Time Series Analysis
Lecture 3

Autoregressive Models and Moving Average Models

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Moving Average Models

Mathematical Formulation of the Model and Derivation of Properties

The Invertibility Condition

To fix the idea, consider an MA(1) model:

$$y_t = \epsilon_t + \theta\epsilon_{t-1}$$
$$\epsilon_t \sim WN(0, \sigma_w^2)$$

Solve for the shocks (or innovations):

$$\epsilon_t = y_t - \theta\epsilon_{t-1}$$

Using recursive substitution,

$$\epsilon_{t-1} = y_{t-1} - \theta\epsilon_{t-2}$$
$$\epsilon_{t-2} = y_{t-2} - \theta\epsilon_{t-3}$$
$$\epsilon_{t-3} = y_{t-3} - \theta\epsilon_{t-4}$$
$$\vdots$$

The Invertibility Condition

We will obtain

$$y_t = \epsilon_t + \theta y_{t-1} - \theta^2 y_{t-2} + \theta^3 y_{t-3} - \cdots$$

Rearrange and use the backward shift operator:

$$y_t - \theta y_{t-1} - \theta^2 y_{t-2} - \theta^3 y_{t-3} - \cdots = \epsilon_t$$

$$(1 - \theta B - \theta^2 B^2 - \theta^3 B^3 - \cdots) y_t = \epsilon_t$$

Therefore, the infinite autoregressive representation of a MA(1) model can be expressed as

$$\frac{1}{1 + \theta B} = \epsilon_t$$

Expressed in backward shift operator, the infinite sequence of $\{\theta^i\}_{i=1}^{\infty}$ converges if $|\theta| < 1$, which is the invertible condition for the MA(1) model.

Mean and Variance of

- The mean of the MA(q) model has a constant mean equal to

$$E(x_t) = \sum_{j=0}^{q} \theta_j E\left(w_{t-j}\right) = 0 \tag{3.0.4}$$

where $\theta_0 = 1$,

It also has a constant variance with the following form:

$$Var(x_t) = \sum_{j=0}^{q} \theta_j Var\left(w_{t-j}\right) = \left(1 + \beta_1 + \cdots + \beta_q\right) \sigma_\omega^2 \tag{3.0.5}$$

Covariance

Using the definition of covariance

$$\gamma(h) = cov(x_{t+h}, x_t) = cov\left(\sum_{j=0}^{q} \theta_j w_{t+h-j}, \sum_{k=0}^{q} \theta_k w_{t-k}\right) \tag{3.0.6}$$

and the variance derived above, we can come up with the autocorrelation function, for $k \geq 0$:

$$\rho(k) = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\sum_{i=0}^{q-k} \beta_i \beta_{i+k}}{\sum_{i=0}^{q} \beta_i^2} & \text{for } k = 1, \ldots, q \\ 0 & \text{for } k > q \end{cases} \tag{3.0.7}$$

where $\beta_0 = 1$.

A MA(2) Model

To get a feel of the above formulas, consider a simple **MA(q)** model: MA(2)

This process has mean μ and variance $\sigma_\omega^2(1+\beta_1^2+\beta_2^2)$, and the autocorrelation

$$\rho(k) = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\beta_1(1+\beta_2)}{1+\beta_1^2+\beta_2^2} & \text{for } k = 1 \\ \frac{\beta_2}{1+\beta_1^2+\beta_2^2} & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases} \qquad (3.0.9)$$

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