Live Session 1 - Discrete Response Models

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Required Readings: BL2015: Ch. 1 (Skip Sections 1.2.6 and 1.2.7), Appendix B.3 and B.5

Note: For the first five lectures of the course we follow the book very closely to allow for multiple touch points - async video lectures, assigned readings, live session examples, and homework on the same concepts and techniques.

Topics covered in this lecture:

- An introduction to categorical data, Bernoulli probability model, and Binomial probability model
- Computing the probability of binomial probability model
- Simulating a binomial probability model
- Estimating the Binomial probability model using maximum likelihood estimation (MLE)
- Confidence intervals:
  - Wald confidence interval
  - Alternative confidence intervals
- Hypothesis test for the probability of success
- The case of two binary variable
  - Contingency tables
  - The notions of relative risks, odds, and odds ratios
- Two Binary variables
  - Contingency table
  - MLE
  - C.I.s for the difference of two probabilities
  - Relative Risks
  - Odds
  - Odds ratios (OR)
  - log(OR)
  - Estimation and inference

This week's lecture starts with the simplest case of discrete response modeling, the Binomial probability model, covering both parameter estimation and statistical inference. It points out the importance of proposing the appropriate mathematical framework to model the variable of interest based on the "type" of the variable. In this case, the "type" of the variable of interest is "discrete", rendering continuous probability model inappropriate.

This is the simplest case of modeling categorical variables, as there are only two categories. Therefore, we do not have to worry about whether or not the categories are ordered (or ranked) or count (such as the number of customers arriving at a particular store within a specific hour).

You have already studied both the Bernoulli and Binomial probability models in w203 and perhaps other statistics courses elsewhere. However, many aspects of this model are typically not obvious and the underlying assumptions are often overlooked (note that all probability and statistical models come with assumptions). In this course, we will study how to apply theoretical statistical models in practice.

The Binomial probability model is a reasonable model for distribution of an event happened (or "success") in a given number of trails, so long as the assumptions listed on page 3 and 4 of our textbook *"Analysis of Categorical Data with R"* are not violated.

The book lists several examples and discusses in each cases whether each of the assumptions is reasonable

Another (perhaps) new aspect of studying Bernoulli and Binominal probability models is to model it in R using the R built-in *dbinom(n,size,prob)* function

We also study how to simulate a probability model and use simulation to evaluate how well procedures perform when these assumptions are violated. (Pages 7 and 8 of the book describe this)

A lot of time is spent on discussing the confidence interval of this model, a discussion typically not covered in the elementary statistics courses.

As you will see, we use simulation extensively in this course, as simulation is one of the best ways to "get a feel" of a probability model and gain an understanding of its behavior

A single binomial probability model is then extended to two binomial probability model, introducing the notion of a contingency table. Also introduced is the formulation of a likelihood function and the method of maximum likelihood, a version of which was taught in the parameter estimation module in w203.

With more than one binomial random variables, the concepts of relative risk, odds, and odds ratios become very powerful, as the meaning of the difference between probability of success, $\pi_1 - \pi_2$, is a function of the magnitude of π_1 and π_2.

All of these concepts are introduced as the preparation for the study of the regression model of categorical response data, with binary response being the simplest case.

In the live session today, I want to focus on (1) the statistical analysis workflow, (2) confidence intervals, and (3) the concepts of odds ratios.

Statistical Analysis Workflow Revisit

- Postulate a statistical model that conforms with the underlying (business, policy, scientific, etc.) question being asked
- Estimate the parameter of the statistical model
- Check model assumptions
- Conduct statistical inference

In the case of the success parameter π from the Bernoulli distribution, we use MLE:

$$\begin{aligned} L(\pi|y_1 \cdots y_n) &= P(Y_1 = y_1, \ldots, Y_n = y_n) \\ &= P(Y_1 = y_1) \times \cdots \times P(Y_n = y_n) \\ &= \prod_{i=1}^{n} P(Y_i = y_i) \\ &= \prod_{i=1}^{n} \pi_i^{y_i} (1 - \pi_i)^{(1-y_i)} \end{aligned}$$

The MLE is $\hat{\pi} = \frac{w}{n}$, where w denotes the number of successes in n trials.

The variance of the estimate is $\hat{Var}(\hat{\pi}) = \frac{\hat{\pi}(1-\hat{\pi})}{n}$

A Summary of the Confidence Intervals for the Probability of Success π

Recall that in w203, we've learned that the *typical form* of a confidence interval for a parameter of a probability model, θ, is:

estimator $\pm$ (distributional value) $\times$ (standard deviation of estimator)

- *What is the frequentist interpretation of a (1-α)100% confidence interval (as we learned in w203)?*

In fact, if we assume that $\hat{pi}$ follow the Normal Distribution, that is, $\hat{\pi} \sim N(\pi, \hat{Var}(\hat{\pi}))$, we have the **Wald confidence interavl** taking the following form:

1. Wald Confidence Interval:

$$\hat{\pi} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$

What "assumption" is required for one to construct the Wald confidence interval?

While one of MLE's properties is that it is asymptotically normal, an implicit assumption to construct this confidence interval is that the normal approximation is a good approximation.

