4.3 A Robust Correlation Estimator

The following theorem will give us a good result about the robust correlation estimator which we define. As described in the beginning of this chapter, this approach is based on the first part of the nonparanormal SKEPTIC algorithm as described by [LHY+12].

Theorem 4.9 (Robust Correlation Estimator).
Let (X, Y) be a $\mathbb{R}^2$-valued random variable that follows an elliptical distribution. Let $\rho = cor(X, Y)$ be the (Pearson-) correlation. Furthermore, let $(X_i, Y_i)_{i\in\{1,\ldots,n\}}$ be a sample of (X, Y), and $\hat{\tau} := \hat{\tau}_n$ be the sample estimator of Kendall's τ, as defined in (21), where we drop the n for notational convenience. We now define an estimator for the (Pearson-) correlation by:

$$\hat{\rho}_\tau := \sin(\frac{\pi}{2}\hat{\tau}).$$

Then, for all $s > 0$:

$$P(|\hat{\rho}_\tau - \rho| > s) \leq 2e^{-\frac{ns^2}{\pi^2}}.$$

Proof. As noted in Remark 4.6, $\hat{\tau}$ is a U-statistics with $r = 2$, n representing the sample size and the function g as defined there. That g is bounded between -1 and 1. We can therefore apply Theorem 4.8, which gives us

$$P(|\hat{\tau} - \tau| \geq s) \leq 2e^{-\frac{2s^2\frac{n}{2}}{2^2}} = 2e^{-\frac{ns^2}{4}}. \tag{143}$$

In the following inequality we use the fact, that for (X, Y) elliptically distributed, we have, from (23) the relationship $\rho = \sin(\frac{\pi}{2}\tau)$. We also use the inequality $|\sin(x) - \sin(y)| \leq |x - y|$. To prove this inequality, let $x, y \in \mathbb{R}$. Then, there exists a $\xi \in [a, b]$ such that

$$\sin(x) = \sin(y) + \sin'(\xi)(x - y) = \sin(y) + \cos(\xi)(x - y).$$

This implies that

$$|\sin(x) - \sin(y)| = |\cos(\xi)||x - y| \leq |x - y|.$$

Now we can prove the main result:

$$\begin{aligned} P(|\hat{\rho}_\tau - \rho| > s) &= P(|\sin(\frac{\pi}{2}\hat{\tau}) - \sin(\frac{\pi}{2}\tau)| > s) \leq \\ &\leq P(|\frac{\pi}{2}\hat{\tau} - \frac{\pi}{2}\tau| > s) = P(|\hat{\tau} - \tau| > \frac{2s}{\pi}) \overset{(143)}{\leq} 2e^{-\frac{n4s^2}{4\pi^2}} = 2e^{-\frac{ns^2}{\pi^2}}. \end{aligned} \tag{144}$$

□