Algebra Practice Exam

M1 MIASHS

September 2025

Solution to Exercise 1 : Subspace of symmetric matrices

We work in $\mathcal{M}_3(\mathbb{R})$ and denote by

$$\mathscr{S}_3(\mathbb{R}) = \{A \in \mathcal{M}_3(\mathbb{R}) : A^\top = A\}$$

the space of real symmetric 3×3 matrices.

(a) $\mathscr{S}_3(\mathbb{R})$ **is a vector subspace of** $\mathcal{M}_3(\mathbb{R})$**.**

— The zero matrix $0_{3\times3}$ satisfies $0^\top = 0$, hence $0 \in \mathscr{S}_3(\mathbb{R})$.

— If $A, B \in \mathscr{S}_3(\mathbb{R})$, then $(A+B)^\top = A^\top + B^\top = A + B$, so $A + B \in \mathscr{S}_3(\mathbb{R})$.

— For any $\lambda \in \mathbb{R}$ and $A \in \mathscr{S}_3(\mathbb{R})$, $(\lambda A)^\top = \lambda A^\top = \lambda A$, so $\lambda A \in \mathscr{S}_3(\mathbb{R})$.

Thus $\mathscr{S}_3(\mathbb{R})$ is a subspace.

(b) **A convenient generating family and an explicit linear decomposition.**
For $1 \le i \le 3$, set

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and for $1 \le i < j \le 3$,

$$F_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let

$$\mathcal{B} = \{E_{11}, E_{22}, E_{33}, F_{12}, F_{13}, F_{23}\}.$$

Spanning. Any $S \in \mathscr{S}_3(\mathbb{R})$ has the form

$$S = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \quad (a, b, c, d, e, f \in \mathbb{R}).$$

Entrywise inspection shows the *natural* linear decomposition

$$\boxed{S = a\,E_{11} + d\,E_{22} + f\,E_{33} + b\,F_{12} + c\,F_{13} + e\,F_{23}.}$$

Hence $S \in \operatorname{Span}\mathcal{B}$, so $\operatorname{Span}\mathcal{B} = \mathscr{S}_3(\mathbb{R})$.

Linear independence. Suppose

$$\alpha\, E_{11} + \beta\, E_{22} + \gamma\, E_{33} + \mu\, F_{12} + \nu\, F_{13} + \rho\, F_{23} = 0_{3\times 3}.$$

Comparing entries :

$$(1,1):\ \alpha = 0, \quad (2,2):\ \beta = 0, \quad (3,3):\ \gamma = 0,$$

$$(1,2) \text{ and } (2,1):\ \mu = 0, \quad (1,3) \text{ and } (3,1):\ \nu = 0, \quad (2,3) \text{ and } (3,2):\ \rho = 0.$$

Thus all coefficients vanish, so $\mathcal{B}$ is linearly independent.
Therefore $\mathcal{B}$ is a basis of $\mathscr{S}_3(\mathbb{R})$.

(c) **Dimension.** Since $\mathcal{B}$ has 6 elements and is a basis, we get

$$\dim\left(\mathscr{S}_3(\mathbb{R})\right) = 6.$$

Bonus (general case n). For $n \geq 1$, define E_{ii} for $1 \leq i \leq n$ and $F_{ij} = E_{ij} + E_{ji}$ for $1 \leq i < j \leq n$. Then

$$\mathcal{B}_n = \{E_{11}, \ldots, E_{nn}\} \ \cup\ \{F_{ij}\ :\ 1 \leq i < j \leq n\}$$

spans $\mathscr{S}_n(\mathbb{R})$ (same entrywise decomposition as above) and is linearly independent by comparing entries. Hence

$$\dim\left(\mathscr{S}_n(\mathbb{R})\right) = n\ +\ \frac{n(n-1)}{2}\ =\ \frac{n(n+1)}{2}.$$

Optional inner-product proof of independence. Equip $\mathcal{M}_3(\mathbb{R})$ with the Frobenius inner product $\langle X, Y\rangle = \operatorname{tr}(X^\top Y)$. The family $\mathcal{B}$ is orthogonal with respect to this product (each basis matrix has disjoint support except for symmetric pairs), hence linearly independent.

