Question 1

full-rank ⇒ invertible

Let $A \in \mathbb{R}^{n\times p}$ $(n \geq p)$ be a full-rank matrix. For some $y \in \mathbb{R}^n$, consider the least squares $\min_{x\in\mathbb{R}^p} \|y - Ax\|$. Then write a solution $\hat{x}$ to the least squares by Q and R, where Q and R are the factors of QR decomposition, i.e. $A = QR$.

$$\|y - Ax\|$$

$$A\hat{x} = \hat{y}$$

$$QR\hat{x} = \hat{y}$$

$$R\hat{x} = Q^T\hat{y}$$

$$\hat{x} = R^{-1}Q^T\hat{y}$$

Question 2

Solve the following linear equation using Gaussian elimination method

$$
\begin{aligned}
3x + 5y - 2z &= 17 \\
2x - 2y + 4z &= 6 \\
4x + 4y - z &= 19
\end{aligned}
$$

$$
\begin{bmatrix} 3 & 5 & -2 & 17 \\ 2 & -2 & 4 & 6 \\ 4 & 4 & -1 & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5/3 & -2/3 & 17/3 \\ 2 & -2 & 4 & 6 \\ 4 & 4 & -1 & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5/3 & -2/3 & 17/3 \\ 0 & -16/3 & 16/3 & -16/3 \\ 0 & -8/3 & 5/3 & -11/3 \end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix} 1 & 5/3 & -2/3 & 17/3 \\ 0 & 1 & -1 & 1 \\ 0 & -8/3 & 5/3 & -11/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5/3 & -2/3 & 17/3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5/3 & -2/3 & 17/3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
$$

$\therefore$ $z = 1$, $y = 2$, $x = 3$.

Question 3

Solve the following linear equation using Gaussian elimination method with partial pivoting

$-\cancel{\frac{3}{14}} \quad + \cancel{\frac{8}{3}}$

$$
\begin{aligned}
2x + 5y - 2z &= 6 \\
2x - 2y + 4z &= 6 \\
3x + 4y - z &= 8
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 5 & -2 & 6 \\ 2 & -2 & 4 & 6 \\ 3 & 4 & -1 & 8 \end{bmatrix} \to \begin{bmatrix} 3 & 4 & -1 & 8 \\ 2 & -2 & 4 & 6 \\ 2 & 5 & -2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & -1/3 & 8/3 \\ 2 & -2 & 4 & 6 \\ 2 & 5 & -2 & 6 \end{bmatrix}
$$

$$
\to \begin{bmatrix} 1 & 4/3 & -1/3 & 8/3 \\ 0 & -14/3 & 14/3 & 2/3 \\ 0 & 7/3 & -4/3 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & -1/3 & 8/3 \\ 0 & 1 & -1 & -\frac{1}{7} \\ 0 & 7/3 & -4/3 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & -1/3 & 8/3 \\ 0 & 1 & -1 & -\frac{1}{7} \\ 0 & 0 & 1 & 1 \end{bmatrix}
$$

$\therefore z = 1, \ y = \frac{6}{7}, \ x = \frac{13}{7}$

Question 4

Let A be some 4 by 4 matrix satisfying

$$\overset{M_3}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.25 & 1 \end{bmatrix}} \overset{M_2}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0.25 & 0 & 1 \end{bmatrix}} \overset{M_1}{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}} A = \overset{A_2}{\begin{bmatrix} 2 & 1 & 0 & -3 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

Then, find A, and explain its LU decomposition with each row operation.

M_1^{-1} : 전체 매트릭스 $\times 2$ & 1행 + 2행 & 1행$\times(-1)$ + 3행 & 1행$\times(-2)$ + 4행

M_2^{-1} : 2행 $\times (-0.5)$ + 3행 & 2행$\times(0.25)$ + 4행

M_3^{-1} : 3행 $\times (0.25)$ + 4행

1) $A = M_1^{-1} M_2^{-1} M_3^{-1} A_2$

$$M_1^{-1} M_2^{-1} M_3^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & -1/8 & -1/8 & 1/2 \end{bmatrix} = L$$

$$A_2 = \begin{bmatrix} 2 & 1 & 0 & -3 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix} = U$$

2) $A = LU$

$$A = \begin{bmatrix} 1 & 1/2 & 0 & -3/2 \\ -1/2 & 1/4 & 2 & 5/4 \\ 1/2 & 1/2 & 5/2 & -1 \\ 1 & 3/8 & -7/8 & -1/2 \end{bmatrix}$$

Question 10

Suppose that for some matrix A, $svd(A)$ in R gives the following result:

```
> svd(A)
$d
[1] 10.9053  6.0228  3.8472

$u
         [,1]     [,2]     [,3]
[1,] -0.48721 -0.18126  0.19771
[2,] -0.16910 -0.76084 -0.61517
[3,] -0.83050  0.10561  0.19620
[4,]  0.21049 -0.61409  0.73755

$v
          [,1]      [,2]      [,3]
[1,] -0.185133 -0.980175 -0.070586
[2,] -0.982195  0.186891 -0.019103
[3,] -0.031916 -0.065792  0.997323
```

Then what are singular values of A? Can we get a determinant of A?

$d의 값들이 singular values 다.

A가 4×3 행렬이기 때문에, det(A)를 얻을 수 없다.

Question 11

Suppose that for some matrix A, $svd(A)$ in R gives the following result:

```
> svd(A)
$d
[1] 17.1189  9.5683  8.4494

$u
           [,1]     [,2]     [,3]
[1,] -0.66926  0.68005  0.29938
[2,] -0.65196 -0.34418 -0.67563
[3,] -0.35642 -0.64736  0.67371

$v
           [,1]     [,2]     [,3]
[1,] -0.66926 -0.68005  0.29938
[2,] -0.65196  0.34418 -0.67563
[3,] -0.35642  0.64736  0.67371
```

Then what are singular values of A? Can we get any information on eigenvalues of A or determinant of A?

$d의 값들이 singular values 다.

A가 symmetric 하다는 것을 알 수 있다.

⇒ determinant를 구할 수 있다.