Summary of the dynamical core

In this section, we enumerate the calculations performed in the dynamical core, although they overlap with the previous descriptions.

Conversion of Horizontal Wind to Vorticity and Divergence

Obtain grid point values of vorticity and divergence from the grid point values of u_{ij}, v_{ij} for horizontal wind. First, we obtain the vorticity and divergence in spectral space, ζ_n^m, D_n^m,

$$ \zeta_n^m = \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\mathrm{i}m\, v_{ij} \cos\varphi_j \{Y_n^{m*}\}_{ij} \frac{w_j}{a(1-\mu_j^{2})} + \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
u_{ij} \cos\varphi_j (1-\mu_j^2) \frac{\partial }{\partial \mu} \{Y_n^{m*}\}_{ij} \frac{w_j}{a(1-\mu_j^{2})} ; , $$

$$ D_n^m = \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\mathrm{i}m\, u_{ij} \cos\varphi_j \{Y_n^{m*}\}_{ij} \frac{w_j}{a(1-\mu_j^{2})} - \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
v_{ij} \cos\varphi_j (1-\mu_j^2) \frac{\partial }{\partial \mu} \{Y_n^{m*}\}_{ij} \frac{w_j}{a(1-\mu_j^{2})} ; . $$

The grid point value is calculated by

$$ \zeta_{ij} = \mathcal{R}\mathbf{e} \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \zeta_n^m {Y_n^m}_{ij} ; , $$

and so on.

Corresponding file & subroutines: [G2Wpush, G2Wtrans, G2Wshift, W2Gpush, W2Gtrans, W2Gshift (xdsphe.F)]

Calculating a virtual temperature

virtual Temperature T_v is ,

$$ T_v = T(1 + \epsilon_v q - l) ; , $$

However, it is $\epsilon_v = R_v/R - 1$ and R_v is the gas constant for water vapor (461 Jkg^{-1}K^{-1}) and R is the gas constant for air (287.04 Jkg^{-1}K^{-1}).

Corresponding file & subroutines: [VIRTMD (dvtmp.F)]

Calculating the pressure gradient term

The pressure gradient term $\nabla \pi = \frac{1}{p_S} \nabla p_S$ is first used to define the π_n^m

$$ \pi_n^m = \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
(\ln p_S)_{ij} \{Y_n^{m *}\}_{ij} w_j ; , $$

to a spectral representation and then ,

$$ \frac{1}{a \cos \varphi} \left(\frac{\partial \pi}{\partial \lambda} \right)_{ij} = \frac{1}{a \cos \varphi} \frac{\mathcal R}{\mathbf{e}} \sum_{m=-N}^{N} \sum_{n=|m|}^{N}
\mathrm{i}m\, \tilde{X}_n^m \{Y_n^m\}_{ij} ; , $$

$$ \frac{1}{a} \left(\frac{\partial \pi}{\partial \varphi} \right)_{ij} =
\frac{1}{a \cos \varphi} \frac{\mathcal R}{\mathbf{e}} \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \pi_n^m (1-\mu^{2}) \frac{\partial }{\partial \mu} \{Y_n^m\}_{ij} ; . $$

Corresponding file & subroutine: [PSDOT (dgdyn.F)]

Diagnosis of vertical flow

Pressure change term, and lead DC,

$$ \frac{\partial \pi}{\partial t}
= - \sum_{k=1}^{K} \left\{ D_k \Delta\sigma_k + (\{\mathbf{v}\}_k \cdot \nabla \pi)\Delta B_k \right\} $$

$$ \frac{(m\dot{\eta})_{k-1/2}}{p_s} = - B_{k-1/2} \frac{\partial \pi}{\partial t} - \sum_{l=k}^{K}\left\{ D_l \Delta\sigma_l + (\{\mathbf{v}\}_l \cdot \nabla \pi)\Delta B_l \right\} $$

and its non-gravity components.

$$ \left(\frac{\partial \pi}{\partial t} \right)^{NG} = - \sum_{k=1}^{K} \{\mathbf{v}\}_k \cdot \nabla \pi \Delta B_k $$

$$ \frac{(m\dot{\eta})^{NG}_{k-1/2}}{p_s} = - B_{k-1/2} \left(\frac{\partial \pi}{\partial t} \right)^{NG} - \sum_{l=k}^{K} \{\mathbf{v}\}_l \cdot \nabla \pi \Delta B_l $$

Corresponding file and subroutine: [PSDOT (dgdyn.F)]

