, "red" spheres radius: $\sqrt{4} - 1$ $\left(= 1\right)$

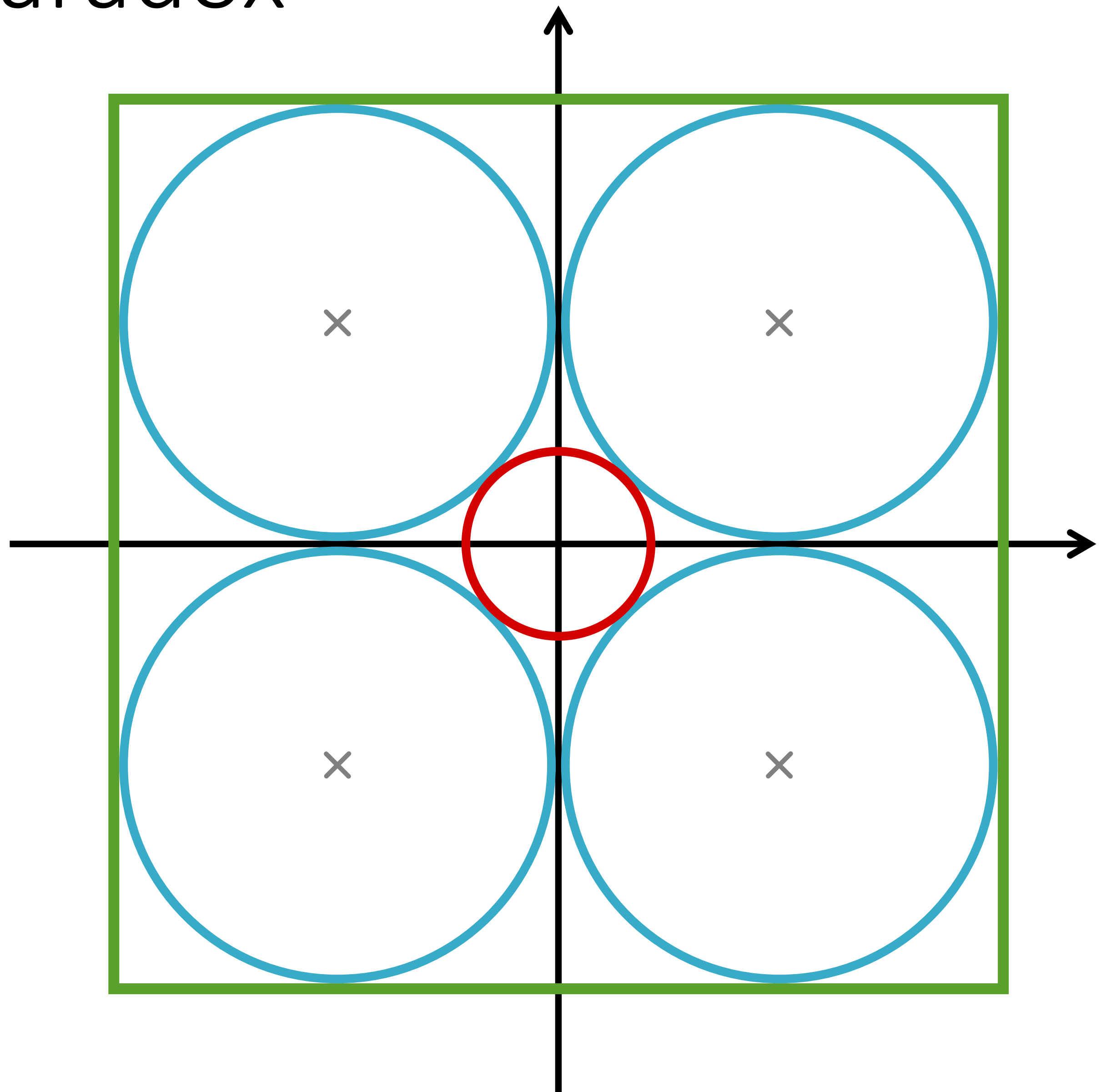
…

- n dimensions: 2^n "blue" spheres, "red" spheres radius: $\sqrt{n} - 1$

$\mathbf{10}$ *dimensions:* $\mathbf{2^{10}}$ = $\mathbf{1024}$ *"blue"* spheres,

