A Basic Study of Dimensionality

A Quantitative Approach

Scientific Data Acquisition and Processing

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Abstract

A brief summary of the experiment.

Keywords

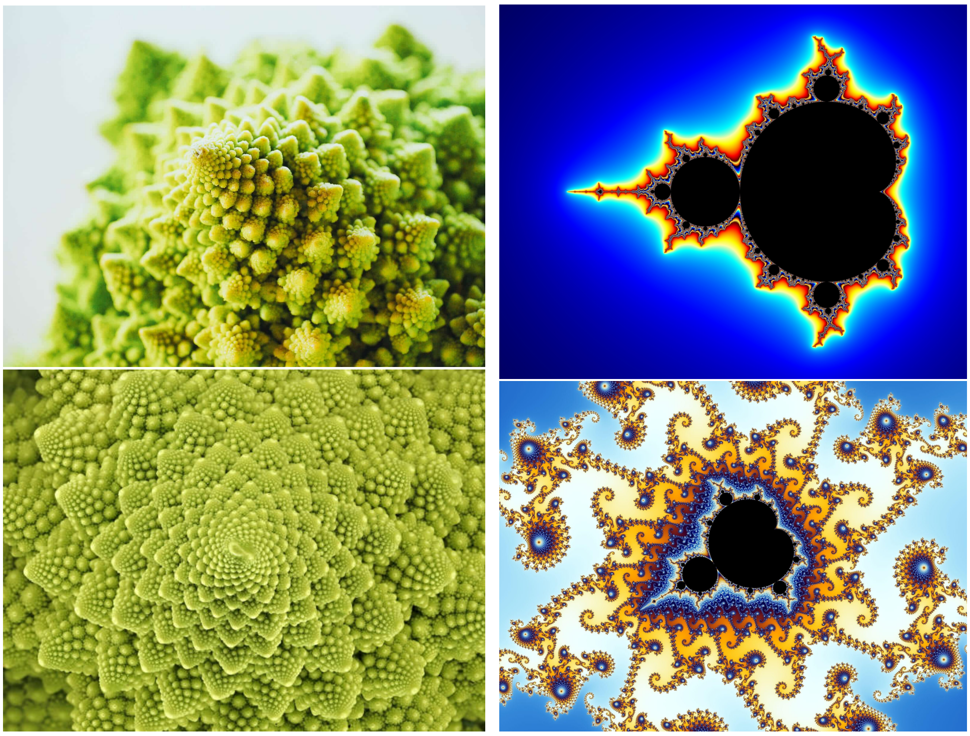
fractal, felf-similarity, fractal dimension, Aluminum foil, spheres, curve, fitting, model, regression, power law, linearization, least squares method.

Contents

1 Introduction

The following experiment aims to demonstrate the fractal nature of a very simple physical object: a small tin foil ball.

Fractal objects are objects which are characterised of the self-similarity property; in simpler words, they look the same when observed at different scales. One of the most famous fractal object is the Mandelbrot set, but we can find these objects also in nature, such as in the structure of a Romanesco Cauliflower, or from a physical point of view; polymers can be regarded as fractals as well [1].



(a) Cauliflower from far (top) and up close (bottom).

(b) Mandelbrot set from far (top) and up close (bottom).

We observe the same level of complexity in the images, as seen from far away and up close as shown in Fig. (1a) and Fig. (1b) above.

From a mathematical point of view, we define a fractal from its changes in terms of mass and volume. For example, we start from an object that we know: a square sheet of paper. In this case we will have the mass distributed following the area of the sheet, furthermore the mass grows as the square of the typical lenght of the sheet (the side of the square which we will call r). The formula will be the following

$$M = Cr^2$$

where C is the surface density of the material of the sheet. We can say that the dimension on the sheet of paper is equal to the exponent 2.

We can repeat the same reasoning with a metal cube, obtaining ultimately the formula

$$M = Dr^3$$

where D is the volume density of the metal used. It has of course 3 dimensions.