Tendency terms due to advection

Momentum advection term:

$$ (A_u)k = (\zeta_k + f) v_k - \left[\frac{(m\dot{\eta})_{k-1/2}}{p_s} \frac{u_{k-1} - u_k}{\Delta\sigma_{k-1}+\Delta\sigma_k} + \frac{(m\dot{\eta})_{k+1/2}}{p_s} \frac{u_k - u_{k+1}}{\Delta\sigma_{k}+\Delta\sigma_{k+1}} \right] $$

$$ - \frac{1}{a\cos\varphi} \frac{\partial \pi}{\partial \lambda}(C_p T_{v,k}\hat{\kappa}-R\bar{T}) + {\mathcal F}_x $$

$$ (A_v)k = - (\zeta_k + f) u_k - \left[\frac{(m\dot{\eta}){k-1/2}}{p_s} \frac{v_{k-1} - v_k}{\Delta\sigma_{k-1}+\Delta\sigma_k} + \frac{(m\dot{\eta}){k+1/2}}{p_s} \frac{v_k - v{k+1}}{\Delta\sigma_{k}+\Delta\sigma_{k+1}} \right] $$

$$ - \frac{1}{a} \frac{\partial \pi}{\partial \varphi}(C_p T_{v,k}\hat{\kappa}-R\bar{T}) + {\mathcal F}_y $$

Temperature advection term:

$$ (uT')_k = u_k(T_k - \bar{T}) $$

$$ (vT')_k = v_k(T_k - \bar{T}) $$

$$ H_k = T_k' D_k - \left[\frac{(m\dot{\eta}){k-1/2}}{p_s} \frac{\hat{T}{k-1/2} - T_k}{\Delta \sigma_l} + \frac{(m\dot{\eta}){k+1/2}}{p_s} \frac{T_k - \hat{T}{k+1/2}}{\Delta \sigma_l} \right] $$

$$ + \hat{\kappa}k {\mathbf{v}}k \cdot \nabla \pi T{v,k} $$

$$ - \alpha_k \sum{l=k}^{K} (D_l \Delta \sigma_l + ({\mathbf{v}}l \cdot \nabla \pi)\Delta B_l) \frac{T{v,k}}{\Delta \sigma_k} $$

$$ - \beta_k \sum_{l=k+1}^{K} (D_l \Delta \sigma_l + ({\mathbf{v}}l \cdot \nabla \pi)\Delta B_l) \frac{T{v,k}}{\Delta \sigma_k} $$

Water vapor advection term:

$$ (uq)_k = u_kq_k $$

$$ (vq)_k = v_kq_k $$

$$ R_k = q_k D_k - \frac{1}{2} \left[\frac{(m\dot{\eta}){k-1/2}}{p_s} \frac{q{k-1} - q_k}{\Delta\sigma_k} + \frac{(m\dot{\eta}){k+1/2}}{p_s} \frac{q_k - q{k+1}}{\Delta\sigma_k} \right] $$

Corresponding file & subroutine [GRTADV, GRUADV (dgdyn.F)]

Transformation of prognostic variables to spectral space

(122) and (123).

Transform $u_{ij}^{t-\Delta t}, v_{ij}^{t-\Delta t}$ to a spectral representation of vorticity and divergence ζ_n^m, D_n^m. Furthermore, transforming the temperature $T^{t-\Delta t}$, specific humidity $q^{t-\Delta t}$, and $\pi = \ln p_S^{t-\Delta t}$ to

$$ X_n^m = \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij} Y_n^{m*}{}_{ij} w_j; , $$

to a spectral representation.

Corresponding file & subroutine: [G2Wpush, G2Wtrans, G2Wshift (xdsphe.F)]

Transformation of tendency terms to spectral space

Tendency Term of Vorticity

$$ \frac{\partial\zeta_n^m}{\partial {t}} = \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\mathrm{i}m (A_v){ij} \cos \varphi_j {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} \ +\frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
(A_u){ij} \cos \varphi_j (1-\mu_j^2) \frac{\partial }{\partial \mu} {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} $$

The non-gravity wave component of the tendency term of the divergence

$$ \left(\frac{\partial{D_n^m}}{\partial {t}} \right)^{NG} = \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\mathrm{i}m (A_u){ij} \cos \varphi_j {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} \ -\frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
(A_v){ij} \cos \varphi_j (1-\mu_j^2) \frac{\partial }{\partial \mu} {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} \ -\frac{n(n+1)}{a^{2}} \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\hat{E}{ij} {Y_n^{m *}}{ij} w_j \ $$

