Linear Algebra for Machine Learning

Paul MINCHELLA, Stéphane CHRÉTIEN
paul.minchella@lyon.unicancer.fr

Motivations for Linear Algebra Review

- Computational engine of mathematics:
  → **Numerical Analysis** *(Finite Elements)*; **Algebraic Geometry** *(Hodge Decomposition)*; **Statistics** *(Covariance Matrix, Data Shape)*
- Data science practitioners: diverse backgrounds
- Refresh key concepts often forgotten (*e.g.*, eigenvalues)

Goal: develop dexterity with

- ✓ Linear Equations, Gaussian Elimination, Matrices
- ✓ Vector Spaces, Transformations, Basis Changes
- ✓ Diagonalization, Webpage Ranking, Covariance
- ✓ Orthogonality, Least Squares, SVD

Motivations for Linear Algebra Review

- Computational engine of mathematics:
  → **Numerical Analysis** *(Finite Elements)*; **Algebraic Geometry** *(Hodge Decomposition)*; **Statistics** *(Covariance Matrix, Data Shape)*
- Data science practitioners: diverse backgrounds
- Refresh key concepts often forgotten (*e.g.*, eigenvalues)

Goal: develop dexterity with

✓ Linear Equations, Gaussian Elimination, Matrices

✓ Vector Spaces, Transformations, Basis Changes

✓ Diagonalization, Webpage Ranking, Covariance

✓ Orthogonality, Least Squares, SVD

Example: Smoking in Smallville

Each year: 30% of nonsmokers start smoking, 20% of smokers quit. Initial population: 8000 smokers, 2000 nonsmokers. Questions:

- Numbers after 100 years?
- Numbers after n years?
- Is there a stable equilibrium?

Core points (why and how)

- **Goal**: reduce to fewer equations/variables via elimination.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

- **Basic elimination step (column 1):**
  1. Pivot: swap so $a_{11} \neq 0$.
  2. Normalize: $L_1 \leftarrow \frac{1}{a_{11}} L_1$.
  3. Zero below: for $i \geq 2$, $L_i \leftarrow L_i - a_{i1} L_1$.
- Iterate on remaining submatrix; back-substitute or continue to RREF.

Operations (preserve solution set of $Ax = b$):

Reason: Each operation equals left-multiplication by an invertible *elementary matrix* E, hence

$$Ax = b \iff (EA)x = Eb.$$

- Row swap: $L_i \leftrightarrow L_j$
- Scale: $L_i \leftarrow \lambda L_i$, $\lambda \neq 0$
- Row add: $L_i \leftarrow L_i + \lambda L_j$

Exercise

Solve the system

$$\begin{cases} x + y + z = 3 \\ 2x + y = 7 \\ 3x + 2z = 5 \end{cases}$$

Definition

A *matrix* is an $m \times n$ array of elements, where m is the number of rows and n is the number of columns.

$$A \in \mathcal{M}_{m\times n}(\mathbb{K}) \quad \text{(matrix with entries in a field } \mathbb{K}\text{, } \textit{e.g.}\text{, } \mathbb{R}, \mathbb{C}\text{).}$$

We also write $A \in \mathbb{R}^{m\times n}$ for $\mathbb{K} = \mathbb{R}$.

Key properties of $\mathcal{M}_{m\times n}(\mathbb{K})$

- Vector space over $\mathbb{K}$: addition and scalar multiplication are defined entrywise.
- Dimension: $\dim \mathcal{M}_{m\times n}(\mathbb{K}) = mn$.
- Matrix multiplication defined if inner dimensions match $(A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{n\times p})$.
- Multiplication is associative but **not commutative** in general.
- Identity matrix $I_n \in \mathcal{M}_{n\times n}(\mathbb{K})$: $AI_n = I_m A = A$.
- Invertibility only for square matrices $A \in \mathcal{M}_{n\times n}(\mathbb{K})$, with $\det(A) \neq 0$.

Idea

Matrix multiplication corresponds to composing two linear systems:

$$C = AB \quad \Longleftrightarrow \quad \text{Apply } B \text{ first, then } A.$$

Example (two 2×2 systems)

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}.$$

First system (apply B to $\begin{pmatrix} x \\ y \end{pmatrix}$):

$$\begin{cases} z = 2x + 0y \\ w = x + 3y \end{cases}$$

Second system (apply A to $\begin{pmatrix} z \\ w \end{pmatrix}$):

$$\begin{cases} u = z + 2w = 4x + 6y \\ v = w = 1x + 3y \end{cases}$$

Overall transformation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$ is given by

$$C = AB = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}.$$

Definition: Dot Product

For vectors $\mathbf{v} = [a_1, \ldots, a_n]$ and $\mathbf{w} = [b_1, \ldots, b_n]$, the dot product is

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} a_i b_i, \quad \text{and} \quad |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

By the law of cosines, vectors $\mathbf{v}, \mathbf{w}$ are orthogonal iff $\mathbf{v} \cdot \mathbf{w} = 0$.

Matrix Multiplication

Let $A \in \mathcal{M}_{m \times n}(\mathbb{K})$ and $B \in \mathcal{M}_{p \times q}(\mathbb{K})$. Matrix multiplication AB is defined when $n = p$. If $(AB)_{ij}$ denotes the (i, j) entry, then

$$(AB)_{ij} = \mathrm{row}_i(A) \cdot \mathrm{col}_j(B).$$

Interpretation: Each matrix represents a linear map in a chosen basis. Therefore, multiplication of matrices (composition of linear maps) and the dot product (row · column) only make sense within the same basis. We will formalize this with the notions of *linear maps* and *basis*, introduced next.

Exercise

$$\begin{bmatrix} 2 & 7 \\ 3 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 37 & 46 & 55 & 64 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Fill in the missing entries.

Definitions

- Transpose: $(A^\top)_{ij} = A_{ji}$.
  - Linearity: $(\alpha A + \beta B)^\top = \alpha A^\top + \beta B^\top$.
  - Involution: $(A^\top)^\top = A$.
  - Product rule: $(AB)^\top = B^\top A^\top$.
- A is **symmetric** if $A^\top = A$.
- A is **diagonal** if $A_{ij} \neq 0 \Rightarrow i = j$.
  - If A, B diagonal $\in \mathcal{M}_{n\times n}(\mathbb{K})$: $AB = BA, \quad (AB)_{ii} = a_{ii}\, b_{ii}$.
- Identity: $I_n = \mathrm{diag}(1, \ldots, 1)$, $I_n A = A = A I_m$.
- Inverse: $A \in \mathcal{M}_{n\times n}$ is invertible $\iff \exists\, B$ s.t. $BA = AB = I_n$. Denote $A^{-1} = B$.

Definitions

- Transpose: $(A^\top)_{ij} = A_{ji}$.
    - Linearity: $(\alpha A + \beta B)^\top = \alpha A^\top + \beta B^\top$.
    - Involution: $(A^\top)^\top = A$.
    - Product rule: $(AB)^\top = B^\top A^\top$.
- A is **symmetric** if $A^\top = A$.
- A is **diagonal** if $A_{ij} \neq 0 \Rightarrow i = j$.
    - If A, B diagonal $\in \mathcal{M}_{n\times n}(\mathbb{K})$: $AB = BA, \quad (AB)_{ii} = a_{ii}\, b_{ii}$.
- Identity: $I_n = \operatorname{diag}(1, \ldots, 1), \; I_n A = A = A I_m$.
- Inverse: $A \in \mathcal{M}_{n\times n}$ is invertible $\iff \exists\, B$ s.t. $BA = AB = I_n$. Denote $A^{-1} = B$.

Exercise

- Find 2×2 matrices (A, B) with $AB \neq BA$.
- Show $(AB)^\top = B^\top A^\top$. Deduce that $A^\top A$ is symmetric.

Definitions

- Transpose: $(A^\top)_{ij} = A_{ji}$.
  - Linearity: $(\alpha A + \beta B)^\top = \alpha A^\top + \beta B^\top$.
  - Involution: $(A^\top)^\top = A$.
  - Product rule: $(AB)^\top = B^\top A^\top$.
- A is **symmetric** if $A^\top = A$.
- A is **diagonal** if $A_{ij} \neq 0 \Rightarrow i = j$.
  - If A, B diagonal $\in \mathcal{M}_{n\times n}(\mathbb{K})$: $AB = BA, \quad (AB)_{ii} = a_{ii}\, b_{ii}$.
- Identity: $I_n = \mathrm{diag}(1, \ldots, 1)$, $I_n A = A = A I_m$.
- Inverse: $A \in \mathcal{M}_{n\times n}$ is invertible $\iff \exists\, B$ s.t. $BA = AB = I_n$. Denote $A^{-1} = B$.

Exercise

- Find 2×2 matrices (A, B) with $AB \neq BA$.
- Show $(AB)^\top = B^\top A^\top$. Deduce that $A^\top A$ is symmetric.