Now, let's follow the same reasoning in our experiment. Since the physical data we collect in the laboratory are the mass and the linear dimension of the system (the tin foil ball), the unknown parameters will be the general density k and the dimensionality exponent α, which is the fractal dimension of the system.

$$M_{\text{experimental}} = kr^{\alpha}_{\text{experimental}} \tag{1}$$

We anticipate that, in the following experiment, we show how a rolled up ball of tin foil can be considered a fractal object, and our main goal will be to determine its fractal dimension.

2 Materials and Methods

2.1 Equipment and Tools

- Precision balance
- Caliber
- Micrometer

- Drawing rule and square
- Scissors
- Aluminum foil

2.2 Experimental Procedure

We begin by checking that we have everything at our disposal. In order to do so, we set up the laboratory as shown in Fig. (2). Then we start off from the aluminum foil and we cut, using the scissors, a few square sheets of tin foil. We had to be careful in cutting squares as perfect as possible in order to reduce the error in the measurementes; to do so, we make use of both the ruler and the square. We cut squares of linear dimensions of 2 mm, 5 mm, 8 mm, 11 mm, 14 mm, 17 mm, 20 mm, 23 mm, 26 mm and 29 mm.

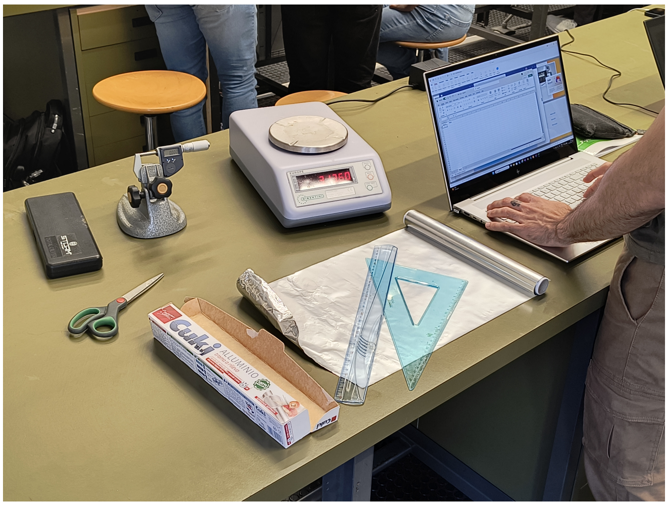


Figure 2: Our laboratory setup

Next, for each square, we measure the mass three times using the precision balance. We also make measurements of the different lenghts of the square as represented in Fig (3). Finally, we collect all data in an excel spreadsheet which we show in the next section.

Now, we roll up every tin foil square into a sphere trying to have each ball at the same density as shown in Fig. (4). Since we do this operation by hand, we can only have an idea of its density. This is the step in the

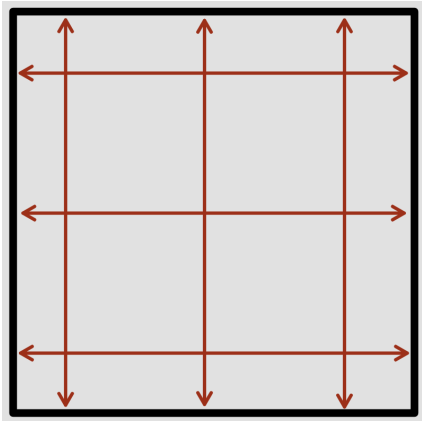


Figure 3: All the direction in which we measured the lenght of the side of the aluminum foil squares