***"red" radius:* $\sqrt{\mathbf{10}} - \mathbf{1} \approx \mathbf{2.16} \boxed{> \mathbf{2}}$**

4 circles paradox



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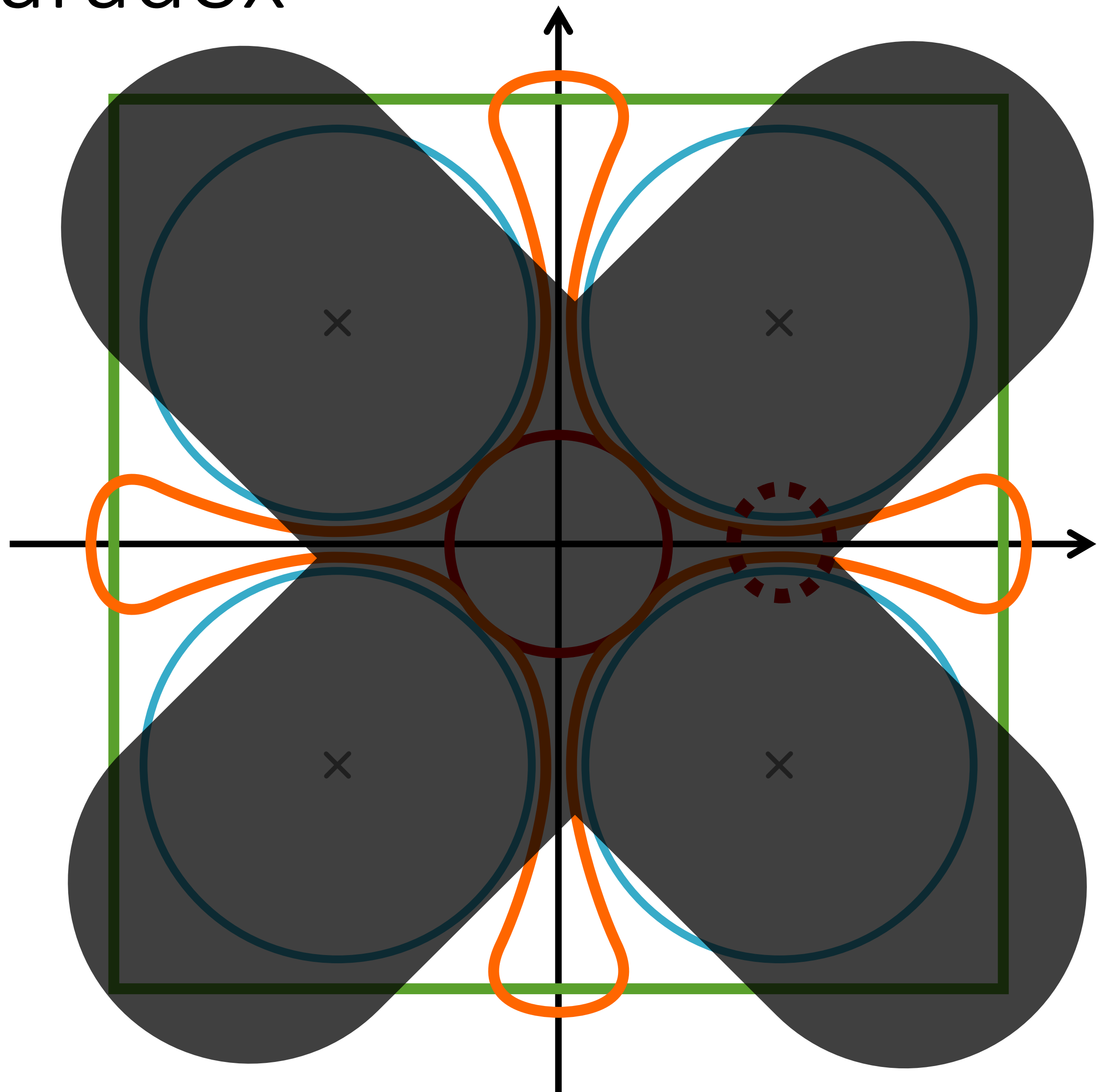

