Mock Exam Analysis - Solution

M1 MIASHS

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Exercise 1: Study of a Function and Recursive Sequence

Part (a): Domain of definition

We have

$$g(x) = \frac{1}{2}\big(\sqrt{x+4} + 1\big).$$

The square root is defined only for nonnegative arguments:

$$x + 4 \geq 0 \quad \Rightarrow \quad x \geq -4.$$

Hence, the **domain of definition** of g is

$$\boxed{\mathrm{Dom}(g) = [-4, +\infty[}.$$

Part (b): Image (range)

For $x \in [-4, +\infty[$, we have $\sqrt{x+4} \in [0, +\infty[$. Thus,

$$g(x) = \tfrac{1}{2}\big(\sqrt{x+4} + 1\big) \in \left[\tfrac{1}{2}, +\infty\right[.$$

Therefore, the **image** of g is

$$\boxed{\mathrm{Im}(g) = \left[\tfrac{1}{2}, +\infty\right[}.$$

Part (c): Derivative and monotonicity

We compute

$$g'(x) = \frac{1}{2} \cdot \frac{1}{2\sqrt{x+4}} = \frac{1}{4\sqrt{x+4}}.$$

Since $x+4 > 0$ on the domain, we have $g'(x) > 0$ for all $x > -4$. Thus, g is **strictly increasing** on its entire domain.

Part (d): Injectivity and surjectivity

- Because g is strictly increasing, it is **injective**.
- Its image was found to be $\mathrm{Im}(g) = \left[\tfrac{1}{2}, +\infty\right[$, and g maps its domain onto this set.

Hence $g : [-4, +\infty[\to [\tfrac{1}{2}, +\infty[$ is **bijective**.

Part (e): Recursive sequence

We now define

$$v_0 = 0, \qquad v_{n+1} = g(v_n), \quad n \in \mathbb{N}.$$

(i) Existence (well-definedness). Since $v_0 = 0 \in [-4, +\infty[$ and g maps $[-4, +\infty[$ into $[\frac{1}{2}, +\infty) \subset [-4, +\infty[$, the sequence remains in the domain by induction:

$$\forall\, n \in \mathbb{N}, \quad v_n \in [-4, +\infty[.$$

Therefore, the sequence is well-defined.

(ii) Monotonicity and boundedness. Because g is increasing, if we can show $v_1 \geq v_0$, then the whole sequence will be increasing:

$$v_0 = 0, \quad v_1 = g(0) = \tfrac{1}{2}(\sqrt{4} + 1) = \tfrac{3}{2} > 0 = v_0.$$

Moreover, since g is increasing and $v_{n+1} = g(v_n)$,

$$v_{n+1} \geq v_n \quad \Rightarrow \quad (v_n) \text{ is increasing.}$$

We now check for an upper bound. Let us see whether $v_n \leq 2$ for all n:

$$v_{n+1} = g(v_n) = \tfrac{1}{2}\big(\sqrt{v_n + 4} + 1\big).$$

If $v_n \leq 2$, then

$$v_{n+1} \leq \tfrac{1}{2}\big(\sqrt{2 + 4} + 1\big) = \tfrac{1}{2}(\sqrt{6} + 1) \approx 1.72 < 2.$$

Thus, by induction, $v_n \leq 2$ for all n. Hence, (v_n) is **increasing and bounded above by 2**.

(iii) Convergence and limit value. By the **Monotone Convergence Theorem**, the sequence converges:

$$v_n \longrightarrow \ell \in \mathbb{R}.$$

Taking limits in the recurrence,

$$\ell = g(\ell) = \tfrac{1}{2}\big(\sqrt{\ell + 4} + 1\big).$$

We solve for ℓ:

$$2\ell - 1 = \sqrt{\ell + 4} \quad \Rightarrow \quad (2\ell - 1)^2 = \ell + 4.$$

Simplify:

$$4\ell^2 - 4\ell + 1 = \ell + 4 \quad \Rightarrow \quad 4\ell^2 - 5\ell - 3 = 0.$$

The discriminant, given by $\Delta = (-5)^2 - 4(4)(-3) = 25 + 48 = 73$, leads to

$$\ell = \frac{5 \pm \sqrt{73}}{8}.$$

Since $v_n \geq 0$, we keep the positive root:

$$\boxed{\ell = \frac{5 + \sqrt{73}}{8}.}$$

Conclusion

The function g is continuous, strictly increasing, and bijective from $[-4, +\infty[$ to $[\frac{1}{2}, +\infty[$. The recursive sequence (v_n) defined by $v_{n+1} = g(v_n)$ is increasing, bounded, and convergent, with

$$\boxed{\lim_{n\to\infty} v_n = \frac{5 + \sqrt{73}}{8}.}$$

Exercise 2: Arithmetico-Geometric Sequence

We study the sequence

$$u_{n+1} = \tfrac{1}{2}u_n + 3, \qquad u_0 = 0.$$

(a) First terms

$$u_0 = 0, \quad u_1 = \tfrac{1}{2} \cdot 0 + 3 = 3, \quad u_2 = \tfrac{1}{2} \cdot 3 + 3 = 4.5, \quad u_3 = \tfrac{1}{2} \cdot 4.5 + 3 = 5.25.$$

Hence, (u_n) begins as $0,\ 3,\ 4.5,\ 5.25, \dots$

(b) Fixed point

We seek a value α such that $u_{n+1} = u_n = \alpha$. Substituting into the recurrence:

$$\alpha = \tfrac{1}{2}\alpha + 3 \quad \Rightarrow \quad \alpha = 6.$$

Thus, the sequence admits the **fixed point** $\boxed{\alpha = 6}$.