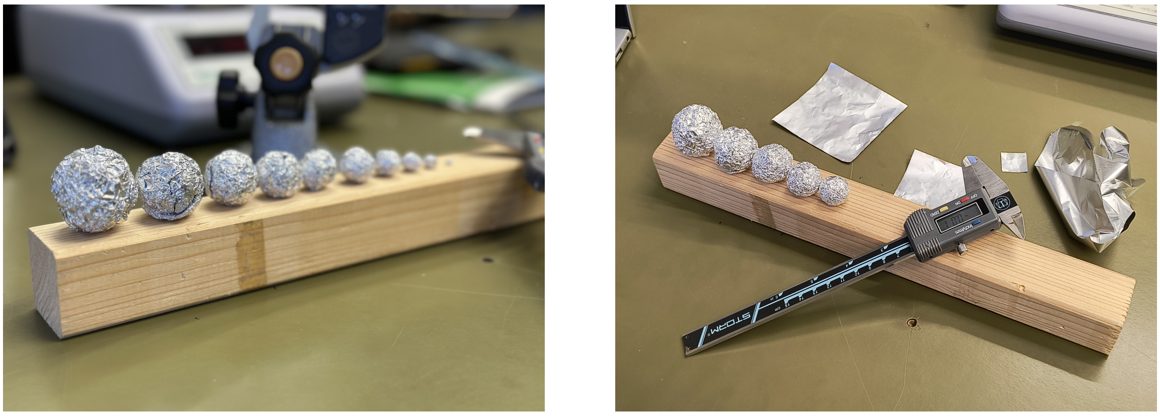


Figure 4: Some of the tin foil balls we made, the caliber, and some squares of tin foil in the background.

Once finished, our last measurent was to collect the information of the linear dimension (its diameter) of each ball. We use the caliber or the micrometer (depending on the size of the ball) to measure the diameter of the ball along three different axis. We put this data in an excel spreadsheet too.

3 Results

3.1 Part 1: Aluminum Foil Squares

The data collected from the aluminum foil squares is shown in Tables (1) and (2); for instance, as explained in the preceding section, for each square, we measured six times its linear dimension along different axis in order to reduce the error in the measurements. In the same way, while measuring the mass of the squares, we repeated the same procedure three times and the related data is shown in Table (2).

Square (cm)	$\overline{L} \pm \Delta L$ (cm)	$\overline{m} \pm \Delta m$ (g)
29x29	29.02 ± 0.11	2.80 ± 0.01
26x26	26.02 ± 0.13	2.28 ± 0.01
23x23	23.00 ± 0.10	1.77 ± 0.01
20x20	20.02 ± 0.11	1.34 ± 0.01
17x17	17.02 ± 0.11	0.97 ± 0.01
14x14	14.00 ± 0.12	0.66 ± 0.01
11x11	11.00 ± 0.10	0.40 ± 0.01
8x8	7.98 ± 0.11	0.21 ± 0.01
5x5	5.00 ± 0.10	0.08 ± 0.01
3x3	3.00 ± 0.10	0.02 ± 0.01

Table 1: Measurements of Squares with Average Length and Mass with their respective uncertainties

3.2 Part 2: Crumpled Aluminum foils

Square (cm)	$\overline{D} \pm \Delta D$ (mm)
29x29	35.348 ± 0.903
26x26	29.245 ± 0.881
23x23	25.852 ± 0.327
20x20	23.304 ± 1.319
17x17	20.240 ± 1.177
14x14	18.170 ± 0.720
11x11	14.950 ± 0.824
8x8	10.643 ± 0.386
5x5	7.535 ± 0.305
3x3	3.117 ± 0.212

Table 2: Diameters values for each aluminum ball with their respective uncertainties

For the second part of the experiment, after having rolled up all the squares into balls, we proceeded to measure each diameter as explained in the previous section. The data collected from this measurements is reported in Table (2).

3.3 Remarks on the Data processing

We also compute the natural logarithm of the mean values of the mass, the diameter and the lenght shown in the tables above. The related errors are calculated using traditional methods of error propagation [2], having taking into account the uncertainties of both casual and systematic nature. The results are shown in Table (3). These quantities will be useful in the following sections for the computation of the fractal dimension of the system using the well known Least Squares method.

