Extreme Correlation of International Equity Markets

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ABSTRACT

Testing the hypothesis that international equity market correlation increases in volatile times is a difficult exercise and misleading results have often been reported in the past because of a spurious relationship between correlation and volatility. Using "extreme value theory" to model the multivariate distribution tails, we derive the distribution of extreme correlation for a wide class of return distributions. Empirically, we reject the null hypothesis of multivariate normality for the negative tail, but not for the positive tail. We also find that correlation is not related to market volatility per se but to the market trend. Correlation increases in bear markets, but not in bull markets.

INTERNATIONAL EQUITY MARKET CORRELATION has been widely studied. Previous studies[1] suggest that correlation is larger when focusing on large absolute-value returns, and that this seems more important in bear markets. The conclusion that international correlation is much higher in periods of volatile markets (large absolute returns) has indeed become part of the accepted wisdom among practitioners and the financial press. However, one should exert great care in testing such a proposition. The usual approach is to condition the estimated correlation on the observed (or ex post) realization of market returns. Unfortunately correlation is a complex function of returns and such tests can lead to wrong conclusions, unless the null hypothesis and

[1] See Lin, Engle, and Ito (1994), Erb, Harvey, and Viskanta (1994), Longin and Solnik (1995), Karolyi and Stulz (1996), Solnik, Bourcrelle, and Le Fur (1996), De Santis and Gérard (1997), Ramchmand and Susmel (1998), Ang and Bekaert (1999), and Das and Uppal (1999).

its statistics are clearly specified. To illustrate our point, let us consider a simple example where the distribution of returns on two markets (say the United States and the United Kingdom) is multivariate normal with zero mean, unit standard deviation, and a constant correlation of 0.50. Let us split the sample in two fractiles (50 percent) based on absolute values of U.S. returns. The first fractile consists of "small" returns (absolute returns lower than 0.674) and the second fractile consists of "large" returns (absolute returns higher than 0.674). Under the assumption of bivariate normality with constant correlation, the conditional correlation[2] of small returns is 0.21 and the conditional correlation of large returns is 0.62. It would be wrong to infer from this large difference in conditional correlation that correlation differs between volatile and tranquil periods, as correlation is constant and equal to 0.50 by assumption. Boyer, Gibson, and Loretan (1999) further show that conditional correlation is highly nonlinear in the level of return on which it is conditioned. They also indicate that a similar problem exists when the true data-generating process is not multivariate normal but follows a GARCH model.

An obvious implication is that one cannot conclude that the "true" correlation is changing over time by simply comparing estimated correlations conditional on different values of one (or both) return variable. First, the distribution of the conditional correlation that is expected under the null hypothesis (e.g., a multivariate normal distribution) must be clearly specified in order to test whether correlation increases in periods of volatile markets. This has not been done so far.

In this paper, we study the conditional correlation structure of international equity returns and derive a formal statistical method, based on extreme value theory. We can derive the asymptotic distribution of conditional tail correlation, which is not possible for other parts of the distribution of the conditional correlation. Extreme value theory only provides asymptotic results, but it offers the benefit that its asymptotic results hold for a wide range of parametric distributions of returns, not only the multivariate normal. An attractive feature of the methodology is that the asymptotic tail distribution is characterized by very few parameters regardless of the actual distribution.

A first contribution of this paper is to provide a method to formally test whether these correlations deviate from what would be expected under multivariate normality. More importantly, this paper contributes to the debate on market correlations in periods of extreme returns by providing a stark empirical distinction between bear and bull markets. High volatility per se (i.e., large absolute returns) does not seem to lead to an increase in conditional correlation. Correlation is mainly affected by the market trend. We find that it is only in bear markets that conditional correlation strongly increases;

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[2] Our results are obtained from simulations of a multivariate normal distribution and can be easily replicated. Forbes and Rigobon (1998) and Boyer, Gibson, and Loretan (1999) provide some analytical derivations.

conditional correlation does not seem to increase in bull markets. Our empirical distinction between bear and bull markets has potential implications for asset allocation and portfolio construction, but we do not explore them here. Although we do not suggest the exact time-varying distribution that should be used, our results lead to the rejection of a large class of models that would be inconsistent with our findings. This is the case of the multivariate normal distribution with constant volatility and correlation. It is also the case of a multivariate GARCH process with time-varying volatilities but constant correlation, in which extreme returns can be generated by *different* volatility regimes. Furthermore, Ang and Bekaert (1999) show that a fairly general asymmetric GARCH[3] also cannot reproduce the asymmetric correlations that we document. On the other hand, regime-switching models as proposed by Das and Uppal (1999) or Ang and Bekaert (1999) could be consistent with our empirical findings. The asymmetric correlation pattern should become a key property for any multivariate equity return model to match.

The paper is organized as follows: the first section presents some theoretical results about the extremes of univariate and multivariate random processes. It summarizes the main results of extreme value theory and draws the implications for the correlation of extreme returns. The second section deals with the econometric methodology, the third section presents the empirical results, and the fourth section concludes.

I. Correlation of Extreme Returns: Theory

Extreme value theory involves two modeling aspects: the tails of the marginal distributions and the dependence structure of extreme observations.

A. *The Univariate Case: Modeling of the Distribution Tails*

Let us call R the return on a portfolio and F_R the cumulative distribution function of R. The lower and upper endpoints of the associated density function are denoted by (l, u). For example, for a variable distributed normally, $l = -\infty$ and $u = +\infty$. In this paper, extreme returns are defined in terms of exceedances with reference to a threshold denoted by θ. For example, positive θ-exceedances correspond to all observations of R greater than the threshold θ (results for negative exceedances can be deduced from those for positive exceedances by consideration of symmetry). A return R is higher than θ with probability p and lower than θ with probability $1 - p$. The probability p is linked to the threshold θ and the distribution of returns F_R by the relation: $p = 1 - F_R(\theta)$. We focus on the case $(R > \theta)$ which defines the (right) tail of the distribution of returns.

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[3] Our simulations lead to similar conclusions. Analytical results cannot be derived except for the simplest distributions (normal).

The cumulative distribution of θ-exceedances, denoted by F_R^θ and equal to $(F_R(x) - F_R(\theta))/(1 - F_R(\theta))$ for $x > \theta$, is exactly known if the distribution of returns F_R is known. However, in most financial applications, the distribution of returns is not precisely known and, therefore, neither is the exact distribution of return exceedances. For empirical purposes, the *asymptotic* behavior of return exceedances needs to be studied. Extreme value theory addresses this issue by determining the possible nondegenerate limit distributions of exceedances as the threshold θ tends to the upper point u of the distribution. In statistical terms, a limit cumulative distribution function denoted by G_R^θ satisfies the following condition: $\lim_{\theta \to u} \sup_{\theta < x < u} |F_R^\theta(x) - G_R^\theta(x)| = 0$. Balkema and De Haan (1974) and Pickands (1975) show that the generalized Pareto distribution (GPD) is the only nondegenerate distribution that approximates the distribution of return exceedances F_R^θ. The limit distribution function G_R^θ is given by

$$G_R^\theta(x) = 1 - (1 + \xi \cdot (x - \theta)/\sigma)_+^{1/\xi}, \tag{1}$$

where σ, the dispersion parameter, depends on the threshold θ and the distribution of returns F_R, and ξ, the tail index, is intrinsic to the distribution of returns F_R (the + operator gives the positive part of the expression in parentheses).

The tail index ξ gives a precise characterization of the tail of the distribution of returns. Distributions with a power-declining tail (fat-tailed distributions) correspond to the case $\xi > 0$, distributions with an exponentially declining tail (thin-tailed distributions) to the case $\xi = 0$, and distributions with no tail (finite distributions) to the case $\xi < 0$.

For a particular return distribution, the parameters of the limit distribution can be computed (see Embrechts, Klüppelberg, and Mikosch (1997)). For example, the normal and log-normal distributions commonly used in finance lead to a GPD with $\xi = 0$. The Student-t distributions and stable Paretian laws lead to a GPD with $\xi > 0$ and the uniform distribution belongs to a GPD with $\xi < 0$. The extreme value theorem has also been extended to processes which are not i.i.d. Leadbetter, Lindgren, and Rootzén (1983) consider various processes based on the normal distribution: autocorrelated normal processes, discrete mixtures of normal distributions and mixed diffusion jump processes. All have thin tails so that they lead to a GPD with $\xi = 0$. De Haan et al. (1989) show that if returns follow the GARCH process, then the extreme return has a GDP with $\xi < 0.5$.

