Robotics I: Mechanics

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Contents

Preface

In Chapter 1, we introduce points and vectors, which are not the same, although they are intimately related. We describe their similarities and differences, and we describe how to parameterize them numerically.

In Chapter 2, we discuss the interrelated concepts of rotation and orientation. We describe how to express the orientation of one coordinate system with respect to another. We describe how to express a given vector with respect to different coordinate systems. We describe how to express the rotation of a coordinate system, which is likely attached to some rigid body, about an arbitrary axis of rotation. We present a variety of ways to conceptualize and parameterize orientations, and we discuss their respective strengths and weaknesses.

In Chapter 3, we discuss the concept of spatial transformations, which generalizes the topics from Chapter 2 to include both rotations and translations. We describe how to express a given point with respect to different coordinate systems that are rotated and/or translated with respect to each other. We also describe how to express the general motions of a rigid body, which can include rotations and/or translations, with respect to any other frame of reference.

In Chapter 4, we describe different types of robots and their construction. Understanding how a robot is capable of moving is critical to the mathematical formalisms that will follow.

In Chapter 5, we describe the forward kinematics of serial (a.k.a. open-chain) robots, which answers the question: Given a set of robot joint poses, what is the pose of the robot's end-effector?

In Chapter 6, we discuss how to conceive of the 6-DOF velocity of a rigid body, which comprises 3-DOF translational velocity and 3-DOF angular velocity, in general

In Chapter 7, we discuss velocity kinematics of serial robots, which describes how joint velocities are related to the velocity of the robot's end-effector. To this end, we introduce the concept of a robot's Jacobian.

In Chapter 8, we apply the tools developed in Chapter 7 to characterize the static loads that a robot's end-effector can support. This includes loads carried by the robot's joint actuators, as well as loads carried by the structure of the robot itself.

In Chapter 9, we discuss the inverse-velocity problem (i.e., Given some desired end-effector velocity, what joint velocities should we command?) and the inverse-statics problem (i.e., Given some desired force and torque applied by the end-effector on the environment, what joint torques should we command?). To tackle these problems, we introduce the interrelated topics of singularities in our robot's Jacobian and the workspace of our robot.

In Chapter 10, we describe the inverse kinematics of serial robots, which answers the question: Given a desired pose of the robot's end-effector, what robot joint angles should we command?

In Chapter 11, we consider the forward and inverse kinematics of parallel (a.k.a. closed-chain) robots. For these robots, the inverse kinematics are simple and unique, whereas the forward kinematics are complicated and often not unique; this is the opposite of the case of serial robots.

In Chapter 12, we describe how to generated smooth trajectories to move a robot from some initial pose to some desired final pose over some desired duration of time.

Finally, in Chapter 13, we describe how to conceptualize the dynamics of a robot. We describe both the

Lagrangian and Newton-Euler frameworks.

The development in these notes assumes some prerequisite knowledge of mathematics. A review of important topics can be found in the appendices.

Chapter 1

Points and Vectors

Points and vectors are not the same, although they are intimately related. In this chapter, we introduce points and vectors, including their similarities and differences, and we describe how to parameterize them numerically.

The material in this chapter assumes some knowledge of linear algebra, which can be reviewed in Appendix A.1.

1.1 Points and Vectors

A *point* is a specific location on an object. The object may be stationary, such as the floor of a room, or it may be moving (or at least have the potential to move), such as the *end-effector* (e.g., gripper, hand) of a robotic manipulator. A point exists independently of how we describe it—that is, before we ever attempt to represent it numerically. We will denote points by upper-case italic letters with optional subscripts, e.g., P_j, where any subscript j just identifies a particular instance.

Consider Figure 1.1(A), which represents a planar table top where distinctive points P_1 and P_2 have been identified. As we can see, these points exist without any numbers attributed to them. In this particular example, both points happen to be attached to the same object.

A *vector* represents a quantity with a direction and a magnitude, but not inherently tied to any specific location. In robotics, we will use vectors to represent quantities such as velocities and forces, for example. We have to specify a dimension to describe a vector's magnitude, or length; for example, we will commonly use units m/s to describe velocities, and units N to describe forces. But as with points, a vector's direction exists independently of how we describe it. We will denote vectors by lower-case bold letters with optional subscripts, e.g., $\mathbf{v}_j$, where any subscript j just identifies a particular instance.

Consider Figure 1.1(B), which depicts two vectors. We can see that one vector points up and the other points to the right, and thus the vectors are perpendicular (which is also called orthogonal). We can also see that one vector is twice as long as the other. We can see all of this without have having specified any particular dimension with which to describe the vectors' lengths, and without having decided on a way to represent the vectors' directions numerically.

A vector also represents a displacement between two points. For example, given points P_1 and P_2, the vector $\mathbf{v} = P_2 - P_1$ defines the distance and direction from P_1 to P_2. Alternatively, we can think of getting to P_2 by starting at P_1 and then moving along vector $\mathbf{v}$: $P_2 = P_1 + \mathbf{v}$. Figure 1.2(A) represents our table top from Figure 1.1(A), where we have identified such a displacement vector $\mathbf{v}$. As with other vectors, this vector has a direction and a length independently of how we describe it. This vector will require a dimension

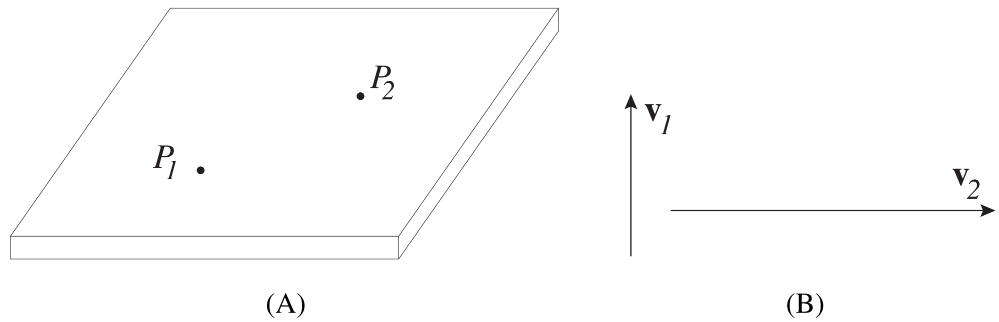


Figure 1.1: (A) Points P_1 and P_2 on a table top. (B) Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in free space.

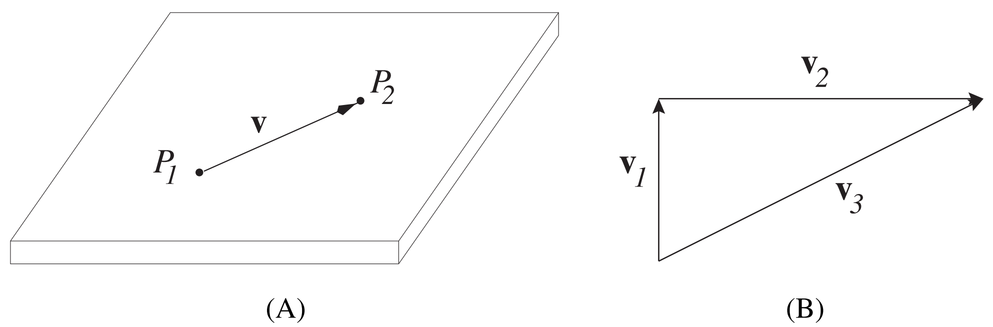


Figure 1.2: (A) Points P_1 and P_2 and the displacement vector $\mathbf{v}$ between them, with $P_2 = P_1 + \mathbf{v}$. (B) Vectors $\mathbf{v}_1$ and and $\mathbf{v}_2$ can be summed to create $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ by aligning the arrows head-to-tail.

to describe its length; for example, we will commonly use units m. Only when added to a preexisting point is there an association of a vector with a unique point.

Two vectors can also be added together. Figure 1.2(B) represents our free vectors from Figure 1.1(B), aligned such that the head of the arrow representing vector $\mathbf{v}_1$ touches the tail of the arrow representing the vector $\mathbf{v}_2$. The vector $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ is uniquely defined as shown, and exists independently of how we describe it. Vector subtractions follows directly, e.g., $\mathbf{v}_1 = \mathbf{v}_3 - \mathbf{v}_2$.

Often we will specify points in terms of displacements from a fixed point O_i, called origin i. Once we specify some origin, then every other point can be represented by a unique vector from that origin. For example, in Figure 1.3, point P_2 is displaced from origin O_0 by $\mathbf{p}_{02}$, and from O_1 by $\mathbf{p}_{12}$, such that $\mathbf{p}_{02} = P_2 - O_0$ and $\mathbf{p}_{12} = P_2 - O_1$.

1.2 Vector Spaces and Bases

A vector space is composed of vectors, whose linear combinations are also vectors. Vectors in a three-dimensional (3D) space, such as the world that we inhabit, have three components. We will write $\mathbf{v} \in \mathcal{R}^3$, meaning that it requires three real numbers to represent a vector numerically. Given any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{R}^3$, then $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ is also a vector in $\mathcal{R}^3$, where a_1 and a_2 are arbitrary scalars. We will denote scalars by lower-case italic letters with optional subscripts, e.g., a_j, where any subscript just identifies a particular instance.

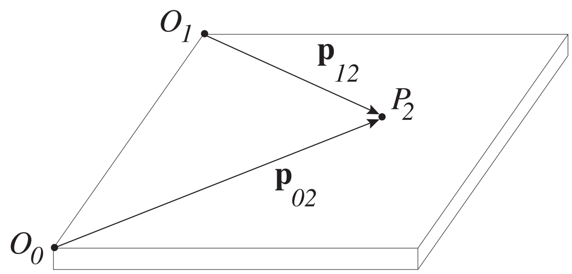


Figure 1.3: Displacement of point P_2 relative to origin O_0 and origin O_1, resulting in two distinct vector displacements $\mathbf{p}_{02} = P_2 - O_0$ and $\mathbf{p}_{12} = P_2 - O_1$, respectively.

Vectors are usually expressed in terms of other vectors that form a *basis* for the vector space. A basis is a set of vectors that spans the vector space; that is to say, any vector can be expressed as a linear combination of the basis vectors. There are as many vectors in a basis as the dimension of the space. For 3D space $\mathcal{R}^3$, any three vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ that are linearly independent (meaning that no basis vector can be represented as a linear combination of the remaining basis vectors) can form a basis. Thus any vector $\mathbf{v}$ in 3D space can be represented as:

$$\mathbf{v} = b_1\mathbf{b}_1 + b_2\mathbf{b}_2 + b_3\mathbf{b}_3 \tag{1.1}$$

for some unique scalars b_1, b_2, and b_3.

Of all possible bases, the most useful ones are *orthonormal bases*, i.e., bases in which the vectors are unit vectors that are mutually orthogonal (meaning that the dot product of any two basis vectors is 0). We will denote the three vectors in an orthonormal basis j as $\mathbf{x}_j$, $\mathbf{y}_j$, and $\mathbf{z}_j$, where the right-hand rule determines that $\mathbf{z}_j = \mathbf{x}_j \times \mathbf{y}_j$. A vector $\mathbf{v}$ in terms of the orthonormal basis j will then be written as:

$$\mathbf{v} = {}^j v_x \mathbf{x}_j + {}^j v_y \mathbf{y}_j + {}^j v_z \mathbf{z}_j \tag{1.2}$$

The unique scalars ${}^j v_x$, ${}^j v_y$, and ${}^j v_z$ are the coordinates of $\mathbf{v}$ with respect to orthonormal basis j. The superscript j indicates which orthonormal basis is the reference. For a different orthonormal basis k, the same vector will have different coordinates:

$$\mathbf{v} = {}^k v_x \mathbf{x}_k + {}^k v_y \mathbf{y}_k + {}^k v_z \mathbf{z}_k \tag{1.3}$$

The basis k coordinates ${}^k v_x$, ${}^k v_y$, and ${}^k v_z$ will be different from the basis j coordinates because the basis vectors are different. A key point is that the vector $\mathbf{v}$ is the same in both cases, only its description in terms of a particular basis differs.

Example 1.1: In Figure 1.4(A), a particular vector $\mathbf{v}$ has been constructed such that $\mathbf{v} = \mathbf{x}_1 + \mathbf{y}_1$ with respect to orthonormal basis 1. Hence its coordinates are ${}^1 v_x = 1$, ${}^1 v_y = 1$, and ${}^1 v_z = 0$. In Figure 1.4(B), there is a different orthonormal basis 2. Then $\mathbf{v} = \mathbf{x}_2 - \mathbf{y}_2$, and its associated coordinates are ${}^2 v_x = 1$, ${}^2 v_y = -1$, and ${}^2 v_z = 0$.

Just like any other vectors, the basis j vectors can be represented in terms of some other basis, say k. Then

$$\begin{aligned}
\mathbf{x}_j &= {}^k x_{jx}\, \mathbf{x}_k + {}^k x_{jy}\, \mathbf{y}_k + {}^k x_{jz}\, \mathbf{y}_z \\
\mathbf{y}_j &= {}^k y_{jx}\, \mathbf{x}_k + {}^k y_{jy}\, \mathbf{y}_k + {}^k y_{jz}\, \mathbf{y}_z \\
\mathbf{z}_j &= {}^k z_{jx}\, \mathbf{x}_k + {}^k z_{jy}\, \mathbf{y}_k + {}^k z_{jz}\, \mathbf{y}_z
\end{aligned} \tag{1.4}$$

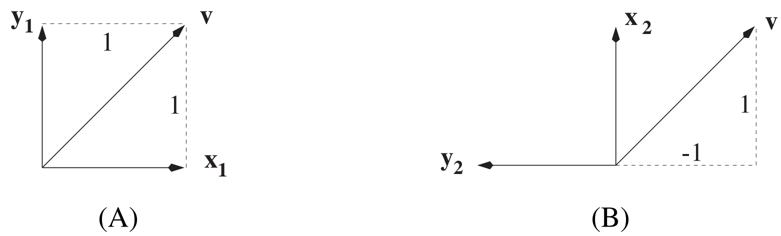


Figure 1.4: (A) The vector $\mathbf{v}$ expressed in basis 1; $\mathbf{z}_1$ points out of the page, per the right-hand rule. (B) The same vector $\mathbf{v}$ expressed in basis 2; $\mathbf{z}_z$ points out of the page, per the right-hand rule.

where the double subscripts jx, jy, and jz refer to the x, y, and z coordinates of the respective basis j vectors. Thus ${}^k x_{jx}$, ${}^k x_{jy}$, and ${}^k x_{jz}$ are the coordinates of $\mathbf{x}_j$ with respect to basis k, and so forth.

Example 1.2: For the simple case shown in Figure 1.4, we can see by inspection that $\mathbf{x}_2 = \mathbf{y}_1$, $\mathbf{y}_2 = -\mathbf{x}_1$, and $\mathbf{z}_2 = \mathbf{z}_1$. Hence the coordinates of ${}^1\mathbf{x}_2$ are ${}^1x_{2x} = 0$, ${}^1x_{2y} = 1$, and ${}^1x_{2z} = 0$, the coordinates of ${}^1\mathbf{y}_2$ are ${}^1y_{2x} = -1$, ${}^1y_{2y} = 0$, and ${}^1y_{2z} = 0$, and the coordinates of ${}^1\mathbf{z}_2$ are ${}^1z_{2x} = 0$, ${}^1z_{2y} = 0$, and ${}^1z_{2z} = 1$.

What are the coordinates of the basis vectors with respect to their own basis? Clearly

$$
\begin{aligned}
\mathbf{x}_j &= (1)\mathbf{x}_j + (0)\mathbf{y}_j + (0)\mathbf{z}_j \\
\mathbf{y}_j &= (0)\mathbf{x}_j + (1)\mathbf{y}_j + (0)\mathbf{z}_j \\
\mathbf{z}_j &= (0)\mathbf{x}_j + (0)\mathbf{y}_j + (1)\mathbf{z}_j
\end{aligned}
\tag{1.5}
$$

Hence ${}^jx_{jx} = 1$, ${}^jx_{jy} = 0$, and ${}^jx_{jz} = 0$ are the coordinates of $\mathbf{x}_j$, ${}^jy_{jx} = 0$, ${}^jy_{jy} = 1$, and ${}^jy_{jz} = 0$ are the coordinates of $\mathbf{y}_j$, and so forth.

Up to this point, we have yet to represent our vectors numerically. When actually evaluating any vector equation, for example (1.2), all vectors have to be expressed with respect to the same basis. We did not state what basis the vectors $\mathbf{x}_j$, $\mathbf{y}_j$, and $\mathbf{z}_j$ in (1.2) were being referred to. We explicitly indicate the reference basis, say k, with respect to which all vectors are being expressed by a left superscript on the vectors:

$$
{}^k\mathbf{v} = {}^jv_x\,{}^k\mathbf{x}_j + {}^jv_y\,{}^k\mathbf{y}_j + {}^jv_z\,{}^k\mathbf{z}_j \tag{1.6}
$$

Note that the j coordinates did not change, only the expression of the three vectors.

The simplest case to evaluate is when $j = k$. For this case, the coordinates of a vector will be written as a column vector. First, consider the coordinates of basis vectors with respect to themselves, rewritten from (1.6):

$$
{}^j\mathbf{x}_j = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad {}^j\mathbf{y}_j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad {}^j\mathbf{z}_j = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.7}
$$

This familiar result is known as the standard basis. It is true of any set of basis vectors expressed with respect to themselves. Thus for a different basis k, then ${}^k\mathbf{x}_k = \mathbf{i}$, and so forth. Note that when a set of orthonormal basis vectors is expressed in terms of a different basis, for example ${}^k\mathbf{x}_j$, ${}^k\mathbf{y}_j$, and ${}^k\mathbf{z}_j$, they are no longer the standard basis.

Next, the coordinates of an arbitrary vector expressed in basis j will be written as:

$$^j\mathbf{v} = {^j}v_x\, {^j}\mathbf{x}_j + {^j}v_y\, {^j}\mathbf{y}_j + {^j}v_z\, {^j}\mathbf{z}_j = \left[\begin{array}{ccc} {^j}\mathbf{x}_j & {^j}\mathbf{y}_j & {^j}\mathbf{z}_j \end{array}\right] \left[\begin{array}{c} {^j}v_x \\ {^j}v_y \\ {^j}v_z \end{array}\right] = \left[\begin{array}{c} {^j}v_x \\ {^j}v_y \\ {^j}v_z \end{array}\right] \tag{1.8}$$

since

$$\left[\begin{array}{ccc} {^j}\mathbf{x}_j & {^j}\mathbf{y}_j & {^j}\mathbf{z}_j \end{array}\right] = \mathbf{I}$$

is the 3-by-3 identity matrix. Thus the vector ${^j}\mathbf{v}$ referred to basis j is represented as a 3-by-1 column vector of its coordinates. An analogous result holds for vectors of a different dimension.

Example 1.3: From Figure 1.4, we would write that

$$^1\mathbf{v} = \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right], \qquad {^2}\mathbf{v} = \left[\begin{array}{c} 1 \\ -1 \\ 0 \end{array}\right]$$

We can also write that ${^1}\mathbf{v} = [1\ 1\ 0]^T$ and ${^2}\mathbf{v} = [1\ \ -1\ 0]^T$, where $\mathbf{v}^T$ means the transpose of vector $\mathbf{v}$. In this context, the transpose operator converts a 1-by-3 row vector into a 3-by-1 column vector.

We are now in a position to define the length of a vector and the dot product. The dot product of two vectors ${^j}\mathbf{v}$ and ${^j}\mathbf{w}$ is:

$$^j\mathbf{v} \cdot {^j}\mathbf{w} = ({^j}\mathbf{v})^T {^j}\mathbf{w} = \left[\begin{array}{ccc} {^j}v_x & {^j}v_y & {^j}v_z \end{array}\right] \left[\begin{array}{c} {^j}w_x \\ {^j}w_y \\ {^j}w_z \end{array}\right] = {^j}v_x\, {^j}w_x + {^j}v_y\, {^j}w_y + {^j}v_z\, {^j}w_z \tag{1.9}$$

This definition of the dot product generalizes to vector spaces of any dimensions. Although the dot product is evaluated with respect to a particular basis, the value of the dot product is the same irrespective of the basis. This is because the dot product represents the projection of $\mathbf{w}$ onto $\mathbf{v}$, and this projection always has the same value. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if their dot product is zero: $\mathbf{v} \cdot \mathbf{w} = 0$.

The length $||{^j}\mathbf{v}||$ of a vector is defined in terms of the dot product:

$$||{^j}\mathbf{v}|| = \sqrt{{^j}\mathbf{v} \cdot {^j}\mathbf{v}} = \sqrt{({^j}v_x)^2 + ({^j}v_y)^2 + ({^j}v_z)^2} \tag{1.10}$$

This is the *Euclidean norm* of $\mathbf{v}$. The length of a vector does not depend on the basis; it's always the same. Hence $||{^j}\mathbf{v}|| = ||{^k}\mathbf{v}||$ for different bases j and k.

A unit vector has a length of 1. Any vector can be converted to a unit vector by dividing by its length: $\hat{\mathbf{v}} = \mathbf{v}/||\mathbf{v}||$, which is used when we are interested in representing the direction of the vector without concern for its length.

Example 1.4: For the vector $\mathbf{v}$ of Figure 1.4, we have that $||{^1}\mathbf{v}|| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $||{^2}\mathbf{v}|| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. The length is the same, regardless of the choice of basis. We can form unit vectors by dividing by the length:

$$^1\hat{\mathbf{v}} = \frac{1}{\sqrt{2}} \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{array}\right], \qquad {^2}\hat{\mathbf{v}} = \frac{1}{\sqrt{2}} \left[\begin{array}{c} 1 \\ -1 \\ 0 \end{array}\right] = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{array}\right]$$

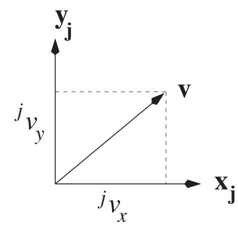


Figure 1.5: The projections of a vector $\mathbf{v}$ onto axes j are its coordinates ${}^j v_1$ and ${}^j v_2$.

An orthonormal basis $\mathbf{x}_j$, $\mathbf{y}_j$, $\mathbf{z}_j$ consists of orthogonal ($\mathbf{x}_j \cdot \mathbf{y}_j = 0$, $\mathbf{x}_j \cdot \mathbf{z}_j = 0$, $\mathbf{y}_j \cdot \mathbf{z}_j = 0$) and unit ($||\mathbf{x}_j|| = ||\mathbf{y}_j|| = ||\mathbf{z}_j|| = 1$) vectors.

