Algebra Lecture Notes – Online
M1 MIASHS

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1 Types of Linear Maps

Let V, W be real vector spaces and let f be a linear map from V to W, i.e., $f \in \mathcal{L}(V, W)$. In particular, we will consider $f : \mathbb{R}^n \to \mathbb{R}^n$, or more simply $f : \mathbb{R}^3 \to \mathbb{R}^3$.

Reminder. A map $f : V \to W$ is *linear* if and only if, for all $u, v \in V$ and all scalars $\lambda \in \mathbb{R}$,

$$f(u + \lambda v) \; = \; f(u) + \lambda f(v).$$

Exam tip. You may be asked to show that a given map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is linear. The expected approach:

1. Take two generic vectors in $\mathbb{R}^3$:
$$u = (u_1, u_2, u_3), \qquad v = (v_1, v_2, v_3).$$
2. Consider $u + \lambda v$:
$$u + \lambda v = (\, u_1 + \lambda v_1,\; u_2 + \lambda v_2,\; u_3 + \lambda v_3 \,).$$
3. Apply f to this vector: compute $f(u + \lambda v)$ explicitly using the definition of f.
4. Separate terms to identify $f(u)$ and $\lambda f(v)$.
5. Conclude that
$$f(u + \lambda v) = f(u) + \lambda f(v).$$

Types of morphisms. We distinguish several important properties of a linear map $f : V \to W$: injectivity, surjectivity, bijectivity (isomorphism), and the notions of endomorphism and automorphism.

- $f : V \to W$ is **injective** if, for $u, v \in V$,
$$f(u) = f(v) \implies u = v.$$
In other words, distinct vectors of V cannot share the same image in W. (Equivalent: $\ker(f) = \{0_V\}$.)
- It is **surjective** if every vector of W is reached by f, i.e.,
$$\forall w \in W, \;\; \exists v \in V \text{ s.t. } f(v) = w.$$
Equivalently, $\mathrm{Im}(f) = W$.
- It is **bijective** if it is both injective and surjective. In that case
$$\forall w \in W, \;\; \exists!\, v \in V \text{ s.t. } f(v) = w.$$
Then f admits an inverse $f^{-1} : W \to V$ (corresponding to A^{-1} when A denotes the matrix of f in the canonical basis). A bijective linear map is an **isomorphism**.
- If $V = W$, then f is an **endomorphism**. If moreover f is an isomorphism, it is an **automorphism**. We denote $\mathrm{Aut}(V) = \{f \in \mathcal{L}(V) \mid f \text{ is bijective}\}$.

2 Kernel and Image

Definitions. Let $f : V \to W$ be linear.

- The **kernel** of f is
$$\ker(f) = \{v \in V \mid f(v) = 0_W\}.$$
It is a subspace of V.
- The **image** of f is
$$\mathrm{Im}(f) = \{f(v) \mid v \in V\} \subseteq W.$$
It is a subspace of W.

Link with injectivity and surjectivity.

- f is **injective** iff $\ker(f) = \{0_V\}$.
- f is **surjective** iff $\mathrm{Im}(f) = W$.

Example. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$f(x, y) = (2x + y,\ x + y).$$

- Its kernel is
$$\ker f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = (0, 0)\}.$$
Solve
$$\begin{cases} 2x + y = 0 \\ x + y = 0 \end{cases} \implies x = 0,\ y = 0.$$
Hence
$$\ker f = \{(0, 0)\},$$
so f is **injective**.
- The image is
$$\mathrm{Im}(f) = \{(2x + y,\ x + y) \mid (x, y) \in \mathbb{R}^2\}.$$
Note that
$$(2x + y,\ x + y) = x\,(2, 1) + y\,(1, 1).$$
Since $(2, 1)$ and $(1, 1)$ are linearly independent, they form a basis of $\mathbb{R}^2$. Therefore
$$\mathrm{Im}(f) = \mathrm{Span}\{(2, 1), (1, 1)\} = \mathbb{R}^2,$$
and f is **surjective**.

Conclusion. f is both injective and surjective, hence a **linear bijection** (an **isomorphism**).

Rank.

