# Analysis Project

M1 MIASHS

October 2025

**Individual or pair work**
**The *quality of writing* will be strongly evaluated!**

## General Guidelines

- **Expected deliverables**:
  1. **PDF report** written in LaTeX (Overleaf or equivalent). The **theoretical answers** must be written with mathematical rigor; numerical results (tables and figures) should be inserted and discussed.
  2. **Python notebook** (`.ipynb`) containing the computations and plots. The code remains in the notebook and **must not** be explained line by line in the report.

→ **Avoid referring to the code in the report**: describe the *methodology*, the *results*, the *tests*, and their *interpretation* (not the loops, imports, etc.).

- **Figures and tables**: provide informative captions, clearly labeled axes and units, and cite them in the text (e.g., "see Fig. 1"). Figures must be sufficiently readable (size, font).
- **Reproducibility**: the notebook must allow for reproducing all numerical results and figures shown in the report (set a random seed if relevant).

## Exercise 1: Numerical Sequences and Convergence

We consider the sequence $(u_n)_{n\geq 0}$ defined by

$$u_0 = 0, \qquad u_{n+1} = \sqrt{u_n + 2}, \quad \forall n \in \mathbb{N}.$$

### Part A — Theory (formal reasoning)

1. (*Existence*) Prove by induction that $u_n \geq 0$ for all $n \in \mathbb{N}$.
2. (*Monotonicity and upper bound*) Show that $(u_n)_{n\geq 0}$ is increasing and bounded above by 2.
3. (*Convergence*) Conclude that $(u_n)_{n\geq 0}$ converges, and cite the appropriate theorem.
4. (*Characterization of the limit*) Let $\ell = \lim\limits_{n\to\infty} u_n$. Show that

   $$\ell = \sqrt{\ell + 2}.$$

   Determine all possible values of $\ell$ and specify which one is actually reached by the sequence $(u_n)_{n\geq 0}$.

5. (*Rate of convergence – local contraction*) Let $f(x) = \sqrt{x+2}$. On $[0, 2]$, check that $f$ is $C^1$ and compute $f'(x)$. Deduce:

   (a) a **global bound** of the contraction rate $q = \sup_{x\in[0,2]} |f'(x)|$;

   (b) the **asymptotic rate** $|f'(\ell)|$. This leads to the linearized *error relation* $e_{n+1} \approx f'(\ell)\, e_n$ with $e_n = u_n - \ell$.

   **Why do we focus on $f'(\ell)$?** When $n$ becomes large, the terms $u_n$ are very close to the limit $\ell$, so it makes sense to study the sequence in a neighborhood of this point. Expanding $f$ in a first-order Taylor series around $\ell$, we write:

$$f(u_n) \underset{n\to\infty}{=} f(\ell) + f'(\ell)\,(u_n - \ell) + o(u_n - \ell),$$

   with $f(\ell) = \ell$ since $\ell$ is a fixed point. Hence:

$$u_{n+1} - \ell \underset{n\to\infty}{=} f'(\ell)\,(u_n - \ell) + o(u_n - \ell),$$

   or, setting $e_n = u_n - \ell$,

$$e_{n+1} \underset{n\to\infty}{=} f'(\ell)\, e_n + o(e_n).$$

   Thus, in a neighborhood of $\ell$, the remainder term $o(e_n)$ becomes negligible compared to $e_n$, giving the linearized approximation:

$$e_{n+1} \approx f'(\ell)\, e_n.$$

   Hence, the study of the asymptotic rate $|f'(\ell)|$ and the error relation $e_{n+1} \approx f'(\ell)e_n$ allows us to **quantify the speed of convergence** of the sequence toward its limit. This linearized relation shows that $f'(\ell)$ governs the local rate of convergence: if $|f'(\ell)| < 1$, the sequence is contractive and converges toward $\ell$.

## Part B — Numerical exploration (Python Notebook, results integrated into the report)

**Objective**: Illustrate and quantify the convergence of $(u_n)_{n\geq 0}$; compare theoretical and numerical behavior. Unless explicitly stated otherwise, the **theoretical justification is not required** in this Part B.

1. **Computation and value table.** Compute $(u_n)_{n\geq 0}$ for $n = 0, \dots, N$ with a reasonable $N$ (e.g. $N = 20$) and initial conditions

$$u_0 \in \{\, 0,\ 1.5,\ 2.5,\ -1 \,\}.$$

   (i) Identify for which initial values the definition of the sequence is valid;

   (ii) present a synthetic table (a few iterations) in the report;

   (iii) plot $n \mapsto u_n$.

2. **Global contraction bound.** On $[0, 2]$, verify numerically that

$$|u_{n+1} - \ell| \leq q\, |u_n - \ell| \quad \text{where } q \text{ is the constant found in Part A,}$$

   at least from a certain index onward. Illustrate this using a figure (for example, plot $|u_{n+1} - \ell|$ versus $|u_n - \ell|$, along with the line $y = q\, x$ on the same graph).

3. **Graphical analysis of the associated function.** It can be shown that the sequence $(u_n)_{n\geq 0}$ admits an explicit expression:

$$u_n = g(n), \qquad g(x) = 2\cos\left(\frac{\pi}{2^{x+1}}\right).$$

This continuous function extends the discrete sequence $(u_n)_{n\geq 0}$ to all real $x \geq 0$.