Let's reexamine the meaning of the coordinates of a vector, in view of the dot product and its definition as a projection (Figure 1.5). Take the dot product of (1.6) with ${}^k\mathbf{x}_j$:

$$ {}^k\mathbf{v} \cdot {}^k\mathbf{x}_j = {}^j v_x \, {}^k\mathbf{x}_j \cdot {}^k\mathbf{x}_j + {}^j v_y \, {}^k\mathbf{y}_j \cdot {}^k\mathbf{x}_j + {}^j v_z \, {}^k\mathbf{z}_j \cdot {}^k\mathbf{x}_j = {}^j v_x \tag{1.11} $$

since ${}^k\mathbf{x}_j \cdot {}^k\mathbf{x}_j = 1$, ${}^k\mathbf{y}_j \cdot {}^k\mathbf{x}_j = 0$, and ${}^k\mathbf{z}_j \cdot {}^k\mathbf{x}_j = 0$. Hence the coordinate ${}^j v_x$ represents the projection of $\mathbf{v}$ onto $\mathbf{x}_j$, and is independent of the basis k with respect to which the vectors in (1.6) are expressed; this independence was already evident in (1.2). Similarly, ${}^j v_y$ is the projection of $\mathbf{v}$ onto $\mathbf{y}_j$, and ${}^j v_z$ is the projection of $\mathbf{v}$ onto $\mathbf{z}_j$.

1.3 Coordinate Systems

The combination of an origin O_j with a set of orthonormal basis vectors $\mathbf{x}_j$, $\mathbf{y}_j$, $\mathbf{z}_j$ is called *coordinate system* j, or *frame* j. The basis vectors $\mathbf{x}_j$, $\mathbf{y}_j$, $\mathbf{z}_j$ are called the axes of coordinate system j. The axes are pictured as displacements from the origin to implied points $X_j = O_j + \mathbf{x}_j$, $Y_j = O_j + \mathbf{y}_j$, and $Z_j = O_j + \mathbf{z}_j$ at the tips of the arrows representing the axes.

Consider Figure 1.6, which depicts the same scenario depicted in Figure 1.3, but now the two origins are associated with complete coordinate systems. The displacement vectors from the respective origins can now be expressed with respect to the basis vectors of the coordinate system, which enables them to each be parameterized using three real numbers: $\mathbf{p}_{02} = [2 \;\; 2 \;\; 0]^T$ and $\mathbf{p}_{12} = [0 \;\; 1 \;\; 2]^T$.

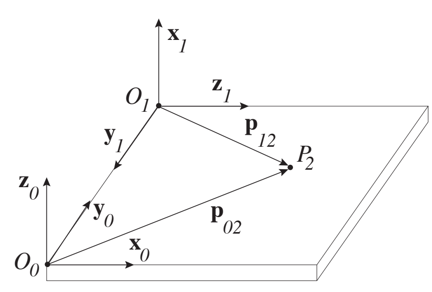


Figure 1.6: Representation of point P_2 in coordinate system 0, defined by the vector $\mathbf{p}_{02} = P_2 - O_0$, and representation of point P_2 in coordinate system 1 defined by the vector $\mathbf{p}_{12} = P_2 - O_1$.

1.4 Homogeneous Coordinates

We can summarize the rules for adding and subtracting points and vectors from Section 1.1 as follows:

- A vector plus/minus a vector is a vector.
- A point plus/minus a vector is a point.
- A point minus a point is a vector.
- A point plus a point is undefined.
- A vector minus a point is undefined.

We've also seen that, once we specify a given coordinate system, a vector in 3D space is represented by three real numbers, but so is a point (via the displacement vector to the point from the origin) making them sometimes difficult to distinguish.

Homogeneous coordinates enable us to distinguish vectors from points while also encoding the addition/subtraction rules above by appending a fourth numerical value: for vectors we append a 0, and for points we append a 1. In homogeneous coordinates, the vector $\mathbf{v}_i$, expressed in frame j, becomes

$$ {}^j\boldsymbol{V}_i = \begin{bmatrix} {}^j\mathbf{v}_i \\ 0 \end{bmatrix}, \tag{1.12} $$

which we represent with an upper-case letter in a slanted bold font. The displacement vector from origin O_j to point P_i, expressed in frame j, then becomes

$$ \begin{aligned} {}^j\boldsymbol{P}_i &= \begin{bmatrix} {}^j(P_i - O_j) \\ 0 \end{bmatrix} && (1.13) \\ &= \begin{bmatrix} {}^j\mathbf{p}_{ji} \\ 0 \end{bmatrix}. && (1.14) \end{aligned} $$

The point P_i, expressed in frame j, becomes

$$ \begin{aligned} {}^jP_i &= \begin{bmatrix} {}^j(P_i - O_j) \\ 1 \end{bmatrix} && (1.15) \\ &= \begin{bmatrix} {}^j\mathbf{p}_{ji} \\ 1 \end{bmatrix}, && (1.16) \end{aligned} $$

which we represent with an upper-case letter in a slanted (not bold) font, where the existence of the superscript signals that the point is being represented in homogeneous coordinates.

If we consider the addition of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in homogeneous coordinates, expressed in frame j, the resulting 0 entry indicates that the result is some new vector:

$$ \begin{aligned} {}^j\boldsymbol{V}_1 + {}^j\boldsymbol{V}_2 &= \begin{bmatrix} {}^j\mathbf{v}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} {}^j\mathbf{v}_2 \\ 0 \end{bmatrix} && (1.17) \\ &= \begin{bmatrix} {}^j\mathbf{v}_1 + {}^j\mathbf{v}_2 \\ 0 \end{bmatrix} && (1.18) \\ &= \begin{bmatrix} {}^j\mathbf{v}_3 \\ 0 \end{bmatrix} = {}^j\boldsymbol{V}_3 && (1.19) \end{aligned} $$

Similarly, if we consider the addition of vector $\mathbf{v}_1$ with point P_1, expressed in frame j, the resulting 1 entry indicates that the result is some new point:

$$ {}^jP_1 + {}^j\boldsymbol{V}_1 = \left[\begin{array}{c} {}^j\mathbf{p}_{j1} \\ 1 \end{array} \right] + \left[\begin{array}{c} {}^j\mathbf{v}_1 \\ 0 \end{array} \right] \tag{1.20} $$

$$ = \left[\begin{array}{c} {}^j\mathbf{p}_{j1} + {}^j\mathbf{v}_1 \\ 1 \end{array} \right] \tag{1.21} $$

$$ = \left[\begin{array}{c} {}^j\mathbf{p}_{j2} \\ 1 \end{array} \right] = {}^jP_2 \tag{1.22} $$

If we were to attempt to sum two points, the resulting 2 entry would indicate that we have made something that is undefined. Similarly, if we were to attempt to subtract a point from a vector, the resulting -1 entry would indicate that we have made something that is undefined.

We will return to the use of homogeneous coordinates in Chapter 3.

Chapter 2

Rotation and Orientation

In this chapter, we discuss the interrelated concepts of rotation and orientation. We describe how to express the orientation of one coordinate system with respect to another. We describe how to express a given vector with respect to different coordinate systems. We describe how to express the rotation of a coordinate system, which is likely attached to some rigid body, about an arbitrary axis of rotation. We present a variety of ways to conceptualize and parameterize orientations, and we discuss their respective strengths and weaknesses.

The material in this chapter assumes some knowledge of linear algebra, which can be reviewed in Appendix A.2.

2.1 Rotation Matrices

2.1.1 Coordinate-Axis Form of a Rotation Matrix

Let's consider how to relate two different coordinate axes, say axes 1 and axes 0. We can express each of the three axes of frame 1 (which are vectors) with respect to frame 0:

$$\begin{aligned} {}^0\mathbf{x}_1 &= {}^0x_{1x}\,{}^0\mathbf{x}_0 + {}^0x_{1y}\,{}^0\mathbf{y}_0 + {}^0x_{1z}\,{}^0\mathbf{z}_0 \\ {}^0\mathbf{y}_1 &= {}^0y_{1x}\,{}^0\mathbf{x}_0 + {}^0y_{1y}\,{}^0\mathbf{y}_0 + {}^0y_{1z}\,{}^0\mathbf{z}_0 \\ {}^0\mathbf{z}_1 &= {}^0z_{1x}\,{}^0\mathbf{x}_0 + {}^0z_{1y}\,{}^0\mathbf{y}_0 + {}^0z_{1z}\,{}^0\mathbf{z}_0 \end{aligned} \tag{2.1}$$

That is to say, the coordinates of the $\mathbf{x}$ axis of frame 1 expressed in frame 0, which we denote as ${}^0\mathbf{x}_1$, are $[{}^0x_{1x}\ {}^0x_{1y}\ {}^0x_{1z}]^T$, and so forth. We can rewrite (2.1) as

$$\begin{aligned} \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix} &= \begin{bmatrix} {}^0\mathbf{x}_0 & {}^0\mathbf{y}_0 & {}^0\mathbf{z}_0 \end{bmatrix} \begin{bmatrix} {}^0x_{1x} & {}^0y_{1x} & {}^0z_{1x} \\ {}^0x_{1y} & {}^0y_{1y} & {}^0z_{1y} \\ {}^0x_{1z} & {}^0y_{1z} & {}^0z_{1z} \end{bmatrix} = \begin{bmatrix} {}^0x_{1x} & {}^0y_{1x} & {}^0z_{1x} \\ {}^0x_{1y} & {}^0y_{1y} & {}^0z_{1y} \\ {}^0x_{1z} & {}^0y_{1z} & {}^0z_{1z} \end{bmatrix} \\ &\doteq {}^0\mathbf{R}_1 \end{aligned} \tag{2.2}$$

where ${}^0\mathbf{R}_1$ is the *rotation matrix* from coordinate axes 1 to coordinate axes 0. We call this the coordinate-axis form of a rotation matrix. It is a unique representation of the orientation of coordinate system 1 with respect to coordinate system 0, as its columns are each of the three axes of coordinate system 1 expressed with respect to coordinate system 0. Because the axes of coordinate system 1 make up the columns of ${}^0\mathbf{R}_1$, it is clear that rotation matrices are orthogonal matrices: their columns are mutually orthogonal and have unit length. In general, ${}^i\mathbf{R}_j$ will denote the rotation matrix from coordinate axes j to i.

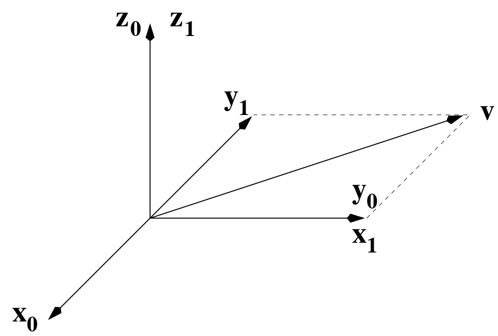


Figure 2.1: Relation of frame 1 to frame 0, where $\mathbf{x}_1 = \mathbf{y}_0$, $\mathbf{y}_1 = -\mathbf{x}_0$, and $\mathbf{z}_1 = \mathbf{z}_0$. The vector $\mathbf{v}$ can be expressed in either frame.

Example 2.5: In Figure 2.1, the axes of frame 1 have a simple relation to the axes of frame 0. Therefore the rotation matrix can simply be written by inspection:

$$ {}^0\mathbf{R}_1 = \left[\ {}^0\mathbf{x}_1 \quad {}^0\mathbf{y}_1 \quad {}^0\mathbf{z}_1\ \right] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

2.1.2 Direction-Cosine Form of a Rotation Matrix

Another view about the nature of a rotation matrix is obtained by taking the dot product of each equation in (2.1) with each of ${}^0\mathbf{x}_0, {}^0\mathbf{y}_0, {}^0\mathbf{z}_0$. For example,

$$ \begin{aligned} {}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_0 &= \left({}^0x_{1x}\,{}^0\mathbf{x}_0 + {}^0x_{1y}\,{}^0\mathbf{y}_0 + {}^0x_{1z}\,{}^0\mathbf{z}_0\right) \cdot {}^0\mathbf{x}_0 \\ &= {}^0x_{1x}\left({}^0\mathbf{x}_0 \cdot {}^0\mathbf{x}_0\right) + {}^0x_{1y}\left({}^0\mathbf{y}_0 \cdot {}^0\mathbf{x}_0\right) + {}^0x_{1z}\left({}^0\mathbf{z}_0 \cdot {}^0\mathbf{x}_0\right) \\ &= {}^0x_{1x} \end{aligned} $$

since ${}^0\mathbf{x}_0 \cdot {}^0\mathbf{x}_0 = 1$ due to all axes being unit length, and since ${}^0\mathbf{y}_0 \cdot {}^0\mathbf{x}_0 = 0$ and ${}^0\mathbf{z}_0 \cdot {}^0\mathbf{x}_0 = 0$ because of orthogonality. The result of this process is that

$$ {}^0\mathbf{R}_1 \doteq \begin{bmatrix} {}^0x_{1x} & {}^0y_{1x} & {}^0z_{1x} \\ {}^0x_{1y} & {}^0y_{1y} & {}^0z_{1y} \\ {}^0x_{1z} & {}^0y_{1z} & {}^0z_{1z} \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_0 & {}^0\mathbf{y}_1 \cdot {}^0\mathbf{x}_0 & {}^0\mathbf{z}_1 \cdot {}^0\mathbf{x}_0 \\ {}^0\mathbf{x}_1 \cdot {}^0\mathbf{y}_0 & {}^0\mathbf{y}_1 \cdot {}^0\mathbf{y}_0 & {}^0\mathbf{z}_1 \cdot {}^0\mathbf{y}_0 \\ {}^0\mathbf{x}_1 \cdot {}^0\mathbf{z}_0 & {}^0\mathbf{y}_1 \cdot {}^0\mathbf{z}_0 & {}^0\mathbf{z}_1 \cdot {}^0\mathbf{z}_0 \end{bmatrix} \tag{2.3} $$

Suppose α_{11} is the angle between $\mathbf{x}_0$ and $\mathbf{x}_1$ (Figure 2.2), where the 11 subscript on α_{11} simply indicates the row and column of the matrix entry that we are referring to. Then

$$ {}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_0 = ||{}^0\mathbf{x}_1||\ ||{}^0\mathbf{x}_0|| \cos\alpha_{11} = \cos\alpha_{11} = {}^0x_{1x} $$

since the axes all have unit length. Hence ${}^0x_{1x}$ is the *direction cosine* between $\mathbf{x}_1$ and $\mathbf{x}_0$.

Similarly, every other dot product and associated coordinate in (2.3) represents a direction cosine between corresponding axes. For example, ${}^0x_{1y}$ is the direction cosine between $\mathbf{x}_1$ and $\mathbf{y}_0$, and ${}^0y_{1x}$ is the direction cosine between $\mathbf{y}_1$ and $\mathbf{x}_0$. Therefore (2.3) is the *direction-cosine form* of a rotation matrix, and represents another important view of what a rotation matrix is. The relation to the coordinate-axes form is that the direction cosines are the components of coordinate axes 1. The two forms represent the same rotation matrix, it's the interpretation that differs.

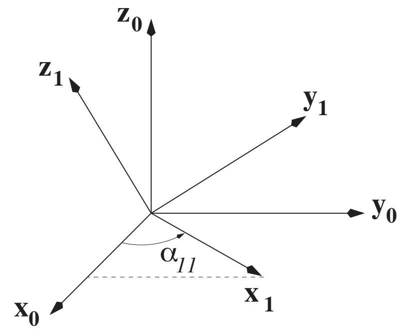


Figure 2.2: Relation of axes 1 to axes 0. α_{11} is the direction cosine between $\mathbf{x}_0$ and $\mathbf{x}_1$. The dotted line indicates the projection of $\mathbf{x}_1$ onto $\mathbf{x}_0$, which is $\cos\alpha_{11}$.

2.1.3 Coordinate Transformation of a Vector

Suppose a particular vector $\mathbf{v}$ is expressed with respect to frame 1:

$$ {}^1\mathbf{v} = {}^1v_x\,{}^1\mathbf{x}_1 + {}^1v_y\,{}^1\mathbf{y}_1 + {}^1v_z\,{}^1\mathbf{z}_1 = \begin{bmatrix} {}^1v_x \\ {}^1v_y \\ {}^1v_z \end{bmatrix} \tag{2.4} $$

What is the relation between ${}^1\mathbf{v}$ and ${}^0\mathbf{v}$? That is, how do we express the same vector $\mathbf{v}$, but now with respect to frame 0? The coordinates of $\mathbf{v}$ along the $\mathbf{x}_1$, $\mathbf{y}_1$, and $\mathbf{z}_1$ axes haven't changed, but we now express those axes with respect to frame 0:

$$ \begin{aligned} {}^0\mathbf{v} &= {}^1v_x\,{}^0\mathbf{x}_1 + {}^1v_y\,{}^0\mathbf{y}_1 + {}^1v_z\,{}^0\mathbf{z}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix} \begin{bmatrix} {}^1v_x \\ {}^1v_y \\ {}^1v_z \end{bmatrix} \\ &= {}^0\mathbf{R}_1\,{}^1\mathbf{v} \end{aligned} \tag{2.5} $$

The rotation matrix ${}^0\mathbf{R}_1$ converts a representation of a vector $\mathbf{v}$ in frame 1 to a representation in frame 0: ${}^0\mathbf{v} = {}^0\mathbf{R}_1{}^1\mathbf{v}$.

Note how the superscript on ${}^1\mathbf{v}$ aligns with, and cancels, the subscript on ${}^0\mathbf{R}_1$ in this notation. Recall that for vectors the (optional) subscript is simply part of the name of the vector. This is not true of rotation matrices. For a rotation matrix, the subscript and superscript are collectively used to indicate its function.

Example 2.6: In Figure 2.1, ${}^1\mathbf{v} = [1\ 1\ 0]^T$. Then

$$ {}^0\mathbf{v} = {}^0\mathbf{R}_1\,{}^1\mathbf{v} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. $$

2.1.4 Composition of Rotation Matrices

Consider three sets of coordinate axes 0, 1, and 2, and suppose $\mathbf{v}$ is any vector. Then

$$
\begin{aligned}
{}^1\mathbf{v} &= {}^1\mathbf{R}_2{}^2\mathbf{v} \\
{}^0\mathbf{v} &= {}^0\mathbf{R}_1{}^1\mathbf{v} = {}^0\mathbf{R}_1{}^1\mathbf{R}_2{}^2\mathbf{v} = {}^0\mathbf{R}_2{}^2\mathbf{v}
\end{aligned}
$$

where ${}^0\mathbf{R}_2 = {}^0\mathbf{R}_1{}^1\mathbf{R}_2$. Note how the superscript on ${}^1\mathbf{R}_2$ aligns with, and cancels, the subscript on ${}^0\mathbf{R}_1$ in this notation.

2.1.5 Inverse of a Rotation Matrix

Similar to (2.1), we can express coordinate axes 0 relative to coordinate axes 1:

$$
\begin{aligned}
{}^1\mathbf{x}_0 &= {}^1x_{0x}\,{}^1\mathbf{x}_1 + {}^1x_{0y}\,{}^1\mathbf{y}_1 + {}^1x_{0z}\,{}^1\mathbf{z}_1 \\
{}^1\mathbf{y}_0 &= {}^1y_{0x}\,{}^1\mathbf{x}_1 + {}^1y_{0y}\,{}^1\mathbf{y}_1 + {}^1y_{0z}\,{}^1\mathbf{z}_1 \\
{}^1\mathbf{z}_0 &= {}^1z_{0x}\,{}^1\mathbf{x}_1 + {}^1z_{0y}\,{}^1\mathbf{y}_1 + {}^1z_{0z}\,{}^1\mathbf{z}_1
\end{aligned}
\tag{2.6}
$$

and write the direction-cosine form of the rotational transformation ${}^1\mathbf{R}_0$ similar to (2.3):

$$
{}^1\mathbf{R}_0 \doteq \begin{bmatrix} {}^1x_{0x} & {}^1y_{0x} & {}^1z_{0x} \\ {}^1x_{0y} & {}^1y_{0y} & {}^1z_{0y} \\ {}^1x_{0z} & {}^1y_{0z} & {}^1z_{0z} \end{bmatrix} = \begin{bmatrix} {}^1\mathbf{x}_0 \cdot {}^1\mathbf{x}_1 & {}^1\mathbf{y}_0 \cdot {}^1\mathbf{x}_1 & {}^1\mathbf{z}_0 \cdot {}^1\mathbf{x}_1 \\ {}^1\mathbf{x}_0 \cdot {}^1\mathbf{y}_1 & {}^1\mathbf{y}_0 \cdot {}^1\mathbf{y}_1 & {}^1\mathbf{z}_0 \cdot {}^1\mathbf{y}_1 \\ {}^1\mathbf{x}_0 \cdot {}^1\mathbf{z}_1 & {}^1\mathbf{y}_0 \cdot {}^1\mathbf{z}_1 & {}^1\mathbf{z}_0 \cdot {}^1\mathbf{z}_1 \end{bmatrix}
\tag{2.7}
$$

Compare the $1, 1$ elements of (2.7) and (2.3). Since the value of the dot product does not depend on the basis, ${}^1\mathbf{x}_0 \cdot {}^1\mathbf{x}_1 = {}^0\mathbf{x}_1 \cdot {}^0\mathbf{x}_0$. Similar comparisons with the other elements shows that

$$
{}^1\mathbf{R}_0 = {}^0\mathbf{R}_1^T
\tag{2.8}
$$

Apply ${}^1\mathbf{R}_0$ to convert a vector $\mathbf{v}$ represented in frame 0 to a representation in frame 1, and then apply (2.5):

$$
{}^1\mathbf{v} = {}^1\mathbf{R}_0{}^0\mathbf{v} = {}^1\mathbf{R}_0{}^0\mathbf{R}_1{}^1\mathbf{v}
\tag{2.9}
$$

This can only be true for an arbitrary vector ${}^1\mathbf{v}$ if

$$
{}^1\mathbf{R}_0{}^0\mathbf{R}_1 = \mathbf{I}.
\tag{2.10}
$$

In view of (2.8),

$$
{}^1\mathbf{R}_0 = {}^0\mathbf{R}_1^{-1} = {}^0\mathbf{R}_1^T
\tag{2.11}
$$

This is an important result: the inverse of a rotation matrix is its transpose. This is actually true of any orthogonal matrix, of which a rotation matrix is one.

