

Homework 1

Deadline: 06-09-2022

Question 1. A biased four-sided die is rolled, and the down face is a random variable X described by the following probability mass function (pmf).

$$p(x) = \begin{cases} \frac{x}{10}, & x = 1, 2, 3, 4. \\ 0, & \text{otherwise} \end{cases}$$

Given the random variable X a biased coin is flipped and the random variable Y is 1 or zero according to whether the coin shows heads or tails. The conditional pmf is

$$p(y|x) = \left(\frac{x+1}{2x}\right)^y \left(1 - \frac{x+1}{2x}\right)^{1-y},$$

where $y \in \{0, 1\}$.

- Find the expectation $E(X)$ and the variance $Var(X)$. [5 pts]
- Find the conditional pmf $p(x|y)$. [5 pts]
- Find the conditional expectation $E[X|Y = 1]$; i.e., the expectation with respect to the conditional pmf $p(x|y = 1)$. [5 pts]

Question 2. Show that if two variables x and y are independent, then their covariance is zero. [5 pts]

Question 3. Let X be a random variable on $X = \{a, b, c\}$ with the probability mass function $p(x)$. Let $p(a) = 0.1$, $p(b) = 0.2$, and $p(c) = 0.7$ and some function $f(x)$ be

$$f(x) = \begin{cases} 10; & x = a \\ 5; & x = b \\ \frac{10}{7}; & x = c \end{cases}$$

- What is $E[f(X)]$? [5 pts]
- What is $E[1/p(X)]$? [5 pts]

Question 4. Let $X_1, X_2, X_3, \dots, X_n$ be a random sample from a normal distribution with unknown mean μ and variance σ^2 .

- Find maximum likelihood estimators of mean μ and variance σ^2 . [10 pts]
- Are the MLEs unbiased for their respective parameters? [10 pts]

Question 5. Show that if two random variables X and Y are independent, then $E[XY] = E[X]E[Y]$ – [5 pts]

Question 6. Conditional independence

- a) Let $H \in \{1, \dots, K\}$ be a discrete random variable, and let e_1 and e_2 be the observed values of two other random variables E_1 and E_2 . Suppose we wish to calculate the vector $P(H|e_1, e_2) = (P(H = 1|e_1, e_2), \dots, P(H = K|e_1, e_2))$ [5 pts]

Which of the following sets of numbers are sufficient for the calculation?

- i. $P(e_1, e_2), P(H), P(e_1|H), P(e_2|H)$
- ii. $P(e_1, e_2), P(H), P(e_1, e_2|H)$
- iii. $P(e_1|H), P(e_2|H), P(H)$

- b) Now suppose we now assume $E_1 \perp E_2|H$ (i.e., E_1 and E_2 are conditionally independent given H).

Which of the above 3 sets are sufficient now? [5 pts]

Show your calculations as well as giving the final result. Hint: use Bayes rule.

Question 7. Show that the variance of a sum is $\text{var}[X + Y] = \text{var}[X] + \text{var}[Y] + 2\text{cov}[X, Y]$, where $\text{cov}[X, Y]$ is the covariance between X and Y . [10 pts]

Question 8. Let $D = \{x_1, x_2, x_3, \dots, x_n\}$ be random sample with joint probability

$$p(x|\theta_0) = \begin{cases} e^{-\sum_{i=1}^n (x_i - \theta_0)}, & x \geq \theta_0. \\ 0, & \text{otherwise} \end{cases}$$

Determine θ_{MLE} the maximum likelihood estimate of θ_0 . [15 pts]

Question 9. Suppose A is a real $m \times n$ matrix.

- a) Prove that the symmetric matrix $A^T A$ has the property $x^T (A^T A) x \geq 0$ for every vector x in R^n . [5 pts]
- b) According to part a), the matrix $A^T A$ is positive semidefinite at least and possibly positive definite. Under what condition A on is $A^T A$ positive definite? [5 pts]