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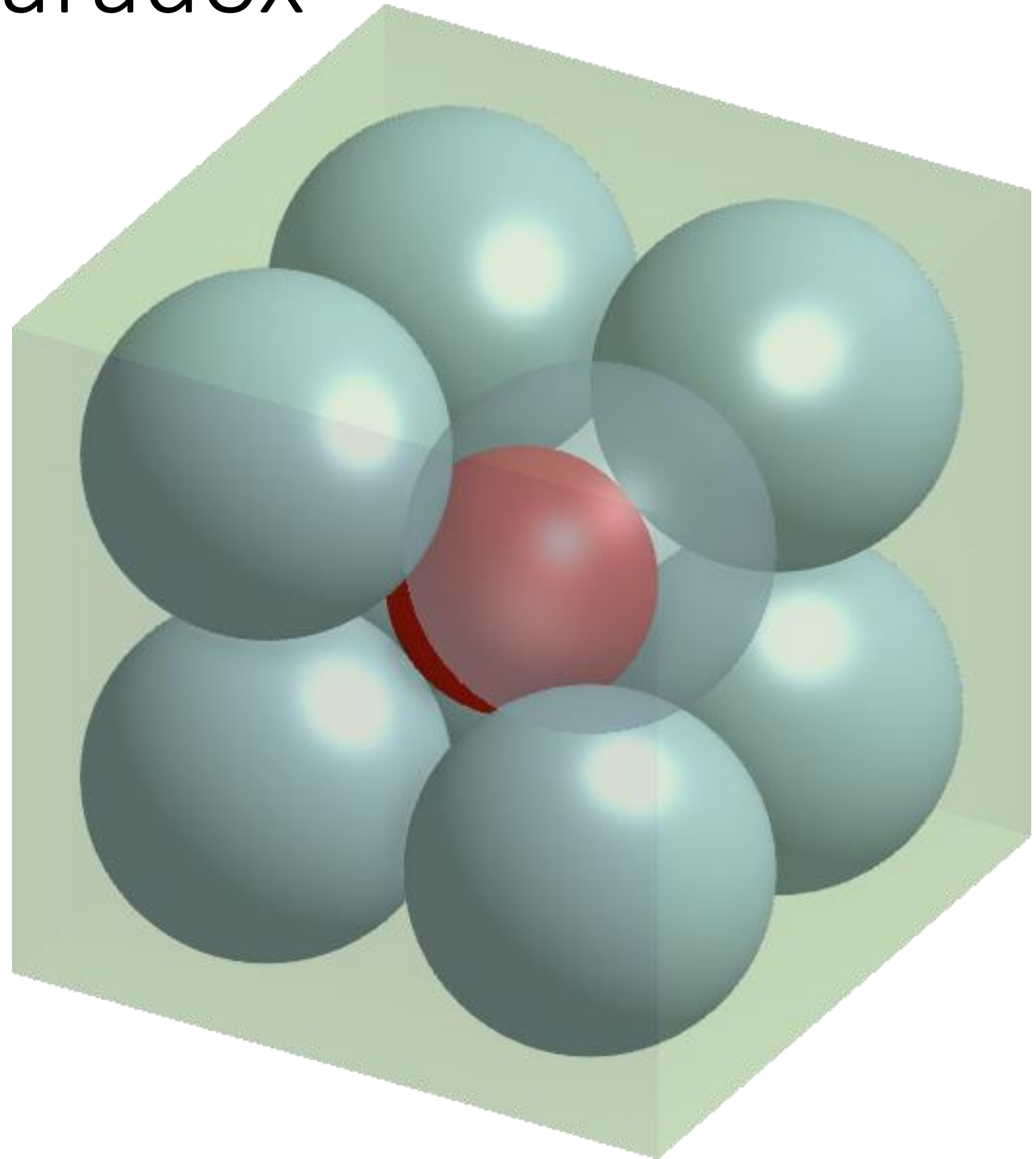