What are the problems with this confidence interval?

Regardless of whether or not the above assumption is satisfied, this confidence interval suffers from some problems:

1. The interval can exceed the (0,1) interval, which would not be sensible for probability
2. When $w = 0$ or $w = 1$, the confidence interval degenerates into a single point equal to $\hat{\pi}$
3. The coverage is not necessarily equal to the stated level $1 - \alpha$

Why does a confidence interval method not actually achieve its stated confidence level of binomial random variable?

Example: Computation of Wald Confidence Interval

Let's pause for a minute before writing code.

Suppose we have a Bernoulli probability model with $\pi = 0.6$. In addition, suppose that we "think" that the sample of observations (that is, our data) come from this (theoretical) distribution.

```
set.seed(23951)
n_sim = 1
n_trials = 10
p = 0.6
bin <- rbinom(n=n_sim, size = n_trials, prob = p)
bin
```

```
## [1] 6
```

From the simulation, we get a total of 6 successes from 10 draws. Our hypothetical value of the true probability of success is $\pi = 0.6$. Let's use this information to derive the Wald confidence interval.

Recall the Wald Confidence Interval's formula before writing done the code.

$$\hat{\pi} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$

$w = 6$ and $n = 10$

```
w = 6
n = 10
alpha = 0.05

pi.hat = w/n # Recall that this is an MLE for pi in Bernoulli probability model
var.wald = pi.hat*(1-pi.hat)/n

wald.CI_lower.bound = pi.hat - qnorm(p = 1-alpha/2)*sqrt(var.wald)
wald.CI_upper.bound = pi.hat + qnorm(p = 1-alpha/2)*sqrt(var.wald)

round(data.frame(pi.hat, var.wald, wald.CI_lower.bound,wald.CI_upper.bound),4)
```

```
##   pi.hat var.wald wald.CI_lower.bound wald.CI_upper.bound
## 1    0.6    0.024              0.2964              0.9036
```

Is this a "good" confidence interval or not? What do we mean by "good"? Does it achieve its stated confidence level?

- we have never really questioned the validity of confidence interval in elemenary statistics courses as well as w203? Why now?
- As it turns out, Wald confidence interval does not perform well when it comes to achieving the true confidence level, say $95\% = 1 - \alpha$ where $\alpha = 0.05$. Let's examine the reason behind it.
- Binomial random variables are discrete random variables, meaning that it can only take a finite number of values; given the number of trails n, there are only $n + 1$ possible intervals, corresponding to $w \in 0, 1, \ldots, n$.
- Given the true parameter value π, some of these $n + 1$ intervals contain π, obviously, and some don't.
- More formally, the true confidence level at a given parameter value π, called it $C(\pi)$, following the book's notation, is the sum of the binomial probabilities for all the $n + 1$ intervals that actually contain π:

$$C(\pi) = \sum_{i=1}^{n} I(w) \binom{n}{w} \pi^w (1-\pi)^{n-w}$$

where $I(w) = 1$ if the interval contains π and 0 otherwise.

Let's do some computation to solidify our intuition. Let's continue to use the numbers from the example above: $n = 10$ and $\pi = 0.6$

```
pi = 0.6
alpha = 0.05
n = 10
w = 0:n

wald.CI.true.coverage = function(pi, alpha=0.05, n) {

  w = 0:n

  pi.hat = w/n
  pmf = dbinom(x=w, size=n, prob=pi)

  var.wald = pi.hat*(1-pi.hat)/n
  wald.CI_lower.bound = pi.hat - qnorm(p = 1-alpha/2)*sqrt(var.wald)
  wald.CI_upper.bound = pi.hat + qnorm(p = 1-alpha/2)*sqrt(var.wald)
```