To summarize the univariate case, extreme value theory shows that the distribution of return exceedances can only converge toward a generalized Pareto distribution. This result is robust as it is also obtained for non-i.i.d. return processes commonly used in finance. Hence, for a given threshold, the distribution tail in the univariate case is perfectly described by three parameters: the tail probability, the dispersion parameter, and the tail index.

B. Multivariate Case: Modeling of the Dependence Structure

Let us consider a q-dimensional vector of random variables denoted $R = (R_1, R_2, \ldots, R_q)$. Multivariate return exceedances correspond to the vector of univariate return exceedances defined with a q-dimensional vector of thresholds $\theta = (\theta_1, \theta_2, \ldots, \theta_q)$. As for the univariate case, when the return distribution is not exactly known, we need to consider asymptotic results. The possible limit nondegenerate distributions G_R^θ satisfying the limit condition must satisfy two properties:[4]

1. Its univariate marginal distributions $G_{R_1}^{\theta_1}, [G_{R_2}^{\theta_2}, \ldots, G_{R_q}^{\theta_q}$ are generalized Pareto distributions.
2. There exists a function called the dependence function denoted by D_{G_R}, which maps from $\mathfrak{R}^q$ into $\mathfrak{R}$, and satisfies the following condition:

$$G_R^\theta(x_1, x_2, \ldots, x_q) = \exp(-D_{G_R}(-1/\log G_{R_1}^{\theta_1}(x_1), -1/\log G_{R_2}^{\theta_2}(x_2),$$
$$\ldots, -1/\log G_{R_q}^{\theta_q}(x_q))). \qquad (2)$$

As in the univariate case, the generalized Pareto distribution plays a central role. However, unlike the univariate case, the multivariate asymptotic distribution is not completely specified as the shape of the dependence function D_{G_R} is not known.

When the components of the multivariate distribution of extreme returns are asymptotically independent, the dependence function D_{G_R} is characterized by:

$$D_{G_R}(y_1, y_2, \ldots, y_q) = \left(\frac{1}{y_1} + \frac{1}{y_2} + \ldots + \frac{1}{y_q} \right), \qquad (3)$$

where $y_i = -1/\log G_{R_i}^{\theta_i}(x_i)$. Actually, asymptotic independence of extreme returns is reached in many cases. Of course, when the components of the return distribution themselves are independent, exact independence of extreme returns is obtained. But more surprisingly, asymptotic independence is often reached when the components of the return distribution are not independent. An important example is the multivariate normal distribution (see Galambos (1978, pp. 257–258) and Embrechts, McNeil, and Straumann (1998)).

B.1. Asymptotic Independence and Multivariate Normality

If all correlation coefficients between any two components of a multivariate normal process are different from ± 1, then the return exceedances of all variables tend to independence as the threshold used to define the tails tends to the upper endpoint of the distribution of returns ($+\infty$ for the normal

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[4] See Ledford and Tawn (1997). A general presentation of multivariate extreme value theory can be found in Galambos (1978) and Resnick (1987). Specific results for the bivariate case are given in Tawn (1988).

distribution). In particular, the asymptotic correlation of extreme returns is equal to zero. For example, considering a bivariate normal process with standard mean and variance and a correlation of 0.80, the correlation is equal to 0.48 for return exceedances one standard deviation away from the mean, 0.36 for return exceedances two standard deviations away from the mean, 0.24 for return exceedances three standard deviations away from the mean and 0.14 for return exceedances four standard deviations away from the mean. It goes to zero for extreme returns.

At first, the result of asymptotic independence may seem counterintuitive and at odds with the traditional view of bivariate normality.[5] It all depends on how conditioning is conducted. A slight difference is introduced by conditioning on values in the two series, as done in extreme value theory, or on values in a single series, as done in the introduction of this paper and in most empirical studies. But the major source of difference comes from the conditioning on absolute values (two-sided) versus the conditioning on signed values (one-sided). If we condition on the absolute value of realized returns, the conditional correlation of a bivariate normal distribution trivially increases with the threshold, as mentioned in the introduction. As the normal distribution is symmetric, the truncated distribution retains the same mean as the total distribution. But a large positive (respectively negative) return in one series tends to be associated with a large positive (respectively negative) return in the other series, so the estimated conditional correlation is larger than the "true" constant correlation. Conditional correlation increases with the threshold (see also Forbes and Rigobon (1998) and Boyer et al. (1999)). Here, we condition on signed extremes (e.g, positive or negative). The mean of the truncated distribution is not equal to the mean of the total distribution. As indicated above, the conditional correlation of a multivariate normal distribution decreases with the threshold and reaches zero for extreme returns. A false intuition would be that extreme returns in two series appear highly correlated as they are large compared with the mean of all returns. Extreme value theory says that two extreme returns are not necessarily correlated, as they may not always be large compared with the mean of extreme returns.

B.2. The General Case

For the general case with asymptotically dependent components for the multivariate distribution of extreme returns, the form of the dependence function is not known, and it has to be modeled.[6] A model commonly used in the literature is the logistic function proposed by Gumbel (1961).[7] The dependence function denoted by D_l is given by

$$D_l(y_1, y_2, \ldots, y_q) = (y_1^{-1/\alpha} + y_2^{-1/\alpha} + \ldots + y_q^{-1/\alpha})^\alpha, \tag{4}$$

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[5] We are grateful to an anonymous referee for providing useful insights on this issue.
[6] The properties of the asymptotic distribution can be worked only out in very special cases.
[7] See also Tawn (1988) and Straetmans (1998).

where parameter α, controls the level of dependence between extreme returns. In the bivariate case ($q = 2$), the correlation coefficient ρ of extremes is related to the coefficient α by $\rho = 1 - \alpha^2$ (Tiago de Oliveira, 1973). The special cases $\alpha = 1$ and $\alpha = 0$ correspond respectively to asymptotic independence ($\rho = 0$) and total dependence ($\rho = 1$).

Although arbitrary, the logistic model used in engineering studies presents several advantages: It includes the special cases of asymptotic independence and total dependence, and it is parsimonious, as only one parameter is needed to model the dependence among extremes. An attractive feature of the methodology is that the asymptotic tail distribution is characterized by very few parameters regardless of the actual conditional distribution.

To summarize the multivariate case, extreme value theory shows that the distribution of extreme returns can only converge toward a distribution characterized by generalized Pareto marginal distributions and a dependence function. The shape of this function is not well defined. Consistent with the existing literature, we use the logistic function to model the dependence between extreme returns of different markets. The case where returns are multivariate normal leads to a limit case of the logistic function where the asymptotic correlation of extreme returns is equal to zero. We estimate the dependence function and test whether the correlation of extreme returns is equal to zero.

II. Correlation of Extreme Returns: Estimation Procedure

The choice of the threshold value is first discussed. The estimation method for the parameters of the model is then presented.

A. Optimal Threshold Values

The theoretical result about the limit distribution of return exceedances exactly holds when the threshold θ goes to the upper endpoint u of the distribution of returns. In practice, as the database contains a finite number of return observations, the threshold used for the estimation of the model is finite. The choice of its value is a critical issue. On the one hand, choosing a high value for θ leads to few observations of return exceedances and implies inefficient parameter estimates with large standard errors. On the other hand, choosing a low value for θ leads to many observations of return exceedances, but induces biased parameter estimates, as observations not belonging to the tails are included in the estimation process. To optimize this trade-off between bias and inefficiency, we use a Monte Carlo simulation method. Return time series are simulated from a known distribution for which the tail index can be computed. For each time series, the tail index value is estimated for different threshold levels. The choice of the optimal value is based on the mean square error (MSE) criterion, which allows one to take into account the trade-off between bias and inefficiency. The procedure is detailed in Appendix 1.

B. *Modeling of the Tails of the Marginal Distributions*

The model presented in the previous section is multivariate. In the empirical study, we deal with bivariate models. This choice is justified by a theoretical result that demonstrates that multivariate independence can be tested using bivariate pairs of variables (see Tiago de Oliveira (1962) and Reiss (1989, pp. 234–237)).

Following Davison and Smith (1990) and Ledford and Tawn (1997), the limiting result about the distribution of exceedances presented in Section I is taken to derive a model of the tails of each marginal distribution. Considering return exceedances defined from returns R_1 and R_2 in two markets with thresholds θ_1 and θ_2, the tail of the distribution of each return R_i denoted by $F_{R_i}^{\theta_i}$ for $i = 1$ and 2 is modeled as follows:

$$F_{R_i}^{\theta_i}(x_i) = (1 - p_i) + p_i \cdot G_{R_i}^{\theta_i}(x_i) = 1 - p_i \cdot (1 + \xi_i \cdot (x_i - \theta_i)/\sigma_i)_+^{-1/\xi_i}, \qquad (5)$$

which simply expresses that a return R_i either does not belong to the tail with probability $1 - p_i$ or is drawn from the limit univariate distribution $G_{R_i}^{\theta_i}$ of positive return θ_i-exceedances with probability p_i. In other words, for a return that does not exceed the threshold θ_i the only relevant information it conveys to the model is that it occurs below the threshold, not its actual value. In the construction of the likelihood function, a return R_i below θ_i is considered as censored at the threshold.