- The **rank** of a matrix A (or a linear map f) is the dimension of its image:
$$\mathrm{rg}(A) = \dim \mathrm{Im}(A) \quad \text{or} \quad \mathrm{rg}(f) = \dim \mathrm{Im}(f).$$
- The **rank–nullity theorem** states that, for any linear map $f : V \to W$,
$$\dim \ker f + \dim \mathrm{Im}\, f = \dim V.$$
- In the case $V = \mathbb{R}^2$:
$$\dim \ker f + \mathrm{rg}(f) = \dim \mathbb{R}^2 = 2.$$

Exercise from the slides. Consider the linear map f defined by

$$f:(x,y)\mapsto(x+y,\ 2x+2y).$$

1. The matrix A representing f in the canonical basis is

$$A=\begin{pmatrix}1&1\\2&2\end{pmatrix},$$

since

$$f(e_1)=f(1,0)=(1,2)=1\,e_1+2\,e_2,\qquad f(e_2)=f(0,1)=(1,2)=1\,e_1+2\,e_2.$$

Reminder. If the matrix of a linear map is not invertible, then the map cannot be bijective.

First reflex: compute $\det A$.

$$\det A=0\quad\Longrightarrow\quad A\text{ non-invertible}\quad\Longrightarrow\quad f\text{ is neither injective nor surjective.}$$

2. What about $\ker f\equiv\ker A$ and $\operatorname{Im} f\equiv\operatorname{Im} A$?

Kernel (find a basis). By definition,

$$\ker A=\left\{(x,y)\in\mathbb{R}^2\;\middle|\;A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

Solve the system:

$$(x,y)\in\ker A\iff\begin{cases}x+y=0,\\2x+2y=0\end{cases}\iff x+y=0.$$

Thus $y=-x$, and

$$\boxed{\ker A=\operatorname{Vect}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}}.$$

Image. For all $(x,y)\in\mathbb{R}^2$,

$$f(x,y)=(x+y,\ 2x+2y)=x(1,2)+y(1,2)=(x+y)(1,2).$$

In matrix form:

$$A\begin{pmatrix}x\\y\end{pmatrix}=(x+y)\begin{pmatrix}1\\2\end{pmatrix}.$$

Hence

$$\boxed{\operatorname{Im}(A)\equiv\operatorname{Im}(f)=\operatorname{Vect}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}},\qquad\dim\operatorname{Im} f=\operatorname{rg}(f)=1.$$

Rank–nullity.

$$\dim\ker A+\operatorname{rg}(A)=\dim\mathbb{R}^2=2.$$

3 Matrix of a Linear Map in a Given Basis

Exercise. Represent $T : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (x + z,\ y,\ 2z)$ by its matrix M in the **canonical basis**. Compute

$$T(e_1) = T(1,0,0) = (1,0,0), \quad T(e_2) = (0,1,0), \quad T(e_3) = (1,0,2).$$

Thus

$$\boxed{M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}.$$

Change of Basis — Application

We first represent the linear map in basis $\mathcal{B}_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ (the **canonical basis**). We want its representation in the basis

$$\mathcal{B}_2 = \{(1,1,1),\ (1,-1,1),\ (2,1,1)\}.$$

To obtain the change-of-basis matrix, we perform Gaussian elimination on the augmented matrix

$$[\mathcal{B}_2 \mid \mathcal{B}_1]\,.$$

After row operations, we get the matrix

$$\boxed{P_{1\to 2} = P_{2\,1} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \end{pmatrix}}.$$

Let the vector originally expressed in $\mathcal{B}_1$ be $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. Then

$$[\mathbf{b}]_{\mathcal{B}_2} = P_{2\,1}\,[\mathbf{b}]_{\mathcal{B}_1} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}.$$

4 Eigenvalues, Eigenvectors, Eigenspaces

A simple example. For $f(x,y) = (x, 2y)$ in the canonical basis,

$$f(e_1) = 1 \cdot e_1, \qquad f(e_2) = 2 \cdot e_2.$$

In general

We seek scalars λ_i and nonzero vectors v_i such that

$$f(v_i) = \lambda_i v_i \quad \Longleftrightarrow \quad A v_i = \lambda_i v_i.$$

Each λ_i is an *eigenvalue* and each $v_i \neq 0$ an *eigenvector* associated with λ_i. The family $\{v_i\}$ is an **eigenbasis**.

Let A be the matrix of f in the canonical basis. For $v \neq 0$ we write

$$Av = \lambda v \quad \Longleftrightarrow \quad Av - \lambda v = 0 \quad \Longleftrightarrow \quad (A - \lambda I)v = 0.$$

Since $v \neq 0$, the linear map represented by $A - \lambda I$ is not injective; its kernel is nontrivial.

The **eigenspace** associated with λ is the kernel

$$E_\lambda = \ker(A - \lambda I).$$

The **characteristic polynomial** of A is

$$\chi_A(\lambda) = \det(\lambda I - A).$$

Its roots are precisely the eigenvalues of A, because $\det(\lambda I - A) = 0$ iff $\lambda I - A$ is non-invertible, equivalently $\exists\, v \neq 0$ such that $(\lambda I - A)v = 0$, i.e., $Av = \lambda v$. The set of eigenvalues is the **spectrum** $\mathrm{Sp}(A)$.