Matrix form of a linear system

A system of n linear equations in m unknowns is written as

Smoking in Smallville

Let $(n_t, s_t)^\top =$ (# nonsmokers , # smokers) at year t. Transition rule:

$$\begin{bmatrix} n_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} n_t \\ s_t \end{bmatrix}.$$

By iteration:

$$\begin{bmatrix} n_t \\ s_t \end{bmatrix} = \left(\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \right)^t \begin{bmatrix} n_0 \\ s_0 \end{bmatrix}.$$

To study $t \gg 0$, we need to compute A^t with

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}.$$

Key Trick: Diagonalization

There exists a change-of-basis matrix B such that

$$B\,A\,B^{-1} = D \quad \text{(diagonal)}.$$

Then, for any integer m, we get

$$A^m = B^{-1}\,D^m\,B,$$

and computing D^m is easy (just raise diagonal entries).

$\Rightarrow$ Expensive repeated multiplications become trivial if A is diagonalizable.

Number Sets

- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, non-negative integers.
- **Integers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, all negative and positive whole numbers, including 0.
- **Rational numbers:** $\mathbb{Q} = \left\{\frac{p}{q} \,:\, p \in \mathbb{Z},\ q \in \mathbb{Z}^*,\ q \neq 0\right\}$, ratios of integers.
- **Real numbers:** $\mathbb{R}$ is the *completion* of $\mathbb{Q}$; a totally ordered complete field. π, e, $\sqrt{2}$, $\varphi = \frac{1+\sqrt{5}}{2}$ are real numbers that are irrational.
- **Complex numbers:** $\mathbb{C} = \{x + \mathrm{i}y :\ x, y \in \mathbb{R},\ \mathrm{i}^2 = -1\}$, an algebraic extension of $\mathbb{R}$.

Hierarchy

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Binary Operation (Internal Law)

Let E be a set. An *internal binary operation* on E is a map

$$\star : E \times E \to E, \qquad (x, y) \mapsto x \star y.$$

Examples: $+$ and $\times$ on $\mathbb{Z}$.

Group

A *group* is a pair $(G, \star)$ where $\star$ is an internal binary operation satisfying:

- **Associativity:** $(x \star y) \star z = x \star (y \star z)$.
- **Identity element:** there exists $e \in G$ such that $x \star e = e \star x = x$.
- **Inverse:** every $x \in G$ has an inverse x^{-1} with $x \star x^{-1} = e$.

If $x \star y = y \star x$ for all x, y, the group is *abelian*.

Name some familiar examples!

Example

$(\mathbb{Z}, +)$ is an abelian group. $(\mathbb{Z}, \times)$ is *not* a group: not every integer has a multiplicative inverse in $\mathbb{Z}$.

Ring

A *ring* $(A, +, \times)$ is a set with two operations:

- $(A, +)$ is an abelian group.
- $\times$ is associative and has a multiplicative identity 1.
- $\times$ distributes over $+$.

Field

A *field* is a ring $(K, +, \times)$ in which every nonzero element has a multiplicative inverse.

Examples

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields. $\mathbb{Z}$ is a ring but not a field.

Problem

Define, for $a, b \in \mathbb{Z}$, the operation

$$a \star b \;=\; a + b + 1.$$

1. Show that $\star$ is an *internal* binary operation on $\mathbb{Z}$.
2. Check associativity and commutativity of $\star$.
3. Determine the identity element e for $\star$.
4. For a given $a \in \mathbb{Z}$, find the inverse of a with respect to $\star$.
5. Conclude: is $(\mathbb{Z}, \star)$ a group? Is it abelian?

Your solution?

To fully grasp and master the notion of matrices, we need to introduce the formal framework that governs them: ***vector spaces*** *and* ***linear maps****.*

Definition (Vector Space)

Let $\mathbb{K}$ be a field (*e.g.*, $\mathbb{R}$ or $\mathbb{C}$). A *vector space over* $\mathbb{K}$ is a set V equipped with:

- an internal operation $+$ (vector addition)
$$(x, y) \mapsto x + y,$$
- an external operation (scalar multiplication)
$$\mathbb{K} \times V \to V, \qquad (\lambda, x) \mapsto \lambda x,$$

such that the following axioms hold for all $x, y, z \in V$ and $\lambda, \mu \in \mathbb{K}$:

1. $(V, +)$ is an abelian group (associativity, commutativity, neutral element 0, inverses).
2. Scalar compatibility: $(\lambda\mu)x = \lambda(\mu x)$.
3. Neutral element of scalars: $1_{\mathbb{K}} x = x$.
4. Distributivity: $(\lambda + \mu)x = \lambda x + \mu x$, and $\lambda(x + y) = \lambda x + \lambda y$.

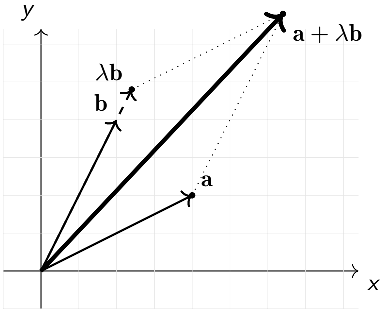
Examples of Vector Spaces

- $\mathbb{K}^n$ (the n-tuples of scalars from $\mathbb{K}$) is a vector space. Indeed:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ \vdots \\ a_n + \lambda b_n \end{pmatrix} \in \mathbb{K}^n$$

- The set $\mathbb{R}[X]$ of real-coefficient polynomials is a vector space over $\mathbb{R}$.
- The set $\mathcal{C}^0([a,b],\mathbb{R})$ of continuous functions from $[a,b]$ to $\mathbb{R}$ is a vector space.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \end{pmatrix} \in \mathbb{R}^2$$



Definition (Subspace)

Let V be a vector space over $\mathbb{K}$. A subset $F \subset V$ is a *subspace* if:

- $0_V \in F$,
- $\forall x, y \in F$, then $x + y \in F$ (closed under addition),
- $\forall \lambda \in \mathbb{K},\ x \in F$, then $\lambda x \in F$ (closed under scalar multiplication).

Definition (Subspace)

Let V be a vector space over $\mathbb{K}$. A subset $F \subset V$ is a *subspace* if:

- $0_V \in F$,
- $\forall x, y \in F$, then $x + y \in F$ (closed under addition),
- $\forall \lambda \in \mathbb{K},\ x \in F$, then $\lambda x \in F$ (closed under scalar multiplication).

Examples

- In $\mathbb{R}^3$, the set of vectors $(x, y, 0)$ is a subspace.
- $\{0\}$ and V are always subspaces (trivial subspaces).

Definition (Subspace)

Let V be a vector space over $\mathbb{K}$. A subset $F \subset V$ is a *subspace* if:

- $0_V \in F$,
- $\forall x, y \in F$, then $x + y \in F$ (closed under addition),
- $\forall \lambda \in \mathbb{K},\ x \in F$, then $\lambda x \in F$ (closed under scalar multiplication).

Examples

- In $\mathbb{R}^3$, the set of vectors $(x, y, 0)$ is a subspace.
- $\{0\}$ and V are always subspaces (trivial subspaces).

Properties

- If F, G are subspaces of V, then $F \cap G$ is also a subspace.
- The **sum** of subspaces is

$$F + G = \{x + y : x \in F,\ y \in G\},$$

which is again a subspace.

Span of a Set

Given a subset $A \subset V$, the set of all finite linear combinations of elements of A forms a subspace of V, denoted

$$\mathrm{Span}(A) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid k \in \mathbb{N},\ v_i \in A,\ \lambda_i \in \mathbb{K}\}.$$

It is called the *subspace spanned by A*.

Span of a Set

Given a subset $A \subset V$, the set of all finite linear combinations of elements of A forms a subspace of V, denoted

$$\mathrm{Span}(A) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid k \in \mathbb{N},\ v_i \in A,\ \lambda_i \in \mathbb{K}\}.$$

It is called the *subspace spanned by A*.

Linear Independence and Generating Set

- A family $(v_1, \ldots, v_p)$ is *linearly independent* if the only relation

$$\lambda_1 v_1 + \cdots + \lambda_p v_p = 0 \Rightarrow \forall\, i,\ \lambda_i = 0.$$

- It is a *generating set* of V if

$$\mathrm{Span}(v_1, \ldots, v_p) = V.$$

Definition

Let V be a vector space over a field $\mathbb{K}$.

- A **basis** of V is a family of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ in V that is
  1. linearly independent,
  2. and generates V (*i.e.*, $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$).

- The **dimension** of V, denoted $\dim(V)$, is the number of vectors in any basis of V.

Remark. The definition is well-posed: every two bases of a finite-dimensional vector space V have the same number of elements.

Theorem (Dimension of a Subspace)

Let F be a subspace of a finite-dimensional vector space V. Then

$$\dim(F) \leq \dim(V).$$

Moreover, if $F \neq V$, the inequality is strict.

Grassmann Formula

If F, G are finite-dimensional subspaces of V, then

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

Exercise

Consider the set

$$\mathcal{F} = \{[1,0],\ [0,1],\ [2,3]\} \subset \mathbb{R}^2.$$

- Is $\mathcal{F}$ linearly independent?
- Is $\mathcal{F}$ a generating family of $\mathbb{R}^2$?
- What is the cardinality of a maximal linearly independent subfamily of $\mathcal{F}$, *i.e.*, the dimension of $\mathrm{Span}(\mathcal{F})$?