C. *Modeling of the Dependence Structure*

Following Ledford and Tawn (1997), the dependence function associated with the distribution of returns F_R is modeled with the logistic function D_l given by equation (4). The model F_R^{θ} of the bivariate distribution of return exceedances is given by

$$F_R^{\theta}(x_1, x_2) = \exp(-D_l(-1/\log F_{R_1}^{\theta_1}(x_1), -1/\log F_{R_2}^{\theta_2}(x_2))). \qquad (6)$$

For given thresholds θ_1 and θ_2, the bivariate distribution of return exceedances is then described by seven parameters: the tail probabilities (p_1 and p_2), the dispersion parameters (σ_1 and σ_2) and the tail indexes (ξ_1 and ξ_2) for each variable, and the dependence parameter of the logistic function (α) or equivalently the correlation of extreme returns (ρ). The parameters of the model are estimated by the maximum likelihood method. Details of the construction of the likelihood function are given in Appendix 2.

III. **Correlation of Extreme Returns: Empirical Evidence**

We estimate the multivariate distribution of return exceedances and test the null hypothesis of normality focusing on the correlation of extreme returns.

A. Data

We use monthly equity index returns for five countries: the United States (US), the United Kingdom (UK), France (FR), Germany (GE), and Japan (JA). Data for the period January 1959 to December 1996 (456 observations) come from Morgan Stanley Capital International (MSCI). A description of the data can be found in Longin and Solnik (1995).

B. Threshold Values

We consider return exceedances defined with various predetermined threshold levels: ± 0 percent, ± 3 percent, ± 5 percent, ± 8 percent, and ± 10 percent (percentage points) away from the empirical mean of each country. In selecting large thresholds, we are constrained by the fact that there are very few monthly observations below -10 percent or above $+10$ percent.

We also consider return exceedances defined with optimal thresholds (see Appendix 1). Optimal threshold values are different for the left tail and the right tail of the return distribution. For example, considering the United States, it is optimal to use 25 negative tail observations corresponding to a threshold of -6.12 percent for the left tail, and 18 positive tail observations defining a threshold of $+7.21$ percent for the right tail. Optimal threshold values also depend on the country. For example, considering the left tail, the following numbers of negative tail observations with the corresponding threshold values in parentheses are: 25 (-6.12 percent) for the United States, 16 (-9.68 percent) for the United Kingdom, 18 (-8.38 percent) for France, 16 (-7.84 percent) for Germany, and 16 (-8.53 percent) for Japan. On average, around 20 to 30 tail observations are used representing a proportion of 4–5 percent of the total number of return observations (456).

C. Estimation of the Parameters of the Model

We use a bivariate framework, looking at the correlation of the U.S. market with the other four markets separately. Hence, we have four country pairs: US/UK, US/FR, US/GE, and US/JA. We start with a maximum-likelihood univariate estimation for each country. The estimated parameters, plus the sample unconditional correlation, are then used as starting values in the maximum-likelihood bivariate estimation.

Tables I to IV present the estimation of the bivariate distribution of return exceedances of predetermined and optimal values for the threshold θ. Estimated coefficients are presented in Panel A for negative return exceedances (return lower than the threshold θ) and in Panel B for positive return exceedances (returns higher than the threshold θ). The estimate of the tail probability p is close to the empirical probability of returns being lower or higher than the threshold considered. For example, the estimated value of the probability p^{US} of U.S. monthly returns lower than $\theta = -3$ percent is equal to 0.194 with a standard error of 0.018 whereas, over the period January 1959 to December 1996, there are 86 out of 456 monthly returns under

Table I
Estimation of the Bivariate Distribution of U.S. and U.K. Return Exceedances

This table gives the maximum likelihood estimates of the parameters of the bivariate distribution of U.S. and U.K. return exceedances (Panel A for negative return exceedances and Panel B for positive return exceedances). Return exceedances are defined with a threshold θ. Both fixed and optimal levels are used for θ. Fixed levels (defined as percentage points) are: 0 percent, ± 3 percent, ± 5 percent, ± 8 percent, and ± 10 percent away from the empirically observed means of monthly returns (the same value of θ is then taken for the two countries: $\theta = \theta^{US} = \theta^{UK}$). Optimal levels are computed by the procedure described in Appendix 1. They are given for the United States and the United Kingdom on the last line of each panel. Seven parameters are estimated: the tail probability p, the dispersion parameter σ, the tail index ξ for each country, and the correlation of return exceedances ρ of the logistic function used to model the dependence between extreme returns. Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}$ is also tested. Two cases are considered: the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances of thresholds tending to infinity, denoted by ρ_{nor}^{asy}, is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold θ, denoted by $\rho_{nor}^{f.s.}(\theta)$, is computed by simulation assuming that monthly returns follow a bivariate-normal distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. Both a likelihood ratio test (LR test) between the constrained model ($\rho = \rho_{nor}^{asy} = 0$ in the asymptotic case and $\rho = \rho_{nor}^{f.s.}(\theta)$ in the finite-sample case) and the unconstrained model, and a Wald test (W test) on the correlation coefficient are carried out. The p values of the tests are given below in brackets.

Threshold	Parameters of the Model							$H_0: \rho = \rho_{nor}^{asy} = 0$		$H_0: \rho = \rho_{nor}^{f.s.}(\theta)$	
θ	p^{US}	σ^{US}	ξ^{US}	p^{UK}	σ^{UK}	ξ^{UK}	$\rho^{US/UK}$	LR test	W test	LR test	W test
				Panel A: Negative Return Exceedances							
-10%	0.016	1.480	0.672	0.040	3.188	0.246	0.676	36.465	5.160	11.382	3.573
	(0.006)	(0.962)	(0.697)	(0.009)	(1.238)	(0.311)	(0.131)	[0.000]	[0.000]	[0.001]	[0.000]
-8%	0.034	2.733	0.149	0.062	3.520	0.146	0.600	44.138	5.310	7.653	2.708
	(0.008)	(0.901)	(0.240)	(0.011)	(1.074)	(0.244)	(0.113)	[0.000]	[0.000]	[0.006]	[0.007]
-5%	0.106	2.349	0.157	0.160	2.997	0.151	0.553	73.143	7.681	5.243	2.236
	(0.014)	(0.490)	(0.154)	(0.017)	(0.527)	(0.129)	(0.072)	[0.000]	[0.000]	[0.022]	[0.025]

−3%	0.194 (0.018)	3.075 (0.403)	0.013 (0.080)	0.194 (0.018)	3.698 (0.459)	0.037 (0.083)	0.579 (0.055)	102.882 [0.000]	10.527 [0.000]	6.261 [0.012]	2.564 [0.010]
0%	0.476 (0.023)	3.437 (0.276)	−0.044 (0.043)	0.476 (0.023)	4.195 (0.372)	−0.017 (0.060)	0.530 (0.045)	120.057 [0.000]	11.778 [0.000]	0.484 [0.487]	0.444 [0.657]
−6.12 −9.68%	0.056 (0.011)	2.428 (0.681)	0.178 (0.219)	0.036 (0.009)	2.951 (1.374)	0.286 (0.414)	0.578 (0.121)	36.393 [0.000]	4.777 [0.000]	4.606 [0.032]	2.012 [0.044]
Panel B: Positive Return Exceedances											
0%	0.534 (0.023)	3.402 (0.262)	−0.156 (0.045)	0.524 (0.023)	3.934 (0.262)	0.012 (0.038)	0.415 (0.051)	57.186 [0.000]	8.137 [0.000]	4.297 [0.038]	−1.863 [0.062]
+3%	0.217 (0.019)	1.911 (0.309)	0.105 (0.127)	0.254 (0.020)	2.853 (0.363)	0.159 (0.081)	0.353 (0.065)	35.244 [0.000]	5.431 [0.000]	2.045 [0.153]	−1.308 [0.191]
+5%	0.072 (0.012)	3.196 (0.732)	−0.199 (0.153)	0.132 (0.016)	2.475 (0.492)	0.310 (0.142)	0.360 (0.090)	24.018 [0.000]	4.000 [0.000]	0.205 [0.651]	−0.356 [0.722]
+8%	0.023 (0.007)	2.889 (1.261)	−0.274 (0.302)	0.046 (0.010)	2.483 (0.951)	0.548 (0.325)	0.293 (0.140)	9.775 [0.002]	2.093 [0.036]	0.001 [0.994]	−0.007 [0.994]
+10%	0.013 (0.005)	0.976 (0.911)	0.317 (0.687)	0.023 (0.007)	4.468 (2.444)	0.390 (0.458)	0.189 (0.173)	4.521 [0.033]	1.092 [0.275]	0.010 [0.920]	−0.110 [0.913]
+7.21% +6.70%	0.039 (0.009)	3.020 (0.925)	−0.256 (0.191)	0.096 (0.015)	2.449 (0.546)	0.423 (0.364)	0.226 (0.120)	7.638 [0.006]	1.886 [0.059]	0.612 [0.434]	−0.714 [0.475]