The inverse of the composition of rotation matrices is related to the transpose of the composition of matrices:

$$
{}^0\mathbf{R}_2^{-1} = {}^0\mathbf{R}_2^T = \left({}^0\mathbf{R}_1{}^1\mathbf{R}_2\right)^T = \left({}^1\mathbf{R}_2^T\right)\left({}^0\mathbf{R}_1^T\right) = \left({}^1\mathbf{R}_2^{-1}\right)\left({}^0\mathbf{R}_1^{-1}\right) = {}^2\mathbf{R}_1{}^1\mathbf{R}_0 = {}^2\mathbf{R}_0
\tag{2.12}
$$

2.2 Rotation

2.2.1 Operators

Until this point, we have considered the rotation matrix ${}^0\mathbf{R}_1$ as describing the static coordinate system 1, and more specifically the axes of coordinate system 1 with respect to the static coordinate system 0. We also considered ${}^0\mathbf{R}_1$ as implementing the coordinate transformation of some static vector $\mathbf{v}$, originally represented as ${}^1\mathbf{v}$ (i.e., with respect to frame 1), to a representation ${}^0\mathbf{v}$ (i.e., with respect to frame 0).

We can also conceptualize coordinate system 1 as having been initially aligned with coordinate system 0, and then having been rotated to achieve its current orientation. We can conceptualize orientations in this way, whether or not such a rotation ever actually occurred.

When conceiving of rotation matrices in this way, the coordinate transformation ${}^0\mathbf{v} = {}^0\mathbf{R}_1{}^1\mathbf{v}$ also takes on a new meaning. We can now think of the vector $\mathbf{v}$ having been rigidly embedded in frame 1 (which is why we call it ${}^1\mathbf{v}$), and then rotated with the rest of the frame into its current orientation with respect to frame 0, which we call ${}^0\mathbf{v}$. This is referred to as the rotation matrix acting as an operator. A simple way to understand this is to picture the axes of frame 1 themselves; for example, consider ${}^1\mathbf{v} = {}^1\mathbf{x}_1 = [1\ 0\ 0]^T$ being rotated by ${}^0\mathbf{R}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 \end{bmatrix}$, resulting in ${}^0\mathbf{v} = {}^0\mathbf{R}_1{}^1\mathbf{v}_1 = {}^0\mathbf{x}_1$.

To highlight this distinct way of conceptualizing rotating matrices, we will use a distinct notation $\mathbf{R}(\mathbf{k}, \theta)$, where $\mathbf{k}$ (a unit vector) is some axis of rotation and θ is the amount of rotation about that axis; the direction of $\mathbf{k}$ defines the positive sign convention of θ via the right-hand rule. Note that we have lost the subscript and superscript on $\mathbf{R}$, although it still represents the orientation of some frame j with respect to some other frame i.

2.2.2 Rotation about Principal Axes

There are three elemental rotation matrices that result from rotation operations about the $\mathbf{x}$, $\mathbf{y}$, or $\mathbf{z}$ axes, with the two frames initially aligned. For example, rotation about the $\mathbf{z}$ axis is shown in Figure 2.3(A), where $\mathbf{z}_1 = \mathbf{z}_0$, and frame 1 has been rotated by the angle θ about the common $\mathbf{z}$ axis. Then

$$\begin{aligned} \mathbf{x}_1 &= (\cos\theta)\mathbf{x}_0 + (\sin\theta)\mathbf{y}_0 \\ \mathbf{y}_1 &= -(\sin\theta)\mathbf{x}_0 + (\cos\theta)\mathbf{y}_0 \\ \mathbf{z}_1 &= \mathbf{z}_0 \end{aligned} \tag{2.13}$$

and therefore from (2.3)

$${}^0\mathbf{R}_1 = \begin{bmatrix} \mathrm{c}\theta & -\mathrm{s}\theta & 0 \\ \mathrm{s}\theta & \mathrm{c}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \doteq \mathbf{R}(\mathbf{z}\theta) \tag{2.14}$$

Note that we have adopted a shorthand notation $\mathrm{c}\theta = \cos\theta$ and $\mathrm{s}\theta = \sin\theta$, which we will use going forward. For elemental rotation about the $\mathbf{z}$ axis by some angle θ, the rotation matrix will henceforth be designated as $\mathbf{R}(\mathbf{z}\theta)$. The positive sign convention for θ should be interpreted using the right-hand rule: when rotating about the $\mathbf{z}$ axis, point the thumb of your right hand in the direction of $\mathbf{z}$, and your fingers curl in the direction of positive θ.

Similar expressions derive for $\mathbf{R}(\mathbf{y}\theta)$ and $\mathbf{R}(\mathbf{x}\theta)$, as depicted in Figures 2.3(B) and 2.3(C), respectively,

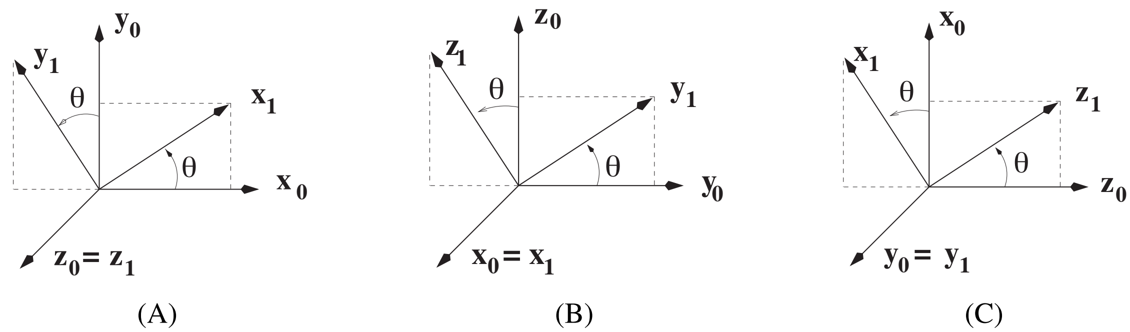


Figure 2.3: Rotation by θ about the (A) $\mathbf{z}$, (B) $\mathbf{x}$, and (C) $\mathbf{y}$ axes.

are left as an exercise.

$$\mathbf{R}(\mathbf{x}\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \tag{2.15}$$

$$\mathbf{R}(\mathbf{y}\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \tag{2.16}$$

Example 2.7: Frame 0 is located at the base of the robot in Figure 2.4. Frame 1 is located at the corner of a square table that is 2 m on each side and 1 m off the ground; it is displaced by 1.5 m along $\mathbf{y}_0$, and rotated by $\pi/2$ about the $\mathbf{z}$ axis relative to frame 0. Hence

$${}^0\mathbf{R}_1 = \mathbf{R}\left(\frac{\pi}{2}\mathbf{z}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Frame 2 is at the corner of a block, situated in the middle of the table, and rotated by $-\pi/2$ about the $\mathbf{z}$ axis relative to frame 1. Hence

$${}^1\mathbf{R}_2 = \mathbf{R}\left(-\frac{\pi}{2}\mathbf{z}\right) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Frame 3 locates a camera, which sits 3 m directly above the block; its orientation relative to frame 2 cannot be described by a single elemental rotation. However, the rotation matrix ${}^2\mathbf{R}_3$ can be simply written by inspection using the coordinate-axis form (2.2), since $\mathbf{x}_3 = \mathbf{y}_2$, $\mathbf{y}_3 = \mathbf{x}_2$, and $\mathbf{z}_3 = -\mathbf{z}_2$:

$${}^2\mathbf{R}_3 = \left[\ {}^2\mathbf{x}_3 \quad {}^2\mathbf{y}_3 \quad {}^2\mathbf{z}_3\ \right] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Similarly, ${}^0\mathbf{R}_1$ and ${}^0\mathbf{R}_2$ could have been written by inspection, without solving (2.14).

Note that, in this example, we have not needed to concern ourselves with the positions of the coordinate systems (i.e., the positions of their origins) because we are only interested in the relative orientations of the three frames. Considering both orientations and positions is the topic of Chapter 3.

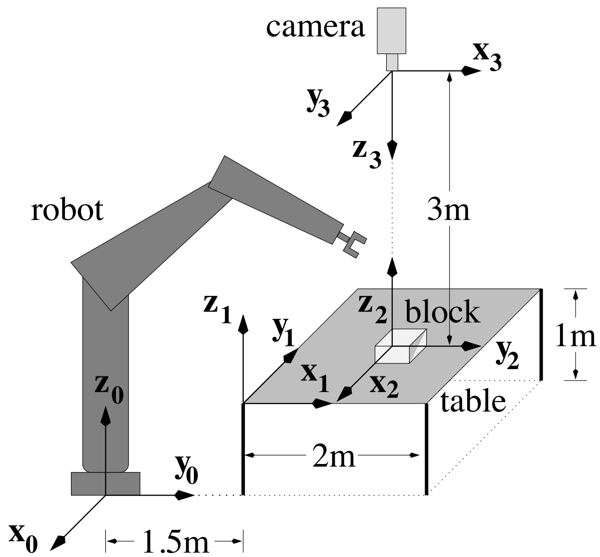


Figure 2.4: Relative locations of a robot, table, block, and camera.

2.2.3 Rotation about Arbitrary Vectors

It is possible to rotate about any arbitrary axis $\mathbf{k}$ (a unit vector) by an angle θ, using the right-hand rule to define the positive direction of θ. That is, we want to compute $\mathbf{R}(\mathbf{k}\theta)$ in general. There are two different, but largely equivalent, ways to formalize such a rotation: the angle-axis method and the matrix-exponential method.

Angle-Axis Method

We want to rotate a vector $\mathbf{v}$ about $\mathbf{k}$ to yield $\mathbf{v}'$, where θ is measured as the angle between projections onto the plane whose normal is $\mathbf{k}$, as depicted in Figure 2.5 (assuming we've selected some $\mathbf{v}$ that is not parallel with $\mathbf{k}$). To proceed, form a right-handed coordinate system as follows.

$\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$:	component of $\mathbf{v}$ $\|$ to $\mathbf{k}$.
$\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$:	component of $\mathbf{v}$ $\perp$ to $\mathbf{k}$.
$\mathbf{k}\times\mathbf{v}$:	component $\perp$ to the other two.

The orthogonality of these three vectors may be simply verified. Note that $\mathbf{k}\times\mathbf{v} = \mathbf{k}\times(\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v}))$. Hence the 2nd and 3rd vectors are equal in length:

$$||\mathbf{k}\times\mathbf{v}|| = ||\mathbf{k}\times(\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v}))|| = ||\mathbf{k}||\,||\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})||\sin\frac{\pi}{2} = ||\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})|| \tag{2.17}$$

Next, rotate $\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$ by θ about $\mathbf{k}$ to give a vector $\mathbf{r}$ in the plane spanned by $\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$ and $\mathbf{k}\times\mathbf{v}$. If we knew $\mathbf{r}$, then we would have:

$$\mathbf{v}' = \mathbf{r}+\mathbf{k}(\mathbf{k}\cdot\mathbf{v}) \tag{2.18}$$

Note that $||\mathbf{r}|| = ||\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})|| = ||\mathbf{k}\times\mathbf{v}||$, since a rotational operator does not change a vector's length. Since $\mathbf{r}$ lies in the plane spanned by vectors $\mathbf{v}-\mathbf{k}(\mathbf{k}\cdot\mathbf{v})$ and $\mathbf{k}\times\mathbf{v}$, we can write $\mathbf{r}$ as a linear sum of these

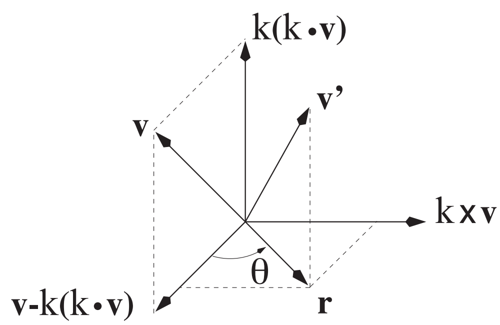


Figure 2.5: Vector $\mathbf{v}$ is rotated by θ about $\mathbf{k}$ to yield $\mathbf{v}'$.

two vectors:

$$\mathbf{r} = (\mathbf{v} - \mathbf{k}(\mathbf{k}\cdot\mathbf{v}))\,\mathrm{c}\theta + (\mathbf{k}\times\mathbf{v})\,\mathrm{s}\theta \tag{2.19}$$

Substituting (2.19) into (2.18),

$$\begin{aligned}\mathbf{v}' &= (\mathbf{v} - \mathbf{k}(\mathbf{k}\cdot\mathbf{v}))\,\mathrm{c}\theta + (\mathbf{k}\times\mathbf{v})\,\mathrm{s}\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})\\ &= \mathbf{v}\mathrm{c}\theta + \mathbf{k}\times\mathbf{v}\mathrm{s}\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\mathrm{c}\theta)\end{aligned} \tag{2.20}$$

To rewrite the result (2.20) as a matrix/vector operation, we will employ the outer product of two vectors:

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{v}) = (\mathbf{k}\mathbf{k}^T)\mathbf{v} \tag{2.21}$$

where the outer product $\mathbf{k}\mathbf{k}^T$ is a 3-by-3 matrix. We will also employ the matrix representation $\mathbf{S}(\mathbf{k})\mathbf{v}$ of the cross product $\mathbf{k}\times\mathbf{v}$:

$$\mathbf{k}\times\mathbf{v} = \begin{vmatrix} k_1 & k_2 & k_3 \\ v_1 & v_2 & v_3 \\ {}^j\mathbf{x}_j & {}^j\mathbf{y}_j & {}^j\mathbf{z}_j \end{vmatrix} = \begin{bmatrix} k_2v_3 - k_3v_2 \\ k_3v_1 - k_1v_3 \\ k_1v_2 - k_2v_1 \end{bmatrix} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{S}(\mathbf{k})\mathbf{v} \tag{2.22}$$

where the skew-symmetric matrix $\mathbf{S}(\mathbf{k})$ represents the cross product by $\mathbf{k}$:

$$\mathbf{S}(\mathbf{k}) = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \tag{2.23}$$

Substituting (2.21) and (2.22) into (2.20),

$$\begin{aligned}\mathbf{v}' &= (\mathbf{I}\mathrm{c}\theta)\mathbf{v} + \mathbf{S}(\mathbf{k})\mathbf{v}\mathrm{s}\theta + (\mathbf{k}\mathbf{k}^T)\mathbf{v}(1-\mathrm{c}\theta)\\ &= \left(\mathbf{I}\mathrm{c}\theta + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{k}\mathbf{k}^T(1-\mathrm{c}\theta)\right)\mathbf{v}\\ &= \mathbf{R}(\mathbf{k}\theta)\mathbf{v}\end{aligned}$$

where $\mathbf{I}$ is the 3-by-3 identity matrix. The equivalent matrix operation, which is known as Rodrigues' formula, is:

$$\begin{aligned}\mathbf{R}(\mathbf{k}\theta) &= \mathbf{I}\mathrm{c}\theta + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{k}\mathbf{k}^T(1-\mathrm{c}\theta) \\ &= \begin{bmatrix} k_1^2\mathrm{v}\theta + \mathrm{c}\theta & k_1k_2\mathrm{v}\theta - k_3\mathrm{s}\theta & k_1k_3\mathrm{v}\theta + k_2\mathrm{s}\theta \\ k_1k_2\mathrm{v}\theta + k_3\mathrm{s}\theta & k_2^2\mathrm{v}\theta + \mathrm{c}\theta & k_2k_3\mathrm{v}\theta - k_1\mathrm{s}\theta \\ k_1k_3\mathrm{v}\theta - k_2\mathrm{s}\theta & k_2k_3\mathrm{v}\theta + k_1\mathrm{s}\theta & k_3^2\mathrm{v}\theta + \mathrm{c}\theta \end{bmatrix}\end{aligned} \tag{2.24}$$

where the versine of θ is abbreviated $\mathrm{v}\theta = 1 - \mathrm{c}\theta$.

Matrix-Exponential Method

Analogous to the series expansion of the exponential e^a with scalar a, the matrix exponential of a square matrix $\mathbf{A}$ is defined as:

$$e^{\mathbf{A}} = \sum_{i=1}^{\infty} \frac{1}{i!}\mathbf{A}^i = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{4!}\mathbf{A}^4 + \cdots \tag{2.25}$$

Consider $\mathbf{A} = \mathbf{S}(\mathbf{k}\theta) = \mathbf{S}(\mathbf{k})\theta$ (i.e., a skew-symmetrix matrix), where $\mathbf{k}$ is still our unit-vector axis of rotation, and θ (which must be in units rad) is still the amount of rotation, as in Section 2.2.3. Then

$$e^{\mathbf{S}(\mathbf{k}\theta)} = e^{\mathbf{S}(\mathbf{k})\theta} = \mathbf{I} + \theta\mathbf{S}(\mathbf{k}) + \frac{\theta^2}{2!}\mathbf{S}(\mathbf{k})^2 + \frac{\theta^3}{3!}\mathbf{S}(\mathbf{k})^3 + \frac{\theta^4}{4!}\mathbf{S}(\mathbf{k})^4 + \cdots \tag{2.26}$$

This infinite series can be simplified by the relation $\mathbf{S}(\mathbf{k})^3 = -\mathbf{S}(\mathbf{k})$, which simplifies the odd power terms in the series; for example, $\mathbf{S}(\mathbf{k})^5 = \mathbf{S}(\mathbf{k})$, $\mathbf{S}(\mathbf{k})^7 = -\mathbf{S}(\mathbf{k})$, $\mathbf{S}(\mathbf{k})^9 = \mathbf{S}(\mathbf{k})$, etc., showing that the minus sign alternates. For the even powers, $\mathbf{S}(\mathbf{k})^4 = -\mathbf{S}(\mathbf{k})^2$, $\mathbf{S}(\mathbf{k})^6 = \mathbf{S}(\mathbf{k})^2$, etc., with the minus sign again alternating. Substituting both results into the series expansion and collecting terms yields

$$e^{\mathbf{S}(\mathbf{k}\theta)} = e^{\mathbf{S}(\mathbf{k})\theta} = \mathbf{I} + \mathbf{S}(\mathbf{k})\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) + \mathbf{S}(\mathbf{k})^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right) \tag{2.27}$$

The two series in parentheses are the series expansions for sine and cosine:

$$\mathrm{s}\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{2.28}$$

$$\mathrm{c}\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{2.29}$$

Thus,

$$e^{\mathbf{S}(\mathbf{k}\theta)} = e^{\mathbf{S}(\mathbf{k})\theta} = \mathbf{I} + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{S}(\mathbf{k})^2(1-\mathrm{c}\theta) \tag{2.30}$$

This is seen to be equivalent to the angle-axis matrix formula (2.24) by substituting that $\mathbf{S}(\mathbf{k})^2 = \mathbf{k}\mathbf{k}^T - \mathbf{I}$:

$$e^{\mathbf{S}(\mathbf{k}\theta)} = e^{\mathbf{S}(\mathbf{k})\theta} = \mathbf{I} + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + (\mathbf{k}\mathbf{k}^T - \mathbf{I})(1-\mathrm{c}\theta) \tag{2.31}$$

$$= \mathbf{I}\mathrm{c}\theta + \mathbf{S}(\mathbf{k})\mathrm{s}\theta + \mathbf{k}\mathbf{k}^T(1-\mathrm{c}\theta) \tag{2.32}$$

Thus

$$\mathbf{R}(\mathbf{k}\theta) = e^{\mathbf{S}(\mathbf{k}\theta)} = e^{\mathbf{S}(\mathbf{k})\theta} \tag{2.33}$$

It's easy to show that $\mathbf{R}(\mathbf{k}\theta)\mathbf{k} = \mathbf{k}$, i.e., the vector representing the axis of rotation (or, more generally, any vector parallel to the axis of rotation) is unaffected by the rotation.

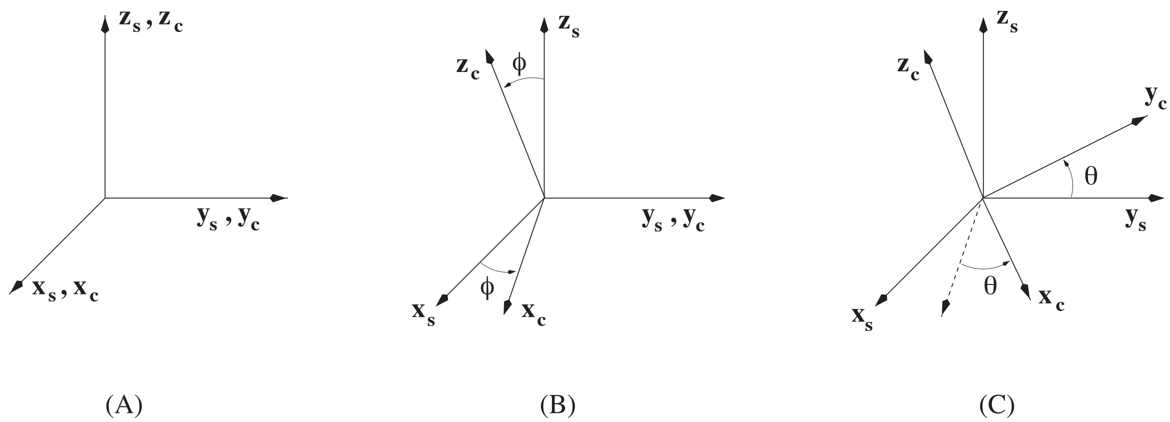


Figure 2.6: Composition of elementary rotations as a coordinate transformation. (A) Initially, static axes s and current axes c are overlapping. (B) A rotation of ϕ of the current axes about the $\mathbf{y}_c$ axis. (C) A rotation of θ of the current axes about the $\mathbf{z}_c$ axis, resulting in the final orientation.

2.2.4 Composition of Rotations

Consider three sets of coordinate axes 0, 1, and 2, where ${}^0\mathbf{R}_2 = {}^0\mathbf{R}_1{}^1\mathbf{R}_2$. We may also want to consider the orientation of one frame with respect to another as a composition of elementary rotations, e.g., ${}^0\mathbf{R}_2 = \mathbf{R}(\mathbf{y}\phi)\mathbf{R}(\mathbf{z}\theta)$. We can conceive of these rotations as occurring in two different ways: as coordinate transformations or as operators.

As coordinate transformations, we interpret these rotations from left to right, with rotations occurring about the current frame (Figure 2.6):

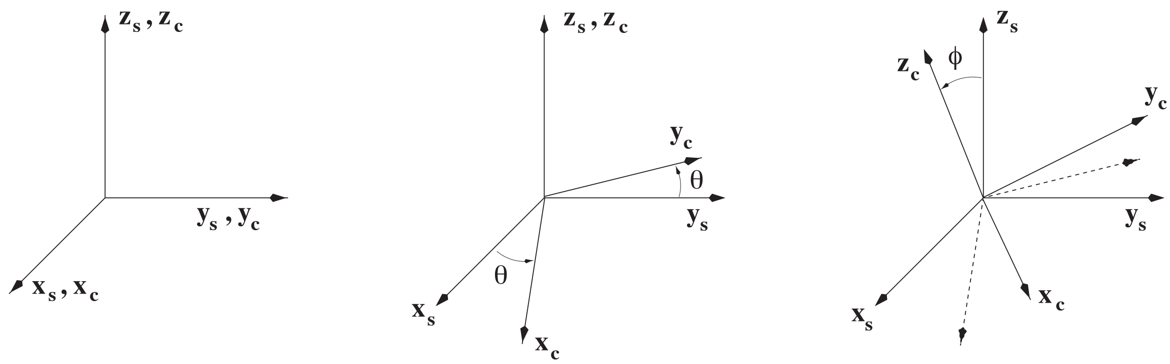
1. Assume initially the static coordinate frame s and the current coordinate frame c coincide.
2. Rotate frame c about $\mathbf{y}_c$ by ϕ.
3. Rotate frame c about $\mathbf{z}_c$ by θ.