Solution to Exercise 2 : Non-invertible linear map on $\mathbb{R}^3$

We denote by $(e_i)_{i=1,\ldots,3}$ the canonical basis vectors of $\mathbb{R}^3$, i.e. the vectors with all components zero except for a 1 in the i-th position. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(x, y, z) = (x + y - z,\ 2x + 2y + z,\ 3x + 3y).$$

(a) **Linearity.** For all $(x, y, z), (x', y', z') \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$,

$$f(\alpha(x, y, z) + \beta(x', y', z')) = \alpha f(x, y, z) + \beta f(x', y', z').$$

This follows by distributing α, β across the coordinate-wise linear formulas. Hence $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$.

(b) **Matrix in the canonical basis.** Since $f(e_1) = (1, 2, 3)$, $f(e_2) = (1, 2, 3)$, $f(e_3) = (-1, 1, 0)$, the associated matrix is

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 3 & 3 & 0 \end{pmatrix}.$$

(c) **Invertibility / injectivity / surjectivity.** By denoting $A = [C_1 \mid C_2 \mid C_3]$, we can clearly read from the matrix A that columns C_1 and C_2 are linearly dependent (in fact, $C_1 = C_2$). Therefore, the matrix is necessarily not invertible, and this linear dependence directly implies $\det(A) = 0$.

Let us check this explicitly by expanding the determinant along column 3 :

$$\det(A) = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 3 & 3 & 0 \end{vmatrix} \underset{\text{dev. } C_3}{=} (-1)\cdot(-1)^{1+3}\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} + (1)\cdot(-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + (0)\cdot(-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}.$$

Each 2×2 minor is zero (two identical columns), hence $\boxed{\det(A) = 0}$. Consequently, A is **not** invertible. It follows that f is neither injective (kernel nontrivial) nor surjective (rank < 3), and thus not bijective.

Kernel of f To determine a basis of $\ker(f)$, we take a vector $(x, y, z) \in \mathbb{R}^3$ and write the equations that its coordinates must satisfy. Thus,

$$(x, y, z) \in \ker(f) \iff \begin{cases} x + y - z = 0 \\ 2x + 2y + z = 0 \\ 3x + 3y = 0 \end{cases}$$

$$\iff \begin{cases} x + y - z = 0 \\ 3x + 3y = 0 \quad R2 \leftarrow R2 + R1 \\ 3x + 3y = 0 \end{cases}$$

$$\iff \begin{cases} z = x + y = 0 \\ y = -x \end{cases}$$

Hence every vector in the kernel has the form

$$(x, y, z) = (x, -x, 0) = x\,(1, -1, 0), \qquad x \in \mathbb{R}.$$

Therefore

$$\boxed{\ker(f) = \operatorname{Span}\{(1, -1, 0)\}}, \qquad \dim\ker(f) = 1.$$

Image of f To describe $\operatorname{Im}(f)$, let us compute $f(x, y, z)$ for a general vector $(x, y, z) \in \mathbb{R}^3$:

$$f(x, y, z) = (x + y - z,\ 2x + 2y + z,\ 3x + 3y).$$

We can rearrange the expression by grouping terms :

$$f(x, y, z) = (x + y)(1, 2, 3) \ +\ z\,(-1, 1, 0).$$

Thus we have written $f(x, y, z)$ as a linear combination of the two *fixed* vectors

$$v_1 = (1, 2, 3), \qquad v_2 = (-1, 1, 0).$$

It follows that for any $(x, y, z) \in \mathbb{R}^3$, the image $f(x, y, z)$ belongs to the subspace generated by v_1 and v_2. Therefore

$$\boxed{\operatorname{Im}(f) = \operatorname{Span}\{(1, 2, 3),\ (-1, 1, 0)\}}.$$

Since v_1 and v_2 are linearly independent (easy to check, as one is not a multiple of the other), we conclude that

$$\dim \operatorname{Im}(f) = \operatorname{rank}(f) = 2.$$

(e) **Equations characterizing** $\ker(f)$ **and subspace proof.**
We first prove directly that $\ker(f)$ is a vector subspace of $\mathbb{R}^3$ by checking the two defining properties.

<u>Zero vector in the kernel.</u>

$$f(0,0,0) = (0+0-0,\ 2\cdot 0+2\cdot 0+0,\ 3\cdot 0+3\cdot 0) = (0,0,0),$$

hence $0_{\mathbb{R}^3} \in \ker(f)$.