Table II
Estimation of the Bivariate Distribution of U.S. and French Return Exceedances

This table gives the maximum likelihood estimates of the parameters of the bivariate distribution of U.S. and French return exceedances (Panel A for negative return exceedances and Panel B for positive return exceedances). Return exceedances are defined with a threshold θ. Both fixed and optimal levels are used for θ. Fixed levels (defined as percentage points) are: 0 percent, ± 3 percent, ± 5 percent, ± 8 percent, and ± 10 percent away from the empirically observed means of monthly returns (the same value of θ is then taken for the two countries: $\theta = \theta^{US} = \theta^{FR}$). Optimal levels are computed by the procedure described in Appendix 1. They are given for the United States and France on the last line of each panel. Seven parameters are estimated: the tail probability p, the dispersion parameter σ, the tail index ξ for each country, and the correlation of return exceedances ρ of the logistic function used to model the dependence between extreme returns. Standard errors are given below in parentheses. The null hypothesis of normality H_0: $\rho = \rho_{nor}$ is also tested. Two cases are considered: the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a given finite threshold θ, denoted by $\rho_{nor}^{asy}(\theta)$, is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold θ, denoted by $\rho_{nor}^{f.s.}(\theta)$, is computed by simulation assuming that monthly returns follow a bivariate-normal distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. Both a likelihood ratio test (LR test) between the constrained model ($\rho = \rho_{nor}^{asy} = 0$ in the asymptotic case and $\rho = \rho_{nor}^{f.s.}(\theta)$ in the finite-sample case) and the unconstrained model, and a Wald test (W test) on the correlation coefficient are carried out. The p values of the tests are given below in brackets.

Threshold	Parameters of the Model							H_0: $\rho = \rho_{nor}^{asy} = 0$		H_0: $\rho = \rho_{nor}^{f.s.}(\theta)$	
θ	p^{US}	σ^{US}	ξ^{US}	p^{FR}	σ^{FR}	ξ^{FR}	$\rho^{US/FR}$	LR test	W test	LR test	W test
				Panel A: Negative Return Exceedances							
-10%	0.016	1.542	0.744	0.029	4.275	0.072	0.845	55.227	9.826	25.610	7.826
	(0.006)	(1.062)	(0.612)	(0.008)	(2.226)	(0.400)	(0.086)	[0.000]	[0.000]	[0.000]	[0.000]
-8%	0.035	2.459	0.188	0.057	3.130	0.137	0.617	44.458	5.274	10.268	3.197
	(0.009)	(0.800)	(0.202)	(0.011)	(0.988)	(0.243)	(0.117)	[0.000]	[0.000]	[0.001]	[0.001]
-5%	0.111	2.113	0.150	0.186	2.582	0.107	0.474	59.431	6.237	5.169	2.145
	(0.015)	(0.404)	(0.128)	(0.018)	(0.394)	(0.106)	(0.076)	[0.000]	[0.000]	[0.023]	[0.032]

-3%	0.202 (0.019)	3.204 (0.446)	0.010 (0.084)	0.307 (0.021)	3.367 (0.361)	0.005 (0.071)	0.512 (0.061)	79.326 [0.000]	8.393 [0.000]	4.968 [0.026]	2.492 [0.013]
0%	0.437 (0.023)	3.652 (0.310)	-0.052 (0.049)	0.504 (0.023)	5.020 (0.384)	-0.128 (0.041)	0.493 (0.048)	96.261 [0.000]	10.271 [0.000]	0.536 [0.464]	1.146 [0.252]
-6.12 -8.38%	0.053 (0.010)	2.384 (0.657)	0.172 (0.200)	0.040 (0.009)	3.256 (1.289)	0.167 (0.333)	0.652 (0.103)	53.308 [0.000]	6.311 [0.000]	14.156 [0.000]	3.958 [0.000]

Panel B: Positive Return Exceedances

0%	0.525 (0.023)	3.436 (0.270)	-0.158 (0.047)	0.496 (0.023)	5.589 (0.429)	-0.215 (0.040)	0.347 (0.056)	36.513 [0.000]	6.196 [0.000]	2.886 [0.089]	-1.625 [0.104]
+3%	0.216 (0.019)	1.904 (0.310)	0.104 (0.128)	0.311 (0.022)	3.570 (0.391)	0.069 (0.067)	0.264 (0.066)	20.709 [0.000]	4.000 [0.000]	2.031 [0.154]	-1.455 [0.146]
+5%	0.071 (0.012)	3.186 (0.734)	-0.201 (0.156)	0.186 (0.018)	2.660 (0.429)	0.074 (0.116)	0.247 (0.088)	12.535 [0.000]	2.807 [0.005]	0.512 [0.474]	-0.727 [0.467]
+8%	0.024 (0.007)	2.803 (1.217)	-0.277 (0.293)	0.061 (0.011)	3.037 (0.868)	0.029 (0.213)	0.134 (0.112)	3.658 [0.056]	1.196 [0.232]	0.819 [0.365]	-0.973 [0.330]
+10%	0.013 (0.005)	0.986 (0.872)	0.320 (0.697)	0.034 (0.008)	2.006 (0.995)	0.348 (0.433)	0.159 (0.151)	3.699 [0.054]	1.053 [0.292]	0.006 [0.938]	-0.086 [0.931]
+7.21 +9.90%	0.041 (0.010)	2.878 (0.909)	-0.263 (0.207)	0.041 (0.009)	2.840 (1.116)	0.095 (0.317)	0.116 (0.105)	3.068 [0.080]	1.111 [0.267]	0.695 [0.405]	-0.874 [0.382]

Table III
Estimation of the Bivariate Distribution of U.S. and German Return Exceedances

This table gives the maximum likelihood estimates of the parameters of the bivariate distribution of U.S. and German return exceedances (Panel A for negative return exceedances and Panel B for positive return exceedances). Return exceedances are defined with a threshold θ. Both fixed and optimal levels are used for θ. Fixed levels (defined as percentage points) are: 0 percent, ± 3 percent, ± 5 percent, ± 8 percent, and ± 10 percent away from the empirically observed means of monthly returns (the same value of θ is then taken for the two countries: $\theta = \theta^{US} = \theta^{GE}$). Optimal levels are computed by the procedure described in Appendix 1. They are given for the United States and Germany on the last line of each panel. Seven parameters are estimated: the tail probability p, the dispersion parameter σ, the tail index ξ for each country and the correlation of return exceedances ρ of the logistic function used to model the dependence between extreme returns. Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}^{asy}$ is also tested. Two cases are considered: the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances of thresholds tending to infinity, denoted by ρ_{nor}^{asy}, is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold θ, denoted by $\rho_{nor}^{f.s.}(\theta)$, is computed by simulation assuming that monthly returns follow a bivariate-normal distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. Both a likelihood ratio test (LR test) between the constrained model ($\rho = \rho_{nor}^{asy} = 0$ in the asymptotic case and $\rho = \rho_{nor}^{f.s.}(\theta)$ in the finite-sample case) and the unconstrained model, and a Wald test (W test) on the correlation coefficient are carried out. The p values of the tests are given below in brackets.

