

Horizontal discretization

The horizontal discretization is based on the spectral transformation method (Bourke, 1988). The differential terms for longitude and latitude are evaluated by the orthogonal function expansion, while the non-linear terms are calculated on the grid.

Spectral Expansion.

As an expansion function, the spherical harmonic functions $Y_n^m(\lambda, \mu)$, which are eigenfunction of Laplacian on a sphere, are used. However, $\mu \equiv \sin \varphi$ is used. Y_n^m satisfies the following equation,

$$\nabla_{\sigma}^2 Y_n^m(\lambda, \mu) = -\frac{n(n+1)}{a^2} Y_n^m(\lambda, \mu)$$

Using the Associated Legendre function P_n^m it is written as follows.

$$Y_n^m(\lambda, \mu) = P_n^m(\mu) e^{im\lambda}$$

However, it is $n \geq |m|$.

The expansion by spherical harmonic functions is ,

$$Y_{n\ ij}^m \equiv Y_n^m(\lambda_i, \mu_j)$$

When I write ,

$$X_{ij} \equiv X(\lambda_i, \mu_j) = \mathcal{R}e \sum_{m=-N}^N \sum_{n=|m|}^N X_n^m Y_{n\ ij}^m,$$

The inverse of that is ,

$$X_n^m = \frac{1}{4\pi} \int_{-1}^1 d\mu \int_0^{2\pi} d\lambda X(\lambda, \mu) Y_n^m(\lambda, \mu) \\ = \frac{1}{4\pi} \sum_{i=1}^I \sum_{j=1}^J X_{ij} Y_{n\ ij}^m w_j$$

The formula is expressed as To evaluate by replacing the integral with a sum, we use the Gauss trapezoidal formula for the λ integral and the Gauss-Legendre integral formula for the μ integral. μ_j is the Gauss latitude and w_j is the Gaussian weights. Also, λ_i is a grid of evenly spaced Gaussian weights.

Using spectral harmonics transformation, the grid point values of the terms containing the derivatives can be calculated as follows.

$$\left(\frac{\partial X}{\partial \lambda} \right)_{ij} = \{\mathcal{R}e\} \sum_{m=-N}^N \sum_{n=|m|}^N \mathrm{i} m X_n^m Y_{n\ ij}^m$$

$$\left(\cos \varphi \frac{\partial X}{\partial \varphi} \right)_{ij} = \{\mathcal{R}e\} \sum_{m=-N}^N \sum_{n=|m|}^N X_n^m (1 - \mu^2) \frac{\partial}{\partial \mu} Y_{n\ ij}^m$$

Furthermore, the grid point values of u, v can be obtained from the spectral components of ζ and D as follows

$$u_{ij} = \frac{1}{\cos\varphi} \sum_{m=-N}^N \sum_{\substack{n=|m| \\ n \neq 0}}^N \left\{ \frac{a}{n(n+1)} \zeta_n^m (1-\mu^2) \frac{\partial}{\partial \mu} \{Y_n^m\}_{ij} - \frac{\mathbf{m} \cdot \mathbf{a}}{n(n+1)} D_n^m \{Y_n^m\}_{ij} \right\}$$

$$v_{ij} = \frac{1}{\cos\varphi} \sum_{m=-N}^N \sum_{\substack{n=|m| \\ n \neq 0}}^N \left\{ -\frac{\mathbf{m} \cdot \mathbf{a}}{n(n+1)} \zeta_n^m \{Y_n^m\}_{ij} - \frac{a}{n(n+1)} D_n^m (1-\mu^2) \frac{\partial}{\partial \mu} \{Y_n^m\}_{ij} \right\}$$

The derivative appearing in the advection term of the equation is calculated as

$$\left(\frac{1}{a \cos\varphi} \frac{\partial A}{\partial \lambda} \right) n^m = \frac{1}{4\pi} \int_{-1}^1 d\mu \int_0^\pi d\lambda \frac{1}{a \cos\varphi} \frac{\partial A}{\partial \lambda} Y_n^m \backslash = \frac{1}{4\pi} \int_{-1}^1 d\mu \int_0^\pi d\lambda \, \mathbf{m} \cdot \mathbf{A} \cos\varphi \frac{1}{a(1-\mu^2)} Y_n^m \backslash = \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \mathbf{m} \cdot \mathbf{A}_{ij} \cos\varphi_j \{Y_n^m\}_{ij} \frac{w_j}{a(1-\mu_j^2)}$$

$$\left(\frac{1}{a \cos\varphi} \frac{\partial}{\partial \varphi} (A \cos\varphi) \right) n^m = \frac{1}{4\pi} \int_{-1}^1 d\mu \int_0^\pi d\lambda \frac{\partial}{\partial \mu} (A \cos\varphi) Y_n^m \backslash = -\frac{1}{4\pi} \int_{-1}^1 d\mu \int_0^\pi d\lambda A \cos\varphi \frac{\partial}{\partial \mu} Y_n^m \backslash = -\frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J A_{ij} \cos\varphi_j (1-\mu_j^2) \frac{\partial}{\partial \mu} \{Y_n^m\}_{ij} \frac{w_j}{a(1-\mu_j^2)}$$

Furthermore,

$$(\nabla_\sigma^2 X)_n^m = -\frac{n(n+1)}{a^2} X_n^m$$

to be used for the evaluation of the ∇^2 section.