<u>Closed under linear combinations.</u> Let $v_1, v_2 \in \ker(f)$ and let $\alpha, \lambda \in \mathbb{R}$. Since f is linear,

$$f(\alpha v_1 + \lambda v_2) = \alpha\, f(v_1) + \lambda\, f(v_2) = \alpha\, 0 + \lambda\, 0 = 0,$$

so $\alpha v_1 + \lambda v_2 \in \ker(f)$. Therefore $\ker(f)$ is a vector subspace of $\mathbb{R}^3$.

Equational form (from part (d)). Starting from the definition,

$$\begin{aligned}
(x,y,z) \in \ker(f) \quad &\iff \quad \begin{cases} x+y-z=0 \\ 2x+2y+z=0 \\ 3x+3y=0 \end{cases} \\
&\iff \quad \begin{cases} x+y-z=0 \\ 3x+3y=0 \quad (\text{e.g. } R_2 \leftarrow R_2+R_1) \\ 3x+3y=0 \end{cases} \\
&\iff \quad \begin{cases} y=-x \\ z=0 \end{cases}
\end{aligned}$$

Hence every vector in the kernel has the form $(x,y,z) = (t,-t,0) = t\,(1,-1,0)$ with $t \in \mathbb{R}$. In particular,

$$\boxed{\ker(f) = \operatorname{Span}\{(1,-1,0)\}}, \qquad \dim \ker(f) = 1.$$

Consistency with the subspace proof. Since any linear combination $\alpha(1,-1,0)+\lambda(1,-1,0) = (\alpha+\lambda)(1,-1,0)$ remains of the form $t(1,-1,0)$, we indeed have $\alpha v_1 + \lambda v_2 \in \ker(f)$ for all $v_1, v_2 \in \ker(f)$ and $\alpha, \lambda \in \mathbb{R}$.

(f) **Rank theorem.** We found $\dim \ker(f) = 1$ and $\operatorname{rg}(f) = 2$, so

$$\dim \ker(f) + \operatorname{rg}(f) = 1 + 2 = 3 = \dim \mathbb{R}^3,$$

which verifies the rank theorem.

Solution to Exercise 3 : Diagonalization and spectral theorem

We consider

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

(a) **Symmetry.**

$$S^\top = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix} = S,$$

so S is symmetric.

(b) **Checking the given eigenpair.**

$$S\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + 1\cdot 1 + 2\cdot(-1) \\ 1\cdot 1 + 1\cdot 1 + 0\cdot(-1) \\ 2\cdot 1 + 0\cdot 1 + 4\cdot(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Hence $\lambda = 2$ is an eigenvalue and $(1, 1, -1)$ is an associated eigenvector.

(c) **Injectivity / surjectivity / bijectivity.** In finite dimension, these are equivalent to invertibility. Let us compute $\det(S)$ (expansion along the first row) :

$$\det(S) = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 3\cdot 4 - 1\cdot 4 + 2\cdot(-2) = 4 \neq 0.$$

Thus S is invertible, hence injective, surjective, and bijective on $\mathbb{R}^3$.

(d) **Diagonalizability, characteristic polynomial, spectrum, eigenspaces.**

Since S is symmetric, it is diagonalizable over $\mathbb{R}$ with an orthonormal eigenbasis (spectral theorem). We can find all eigenvalues without expanding a full 3×3 determinant by using the invariants $\mathrm{Tr}(S)$ and $\det(S)$ together with the known eigenvalue from (b).

Step 1 : compute trace and determinant (already done).

$$\mathrm{Tr}(S) = 3 + 1 + 4 = 8, \qquad \det(S) = 4.$$

Step 2 : set up the system with the eigenvalues. Let the eigenvalues of S be denoted by $\lambda_0, \lambda_1, \lambda_2$, where we know from (b) that $\lambda_0 = 2$. (They are *not* necessarily all distinct.) Using $\mathrm{Tr}(S) = \lambda_0 + \lambda_1 + \lambda_2$ and $\det(S) = \lambda_0\lambda_1\lambda_2$, we write

$$\begin{cases} \lambda_0 + \lambda_1 + \lambda_2 = \mathrm{Tr}(S) = 8, \\ \lambda_0\,\lambda_1\,\lambda_2 = \det(S) = 4 \end{cases} \iff \begin{cases} 2 + \lambda_1 + \lambda_2 = 8, \\ 2\,\lambda_1\,\lambda_2 = 4 \end{cases} \iff \begin{cases} \lambda_1 + \lambda_2 = 6, \\ \lambda_1\lambda_2 = 2. \end{cases}$$