Threshold	Parameters of the Model							$H_0: \rho = \rho_{nor}^{asy} = 0$		$H_0: \rho = \rho_{nor}^{f.s.}(\theta)$	
θ	p^{US}	σ^{US}	ξ^{US}	p^{GE}	σ^{GE}	ξ^{GE}	$\rho^{US/GE}$	LR test	W test	LR test	W test
				Panel A: Negative Return Exceedances							
-10%	0.016	1.533	0.554	0.024	5.802	-0.147	0.656	29.221	4.100	11.966	3.375
	(0.006)	(1.524)	(1.255)	(0.007)	(4.939)	(0.877)	(0.160)	[0.000]	[0.000]	[0.001]	[0.001]
-8%	0.031	2.476	0.185	0.053	2.090	0.404	0.512	34.515	4.096	8.365	2.576
	(0.008)	(0.859)	(0.206)	(0.010)	(0.869)	(0.375)	(0.125)	[0.000]	[0.000]	[0.004]	[0.010]
-5%	0.110	2.432	0.129	0.132	3.083	0.092	0.507	58.560	6.418	10.012	3.127
	(0.015)	(0.521)	(0.142)	(0.016)	(0.551)	(0.126)	(0.079)	[0.000]	[0.000]	[0.002]	[0.002]

-3%	0.201 (0.019)	2.786 (0.346)	0.016 (0.065)	0.254 (0.020)	2.884 (0.404)	0.090 (0.105)	0.440 (0.063)	63.744 [0.000]	6.984 [0.000]	4.596 [0.032]	2.048 [0.041]
0%	0.503 (0.023)	3.176 (0.245)	-0.034 (0.041)	0.489 (0.023)	4.155 (0.347)	-0.065 (0.050)	0.435 (0.049)	81.706 [0.000]	8.878 [0.000]	1.488 [0.223]	1.204 [0.229]
-6.12 -7.84%	0.060 (0.015)	2.367 (0.652)	0.153 (0.187)	0.043 (0.009)	3.102 (1.910)	0.260 (0.674)	0.482 (0.124)	32.161 [0.000]	3.891 [0.000]	5.645 [0.018]	2.210 [0.027]
Panel B: Positive Return Exceedances											
0%	0.511 (0.023)	3.593 (0.295)	-0.172 (0.048)	0.511 (0.023)	4.434 (0.397)	-0.140 (0.065)	0.276 (0.060)	19.448 [0.000]	4.600 [0.000]	4.115 [0.042]	-1.667 [0.096]
+3%	0.229 (0.020)	1.766 (0.275)	0.122 (0.117)	0.257 (0.020)	3.628 (0.482)	-0.105 (0.099)	0.165 (0.066)	7.465 [0.006]	2.500 [0.012]	4.774 [0.029]	-2.212 [0.027]
+5%	0.068 (0.011)	3.376 (0.823)	-0.219 (0.172)	0.143 (0.016)	3.320 (0.628)	-0.093 (0.146)	0.189 (0.089)	6.098 [0.014]	2.124 [0.034]	0.637 [0.425]	-0.798 [0.425]
+8%	0.026 (0.008)	2.670 (1.127)	-0.226 (0.328)	0.053 (0.010)	4.069 (1.192)	-0.373 (0.223)	0.020 (0.104)	0.001 [0.998]	0.192 [0.848]	2.482 [0.115]	-1.635 [0.102]
+10%	0.014 (0.006)	0.939 (0.812)	0.376 (0.795)	0.031 (0.009)	3.928 (1.574)	-0.511 (0.302)	0.000 (0.370)	0.001 [0.999]	0.000 [0.999]	1.545 [0.214]	-0.314 [0.754]
+7.21 +9.01%	0.040 (0.009)	3.174 (0.950)	-0.243 (0.176)	0.042 (0.009)	4.888 (1.510)	-0.533 (0.242)	0.078 (0.104)	0.7484 [0.387]	0.757 [0.449]	0.740 [0.390]	-0.904 [0.366]

Table IV
Estimation of the Bivariate Distribution of U.S. and Japanese Return Exceedances

This table gives the maximum likelihood estimates of the parameters of the bivariate distribution of U.S. and Japanese return exceedances (Panel A for negative return exceedances and Panel B for positive return exceedances). Return exceedances are defined with a threshold θ. Both fixed and optimal levels are used for θ. Fixed levels (defined as percentage points) are: 0 percent, ± 3 percent, ± 5 percent, ± 8 percent, and ± 10 percent away from the empirically observed means of monthly returns (the same value of θ is then taken for the two countries: $\theta = \theta^{US} = \theta^{JA}$). Optimal levels are computed by the procedure described in Appendix 1. They are given for the United States and Japan on the last line of each panel. Seven parameters are estimated: the tail probability p, the dispersion parameter σ, the tail index ξ for each country and the correlation of return exceedances ρ of the logistic function used to model the dependence between extreme returns. Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}$ is also tested. Two cases are considered: the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a given finite threshold θ, denoted by ρ_{nor}^{asy}, is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold θ, denoted by $\rho_{nor}^{f.s.}(\theta)$, is computed by simulation assuming that monthly returns follow a bivariate-normal distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. Both a likelihood ratio test (LR test) between the constrained model ($\rho = \rho_{nor}^{asy} = 0$ in the asymptotic case and $\rho = \rho_{nor}^{f.s.}(\theta) = 0$ in the finite-sample case) and the unconstrained model, and a Wald test (W test) on the correlation coefficient are carried out. The p values of the tests are given below in brackets.

Threshold	Parameters of the Model							$H_0: \rho = \rho_{nor}^{asy} = 0$		$H_0: \rho = \rho_{nor}^{f.s.}(\theta)$	
θ	p^{US}	σ^{US}	ξ^{US}	p^{JA}	σ^{JA}	ξ^{JA}	$\rho^{US/JA}$	LR test	W test	LR test	W test
				Panel A: Negative Return Exceedances							
-10%	0.016	1.581	0.762	0.036	3.346	-0.073	0.400	13.262	2.516	7.380	2.145
	(0.006)	(1.124)	(0.667)	(0.008)	(1.044)	(0.186)	(0.159)	[0.000]	[0.012]	[0.007]	[0.032]
-8%	0.034	2.742	0.169	0.064	3.791	-0.095	0.309	13.072	2.512	3.934	1.715
	(0.008)	(0.959)	(0.238)	(0.011)	(0.970)	(0.160)	(0.123)	[0.000]	[0.012]	[0.047]	[0.086]
-5%	0.100	2.356	0.178	0.158	3.215	0.018	0.326	25.036	4.025	4.888	2.123
	(0.014)	(0.520)	(0.164)	(0.017)	(0.562)	(0.127)	(0.081)	[0.000]	[0.000]	[0.027]	[0.034]

−3%	0.195 (0.019)	3.085 (0.432)	0.014 (0.084)	0.239 (0.020)	4.092 (0.518)	−0.082 (0.084)	0.298 (0.069)	25.644 [0.000]	4.319 [0.000]	1.404 [0.236]	1.304 [0.192]
0%	0.487 (0.023)	3.516 (0.296)	−0.047 (0.047)	0.491 (0.023)	4.385 (0.378)	−0.096 (0.055)	0.281 (0.056)	29.781 [0.000]	5.018 [0.000]	0.040 [0.842]	0.375 [0.708]
−6.12 −8.53%	0.050 (0.009)	2.660 (0.759)	0.158 (0.173)	0.043 (0.009)	4.006 (1.187)	−0.149 (0.170)	0.311 (0.117)	14.625 [0.000]	2.669 [0.008]	3.127 [0.077]	1.511 [0.131]

Panel B: Positive Return Exceedances

0%	0.511 (0.023)	3.256 (0.252)	−0.145 (0.046)	0.509 (0.023)	4.530 (0.381)	−0.174 (0.053)	0.171 (0.056)	10.743 [0.001]	3.000 [0.003]	1.563 [0.211]	−1.561 [0.118]
+3%	0.208 (0.019)	1.829 (0.291)	0.112 (0.121)	0.250 (0.020)	3.713 (0.509)	−0.105 (0.100)	0.153 (0.062)	7.695 [0.006]	2.468 [0.014]	0.460 [0.497]	−0.887 [0.375]
+5%	0.070 (0.012)	3.270 (0.793)	−0.199 (0.177)	0.136 (0.016)	3.927 (0.733)	−0.174 (0.140)	0.183 (0.088)	6.100 [0.014]	2.080 [0.038]	0.126 [0.723]	0.330 [0.742]
+8%	0.025 (0.007)	2.956 (1.270)	−0.292 (0.324)	0.059 (0.011)	3.875 (1.014)	−0.306 (0.185)	0.072 (0.112)	0.631 [0.427]	0.643 [0.520]	0.038 [0.845]	−0.232 [0.816]
+10%	0.014 (0.005)	1.040 (0.866)	0.324 (0.757)	0.033 (0.008)	3.540 (1.274)	−0.365 (0.269)	0.091 (0.143)	0.645 [0.422]	0.636 [0.525]	0.055 [0.815]	0.224 [0.823]
+7.21 +10.27%	0.037 (0.008)	3.257 (0.793)	−0.208 (0.251)	0.039 (0.009)	3.286 (1.142)	−0.276 (0.269)	0.077 (0.099)	1.032 [0.310]	0.788 [0.430]	0.175 [0.675]	−0.569 [0.569]

−3 percent, leading to an empirical frequency of 0.189. The dispersion parameter and the tail index are not estimated with great precision. The sign of the tail index for high threshold values gives some indication regarding the type of asymptotic distribution of extreme returns: the estimates of the tail index are mostly positive for the U.S., the U.K., and the French markets,[8] and mostly negative for the German and Japanese markets. However, none of these results can be considered statistically significant.