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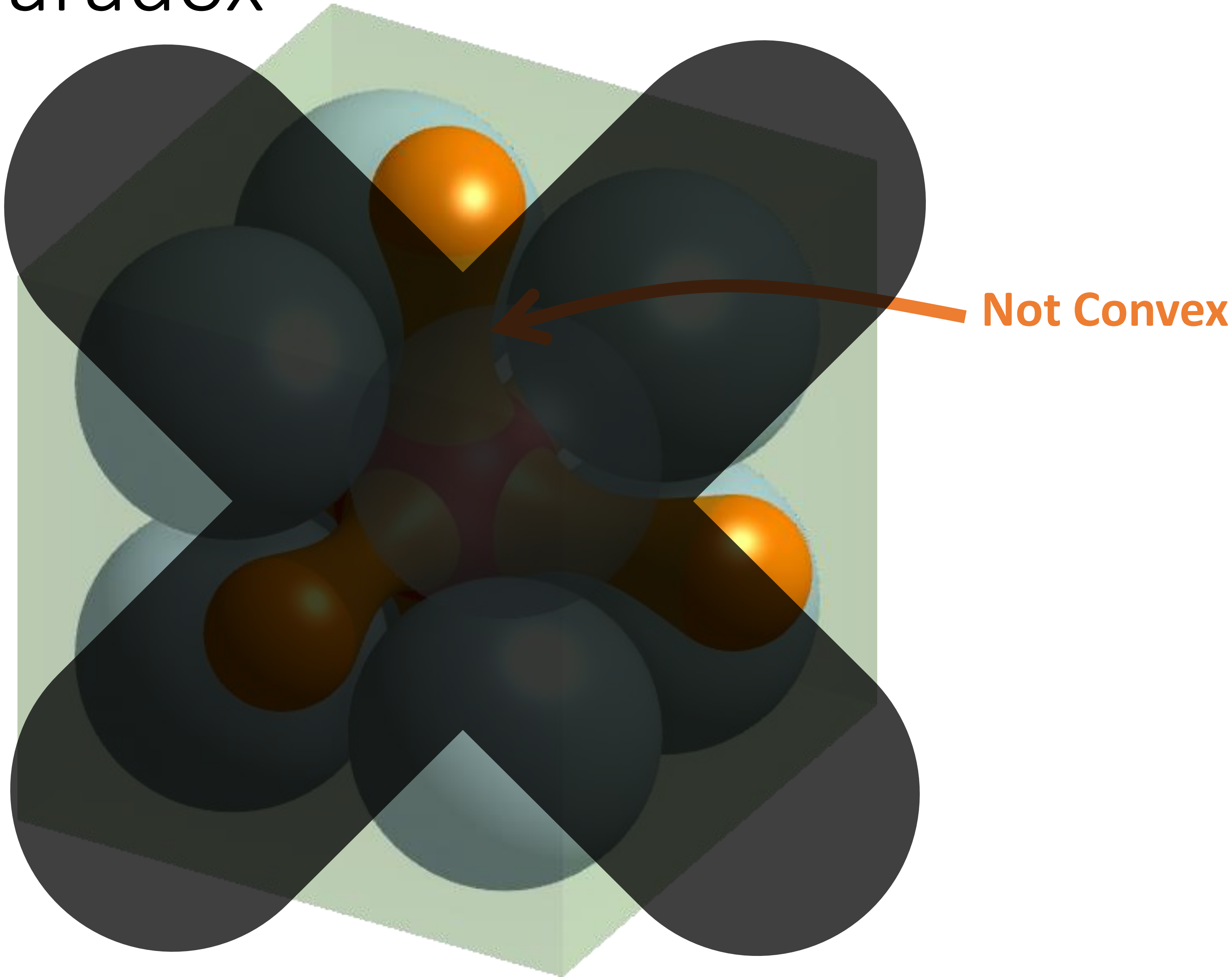