Exercise

Consider the set

$$\mathcal{F} = \{[1,0],\ [0,1],\ [2,3]\} \subset \mathbb{R}^2.$$

- Is $\mathcal{F}$ linearly independent?
- Is $\mathcal{F}$ a generating family of $\mathbb{R}^2$?
- What is the cardinality of a maximal linearly independent subfamily of $\mathcal{F}$, *i.e.*, the dimension of $\mathrm{Span}(\mathcal{F})$?

Canonical Example

In $\mathbb{R}^n$, the canonical family

$$e_1 = (1, 0, \dots, 0),\ \dots,\ e_j = (0, \dots, \overset{\text{index } j}{1}, \dots, 0),\ \dots,\ e_n = (0, \dots, 0, 1)$$

is a basis of $\mathbb{R}^n$, and its dimension is n.

Theorem (Incomplete Basis Theorem)

Let V be a finite-dimensional vector space with $\dim(V) = n$. Suppose

$$(v_1, \dots, v_p), \quad p < n,$$

is a linearly independent family of vectors in V.
Then there exist additional vectors $v_{p+1}, \dots, v_n \in V$ such that

$$(v_1, \dots, v_p, v_{p+1}, \dots, v_n)$$

is a basis of V.
*In other words, **any linearly independent family can be extended to a basis**.*

Theorem (Incomplete Basis Theorem)

Let V be a finite-dimensional vector space with $\dim(V) = n$. Suppose

$$(v_1, \dots, v_p), \quad p < n,$$

is a linearly independent family of vectors in V.
Then there exist additional vectors $v_{p+1}, \dots, v_n \in V$ such that

$$(v_1, \dots, v_p, v_{p+1}, \dots, v_n)$$

is a basis of V.
In other words, ***any linearly independent family can be extended to a basis****.*

Example

In $\mathbb{R}^3$, consider $\{\,(1,0,0),\ (0,1,0)\,\}$. This family is linearly independent but not a basis (only $p = 2 < n = 3$).
Adding $(0,0,1)$ yields

$$\{(1,0,0),\ (0,1,0),\ (0,0,1)\},$$

which forms the canonical basis of $\mathbb{R}^3$.

Theorem (Extracted Basis Theorem)

Let V be a finite-dimensional vector space with $\dim(V) = n$. Suppose

$$(v_1, \ldots, v_p), \quad p \geq n,$$

is a generating family of V.
Then there exists a subfamily

$$(v_{i_1}, \ldots, v_{i_n})$$

that forms a basis of V.
*In other words, **any generating family contains a basis**.*

Example

In $\mathbb{R}^3$, consider the generating family

$$\{(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,1)\}.$$

This set spans $\mathbb{R}^3$. By removing the redundant vector $(1,1,1)$, we obtain the canonical basis

$$\{(1,0,0),\ (0,1,0),\ (0,0,1)\},$$

Conceptual Motivation

A **linear transformation** is a mapping between vector spaces that preserves their structure. It takes a vector as input and produces another vector, in such a way that:

- vector addition is preserved,
- scalar multiplication is preserved.

These maps are fundamental because they capture the essence of “structure-preserving” operations in linear algebra.

Conceptual Motivation

A **linear transformation** is a mapping between vector spaces that preserves their structure. It takes a vector as input and produces another vector, in such a way that:

- vector addition is preserved,
- scalar multiplication is preserved.

These maps are fundamental because they capture the essence of "structure-preserving" operations in linear algebra.

Definition

Let V and W be vector spaces. A map $T : V \to W$ is a *linear transformation* if

$$T(c\mathbf{v}_1 + \mathbf{v}_2) = c\,T(\mathbf{v}_1) + T(\mathbf{v}_2), \quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V,\ c \in \mathbb{K}.$$

Conceptual Motivation

A **linear transformation** is a mapping between vector spaces that preserves their structure. It takes a vector as input and produces another vector, in such a way that:

- vector addition is preserved,
- scalar multiplication is preserved.

These maps are fundamental because they capture the essence of "structure-preserving" operations in linear algebra.

Definition

Let V and W be vector spaces. A map $T : V \to W$ is a *linear transformation* if

$$T(c\mathbf{v}_1 + \mathbf{v}_2) = c\,T(\mathbf{v}_1) + T(\mathbf{v}_2), \quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V,\ c \in \mathbb{K}.$$

Exercise

Check whether the map T is a linear transformation, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

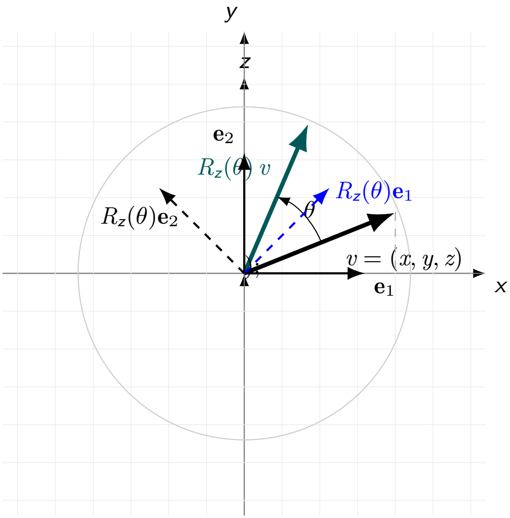
$$T(x, y) = (2x + y,\ -x + 3y)$$

Canonical Examples

Let $V = \mathbb{R}^3$.

- **Identity** $Id : \mathbb{R}^3 \to \mathbb{R}^3$, $Id(\mathbf{x}) = \mathbf{x}$. Matrix: I_3.
- **Homothety (Scaling)** $H_\alpha(\mathbf{x}) = \alpha\,\mathbf{x}$ for $\alpha \in \mathbb{R}$. Matrix: αI_3.
- **Rotation about the *z*-axis by angle θ**

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_z(\theta)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}.$$



$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

Definitions

Let V, W be vectors spaces.

- $\mathcal{L}(V, W) :=$ set of linear maps $T : V \to W$.
- **Endomorphism**: $T \in \mathcal{L}(V, V)$.
- **Isomorphism**: bijective linear map $T \in \mathcal{L}(V, W)$.
- **Automorphism**: bijective endomorphism $T \in \mathcal{L}(V, V)$.

Algebraic Structure

On $\mathcal{L}(V) := \mathcal{L}(V, V)$,

- With pointwise addition $+$ and composition $\circ$, $(\mathcal{L}(V), +, \circ)$ is a (not-necessarily commutative) **ring** with identity Id.
- Distributivity: $T \circ (S_1 + S_2) = T \circ S_1 + T \circ S_2$ and $(S_1 + S_2) \circ T = S_1 \circ T + S_2 \circ T$.

Definitions

For $T \in \mathcal{L}(V, W)$:

$$\ker(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}, \qquad \mathrm{Im}(T) := \{T(\mathbf{v}) \mid \mathbf{v} \in V\} \subseteq W.$$

Key Properties

- T is injective $\iff \ker(T) = \{\mathbf{0}_V\}$.
- If $(\mathbf{v}_i)_{i \in I}$ generates V, then $\mathrm{Im}(T) = \mathrm{Span}\big(T(\mathbf{v}_i) \;:\; i \in I\big)$.

Rank–nullity theorem

Let $T \in \mathcal{L}(V, W)$ with $\dim(V) < \infty$. Then

$$\underbrace{\dim \ker(T)}_{\text{nullity}} + \underbrace{\dim \operatorname{Im}(T)}_{\text{rank}} = \dim(V).$$

Consequences

- T injective $\iff \dim \ker(T) = 0 \iff \operatorname{rank}(T) = \dim(V)$.
- If $\dim(W) < \infty$, then T surjective $\iff \operatorname{rank}(T) = \dim(W)$.
- If $\dim(V) = \dim(W)$, then: injective $\iff$ surjective $\iff$ T is an isomorphism.

Setup

Let $T \in \mathcal{L}(\mathbb{R}^n) \equiv \mathbb{R}^{n\times n}$ be endomorphism with $\operatorname{rank}(T) = 1$. Then there exist $u, v \in \mathbb{R}^n$ such that

$$T = v\,u^\top \quad \textit{(i.e., } T(x) = (u^\top x)\,v \text{ for all } x \in \mathbb{R}^n).$$

Image and Kernel

Since $u^\top x$ is a scalar,

$$\operatorname{Im}(T) = \operatorname{span}\{v\}, \qquad \ker(T) = \{x \in \mathbb{R}^n :\; u^\top x = 0\} = u^\perp.$$

In particular, $\dim \operatorname{Im}(T) = 1$ and $\dim \ker(T) = n - 1$ (rank–nullity).