Here, frame 0 is the “static” frame, and frame 2 is the final orientation of the “current” frame. When thinking in terms of coordinate transformations, the intermediate frame is meaningful, i.e., the “current” frame in Figure 2.6(B) is frame 1.

As operators, we interpret these rotations from right to left, with rotations occurring about the static frame (Figure 2.7):

1. Assume initially the static coordinate frame s and the current coordinate frame c coincide.
2. Rotate frame c about $\mathbf{z}_s$ by θ.
3. Rotate frame c about $\mathbf{y}_s$ by ϕ.

Again, frame 0 is the “static” frame, and frame 2 is the final orientation of the “current” frame. However, when thinking in terms of operations, the intermediate frame is not meaningful, i.e., the “current” frame in Figure 2.7(B) is not frame 1. There are often cases in which we are not interested in the intermediate frame(s), just the final frame due to the composition of rotations.



(A) (B) (C)

Figure 2.7: Composition of elementary rotations as a operation. (A) Initially, static axes s and current axes c are overlapping. (B) A rotation of θ of the current axes about the $\mathbf{z}_s$ axis. (C) A rotation of ϕ of the current axes about the $\mathbf{y}_s$ axis, resulting in the final orientation.

Note that, in general, 3D rotations do not commute, meaning that the order of rotations matters. Therefore

$$\mathbf{R}(\mathbf{y}\phi)\mathbf{R}(\mathbf{z}\theta) \neq \mathbf{R}(\mathbf{z}\theta)\mathbf{R}(\mathbf{y}\phi)$$

Differential rotations, however, do commute. That is, in the limit as both $\phi \to 0$ and $\theta \to 0$, the order of the rotations becomes less and less important.

2.2.5 No Rotation and Complete Rotation

Regardless of the axis of rotation, no rotation (i.e., $\theta = 0$) or complete rotations (i.e., $\theta = 2\pi i$ for any integer i) always results in $\mathbf{R}(\mathbf{k}\theta) = \mathbf{I}$, the 3-by-3 identity matrix.

2.3 Representations of Orientation

Although a rotation matrix $\mathbf{R} = \{r_{ij}\}$ is a unique representation of orientation, with 9 numbers it is a redundant representation, because elements are related. Consider the columns $\mathbf{r}_i$ of $\mathbf{R}$:

$$\mathbf{r}_i = \begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} \qquad \text{for } i = 1, 2, 3 \tag{2.34}$$

Because the columns (and rows) of a rotation matrix are orthonormal,

$$\mathbf{r}_i^T \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This comprises 6 constraint equations, hence only $9 - 6 = 3$ elements of $\mathbf{R}$ are independent.

There are a variety of 3-number orientation representations, such as the angle-axis representation and Euler angles, which we will detail below. Yet each 3-number representation has some distinctive drawback. As a result, we will also examine the 4-number representation called a quaternion that addresses deficiencies in the 3-number representations.

2.3.1 Angle-Axis Representation

Any rotation matrix can be represented as some rotation θ about some axis $\mathbf{k}$. We show this by construction, given the previous expression (2.24) for $\mathbf{R}(\mathbf{k}\theta)$. Let $\mathbf{R} = \{r_{ij}\}$ be an arbitrary 3D rotation matrix with elements r_{ij}. Equate $\mathbf{R} = \mathbf{R}(\mathbf{k}\theta)$ to solve for θ and $\mathbf{k}$. If $\mathbf{R} = \mathbf{I}$ (i.e., no rotation), then $\theta = 0$ and $\mathbf{k}$ is undefined, so let's assume $\mathbf{R} \neq \mathbf{I}$.

First, solve for θ by taking the trace of $\mathbf{R}$:

$$\begin{aligned}
\mathrm{Tr}(\mathbf{R}) &= r_{11} + r_{22} + r_{33} \\
&= k_1^2 \mathrm{v}\theta + \mathrm{c}\theta + k_2^2 \mathrm{v}\theta + \mathrm{c}\theta + k_3^2 \mathrm{v}\theta + \mathrm{c}\theta \\
&= (k_1^2 + k_2^2 + k_3^2)\mathrm{v}\theta + 3\mathrm{c}\theta \\
&= 1 + 2\mathrm{c}\theta
\end{aligned} \tag{2.35}$$

$$\mathrm{c}\theta = \frac{\mathrm{Tr}(\mathbf{R}) - 1}{2} \tag{2.36}$$

An independent estimate of $\mathrm{s}\theta$ is obtained by differencing diagonally opposite elements of $\mathbf{R}$:

$$\begin{aligned}
r_{32} - r_{23} &= k_2 k_3 \mathrm{v}\theta + k_1 \mathrm{s}\theta - (k_2 k_3 \mathrm{v}\theta - k_1 \mathrm{s}\theta) \\
&= 2k_1 \mathrm{s}\theta \\
r_{13} - r_{31} &= k_1 k_3 \mathrm{v}\theta + k_2 \mathrm{s}\theta - (k_1 k_3 \mathrm{v}\theta - k_2 \mathrm{s}\theta) \\
&= 2k_2 \mathrm{s}\theta \\
r_{21} - r_{12} &= k_1 k_2 \mathrm{v}\theta + k_3 \mathrm{s}\theta - (k_1 k_2 \mathrm{v}\theta - k_3 \mathrm{s}\theta) \\
&= 2k_3 \mathrm{s}\theta
\end{aligned} \tag{2.37}$$

Hence

$$4(k_1^2 + k_2^2 + k_3^2)\mathrm{s}\theta^2 = (r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2$$

$$\mathrm{s}\theta = \pm\frac{1}{2}\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2} \tag{2.38}$$

We may now use the 4-quadrant arctangent function $\theta = \mathtt{atan2}(\mathrm{s}\theta, \mathrm{c}\theta)$, using independent estimates of $\mathrm{c}\theta$ (2.36) and $\mathrm{s}\theta$ (2.38). Two solutions exist for θ, corresponding to the two solutions for $\mathrm{s}\theta$ in (2.38).

Why don't we apply the arccos in (2.36) or the arcsin in (2.38) to estimate θ? Besides the issue of uniqueness, which the 4-quadrant arctan gives, there is an issue of poor numerical robustness. The derivative of $\cos\theta$ is, of course, $-\sin\theta$, and when $\theta = 0$ or π the slope of the cosine curve is 0. This means that changes in angle have a small effect on the value of the cosine. If there is error in the knowledge of robot position, which is quite likely since it is often derived from sensor data, then the angle between these vectors will be derived quite inaccurately. Similarly, for arcsin the angle is inaccurately derived near $\theta = \pm\pi/2$. The derivative of $\tan\theta$ is $\sec^2\theta$, and the slope of the tangent curve is never less than 1. Thus the arctangent has good numerical sensitivity everywhere.

We can now solve for $\mathbf{k}$ from (2.37):

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2\mathrm{s}\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{2.39}$$

This equation is clearly undefined when $\mathrm{s}\theta = 0$ (e.g., $\theta = \pm\pi$). However, the equation is correct in the limit as $\mathrm{s}\theta \to 0$, so in the event that $\mathrm{s}\theta = 0$, we can rotate the original $\mathbf{R}$ by a differential amount—which, in practice, is some tiny amount—about any axis (i.e., $\mathbf{R}(\mathbf{k}\epsilon)$), and then compute $\mathbf{k}$ for this new $\mathbf{R}' = \mathbf{R}(\mathbf{k}\epsilon)\mathbf{R}$; the resulting new θ should be negligibly different from the previous calculation that led to $\mathrm{s}\theta = 0$.

Note that

$$\mathbf{R}(\mathbf{k}\theta) = \mathbf{R}((-\mathbf{k})(-\theta)) \tag{2.40}$$

The rotation matrix $\mathbf{R}((-\mathbf{k})(-\theta))$ would have resulted if we had chosen the other of the two solutions for $\mathrm{s}\theta$ in (2.38). Also note that

$$\mathbf{R}(\mathbf{k}(-\theta)) = \mathbf{R}(-\mathbf{k}\theta) = \mathbf{R}(\mathbf{k}\theta)^T = \mathbf{R}(\mathbf{k}\theta)^{-1}. \tag{2.41}$$

One potential problem with the angle-axis representation is that can return discontinuous solutions for the angle θ and axis $\mathbf{k}$ even when the rotation matrix $\mathbf{R}$ is changed continuously.

Example 2.8: Consider the following rotation matrix, where the relative orientation of frame 1 to frame 0 is shown in Figure 2.8.

$${}^0\mathbf{R}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

From (2.36) and (2.38), using the positive value for $\mathrm{s}\theta$,

$$\mathrm{c}\theta = -1/2, \quad \mathrm{s}\theta = \sqrt{3}/2, \quad \theta = \text{atan2}(\sqrt{3}/2, -1/2) = 2\pi/3 = 120^\circ$$

Hence, from (2.39),

$$\mathbf{k} = \frac{1}{2\left(\frac{\sqrt{3}}{2}\right)} \begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

It's easy to verify that $\mathbf{k}$ is a unit vector. Thus, if we imagine a coordinate system originally aligned with frame 0, and we were to rotate it by 120° about axis $\mathbf{k}$, it would result in coordinate system 1 in Figure 2.8. If we would have selected the negative value for $\mathrm{s}\theta$, $\mathrm{s}\theta = -\sqrt{3}/2$, it would have resulted in $\theta = -2\pi/3 = -120^\circ$, and $\mathbf{k}$ antiparallel with the one calculated above.

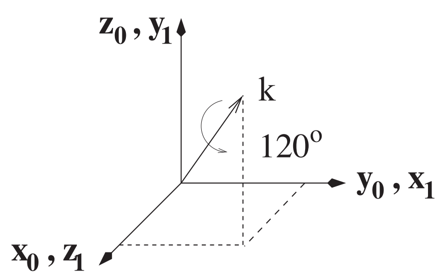


Figure 2.8: Equivalent angle-axis rotation.

2.3.2 Euler Angles

Euler angles are a common, although not particularly good, 3-parameter representation of rotation. A set of Euler angles are any nonredundant set of three successive rotations about (current) principle axes. There are 12 different Euler-angle systems, depending on the axes chosen: XYX, XYZ, XZX, XZY, YXY, YXZ, YZX, YZY, ZXY, ZXZ, ZYX, ZYZ. One of the most common sets is the ZYZ Euler angles (ϕ, θ, ψ):

$$\mathbf{R}(\phi,\theta,\psi) \doteq \mathbf{R}(\mathbf{z}\phi)\mathbf{R}(\mathbf{y}\theta)\mathbf{R}(\mathbf{z}\psi) \tag{2.42}$$

$$= \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi & 0 \\ \mathrm{s}\phi & \mathrm{c}\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & \mathrm{s}\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}\psi & -\mathrm{s}\psi & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi & -\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi & \mathrm{c}\phi\mathrm{s}\theta \\ \mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi & -\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta \\ -\mathrm{s}\theta\mathrm{c}\psi & \mathrm{s}\theta\mathrm{s}\psi & \mathrm{c}\theta \end{bmatrix} \tag{2.43}$$

where Figure 2.9 shows left-to-right evaluation of (2.42) to derive the final coordinate system 3 beginning from frame 0, with the intermediate frames shown as well. Again, the Euler-angle transformation (2.42) is considered a coordinate transformation, hence the left-to-right evaluation. It should be thought of a follows:

- With the current frame initially aligned with frame 0 (i.e., the static frame), rotate about the $\mathbf{z}$ axis of the current frame (which, at this stage, is the same as the $\mathbf{z}$ axis of frame 0) by ϕ. This rotation changes the direction of the current $\mathbf{y}$ axis if $\phi \neq 0$.
- Next, rotate about the current $\mathbf{y}$ axis by θ. This rotation changes the direction of the current $\mathbf{z}$ axis if $\theta \neq 0$.
- Finally, rotate about the current $\mathbf{z}$ axis by ψ.

Given an arbitrary matrix $\mathbf{R} = \{r_{ij}\}$, we can extract the ZYZ Euler angles. To find ϕ, use the 13 and 23 elements of (2.43):

$$r_{13} = \mathrm{c}\phi\mathrm{s}\theta$$
$$r_{23} = \mathrm{s}\phi\mathrm{s}\theta$$

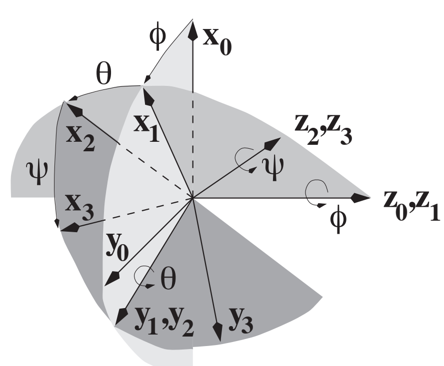


Figure 2.9: ZYZ Euler angles ϕ, θ, ψ.

Hence we get two solutions for ϕ:

$$\tan\phi = \frac{r_{23}}{r_{13}}$$
$$\phi = \tan^{-1}\left(\frac{r_{23}}{r_{13}}\right) \tag{2.44}$$

As will be seen below, given one of these two solutions of ϕ, the other Euler angles will be determined uniquely. This means that there are two possible Euler angle sets corresponding to a given rotation matrix, but these sets are related.

When $r_{13} = r_{23} = 0$, then $\theta = 0$ or $\theta = \pi$ and the solution is ill-defined. Then

$$\mathbf{R}(\phi, 0, \psi) = \mathbf{R}(\mathbf{z}\phi)\ \mathbf{I}\ \mathbf{R}(\mathbf{z}\psi) = \mathbf{R}(\mathbf{z}(\phi + \psi)) \tag{2.45}$$

and we cannot resolve ϕ and ψ uniquely. This is a degeneracy, as ϕ and ψ correspond to the same rotation. We can only determine the sum:

$$\phi + \psi = \text{atan2}(r_{21}, r_{11}) \tag{2.46}$$

One convention is to choose $\phi = 0$.

This degeneracy of the Euler angles is one of the reasons that the use of Euler angles is currently not preferred. Each of the 12 different Euler-angle systems has some degeneracy, analogous to this one.

To find θ given ϕ, we have

$$r_{33} = \mathrm{c}\theta$$

To find sθ independently, we use r_{13} and r_{23}, but not directly:

$$\begin{aligned} r_{13}\mathrm{c}\phi + r_{23}\mathrm{s}\phi &= (\mathrm{c}\phi\mathrm{s}\theta)\mathrm{c}\phi + (\mathrm{s}\phi\mathrm{s}\theta)\mathrm{s}\phi \\ &= (\mathrm{c}^2\phi + \mathrm{s}^2\phi)\mathrm{s}\theta \\ &= \mathrm{s}\theta \end{aligned}$$

Hence

$$\theta = \text{atan2}(r_{13}\mathrm{c}\phi + r_{23}\mathrm{s}\phi, r_{33})$$

To find ψ given ϕ and θ, note that

$$\begin{aligned} r_{21}\mathrm{c}\phi - r_{11}\mathrm{s}\phi &= (\mathrm{s}\phi\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{s}\psi)\mathrm{c}\phi - (\mathrm{c}\phi\mathrm{c}\theta\mathrm{c}\psi - \mathrm{s}\phi\mathrm{s}\psi)\mathrm{s}\phi \\ &= \mathrm{c}^2\phi\mathrm{s}\psi + \mathrm{s}^2\phi\mathrm{s}\psi \\ &= \mathrm{s}\psi \\ r_{22}\mathrm{c}\phi - r_{12}\mathrm{s}\phi &= (-\mathrm{s}\phi\mathrm{c}\theta\mathrm{s}\psi + \mathrm{c}\phi\mathrm{c}\psi)\mathrm{c}\phi - (-\mathrm{c}\phi\mathrm{c}\theta\mathrm{s}\psi - \mathrm{s}\phi\mathrm{c}\psi)\mathrm{s}\phi \\ &= \mathrm{c}^2\phi\mathrm{c}\psi + \mathrm{s}^2\phi\mathrm{c}\psi \\ &= \mathrm{c}\psi \end{aligned}$$

Hence

$$\psi = \text{atan2}(r_{21}\text{c}\phi - r_{11}\text{s}\phi, r_{22}\text{c}\phi - r_{12}\text{s}\phi)$$

<u>**Example 2.9:**</u> Let us construct a rotation matrix $\mathbf{R}$ using ZYZ Euler angles $0, \pi/2, \pi/2$:

$$\mathbf{R}(0, \pi/2, \pi/2) = \mathbf{R}(0\mathbf{z})\mathbf{R}\left(\frac{\pi}{2}\mathbf{y}\right)\mathbf{R}\left(\frac{\pi}{2}\mathbf{z}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Now let's reconstruct the Euler angles. The first Euler angle ϕ is:

$$\phi = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

where we have taken the positive solution 0 rather than π. Then

$$\theta = \text{atan2}(1, 0) = \pi/2$$

$$\psi = \text{atan2}(1, 0) = \pi/2$$

Thus we have extracted the correct Euler angles.

2.3.3 Roll, Pitch, Yaw Angles

The roll, pitch, yaw (RPY) angles (Figure 2.10) are successive rotations about the fixed XYZ axes, in that order:

$$\mathbf{R}(\phi, \theta, \psi) \doteq \mathbf{R}(\mathbf{z}\phi)\mathbf{R}(\mathbf{y}\theta)\mathbf{R}(\mathbf{x}\psi) \tag{2.47}$$

where (2.47) represents an operator. Note that the order is from right to left because of the rotation about the fixed axes.

Note that RPY angles give the exact same result as ZYX Euler angles, although they are conceptualized differently. In a process similar that of Section 2.3.2, we can extract the RPY angles from any rotation matrix $\mathbf{R}$.

RPY angles are particularly useful for differential rotations (which commute) about a nominal configuration of $\phi = \theta = \psi = 0$, because these rotations are about orthogonal axes. This is also true of the XYZ, XZY, YXZ, YZX, and ZXY Euler angles; in each, no axis is repeated.

PRY angles are commonly used to describe mobile robots on the ground, sea, air, and space. The convention is to:

- Point the $\mathbf{x}$ axis forward and define roll as rotation ψ about this nominal forward axis in the static frame. Positive ψ corresponds to a clockwise roll, and negative ψ corresponds to a counterclockwise roll.

- Point the $\mathbf{y}$ axis to the right and define as pitch as the rotation θ about this nominar rightward axis in the static frame. Positive θ corresponds to the nose of the craft pitching upward, and negative θ corresponds to the nose of the craft pitching downward.

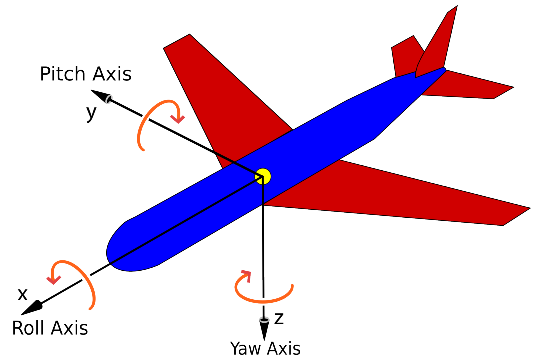


Figure 2.10: The roll, pitch, yaw axes.

- Point the $\mathbf{z}$ axis downward (due to the right-hand rule) and define yaw as rotation ϕ about this vertical axis in the static frame. Positive ϕ corresponds to the craft steering to the right, and negative ϕ corresponds to the craft steering to the left.

RPY angles suffer from the same type of degeneracy as Euler angles. For example, a roll of $\phi = 90°$ followed by a pitch of $\theta = 90°$ (with no final yaw) leads to the exact same orientation as a pitch of $\theta = 90°$ followed by a yaw of $\phi = -90°$ (with no initial roll). Thus, when given a rotation matrix representing such an orientation, there is no way to uniquely determine which RPY angles it corresponds to.

2.3.4 Quaternions

It appears that any three-parameter representation of orientation has some problem, and hence redundant four-parameter representations have been proposed. Quaternions are a popular four-parameter substitute for rotation matrices. We write:

$$\boldsymbol{q} = q_0 + \mathbf{q} \qquad \text{where } q_0 = \cos\left(\frac{\theta}{2}\right), \quad \mathbf{q} = \mathbf{k}\sin\left(\frac{\theta}{2}\right) \tag{2.48}$$

The notation $\boldsymbol{q}$ denotes a quaternion, where q_0 is a scalar part and $\mathbf{q}$ is a vector part. A quaternion will be denoted by a slanted bold-font lower-case letter, such as $\boldsymbol{q}$. A quaternion comprises four parameters: whether θ and the three parameters in $\mathbf{k}$, or q_0 and the three parameters in $\mathbf{q}$. We interpret $\mathbf{k}$ as the axis of rotation, and θ as the amount of rotation, as we have earlier. We can immediately see the potential benefit of the quaternion: using the q_0 term, we can keep track of the sign of θ, independent of the axis of rotation.

The sum in (2.48) is purely a formal way to designate that a quaternion comprises both components; we don't actually add the scalar part to the vector part. We have to adjust to this type of nonconventional notation when working with quaternions. There is one constraint equation for quaternions that represent rotations:

$$\boldsymbol{q} \cdot \boldsymbol{q} \doteq q_0^2 + \mathbf{q} \cdot \mathbf{q} = 1 \tag{2.49}$$

Such $\boldsymbol{q}$ are called unit quaternions. Scalars and vectors are special cases of quaternions:

- If $q_0 = 0$, then $\boldsymbol{q} = \mathbf{q}$ is a vector, i.e., a vector is a quaternion with a zero scalar part.
- If $\mathbf{q} = 0$, then $\boldsymbol{q} = q_0$ is a scalar, i.e., a scalar is a quaternion with a zero vector part.