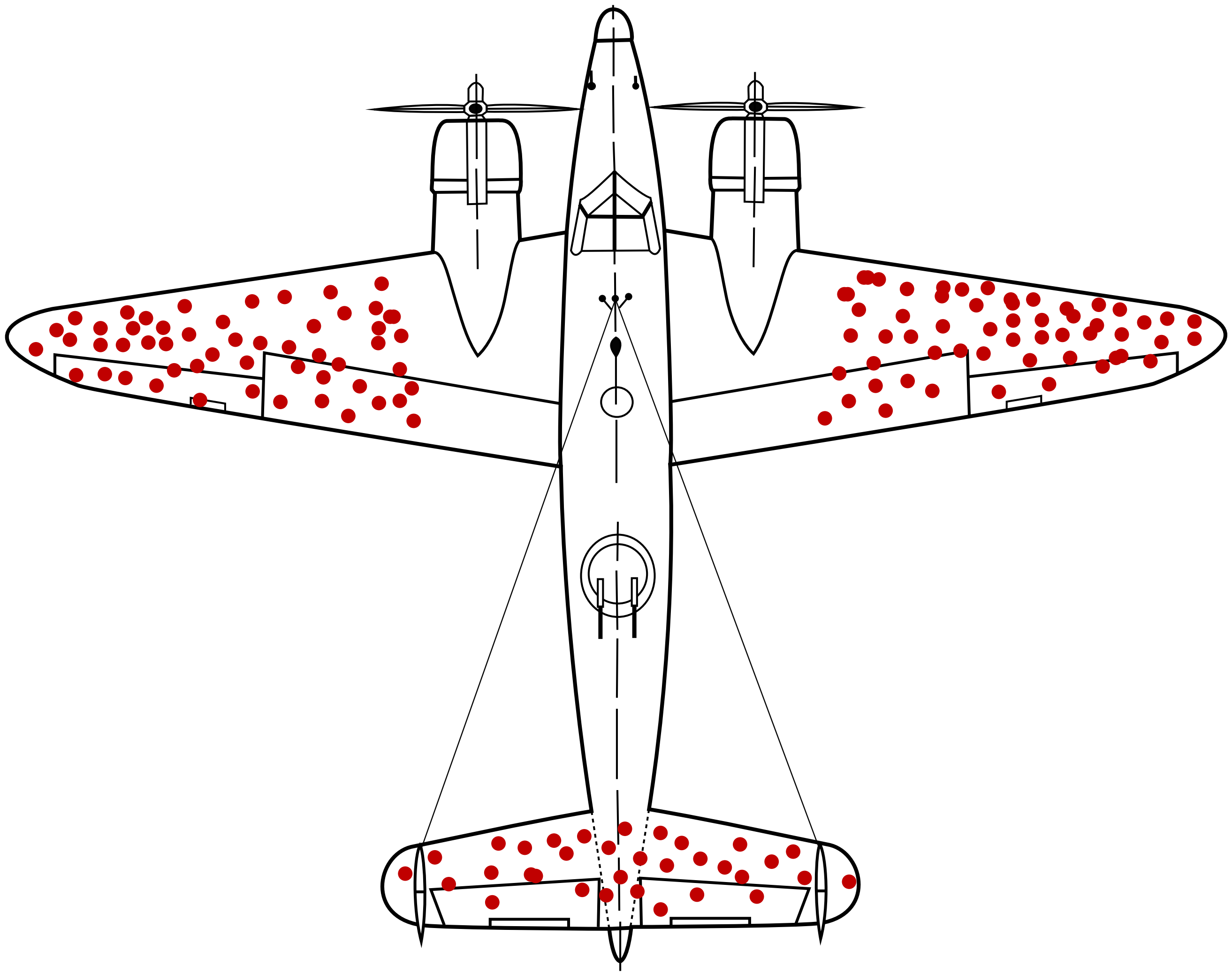
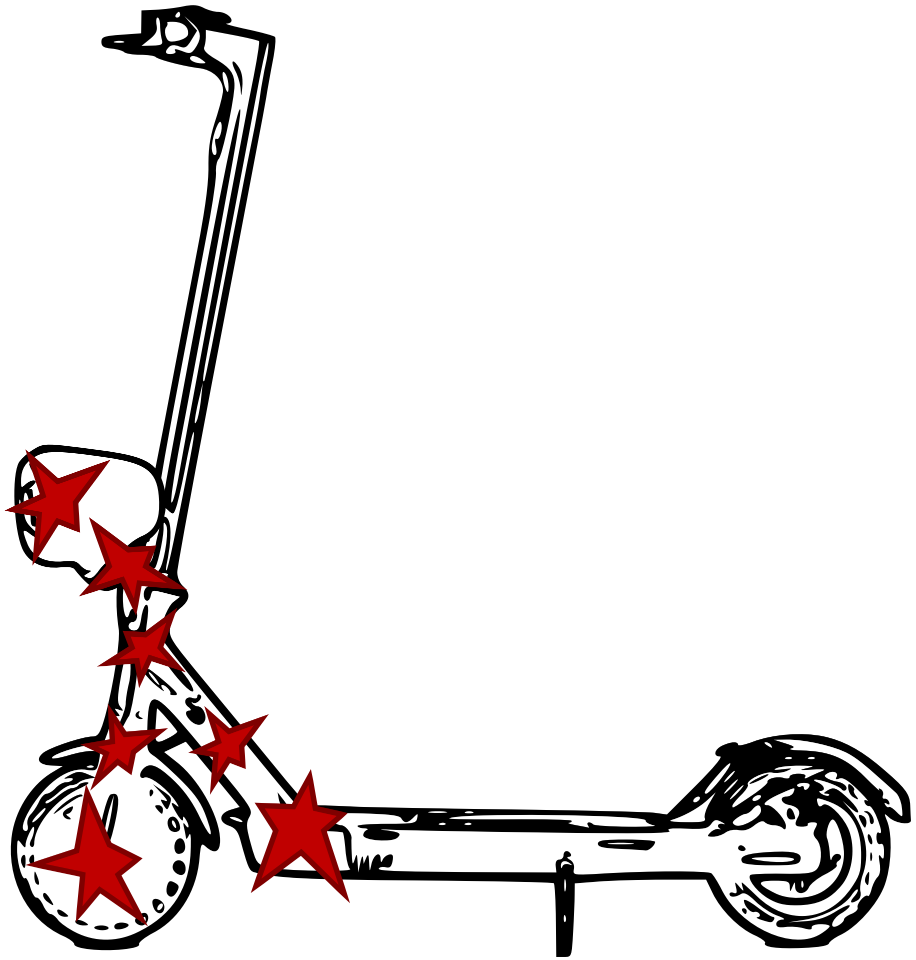