Quick Properties

- $T^2 = (u^\top v)\,T$ (so T is diagonalizable with eigenvalues $\{0,\; u^\top v\}$).
- $\operatorname{tr}(T) = u^\top v$ and $\det(T) = 0$.
- If $u^\top v = 1$ and v is a unit vector, then T is the orthogonal projection onto $\operatorname{span}\{v\}$ along $u^\perp$.

Note: *The subspace denoted by the orthogonal complement symbol* $^\perp$ *will be formally introduced in the final chapter of this course.*

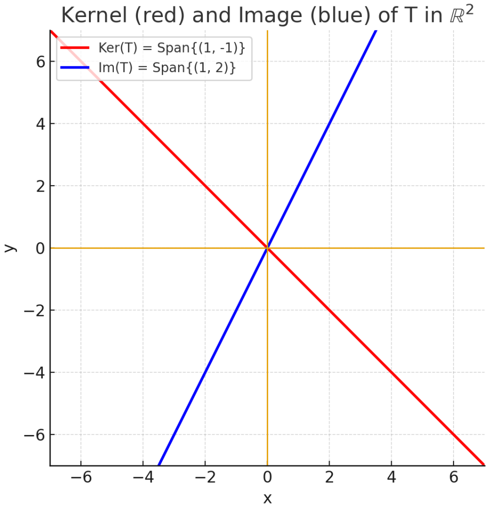
In $\mathbb{R}^2$

Consider the following linear map:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x + y,\ 2x + 2y)$$

1. What type of morphism is T (*e.g.*,, injective, surjective, isomorphism) ? Prove the linear aspect.
2. Determine $\ker(T)$ and $\mathrm{Im}(T)$. Is the rank-null theorem verified ?
3. Plot $\ker(T)$ in red and $\mathrm{Im}(T)$ in blue in the Cartesian plane, and provide their equations. Interpret your result.

Geometric interpretation



- Every vector $v \in \mathbb{R}^2$ is sent *onto the blue line*: $T(v) \in \mathrm{im}(T) = \mathrm{Span}\{(1,2)\}\dots$
- ... except vectors on the red line that are mapped to the origin: $T(v) = 0 \iff v \in \ker(T) = \mathrm{Span}\{(1,-1)\}$.

Hence T is a rank-1 linear map that *collapses the plane* onto $\mathrm{Span}\{(1,2)\}$ along $\mathrm{Span}\{(1,-1)\}$.

Definitions

Let U, V be subspaces of a vector space E.

$$U + V = \{u + v : \; u \in U, \; v \in V\}.$$

We say that E **is the direct sum of** U **and** V, written $E = U \oplus V$, if

$$E = U + V \quad \text{and} \quad U \cap V = \{0\}.$$

In this case, every decomposition $x = u + v$ is *unique*.

Criteria and Dimensions

For finite-dimensional spaces:

$$E = U \oplus V \iff E = U + V \text{ and } U \cap V = \{0\}. \qquad \dim(U \oplus V) = \dim U + \dim V.$$

Rank-nullity theorem

Let $T : E \to E$ be linear with $\dim E < \infty$.

$$\dim \ker T + \underbrace{\dim \operatorname{im} T}_{\mathrm{rg}(T)} = \dim E.$$

If moreover $\ker T \cap \operatorname{im} T = \{0\}$, then

$$E = \ker T \oplus \operatorname{im} T.$$

Useful cases:

- If T is a **projection** ($T^2 = T$), then $E = \ker T \oplus \operatorname{im} T$.
- If T is **symmetric** (real matrix A with $A^\top = A$), then $(\operatorname{im} T)^\perp = \ker T$ and thus $E = \ker T \overset{\perp}{\oplus} \operatorname{im} T$.

Check the previous example to see an application of this theorem with $\mathbb{R}^2 = \ker T \oplus \operatorname{im} T$!

Definition

Let $T : V \to W$ be a linear transformation, where V and W are vector spaces with bases $\mathcal{B}_V = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\mathcal{B}_W = \{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$, respectively.
The **matrix of T with respect to these bases** is the $m \times n$ matrix

$$M_{ij} \text{ such that } T(\mathbf{e}_j) = \sum_{i=1}^{m} M_{ij}\,\mathbf{f}_i.$$

Equivalently, the j-th column of the matrix is the coordinate vector of $T(\mathbf{e}_j)$ in the basis $\mathcal{B}_W$.

Examples

What are the matrix of the following linear maps?

- $\mathbb{R}^2 \to \mathbb{R}^2$: $T(x, y) = (2x + y,\ -x + 3y)$
- $\mathbb{R}^3 \to \mathbb{R}^3$: $T(x, y, z) = (x + z,\ y,\ 2z)$
- $\mathbb{R}^2 \to \mathbb{R}^3$: $T(x, y) = (x,\ y,\ x + y)$

Key Idea in 3D

A linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ associates each vector with another vector.

- If T is **bijective** (invertible), its image is all of $\mathbb{R}^3$. Any direction in space can be reached.
- If T is **not bijective**, the image has lower dimension: a plane (dim 2), a line (dim 1), or just $\{0\}$ (dim 0).

Key Idea in 3D

A linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ associates each vector with another vector.

- If T is **bijective** (invertible), its image is all of $\mathbb{R}^3$. Any direction in space can be reached.
- If T is **not bijective**, the image has lower dimension: a plane (dim 2), a line (dim 1), or just $\{0\}$ (dim 0).

Example from Physics

1. **Projection of a 3D object onto a screen**
   - $\mathbb{R}^3 \to$ a plane in $\mathbb{R}^3$.
   - Linear but not bijective: depth information is lost.
2. **Meteorology: wind coordinate change**
   - Convert wind (u, v, w) into rotated/polar coordinates.
   - Invertible linear map (rotation, change of basis).
3. **Mechanics/Thermodynamics: unit conversion**
   - Joules $\leftrightarrow$ kilocalories, or forces $\leftrightarrow$ stresses.
   - Invertible linear map (diagonal scaling matrix).

Motivation

A linear transformation is an abstract object, but once a basis is chosen it can be represented by a matrix. Different choices of basis yield different matrix representations. This motivates the study of the **change of basis**, a **bijective** correspondence between the coordinates of the same vector expressed in different bases.

Setup

Let $\mathcal{B}_1 = (b_1, b_2, b_3)$ and $\mathcal{B}_2 = (c_1, c_2, c_3)$ be bases of $\mathbb{R}^3$. We want the change-of-basis matrix P_{21} that converts coordinates from $\mathcal{B}_1$ to $\mathcal{B}_2$:

$$[\mathbf{x}]_{\mathcal{B}_2} = P_{21}\, [\mathbf{x}]_{\mathcal{B}_1}.$$

Linear system

Express each old basis vector $b_j \in \mathbb{R}^3$ as a linear combination of the new basis vectors c_i:

$$\begin{cases} b_1 = \alpha_{11} c_1 + \alpha_{21} c_2 + \alpha_{31} c_3 & \leftarrow \textit{This is a system of 3 equations!} \\ b_2 = \alpha_{12} c_1 + \alpha_{22} c_2 + \alpha_{32} c_3 & \leftarrow \textit{same!} \\ b_3 = \alpha_{13} c_1 + \alpha_{23} c_2 + \alpha_{33} c_3 & \leftarrow \textit{same!} \end{cases}$$

Matrix form (the matrix to invert)

Let $C = [c_1 \ c_2 \ c_3]$, $\quad B = [b_1 \ b_2 \ b_3]$, $\quad A = (\alpha_{ij})_{1 \leq i, j \leq 3}$. Then the three systems above compactly read

$$CA = B \quad \Longrightarrow \quad A = C^{-1} B \quad \text{(since } C \text{ is invertible).}$$

The change-of-basis matrix is precisely

$$P_{21} :\equiv A = C^{-1} B.$$

Augmented-matrix viewpoint

Gaussian elimination on the augmented matrix *(where the first block contains the column vectors of $\mathcal{B}_2$ and the second block those of $\mathcal{B}_1$)* simultaneously solves the three systems:

$$\left[\, \mathcal{B}_2 \mid \mathcal{B}_1 \,\right] = \left[\, C \mid B \,\right] \;\sim\; \left[\, I_d \mid C^{-1}B \,\right] = \left[\, I_d \mid P_{21} \,\right].$$

Example (to be solved)

Let $B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B_2 = \{(1,1,1),(1,-1,1),(2,1,1)\}$. Find the coordinates of $\mathbf{b} = (2,1,-1)$ in basis B_2.