Results for the correlation coefficient of return exceedances are particularly interesting: The correlation seems to be influenced both by the size and the sign of the thresholds used to define the extremes. It is also different from the usual correlation, that is to say the correlation computed using all the observations of returns. We will describe the results using the US/UK pair as an example. The usual correlation of monthly returns is equal to 0.519 for the US/UK pair. The correlation of return exceedances tends to increase when we look at negative return exceedances defined with lower thresholds: It is equal to 0.530 for $\theta = -0$ percent (negative semicorrelation), 0.579 for $\theta = -3$ percent, 0.553 for $\theta = -10$ percent (Table I, Panel A). On the other hand, correlation tends to decrease with the level of the threshold when we look at positive return exceedances: It is equal to 0.415 for $\theta = +0$ percent (positive semicorrelation), 0.353 for $\theta = +3$ percent, 0.360 for $\theta = +5$ percent, 0.293 for $\theta = +8$ percent, and only 0.189 for $\theta = +10$ percent (Table I, Panel B). The correlation θ goes up with the absolute size of the threshold if it is negative and goes down with the threshold if positive. This is illustrated graphically on Figure 1, which depicts the relation between the correlation of return exceedances and the threshold used to define them. The solid line indicates the estimated correlation as a function of the threshold. It starts at the (negative or positive) semicorrelation for a threshold of $\theta = -0$ percent or $\theta = +0$ percent. A similar conclusion obtains for the other country pairs as seen in Tables II, III, and IV and Figures 2, 3, and 4.

The asymmetry between negative and positive return exceedances is confirmed by results obtained with optimal thresholds. As shown on the last lines of Tables I to IV, for all country pairs, the correlation between negative return exceedances is always greater than the correlation between positive return exceedances. On average, the former is equal to 0.505 whereas the latter is equal to 0.124. The difference is statistically significant at the 5 percent confidence level in three cases out of four (US/UK, US/FR, and US/GE). For example, considering the US/UK pair, the correlation between negative return exceedances (with the standard error in parentheses) is equal to 0.578 (0.121) whereas the correlation between positive return exceedances is equal to 0.226 (0.120). The value of a t test between the two correlation coefficients is equal to 2.066 with a p value of 0.039 (independence between negative and positive return exceedances is assumed to compute the t test).

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[8] Jansen and De Vries (1991), Loretan and Phillips (1994), and Longin (1996 and 2000) obtained similar results in univariate studies for the United States.
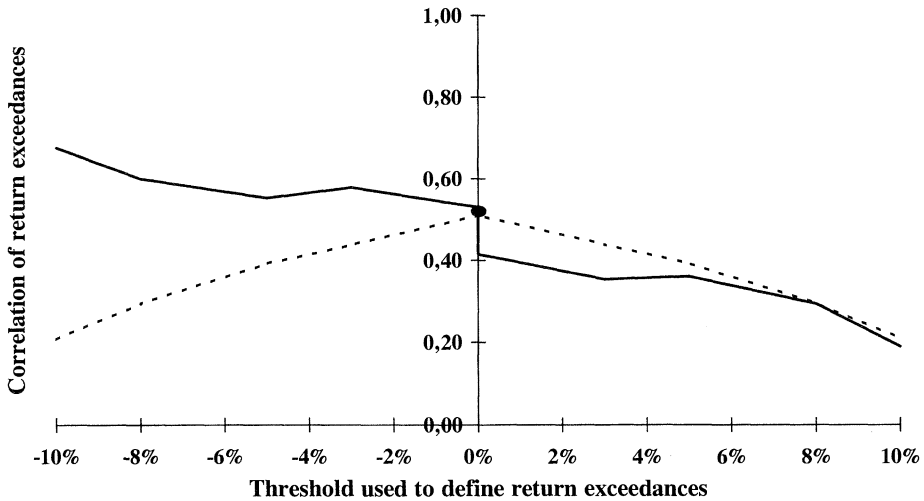
Figure 1. Correlation between U.S. and U.K. return exceedances. This figure represents the correlation structure of return exceedances between the United States and the United Kingdom. The solid line represents the correlation between actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see results in Table I). The dotted line represents the theoretical correlation between simulated normal return exceedances, ρ_{nor}, assuming a multivariate-normal return distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. The value of the threshold θ used to define return exceedances ranges from -10 percent to $+10$ percent (percentage points). For a given estimation, the same value of θ is taken for both countries: $\theta = \theta^{US} = \theta^{UK}$. The usual correlation using all returns is represented by a large dot on the vertical axis.

D. Test of Normality

We also test the null hypothesis of normality H_0: $\rho = \rho_{nor}$, where ρ_{nor} stands for the correlation between *normal* return exceedances. Under the null hypothesis of normality, this correlation coefficient tends to zero as the threshold value goes to infinity (see Section I). As we work with a finite sample, we can only use finite threshold values. Two cases are then formally considered: the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances of thresholds tending to infinity, denoted by ρ_{nor}^{asy}, is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold θ, denoted by $\rho_{nor}^{f.s.}(\theta)$, is computed by simulation. We compute the correlation between normal return exceedances for the predetermined threshold values considered above and for optimal threshold values. This is done by using a simulated bivariate normal process with means and covariance matrix equal to their empirical counterparts. Given these parameters, which fully describe a multivariate normal process, there is only one theoretical value for the correlation of return exceedances at a given threshold level. As indicated in the theoretical section, this "normal" correlation coefficient decreases with
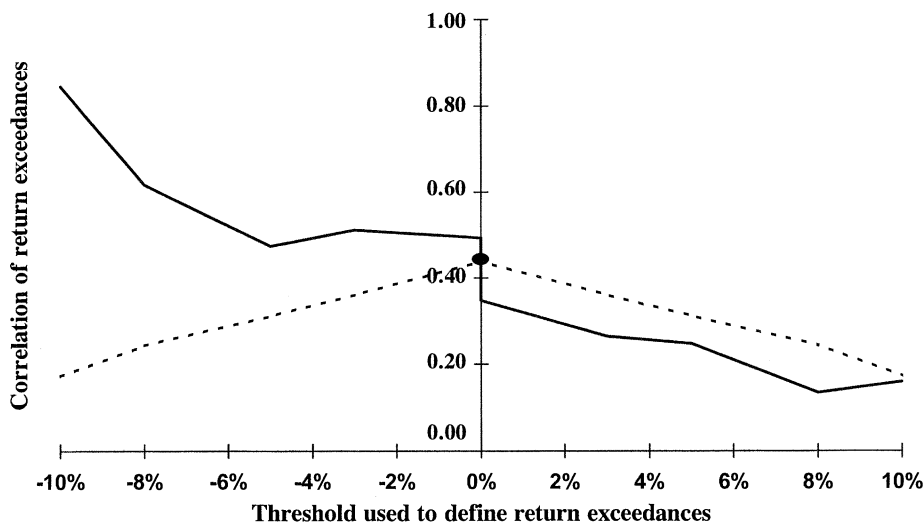
Figure 2. Correlation between U.S. and French return exceedances. This figure represents the correlation structure of return exceedances between the United States and France. The solid line represents the correlation between actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see results in Table II). The dotted line represents the theoretical correlation between simulated normal return exceedances, ρ_{nor}, assuming a multivariate-normal return distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. The value of the threshold θ used to define return exceedances ranges from -10 percent to $+10$ percent (percentage points). For a given estimation, the same value of θ is taken for both countries: $\theta = \theta^{US} = \theta^{FR}$. The usual correlation using all returns is represented by a large dot on the vertical axis.

the absolute size of the threshold. For example, for the US/UK pair, the normal correlation of positive return exceedances computed numerically decreases with the threshold: It is equal to 0.51 for $\theta = +0$ percent, 0.44 for $\theta = +3$ percent, 0.39 for $\theta = +5$ percent, 0.29 for $\theta = +8$ percent, and only 0.21 for $\theta = +10$ percent. In each figure, the dotted line plots the normal correlation as a function of the threshold. As seen in Figure 1, the US/UK correlation of return exceedances is close to its normal value for positive thresholds, but is markedly larger for negative thresholds.