Example 2.10: Let $\boldsymbol{q}_x$ be the quaternion with $\mathbf{k} = \mathbf{x}_0$ and $\theta = \pi/2$. Then

$$\boldsymbol{q}_x = \cos\frac{\pi}{4} + \mathbf{x}_0 \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathbf{x}_0$$

Let $\boldsymbol{q}_z$ be the quaternion with $\mathbf{k} = \mathbf{z}_0$ and $\theta = \pi/2$. Then

$$\boldsymbol{q}_z = \cos\frac{\pi}{4} + \mathbf{z}_0 \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \mathbf{z}_0\frac{\sqrt{2}}{2}$$

Given two quaternions $\boldsymbol{p} = p_0 + \mathbf{p}$ and $\boldsymbol{q} = q_0 + \mathbf{1}$, the composition of two quaternions is written as the two quaternions juxtaposed, and is defined as follows:

$$\begin{aligned}\boldsymbol{pq} &= (p_0 + \mathbf{p})(q_0 + \mathbf{q}) \\ &= p_0q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{pq} \\ &= p_0q_0 + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q}\end{aligned}$$

where we define two vectors $\mathbf{p}$ and $\mathbf{q}$ juxtaposed as:

$$\mathbf{pq} = \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q} \tag{2.50}$$

The resulting quaternion is rewritten to group the scalar and vector parts together:

$$\boldsymbol{pq} = (p_0q_0 - \mathbf{p} \cdot \mathbf{q}) + (p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}) \tag{2.51}$$

Example 2.11: The composition $\boldsymbol{q}_x\boldsymbol{q}_z$ is:

$$\begin{aligned}\boldsymbol{q}_x\boldsymbol{q}_z &= \left(\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\mathbf{x}_0\right) \cdot \left(\frac{\sqrt{2}}{2}\mathbf{z}_0\right)\right) \\ &\quad + \left(\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{z}_0\right) + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{x}_0\right) + \left(\frac{\sqrt{2}}{2}\mathbf{x}_0\right) \times \left(\frac{\sqrt{2}}{2}\mathbf{z}_0\right)\right) \\ &= \frac{1}{2} + \left(\frac{1}{2}\mathbf{z}_0 + \frac{1}{2}\mathbf{x}_0 + \frac{1}{2}(-\mathbf{y}_0)\right) \\ &= \frac{1}{2} + \frac{1}{2}(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0)\end{aligned}$$

The inverse of $\boldsymbol{q}$ is denoted as $\boldsymbol{q}^*$, where

$$\boldsymbol{q}^* = q_0 - \mathbf{q} \tag{2.52}$$

Then

$$\begin{aligned}\boldsymbol{qq}^* &= (q_0q_0 - \mathbf{q} \cdot (-\mathbf{q})) + (q_0(-\mathbf{q}) + q_0\mathbf{q} + \mathbf{q} \times (-\mathbf{q})) \\ &= q_0^2 + \mathbf{q} \cdot \mathbf{q} \\ &= 1\end{aligned}$$

since we only deal with unit quaternions (2.49). We can therefore define the dot product of two quaternions as the equivalent composition:

$$\boldsymbol{q} \cdot \boldsymbol{q} = \boldsymbol{q}\boldsymbol{q}^*$$

Representing rotations by quaternion compositions

Consider a vector $\mathbf{v}$ (i.e., a quaternion $\boldsymbol{v}$ with a zero scalar part, $v_0 = 0$) composed on the right with a quaternion $\boldsymbol{q}^*$ as follows:

$$\begin{aligned}
\mathbf{v}\boldsymbol{q}^* &= (0 + \mathbf{v})(q_0 - \mathbf{q}) \\
&= \mathbf{v} \cdot \mathbf{q} + (q_0\mathbf{v} - \mathbf{v} \times \mathbf{q})
\end{aligned}$$

Now compose this result on the left with $\boldsymbol{q}$:

$$\begin{aligned}
\boldsymbol{q}\mathbf{v}\boldsymbol{q}^* &= (q_0 + \mathbf{q})(\mathbf{v} \cdot \mathbf{q} + (q_0\mathbf{v} - \mathbf{v} \times \mathbf{q})) \\
&= (q_0(\mathbf{v} \cdot \mathbf{q}) - \mathbf{q} \cdot (q_0\mathbf{v} - \mathbf{v} \times \mathbf{q})) + (q_0(q_0\mathbf{v} - \mathbf{v} \times \mathbf{q}) + (\mathbf{v} \cdot \mathbf{q})\mathbf{q} + \mathbf{q} \times (q_0\mathbf{v} - \mathbf{v} \times \mathbf{q})) \\
&= q_0(\mathbf{v} \cdot \mathbf{q}) - q_0(\mathbf{q} \cdot \mathbf{v}) + q_0^2\mathbf{v} - q_0\mathbf{v} \times \mathbf{q} + (\mathbf{v} \cdot \mathbf{q})\mathbf{q} + q_0\mathbf{q} \times \mathbf{v} - \mathbf{q} \times (\mathbf{v} \times \mathbf{q}) \\
&= q_0^2\mathbf{v} - q_0\mathbf{v} \times \mathbf{q} + (\mathbf{v} \cdot \mathbf{q})\mathbf{q} + q_0\mathbf{q} \times \mathbf{v} - \mathbf{v}(\mathbf{q} \cdot \mathbf{q}) + \mathbf{q}(\mathbf{q} \cdot \mathbf{v}) \\
&= (q_0^2 - \mathbf{q} \cdot \mathbf{q})\mathbf{v} + 2q_0\mathbf{q} \times \mathbf{v} + 2\mathbf{q}(\mathbf{q} \cdot \mathbf{v}) \qquad (2.53)
\end{aligned}$$

It will now be shown that this result is equivalent to a rotation of $\mathbf{v}$ by the quaternion $\boldsymbol{q}$. Substitute for $q_0 = \cos(\theta/2)$ and $\mathbf{q} = \mathbf{k}\sin(\theta/2)$:

$$\begin{aligned}
\boldsymbol{q}\mathbf{v}\boldsymbol{q}^* &= \left(\cos^2\left(\frac{\theta}{2}\right) - \mathbf{k}^2\sin^2\left(\frac{\theta}{2}\right)\right)\mathbf{v} + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathbf{k} \times \mathbf{v} + 2\sin^2\left(\frac{\theta}{2}\right)\mathbf{k}(\mathbf{k} \cdot \mathbf{v}) \\
&= \mathbf{v}\cos\theta + \mathbf{k} \times \mathbf{v}\sin\theta + \mathbf{k}(\mathbf{k} \cdot \mathbf{v})(1 - \cos\theta) \qquad (2.54)
\end{aligned}$$

This is the same as a rotation matrix expressed in angle-axis form $\mathbf{R}_a(\theta)\mathbf{v}$ in (2.20).

Example 2.12: Suppose we wish to rotate the vector $\mathbf{x}_0$ by the quaternion $\boldsymbol{q}_z$, defined in the earlier example, representing a rotation about the $\mathbf{z}$ axis by an angle $\theta = \pi/2$. Then

$$\begin{aligned}
\boldsymbol{q}_z\mathbf{x}_0\boldsymbol{q}_z^* &= \left(\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{\sqrt{2}}{2}\mathbf{z}_0 \cdot \frac{\sqrt{2}}{2}\mathbf{z}_0\right)\mathbf{x}_0 + 2\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{z}_0 \times \mathbf{x}_0\right) + 2\frac{\sqrt{2}}{2}\mathbf{z}_0\left(\frac{\sqrt{2}}{2}\mathbf{z}_0 \cdot \mathbf{x}_0\right) \\
&= \mathbf{0} + \mathbf{y}_0 + \mathbf{0} \\
&= \mathbf{y}_0
\end{aligned}$$

One can verify the same result as $\mathbf{R}_z(\pi/2)\mathbf{x}_0$.

Quaternions can be composed just like rotation matrices.

Example 2.13: Consider the quaternion composition $\boldsymbol{q}_x\boldsymbol{q}_z$. Then from the previous example and regrouping,

$$\boldsymbol{q}_x\boldsymbol{q}_z\mathbf{x}_0(\boldsymbol{q}_x\boldsymbol{q}_z)^* = \boldsymbol{q}_x\boldsymbol{q}_z\mathbf{x}_0\boldsymbol{q}_z^*\boldsymbol{q}_x^* = \boldsymbol{q}_x(\boldsymbol{q}_z\mathbf{x}_0\boldsymbol{q}_z^*)\boldsymbol{q}_x^* = \boldsymbol{q}_x\mathbf{y}_0\boldsymbol{q}_x^* = 2\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{x}_0 \times \mathbf{y}_0\right) = \mathbf{z}_0$$

Alternatively, we may compose $\boldsymbol{q}_x\boldsymbol{q}_z$ directly, as in a previous example, then apply to $\mathbf{x}_0$:

$$\begin{aligned}
(\boldsymbol{q}_x\boldsymbol{q}_z)\mathbf{x}_0(\boldsymbol{q}_x\boldsymbol{q}_z)^* &= \left(\left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0\right)\cdot\frac{1}{2}\left(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0\right)\right)\mathbf{x}_0 \\
&\quad + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\left(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0\right)\times\mathbf{x}_0\right) + 2\left(\frac{1}{2}\right)\left(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0\right)\left(\frac{1}{2}\left(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0\right)\cdot\mathbf{x}_0\right) \\
&= -\frac{1}{2}\mathbf{x}_0 + \frac{1}{2}(\mathbf{y}_0 + \mathbf{z}_0) + \frac{1}{2}(\mathbf{z}_0 + \mathbf{x}_0 - \mathbf{y}_0) \\
&= \mathbf{z}_0
\end{aligned}$$

One can verify the same result from $\mathbf{R}_x(\pi/2)\mathbf{R}_z(\pi/2)\mathbf{x}_0$.

Converting from a quaternion to a rotation matrix

We can find the equivalent rotation matrix $\mathbf{R} = \{r_{ij}\}$ directly from the components $\boldsymbol{q} = q_0 + \mathbf{q}$ of a quaternion. Rewrite (2.53) to extract the rotation matrix:

$$\mathbf{R} = (q_0^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{I} + 2q_0\mathbf{S}(\mathbf{q}) + 2\mathbf{q}\mathbf{q}^T \tag{2.55}$$

From (2.49), we can substitute for the left coefficient:

$$\begin{aligned}
q_0^2 - \mathbf{q}\cdot\mathbf{q} &= (1 - \mathbf{q}\cdot\mathbf{q}) - \mathbf{q}\cdot\mathbf{q} \\
&= 1 - 2\mathbf{q}\cdot\mathbf{q} \\
&= 1 - 2(q_1^2 + q_2^2 + q_3^2)
\end{aligned}$$

where $\mathbf{q} = [q_1\ q_2\ q_3]^T$. For the r_{11} element,

$$\begin{aligned}
r_{11} &= 1 - 2(q_1^2 + q_2^2 + q_3^2) + 2q_1^2 \\
&= 1 - 2q_2^2 - 2q_3^2
\end{aligned}$$

For the r_{12} element,

$$r_{12} = -2q_0q_3 + 2q_1q_2$$

In a similar manner, we can find all elements r_{ij} of $\mathbf{R}$ to yield:

$$\mathbf{R} = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 1 - 2q_1^2 - 2q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix} \tag{2.56}$$

Converting from a rotation matrix to a quaternion

Given a rotation matrix with elements $\mathbf{R} = \{r_{ij}\}$, we can find the equivalent quaternion $\boldsymbol{q} = q_0 + \mathbf{q}$. From (2.56),

$$\begin{aligned} r_{11} + r_{22} + r_{33} &= (1 - 2q_2^2 - 2q_3^2) + (1 - 2q_1^2 - 2q_3^2) + (1 - 2q_1^2 - 2q_2^2) \\ &= 3 - 4(q_1^2 + q_2^2 + q_3^2) \\ &= 3 - 4(1 - q_0^2) \\ &= 4q_0^2 - 1 \end{aligned}$$

Hence

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

We take the positive value for the square root, so as to choose

$$0 \leq q_0 = \cos\left(\frac{\theta}{2}\right) \leq 1 \quad \Longleftrightarrow \quad -\frac{\pi}{2} \leq \frac{\theta}{2} \leq \frac{\pi}{2} \quad \Longleftrightarrow \quad -\pi \leq \theta \leq \pi$$

Next, solve for q_1, q_2, q_3 from

$$\begin{aligned} r_{32} - r_{23} &= 4q_0q_1 \\ r_{13} - r_{31} &= 4q_0q_2 \\ r_{21} - r_{12} &= 4q_0q_3 \end{aligned} \tag{2.57}$$

This procedure is similar to extracting $\mathbf{k}$ and θ from $\mathbf{R} = \mathbf{R}(\mathbf{k}\theta)$ in Section 2.3.1.

There are two problems.

1. If $\theta \approx 0$, then $\mathbf{q} = \mathbf{k}\sin(\theta/2)$ is poorly defined.

2. If $|\theta| \approx \pi$, then $\mathbf{q}$ is well defined, but $q_0 = 0$ and you can't divide by it in (2.57).

We sketch Salamin's [1] procedure for handling the case $|\theta| \approx \pi$. Similar to above, it may be shown that:

$$\begin{aligned} q_0^2 &= \frac{1}{4}(1 + r_{11} + r_{22} + r_{33}) \\ q_1^2 &= \frac{1}{4}(1 + r_{11} - r_{22} - r_{33}) \\ q_2^2 &= \frac{1}{4}(1 - r_{11} + r_{22} - r_{33}) \\ q_3^2 &= \frac{1}{4}(1 - r_{11} - r_{22} + r_{33}) \end{aligned}$$

Also

$$q_0q_1 = \frac{1}{4}(r_{32} - r_{23})$$

$$q_0q_2 = \frac{1}{4}(r_{13} - r_{31})$$

$$q_0q_3 = \frac{1}{4}(r_{21} - r_{12})$$

$$q_1q_2 = \frac{1}{4}(r_{12} + r_{21})$$

$$q_1q_3 = \frac{1}{4}(r_{13} + r_{31})$$

$$q_2q_3 = \frac{1}{4}(r_{23} + r_{32})$$

Choose the largest of the q_i^2 (at least one of them must be significantly different from 0). Then solve from the q_iq_j products for the other q_j's. This will successfully handle the case $|\theta| \approx \pi$.

Normalization

With many successive rotations, computational errors may build up that cause quaternions to become of non-unit length. The correction is easy:

$$\boldsymbol{q}' = \frac{\boldsymbol{q}}{\sqrt{\boldsymbol{q} \cdot \boldsymbol{q}}}$$

By contrast, normalization of rotation matrices that become nonorthogonal is much more involved. We need to find the $\mathbf{R}'$ such that:

$$f = \sum_{i,j}(r'_{ij} - r_{ij})^2 \text{ is a minimum subject to } \mathbf{R}'(\mathbf{R}')^T = \mathbf{I}.$$

The answer is $\mathbf{R}' = \mathbf{R}(\mathbf{R}^T\mathbf{R})^{-\frac{1}{2}}$ [1].

Chapter 3

Spatial Transformations

In robotics, we are interested in describing the entire pose of the robot (or some specific part of the robot), which comprises a 3D orientation and a 3D position. To describe where objects in the environment are, we will attach coordinate systems to them and relate the various coordinate systems. We will also be attaching coordinate systems to each link of a robotic manipulator, in order to deduce joint angles and to relate the position of a manipulator relative to objects.

How coordinate systems are defined and attached to objects is a matter of convenience. For example, consider the two trapezoids in arbitrary placement in Figure 3.1(A), where the origins have been placed in corresponding corners and the axes on corresponding sides. In this general case, the origins O_1 and O_2 are different, as are the axes. The position and orientation of each trapezoid is considered to be synonymous with the position and orientation of its coordinate system. In Figure 3.1(B), trapezoid 2 has been rotated such that $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{y}_1 = \mathbf{y}_2$. When the axes are aligned like this, we say that the two trapezoids are in the same orientation; however, they are not in the same position becasue their origins differ. Furthermore, when we say something like $\mathbf{x}_1 = \mathbf{x}_2$, we are not saying that these vectors are pointing to the same point. What we are saying is that they are pointing in the same direction, because they are vectors. In Figure 3.1(C), trapezoid 2's origin has been made coincident with trapezoid 1's origin, and trapezoid 2 has been rotated so that $\mathbf{x}_2 = \mathbf{y}_1$. Their position is now the same because $O_1 = O_2$. Their orientations differ by a 90° rotation about the common $\mathbf{z}$ axes.

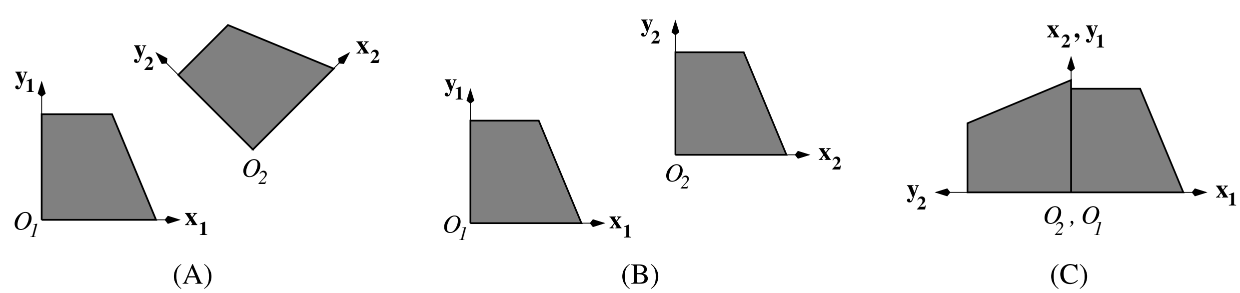


Figure 3.1: (A) Two trapezoids with attached coordinate systems in an arbitrary general position relative to each other. (B) The two trapezoids have the same orientation but different positions, which is a pure translation. (C) The two trapezoids are in the same position but have different orientations, which is a pure rotation. In each image, the $\mathbf{z}_1$ and $\mathbf{z}_2$ axes point out of the page, per the right-hand rule.

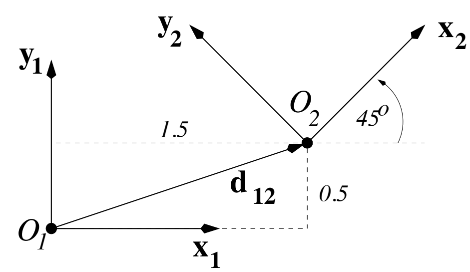


Figure 3.2: Trapezoid 2 is displaced by ${}^1\mathbf{d}_{12} = [1.5 \;\; 0.5 \;\; 0]^T$ from trapezoid 1, and rotated by 45° about the $\mathbf{z}_2$ axis. The $\mathbf{z}_1$ and $\mathbf{z}_2$ axes point out of the page, based on the right-hand rule.

Example 3.14: Suppose in Figure 3.1(A) that axes 2 are rotated by 45° about the $\mathbf{z}$ axis, and thus with respect to axes 1, and that the origin O_2 is displaced from origin O_1 by ${}^1\mathbf{d}_{12} = {}^1(O_2 - O_1) = 1.5\ {}^1\mathbf{x}_1 + 0.5\ {}^1\mathbf{y}_1 = [1.5\ 0.5\ 0]^T$ (see Figure 3.2(A)). The same linear displacement ${}^1\mathbf{d}_{12}$ also applies to Figure 3.1(B). What are the coordinates of $\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2$ with respect to axes 1 in the three figures?

1. For Figure 3.1(C), ${}^1\mathbf{x}_2 = [0\ 1\ 0]^T$, ${}^1\mathbf{y}_2 = [-1\ 0\ 0]^T$, and ${}^1\mathbf{z}_2 = [0\ 0\ 1]^T$. This is no different from Chapter 2.

2. For Figure 3.1(B), ${}^1\mathbf{x}_2 = [1\ 0\ 0]^T$, ${}^1\mathbf{y}_2 = [0\ 1\ 0]^T$, and ${}^1\mathbf{z}_2 = [0\ 0\ 1]^T$ since axes 2 are parallel to axes 1. Note that the displacement ${}^1\mathbf{d}_{12}$ between origins makes no difference, because we are comparing vectors, not points.

3. The same is true for Figure 3.1(A), where

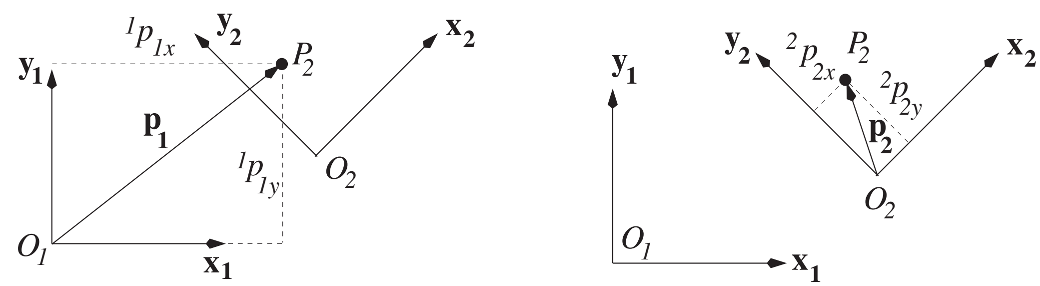
$$ {}^1\mathbf{x}_2 = \sqrt{2}/2\ {}^1\mathbf{x}_1 + \sqrt{2}/2\ {}^1\mathbf{y}_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \tag{3.1} $$

$$ {}^1\mathbf{y}_2 = -\sqrt{2}/2\ {}^1\mathbf{x}_1 + \sqrt{2}/2\ {}^1\mathbf{y}_1 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \tag{3.2} $$

$$ {}^1\mathbf{z}_2 = {}^1\mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.3} $$

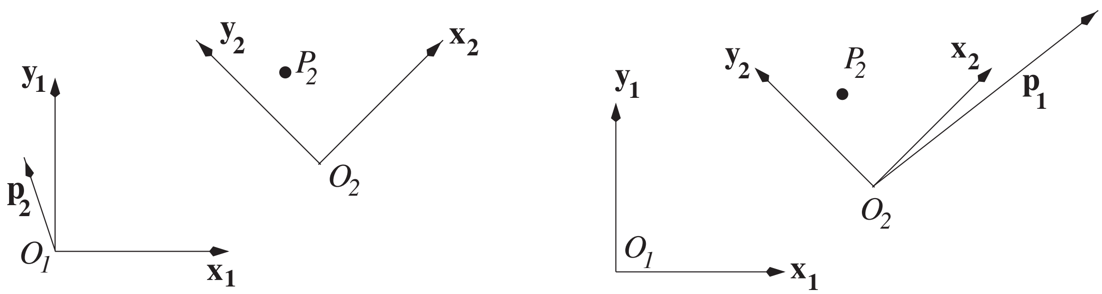
Hence, when comparing vectors in coordinate systems with different origins, we can just pretend the origins coincide in the derivation.