Multiplicities. The *algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic polynomial. The *geometric multiplicity* of λ is $\dim E_\lambda = \dim\ker(A-\lambda I)$.

Characterization. When the sum of geometric multiplicities equals n and each equals the corresponding algebraic multiplicity, the matrix is diagonalizable.

Example: diagonalizing a 2×2 matrix

$$A = \begin{pmatrix} 7 & 2 \\ 3 & 8 \end{pmatrix}.$$

Then

$$\lambda I - A = \begin{pmatrix} \lambda-7 & -2 \\ -3 & \lambda-8 \end{pmatrix}, \quad \chi_A(\lambda) = \det(\lambda I - A) = (\lambda-7)(\lambda-8) - 6 = \lambda^2 - 15\lambda + 50.$$

The discriminant is $\Delta = 225 - 200 = 25$, hence

$$\lambda_- = \frac{15-\sqrt{25}}{2} = 5, \qquad \lambda_+ = \frac{15+\sqrt{25}}{2} = 10.$$

Thus

$$\mathrm{Sp}(A) = \{5, 10\},$$

and A is diagonalizable (two distinct eigenvalues).

Eigenvectors / eigenspaces. *Case* $\lambda_+ = 10$.

$$A - 10I = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}.$$

Solving $(A-10I)\binom{x}{y} = \binom{0}{0}$ yields $y = \frac{3}{2}x$, so

$$E_{10} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

Case $\lambda_- = 5$.

$$A - 5I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}.$$

Solving $(A-5I)\binom{x}{y} = \binom{0}{0}$ gives $y = -x$, so

$$E_5 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Remark. It is conventional to study $\det(\lambda I - A)$ rather than $\det(A - \lambda I)$ so that the leading coefficient is $+1$.

5 Diagonalization

We can sometimes find a basis in which the matrix of f is diagonal,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3).$$

Let A be the matrix of f in the canonical basis. If P is the (invertible) change-of-basis matrix from the canonical basis to an eigenbasis, then

$$A = PDP^{-1}.$$

Key properties.

- For any integer $m \geq 0$,
$$D^m = \mathrm{diag}(\lambda_1^m, \lambda_2^m, \lambda_3^m).$$
- By conjugation,
$$A^m = PD^mP^{-1}.$$
- The *trace* of a matrix A is the sum of its diagonal entries:
$$\mathrm{Tr}(A) = \sum_{i=1}^{3} a_{ii}.$$
Trace is invariant under change of basis, hence
$$\mathrm{Tr}(A) = \mathrm{Tr}(D).$$
Since D is diagonal with eigenvalues on the diagonal,
$$\mathrm{Tr}(A) = \sum_{i=1}^{3} \lambda_i$$
(sum of eigenvalues with algebraic multiplicity).
- For the determinant:
$$\det(A) = \det(D) = \prod_{i=1}^{3} \lambda_i.$$

Are the following diagonalizable?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 7 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 7 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$$

Case A. A is upper triangular, so its characteristic polynomial reads off the diagonal:

$$\chi_A(\lambda) = \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 7 & 1 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 7)(\lambda - 2).$$

Eigenvalues are $1, 7, 2$ (all distinct), hence A **is diagonalizable**.

Case B. B is upper triangular, so

$$\chi_B(\lambda) = (\lambda - 7)^3.$$

The only eigenvalue is $\lambda = 7$ with algebraic multiplicity 3. The geometric multiplicity is $\dim E_7 = \dim\ker(B - 7I)$:

$$B - 7I = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (B - 7I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 3z \\ -z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $z = 0$, $y = 0$, and x is free, so $\dim E_7 = 1$. Since $1 < 3$, B **is not diagonalizable**. (A quick check: if B were diagonalizable with only eigenvalue 7, then $B = P(7I)P^{-1} = 7I$, which is false.)

Case C. The rows of C are colinear: row 2 is the opposite of row 1, row 3 equals row 1. Hence $\mathrm{rg}(C) = 1$. For all $(x, y, z) \in \mathbb{R}^3$,

$$C \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ -x - 2y - z \\ x + 2y + z \end{pmatrix} = (x + 2y + z) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore $\mathrm{Im}(C) = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ and $\mathrm{rg}(C) = 1$. By rank–nullity,

$$\dim \ker(C) = 3 - \mathrm{rg}(C) = 2 \quad \Longrightarrow \quad 0 \text{ is an eigenvalue with geometric multiplicity at least 2.}$$

The trace is

$$\mathrm{Tr}(C) = 1 + (-2) + 4 = 3.$$

Since the sum of eigenvalues (with multiplicity) equals the trace and 0 appears at least twice, the spectrum is

$$\chi_C(\lambda) = \lambda^2(\lambda - 3), \qquad \mathrm{Sp}(C) = \{0, 0, 3\}.$$

Here algebraic and geometric multiplicities match, so C **is diagonalizable** (though non-invertible).