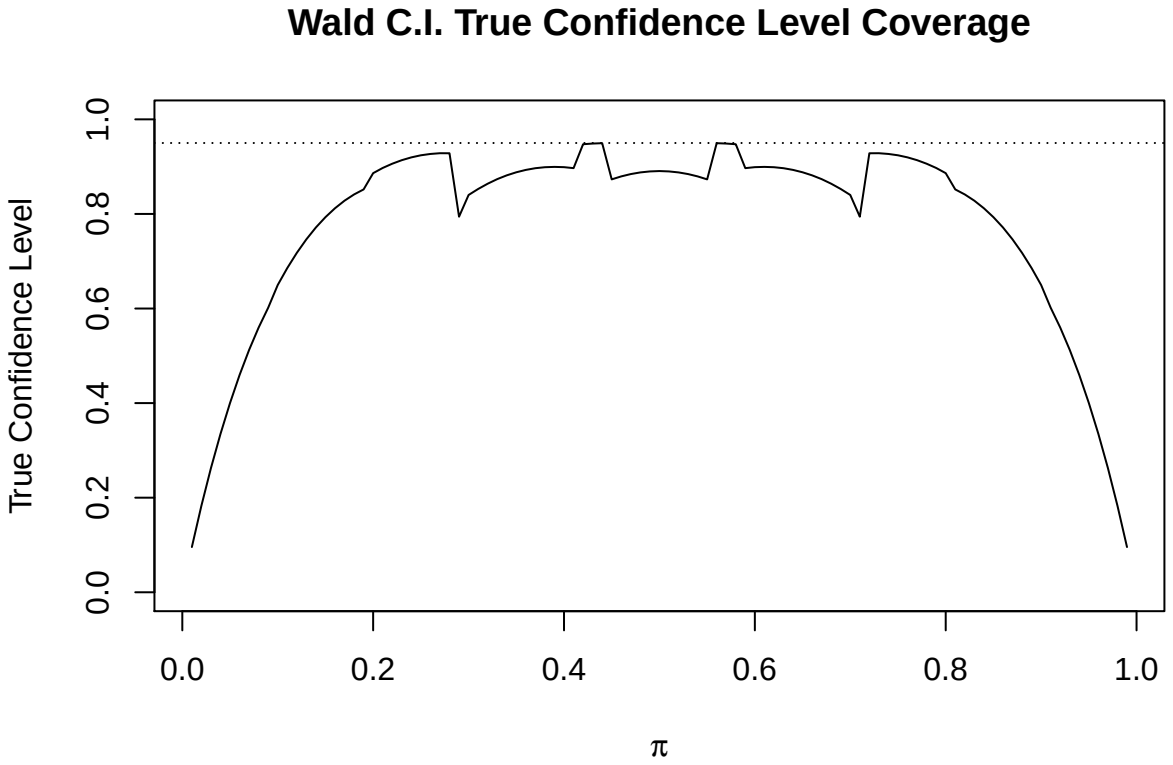
```
  covered.pi = ifelse(test = pi>wald.CI_lower.bound, yes = ifelse(test = pi<wald.CI_upper.bound, yes=1,

  wald.CI.true.coverage = sum(covered.pi*pmf)

  wald.df = data.frame(w, pi.hat, round(data.frame(pmf, wald.CI_lower.bound,wald.CI_upper.bound),4), co

  return(wald.df)
}

wald.df = wald.CI.true.coverage(pi=0.6, alpha=0.05, n=10)
wald.CI.true.coverage.level = sum(wald.df$covered.pi*wald.df$pmf)

# Let's compute the ture coverage for a sequence of pi
pi.seq = seq(0.01,0.99, by=0.01)
wald.CI.true.matrix = matrix(data=NA,nrow=length(pi.seq),ncol=2)
counter=1
for (pi in pi.seq) {
    wald.df2 = wald.CI.true.coverage(pi=pi, alpha=0.05, n=10)
    #print(paste('True Coverage is', sum(wald.df2$covered.pi*wald.df2$pmf)))
    wald.CI.true.matrix[counter,] = c(pi,sum(wald.df2$covered.pi*wald.df2$pmf))
    counter = counter+1
}
str(wald.CI.true.matrix)
```

```
##  num [1:99, 1:2] 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.1 ...
```

```
wald.CI.true.matrix[1:5,]
```

```
##      [,1]   [,2]
## [1,] 0.01 0.0956
## [2,] 0.02 0.1828
## [3,] 0.03 0.2624
## [4,] 0.04 0.3347
## [5,] 0.05 0.4002
```

```
# Plot the true coverage level (for given n and alpha)
plot(x=wald.CI.true.matrix[,1],
     y=wald.CI.true.matrix[,2],
     ylim=c(0,1),
     main = "Wald C.I. True Confidence Level Coverage", xlab=expression(pi),
     ylab="True Confidence Level",
     type="l")
abline(h=1-alpha, lty="dotted")
```



There has been a lot of research on finding an interval for π, the earliest of these studies began in the early 20^{th} century. However, the mathematical details go outside of the scope of this course. Interested readers can refer to the papers referenced in Chapter 1 of the book. Here, we will only summarize some of the findings from this literature. In practice, these intervals can be computed using functions from various R libraries. Even in that case, it is instructive to construt these intervals "manually", since not all statistical or programming languages come with these readily-available functions.

Alternatives: For $n < 40$, use *Wilson interval*. For $n \geq 40$, use *Agresti-Coull interval*. Note that even for $n < 40$, the *Agresti-Coull interval* is still generally better than the *Wald interval*.

2. Wilson "score" Interval:

$$\tilde{\pi} \pm \frac{Z_{1-\frac{\alpha}{2}} n^{1/2}}{n + Z^2_{1-\frac{\alpha}{2}}} \sqrt{\hat{\pi}(1-\hat{\pi}) + \frac{Z^2_{1-\frac{\alpha}{2}}}{4n}}$$

where $\tilde{\pi} = \frac{w + \frac{1}{2} Z^2_{1-\frac{\alpha}{2}}}{n + Z^2_{1-\frac{\alpha}{2}}}$, which can be considered as an "adjusted" estimate of π.

One of the advantages of this interval is that it is bounded between 0 and 1.

3. Agresti-Coull interval:

$$\tilde{\pi} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\tilde{\pi}(1-\tilde{\pi})}{n + Z^2_{1-\frac{\alpha}{2}}}}$$

- Agresti and Caffo (2000), based on their examination of various types of CIs, recommended that adding one success and one failure for each group results in an interval that does a reasonable job.