Algebra Practice Exam

M1 MIASHS

September 2025

Exercise 1 : Subspace of symmetric matrices (case $n = 3$)

Let $\mathcal{M}_3(\mathbb{R})$ be the vector space of real 3×3 matrices and

$$\mathscr{S}_3(\mathbb{R}) = \{A \in \mathcal{M}_3(\mathbb{R}) : A^\top = A\}$$

the set of real symmetric 3×3 matrices.

(a) Show that $\mathscr{S}_3(\mathbb{R})$ is a vector subspace of $\mathcal{M}_3(\mathbb{R})$.

(b) For $1 \leq i \leq 3$, let E_{ii} denote the matrix of $\mathcal{M}_3(\mathbb{R})$ whose only nonzero entry is 1 at position (i, i). For $1 \leq i < j \leq 3$, let $F_{ij} = E_{ij} + E_{ji}$, where E_{ij} has 1 at position (i, j) and 0 elsewhere.

Write explicitly the matrices $E_{11},\ E_{22},\ E_{33},\ F_{12},\ F_{13},\ F_{23}$. Show that the family

$$\mathcal{B} = \{E_{11}, E_{22}, E_{33}, F_{12}, F_{13}, F_{23}\}$$

is a basis of $\mathscr{S}_3(\mathbb{R})$ (prove that it spans $\mathscr{S}_3(\mathbb{R})$ and that it is linearly independent).

(c) Deduce the dimension of $\mathscr{S}_3(\mathbb{R})$.

Bonus info (general case n). For $n \geq 1$, define E_{ii} for $1 \leq i \leq n$ and $F_{ij} = E_{ij} + E_{ji}$ for $1 \leq i < j \leq n$. Then

$$\mathcal{B}_n = \{E_{11}, \ldots, E_{nn}\} \ \cup \ \{F_{ij} \ : \ 1 \leq i < j \leq n\}$$

is a basis of $\mathscr{S}_n(\mathbb{R})$, and

$$\dim\left(\mathscr{S}_n(\mathbb{R})\right) = \frac{n(n+1)}{2}.$$

Exercise 2 : Non-invertible linear map on $\mathbb{R}^3$

Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined, for $(x, y, z) \in \mathbb{R}^3$, by

$$f(x, y, z) = (\, x + y - z,\ 2x + 2y + z,\ 3x + 3y\,).$$

(a) Show that f is a linear map (*i.e.*, $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$).

(b) Determine the matrix A associated with f in the canonical basis of $\mathbb{R}^3$.

(c) Is the matrix A invertible ? Consequently, can we say whether f is injective, surjective, bijective ?

(d) Determine $\ker(f)$ and $\mathrm{Im}(f)$, and show that they are vector subspaces of $\mathbb{R}^3$.

(e) Which equation(s) characterize $(x, y, z) \in \ker(f)$? Show that it is a vector subspace of $\mathbb{R}^3$.

(f) Verify the rank theorem in this example : $\dim\ker(f) + \mathrm{rg}(f) = 3$.

Exercise 3 : Diagonalization and spectral theorem

Consider the matrix

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

(a) Check that S is symmetric.

(b) Verify that $S \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

(c) Determine whether S is injective, surjective, bijective (in finite dimension, relate this to invertibility of S).

(d) Is S necessarily diagonalizable ? Justify your answer. Compute the characteristic polynomial of S, its spectrum $\mathrm{Sp}(S)$, and bases of each eigenspace E_λ.

(e) Deduce an orthogonal matrix Q (its columns being orthonormal eigenvectors) and a diagonal matrix D such that $S = QDQ^\top$.

Exercise 4 : Stochastic Chain in Dimension 3

A traveler moves each day between three cities : Paris (P), Lyon (L), and Marseille (M). We model this movement as a random process whose transition probabilities are given by the following rules (read as "*from city j to city i*") :

— If the traveler is in **Paris**, the next day he stays in Paris with probability $\frac{1}{2}$ and goes to Lyon with probability $\frac{1}{2}$ (never directly to Marseille).

— If the traveler is in **Lyon**, the next day he goes to Paris with probability $\frac{1}{4}$, stays in Lyon with probability $\frac{1}{4}$, and goes to Marseille with probability $\frac{1}{2}$.

— If the traveler is in **Marseille**, the next day he goes to Lyon with probability 1 (never directly to Paris, and he never stays in Marseille).

We denote

$$v_t = \begin{pmatrix} p_t \\ \ell_t \\ m_t \end{pmatrix}$$

the probability vector of being in Paris, Lyon, and Marseille at day t, respectively (with $p_t, \ell_t, m_t \geq 0$ and $p_t + \ell_t + m_t = 1$).

(a) From the rules above, write the linear system relating v_{t+1} to v_t (component-wise equations for $p_{t+1}, \ell_{t+1}, m_{t+1}$).

(b) Deduce the transition matrix A (column-stochastic) such that $v_{t+1} = A\, v_t$.

(c) Verify that A is indeed column-stochastic (nonnegative coefficients and each column sums to 1).

(d) Determine the spectrum $\mathrm{Sp}(A)$ and, for each eigenvalue, a basis of the associated eigenspace.

(e) Construct a matrix P of linearly independent eigenvectors and a diagonal matrix D such that $A = PDP^{-1}$.

(f) Compute P^{-1} by Gaussian elimination on the augmented system $(P \mid I_3) \sim (I_3 \mid P^{-1})$.

(g) Deduce, for the initial state

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

the limit

$$v_\infty = \left(\lim_{t \to \infty} A^t \right) v_0.$$

Interpret this result (stationary distribution in the long run).