$\ln(L) \pm \Delta\ln(L)$	$\ln(m) \pm \Delta\ln(m)$	$\ln(D) \pm \Delta\ln(D)$
3.368 ± 0.004	1.028 ± 0.004	3.565 ± 0.253
3.259 ± 0.005	0.824 ± 0.004	3.376 ± 0.261
3.135 ± 0.004	0.571 ± 0.006	3.252 ± 0.100
2.997 ± 0.005	0.293 ± 0.007	3.149 ± 0.419
2.834 ± 0.006	-0.030 ± 0.010	3.008 ± 0.391
2.639 ± 0.008	-0.416 ± 0.015	2.900 ± 0.248
2.398 ± 0.009	-0.916 ± 0.025	2.705 ± 0.305
2.077 ± 0.014	-1.561 ± 0.048	2.365 ± 0.163
1.609 ± 0.020	-2.526 ± 0.125	2.020 ± 0.151
1.099 ± 0.033	-3.912 ± 0.500	1.137 ± 0.187

Table 3: Natural Logarithms and Their Errors

4 Discussion and Analysis

4.1 Aluminum Foil Squares

Once the data has been fully analysed, we can start by discussing the main results obtained from the experiment. In particular, studying the plot shown in Fig. (5) which describes the relationship between the mass of the aluminum squares and their linear dimension, we can easily observe (as expected) that the data is well represented by a power law.

The power law obtained from the data is given by the the equation $M = 0.0034L^{1.9928}$ where M corresponds to the mass of the aluminum squares and L to its linear dimension. The exponent of the power law in equation (1) is very close to 2, which is the expected value for a two-dimensional object; infact, we obtain a value of $\alpha = 1.99 \pm 0.01$ for the

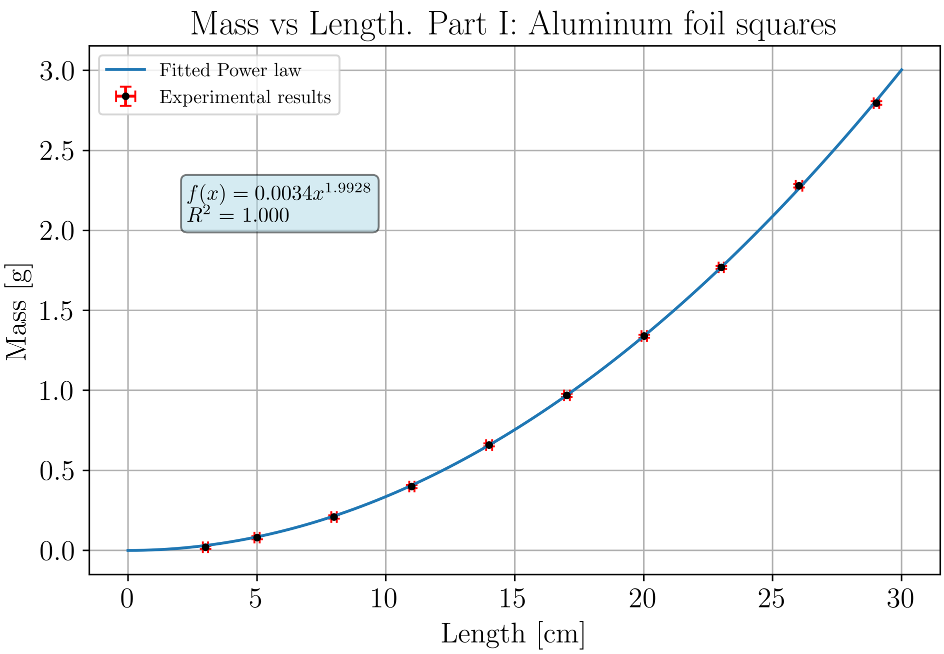


Figure 5: Mass vs Length of the Aluminum Squares

exponent α and for the parameter k the fit provided a value of $k = 0.0034 \pm 0.0001$. This law was obtained numerically using the Python library `scipy` and the function `curve_fit`, which uses a non-linear least squares method to compute these parameters. This is a very satisfactory result supported by the associated R^2 parameter, which validates the fit of the data to the expected power law.