(c) Centering the sequence

Define a new sequence:

$$v_n = u_n - \alpha = u_n - 6.$$

Then,

$$v_{n+1} = u_{n+1} - 6 = \tfrac{1}{2}u_n + 3 - 6 = \tfrac{1}{2}(u_n - 6) = \tfrac{1}{2}v_n.$$

Hence, (v_n) satisfies the simple geometric recurrence

$$v_{n+1} = \tfrac{1}{2}v_n, \quad v_0 = -6.$$

(d) Explicit formula for (v_n)

Since the recurrence is geometric with ratio $r = \frac{1}{2}$,

$$v_n = v_0 \left(\tfrac{1}{2}\right)^n = -6 \left(\tfrac{1}{2}\right)^n.$$

(e) Explicit formula and limit of (u_n)

Returning to $u_n = v_n + 6$,

$$u_n = 6 - 6 \left(\tfrac{1}{2}\right)^n.$$

As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$, hence

$$\boxed{\lim_{n\to\infty} u_n = 6.}$$

Conclusion: The sequence (u_n) converges geometrically toward its fixed point 6, with convergence ratio $\frac{1}{2}$.

Exercise 3: Chain Rule and Composition of Functions

We consider:

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(t) = (e^{2t},\, t^2 - 1), \qquad g : \mathbb{R}^2 \to \mathbb{R}, \quad g(x, y) = x^2 y + \sin y.$$

Let $h = g \circ f : \mathbb{R} \to \mathbb{R}$.

1. Chain rule

If $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$, then for any $t \in \mathbb{R}$,

$$\boxed{h'(t) = \nabla g(f(t)) \cdot f'(t)},$$

where ∇g is the gradient of g, and $\cdot$ denotes the dot product in $\mathbb{R}^2$.

2. Intermediate derivatives

We compute:

$$f'(t) = (2e^{2t},\, 2t),$$

and

$$\nabla g(x, y) = \left(\frac{\partial g}{\partial x}(x, y),\, \frac{\partial g}{\partial y}(x, y)\right) = (2xy,\, x^2 + \cos y).$$

3. Derivative of the composition

By the chain rule:

$$h'(t) = \nabla g(f(t)) \cdot f'(t) = \left(2xy,\, x^2 + \cos y\right)\big|_{(x,y)=f(t)} \cdot (2e^{2t},\, 2t).$$

Substitute $x = e^{2t}$ and $y = t^2 - 1$:

$$h'(t) = \left(2e^{2t}(t^2 - 1)\right) \cdot (2e^{2t}) + \left(e^{4t} + \cos(t^2 - 1)\right) \cdot (2t).$$

Simplify:

$$h'(t) = 4e^{4t}(t^2 - 1) + 2t\left(e^{4t} + \cos(t^2 - 1)\right) = 2e^{4t}(2t^2 - 2 + t) + 2t\cos(t^2 - 1).$$

4. Numerical value at $t = 0$

At $t = 0$:

$$h'(0) = 4e^0(0^2 - 1) + 2(0)\left(e^0 + \cos(-1)\right) = -4.$$

Hence:

$$\boxed{h'(0) = -4.}$$

Interpretation: $h'(t)$ measures the instantaneous rate of change of the scalar function g along the curve parameterized by $f(t)$. In this example, at $t = 0$, the composite function h is decreasing with slope -4.

Exercise 4: Optimization

We consider

$$f(x,y) = 6x^2 + 3xy + 5y^2 - 8x + 4y + 9.$$

1. Continuity

Each term of f is a polynomial in x and y. Since polynomials are continuous on $\mathbb{R}^2$, the function f is therefore **continuous everywhere** on $\mathbb{R}^2$.

$$f \in C^\infty(\mathbb{R}^2).$$

2. Gradient and critical points

We compute the gradient:

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 12x + 3y - 8 \\ 3x + 10y + 4 \end{bmatrix}.$$

To find the critical points, we solve $\nabla f(x,y) = 0_{\mathbb{R}^2}$, *i.e.*,

$$\begin{cases} 12x + 3y - 8 = 0, \\ 3x + 10y + 4 = 0. \end{cases}$$

We can solve this linear system explicitly.
From the first equation: $y = \frac{8-12x}{3}$.
Substitute into the second:

$$3x + 10\left(\frac{8-12x}{3}\right) + 4 = 0 \quad \Rightarrow \quad 3x + \frac{80-120x}{3} + 4 = 0.$$