Step 3 : solve for the two remaining eigenvalues. Thus λ_1, λ_2 are the roots of the quadratic

$$X^2 - (\lambda_1 + \lambda_2)X + \lambda_1\lambda_2 = X^2 - 6X + 2,$$

whence

$$\lambda_1 = 3 - \sqrt{7}, \qquad \lambda_2 = 3 + \sqrt{7}.$$

$$\boxed{\mathrm{Sp}(S) = \{\,2,\ 3-\sqrt{7},\ 3+\sqrt{7}\,\}}, \qquad \chi_S(X) = (X-2)(X^2-6X+2).$$

Step 2 : eigenspaces.
Since the three eigenvalues are distinct, each eigenspace is one-dimensional
For $\lambda = 2$: no need to compute. We already know from question (b) an eigenvector. We only need one because $\dim(E_2) = 1$ (all eingevalues are distincts) so

$$\boxed{E_2 = \mathrm{Span}\{(1,1,-1)\}}.$$

But we can verify this by computing :

$$(x,y,z) \in E_2 = \ker(S-2I) \iff \begin{cases} (3-2)x + 1\cdot y + 2\cdot z = 0 \\ 1\cdot x + (1-2)y + 0\cdot z = 0 \\ 2\cdot x + 0\cdot y + (4-2)z = 0 \end{cases} \iff \begin{cases} x+y+2z = 0 \\ x-y = 0 \\ 2x+2z = 0 \end{cases}$$

From $x - y = 0$ *we get* $y = x$*. From* $2x + 2z = 0$ *we get* $z = -x$*. The first equation is then automatically satisfied :*

$$x + y + 2z = x + x + 2(-x) = 0.$$

Hence

$$(x,y,z) = (t,t,-t) = t(1,1,-1), \quad t \in \mathbb{R}, \quad \Rightarrow \quad E_2 = \mathrm{Span}\{(1,1,-1)\} \quad \checkmark$$

For $\lambda = 3+\sqrt{7}$ (write $\lambda_+ = 3+\sqrt{7}$) :

$$(x,y,z) \in E_{\lambda_+} \iff \begin{cases} (3-\lambda_+)x + y + 2z = 0 \\ x + (1-\lambda_+)y = 0 \\ 2x + (4-\lambda_+)z = 0 \end{cases}$$

From the third equation we get $x = \dfrac{\lambda_+ - 4}{2} z = \dfrac{-1+\sqrt{7}}{2} z$. Plugging in the second equation gives

$$\frac{-1+\sqrt{7}}{2} z + (1-\lambda_+)y = 0 \iff y = \frac{\lambda_+ - 4}{2(\lambda_+ - 1)} z = \frac{-1+\sqrt{7}}{2(2+\sqrt{7})} z = \frac{3-\sqrt{7}}{2} z,$$

(where we rationalized the denominator). Thus we can choose $z = 2$ to avoid fractions and obtain the eigenvector

$$v_+ = (-1+\sqrt{7},\ 3-\sqrt{7},\ 2), \qquad \Rightarrow \qquad \boxed{E_{3+\sqrt{7}} = \mathrm{Span}\{(-1+\sqrt{7},\ 3-\sqrt{7},\ 2)\}}.$$

For $\lambda = 3-\sqrt{7}$ (write $\lambda_- = 3-\sqrt{7}$) :

$$(x,y,z) \in E_{\lambda_-} \iff \begin{cases} (3-\lambda_-)x + y + 2z = 0 \\ x + (1-\lambda_-)y = 0 \\ 2x + (4-\lambda_-)z = 0 \end{cases}$$

From the third equation $x = \dfrac{\lambda_- - 4}{2} z = \dfrac{-1-\sqrt{7}}{2} z$. From the second,

$$\frac{-1-\sqrt{7}}{2} z + (1-\lambda_-)y = 0 \iff y = \frac{\lambda_- - 4}{2(\lambda_- - 1)} z = \frac{-1-\sqrt{7}}{2(2-\sqrt{7})} z = \frac{3+\sqrt{7}}{2} z.$$

Choosing $z = 2$ yields the eigenvector

$$v_- = (-(1+\sqrt{7}),\ 3+\sqrt{7},\ 2), \qquad \Rightarrow \qquad \boxed{E_{3-\sqrt{7}} = \mathrm{Span}\{(-(1+\sqrt{7}),\ 3+\sqrt{7},\ 2)\}}.$$

Since the three eigenvalues are distinct, each eigenspace is one-dimensional and they are pairwise orthogonal (symmetry of S).