Formal tests of the null hypothesis of normality are provided in the last columns of Tables I to IV. First, a likelihood ratio test between the constrained model (corresponding to normality) and the unconstrained model is carried out. Second, a Wald test on the correlation coefficient is done. For a given threshold, the Wald test compares the estimated correlation of return exceedances to its theoretical value under the hypothesis of normal returns. Both the asymptotic and finite-sample cases are considered. For all country pairs, the null hypothesis of normality is always rejected for high negative thresholds at the 5 percent confidence level. Taking as an example the pair US/UK and the threshold $\theta = -5$ percent, the likelihood ratio test strongly
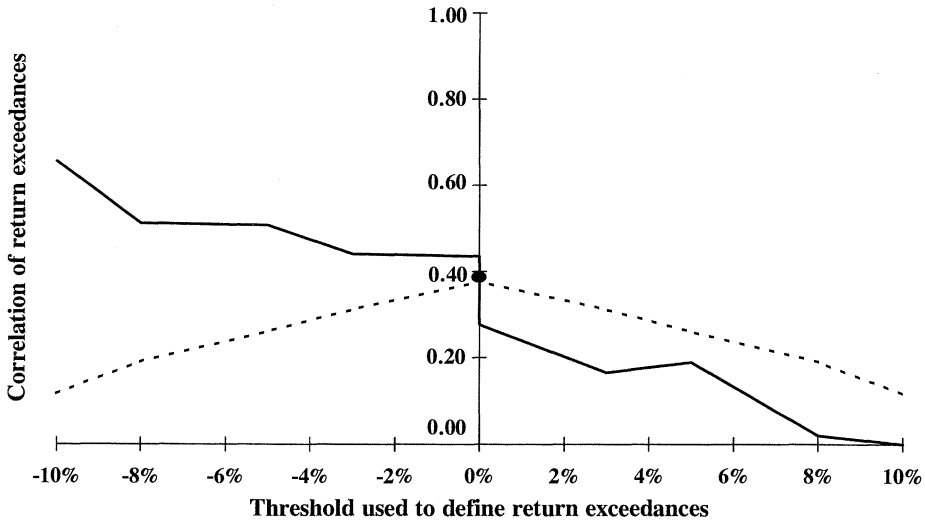
Figure 3. Correlation between U.S. and German return exceedances. This figure represents the correlation structure of return exceedances between the United States and Germany. The solid line represents the correlation between actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see results in Table III). The dotted line represents the theoretical correlation between simulated normal return exceedances, ρ_{nor}, assuming a multivariate-normal return distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. The value of the threshold θ used to define return exceedances ranges from -10 percent to $+10$ percent (percentage points). For a given estimation, the same value of θ is taken for both countries: $\theta = \theta^{US} = \theta^{GE}$. The usual correlation using all returns is represented by a large dot on the vertical axis.

rejects the null hypothesis of normality. The test value is equal to 73.143 with a negligible p value for the asymptotic case, and equal to 5.243 with a p value equal to 0.022 for the finite-sample case (Table I, Panel A). Similarly, the Wald test on the correlation coefficient itself strongly rejects the null hypothesis of normality. The test value is equal to 7.681 with a negligible p value for the asymptotic case, and equal to 2.236 with a p value equal to 0.025 for the finite-sample case. The difference in correlation is economically large (0.55 instead of 0.39) and statistically significant (a similar conclusion is obtained when exceedance returns are defined with optimal thresholds). This phenomenon is illustrated graphically for each pair of countries in Figures 1 to 4. For high negative threshold values, the solid line representing the estimated correlation of return exceedances moves away from the dotted line representing the theoretical correlation under normality. It should be noted that this result does not depend on one outlier, such as the October 1987 crash. Over the 38-year span, the British market, for example, had 29 monthly returns below -8 percent and 19 below -10 percent.

To summarize, the correlation structure of large returns is asymmetric. Correlation tends to decrease with the absolute size of the threshold for
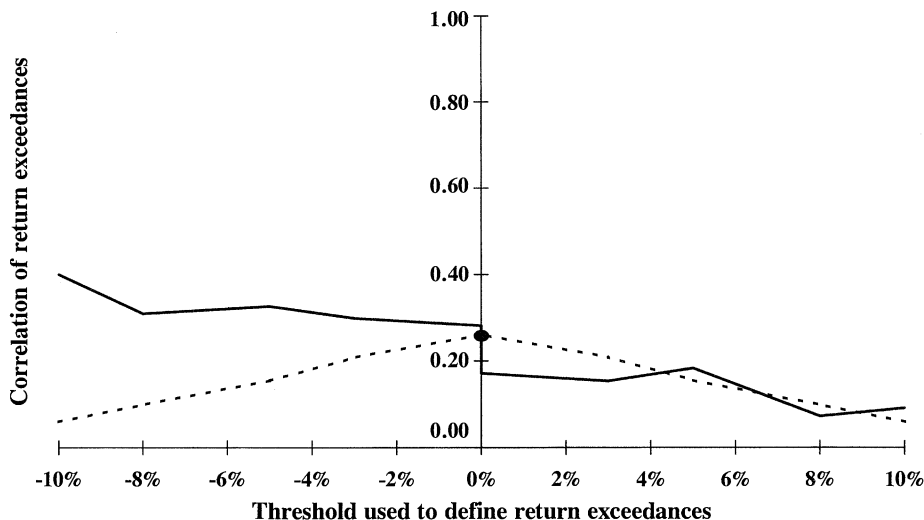
Figure 4. Correlation between U.S. and Japanese return exceedances. This figure represents the correlation structure of return exceedances between the United States and Japan. The solid line represents the correlation between actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see results in Table IV). The dotted line represents the theoretical correlation between simulated normal return exceedances, ρ_{nor}, assuming a multivariate-normal return distribution with parameters equal to the empirically observed means and covariance matrix of monthly returns. The value of the threshold θ used to define return exceedances ranges from -10 percent to $+10$ percent (percentage points). For a given estimation, the same value of θ is taken for both countries: $\theta = \theta^{US} = \theta^{JA}$. The usual correlation using all returns is represented by a large dot on the vertical axis.

positive returns, as expected in the case of multivariate normality, but tends to increase for negative returns. So the probability of having large losses simultaneously on two markets is much larger than would be suggested under the assumption of multivariate normality. It appears that it is a bear market, rather than volatility per se, that is the driving force in increasing international correlation.

IV. Conclusion

We use extreme value theory to study the dependence structure of international equity markets. We explicitly model the multivariate distribution of large returns (beyond a given threshold) and estimate the correlation for increasing threshold levels. Under the assumption of multivariate normality with constant correlation, the correlation of large returns (beyond a given threshold) should asymptotically go to zero as the threshold level increases. This is not the case in our estimation based on 38 years of monthly data for the five largest stock markets, at least for large negative returns. The correlation of large negative returns does not converge to zero, but tends to

increase with the threshold level, and rejection of multivariate normality is highly significant statistically. On the other hand, the correlation of large positive returns tends to decrease and to converge to zero with the threshold level, and the assumption of multivariate normality cannot be rejected. In other words, our results favor the explanation that correlation increases in bear markets, but not in bull markets.

The conclusion that volatility per se does not affect correlation in bull markets is at odds with some previous findings. One explanation provided above is that the null hypothesis of multivariate normality with constant correlation must be properly specified when conditioning on some realized level of return or volatility. Under the assumption of multivariate normality (with constant correlation), correlation conditioned on the level of volatility (absolute value of return) is expected to markedly increase with the level of volatility. So, tests of normality should model this feature in the null hypothesis. Here, we focus on the tail of the distribution whose asymptotic properties can be modeled and we derive a formal statistical method, based on extreme value theory, to test whether the correlation of large returns is higher than expected under the assumption of multivariate normality. An attractive feature of the methodology is that the asymptotic tail distribution is characterized by very few parameters regardless of the actual distribution. Asymptotic conditional correlation should be equal to zero for a wide class of return distributions. Although we do not suggest the exact time-varying distribution that should be used, our results lead to the rejection of a large class of models that would be inconsistent with our findings. This is the case of the multivariate normal distribution. It is also the case of a multivariate GARCH with constant correlation. Simulations for such a model calibrated to the data show that the conditional correlation goes to zero for extreme returns. More importantly, Ang and Bekaert (1999) show that a fairly general asymmetric GARCH also cannot reproduce the asymmetric correlations that we document. Although GARCH models seem ill suited to derive implications for bear and bull markets that are consistent with our findings, other models can. For example Ang and Bekaert (1999, p. 17) indicate that a regime-switching, return-generating process is able to reproduce our asymmetric findings. The disadvantage of our approach is that we do not explicitly specify the class of return-generating processes that are rejected. The advantage of our approach is that the empirical results do not depend on a specific return-generating process and are therefore fairly robust.