Method 1: Direct System

Seek scalars α, β, γ such that

$$\alpha(1,1,1) + \beta(1,-1,1) + \gamma(2,1,1) = (2,1,-1).$$

This gives the system:

$$\begin{cases} \alpha + \beta + 2\gamma = 2, \\ \alpha - \beta + \gamma = 1, \\ \alpha + \beta + \gamma = -1. \end{cases}$$

Solution: $\alpha = -3, \ \beta = -1, \ \gamma = 3.$

$$[\mathbf{b}]_{B_2} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}.$$

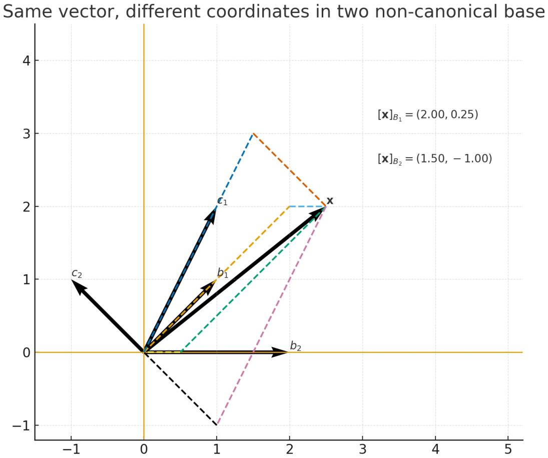
Method 2: Gauss–Jordan

Form $C = [c_1 \ c_2 \ c_3]$ and compute C^{-1}:

$$C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \end{bmatrix}.$$

Thus $P_{12} = C^{-1}$ and

$$[\mathbf{b}]_{B_2} = P_{12}[\mathbf{b}]_{B_1} = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}.$$



Exercise. Knowing

$$\mathcal{B}_1 = \{b_1,\ b_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

$$\mathcal{B}_2 = \{c_1,\ c_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$[\mathbf{x}]_{\mathcal{B}_1} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix}$$

express the change-of-basis matrix P_{21} and find the coordinates of $\mathbf{x}$ in the basis $\mathcal{B}_2$.

$$[\mathbf{x}]_{\mathcal{B}_2} = ?$$

Smallville Recap

We want to analyze the long-term behavior of the system:

- What happens after n years?
- Does the system converge to an equilibrium state?

Key idea: a matrix represents a linear transformation in some basis. Choosing a "better" basis can simplify the representation. Goal: compute

$$\lim_{t\to\infty} A^t.$$

Definition: Eigenvalues and Eigenvectors

Let $T : V \to V$, with $V = \mathbb{R}^n$. If there exists a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that

$$T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i, \quad \lambda_i \in \mathbb{R},$$

then the matrix of T in basis B is diagonal:

$$M_{BB} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Next Step

Given a matrix A representing T, we must find vectors $\mathbf{v}$ and scalars λ such that

$$A\mathbf{v} = \lambda \mathbf{v} \iff (A - \lambda I_n) \cdot \mathbf{v} = 0.$$

Key Idea

For a matrix A, an eigenvalue λ must satisfy

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \quad \text{for some nonzero vector } \mathbf{v}.$$

This means $\ker(A - \lambda I) \neq \{\mathbf{0}\}$. Equivalently, since a square matrix has a nontrivial kernel iff its determinant vanishes:

$$\underbrace{\det(A - \lambda I)}_{\text{Characteristic polynomial } \chi_A(\lambda)} = 0.$$

- The roots of $\chi_A(\lambda)$ are the **eigenvalues** of A.
- The corresponding nonzero solutions $\mathbf{v}$ are the **eigenvectors**.
- The set of eigenvalues of A is called the **spectrum** of A, denoted by $\mathrm{Sp}(A)$.

Eigenspace

Given an eigenvalue λ_i, the **eigenspace** associated with λ_i is

$$E_{\lambda_i} \;=\; \ker(A - \lambda_i I) = \{\, \mathbf{v} \in \mathbb{R}^n \;\mid\; A\mathbf{v} = \lambda_i \mathbf{v} \,\}.$$

It is a subspace of $\mathbb{R}^n$, spanned by all eigenvectors corresponding to λ_i.

Remark

For each eigenvalue λ_i of a matrix A, there are two notions of multiplicity:

- The **algebraic multiplicity** of λ_i is its multiplicity as a root of the characteristic polynomial $\chi_A(\lambda)$.
- The **geometric multiplicity** of λ_i is the dimension of its eigenspace $\dim E_{\lambda_i}$.

These satisfy

$$1 \;\leq\; \dim E_{\lambda_i} \;\leq\; \text{algebraic multiplicity of } \lambda_i.$$

*A matrix is **diagonalizable** precisely when, for **each eigenvalue**, the geometric and algebraic **multiplicities coincide**.*

Example 1: Eigenvalues

Let

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}.$$

Compute $\det(A - \lambda I)$ and find the eigenvalues of A.

Example 2: Eigenvectors

For each eigenvalue λ of A, solve

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Find a basis for the eigenspace corresponding to λ.

Example 1: Eigenvalues

Let

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}.$$

Compute $\det(A - \lambda I)$ and find the eigenvalues of A.

Example 2: Eigenvectors

For each eigenvalue λ of A, solve

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Find a basis for the eigenspace corresponding to λ.

Observation

In Example [3], the eigenvalues appeared on the diagonal of the diagonalized matrix D. Next, we will see how the eigenvectors determine this change of basis.

So, are they ?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 7 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 7 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$$

So, are they ?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 7 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 7 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$$

A is; B is not; C is, even if it is not invertible.

Setup

Let $T: V \to V$ be a linear endomorphism. Suppose B_1 and B_2 are two bases of V. We denote by $M_{B_1 B_1}$ and $M_{B_2 B_2}$ the matrices of T expressed in these bases.

Setup

Let $T : V \to V$ be a linear endomorphism. Suppose B_1 and B_2 are two bases of V. We denote by $M_{B_1 B_1}$ and $M_{B_2 B_2}$ the matrices of T expressed in these bases.

Change of Basis Relation

Let P_{12} be the change-of-basis matrix from B_2 to B_1:

$$\mathbf{v}_{B_1} = P_{12}\,\mathbf{v}_{B_2}.$$

Then the matrices of T in the two bases are related by

$$M_{B_2 B_2} = P_{21}\, M_{B_1 B_1}\, P_{12}, \quad \text{where} \;\; P_{21} = P_{12}^{-1}.$$

Key Point

Two matrices $M_{B_1 B_1}$ and $M_{B_2 B_2}$ representing the same linear transformation in different bases are said to be **similar**. Similarity preserves eigenvalues and many structural properties, but the form of the matrix depends on the chosen basis.

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is **diagonalizable** if there exists a basis B_2 of eigenvectors of A.
In that basis, the matrix of A is diagonal:

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Definition

A matrix $A \in \mathbb{R}^{n\times n}$ is **diagonalizable** if there exists a basis B_2 of eigenvectors of A.
In that basis, the matrix of A is diagonal:

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

Change of Basis Relation

Let $P := M_{B_1 B_2}$ **invertible** be the change-of-basis matrix from the eigenbasis B_2 to the reference basis B_1.
Then

$$A = P\,D\,P^{-1}.$$

Definition

A matrix $A \in \mathbb{R}^{n\times n}$ is **diagonalizable** if there exists a basis B_2 of eigenvectors of A.
In that basis, the matrix of A is diagonal:

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Change of Basis Relation

Let $P := M_{B_1 B_2}$ **invertible** be the change-of-basis matrix from the eigenbasis B_2 to the reference basis B_1.
Then

$$A = P\,D\,P^{-1}.$$

Theorem

For every integer $p \geq 0$*,*

$$A^p = P\,D^p\,P^{-1}.$$

Key Point

Diagonalization transforms a difficult computation (A^p) into a simple one (because simply $D^p = \mathrm{diag}(\lambda_1^p, \ldots, \lambda_n^p)$, just raising eigenvalues to powers).

Characterizations

Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:

- A is diagonalizable $\iff$ there exists a basis of $\mathbb{R}^n$ formed by eigenvectors of A.
- A is similar to a diagonal matrix:
$$\exists\, P \text{ invertible}, \quad P^{-1}AP = D.$$
- The sum of the dimensions of all eigenspaces equals n.

Characterizations

Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

- A is diagonalizable $\iff$ there exists a basis of $\mathbb{R}^n$ formed by eigenvectors of A.
- A is similar to a diagonal matrix:
$$\exists\, P \text{ invertible}, \quad P^{-1}AP = D.$$
- The sum of the dimensions of all eigenspaces equals n.

Practical Conditions

- If A has n distinct eigenvalues $\Rightarrow$ A is diagonalizable.
- If A is symmetric (real entries), then A is diagonalizable (Spectral Theorem).
- Otherwise: compare **algebraic multiplicity** (from characteristic polynomial) and **geometric multiplicity** (dimension of eigenspace). Diagonalizability requires equality for every eigenvalue.

Trace

The **trace** of a square matrix $A \in \mathcal{M}_n(\mathbb{R})$ is

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

If A is diagonalizable, $A = PDP^{-1}$ with $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$\operatorname{tr}(A) = \operatorname{tr}(D) = \lambda_1 + \cdots + \lambda_n.$$

Determinant

The **determinant** of a square matrix $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ is a scalar denoted $\det(A)$.

Formal recursive definition:

- For $n = 1$, $\det([a_{11}]) = a_{11}$.
- For $n \geq 2$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}),$$

where M_{1j} is the $(n-1) \times (n-1)$ submatrix obtained by deleting row 1 and column j.