(e) **Orthogonal diagonalization $S = QDQ^\top$.**
Let us normalize the three eigenvectors.

$$\begin{aligned}\|(1,1,-1)\| &= \sqrt{3},\\ \|v_+\|^2 &= (\sqrt{7}-1)^2 + (3-\sqrt{7})^2 + 2^2 = 28 - 8\sqrt{7},\\ \|v_-\|^2 &= (1+\sqrt{7})^2 + (3+\sqrt{7})^2 + 2^2 = 28 + 8\sqrt{7}.\end{aligned}$$

Set

$$u_2 = \frac{1}{\sqrt{3}}(1,1,-1), \qquad u_+ = \frac{1}{\sqrt{28-8\sqrt{7}}}\,v_+, \qquad u_- = \frac{1}{\sqrt{28+8\sqrt{7}}}\,v_-.$$

Then $Q = [u_+ \ \ u_- \ \ u_2]$ is orthogonal ($Q^\top Q = I_3$) and

$$D = \mathrm{diag}(3+\sqrt{7},\ 3-\sqrt{7},\ 2), \qquad \boxed{S = QDQ^\top}.$$

(Any ordering of the columns of Q must be matched by the same ordering of the diagonal entries of D.)

Solution to Exercise 4 : Stochastic chain in dimension 3

We encode the probabilities at day t by

$$v_t = \begin{pmatrix} p_t \\ \ell_t \\ m_t \end{pmatrix}, \qquad p_t, \ell_t, m_t \geq 0, \quad p_t + \ell_t + m_t = 1.$$

(a) **From the rules to the linear system.**
Reading *incoming* flow for each city from the stated rules :

$$\begin{cases} p_{t+1} = \frac{1}{2}\,p_t \ + \ \frac{1}{4}\,\ell_t \ + \ 0 \cdot m_t \\ \ell_{t+1} = \frac{1}{2}\,p_t \ + \ \frac{1}{4}\,\ell_t \ + \ 1 \cdot m_t \\ m_{t+1} = 0 \cdot p_t \ + \ \frac{1}{2}\,\ell_t \ + \ 0 \cdot m_t \end{cases}.$$

(b) **Transition matrix A with $v_{t+1} = Av_t$.**
Collecting the coefficients gives

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \qquad v_{t+1} = A\,v_t.$$

(c) **Column-stochastic check.**
All entries are ≥ 0 and each column sums to 1 :

$$\mathrm{col}_1 : \ \tfrac{1}{2}+\tfrac{1}{2}+0=1, \quad \mathrm{col}_2 : \ \tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{2}=1, \quad \mathrm{col}_3 : \ 0+1+0=1.$$

Hence A is column-stochastic. In particular, 1 is an eigenvalue.

Indeed,

$$A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}+\frac{1}{2}\\ \frac{1}{2}+\frac{1}{4}+\frac{1}{2}\\ 0+1+0\end{pmatrix}=1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$$

(d) **Spectrum** $\mathrm{Sp}(A)$ **and eigenspaces** E_λ**.**
Characteristic polynomial.

$$\chi_A(\lambda)=\det(\lambda I-A)=\lambda^3-\tfrac{3}{4}\lambda^2-\tfrac{1}{2}\lambda+\tfrac{1}{4}=(\lambda-1)\left(4\lambda^2+\lambda-1\right)/4.$$

Thus the eigenvalues are

$$\boxed{\lambda_0=1, \qquad \lambda_\pm=\frac{-1\pm\sqrt{17}}{8}}$$

(all real).