Suppose that there is an arbitrary point P_2 on trapezoid 2 that we wish to locate (Figure 3.3). The vector $\mathbf{p}_2 = P_2 - O_2$ locates point P_2 relative to origin O_2, while the vector $\mathbf{p}_1 = P_2 - O_1$ locates point P_2 relative



(A) (B)

Figure 3.3: An arbitrary point P_2 on trapezoid 2 is located (A) relative to O_1 by $\mathbf{p}_1$ and (B) relative to O_2 by $\mathbf{p}_2$.



(A) (B)

Figure 3.4: The same vectors from Figure 3.3, expressed with respect to different frames, don't really have any physical significance.

to origin O_1. We can relate either vector to coordinate system 1 or 2:

$$^2\mathbf{p}_2 = {}^2p_{2x}{}^2\mathbf{x}_2 + {}^2p_{2y}{}^2\mathbf{y}_2 + {}^2p_{2z}{}^2\mathbf{z}_2 \tag{3.4}$$

$$^1\mathbf{p}_2 = {}^1p_{2x}{}^1\mathbf{x}_1 + {}^1p_{2y}{}^1\mathbf{y}_1 + {}^1p_{2z}{}^1\mathbf{z}_1 \tag{3.5}$$

$$^2\mathbf{p}_1 = {}^2p_{1x}{}^2\mathbf{x}_2 + {}^2p_{1y}{}^2\mathbf{y}_2 + {}^2p_{1z}{}^2\mathbf{z}_2 \tag{3.6}$$

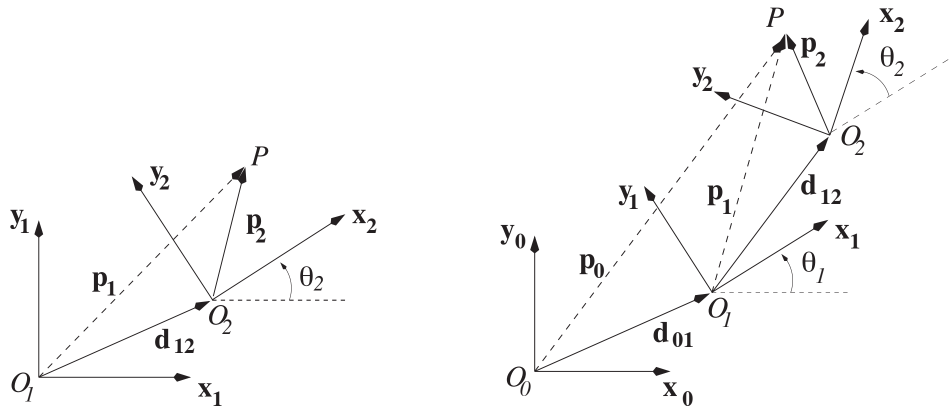
$$^1\mathbf{p}_1 = {}^1p_{1x}{}^1\mathbf{x}_1 + {}^1p_{1y}{}^1\mathbf{y}_1 + {}^1p_{1z}{}^1\mathbf{z}_1 \tag{3.7}$$

The coordinates of each vector are, in general, different from each other. The vectors $^1\mathbf{p}_1$ and $^2\mathbf{p}_2$ have a clear meaning, describing the location of the point P_2 with respect to each of the frames (Figure 3.3), whereas the vectors $^2\mathbf{p}_1$ and $^2\mathbf{p}_1$ don't really have any physical significance (Figure 3.4). We need to develop a consistent methodology to express points, like P_2, with respect to different frames.

3.1 Coordinate Transformations

Let's reconsider the system of Figures 3.2 and 3.3, and define a displacement vector $\mathbf{d}_{12}$ from origin O_1 to O_2, as shown in Figure 3.5(A). In general, $\mathbf{d}_{ij}$ is defined as the vector from O_i to O_j. Clearly, in this example,

$$\mathbf{p}_1 = \mathbf{d}_{12} + \mathbf{p}_2, \tag{3.8}$$



(A) (B)

Figure 3.5: Coordinate transformation between (A) frame 1 and frame 2, and (B) also including frame 0.

regardless of which coordinate system these vectors are expressed in. Often a point will only be known relative to the frame of the body in which it is located. For example, P would be known with respect to frame 2 (i.e., ${}^2\mathbf{p}_2$ would be known). From the previous chapter, the rotation matrix ${}^1\mathbf{R}_2 = \mathbf{R}(\mathbf{z}_2)$ applied to ${}^2\mathbf{p}_2$ yields ${}^1\mathbf{p}_2$:

$$ {}^1\mathbf{p}_2 = {}^1\mathbf{R}_2{}^2\mathbf{p}_2 \tag{3.9} $$

We assume that $_2$ and ${}^1\mathbf{d}_{12}$ are given, i.e., we know where frame 2 is relative to frame 1. To find ${}^1\mathbf{p}_1$ given ${}^2\mathbf{p}_2$, substitute (3.9) into (3.8):

$$ {}^1\mathbf{p}_1 = {}^1\mathbf{d}_{12} + {}^1\mathbf{R}_2{}^2\mathbf{p}_2 \tag{3.10} $$

Equation (3.10) is said to represent a coordinate transformation.

Example 3.15: Let us determine ${}^1\mathbf{p}_1$ in Figure 3.5(A) for $_2 = \pi/4$, ${}^2\mathbf{p}_2 = [1/4 \quad 3/4 \quad 0]^T$, and ${}^1\mathbf{d}_{12} = [3/2 \ \ 1/2 \ \ 0]^T$:

$$ {}^1\mathbf{p}_1 = {}^1\mathbf{d}_{12} + {}^1\mathbf{R}_2{}^2\mathbf{p}_2 $$

$$ = \begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \\ 0 \end{bmatrix} \tag{3.11} $$

$$ = \begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\sqrt{2}/4 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \tag{3.12} $$

$$ = \begin{bmatrix} (6-\sqrt{2})/4 \\ (1+\sqrt{2})/2 \\ 0 \end{bmatrix} \tag{3.13} $$

Given a coordinate transformation from 2 to 1, we can reverse the transformation to go from 1 to 2. Given (3.10), by rearrangement we find:

$$ {}^{2}\mathbf{p}_2 = {}^{1}\mathbf{R}_2{}^{-1}\left({}^{1}\mathbf{p}_1 - {}^{1}\mathbf{d}_{12}\right) \tag{3.14} $$

where again ${}^{1}\mathbf{R}_2^{-1} = {}^{1}\mathbf{R}_2^{T} = {}^{2}\mathbf{R}_1 = \mathbf{R}(-\mathbf{z}\theta)$.

3.1.1 Composition of Coordinate Transformations

Next introduce frame 0, such that frame 1 is displaced from frame 0 by $\mathbf{d}_{01}$ and rotated with respect to frame 0 by an angle $_1$ (Figure 3.5(B)). Suppose point P is located relative to O_2 by $\mathbf{p}_2$, relative to O_1 by $\mathbf{p}_1$, and relative to O_0 by $\mathbf{p}_0$. Then

$$ \mathbf{p}_1 = \mathbf{d}_{12} + \mathbf{p}_2 \tag{3.15} $$

$$ \mathbf{p}_0 = \mathbf{d}_{01} + \mathbf{p}_1 \tag{3.16} $$

Again, suppose $\mathbf{p}_2$ is known relative to frame 2, i.e., as ${}^{2}\mathbf{p}_2$. The location of P relative to frame 1 is:

$$ {}^{1}\mathbf{p}_1 = {}^{1}\mathbf{d}_{12} + {}^{1}\mathbf{R}_2{}^{2}\mathbf{p}_2 \tag{3.17} $$

Utilizing (3.17), the location of P relative to frame 0 is:

$$ {}^{0}\mathbf{p}_0 = {}^{0}\mathbf{d}_{01} + {}^{0}\mathbf{p}_1 \tag{3.18} $$

$$ = {}^{0}\mathbf{d}_{01} + {}^{0}\mathbf{R}_1{}^{1}\mathbf{p}_1 \tag{3.19} $$

$$ = {}^{0}\mathbf{d}_{01} + {}^{0}\mathbf{R}_1\left({}^{1}\mathbf{d}_{12} + {}^{1}\mathbf{R}_2{}^{2}\mathbf{p}_2\right) \tag{3.20} $$

$$ = {}^{0}\mathbf{d}_{01} + {}^{0}\mathbf{R}_1{}^{1}\mathbf{d}_{12} + {}^{0}\mathbf{R}_2{}^{2}\mathbf{p}_2 \tag{3.21} $$

$$ = {}^{0}\mathbf{d}_{01} + {}^{0}\mathbf{d}_{12} + {}^{0}\mathbf{p}_2 \tag{3.22} $$

Equation (3.22) demonstrates the composition of two coordinate transformations. Note again the composition of rotations: ${}^{0}\mathbf{R}_1{}^{1}\mathbf{R}_2 = {}^{0}\mathbf{R}_2$.

3.2 Homogeneous Transformations

A coordinate transformation such as (3.10) separately represents rotations and translations. These transformations can be combined into a single matrix operation by using homogeneous coordinates (which we introduced in Chapter 1), which increasing the dimension of points by 1. A coordinate transformation can now be represented as:

$$ \begin{bmatrix} {}^{0}\mathbf{p}_0 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{0}\mathbf{R}_1{}^{1}\mathbf{p}_1 + {}^{0}\mathbf{d}_{01} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{0}\mathbf{R}_1 & {}^{0}\mathbf{d}_{01} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{1}\mathbf{p}_1 \\ 1 \end{bmatrix} \tag{3.23} $$

or employing the homogeneous coordinate notation,

$$ {}^{0}P = {}^{0}\mathbf{T}_1{}^{1}P \tag{3.24} $$

where $\mathbf{0}^T = [0\ \ 0\ \ 0]$ and ${}^0\mathbf{T}_1$ is the 4-by-4 homogeneous transformation from frame 1 to frame 0. For ${}^0\mathbf{T}_1$, the rotational part ${}^0\mathbf{R}_1$ contains as its three columns ${}^0\mathbf{x}_1$, ${}^0\mathbf{y}_1$, and ${}^0\mathbf{z}_1$. Thus

$$ {}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{x}_1 & {}^0\mathbf{y}_1 & {}^0\mathbf{z}_1 & {}^0\mathbf{d}_{01} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.25} $$

Hence the homogeneous transformation contains the axes of frame 1 relative to frame 0, plus the displacement of origin 1 from origin 0. Because ${}^0\mathbf{T}_1$ contains all the information about the coordinate system 1 relative to coordinate system 0, it itself is often called frame 1.

3.2.1 Transformation of Vectors

The same homogeneous transformations used to transform points from one frame to another can be use to convert vectors (in homogeneous coordinates) from one frame to another:

$$ {}^0\boldsymbol{V} = {}^0\mathbf{T}_1{}^1\boldsymbol{V} = \begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^1\mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_1{}^1\mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{v} \\ 0 \end{bmatrix} \tag{3.26} $$

As we saw earlier, when rotating vectors from one frame to another, we simply ignore the displacement between frames. This is captured inherently by the structure of the homogeneous coordinates.

3.2.2 Composition of Homogeneous Transformations

Coordinate transformations can easily be chained between pairs of coordinate systems. Let ${}^0\mathbf{T}_1$ represent the homogeneous transformation from frame 1 to frame 0 and ${}^1\mathbf{T}_2$ from frame 2 to frame 1. Then the homogeneous transformation from frame 2 to 0 is

$$ {}^0\mathbf{T}_2 = {}^0\mathbf{T}_1{}^1\mathbf{T}_2 \tag{3.27} $$

In this notation, note how the subscript of ${}^0\mathbf{T}_1$ lines up with, and cancels, the superscript of ${}^1\mathbf{T}_2$. Applied to homogeneous coordinates,

$$ {}^0P = {}^0\mathbf{T}_2{}^2P = {}^0\mathbf{T}_1{}^1P \tag{3.28} $$

where ${}^1P = {}^1\mathbf{T}_2{}^2P$. Note how much more compact the homogeneous transformation notation is than (3.22).

Example 3.16: Consider the following relative location of frames 0, 1, and 2 in Figure 3.5(B):

- Frame 1 is displaced from frame 0 by a rotation of 30° about $\mathbf{z}_1$ and a translation of ${}^0\mathbf{d}_{01} = [1\ \ 1\ \ 0]^T$. Then ${}^0\mathbf{R}_1 = \mathbf{R}(30°\mathbf{z})$ and

$$ {}^0\mathbf{R}_1 = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^0\mathbf{d}_{01} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, $$

$$ {}^0\mathbf{T}_1 = \begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 & 1 \\ 1/2 & \sqrt{3}/2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Frame 2 is displaced from frame 1 by a rotation of $60°$ about $\mathbf{z}_2$ and a translation of ${}^1\mathbf{d}_{12} = [1/2 \;\; \sqrt{3}/2 \;\; 0]^T$ expressed in frame 1. Then ${}^1\mathbf{R}_2 = \mathbf{R}(60°\mathbf{z})$ and

$$
{}^1\mathbf{R}_2 = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^1\mathbf{d}_{12} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix},
$$

$$
{}^1\mathbf{T}_2 = \begin{bmatrix} {}^1\mathbf{R}_2 & {}^1\mathbf{d}_{12} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 & 1/2 \\ \sqrt{3}/2 & 1/2 & 0 & \sqrt{3}/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Frame 2 is displaced from frame 0 as the composition of the previous transformations.

$$
{}^0\mathbf{T}_2 = {}^0\mathbf{T}_1 {}^1\mathbf{T}_2 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.29}
$$

Then

$$
{}^0\mathbf{R}_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^0\mathbf{d}_{12} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \tag{3.30}
$$

- Given a point ${}^2P_2 = [1, 1, 0, 1]^T$, then

$$
{}^1P_2 = {}^1\mathbf{T}_2 {}^2P_2 = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 & 1/2 \\ \sqrt{3}/2 & 1/2 & 0 & \sqrt{3}/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (2-\sqrt{3})/2 \\ (1+2\sqrt{3})/2 \\ 0 \\ 1 \end{bmatrix}
$$

$$
{}^0P_2 = {}^0\mathbf{T}_2 {}^2P_2 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}
$$

Example 3.17: Returning to the example of Figure 2.4, the homogeneous transformations between neighboring frames are:

$$^2\mathbf{T}_3 = \begin{bmatrix} ^2\mathbf{x}_3 & ^2\mathbf{y}_3 & ^2\mathbf{z}_3 & ^2\mathbf{d}_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1\mathbf{T}_2 = \begin{bmatrix} ^1\mathbf{x}_2 & ^1\mathbf{y}_2 & ^1\mathbf{z}_2 & ^1\mathbf{d}_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0\mathbf{T}_1 = \begin{bmatrix} ^0\mathbf{x}_1 & ^0\mathbf{y}_1 & ^0\mathbf{z}_1 & ^0\mathbf{d}_{01} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The relationships of frames 2 and 3 to frame 0 are:

$$^0\mathbf{T}_2 = \begin{bmatrix} ^0\mathbf{x}_2 & ^0\mathbf{y}_2 & ^0\mathbf{z}_2 & ^0\mathbf{d}_{02} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0\mathbf{T}_3 = \begin{bmatrix} ^0\mathbf{x}_3 & ^0\mathbf{y}_3 & ^0\mathbf{z}_3 & ^0\mathbf{d}_{03} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2.5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case, the relative coordinate axes and origin displacements can be found from inspection, rather than as the composition of multiple homogeneous transformations.

3.2.3 Inverse of Homogeneous Transformations

The inverse coordinate transformation is again easier to express using homogeneous transformations.

$$^1P = {}^1\mathbf{T}_0\, {}^0P = {}^0\mathbf{T}_1^{-1}\, {}^0P \tag{3.31}$$

$$(^0\mathbf{T}_1)^{-1} = \begin{bmatrix} ^0\mathbf{R}_1^T & -\,^0\mathbf{R}_1^T\, {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} = {}^1\mathbf{T}_0 \tag{3.32}$$

where (3.32) is derived from (3.14). With our notation, $(^0\mathbf{T}_1)^{-1} = {}^1\mathbf{T}_0$: the inverse of the transformation from frame 1 to 0 is the transformation from frame 0 to 1.

The inverse of the composition of transformations may be simply written:

$$^2P = {}^2\mathbf{T}_0{}^0P = {}^2\mathbf{T}_1{}^1\mathbf{T}_0{}^0P = \left({}^1\mathbf{T}_2\right)^{-1}\left({}^0\mathbf{T}_1\right)^{-1}{}^0P = \left({}^0\mathbf{T}_1{}^1\mathbf{T}_2\right)^{-1}{}^0P \tag{3.33}$$

3.2.4 Pure Translations and Pure Rotations

There are two special transformations: pure translations (e.g., see Figure 3.1(B)) and pure rotations (e.g., see Figure 3.1(C)):

- A translational transformation $\mathbf{Trans}$ is the special homogeneous transformation with the form:

$$\mathbf{Trans}(\mathbf{d}) = \begin{bmatrix} \mathbf{I} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.34}$$

  There is no rotation, indicated by $\mathbf{R} = \mathbf{I}$, and there is a displacement $\mathbf{d}$ between origins.

- A rotational transformation $\mathbf{Rot}$ implements a pure rotation without translation:

$$\mathbf{Rot}(\mathbf{R}) = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.35}$$

  There is some rotation, indicated by $\mathbf{R} \neq \mathbf{I}$, and there is no displacement between origins. We can also pass the $\mathbf{Rot}$ operator an axis and angle, where it is understood that

$$\mathbf{Rot}(\mathbf{k}\theta) = \begin{bmatrix} e^{\mathbf{S}(\mathbf{k})\theta} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

A general homogeneous transformation is the composition of a translational with a rotational transformation:

$${}^0\mathbf{T}_1 = \mathbf{Trans}({}^0\mathbf{d}_{01})\mathbf{Rot}({}^0\mathbf{R}_1) = \begin{bmatrix} \mathbf{I} & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} {}^0\mathbf{R}_1 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_1 & {}^0\mathbf{d}_{01} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.36}$$

Note the order. The reverse composition $\mathbf{Rot}({}^0\mathbf{R}_1)\mathbf{Trans}({}^0\mathbf{d}_{01})$ would not give the correct result.

3.2.5 Coordinate Transformations vs. Operators

In Chapter 2, we saw that the composition of rotations could be conceived of as a series of coordinate transformations or a series of operators, depending on the order in which the rotations were performed. Homogeneous transformations have an analogous property. As an example to illustrate these two different interpretations, consider the homogeneous transformation

$${}^0\mathbf{T}_1 = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 & 1 \\ 1/2 & \sqrt{3}/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.37}$$

Consider that coordinate systems 0 and 1 are initially overlapping. The interpretation of ${}^0\mathbf{T}_1$ as a coordinate transformation versus as an operator differs as follows.

- Viewed as a coordinate transform, ${}^0\mathbf{T}_1$ first translates the origin of frame 1 by $\begin{bmatrix} 1 & 1/2 & 0 \end{bmatrix}^T$ (expressed with respect to frame 1, which is aligned with frame 0), then rotates the axes of frame 1 by 30° about $\mathbf{z}_1$.

- Viewed as an operator, ${}^0\mathbf{T}_1$ first rotates frame 1 by 30° about $\mathbf{z}_0$ (which is aligned with $\mathbf{z}_1$), then translates it by $\begin{bmatrix} 1 & 1/2 & 0 \end{bmatrix}^T$ (expressed with respect to frame 0).

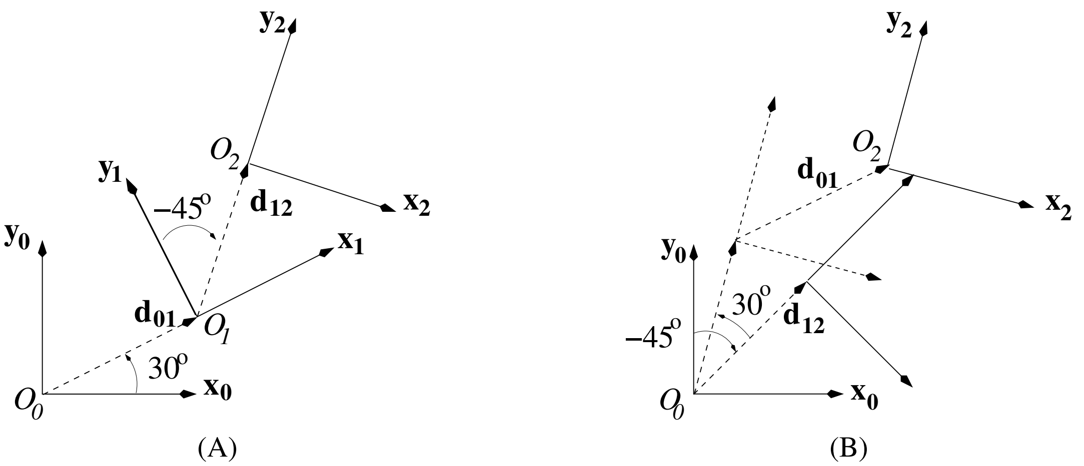


Figure 3.6: The composition ${}^0\mathbf{T}_2 = {}^0\mathbf{T}_1{}^1\mathbf{T}_2$ viewed as (A) a coordinate transformation (left-to-right evaluation about the current frame), or (B) as an operator (right-to-left evaluation about the fixed frame).

There doesn't seem to be such a big difference between these two interpretations for this simple example, but the composition of two general operators illustrates the difference more clearly. Consider the composition of the homogeneous transformations ${}^0\mathbf{T}_2 = {}^0\mathbf{T}_1\ {}^1\mathbf{T}_2$, where

$$ {}^0\mathbf{T}_1 = \mathbf{Trans}\left(\begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}\right)\mathbf{Rot}(30^\circ\mathbf{z}) \tag{3.38}$$

$$ {}^1\mathbf{T}_2 = \mathbf{Trans}\left(\begin{bmatrix} 3/4 \\ 3/4 \\ 0 \end{bmatrix}\right)\mathbf{Rot}(-45^\circ\mathbf{z}) \tag{3.39}$$

$$ {}^0\mathbf{T}_2 = \mathbf{Trans}\left(\begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}\right)\mathbf{Rot}(30^\circ\mathbf{z})\mathbf{Trans}\left(\begin{bmatrix} 3/4 \\ 3/4 \\ 0 \end{bmatrix}\right)\mathbf{Rot}(-45^\circ\mathbf{z}) \tag{3.40}$$

Consider that coordinate systems 0 and 2 are initially overlapping. The interpretation of ${}^0\mathbf{T}_2$ as a coordinate transformation versus as an operator differs as follows.