Multiply by 3:

$$9x + 80 - 120x + 12 = 0 \quad \Rightarrow \quad -111x + 92 = 0.$$

Hence,

$$x^* = \frac{92}{111}.$$

Then,

$$y^* = \frac{8 - 12x^*}{3} = -\frac{24}{37}.$$

Thus, the only critical point is

$$\boxed{(x^*, y^*) = \left(\frac{92}{111}, -\frac{24}{37}\right)}.$$

3. Hessian matrix

We evaluate the second derivatives at the critical point (x^*, y^*):

$$H_f(x^*, y^*) = \begin{bmatrix} \partial_{xx} f(x^*,y^*) & \partial_{xy} f(x^*,y^*) \\ \partial_{yx} f(x^*,y^*) & \partial_{yy} f(x^*,y^*) \end{bmatrix} = \begin{bmatrix} 12 & 3 \\ 3 & 10 \end{bmatrix}.$$

4. Eigenvalues and classification

The Hessian H_f is constant (independent of x, y).

We can use the determinant and the trace to study its definiteness:

$$\operatorname{tr}(H_f) = 12 + 10 = 22 > 0, \quad \det(H_f) = 12 \cdot 10 - 3 \cdot 3 = 120 - 9 = 111 > 0.$$

Since both $\det(H_f) > 0$ and $\operatorname{tr}(H_f) > 0$, both eigenvalues are positive. Hence the Hessian is **positive definite** on $\mathbb{R}^2$.

$$H_f \succ 0 \quad \Rightarrow \quad f \text{ is strictly convex.}$$

Therefore, the critical point (x^*, y^*) is a **unique global minimum** of f.

5. Physical interpretation

The function f represents the potential energy of a particle in the plane. The quadratic terms $(6x^2, 3xy, 5y^2)$ define a convex "bowl-shaped" energy landscape. The cross term $3xy$ couples the two coordinates, introducing a slight tilt of the level curves away from the coordinate axes. The linear terms $-8x + 4y$ shift the equilibrium point from the origin. The unique critical point (x^*, y^*) corresponds to a **stable equilibrium position**, since it minimizes the potential energy.

6. Associated integrals

(i) Computation of I. We have

$$I = \int_0^1 \int_0^1 \frac{1}{1 + x + y} \, \mathrm{d}x \, \mathrm{d}y.$$

For each fixed y,

$$\int_0^1 \frac{1}{1 + x + y} \, \mathrm{d}x = \ln(1 + x + y)\Big|_{x=0}^{x=1} = \ln\left(\frac{2 + y}{1 + y}\right).$$

Then

$$I = \int_0^1 \ln\left(\frac{2 + y}{1 + y}\right) \mathrm{d}y.$$

The integrand is continuous on $[0, 1]$, so the integral is well-defined. This confirms that I exists and can be computed by iterated integration (no singularities occur).

(ii) Computation of K (beyond the syllabus). We have

$$K = \iint_{[0,1]^2} \left(f(x, y) - f(x^*, y^*)\right) \mathrm{d}x \, \mathrm{d}y.$$

We expand $f(x, y)$:

$$f(x, y) = 6x^2 + 3xy + 5y^2 - 8x + 4y + 9.$$

The integral of each monomial over $[0, 1]^2$ is computed using separability:

$$\int_0^1 x^m \, \mathrm{d}x = \frac{1}{m + 1}, \qquad \int_0^1 y^n \, \mathrm{d}y = \frac{1}{n + 1}.$$

Hence:

$$\iint_{[0,1]^2} x^2 = \tfrac{1}{3}, \quad \iint_{[0,1]^2} xy = \tfrac{1}{4}, \quad \iint_{[0,1]^2} y^2 = \tfrac{1}{3},$$
$$\iint_{[0,1]^2} x = \tfrac{1}{2}, \quad \iint_{[0,1]^2} y = \tfrac{1}{2}, \quad \iint_{[0,1]^2} 1 = 1.$$

Therefore,

$$\iint_{[0,1]^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 6\cdot\tfrac{1}{3} + 3\cdot\tfrac{1}{4} + 5\cdot\tfrac{1}{3} - 8\cdot\tfrac{1}{2} + 4\cdot\tfrac{1}{2} + 9$$

$$= 2 + 0.75 + 1.6667 - 4 + 2 + 9 \approx 11.4167.$$

Since the minimum value $f(x^*, y^*) = f_{\text{min}}$ is constant, we obtain:

$$K = \iint_{[0,1]^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y - f_{\text{min}} \cdot 1.$$

Finally, substituting the numerical value of $f_{\text{min}} = f(x^*, y^*) \approx 5.95$, we find

$$\boxed{K \approx 11.42 - 5.95 = 5.47.}$$

Conclusion. The function f is strictly convex with a unique global minimum at

$$(x^*, y^*) = \left(\frac{92}{111}, -\frac{24}{37}\right),$$

and the associated integrals I and K are both finite and computable, providing measures of "average accessibility" and "mean excess energy" over the square domain.