In addition, we can consider a different approach to the problem by using the Least Squares method to compute both parameters of the power law. With this aim, we calculate the natural logarithm of the general law $M = kL^\alpha$, obtaining

$$\ln(M) = \ln(k) + \alpha \ln(L), \tag{2}$$

which clearly describe a linear dependence between the natural logarithm of the mass and the natural logarithm of the linear dimension. Using the data from Table (3), we can plot this relationship and compute the parameters k and α using the Least Squares method. The results are shown in Fig. (6).

As expected, we obtained that the data is well represented by a linear function, whose parameters $\ln(k)$ and α correspond to the slope and the intercept of the line, respectively.

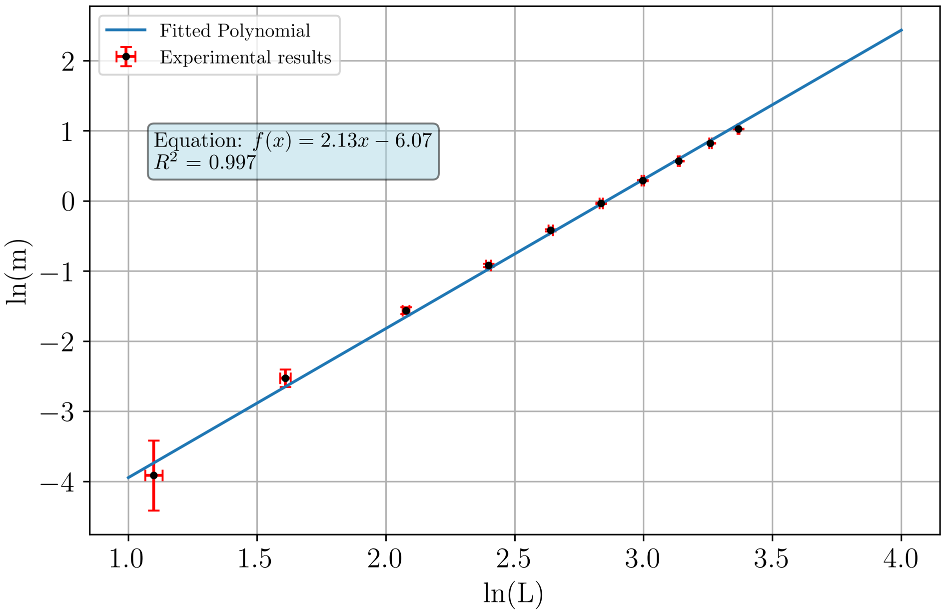


Figure 6: Natural logarithm of Mass vs Natural logarithm of Length of the Aluminum Squares. Alternative approach for the computation of the dimensionality parameters.

The values obtained for these parameters are

$$\ln(k) = -6.07 \pm 0.11 \implies k = 0.0020 \pm 0.0003, \tag{3}$$

$$\alpha = 2.13 \pm 0.04.$$

These values are in perfect agreement with the ones obtained from the previous method, which validates the results obtained from the experiment.

4.2 Crumpled Aluminum Foil Spheres

On the other hand, for the second part of the experiment, we follow the same procedure as before, and now we study the relationship between the mass of the aluminum squares (now crumpled aluminum balls) and their linear dimension: the diameter. The plot shown in Fig. (7) describes this relationship and, as expected, the data is well represented by a power law whose parameters are $\alpha = 1.97 \pm 0.14$ and $k = 0.003 \pm 0.001$. However, the exponent we obtained in this case contradicts our predictions since we expected a value ranging from 2 to 3. This discrepancy could be due to the fact that the balls were not perfectly rolled up, which could have affected the measurements of the diameter and consequently the results obtained from the experiment. Even if the R^2 parameter is very close to 1, we can not consider the results obtained from this part of the experiment as satisfactory. We consider that, one possible way to improve these results, could be collecting more data regarding the diameter of each ball by measuring them across several more different axis.