The non-gravity wave component of the tendency term of temperature

$$ \left(\frac{\partial{T_n^m}}{\partial {t}} \right)^{NG} = - \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\mathrm{i}m (u T'){ij} \cos \varphi_j {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} \ + \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
(v T'){ij} \cos \varphi_j (1-\mu_j^2) \frac{\partial }{\partial \mu} {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} \ + \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\hat{H}{ij} {Y_n^{m *}}{ij} w_j $$

Tendency term of water vapor

$$ \frac{\partial q_n^m}{\partial {t}} = - \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J}
\mathrm{i}m (uq){ij} \cos \varphi_j {Y_n^{m *}}{ij} \frac{w_j}{a(1-\mu_j^{2})} \ + \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J} $$

$$
(vq)_{ij} \cos \varphi_j (1-\mu_j^2) \frac{\partial }{\partial \mu} {Y_n^{m *}}_{ij} \frac{w_j}{a(1-\mu_j^{2})} \ + \frac{1}{I} \sum_{i=1}^{I} \sum_{j=1}^{J} R_{ij} {Y_n^{m *}}_{ij} w_j
$$

Corresponding file & subroutines: [G2Wpush, G2Wtrans, G2Wshift (xdsphe.F)]

Time integration in spectral space

Equations in matrix form

$$ \left\{ (1+2\Delta t {\mathcal D}_H)(1+2\Delta t {\mathcal D}_M) \underline{I}
- (\Delta t)^{2} (\underline{W} \ \underline{h} + (1+2\Delta t {\mathcal D}M) {\mathbf{G}} {\mathbf{C}}^{T}) \nabla^{2}{\sigma} \right\} \overline{ {\mathbf{D}} }^{t} \ = (
1+2\Delta t {\mathcal D}H)(1-\Delta t {\mathcal D}M) {\mathbf{D}}^{t-\Delta t} +\Delta t \left(\frac{\partial {\mathbf{D}}}{\partial t} \right){NG}
\ -\Delta t \nabla^{2}{\sigma}
\left[(1+2\Delta t {\mathcal D}H) {\mathbf{\Phi}}{S} + \underline{W} \left[(1-2\Delta t {\mathcal D}H) {\mathbf{T}}^{t-\Delta t} + \Delta t \left(\frac{\partial {\mathbf{T}}} {\partial t}
\right){NG} \right] \right. \ \left. \hspace*{20mm} + (1+2\Delta t {\mathcal D}H) {\mathbf{G}} \left[\pi^{t-\Delta t} + \Delta t \left(\frac{\partial \pi} {\partial t} \right){NG} \right] \right] \right) . $$

Using LU decomposition, $\bar{D}$ is obtained by solving for

$$
\frac{\partial \mathbf{T}}{\partial t} = \left(\frac{\partial \mathbf{T}}{\partial t} \right)_{NG} - \underline{h}\mathbf{D}
$$

$$
\frac{\partial \pi}{\partial t} = \left(\frac{\partial \pi}{\partial t} \right)_{NG} - \mathbf{C} \cdot \mathbf{D}
$$

Calculate the value of the spectrum in $\partial \mathbf{T}/\partial t$, $\partial \pi/\partial t$ and then calculate the value of the spectrum in $t + \Delta t$ using

$$
\zeta^{t+\Delta t} = \left(\zeta^{t-\Delta t} + 2\Delta t \frac{\partial \zeta}{\partial t} \right) (1+2\Delta t \mathcal{D}_M)^{-1} \ \ D^{t+\Delta t} = 2\bar{D} - D^{t-\Delta t} \ \ T^{t+\Delta t} = \left(T^{t-\Delta t} + 2\Delta t \frac{\partial T}{\partial t} \right)(1+2\Delta t \mathcal{D}_H)^{-1} \ \ q^{t+\Delta t} = \left(q^{t-\Delta t} + 2\Delta t \frac{\partial q}{\partial t} \right)(1+2\Delta t \mathcal{D}_E)^{-1}
$$

Corresponding file & subroutine: [TINTGR (dintg.F)]

Transformation of prognostic variables to grid point Values

Obtain grid values of horizontal wind speed from the spectral values of vorticity and divergence (ζ_n^m, D_n^m) u_{ij}, v_{ij}.