1. Viewed as a coordinate transform, the transformations in ${}^0\mathbf{T}_2$ are interpreted from left to right about the current frame. First ${}^0\mathbf{T}_2$ translates the origin of frame 2 by $\begin{bmatrix} 1 & 1/2 & 0 \end{bmatrix}^T$ (expressed with respect to frame 2, which is aligned with frame 0), then rotates the axes of frame 2 by 30° about $\mathbf{z}_2$, then translates frame 2 by $\begin{bmatrix} 3/4 & 3/4 & 0 \end{bmatrix}^T$ (expressed in frame 2), then rotates frame 2 by -45° about $\mathbf{z}_2$ (Figure 3.6(A)).

2. Viewed as an operator, the transformations in ${}^0\mathbf{T}_2$ are interpreted from right to left about the fixed frame. First ${}^1\mathbf{T}_2$ rotates frame 2 by -45° about $\mathbf{z}_0$, then translates frame 2 by $\begin{bmatrix} 3/4 & 3/4 & 0 \end{bmatrix}^T$ (expressed in frame 0), then rotates frame 2 by 30° about $\mathbf{z}_0$, then translates frame 2 by $\begin{bmatrix} 1 & 1/2 & 0 \end{bmatrix}^T$ (expressed with respect to frame 0).

The result of ${}^0\mathbf{T}_2$ viewed as an operator or a coordinate transformation is the same, but the process of reaching the final placement of frame 2 is different. When viewed as a coordinate transformation, the pose of the intermediate frame after completing the first translation and first rotation (i.e., those due to ${}^0\mathbf{T}_1$) is frame 1. When viewed as an operator, none of the intermediate frames have any significance.

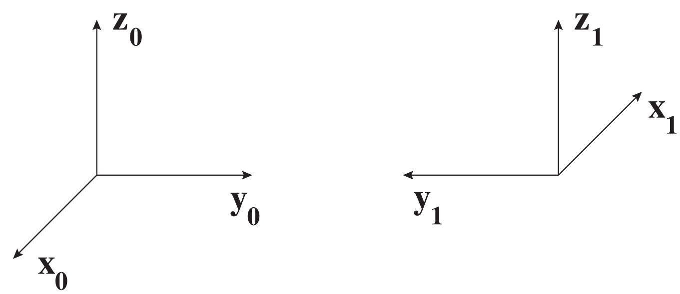


Figure 3.7: Frame 1 is rotated about $\mathbf{z}_0$ by π and then translated along $\mathbf{y}_0$ by a distance of 3.

Example 3.18: Let's consider the transformation between frames 1 and 0 in Figure 3.7. From inspection, the homogeneous transformation between frames is

$$^0\mathbf{T}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can build this transformation as a series of coordinate transforms (i.e., about the current frame, from left to right) in which frame 1, originally aligned with frame 0, was first rotated about $\mathbf{z}$ in frame 1, and then translated by $\begin{bmatrix} 0 & -4 & 0 \end{bmatrix}^T$ in frame 1:

$$^0\mathbf{T}_1 = \mathbf{Rot}\left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix}\right)\mathbf{Trans}\left(\begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\pi & -\sin\pi & 0 & 0 \\ \sin\pi & \cos\pi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternatively, it could be first translated by $\begin{bmatrix} 0 & 4 & 0 \end{bmatrix}^T$ in frame 0, rotated about $\mathbf{z}$ in frame 1:

$$^0\mathbf{T}_1 = \mathbf{Trans}\left(\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}\right)\mathbf{Rot}\left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\pi & -\sin\pi & 0 & 0 \\ \sin\pi & \cos\pi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can also think of this transformation as a series of operators (i.e., about the fixed frame, from right to left) in which frame 1 was first rotated about $\mathbf{z}$ in frame 0, and then translated by $\begin{bmatrix} 0 & 4 & 0 \end{bmatrix}^T$ in frame 0. This ultimately results in the same composition above, although we arrived at it differently.

3.3 Twists and Screws

3.3.1 Twists

In Chapter 2, we saw that any 3-by-3 rotation matrix $\mathbf{R}$ can be expressed as a rotation by some angle θ about some axis $\mathbf{k}$ (a unit vector that encodes both the direction of the axis or rotation and the sign convention for θ), using the matrix exponential of a 3-by-3 skew-symmetric matrix that encodes $\mathbf{k}$ and θ:

$$\mathbf{S}(\mathbf{k}\theta) = \begin{bmatrix} 0 & -k_3\theta & k_2\theta \\ k_3\theta & 0 & -k_1\theta \\ -k_2\theta & k_1\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \theta = \mathbf{S}(\mathbf{k})\theta$$

$$\mathbf{R} = e^{\mathbf{S}(\mathbf{k}\theta)} = e^{\mathbf{S}(\mathbf{k})\theta}$$

There is a 4-by-4 matrix that is analogous to the skew-symmetric matrix above, which we call a twist:

$$\boldsymbol{\Xi}(\mathbf{k}\theta, \mathbf{v}) = \begin{bmatrix} \mathbf{S}(\mathbf{k}\theta) & \mathbf{v} \\ \mathbf{0}^T & 0 \end{bmatrix} \tag{3.41}$$

where $\mathbf{v}$ is any 3-by-1 vector. If $\boldsymbol{\Xi}(\mathbf{k}\theta, \mathbf{v})$ is a 4-by-4 twist, we call the 6-by-1 vector

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{v} \\ \mathbf{k}\theta \end{bmatrix} \tag{3.42}$$

the twist coordinates; it is a more compact way to represent the twist. We can also introduce two functions that enable us to go back and forth between twists and twist coordinates:

$$\boldsymbol{\Xi}^\vee = \boldsymbol{\xi} \tag{3.43}$$

$$\boldsymbol{\xi}^\wedge = \boldsymbol{\Xi} \tag{3.44}$$

The twist coordinates are a generalization of the angle-axis representation of rotations, but for spatial transformations. The physical interpretation lies in screw theory, which is discussed below.

Just as the matrix exponential of any 3-by-3 skew-symmetric matrix is a 3-by-3 rotation matrix, we find that the matrix exponential of any 4-by-4 twist is a 4-by-4 homogeneous transformation matrix:

$$\mathbf{T} = e^{\boldsymbol{\Xi}(\mathbf{k}\theta, \mathbf{v})} \tag{3.45}$$

However, the relationship between the twist $\boldsymbol{\Xi}$ and the resulting homogeneous transformation $\mathbf{T}$ is not so straightforward.

If $\theta = 0$, then

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{v} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{Trans}(\mathbf{d}) \tag{3.46}$$

which is a pure translation, where $\mathbf{v} = \mathbf{d}$ is the displacement vector.

If $\theta \neq 0$, then

$$\mathbf{T} = \begin{bmatrix} e^{\mathbf{S}(\mathbf{k})\theta} & \left(\mathbf{I} - e^{\mathbf{S}(\mathbf{k})\theta}\right)(\mathbf{k} \times \mathbf{v}) + \mathbf{k}\mathbf{k}^T\mathbf{v}\theta \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\mathbf{k}\theta) & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.47}$$

which is a general homogeneous transformation matrix, where the displacement vector $\mathbf{d}$ is a complicated function of $\mathbf{v}$, $\mathbf{k}$, and θ, but the rotation matrix $\mathbf{R}$ is defined as before.

3.3.2 From Homogeneous Transformations to Twists

Given an arbitrary homogeneous transformation

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.48}$$

there is not a unique twist to create it, but we can find one.

If $\mathbf{R} = \mathbf{I}$ and $\mathbf{d} = \mathbf{0}$, this is the trivial case of no transformation. This case doesn't necessitate finding any twist coordinates.

If $\mathbf{R} = \mathbf{I}$ and $\mathbf{d} \neq \mathbf{0}$, this is the case of pure translation. The twist coordinates are

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \tag{3.49}$$

If $\mathbf{R} \neq \mathbf{I}$, which is all cases in which some rotation occurs (i.e., $\theta \neq 0$), we equate

$$\mathbf{R} = e^{\mathbf{S}(\mathbf{k})\theta}$$

and then solve for $\mathbf{k}$ and θ as described in Chapter 2. Then, from (3.47), we have the equation

$$\mathbf{d} = (\mathbf{I} - \mathbf{R})(\mathbf{k} \times \mathbf{v}) + \mathbf{k}\mathbf{k}^T\mathbf{v}\theta \tag{3.50}$$

from which $\mathbf{v}$ is the only unknown. Using the fact that $\mathbf{k} \times \mathbf{v} = \mathbf{S}(\mathbf{k})\mathbf{v}$, we can factor out $\mathbf{v}$,

$$\mathbf{d} = \left((\mathbf{I} - \mathbf{R})\,\mathbf{S}(\mathbf{k}) + \mathbf{k}\mathbf{k}^T\theta\right)\mathbf{v} \tag{3.51}$$

from which we solve

$$\mathbf{v} = \left((\mathbf{I} - \mathbf{R})\,\mathbf{S}(\mathbf{k}) + \mathbf{k}\mathbf{k}^T\theta\right)^{-1}\mathbf{d} \tag{3.52}$$

It can be shown that this matrix inverse always exists if $\mathbf{R} \neq \mathbf{I}$ (i.e., $\theta \neq 0$). Finally, the twist coordinates are

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{v} \\ \mathbf{k}\theta \end{bmatrix} = \begin{bmatrix} \left((\mathbf{I} - \mathbf{R})\,\mathbf{S}(\mathbf{k}) + \mathbf{k}\mathbf{k}^T\theta\right)^{-1}\mathbf{d} \\ \mathbf{k}\theta \end{bmatrix} \tag{3.53}$$

For the special case of pure rotations (i.e., $\mathbf{d} = \mathbf{0}$), the twist coordinates are simply

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{v} \\ \mathbf{k}\theta \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{k}\theta \end{bmatrix} \tag{3.54}$$

3.3.3 Screws

Every rigid-body motion—and thus every homogeneous transformation, conceptualized as an operator—can be realized by a rotation about an axis combined with a translation parallel to that axis. That's the motion of a screw!

The pitch of the screw, h, relates the amount of linear translation to the amount of angular rotation, θ (in units rad). Thus, there is a net translation of $h\theta$ after rotating by θ.

We need to describe the screw axis as a line in space, which is given by any point Q on that line combined with the direction of the axis $\mathbf{k}$. This line is then just a set of points. Alternatively, with respect to any given coordinate system i, we can construct the vector $\mathbf{q} = Q - O_i$, and conceptualize the line as set of vectors (Figure 3.8):

$$\mathcal{L} = \{\mathbf{q} + \lambda\mathbf{k} \quad : \quad \forall\lambda \in \mathcal{R}\} \tag{3.55}$$

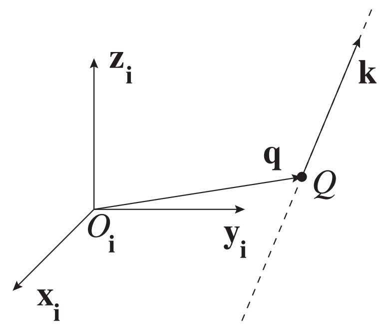


Figure 3.8: Depiction of a screw axis with respect to a coordinate system. Q is any point on the axis, which is represented by the vector $\mathbf{k}$.

3.3.4 From Twists to Screws

How do screws and twists relate? The pitch of the screw is calculated from the twist coordinates as

$$h = \mathbf{k}^T \mathbf{v} \tag{3.56}$$

where we continue to think of $\mathbf{k}$ as a unit vector, which defines the direction and the positive sign convention. We find that $h \to \infty$ as $\theta \to 0$, due to calculation of $\mathbf{v}$ in (3.52). A screw having infinite pitch corresponds to a pure translation. The screw axis is calculated from the twist coordinates as

$$\mathcal{L} = \mathbf{q} + \lambda \mathbf{k} = \mathbf{k} \times \mathbf{v} + \lambda \mathbf{k} \quad : \quad \lambda \in \mathcal{R}, \quad \text{if} \quad \theta \neq 0 \tag{3.57}$$

$$\mathcal{L} = \lambda \mathbf{v} \quad : \quad \lambda \in \mathcal{R}, \quad \text{if} \quad \theta = 0 \tag{3.58}$$

Finally, we call m the magnitude of the screw, which is how much we move a frame along the screw in order result in a given transformation. It is calculated from the twist coordinates as

$$m = \|\theta\|, \quad \text{if} \quad \theta \neq 0 \tag{3.59}$$

$$m = \|\mathbf{v}\|, \quad \text{if} \quad \theta = 0 \tag{3.60}$$

3.3.5 From Screws to Homogeneous Transformations

We can compute the homogeneous transformation matrix from the screw parameters. There are two cases:

- For pure translations along unit-vector $\mathbf{v}/\|\mathbf{v}\|$ by an amount $\|\mathbf{v}\|$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{v} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.61}$$

- For screws that involve rotation about a unit-vector $\mathbf{k}$ by an amount θ

$$\mathbf{T} = \begin{bmatrix} e^{\mathbf{k}\theta} & \left(\mathbf{I} - e^{\mathbf{k}\theta}\right)\mathbf{q} + h\theta\mathbf{k} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.62}$$

  and for the special case of pure rotation, $h = 0$, and thus

$$\mathbf{T} = \begin{bmatrix} e^{\mathbf{k}\theta} & \left(\mathbf{I} - e^{\mathbf{k}\theta}\right)\mathbf{q} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{3.63}$$

3.3.6 From Screws to Twists

We can also compute the twist coordinates of a given screw. There are two cases:

- or pure translations along unit-vector $\mathbf{v}/\|\mathbf{v}\|$ by an amount $\|\mathbf{v}\|$,

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{v}/\|\mathbf{v}\| \\ \mathbf{0} \end{bmatrix} \|\mathbf{v}\| \tag{3.64}$$

- For motions that involve rotation about a unit-vector $\mathbf{k}$ by an amount θ

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{q} \times \mathbf{k} + \mathbf{k}h \\ \mathbf{k} \end{bmatrix} \theta \tag{3.65}$$

and for the special case of pure rotation, $h = 0$, and thus

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{q} \times \mathbf{k} \\ \mathbf{k} \end{bmatrix} \theta \tag{3.66}$$

Example 3.19: Let's consider the pure translation of Figure 3.1(B). Let's say that the displacement is ${}^1\mathbf{d}_{12} = \begin{bmatrix} 1.5 & 0.5 & 0 \end{bmatrix}^T$. For a pure translation, the twist coordinates are

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the twist is

$$\boldsymbol{\Xi}(\mathbf{0}, \mathbf{d}) = \begin{bmatrix} \mathbf{S}(\mathbf{0}) & \mathbf{d} \\ \mathbf{0}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the homogeneous transformation is

$${}^1\mathbf{T}_2 = e^{\boldsymbol{\Xi}(\mathbf{0},\mathbf{d})} = \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^1\mathbf{R}_2 & {}^1\mathbf{d}_{12} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

It is easy to verify that this is correct by inspection. In fact, we could have started by writing ${}^1\mathbf{T}_2$ by inspection, and then could have computed the twist coordinates above.

Example 3.20: Let's consider the pure rotation of Figure 3.1(C). Let's think of defining the screw parameters in frame 1, even though in the case of pure rotation it doesn't really matter, since we conceptualize frames 1 and 2 as being initially aligned. The axis of rotation is the $\mathbf{z}$ axis, which is defined by $\mathbf{k} = \mathbf{z}$ and $\mathbf{q} = \mathbf{0}$ (since the line of the axis passes through the origin O_1). The magnitude of rotation is $\theta = \pi/2$. The pitch is $h = 0$ for a pure rotation. The twist coordinates are

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{0} \\ \mathbf{k}\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \pi/2 \end{bmatrix}$$

and the twist is

$$\boldsymbol{\Xi}(\mathbf{S}(\mathbf{k})\theta, \mathbf{0}) = \begin{bmatrix} \mathbf{S}(\mathbf{k})\theta & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\pi/2 & 0 & 0 \\ \pi/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the homogeneous transformation is

$${}^1\mathbf{T}_2 = e^{\boldsymbol{\Xi}(\mathbf{S}(\mathbf{k}\theta),\mathbf{0})} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to verify that this is correct by inspection. In fact, we could have started by writing ${}^1\mathbf{T}_2$ by inspection, and then computed twist coordinates. Let's try that. From ${}^1\mathbf{T}_2$, we isolate

$$\mathbf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We already know $\mathbf{v} = \mathbf{0}$ since $\mathbf{d} = \mathbf{0}$, and from $\mathbf{R}$ we compute

$$\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \frac{\pi}{2}$$

using the methods of Chapter 2.

Example 3.21: Let's reconsider the transformation between frames 1 and 0 in Figure 3.7. From inspection, the homogeneous transformation between frames is

$${}^0\mathbf{T}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

What is the screw motion that describes frame 1 with respect to frame 0? We begin by using $\mathbf{R}$ to solve for $\theta = 3.1416$ (i.e., $\theta = \pi$), using the method of Chapter 2. This is the case for which our basic equation that we typically use to solve for $\mathbf{k}$ becomes undefined, so we disturb our $\mathbf{R}$ by a tiny amount, e.g.,

$$\mathbf{R}' = \mathbf{R}(0.000001\mathbf{z})\mathbf{R}$$

and then solve for $\mathbf{k}$ (and resolve for θ) from this disturbed $\mathbf{R}'$, using the methods of Chapter 2:

$$\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \theta = 3.1416$$

Next, we solve

$$\mathbf{v} = \left((\mathbf{I} - \mathbf{R})\,\mathbf{S}(\mathbf{k}) + \mathbf{k}\mathbf{k}^T\theta\right)^{-1}\mathbf{d} = \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} \tag{3.67}$$

The axis of the screw is calculated as

$$\mathcal{L} = \mathbf{k} \times \mathbf{v} + \lambda\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \mathbf{q} + \lambda\mathbf{k}$$

$\forall\lambda \in \mathcal{R}$, and the pitch of the screw is calcualted as

$$h = \mathbf{k}^T\mathbf{v} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} = 0$$

We depict this screw axis in Figure 3.9. It is relatively easy to visualize that a frame initially aligned with frame 0, and then rotated about the screw axis by π would in fact result in the location of frame 1.

Example 3.22: Let's return to Figure 3.9, and consider the transformation from frame 2 to frame 0, where frame 2 is a pure translation from frame 1 along the $\mathbf{z}_1$ axis by a distance of 3:

$${}^0\mathbf{T}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Since frame 2 is a pure translation from frame 1, so we could build ${}^0\mathbf{T}_2$ as the composition of ${}^1\mathbf{T}_2 = \mathbf{Trans}({}^1\mathbf{d}_{12})$ with ${}^0\mathbf{T}_1$ from the previous example. However, we can also think of our screw from the previous example as having a pitch of $h = -3/\pi$, resulting in a displacement of -3 along the screw axis due to the rotation of $\theta = \pi$. Let's verify that this is the pitch that we compute from ${}^0\mathbf{T}_2$. Because ${}^0\mathbf{T}_2$ has the same $\mathbf{R}$ as ${}^0\mathbf{T}_1$ from the previous example, it also has the same $\mathbf{k}$ and θ. However, it has a different $\mathbf{d}$ (i.e., $\mathbf{d} = \begin{bmatrix} 0 & 3 & 2 \end{bmatrix}^T$), so we compute

$$\mathbf{v} = \left((\mathbf{I} - \mathbf{R})\,\mathbf{S}(\mathbf{k}) + \mathbf{k}\mathbf{k}^T\theta\right)^{-1}\mathbf{d} = \begin{bmatrix} -1.5 \\ 0 \\ 0.6366 \end{bmatrix} \tag{3.68}$$

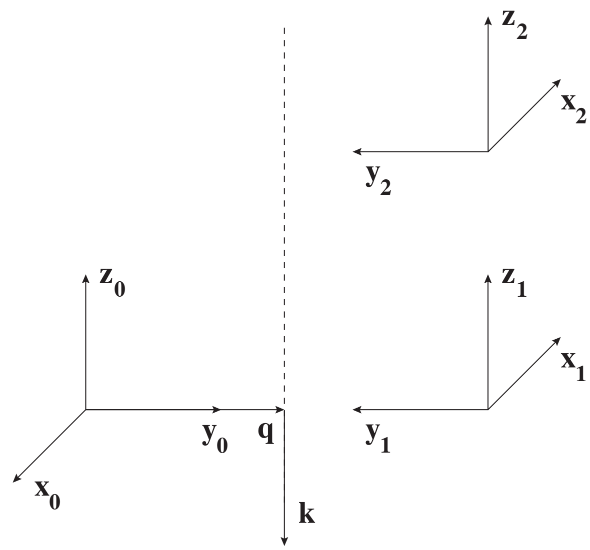


Figure 3.9: Frame 1, initialized at frame 0, is rotated about the screw axis $\mathbf{k}$ by $\theta = \pi$, with a screw pitch of $h = 0$, and with $\|\mathbf{q}\| = 1.5$. Frame 2, initialized at frame 0, is rotated about the screw axis $\mathbf{k}$ by $\theta = \pi$, with a screw pitch of $h = 2/\pi$.

from which we compute

$$h = \mathbf{k}^T \mathbf{v} = -0.6366 = -\frac{2}{\pi}$$

as expected.

Appendix A

Review of Linear Algebra

This appendix provides a review of linear-algebra concepts that are important in robotics. More information can be found in dedicated textbooks on linear algebra, such as Strang [2] or Horn and Johnson [3]. Throughout this appendix, scalars are represented as lower-case italic such as a, vectors are represented by lower-base bold such as $\mathbf{a}$, and matrices are represented as upper-case bold such as $\mathbf{A}$.

A.1 Vector Relations

The convention that vectors are column vectors is adopted here. The n-by-1 vector $\mathbf{a}$ has components

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \tag{A.1}$$

The tranpose operator converts a column vector into a row vector, and vice versa.

$$\mathbf{a}^T = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}, \qquad (\mathbf{a}^T)^T = \mathbf{a} \tag{A.2}$$

A.1.1 Inner Product

Suppose that the n-dimensional vector $\mathbf{b}$ has components

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{A.3}$$

and $\mathbf{a}$ is defined as above. The inner product (also called the dot product) of vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n \tag{A.4}$$

It is clear that the vectors $\mathbf{a}$ and $\mathbf{b}$ must be of the same dimension n for a dot product to be a meaningful concept. Another way to represent the inner product is as the matrix product of the 1-by-n row vector $\mathbf{a}^T$ on the left with an n-by-1 column vector $\mathbf{b}$ on the right.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{A.5}$$

Because the result is a scalar, the inner product is commutative:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}, \qquad \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} \tag{A.6}$$

The inner product is also used to define the length $||\mathbf{a}||$ (also called the Euclidean norm, or 2-norm) of a vector:

$$||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2} \tag{A.7}$$

A unit vector is a vector that has length 1.