The next step would be to assess whether these findings materially affect international portfolio choices. Some recent papers are explicitly using return-generating processes that exhibit a regime-switching correlation increasing with volatility, and they study the portfolio choice implications. Ang and Bekaert (1999) and Das and Uppal (1999) develop different regime-switching models and reach different conclusions about portfolio implications. Ang and Bekaert (1999, p. 30) conclude that "the costs of ignoring regime switching are small for moderate levels of risk aversion," whereas Das and Uppal (1999 abstract) state that "there are substantial differences in the portfolio weights

across regimes." The difference in conclusion may come from the return-generating process postulated, especially how correlation increases in bear and bull markets.

Appendix 1: Computation of Optimal Threshold Levels

An optimal threshold level can be obtained by optimizing the trade-off between bias and inefficiency. To solve this problem, we use a Monte Carlo simulation method inspired by Jansen and de Vries (1991).[9] This appendix describes the procedure in detail.

A particular model for returns is assumed. For each simulated time series of returns, the optimal number of return exceedances (or equivalently the optimal threshold level) is computed. The MSE of simulated optimal numbers of return exceedances is then computed to derive the number of return exceedances for the observed time series. As explained by Theil (1971, pp. 26–32), the MSE criterion allows one to take explicitly into account the two effects of bias and inefficiency. The MSE of S simulated observations $\tilde{X}_s$ of the estimator of a parameter X can be decomposed as follows:

$$\text{MSE}((\tilde{X}_s)_{s=1,S}, X) = (\overline{X} - X)^2 + \frac{1}{S} \sum_{s=1}^{S} (\tilde{X}_s - X)^2,$$

where $\overline{X}$ represents the mean of S simulated observations. The first part of the decomposition measures the bias and the second part the inefficiency.

The procedure can be decomposed in four steps:

1. First we simulate S time series containing T return observations from Student-t distributions with k degrees of freedom, the integer k ranging from 1 to K. The class of the Student-t distributions is chosen to consider different degrees of tail fatness. The lower the degrees of freedom, the fatter the distribution as the tail index ξ is related to k by $\xi = 1/k$. For the simulations, we take: $S = 1,000$, $T = 456$, and $K = 10$.
2. For different numbers n of return exceedances, we obtain a tail index estimate $\tilde{\xi}_s(n, k)$ corresponding to the sth simulated time series and to the Student-t distribution with k degrees of freedom. To identify the optimal number of return exceedances, we focus on the tail index as this parameter models the distribution tails. We choose the values of n ranging from $0.01 \cdot T$ to $0.20 \cdot T$ such that proportions from 1 percent to 20 percent of the total number T of return observations are used in the estimation procedure.
3. For a Student-t distribution with k degrees of freedom and for each number n of return exceedances, we compute the MSE of the S tail

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[9] See also Beirlant, Vynckier, and Teugels (1996) and Huisman et al. (1998).

index estimates, denoted by $\mathrm{MSE}((\tilde{\xi}_s(n,k))_{s=1,S})$. As explained by Jansen and de Vries (1991), there is a U-shaped relation between $\mathrm{MSE}((\tilde{\xi}_s(n,k))_{s=1,S})$ and n, which expresses the trade-off between bias and inefficiency. For high values of n, the inclusion of many observations such that some do not belong to the tail but rather to the center of the distribution makes the bias part of the MSE dominate the inefficiency part. On the other hand, for low values of n, the inclusion of few observations makes the inefficiency part of the MSE dominate the bias part as the tail index is badly estimated. We then select the number of return exceedances that minimizes the MSE. This number, denoted by $n^*(k)$, is optimal for a Student-t distribution with k degrees of freedom.[10]

4. For the K optimal numbers of return exceedances previously obtained by simulation, $(n^*(k))_{k=1,K}$, we compute the tail index estimates of the observed time series of actual returns, denoted by $\tilde{\xi}(n^*(k))$ for k ranging from 1 to K. We then select the number of return exceedances for which the corresponding tail index estimate is statistically the closest to the tail index defined in the simulation procedure, that is to say $1/k$ (we consider the p value of the t test of the following hypothesis: $\tilde{\xi}(n^*(k)) = 1/k$). This number, denoted by n^*, is considered to be the optimal number of return exceedances for the distribution of actual returns. In the estimation of the model, we use the optimal threshold θ^* associated with the optimal number of return exceedances n^*.

Appendix 2: Derivation of the Maximum Likelihood Function

The parameters of the model presented in Section II are estimated by the maximum likelihood method developed by Ledford and Tawn (1997). This appendix presents the construction of the likelihood function in detail.

The method is based on a set of assumptions. Returns are assumed to be independent. The thresholds θ_1 and θ_2 used to select return exceedances (or equivalently the tail probabilities p_1 and p_2) are independent of returns and time. The method is also based on a censoring assumption. For thresholds θ_1 and θ_2, the space of return values is divided into four regions given by $\{A_{jk}; j = I(R_1 > \theta_1), k = I(R_2 > \theta_2)\}$, where I is the indicator function. The method treats return observations below threshold as censored data. Finally, the dependence between extreme returns is modeled using a logistic function denoted by D_l.

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[10] The optimal number of return exceedances is an increasing function of the fatness of the simulated Student-t distribution. For example, it is equal to 64 for a Student-t distribution with one degree of freedom and 25 for a Student-t distribution with five degrees of freedom. The fatter the distribution, the higher the number of return exceedances used in the estimation of the tail index as more extreme observations are available.

The likelihood contribution corresponding to the observation of returns at time t (R_{1t}, R_{2t}) falling in region A_{jk} is denoted by $L_{jk}(R_{1t}, R_{2t})$ and given by

$$L_{00}(R_{1t}, R_{2t}) = F_R^\theta(R_{1t}, R_{2t}) = \exp(-D_l(Y_1, Y_2)),$$

$$L_{01}(R_{1t}, R_{2t}) = \frac{\partial F_R^\theta(R_{1t}, R_{2t})}{\partial R_{2t}} = \exp(-D_l(Y_1, Z_2)) \cdot \frac{\partial D_l}{\partial R_{2t}}(Y_1, Z_2) \cdot K_2,$$

$$L_{10}(R_{1t}, R_{2t}) = \frac{\partial F_R^\theta(R_{1t}, R_{2t})}{\partial R_{1t}} = \exp(-D_l(Z_1, Y_2)) \cdot \frac{\partial D_l}{\partial R_{1t}}(Z_1, Y_2) \cdot K_1,$$

$$L_{11}(R_{1t}, R_{2t}) = \frac{\partial^2 F_R^\theta(R_{1t}, R_{2t})}{\partial R_{1t} \partial R_{2t}} = \exp(-D_l(Z_1, Z_2))$$

$$\times \left(\frac{\partial D_l}{\partial R_{1t}}(Z_1, Z_2) \frac{\partial D_l}{\partial R_{2t}}(Z_1, Z_2) - \frac{\partial^2 D_l}{\partial R_{1t} \partial R_{2t}}(Z_1, Z_2) \right) \cdot K_1 \cdot K_2,$$

where the variables Y_i, Z_i, and K_i for $i = 1$ and 2 are defined by

$$Y_i = -1/\log F_{R_i}^{\theta_i}(\theta_i),$$

$$Z_i = -1/\log F_{R_i}^{\theta_i}(R_{it}),$$

$$K_i = -p_i \cdot \sigma_i^{-1} \cdot (1 + \xi_i \cdot (R_{it} - \theta_i)/\sigma_i)_+^{-(1+\xi_i)/\xi_i} \cdot Z_i^2 \cdot \exp(1/Z_i).$$

The likelihood contribution from the observation of returns at time t (R_{1t}, R_{2t}) for the bivariate distribution of return exceedances described by a set of parameters $\Phi = (p_1, p_2, \sigma_1, \sigma_2, \xi_1, \xi_2, \alpha)$ is given by

$$L(R_{1t}, R_{2t}, \Phi) = \sum_{j,k \in \{0,1\}} L_{jk}(R_{1t}, R_{2t}) \cdot I_{jk}(R_{1t}, R_{2t}),$$

where $I_{jk}(R_{1t}, R_{2t}) = I\{(R_{1t}, R_{2t}) \in A_{jk}\}$. Hence the likelihood for a set of T independent observations of returns is given by

$$L(\{R_{1t}, R_{2t}\}_{t=1,T}, \Phi) = \prod_{t=1}^{T} L(R_{1t}, R_{2t}, \Phi).$$

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