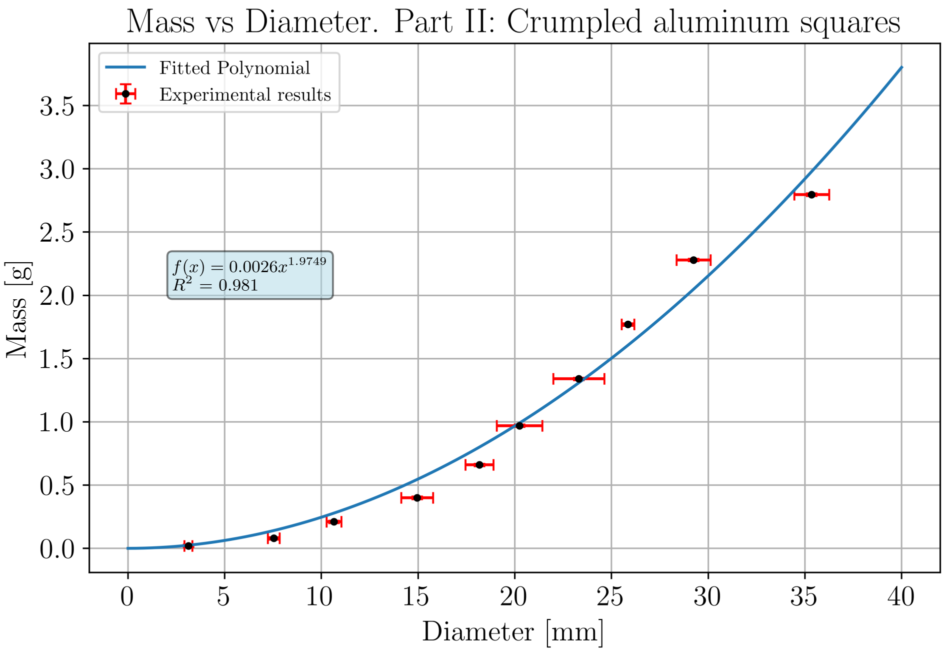


Figure 7: Mass vs Diameter of the Aluminum Balls. As expected, the data is well represented by a power law.

Finally, we compare the previous results with the ones obtained from the Least Squares method. The plot shown in Fig. (8) describes the relationship between the natural logarithm of the mass and the natural logarithm of the diameter of the aluminum balls.

As before, we use the Least Square method to compute the parameters of the power law we are interested in. In Fig. (8), we show the linear law that best fits the data, whose parameters are