(i) Plot on the interval $[0, 10]$ the graph of $g$ in an orthonormal coordinate system. Comment on the global behavior of $g$ (monotonicity, limit, overall shape).

(ii) For $x \geq 0$, compute the derivative $g'(x)$ (symbolic differentiation expected). Choose a point $a \in [0, 10]$ and find the equation of the tangent line to $g$ at $x = a$.

(iii) Write the explicit equation of the tangent $T_a$,

$$y = g'(a)(x - a) + g(a),$$

and plot this tangent on the same graph as $g$.

(iv) Interpret geometrically the slope $g'(a)$: how does it evolve as $a$ increases? What does this tell us about the rate of convergence of the sequence $(u_n)_{n\geq 0}$ toward its limit?

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## Exercise 2: Optimization and Convexity — Theoretical and Interpretative Component

We model the potential energy of a particle in the plane by the quadratic function:

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = 6x^2 + 3xy + 5y^2 - 8x + 4y + 9.$$

This function combines:

- a quadratic term $(6x^2 + 3xy + 5y^2)$ representing the stored energy in the system;
- a coupling term $xy$ describing the interaction between the two spatial directions;
- linear terms $(-8x + 4y)$ representing external forces or sources acting on the particle.

### Part A — Theoretical Analysis

1. (*Continuity*) Justify that $f$ is continuous and differentiable on $\mathbb{R}^2$.
2. (*Existence of an extremum*) On the closed and bounded set $D = [-1, 1]^2$, does $f$ admit a minimum? Explain why.
3. (*Gradient and critical points*) Compute, for all $(x, y) \in \mathbb{R}^2$, the gradient $\nabla f(x, y)$ and explicitly solve the system $\nabla f(x^\star, y^\star) = \mathbf{0}$.
4. (*Hessian matrix*) Compute, for all $(x, y) \in \mathbb{R}^2$, the Hessian matrix $H_f(x, y)$.
5. (*Eigenvalues and classification*) Compute the eigenvalues of $H_f(x^\star, y^\star)$ and determine whether this matrix is positive definite. Deduce the convexity of $f$ and conclude about the nature and uniqueness of the critical point.
6. (*Canonical form*) Show that $f$ can be rewritten as

   $$f(x, y) = f(x^\star, y^\star) + Q(x - x^\star, y - y^\star),$$

   where $Q$ is a positive definite quadratic form. Provide the explicit expression of $Q$.

   ***Motivation.*** This reformulation isolates the purely quadratic contribution around the minimum: the term $Q(x - x^\star, y - y^\star)$ measures the **local stability** of the equilibrium, while $f(x^\star, y^\star)$ represents the minimal energy achieved at equilibrium. The larger the eigenvalues of $H_f$, the steeper the "valley" of the minimum, and the faster the system returns to equilibrium when perturbed.

7. (*Energy along a trajectory*) Suppose now that the particle follows a circular trajectory in the plane:

   $$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (\cos t, \sin t).$$

   We then define the potential energy along the trajectory:

   $$E(t) = f(\gamma(t)).$$

   (i) Express $E(t)$ explicitly as a function of $t$.

   (ii) Compute the derivative $E'(t)$ using the **chain rule** $E'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t)$.

   ***Intuition.*** The derivative $E'(t)$ represents the **instantaneous rate of change of energy** along the circular motion. It measures the infinitesimal work of the force associated with the gradient field of $f$ along the trajectory $\gamma$. If $E'(t) = 0$, the particle is at an equilibrium position along the circle.

## Part B — Numerical Implementation and Visualization

**Objective.** Deepen the understanding of the behavior of $f$ and the trajectory $\gamma$ through graphical representations and numerical calculations. All figures should be inserted into the report and accompanied by qualitative interpretations.

1. **Representation of $\gamma$.** Plot, for $t \in [0, 2\pi]$, the curve representing $\gamma(t)$.

2. **Visualization of the energy surface.** Represent $f(x, y)$ as a 3D surface over the domain $D = [-1, 1]^2$. Identify visually the position of the global minimum $(x^\star, y^\star)$ and verify the symmetry of the energy landscape. Overlay contour lines (level curves) on the ground projection.

3. **Simulation of the circular trajectory.** Implement the trajectory $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. On the $(x, y)$-plane, plot:

   - the contour levels of $f$,
   - the unit circle traversed by the particle,
   - and the current point $\gamma(t)$ moving along the circle.

   The figure should be clear, color-coded, and properly labeled (axes, legend, titles).

4. **Energy along the motion.** Compute and plot the energy curve $t \mapsto E(t) = f(\gamma(t))$ for $t \in [0, 2\pi]$. Identify graphically the points where $E'(t) = 0$ and discuss their physical meaning: do they correspond to energy minima or maxima along the circle?

5. **Study of energy stability.** Discuss the behavior of $E(t)$: amplitude, variations, and positions of extrema. How do these observations relate to the global convexity of $f$?

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**Presentation guidelines.** Figures must include clearly labeled axes, consistent units, descriptive titles, and explicit legends. Comments in the report should connect numerical results with theoretical analysis: convexity, gradient, equilibrium points, and energy variations along the trajectory. No code blocks should be included in the report; only methodology, results, and interpretations are expected.