Mock Exam - Analysis

M1 MIASHS

October 7, 2025

Exercise 1 - Study of a Function and Recursive Sequence

We consider the function

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = \frac{1}{2}\Big(\sqrt{x+4} + 1\Big).$$

a) **Domain of definition.** Find the domain of definition of g.

b) **Image (range).** Determine the image of g, i.e. the set of all possible values of $g(x)$.

c) **Derivative and monotonicity.** Compute $g'(x)$ and study its sign on the domain. Deduce whether g is increasing or decreasing.

d) **Injectivity and surjectivity.** Determine whether g is injective and/or surjective onto its image. Conclude on the bijectivity of g.

e) **Application: Recursive sequence.** We define a sequence by:

$$v_0 = 0, \qquad v_{n+1} = g(v_n), \quad \forall n \in \mathbb{N}.$$

i) Prove that the sequence (v_n) is well-defined, i.e. all its terms belong to the domain of g.

ii) Study the monotonicity and boundedness of (v_n).

iii) Deduce from the Monotone Convergence Theorem that (v_n) converges. Let $\ell = \lim_{n\to\infty} v_n$. Show that ℓ satisfies a fixed-point equation involving g, and determine its exact value.

Hint: Since g is continuous and increasing, the fixed-point method applies.

Exercise 2 - Arithmetico-Geometric Sequence

We consider the sequence (u_n) defined by

$$u_{n+1} = \frac{1}{2}u_n + 3, \qquad u_0 = 0.$$

a) Compute the first three terms of the sequence.

b) Show that the sequence admits a fixed point α, that is, a value satisfying $u_{n+1} = u_n = \alpha$. Find α.

c) Define a new sequence $v_n = u_n - \alpha$. Show that (v_n) satisfies a simpler recurrence relation.

d) Solve explicitly the recurrence relation of (v_n).

e) Deduce the explicit formula for (u_n), and determine its limit as $n \to \infty$.

Hint: This exercise illustrates the standard method for solving affine (arithmetico-geometric) recurrences.

Exercise 3 - Chain Rule and Composition of Functions

We consider the functions

$$f : \mathbb{R} \to \mathbb{R}^2, \qquad f(t) = (e^{2t}, t^2 - 1), \qquad g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x^2 y + \sin y.$$

Let $h = g \circ f : \mathbb{R} \to \mathbb{R}$. For all this exercise, let $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$.

1) **Chain rule.** State the chain rule for the derivative of a composition $h = g \circ f$, where $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$.

2) **Intermediate derivatives.** Compute:

$$f'(t) \quad \text{and} \quad \nabla g(x, y).$$

3) **Derivative of the composition.** Using the chain rule, find the general expression for $h'(t)$.

4) **Numerical value.** Compute $h'(0)$.

Exercise 4 - Optimisation

We model the potential energy of a particle in a plane by the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = 6x^2 + 3xy + 5y^2 \ - \ 8x \ + \ 4y \ + \ 9.$$

This quadratic form (including an xy coupling term and linear forcing terms) is more general while remaining explicitly solvable.

1. **Continuity.** Justify that f is continuous on $\mathbb{R}^2$.

2. **Gradient and critical points.** Compute the gradient $\nabla f(x, y)$ and solve explicitly the system $\nabla f(x, y) = 0_{\mathbb{R}^2}$.

3. **Hessian matrix.** Compute the Hessian matrix $H_f(x, y)$.

4. **Eigenvalues and classification.** Show that $H_f(x, y)$ is positive definite for all $(x, y) \in \mathbb{R}^2$. Deduce that f is strictly convex and conclude about the nature (and uniqueness) of the critical point found in (b).

5. **Physical interpretation.** Discuss the meaning of the global minimum of f, being

$$\min_{\mathbb{R}^2} f,$$

   as a stable equilibrium position, and comment on the combined effect of the xy-coupling term (interaction) and the linear terms (external forcing).

6. **Associated integrals.** For a simple (non-exponential) "accessibility weight" over the square domain, consider

$$I = \iint_D \frac{1}{1 + x + y} \, \mathrm{d}x \, \mathrm{d}y, \qquad D = [0, 1]^2.$$

   (i) Justify that I is well-defined and can be computed using iterated integration.

   (ii) Computation of K **(beyond the syllabus).** As a complement, we connect the quadratic study to an average energy quantity:

$$K = \iint_{[0,1]^2} \left(f(x, y) - \min_{\mathbb{R}^2} f\right) \mathrm{d}x \, \mathrm{d}y.$$

   Express K in terms of integrals of monomials over $[0, 1]$, and conclude with an explicit calculation.