Geometric meaning: $\det(A)$ represents the scaling factor of volumes induced by the linear transformation associated with A, with its sign indicating whether orientation is preserved or reversed.
If A is diagonalizable as above, then

$$\det(A) = \det(D) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

Spectrum and Transpose

Since $\det(A - \lambda I) = \det((A - \lambda I)^\top) = \det(A^\top - \lambda I)$, we have

$$\mathrm{Sp}(A) = \mathrm{Sp}(A^\top).$$

Definition

For a square matrix $A = (a_{ij}) \in M_n(\mathbb{R})$, the trace is defined as:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Intuition and Properties

- **Invariant under change of basis**: $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A)$.
- **Spectral link**: $\mathrm{tr}(A) = \sum_i \lambda_i$ (sum of eigenvalues with multiplicity).
- **Algebraic tool**: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- **Associated norm**: the *Frobenius norm* is defined by $\|A\|_F^2 = \mathrm{tr}(A^\top A)$.
- **Geometric interpretation**: if A is the Jacobian matrix of a linear vector field, then $\mathrm{tr}(A)$ equals its **divergence**.

Historical Note. The term *trace* comes from the German word *Spur* ("footprint, track"), introduced in the 19^{th} century by von Staudt and later used by Frobenius. The idea: the trace is the footprint left by a linear transformation, independent of the chosen basis.

Let's once again consider

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x + y,\ 2x + 2y)$$

We showed in [37] that T is an endomorphism and not invertible.

1. What is the matrix representation of T in the canonical basis of $\mathbb{R}^2$? Let's denote it A.
2. Is A diagonalizable? If so, what are its eigenvalues?
3. Determine a diagonal matrix D and an invertible matrix P (and P^{-1}) such that

$$A = P\,D\,P^{-1}.$$

Diagonalization Method

Let

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}, \quad \begin{bmatrix} n_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 2000 \\ 8000 \end{bmatrix}.$$

Show that

$$\lim_{t \to \infty} A^t = \begin{bmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix}.$$

Hence, what is the equilibrium ?

Solution ?

Diagonalization Method

Let

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}, \quad \begin{bmatrix} n_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 2000 \\ 8000 \end{bmatrix}.$$

Show that

$$\lim_{t\to\infty} A^t = \begin{bmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix}.$$

Hence, what is the equilibrium ?

Solution ?

Observation

Since $\begin{bmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix} \begin{bmatrix} 2000 \\ 8000 \end{bmatrix} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$, even though the initial population had far more smokers, the equilibrium state shifts toward more nonsmokers than at the start.

Please note: ***you must master this type of exercise for the final exam.***

An example you have to work

Let us consider the linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ represented in the canonical basis by the matrix

$$M = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 3 \\ 3 & 3 & 1 \end{pmatrix}.$$

1. Compute the **eigenvalues** of M.
   - $\to$ *Hint: Try to avoid solving the full cubic polynomial. If necessary, look for an obvious root first to help factorize the degree-3 polynomial.*
   - $\to$ *Can you spot an obvious eigenvalue or eigenvector? What happens if you add up all the entries in each row (or in each column)?*
   - $\to$ *What equations can be derived from the trace and the determinant?*
2. For each eigenvalue, determine a basis of the associated **eigenspace**.
3. Deduce whether M is **diagonalizable**, and if so, give an explicit diagonalization.

Eigenspaces: Direct Sums and Diagonalizability

Let $A \in \mathbb{R}^{n\times n}$ an endomorphism.

- If $\lambda_1, \ldots, \lambda_k$ are **distinct** eigenvalues of A, then their eigenspaces are **linearly independent** and

$$E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} = \mathbb{R}^n$$

  Equivalently, $E_{\lambda_i} \cap \left(\sum_{j\neq i} E_{\lambda_j}\right) = \{0\}$ for each i.
- For each eigenvalue λ, $1 \leq \dim E_\lambda \leq$ (algebraic multiplicity of λ).

Why this matters?

→ Direct sums of eigenspaces give **independent invariant directions** where A acts as simple scalings.

→ They provide a basis built from eigenvectors, enabling **decoupling** of linear systems, simple formulas for A^t, e^{tA}, and clear spectral interpretations (trace, determinant).

Implications for diagonalizability

A is **diagonalizable** $\iff$ the eigenspaces span $\mathbb{R}^n$:

$$\mathbb{R}^n = \bigoplus_{\lambda\in\mathrm{Spec}(A)} E_\lambda \iff \sum_\lambda \dim E_\lambda = \dim \mathbb{R}^n,$$

equivalently, for every eigenvalue λ, the **geometric multiplicity** equals the **algebraic multiplicity**.

Definition

A square matrix $Q \in \mathbb{R}^{n\times n}$ is called **orthogonal** if

$$Q^\top Q = QQ^\top = I_n.$$

Equivalently, $Q^{-1} = Q^\top$.

Intuition

An orthogonal matrix represents a linear transformation that:

- preserves inner products and lengths,
- preserves orthogonality,
- is a composition of rotations and reflections.

Remark: Isometry Property

Orthogonal matrices preserve lengths:

$$\|Q\mathbf{x}\|^2 = (Q\mathbf{x})^\top (Q\mathbf{x}) = \mathbf{x}^\top Q^\top Q\mathbf{x} = \mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|^2.$$

Thus, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$.

Why "orthogonal"?

If $Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \dots \ \mathbf{q}_n]$, then

$$\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases},$$

so the column vectors form an **orthonormal basis** of $\mathbb{R}^n$.

Example

The rotation matrix in $\mathbb{R}^2$,

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

is orthogonal: $R(\theta)^\top R(\theta) = I_2$.

Remark

The set of all $n \times n$ orthogonal matrices forms a group under multiplication, called the **orthogonal group** $O(n)$. The subgroup of matrices with determinant 1 is the **special orthogonal group** $SO(n)$, corresponding to pure rotations.

Theorem (Spectral Theorem)

Every real symmetric matrix is diagonalizable. *That is, if $A \in \mathbb{R}^{n\times n}$ and $A^\top = A$, then there exists an orthogonal matrix Q such that*

$$Q^\top A Q = D,$$

where D is diagonal.

Theorem (Spectral Theorem)

Every real symmetric matrix is diagonalizable. *That is, if $A \in \mathbb{R}^{n\times n}$ and $A^\top = A$, then there exists an orthogonal matrix Q such that*

$$Q^\top A Q = D,$$

where D is diagonal.

Example

Consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

- Show that A is symmetric.
- Compute its eigenvalues and eigenvectors.
- Verify that A is diagonalizable via an orthogonal change of basis.

Theorem (Spectral Theorem)

Every real symmetric matrix is diagonalizable. *That is, if $A \in \mathbb{R}^{n\times n}$ and $A^\top = A$, then there exists an orthogonal matrix Q such that*

$$Q^\top A Q = D,$$

where D is diagonal.

Example

Consider

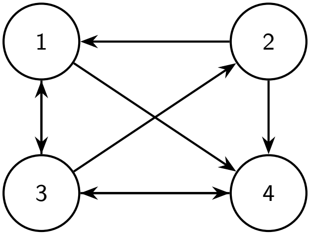
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

- Show that A is symmetric.
- Compute its eigenvalues and eigenvectors.
- Verify that A is diagonalizable via an orthogonal change of basis.

Remark: *Check your result for matrix M in Example [65], observing that $M^\top = M$!*

Motivation

Model the web as a weighted, directed graph: vertices = websites, edges = links. If site j has ℓ_j outgoing links, each outgoing edge carries weight $1/\ell_j$. This yields a column-stochastic transition matrix T; to allow random jumps, add a uniform "teleportation" matrix R.



From Graph to Matrices

With ℓ_j the out-degree of vertex j,

$$T_{ij} = \begin{cases} \frac{1}{\ell_j} & \text{if } j \to i \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, we construct this matrix:

$$T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}.$$

Teleportation and the Google Matrix

The matrix $\mathbf{1}$ denotes the column vector of size n with all entries equal to 1. Hence $\mathbf{1}\mathbf{1}^\top$ is the $n \times n$ matrix of all 1's.

$$R = \frac{1}{n}\,\mathbf{1}\mathbf{1}^\top$$

is therefore the matrix where every entry is $\frac{1}{n}$. This models the fact that with probability p, a user may *randomly jump* to any website, independently of links.

The Google matrix combines both behaviors:

$$G = (1-p)\,T + p\,R \;\in\; \mathbb{R}^{n\times n}, \quad \text{with typical choice } p \approx 0.15.$$

This construction ensures that G is stochastic, irreducible, and aperiodic, so it admits a unique stationary distribution. This stationary distribution reflects the long-term importance (rank) of each page, which is the core of Google's PageRank algorithm.

Ranking Vector

We start with the uniform distribution:

$$\mathbf{v}(0) = \left[\frac{1}{n}, \quad \frac{1}{n}, \quad \ldots, \quad \frac{1}{n}\right]^\top,$$

which represents an equal probability of being at any page initially.
Iterating the process,

$$\mathbf{v}(t+1) = G\,\mathbf{v}(t),$$

converges to the unique fixed point

$$\mathbf{v}_\infty = \left(\lim_{t\to\infty} G^t\right) \cdot \mathbf{v}(0).$$

The vector $\mathbf{v}_\infty$ is the **PageRank vector**: its i-th entry gives the long-term probability of a user visiting page i.
Pages with larger entries are ranked higher in search results.