Without expanding a full 3×3 determinant. Characteristic polynomial and eigenvalues.
Step 1 : compute $\mathrm{Tr}(A)$ and $\det(A)$.

$$A=\begin{pmatrix}\frac{1}{2}&\frac{1}{4}&0\\ \frac{1}{2}&\frac{1}{4}&1\\ 0&\frac{1}{2}&0\end{pmatrix}, \qquad \mathrm{Tr}(A)=\tfrac{1}{2}+\tfrac{1}{4}+0=\boxed{\tfrac{3}{4}}.$$

For the determinant, we can avoid a full 3×3 expansion. Expanding along the first row (since $a_{13}=0$) :

$$\det(A)=\frac{1}{2}\begin{vmatrix}\frac{1}{4}&1\\ \frac{1}{2}&0\end{vmatrix}-\frac{1}{4}\begin{vmatrix}\frac{1}{2}&1\\ 0&0\end{vmatrix}+0=\frac{1}{2}\big(\tfrac{1}{4}\cdot 0-1\cdot\tfrac{1}{2}\big)-\frac{1}{4}\cdot 0=\boxed{-\tfrac{1}{4}}.$$

(Ce calcul cible un cofacteur nul et un mineur simple : pas besoin d'un développement 3×3 complet.)

Step 2 : use trace and determinant with the known eigenvalue $\lambda_0=1$.
Let the eigenvalues of A be $\lambda_0,\lambda_1,\lambda_2$, where we already know (stochastic matrix) that $\lambda_0=1$. They are *not* assumed distinct a priori. By the basic spectral identities $\mathrm{Tr}(A)=\lambda_0+\lambda_1+\lambda_2$ and $\det(A)=\lambda_0\lambda_1\lambda_2$, we obtain

$$\begin{cases}\lambda_0+\lambda_1+\lambda_2=\mathrm{Tr}(A)\\ \lambda_0\lambda_1\lambda_2=\det(A)\end{cases} \iff \begin{cases}1+\lambda_1+\lambda_2=\frac{3}{4}\\ 1\cdot\lambda_1\lambda_2=-\frac{1}{4}\end{cases}$$

$$\iff \boxed{\begin{cases}\lambda_1+\lambda_2=-\frac{1}{4}\\ \lambda_1\lambda_2=-\frac{1}{4}\end{cases}}$$

Step 3 : solve the quadratic for the remaining eigenvalues.

The unknowns λ_1, λ_2 are the roots of

$$X^2 - (\lambda_1 + \lambda_2)X + \lambda_1\lambda_2 \;=\; X^2 + \tfrac{1}{4}X - \tfrac{1}{4} \;=\; 0,$$

or, clearing denominators,

$$\boxed{4X^2 + X - 1 = 0}.$$

By the quadratic formula,

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+16}}{8} = \boxed{\frac{-1 \pm \sqrt{17}}{8}}.$$

These are real numbers, so the full spectrum is real :

$$\boxed{\mathrm{Sp}(A) = \left\{ 1,\ \frac{-1-\sqrt{17}}{8},\ \frac{-1+\sqrt{17}}{8} \right\}}.$$

<u>(Optional) Characteristic polynomial, for checking.</u> Since the eigenvalues are 1, $\frac{-1\pm\sqrt{17}}{8}$, the characteristic polynomial is

$$\chi_A(\lambda) = (\lambda - 1)\Big(\lambda - \tfrac{-1+\sqrt{17}}{8}\Big)\Big(\lambda - \tfrac{-1-\sqrt{17}}{8}\Big) = (\lambda-1)\,\frac{4\lambda^2+\lambda-1}{4} = \boxed{\lambda^3 - \tfrac{3}{4}\,\lambda^2 - \tfrac{1}{2}\,\lambda + \tfrac{1}{4}}.$$

<u>Eigenspace for $\lambda = 1$.</u> We solve $(A - I)x = 0$ with $x = (x, y, z)$:

$$(x,y,z) \in E_1 = \ker(A - I) \quad \Longleftrightarrow \quad \begin{cases} -\frac{1}{2}x + \frac{1}{4}y + 0\cdot z = 0 \\ \frac{1}{2}x - \frac{3}{4}y + 1\cdot z = 0 \\ 0\cdot x + \frac{1}{2}y - 1\cdot z = 0 \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} -2x + y = 0 \\ 2x - 3y + 4z = 0 \\ y - 2z = 0 \end{cases} \qquad \text{(multiply rows by 4)}$$

$$\Longleftrightarrow \quad \begin{cases} y = 2x \\ y = 2z \end{cases} \quad \Longrightarrow \quad y = 2x,\ z = x.$$