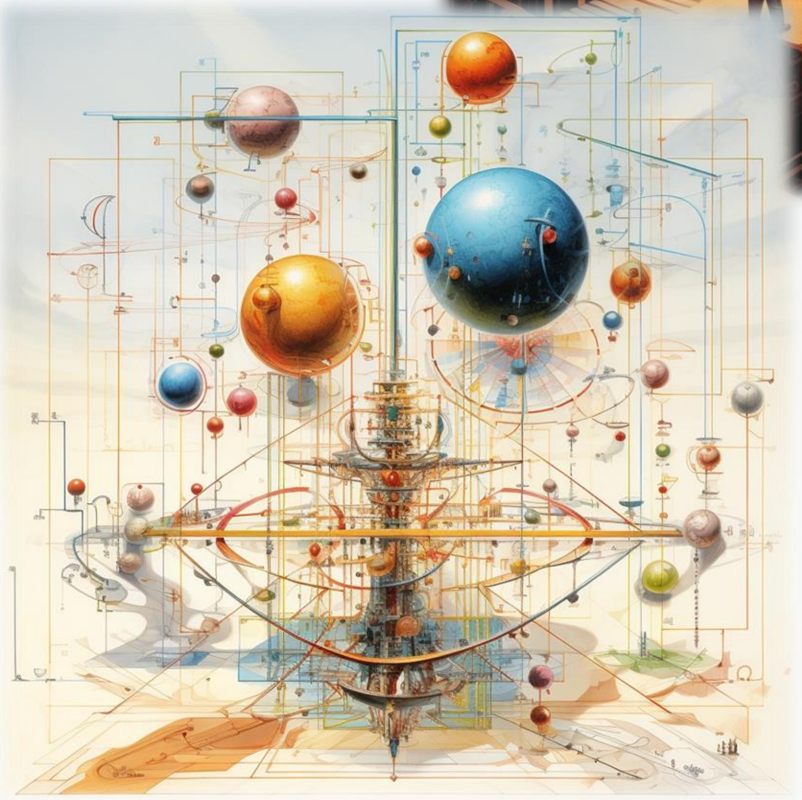
Statistics



Over Interpretation



“high dimensions”
(MidJourney)



References

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- “A world from a sheet of paper” (Tadashi Tokieda), Oxford Mathematics Public Lecture, 14th of June 2023
- “The Legend of Abraham Wald” (Bill Casselman), American Mathematical Society Public Outreach, June 2016
- “MidJourney” V5.2 (Midjourney Inc.), https://www.midjourney.com/, 22nd of June 2023