Theorem (Perron-Frobenius)

If M is a column-stochastic matrix with all entries positive, then:

- 1 *is an eigenvalue of* M*,*
- *the associated eigenvector* $\mathbf{v}_\infty$ *has strictly positive entries,*
- $\mathbf{v}_\infty$ *can be normalized so that its entries sum to* 1*,*
- *the iteration* $M^t\mathbf{v}(0)$ *converges to* $\mathbf{v}_\infty$*.*

Theorem (Perron-Frobenius)

If M is a column-stochastic matrix with all entries positive, then:

- 1 *is an eigenvalue of* M,
- *the associated eigenvector* $\mathbf{v}_\infty$ *has strictly positive entries,*
- $\mathbf{v}_\infty$ *can be normalized so that its entries sum to* 1,
- *the iteration* $M^t\mathbf{v}(0)$ *converges to* $\mathbf{v}_\infty$.

Application to PageRank

The PageRank vector is defined by solving

$$G\mathbf{v} = \mathbf{v},$$

that is, finding the eigenvector of G associated with eigenvalue 1.

Challenge: for the web, n is in the billions. Direct eigenvector computation is infeasible.

Practical solution: approximate $\mathbf{v}_\infty$ by iterating

$$\mathbf{v}(m) = G^m\mathbf{v}(0),$$

for moderate m, until convergence is reached.

Setup

We have k observations of m variables:

$$X = \{p_1, \dots, p_k\}, \quad p_i = (p_{i1}, \dots, p_{im}) \in \mathbb{R}^m.$$

For each coordinate j, let $\mu_j(X)$ be the mean. Define the centered data matrix:

$$N_{ij} = p_{ij} - \mu_j(X).$$

Setup

We have k observations of m variables:

$$X = \{p_1, \dots, p_k\}, \quad p_i = (p_{i1}, \dots, p_{im}) \in \mathbb{R}^m.$$

For each coordinate j, let $\mu_j(X)$ be the mean. Define the centered data matrix:

$$N_{ij} = p_{ij} - \mu_j(X).$$

Questions of Interest

- How is the data spread across directions in $\mathbb{R}^m$?
- Is the variance larger in some directions than others?
- Do subsets of the data cluster in certain patterns?

Setup

We have k observations of m variables:

$$X = \{p_1, \dots, p_k\}, \quad p_i = (p_{i1}, \dots, p_{im}) \in \mathbb{R}^m.$$

For each coordinate j, let $\mu_j(X)$ be the mean. Define the centered data matrix:

$$N_{ij} = p_{ij} - \mu_j(X).$$

Questions of Interest

- How is the data spread across directions in $\mathbb{R}^m$?
- Is the variance larger in some directions than others?
- Do subsets of the data cluster in certain patterns?

Definition

The **covariance matrix** of X is

$$\mathrm{cov}(X) = N^\top N.$$

Example : Centering, Covariance, and Eigenanalysis

Consider the dataset

$$X = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}.$$

Tasks:

1. Compute the coordinate-wise mean $\boldsymbol{\mu} = (\mu_1, \mu_2)$.
2. Form the centered data matrix N with entries

$$N_{ij} = p_{ij} - \mu_j.$$

3. Compute the covariance matrix

$$\mathrm{cov}(X) = N^\top N \quad \text{(optionally normalized by } \tfrac{1}{k} \text{ or } \tfrac{1}{k-1}\text{)}.$$

4. Find the eigenvalues and associated eigenvectors of $\mathrm{cov}(X)$.

Theorem (Variance along Eigenvectors)

Order the eigenvalues of $\mathrm{cov}(X)$ *as*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m.$$

Then the data variance along each direction is proportional to the corresponding eigenvalue, in the direction of the associated eigenvector.

Theorem (Variance along Eigenvectors)

Order the eigenvalues of $\mathrm{cov}(X)$ *as*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m.$$

Then the data variance along each direction is proportional to the corresponding eigenvalue, in the direction of the associated eigenvector.

Interpretation

- The largest eigenvalue $\lambda_{\max}$ indicates the direction of greatest data spread.
- Smaller eigenvalues correspond to directions with less variation.
- For 2D data: eigenvectors give the principal axes of the ellipse approximating the data cloud, and eigenvalues determine their lengths.

Theorem (Variance along Eigenvectors)

Order the eigenvalues of $\mathrm{cov}(X)$ *as*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m.$$

Then the data variance along each direction is proportional to the corresponding eigenvalue, in the direction of the associated eigenvector.

Interpretation

- The largest eigenvalue $\lambda_{\max}$ indicates the direction of greatest data spread.
- Smaller eigenvalues correspond to directions with less variation.
- For 2D data: eigenvectors give the principal axes of the ellipse approximating the data cloud, and eigenvalues determine their lengths.

Key Point: PCA

Theorem [13] is the foundation of **Principal Component Analysis (PCA)**, a fundamental tool in applied mathematics, statistics, and machine learning.

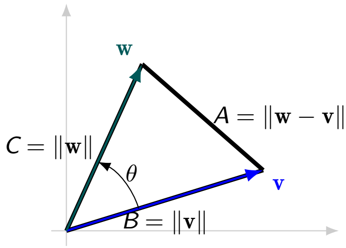
Why orthogonality matters

In data science we approximate: we **minimize distance between model and data**. Squared distances are quadratic, so minimization leads to linear systems. Vector calculus links minimization with **orthogonal projections** onto subspaces.

Law of Cosines (Al-Kashi)

For a triangle with side lengths A, B, C and opposite angles a, b, c,

$$A^2 = B^2 + C^2 - 2BC\cos(c).$$



$$A^2 = B^2 + C^2 - 2BC\cos(\theta)$$

Exercise (warm-up)

Let $\mathbf{v}, \mathbf{w}$ start at the origin and c be the angle between them. Apply the law of cosines to the triangle with sides $A = \|\mathbf{w} - \mathbf{v}\|$, $B = \|\mathbf{v}\|$, $C = \|\mathbf{w}\|$ to show

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos(c).$$

Motivation

Fitting data requires restricting model complexity: *a good fit minimizes the error between data and model, without overfitting.*

Key Point

Least squares = **projection onto a subspace**. Understanding this requires the geometry of orthogonality.

Example: Line of Best Fit in $\mathbb{R}^2$

Data set: $X = \{(1,6),(2,5),(3,7),(4,10)\}$.
We want the line $y = ax + b$ that minimizes the total squared error:

$$\text{error} = \sqrt{\sum_{(x_i,y_i)\in X} (y_i - (ax_i + b))^2}.$$

Minimizing the error is the same as minimizing the content of the square root.
Equivalently, solve the least squares system:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}.$$

Interpretation: we seek the projection of $\begin{bmatrix} 6 & 5 & 7 & 10 \end{bmatrix}^\top$ onto the column space of the matrix.

Ordinary Least Squares in Simple Linear Regression

We consider the model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \qquad i = 1, \ldots, N,$$

where ε_i are random errors with zero mean. The aim is to estimate (β_0, β_1) by minimizing the total squared error.

Derivation of the Optimal Coefficients

We minimize the quadratic error

$$f(\beta_0, \beta_1) = \sum_{i=1}^{N} \left(Y_i - (\beta_0 + \beta_1 X_i)\right)^2.$$

First-order conditions:

$$\begin{cases} \dfrac{\partial f}{\partial \beta_0} = -2 \sum_{i=1}^{N} \left(Y_i - \beta_0 - \beta_1 X_i\right) = 0, \\ \dfrac{\partial f}{\partial \beta_1} = -2 \sum_{i=1}^{N} X_i \left(Y_i - \beta_0 - \beta_1 X_i\right) = 0. \end{cases}$$

Dividing by N and introducing

$$\bar{X} = \tfrac{1}{N} \sum_{i=1}^{N} X_i, \quad \bar{Y} = \tfrac{1}{N} \sum_{i=1}^{N} Y_i,$$

we obtain the system

$$\begin{cases} \bar{Y} = \beta_0 + \beta_1 \bar{X}, \\ \dfrac{1}{N} \sum_{i=1}^{N} X_i Y_i = \beta_0 \bar{X} + \beta_1 \tfrac{1}{N} \sum_{i=1}^{N} X_i^2. \end{cases}$$

Covariance and Variance Forms

Define

$$\mathrm{Cov}(X,Y)=\frac{1}{N}\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y}),\qquad \mathrm{Var}(X)=\frac{1}{N}\sum_{i=1}^{N}(X_i-\bar{X})^2.$$

Useful expansions:

$$\frac{1}{N}\sum(X_i-\bar{X})(Y_i-\bar{Y})=\frac{1}{N}\sum X_iY_i-\bar{X}\bar{Y},$$
$$\frac{1}{N}\sum(X_i-\bar{X})^2=\frac{1}{N}\sum X_i^2-\bar{X}^2.$$

Therefore

$$\hat{\beta}_1=\frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)},\qquad \hat{\beta}_0=\bar{Y}-\hat{\beta}_1\bar{X}.$$

Final

Substituting the first into the second and rearranging yields

$$\hat{\beta}_1 = \frac{\frac{1}{N}\sum_{i=1}^{N} X_i Y_i - \bar{X}\bar{Y}}{\frac{1}{N}\sum_{i=1}^{N} X_i^2 - \bar{X}^2} = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}.$$

Finally,

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}.$$

Remark: Interpretation of $\hat{\beta}_1$

- Numerator $\mathrm{Cov}(X, Y)$: co-variation, the linear effect of X on Y.
- Denominator $\mathrm{Var}(X)$: variability of X itself.
- Hence $\hat{\beta}_1$ measures the **average change in Y per unit change in X**, *i.e.*, the "linear effect" of X normalized by its own variability.