Remark: non-invertibility and eigenspaces. A matrix A can fail to be invertible and yet be diagonalizable. If $\det(A) = 0$, then there exists $v \neq 0$ such that $Av = 0$, i.e., 0 is an eigenvalue and

$$E_0 = \ker(A - 0I) = \ker(A).$$

If 0 is a simple root of the characteristic polynomial, then $\dim E_0 = 1$ and $\ker(A) = \mathrm{Vect}\{v\}$.

6 Orthogonal Matrices and the Spectral Theorem

Definition. A square matrix $Q \in \mathbb{R}^{n \times n}$ is **orthogonal** if

$$Q^\top Q = QQ^\top = I_n,$$

i.e., $Q^{-1} = Q^\top$.

Intuition. An orthogonal matrix represents a linear transformation that

- preserves inner products and lengths,
- preserves orthogonality,
- is a composition of rotations and reflections.

Isometry property. For any vector $\mathbf{x} \in \mathbb{R}^n$,

$$\|Q\mathbf{x}\|^2 = (Q\mathbf{x})^\top (Q\mathbf{x}) = \mathbf{x}^\top Q^\top Q\mathbf{x} = \mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|^2.$$

Why "orthogonal"? If $Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \ldots \ \mathbf{q}_n]$, then

$$\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

i.e., the columns form an **orthonormal basis** of $\mathbb{R}^n$.

Spectral theorem. Every real symmetric matrix is diagonalizable. More precisely, if $A \in \mathbb{R}^{n \times n}$ satisfies $A^\top = A$, then there exists an orthogonal matrix Q such that

$$Q^\top AQ = D,$$

where D is diagonal.

7 Typical Exercise

Consider the matrix

$$M = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 3 \\ 3 & 3 & 1 \end{pmatrix}.$$

Characteristic polynomial, spectrum, eigenvalues. We directly observe that

$$M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus 7 is an eigenvalue of M with associated eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $7 \in \mathrm{Sp}(M)$.

Compute the trace and determinant of M.

$$\mathrm{Tr}(M) = 3 + 3 + 1 = 7.$$

A cofactor expansion yields

$$\det(M) = -28.$$

Let $\mathrm{Sp}(M) = \{7, \lambda_1, \lambda_2\}$. Then

$$\begin{cases} \mathrm{Tr}(M) = 7 + \lambda_1 + \lambda_2, \\ \det(M) = 7 \cdot \lambda_1 \cdot \lambda_2. \end{cases}$$

Hence

$$\begin{cases} \lambda_1 + \lambda_2 = 0, \\ \lambda_1 \lambda_2 = -4. \end{cases}$$

Therefore λ_1, λ_2 are the roots of $X^2 - 4$, i.e.,

$$\lambda_1 = -2, \qquad \lambda_2 = 2.$$

We conclude that

$$\mathrm{Sp}(M) = \{-2, 2, 7\}.$$

As there are three distinct eigenvalues in dimension 3, the matrix is diagonalizable and each eigenspace has dimension 1.

For each eigenvalue, determine its eigenspace. We already have

$$E_7 = \ker(M - 7I) = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Case $\lambda = 2$.

$$M - 2I = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & -1 \end{pmatrix}.$$

Solving $(M - 2I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ gives

$$\begin{cases} x + y + 3z = 0, \\ 3x + 3y - z = 0 \end{cases} \iff \begin{cases} y = 0, \\ x + z = 0, \end{cases}$$

so

$$E_2 = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Case $\lambda = -2$. Similarly,

$$E_{-2} = \mathrm{Vect}\left\{ \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

Change of basis and diagonalization. An eigenvector matrix P is

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}.$$

Then

$$\boxed{M = PDP^{-1}} \quad \text{with} \quad D = \mathrm{diag}(7, 2, -2).$$

Note: Respect the ordering of eigenvalues, which fixes the column order of P.

Link with the spectral theorem. Since M is **symmetric** ($M^\top = M$) with real coefficients, the **spectral theorem** applies:

$$M = QDQ^\top,$$

where Q is orthogonal (the eigenvectors are orthonormalized so that $QQ^\top = Q^\top Q = I$).

Normalizing the eigenvectors of P yields an orthogonal Q (omitted here for brevity).

Exercise. Verify that for any orthogonal Q, one has $Q^{-1} = Q^\top$.