Horizontal Diffusion Term

The horizontal diffusion term is implemented in the form ∇^{N_D} as follows.

$$\begin{aligned} \mathcal{D}(\zeta) &= K_{MH} \left[(-1)^{N_D/2} \nabla^{N_D} - \left(\frac{2}{a^2} \right)^{N_D/2} \right] \zeta, \\ \mathcal{D}(D) &= K_{MH} \left[(-1)^{N_D/2} \nabla^{N_D} - \left(\frac{2}{a^2} \right)^{N_D/2} \right] D, \\ \mathcal{D}(T) &= (-1)^{N_D/2} K_{HH} \nabla^{N_D} T, \\ \mathcal{D}(q) &= (-1)^{N_D/2} K_{EH} \nabla^{N_D} q. \end{aligned}$$

This horizontal diffusion term damps high frequency component occuring aliasing for computational stability. In order to represent selective horizontal diffusion on small scales, $4 \sim 16$ is used as N_D . Here, the extra term for vorticity and divergence diffusion indicates that the term of rigid body rotation in $n = 1$ does not decay.

Spectral representation of equations

1. a series of equations

$$\frac{\partial \pi_m}{\partial t} = - \sum_{k=1}^K (D_n^m)^k \Delta \sigma_k + \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J Z_{ij} Y_n^{m*} w_j, \quad \$\$$$

where,

$$Z \equiv - \sum_{k=1}^K \mathbf{v}_k \cdot \nabla \pi.$$

2. equation of motion

$$\frac{\partial \zeta_n^m}{\partial t} = \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \mathrm{m} (A_v)_{ij} \cos \varphi_{ij} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} + \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J (A_u)_{ij} \cos \varphi_{ij} (1-\mu_j^2) \frac{\partial}{\partial \mu} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} - (\mathcal{D}_M)_n^m \zeta_n^m, \quad \$\$$$

$$\frac{\partial \tilde{D}_n^m}{\partial t} = \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \mathrm{m} (A_u)_{ij} \cos \varphi_{ij} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} - \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J (A_v)_{ij} \cos \varphi_{ij} (1-\mu_j^2) \frac{\partial}{\partial \mu} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} - \frac{n(n+1)}{a^2} \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J E_{ij} Y_n^{m*} w_j + \frac{n(n+1)}{a^2} (\Phi_n^m + C_p) \hat{\kappa}_k \bar{T}_k \pi_n^m - (\mathcal{D}_M)_n^m D_n^m, \quad \$\$$$

where

$$(\mathcal{D}_M)_n^m = K_{MH} \left[\left(\frac{n(n+1)}{a^2} \right)^{N_D/2} - \left(\frac{2}{a^2} \right)^{N_D/2} \right]. \quad \$\$$$

3. thermodynamic equation

$$\frac{\partial T_n^m}{\partial t} = - \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \mathrm{m} u_{ij} T'_{ij} \cos \varphi_{ij} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} + \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J v_{ij} T'_{ij} \cos \varphi_{ij} (1-\mu_j^2) \frac{\partial}{\partial \mu} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} + \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \left(H_{ij} + \frac{Q_{ij} + Q_{diff}}{C_p} \right) Y_n^{m*} w_j - (\tilde{\mathcal{D}}_H)_n^m T_n^m, \quad \$\$$$

where,

$$(\tilde{\mathcal{D}}_H)_n^m = K_{HH} \left(\frac{n(n+1)}{a^2} \right)^{N_D/2}. \quad \$\$$$

4. water vapor formula

$$\frac{\partial q_n^m}{\partial t} = - \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \mathrm{m} u_{ij} q_{ij} \cos \varphi_{ij} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} + \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J v_{ij} q_{ij} \cos \varphi_{ij} (1-\mu_j^2) \frac{\partial}{\partial \mu} Y_n^{m*} \frac{w_j}{a(1-\mu_j^2)} + \frac{1}{l} \sum_{i=1}^l \sum_{j=1}^J \left(\hat{R}_{ij} + S_{q,ij} \right) Y_n^{m*} w_j + (\mathcal{D}_H)_n^m q_n^m \quad \$\$$$

where

$$({\mathcal D}_E)^m = K(EH) \left(\frac{n(n+1)}{a^2} \right)^{N_D/2} .$$