Definition

Subspaces $W, W' \subset V$ are **orthogonal** if

$$\mathbf{w} \cdot \mathbf{w}' = 0 \quad \forall\, \mathbf{w} \in W,\ \mathbf{w}' \in W'.$$

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is **orthonormal** if $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$.
A matrix A is **orthonormal** when its columns are orthonormal vectors.

Fundamental Subspaces of a Matrix

Let $A \in \mathbb{R}^{m \times n}$.

- **Column space (image):**

$$C(A) \equiv \mathrm{Im}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m.$$

  Dimension = **rank** of A.

- **Null space (kernel):**

$$N(A) \equiv \ker(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \subset \mathbb{R}^n.$$

- **Row space:**

$$R(A) = \mathrm{Im}(A^\top) = \{\mathbf{y}^\top A,\ \mathbf{y} \in \mathbb{R}^m\} = [\text{span of row vectors of } A] \subset \mathbb{R}^n.$$

- **Rank–nullity theorem:**

$$n = \dim N(A) + \dim R(A).$$

- **Orthogonal complement:** For a subspace $W \subset V$,

$$W^\perp = \{\mathbf{v} \in V \mid \mathbf{v} \cdot \mathbf{w} = 0,\ \forall\, \mathbf{w} \in W\}.$$

Example

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

- Compute $C(A) = \mathrm{Im}(A)$: span of the column vectors. Is it all of $\mathbb{R}^3$?
- Compute $C^{\perp}$.
- Compute $N(A) = \ker(A)$: solve $A\mathbf{x} = \mathbf{0}$ explicitly.
- Determine $R(A)$: span of row vectors. Compare $\dim R(A)$ with $\dim C(A)$.
- Verify the rank–nullity theorem:
$$n = 3 = \dim N(A) + \dim R(A).$$

Worked Example: Solution

- **Column Space**

$$C(A) = \mathrm{Span}\{(1,2,1)^\top, (2,4,1)^\top\}, \quad \dim = 2 < 3.$$

So $C(A) \neq \mathbb{R}^3$.

- **Orthogonal Complement**

$$C(A)^\perp = \ker(A^\top) = \mathrm{Span}\{(-2,1,0)^\top\}.$$

- **Null Space**

$$N(A) = \ker(A) = \mathrm{Span}\{(1,-2,1)^\top\}, \quad \dim = 1.$$

- **Row Space**

$$R(A) = \mathrm{Span}\{(1,2,3), (1,1,1)\}, \quad \dim = 2.$$

Note $\dim R(A) = \dim C(A) = 2$.

- **Rank–Nullity Theorem**

$$3 = \dim N(A) + \dim R(A) = 1 + 2.$$

Consistency: $\dim C(A)^\perp = 1$, and indeed $(-2,1,0)$ is orthogonal to both generators of $C(A)$.

Exercise

For A as above:

1. Prove $N(A) = R(A)^{\perp}$ and $N(A^{\top}) = C(A)^{\perp}$.
2. Prove any $\mathbf{v} \in V$ decomposes uniquely as $\mathbf{v} = \mathbf{w}' + \mathbf{w}''$ with $\mathbf{w}' \in W$, $\mathbf{w}'' \in W^{\perp}$.
3. Prove that the closest vector in W to $\mathbf{v}$ is exactly $\mathbf{w}'$.

Let A denote the matrix on the left in the last displayed equation in Example [82], and let $\mathbf{b} = [6, 5, 7, 10]^\top$. Then

$$A^\top A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

so

$$\left(A^\top A\right)^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{bmatrix}$$

Continuing with the computation, we have

$$A \cdot \left(A^\top A\right)^{-1} \cdot A^\top = \frac{1}{10} \begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}$$

Putting everything together, we see that indeed

$$A \cdot \left(A^\top A\right)^{-1} \cdot A^\top \cdot \mathbf{b} = \begin{bmatrix} 4.9 \\ 6.3 \\ 7.7 \\ 9.1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3.5 \\ 1.4 \end{bmatrix}$$

where $[3.5, 1.4]$ is the solution we obtained using partials.

Quadratic fit for Example 4.2

Fit a degree–2 model $y = ax^2 + bx + c$ to the data of Example 4.2. Set up the least–squares system

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{A} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}}_{\mathbf{b}}$$

and carry out the same analysis as for the linear fit:

- Derive the normal equations $A^\top A\,\mathbf{y} = A^\top\mathbf{b}$ (with $\mathbf{y} = [c, b, a]^\top$).
- Show that the solution $\mathbf{y}$ gives $\mathbf{b}' = A\mathbf{y}$ equal to the projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$.
- Verify that this agrees with the solution found by minimizing via partial derivatives.

Towards SVD

Real-world problems often involve *non-square* matrices. **Singular Value Decomposition (SVD)** is "diagonalization for non-square matrices" and will generalize these ideas.

Singular Value Decomposition Theoreme

Let M be an $m \times n$ matrix of rank r. There exist matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ with orthonormal columns, and a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ with nonzero entries $\sigma_1, \ldots, \sigma_r$, such that

$$M = U \Sigma V^\top .$$

Singular Value Decomposition Theoreme

Let M be an $m \times n$ matrix of rank r. There exist matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ with orthonormal columns, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonzero entries $\sigma_1, \ldots, \sigma_r$, such that

$$M = U\Sigma V^\top.$$

Key Ideas of the Proof

- $M^\top M$ is symmetric $\Rightarrow$ diagonalizable with orthonormal eigenvectors.
- If $M^\top M\mathbf{v}_i = \lambda_i\mathbf{v}_i$, define singular values $\sigma_i = \sqrt{\lambda_i}$.
- Construct $\mathbf{q}_i = \frac{1}{\sigma_i} M\mathbf{v}_i$; these vectors are orthonormal in $\mathbb{R}^m$.
- Collect $\{\mathbf{q}_i\}$ as columns of U, $\{\mathbf{v}_i\}$ as columns of V.
- Then $U^\top MV = \Sigma$, hence $M = U\Sigma V^\top$.

Singular Value Decomposition Theoreme

Let M be an $m \times n$ matrix of rank r. There exist matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ with orthonormal columns, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonzero entries $\sigma_1, \ldots, \sigma_r$, such that

$$M = U\Sigma V^\top.$$

Key Ideas of the Proof

- $M^\top M$ is symmetric $\Rightarrow$ diagonalizable with orthonormal eigenvectors.
- If $M^\top M\mathbf{v}_i = \lambda_i\mathbf{v}_i$, define singular values $\sigma_i = \sqrt{\lambda_i}$.
- Construct $\mathbf{q}_i = \frac{1}{\sigma_i} M\mathbf{v}_i$; these vectors are orthonormal in $\mathbb{R}^m$.
- Collect $\{\mathbf{q}_i\}$ as columns of U, $\{\mathbf{v}_i\}$ as columns of V.
- Then $U^\top MV = \Sigma$, hence $M = U\Sigma V^\top$.

Interpretation

SVD generalizes diagonalization to non-square matrices. It expresses any matrix as:

(orthogonal change of basis) $\times$ (scaling) $\times$ (orthogonal change of basis).

Example: Computing an SVD

Compute the Singular Value Decomposition of

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

(Hint: start with $M^\top M$ and find eigenvalues/eigenvectors.)

Exercise: Verification and Rank-One Approximation

Check that indeed $M = U\Sigma V^\top$. What is the best rank-one approximation of M?

Example: Decomposition into Rank-One Matrices

Write

$$M = \mathbf{u}_1 \sigma_1 \mathbf{v}_1^\top + \cdots + \mathbf{u}_r \sigma_r \mathbf{v}_r^\top,$$

and interpret this as a decomposition into rank-one matrices. Discuss its use in applications such as image compression.

Exercise: Least Squares via SVD

Show that least squares approximation is an instance of SVD: minimizing $\|M\mathbf{x} - \mathbf{b}\|$ reduces to

$$\mathbf{y} = V^\top \mathbf{x} = \frac{1}{\Sigma} U^\top \mathbf{b}.$$