The inner product $\mathbf{a} \cdot \mathbf{b}$ can also be represented as

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \cos\theta \tag{A.8}$$

where θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. Two vectors are perpendicular (also called orthogonal) when $\theta = \pi/2$ rad, in which case $\mathbf{a}\cdot\mathbf{b} = 0$. Two vectors are parallel when $\theta = 0$ rad, in which case $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|$; they are antiparallel when $\theta = \pi$ rad, in which case $\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\|\|\mathbf{b}\|$.

Any set (i.e., two or more) unit vectors that are perpendicular to each other are called orthonormal vectors. For three-dimensional vectors, special orthonormal vectors known as the standard basis are defined as:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{A.9}$$

A.1.2 Cross Product

The cross product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ of two three-dimensional (i.e., 3-by-1) vectors $\mathbf{a}$ and $\mathbf{b}$ can be represented as the following determinant (see Section A.2.6):

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} \tag{A.10}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{A.11}$$

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel or antiparallel, the resultant vector $\mathbf{c}$ is simply the zero vector. Otherwise, the resultant vector $\mathbf{c}$ is perpendicular to the plane formed by vectors $\mathbf{a}$ and $\mathbf{b}$. The cross product definition follows the right-hand rule: $\mathbf{a}$ is crossed into $\mathbf{b}$ by curling the fingers, and the vector $\mathbf{c}$ is in the direction of the thumb. The direction of crossing is toward the acute angle θ from $\mathbf{a}$ to $\mathbf{b}$. Conversely, the cross product $\mathbf{b} \times \mathbf{a}$ results in a vector $-\mathbf{c}$, i.e., is simply flips the resultant vector by $180°$. This means that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{A.12}$$

The magnitude of the cross product is described by

$$\|\mathbf{c}\| = ||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta \tag{A.13}$$

where, again, θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

A.1.3 Identities the Combine the Dot and Cross Products

Two common relations involving cross and dot products are given below.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \tag{A.14}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{A.15}$$

A.1.4 Outer Product

Matrix multiplication is associative (see Section A.2.2), so the first term on the right side in (A.15) may be rewritten as

$$\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) = \mathbf{b}(\mathbf{a}^T\mathbf{c}) = (\mathbf{b}\mathbf{a}^T)\mathbf{c} \tag{A.16}$$

where the outer product $\mathbf{b}\mathbf{a}^T$ is the 3-by-3 matrix

$$\mathbf{b}\mathbf{a}^T = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} b_1a_1 & b_1a_2 & b_1a_3 \\ b_2a_1 & b_2a_2 & b_2a_3 \\ b_3a_1 & b_3a_2 & b_3a_3 \end{bmatrix} \tag{A.17}$$

The outer product matrix is singular (i.e., it has rank 1) and cannot be inverted. It is distinguished from the inner product of two vectors, $\mathbf{b} \cdot \mathbf{a} = \mathbf{b}^T\mathbf{a}$, which is a scalar.

A.2 Matrix Relations

An m-by-n matrix $\mathbf{A}$ with elements a_{ij} has the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{A.18}$$

The left index i is the row number, of which there are m, and the right index j is the column number, of which there are n. We will write $\mathbf{A} = \{a_{ij}\}$ to indicate matrix $\mathbf{A}$ is made of elements a_{ij}.

Another way of describing a matrix is as a concatenation of column vectors. Define the m-by-1 vector $\mathbf{a}_j$ as the jth column of $\mathbf{A}$, of which there are n. Then

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \quad \text{where} \quad \mathbf{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \tag{A.19}$$

Similarly, a matrix can be described as a concatenation of row vectors.

A square matrix is a matrix that has the same number of rows and columns, i.e., $m = n$.

A.2.1 Matrix Addition

Matrices can be added if they have the same dimensions; for example, if $\mathbf{B} = \{b_{ij}\}$ is also m-by-n, then the result

$$\mathbf{C} = \{c_{ij}\} = \mathbf{A} + \mathbf{B} = \{a_{ij} + b_{ij}\} \tag{A.20}$$

is m-by-n. That is, two matrices are added by performing term-by-term addition.

A.2.2 Matrix Multiplication

Multiplication of a matrix $\mathbf{A}$ by a scalar r merely multiplies each element by the scalar:

$$\mathbf{C} = r\mathbf{A} = \{ra_{ij}\} \tag{A.21}$$

Multiplication of two matrices requires that the number of columns of the left matrix, say $\mathbf{A}$ which is m-by-n, equals the number of rows of the right matrix, say $\mathbf{D}$ which is n-by-p.

$$\mathbf{E} = \mathbf{AD} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{np} \end{bmatrix} = \{e_{ij}\}$$

where element e_{ij} is the multiplication of row i of $\mathbf{A}$, which is 1-by-n, with column j of $\mathbf{D}$, which is n-by-1:

$$e_{ij} = a_{i1}d_{1j} + \cdots + a_{in}d_{nj} \tag{A.22}$$

Right multiplication of a matrix by a vector is simply a special case of multiplication of two matrices in which $p = 1$. Similarly, left multiplication of a matrix by a row vector is simply a special case in which $m = 1$. The inner product can be viewed as a special case in which both $m = 1$ and $p = 1$.

Matrix multiplication is associative, meaning that the order in which the individual matrices are multiplied does not matter:

$$\mathbf{G} = \mathbf{ADF} = (\mathbf{AD})\mathbf{F} = \mathbf{A}(\mathbf{DF}) \tag{A.23}$$

A.2.3 Matrix Transpose

The transpose of a matrix switches rows and columns. For example, for the m-by-n matrix $\mathbf{A} = \{a_{ij}\}$, the transpose $\mathbf{B} = \mathbf{A}^T$ is an n-by-m matrix with elements $b_{ij} = a_{ji}$:

$$\mathbf{B} = \mathbf{A}^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \tag{A.24}$$

The transpose of the product of two matrices reverses the order:

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{A.25}$$

A.2.4 Symmetric and Skew-Symmetric Matrices

A symmetric matrix is one that equals its transpose. Thus $\mathbf{M} = \mathbf{M}^T$ and and $m_{ij} = m_{ji}$, which implies $\mathbf{M}$ is square.

A skew-symmetric matrix $\mathbf{S} = \{s_{ij}\}$ is a square matrix that is equal to the negative of its transpose:

$$\mathbf{S} = -\mathbf{S}^T \qquad \text{or} \qquad s_{ij} = -s_{ji} \tag{A.26}$$

This means that the diagonal elements s_{ii} have to be zero. For a 3-by-3 matrix $\mathbf{S}$, the off-diagonal elements are related by:

$$\mathbf{S} = \begin{bmatrix} 0 & s_{12} & s_{13} \\ s_{21} & 0 & s_{23} \\ s_{31} & s_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{bmatrix} \tag{A.27}$$

Thus, a 3-by-3 skew-symmetric matrix only has three independent terms.

Such skew-symmetric matrices encapsulate the cross product relationship. Factor the cross product (A.11) as a matrix-vector multiplication:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv \mathbf{S}(\mathbf{a})\mathbf{b}$$

where we define a function $\mathbf{S}(\mathbf{a})$ that converts a 3-by-1 vector $\mathbf{a}$ into a skew-symmetric matrix:

$$\mathbf{S}(\mathbf{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{A.28}$$

Conversely, given any skew-symmetric matrix $\mathbf{S}$ such as (A.27) applied to a vector $\mathbf{b}$, the product $\mathbf{Sb}$ can be interpreted as a cross-product

$$\mathbf{Sb} = \mathbf{S}(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b} \tag{A.29}$$

by extracting from $\mathbf{S}$ a vector $\mathbf{a}$ such that

$$\mathbf{a} = \begin{bmatrix} s_{32} \\ s_{13} \\ s_{21} \end{bmatrix} \tag{A.30}$$

Any square matrix $\mathbf{A}$ can be expressed as the sum of a symmetric matrix $\mathbf{M}$ and a skew-symmetric matrix $\mathbf{S}$:

$$\mathbf{A} = \mathbf{M} + \mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) \tag{A.31}$$

A.2.5 Positive-Definite and Positive-Semidefinite Matrices

A positive-definite matrix is a square matrix $\mathbf{A}$ such that

$$\mathbf{a}^T\mathbf{A}\mathbf{a} > 0 \quad \text{for any vector} \quad \mathbf{a} \neq \mathbf{0}, \tag{A.32}$$

and a positive-semidefinite matrix is a square matrix $\mathbf{A}$ such that

$$\mathbf{a}^T\mathbf{A}\mathbf{a} \geq 0 \quad \text{for any vector} \quad \mathbf{a} \neq \mathbf{0}. \tag{A.33}$$

When determining if a matrix $\mathbf{A}$ is positive-definite or positive-semidefinite, only the symmetric part of the matrix, $\mathbf{M}$ (see (A.31)), matters. This is because

$$\mathbf{a}^T\mathbf{A}\mathbf{a} = \mathbf{a}^T(\mathbf{M} + \mathbf{S})\mathbf{a} = \mathbf{a}^T\mathbf{M}\mathbf{a} + \mathbf{a}^T\mathbf{S}\mathbf{a} = \mathbf{a}^T\mathbf{M}\mathbf{a} \tag{A.34}$$

since

$$\mathbf{a}^T\mathbf{S}\mathbf{a} = 0 \quad \text{for any vector} \quad \mathbf{a} \neq \mathbf{0}. \tag{A.35}$$

The necessary and sufficient condition for $\mathbf{A}$ to be positive definite is that all of the eigenvalues of $\mathbf{M}$ are positive (see Section A.3). The necessary and sufficient condition for $\mathbf{A}$ to be positive semidefinite is that all of the eigenvalues of $\mathbf{M}$ are nonnegative.

A.2.6 Matrix Determinant

If a matrix $\mathbf{A}$ is square, it will have a determinant. For a 2-by-2 matrix $\mathbf{A}$, the determinant $\det(\mathbf{A}) = |\mathbf{A}|$ is

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{A.36}$$

For a 3-by-3 matrix $\mathbf{A}$, the determinant is

$$\begin{aligned} \det(\mathbf{A}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned} \tag{A.37}$$

For the determinant of matrices with higher dimensions, please consult a textbook dedicated to linear algebra.

A.2.7 Matrix Inverse

For an n-by-n square matrix $\mathbf{A}$, the inverse $\mathbf{A}^{-1}$ is that matrix which when multiplied against the original matrix yields an n-by-n identity matrix $\mathbf{I}$ with 1's on the entries of the main diagonal and 0's in all other entries:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{A.38}$$

A necessary and sufficient condition for an inverse to exist is that $\det(\mathbf{A}) \neq 0$. If the determinant is zero, it means that the rows or columns of the matrix are linearly dependent, and the matrix is not full rank.

The general formula to derive the inverse may be found in a linear-algebra textbook. The simplest example is the inverse of a 2-by-2 matrix $\mathbf{A}$:

$$\mathbf{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{A.39}$$

The inverse of the product of two matrices acts like the transpose in terms of switching the order of multiplication. If $\mathbf{A}$ and $\mathbf{B}$ are both n-by-n matrices, then

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{A.40}$$

A.3 Eigenvectors and Eigenvalues

Consider an n-by-n square matrix $\mathbf{A}$ that is a linear function mapping an n-by-1 input vector $\mathbf{x}$ to an n-by-1 output vector $\mathbf{y}$:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{A.41}$$

There are certain input vectors $\mathbf{x} = \mathbf{q}_i \neq \mathbf{0}$ that are special in the sense that the resulting output vector is parallel to the input vector (i.e., the output vector is simply input vector multiplied by some scalar λ_i):

$$\mathbf{A}\mathbf{q}_i = \mathbf{q}_i\lambda_i \tag{A.42}$$

We call such vectors eigenvectors, and we call the associated scalar multipliers eigenvalues. The eigenvectors and eigenvalues come as pairs, so we say that the eigenvalue λ_1 is associated with the eigenvector $\mathbf{q}_1$, and so forth.

To solve for eigenvalues and eigenvectors, we rearrange (A.42) as

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{q}_i = \mathbf{0} \tag{A.43}$$

where $\mathbf{I}$ is the identity matrix that has the same number of rows and columns as $\mathbf{A}$. Since $\mathbf{q}_i$ is in the null space of $\mathbf{A} - \lambda_i \mathbf{I}$, we know that this matrix is not full rank, and its determinant is zero:

$$|\mathbf{A} - \lambda_i \mathbf{I}| = 0 \tag{A.44}$$

For a n-by-n matrix, this determinant yields an n^{th}-order polynomial of λ_i; the n roots of this equation are the n eigenvalues. Many numerical packages for root finding exist. Once the eigenvalues of $\mathbf{A}$ are found, the associated eigenvectors are found by solving (A.43), using Guassian elimination. The eigenvectors are not unique in the sense that, once we find an eigenvector $\mathbf{q}_i$, we know that all vectors parallel and antiparallel with it are also eigenvectors with the same associated λ_i (i.e., $r\mathbf{q}_i$ is also an eigenvector for any $r \neq 0$). We think of this entire family of vectors as fundamentally being the same eigenvector. As a result, it is customary (although not necessary) to choose a unit-length vector from this family of vectors as the nominal eigenvector.

An n-by-n matrix can have at most n linearly independent eigenvectors, each of which will have an associated eigenvalue. It is possible that some of the eigenvalues might be repeated. A set of vectors being linearly independent means that none can be constructed as any linear combination of the others. If $\mathbf{A}$ has n linearly independent eigenvectors, we say it is diagonalizable. In such a case, we can combine all of the individual eigenvector-eigenvalue equations together as

$$\mathbf{A}\begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_n\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \tag{A.45}$$

which we can express in the compact form

$$\mathbf{AQ} = \mathbf{Q\Lambda} \tag{A.46}$$

where the columns of $\mathbf{Q}$ are the eigenvectors of $\mathbf{A}$, the diagonal matrix $\mathbf{\Lambda}$ comprises the associated eigenvalues, and $\mathbf{Q}$ is full rank and thus invertible.

Further, if (and only if) $\mathbf{A}$ is symmetric, the eigenvectors (and thus columns of $\mathbf{Q}$) are mutually orthogonal. If they have also each been made unit length, then $\mathbf{Q}$ is an orthogonal matrix (i.e., a square matrix whose columns are all unit length and mutually orthogonal, which is also called an orthonormal matrix). Orthogonal matrices have the useful property that their inverse is equal to their transpose.

A.4 The Singular Value Decomposition

Consider the arbitrary m-by-n matrix $\mathbf{A}$, which can be thought of as linear function mapping the n-by-1 input vector $\mathbf{x}$ to the m-by-1 output vector $\mathbf{y}$, as in (A.41). The matrix $\mathbf{A}$ can always be decomposed using the singular value decomposition (SVD) as

$$\mathbf{A} = \mathbf{U\Sigma V}^{-1} = \mathbf{U\Sigma V}^T, \tag{A.47}$$

where $\mathbf{U}$ is an m-by-m orthogonal matrix whose columns are the ordered unit-length output singular vectors (which are sometimes called the left singular vectors)

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix}, \tag{A.48}$$

$\mathbf{V}$ is an n-by-n orthogonal matrix whose columns are the ordered unit-length input singular vectors (which are sometimes called the right singular vectors)

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}, \tag{A.49}$$

and $\mathbf{\Sigma}$ is an m-by-n matrix that contains the $p = \min(m, n)$ ordered nonnegative singular values of $\mathbf{A}$:

$$\sigma_1 \geq \cdots \geq \sigma_p \geq 0. \tag{A.50}$$

Because $\mathbf{U}$ is orthogonal, its columns form a basis for m-dimensional space, meaning that every m-by-1 vector can be constructed as a unique linear combination of its columns. Similarly, because $\mathbf{V}$ is orthogonal, its columns form a basis for n-dimensional space.

$\mathbf{\Sigma}$ takes different forms, depending on its dimensions. If $m = n$ (for example, let's use $m = n = 3$), then $\mathbf{\Sigma}$ is a diagonal matrix of the form

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{A.51}$$

If $m < n$ (for example, let's use $m = 3$ and $n = 5$), then $\mathbf{\Sigma}$ is "wide" and of the form

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \end{bmatrix} \tag{A.52}$$

where the leftmost m-by-m block is a diagonal matrix containing the singular values, and the rightmost m-by-$(n - m)$ block is all zeros. If $m > n$ (for example, let's use $m = 5$ and $n = 3$), then $\mathbf{\Sigma}$ is "tall" and of the form

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.53}$$

where the upper n-by-n block is a diagonal matrix containing the singular values, and the lower $(m - n)$-by-n block is all zeros.

The first p input and output singular vectors are coupled through the equations

$$\mathbf{A}\mathbf{v}_i = \mathbf{u}_i \sigma_i. \tag{A.54}$$

Note the similarities and differences between (A.54) and the eigenvector-eigenvalue equation (A.42). We see that there is still a notion of special input directions coupled to special output directions, with a scalar multiplier that is paired with each special direction; but unlike with eigenvectors, the special input and output directions are not "the same" in any sense of the word (in general, they can be of different dimension and different units). We can rearrange (A.47) in the form

$$\mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{\Sigma}. \tag{A.55}$$

Note the similarities and differences between (A.55) and (A.46).

The input singular vectors form an orthonormal basis set for the input space. The input vector can be expressed, or constructed, as a unique linear combination of its basis vectors:

$$\mathbf{x} = \mathbf{v}_1\alpha_1 + \cdots + \mathbf{v}_n\alpha_n \tag{A.56}$$

where the α terms are simply scalar coefficients. If $n > m$ (i.e., there are more inputs than outputs), then all input singular vectors $\mathbf{v}_i$ with $i > m$ are null vectors of $\mathbf{A}$, meaning that they result in $\mathbf{A}\mathbf{v}_i = \mathbf{0}$.

The output singular vectors form a orthonormal basis set for the output space. In general, the output vector is also a linear combination of its basis vectors. Utilizing (A.54) and (A.56), the output can expressed as

$$\mathbf{y} = \mathbf{u}_1\sigma_1\alpha_1 + \cdots + \mathbf{u}_p\sigma_p\alpha_p \tag{A.57}$$

Note that $p \leq m$ since $p = \min(m, n)$. If $m > p$ (i.e., $m > n$, indicating that there are more outputs than inputs) the basis vectors $\mathbf{u}_i$ for all $i > p$ do not appear in the output; these are unachievable output directions, and no linear combination of them can be achieved.

If any of the singular values is 0, then the associated output singular vector is unachievable, and the associated input singular vector is a null vector of $\mathbf{A}$. The number of nonzero singular values is equal to the rank of $\mathbf{A}$.

Since the vectors in the basis sets are unit-length, they simply represent directions in the respective spaces, with no sense of magnitude. Because the singular values are ordered from largest to smallest, if we assume that all input directions are equally easy to command, then $\mathbf{u}_1$ represents the output direction that is easiest to generate, $\mathbf{u}_2$ represents the output direction that is second-easiest to generate, and so forth.

If our system has the same number of inputs and outputs, such that $\mathbf{\Sigma}$ is square as in (A.51), then there is a one-to-one correspondence between the input and output singular vectors. If our system has more inputs than outputs ($m < n$), such that $\mathbf{\Sigma}$ is “wide” like in (A.52), then only the first m input singular vectors correspond to respective output singular vectors. The remaining $n - m$ input singular vectors span the null space of the function $\mathbf{A}$. For our example $\mathbf{\Sigma}$ in (A.52), the input $\mathbf{x} = \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5$ results in the output $\mathbf{y} = \mathbf{0}$ regardless of the values of the coefficients α_4 and α_5. If our system has more outputs than inputs ($m > n$), such that $\mathbf{\Sigma}$ is “tall” like in (A.53), then the last $m - n$ output singular vectors form a subspace of the output space that is unachievable using any input. For our example $\mathbf{\Sigma}$ in (A.53), the output $\mathbf{y} = \beta_4\mathbf{u}_4 + \beta_5\mathbf{u}_5$ is unachievable for all values of the coefficients β_4 and β_5.

A.5 The Generalized Inverse

One of the most important uses of the SVD is to generalize the concept of the matrix inverse to matrices that are not invertible. Consider the general linear function in (A.41). Given some input $\mathbf{x}$, it is easy to solve for the unique output $\mathbf{y}$ using matrix multiplication. However, we are often interested in solving the inverse problem: Given some desired output $\mathbf{y}_{\text{des}}$, does an input $\mathbf{x}$ exist to generate the desired output, and if so, is the input unique? If $\mathbf{A}$ is square (i.e., $m = n$) and full rank (i.e., $\det(\mathbf{A}) \neq 0$), then we solve the problem using the matrix inverse:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}_{\text{des}} \tag{A.58}$$

and the solution is unique. However, if $\mathbf{A}$ is square but not full rank, or if $\mathbf{A}$ is not square, the matrix inverse does not exist, and we need to pursue a different solution.

In general, we can use the generalized inverse, also known as the pseudoinverse, to find a solution:

$$\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{y}_{\text{des}} \tag{A.59}$$

The generalized inverse $\mathbf{A}^\dagger$ is an n-by-m matrix that can be found using the SVD of $\mathbf{A}$ given in (A.47). The generalized inverse is defined as

$$\mathbf{A}^\dagger = \mathbf{V}\mathbf{\Sigma}^\dagger\mathbf{U}^{-1} = \mathbf{V}\mathbf{\Sigma}^\dagger\mathbf{U}^T \tag{A.60}$$

where $\mathbf{\Sigma}^\dagger$ is found by taking the transpose of $\mathbf{\Sigma}$ and then inverting all of the nonzero entries. For a specific example:

$$\mathbf{\Sigma} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{\Sigma}^\dagger = \begin{bmatrix} 1/10 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{A}$ is square and full-rank, then $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

If $\mathbf{A}$ is wide (i.e., $m < n$) and has full row rank (i.e., $\sigma_m \neq 0$), then there exists a solution for $\mathbf{x}$ given any $\mathbf{y}_{\text{des}}$, but the solution is not unique. In this case, (A.59) returns the solution for $\mathbf{x}$ in which $\|\mathbf{x}\|$ is minimized. That is, the generalized inverse finds the smallest input that achieves the desired output.

If $\mathbf{A}$ is tall (i.e., $m > n$), or if $\mathbf{A}$ is square or wide but does not have full row rank (i.e., $m \leq n$ and $\sigma_m = 0$), then there is no solution for $\mathbf{x}$ to achieve $\mathbf{y}_{\text{des}}$. In this case, (A.59) returns the solution for $\mathbf{x}$ in which $\|\mathbf{y} - \mathbf{y}_{\text{des}}\|$ is minimized. This is the least-squares solution, which can be thought of as the solution with the smallest error.

An important take-away here is that the singular-value matrix $\mathbf{\Sigma}$ contains all of the information that we need to understand what type of solutions we should expect from our problem, before we ever perform any actual generalized-inverse calculations. We know if a solution exists, and if so, we know if it is unique. In general, we know how to interpret the answer that is returned to us.

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