$$ u_{ij} = \frac{1}{\cos \varphi_j} {\mathcal R}{\mathbf{e}} \sum_{m=-N}^{N} \sum_{\stackrel{n=|m|}{n \neq 0}}^{N} \left\{ \frac{a}{n(n+1)} \zeta_n^m (1-\mu^{2}) \frac{\partial{}}{\partial {\mu}} {Y_n^m}{ij} - \frac{\mathrm{i}m a}{n(n+1)} D_n^m {Y_n^m}{ij} \right\} $$

$$ v_{ij} = \frac{1}{\cos \varphi_j} {\mathcal R}{\mathbf{e}} \sum_{m=-N}^{N} \sum_{\stackrel{n=|m|}{n \neq 0}}^{N} \left\{ - \frac{\mathrm{i}m a}{n(n+1)} \zeta_n^m {Y_n^m}_{ij} - \frac{a}{n(n+1)} \tilde{D}_n^m (1-\mu^{2}) \frac{\partial{}}{\partial {\mu}} {Y_n^m}{ij} \right\} $$

Furthermore,

$$
T_{ij} = \mathcal{R}\mathbf{e} \sum_{m=-N}^{N} \sum_{n=|m|}^{N} T_n^m Y_n^m{}_{ij};,
$$

T_{ij}, π_{ij}, q_{ij}, and so on,

$$ {p_S}_{ij} = \exp \pi_{ij} $$

to calculate.

Corresponding file & subroutines: [W2Gpush, W2Gtrans, W2Gshift (xdsphe.F)]

Diffusion Correction along pressure level

The horizontal diffusion is applied on the surface of $\eta-$plane, but it can cause problems in large slopes, such as transporting water vapor uphill and causing false precipitation at the top of a mountain. To mitigate this problem, corrections have been made for T, q, l to make the diffusion closer to that of the p surface, e.g., for T, q, l.

$$ {\mathcal D}p (T) = (-1)^{N_D/2} K \nabla^{N_D}p T
\simeq (-1)^{N_D/2} K \nabla^{N_D}{\eta} T
- \frac{\partial{\sigma}}{\partial {p}} (-1)^{N_D/2} K \nabla^{N_D}{\eta} p \cdot \frac{\partial{T}}{\partial {\sigma}} $$

$$ = (-1)^{N_D/2} K \nabla^{N_D}{\eta} T
- (-1)^{N_D/2} K \nabla^{N_D}{\eta} \pi \cdot \sigma \frac{\partial{T}}{\partial {\sigma}} $$

$$ = {\mathcal D} (T) - {\mathcal D} (\pi) \sigma \frac{\partial{T}}{\partial {\sigma}} $$

So,

$$
T_k \leftarrow T_k - 2\Delta t \sigma_k \frac{T_{k+1} - T_{k-1}}{\sigma_{k+1} - \sigma_{k-1}} \mathcal{D}(\pi)
$$

and so on. In $\mathcal{D}(\pi)$, the spectral value of π is converted to a grid by multiplying the spectral value of π_n^m by the spectral representation of the diffusion coefficient.

Corresponding file & subroutine: `[CORDIF (ddifc.F)]`

Frictional heat associated with diffusion.

Frictional heat from diffusion is ,

$$ Q_{DIF} = - \left(u_{ij} {\mathcal D}(u)_{ij} + v_{ij} {\mathcal D}(v)_{ij} \right) $$

It is estimated that Therefore,

$$ T_k \leftarrow T_k - \frac{2 \Delta t}{C_p} \left(u_{ij} {\mathcal D}(u)_{ij} + v_{ij} {\mathcal D}(v)_{ij} \right) $$

Corresponding file & subroutine: `[CORDIF (ddifc.F)]`

Horizontal Diffusion and Rayleigh Friction

The coefficients of horizontal diffusion can be expressed spectrally,

$$ \mathcal{D}_M{}_n^m = K_M \left[\left(\frac{n(n+1)}{a^2} \right)^{N_D/2} - \left(\frac{2}{a^2} \right)^{N_D/2} \right] + K_R $$

$$ \mathcal{D}_H{}_n^m = K_M \left(\frac{n(n+1)}{a^2} \right)^{N_D/2} $$

$$ \mathcal{D}_E{}_n^m = K_E \left(\frac{n(n+1)}{a^2} \right)^{N_D/2} $$

K_R is the Rayleigh coefficient of friction. The Rayleigh coefficient of friction is

$$ K_R = K_R^0 \left[1 + \tanh\left(\frac{z - z_R}{H_R} \right) \right] $$

However, the profile is given in the same way as However,

$$ z = -H \ln \sigma $$

The results are approximate to those of $K_R^0 = (30 day)^{-1}$ and $z_R = -H \ln \sigma_{top}$. The standard values are $K_R^0 = (30 day)^{-1}$, $z_R = -H \ln \sigma_{top}$ (σ_{top}: top level of the model), $H = 8000$ m, and $H_R = 7000$ m.