Hence $(x, y, z) = x(1, 2, 1)$ and

$$\boxed{E_1 = \mathrm{Span}\{(1,2,1)\}}.$$

<u>Eigenspaces for $\lambda_\pm$.</u> Set $\lambda_\pm = \dfrac{-1 \pm \sqrt{17}}{8}$. Solving $(A - \lambda_\pm I)x = 0$ yields, after elementary reductions (details below), the eigenvectors

$$v_+ = (-3 - \sqrt{17},\ -1 + \sqrt{17},\ 4), \qquad v_- = (-3 + \sqrt{17},\ -1 - \sqrt{17},\ 4),$$

so that

$$\boxed{E_{\lambda_+} = \mathrm{Span}\{v_+\}, \qquad E_{\lambda_-} = \mathrm{Span}\{v_-\}}.$$

(One clean way to obtain $v_\pm$.) Start from $(A - \lambda I)(x, y, z)^\top = 0$:

$$\begin{cases} (\frac{1}{2} - \lambda)x + \frac{1}{4}y = 0 \\ \frac{1}{2}x + (\frac{1}{4} - \lambda)y + z = 0 \\ \frac{1}{2}y - \lambda z = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} (2 - 4\lambda)x + y = 0 \\ 2x + (1 - 4\lambda)y + 4z = 0 \\ 2y - 4\lambda z = 0 \end{cases}$$

(from multiplying the three equations by 4). For $\lambda_+, \lambda_- \neq 0$ we get from the third equation $y = 2\lambda_+ z$, then from the first $x = \dfrac{y}{4\lambda_+ - 2} = \dfrac{2\lambda_+}{4\lambda_+ - 2}\, z$.

Plugging into the second gives an identity (since λ_+ is a root of $4\lambda_+^2 + \lambda_+ - 1 = 0$). We get the exact same result by remplacing λ_+ by λ_-. Taking $z = 4$ produces exactly the vectors $v_\pm$ above.

(e) **Diagonalization $A = PDP^{-1}$.**
Choose

$$P = \big[\, v_+ \;\mid\; v_- \;\mid\; (1,2,1) \,\big] = \begin{pmatrix} -3-\sqrt{17} & -3+\sqrt{17} & 1 \\ -1+\sqrt{17} & -1-\sqrt{17} & 2 \\ 4 & 4 & 1 \end{pmatrix}, \qquad D = \operatorname{diag}(\lambda_+,\ \lambda_-,\ 1)\,,$$

so that

$$\boxed{A = PDP^{-1}}.$$

(Any permutation of the columns of P must be paired with the same permutation of the diagonal entries of D.)

(f) **Computing P^{-1} by Gaussian elimination.**
Form the augmented matrix $(P \mid I_3)$ and perform row operations until $(I_3 \mid P^{-1})$:

$$(P \mid I_3) \;\sim\; (I_3 \mid P^{-1}).$$

(Students should carry out the row-reduction explicitly. The arithmetic is straightforward though a bit bulky due to square roots ; exact radicals are fine.)

(g) **Long-run limit for $v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and interpretation.**
Using the diagonalization,

$$A^t = PD^tP^{-1}, \qquad v_t = A^t v_0 = PD^tP^{-1}v_0.$$

Since $|\lambda_\pm| < 1$ (indeed $\lambda_\pm = \frac{-1\pm\sqrt{17}}{8} \in (-1,1)$), we have $\lambda_\pm^t \to 0$. Therefore

$$\lim_{t\to\infty} A^t = P\ \operatorname{diag}(0,0,1)\, P^{-1} = \Pi,$$

the rank-one projector onto $E_1 = \operatorname{Span}\{(1,2,1)\}$ along the other eigendirections. Normalizing the eigenvector for $\lambda = 1$ to sum to 1 gives the stationary distribution

$$\pi = \frac{1}{1+2+1}\,(1,2,1)^\top = \begin{pmatrix} \frac14 \\ \frac12 \\ \frac14 \end{pmatrix}.$$

Hence, for $v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$$\boxed{v_\infty := \lim_{t\to\infty} v_t = \pi = \begin{pmatrix} \frac14 \\ \frac12 \\ \frac14 \end{pmatrix}}.$$

Interpretation. Regardless of the initial city, the distribution converges to $\left(\frac14, \frac12, \frac14\right)$: in the long run, the traveler spends half the time in Lyon and a quarter in each of Paris and Marseille.