$$\ln(k) = -6.60 \pm 0.22 \implies k = 0.0010 \pm 0.0003, \tag{4}$$

$$\alpha = 2.16 \pm 0.08$$

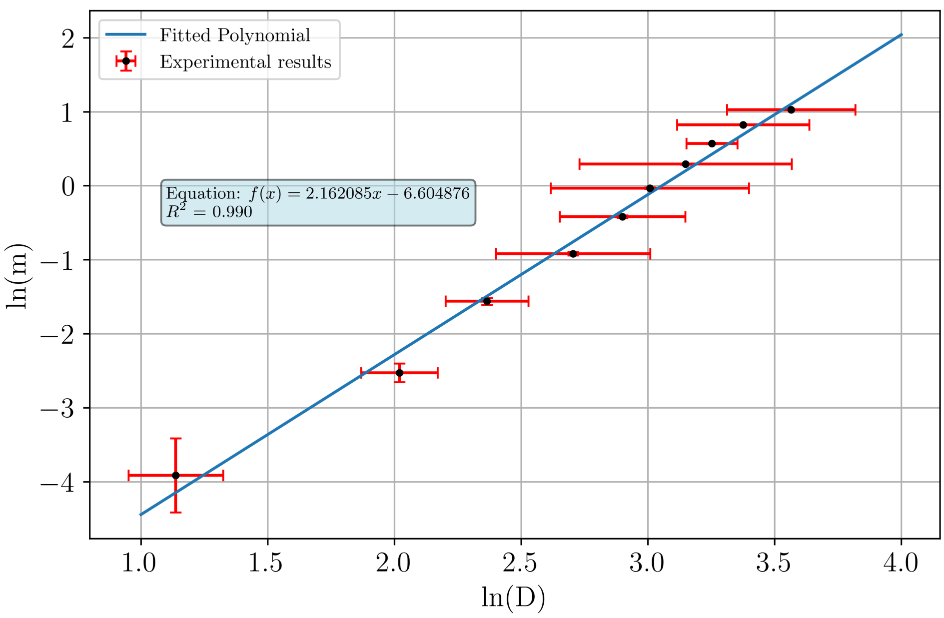


Figure 8: Natural logarithm of Mass vs Natural logarithm of Diameter of the Aluminum Balls. Alternative approach for the computation of the dimensionality parameters.

These results are closer to the ones expected and we consider them more reliable than the ones obtained from the previous method.

4.3 Further analysis and observations

We now make some comments about the results obtained in the previous section. In particular, since the parameters obtained from the power law and the linear regression for the second part of the experiments were not as satisfactory as expected, we decided to analyze the residuals of the experimental data and the data predicted by the models (power law and linear regression). The results are shown in Fig. (9).

As we can see from these plots, in some cases there are some residuals that are not randomly distributed around the zero line (as we expected them to be). We also observed that these residuals correspond in general to the initial and final data points, which interestingly are the ones with the highest uncertainty. We address this issue by excluding these data points from the analysis and repeating the calculations. The results obtained, for our surprise, were noticably better than the previous ones. For instance, for the second part of the experiment we obtained a value of $\alpha = 2.47 \pm 0.07$ for the power law and