Corresponding file & subroutine `[DSETDF (dsetd.F)]`

Time Filter

To reduce numerical mode associated with leap frog scheme, time filter is applied every time step. MIORC6 used modified Asselin time filter (Williams, 2009), which is updated version of Asselin(1972) used previous version of MIROC. Although Asselin time filter attenuate high frequency physical mode, bringing low accuracy of leap frog scheme, current time filter succeeded in suppressing it.

Modified Asselin filter is expressed as following equation

$$ \bar{\bar{X}}^t = \bar{X}^t + \nu\alpha[\bar{\bar{X}}^{t-\Delta t} - 2\bar{X}^t + X^{t+\Delta t}] $$

$$ \bar{X}^{t+\Delta t} = X^{t+\Delta t} + \nu(1 - \alpha)[\bar{\bar{X}}^{t-\Delta t} - 2\bar{X}^t + X^{t+\Delta t}] $$

where bar indicates time filter. The parameters set to $\nu = 0.05$, $\alpha = 0.5$. Assuming $\alpha = 1$, modified Asselin filter is same as Asselin filter.

In the model,

$$ \bar{\bar{X}}^{t*} = (1 - \nu\alpha)^{-1}[(1 - 2\nu\alpha)\bar{X}^t + \nu\alpha\bar{\bar{X}}^{t-\Delta t}] $$

is firstly calculated at `MODULE: [DADVNC]` where transformation of prognostic variableto grid point values. And then, $X^{t-\Delta t} - 2X^t$ is stored. When the $X^{t+\Delta t}$ is obtained later, time filter conduct at `MODULE [TFILT]`,

$$ \bar{\bar{X}}^t = (1 - \nu\alpha)\bar{\bar{X}}^{t*} + \nu\alpha X^{t+\Delta t} $$

$$ \bar{X}^{t+\Delta t} = X^{t+\Delta t} + \nu(1 - \alpha)[\bar{\bar{X}}^{t-\Delta t} - 2\bar{X}^t + X^{t+\Delta t}] $$

Corresponding file & subroutine: `[DADVNC (dadvn.F)]`

Correction for conservation of mass

In the spectral method, the global integral of $\pi = \ln p_S$ is preserved with rounding errors removed, but the preservation of the mass, i.e. the global integral of p_S is not guaranteed. Moreover, a wavenumber break in the spectra sometimes results in negative values of the water vapor grid points. For this reason, we perform a correction to preserve the masses of dry air, water vapor, and cloud water, and to remove the regions with negative water vapor content.

Before entering dynamical calculations, `[FIXMAS]`, the global integrals of water vapor and cloud water are calculated for M_q, M_l.

$$ M_q^0 = \sum_{ijk} q p_S \Delta\lambda_i w_j \Delta\sigma_k \qquad M_l^0 = \sum_{ijk} l p_S \Delta\lambda_i w_j \Delta\sigma_k $$

In the first step of the calculation, the dry mass M_d is calculated and stored.

$$M_d^0 = \sum_{ijk} (1 - q - l) p_S \Delta \lambda_i w_j \Delta \sigma_k$$

After exiting dynamical calculation, [MASFIX], the following procedure is followed.

First, negative water vapor is removed by dividing the water vapor from the grid points immediately below the grid points. Suppose that $q_k < 0$ is used,

$$q_k' = 0 \quad q_{k-1}' = q_{k-1} + \frac{\Delta p_k}{\Delta p_{k-1}} q_k$$

However, this should only be done if it is $q_{k-1}' \geq 0$.

Next, set the value to zero for the grid points not removed by the above procedure.

   3. calculate the global integral value of M_q and multiply the global water vapor content by a fixed percentage so that it is the same as that of M_q^0.

$$q'' = \frac{M_q^0}{M_q} q'$$

   4. correct for dry air mass Likewise calculate M_d,

$$p_S'' = \frac{M_d^0}{M_d} p_S$$

Corresponding file & subroutine: [FIXMAS, MASFIX (dmfix.F)]