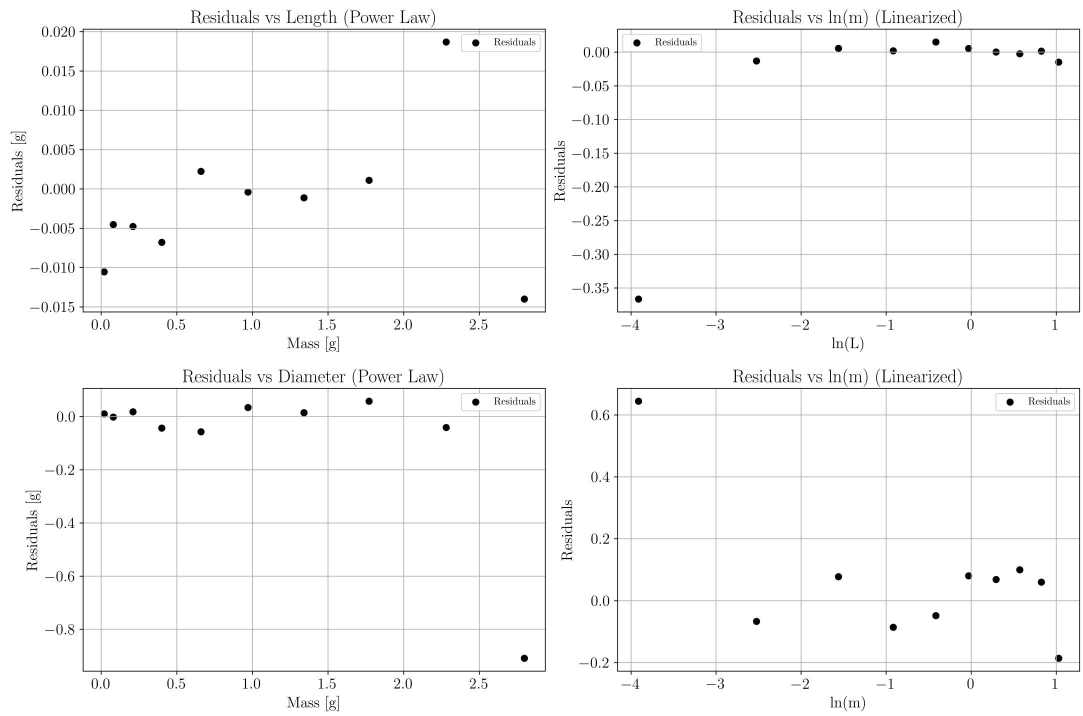


Figure 9: Residuals of the experimental data and the data predicted by the models.

using the linear regression the model provided a value of $\alpha = 2.38 \pm 0.08$. Since for the first part of the experiment our results were in agreement with the expected values, we decided to keep the initial data points during the analysis but as an interesting result, we observed that if we do not include the data from the smallest aluminum square, the linear regression gives a value of alpha of 2.02 ± 0.01, which is closer to the expected value of 2 than the one we obtained in the last section. We finally gather all the results obtained in the following table,

Table 4: Summary of Results obtained from the experiment. The last two rows show the results described in section (4.3).

Part	**Method**	α
Aluminum Square sheets	Power Law	$\alpha = 1.99 \pm 0.01$
Aluminum Square sheets	Linear Regression	$\alpha = 2.13 \pm 0.04$
Aluminum Spherical balls	Power Law	$\alpha = 1.97 \pm 0.14$
Aluminum Spherical balls	Linear Regression	$\alpha = 2.16 \pm 0.08$
Further Results 2.1	Power Law	$\alpha = 2.47 \pm 0.07$
Further Results 2.2	Linear Regression	$\alpha = 2.38 \pm 0.08$

5 Conclusion

The purpose of this experiment was to demonstrate that the fractal dimension of our aluminum tinfoil spheres, which are neither completely solid nor completely hollow, falls between 2 and 3. After following the procedure outlined in (2.2), we obtained all the data, which is shown in the tables (1) and (2). In the first part of the experiment, we measured the mass and various lengths of the aluminum square and, as expected, using the power law and linear regression to fit the data, we obtained a value of α very close to 2, which is the expected value for a two-dimensional object. The problem emerged when applying the same procedure to the aluminum spheres, where we obtained $\alpha = 1.97 \pm 0.14$. This result clearly contradicts our predictions, as we expected a value between 2 and 3. It is possible to justify this discrepancy by noting that the balls were not perfectly rolled up, which could have affected the measurements of the diameter and consequently the results obtained from the experiment. On the other hand, through the linear regression method, we obtained a value of $\alpha = 2.16 \pm 0.08$, which is closer to the expected value. To obtain more reliable results, it is possible to analyze the residuals of the experimental data and the data predicted by the models, shown in Fig. (9). It is easily noticeable from these graphs that the residuals are not randomly distributed around the zero line, and that the residuals corresponding to the initial and final data points are the ones with the highest uncertainty. We addressed this issue by excluding these data points from the analysis and repeating the calculations. The results obtained were noticeably better than the previous ones. Finally, we obtained a value of $\alpha = 2.47 \pm 0.07$ for the power law, and using the linear regression, the model provided a value of $\alpha = 2.38 \pm 0.08$, which is what we expected from the beginning. It is also interesting to note that, in the first part of the experiment, if we do not include the data from the smallest aluminum square, the linear regression gives a value of $\alpha = 2.02 \pm 0.01$, which is closer to the expected value of 2.

References

[1] Michael Rubinstein and Ralph H. Colby. *Polymer Physics*. Jan. 2003.

[2] John Robert Taylor. *An introduction to error analysis*. Univ Science Books, Jan. 1997.