

Coordinates and Transformations

2016 Spring

National Cheng Kung University

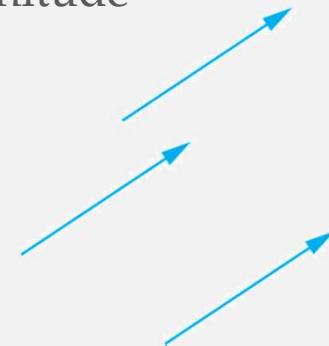
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Shih-Chin Weng 翁士欽 (西基電腦動畫)



Scalars, Pints, & Vectors

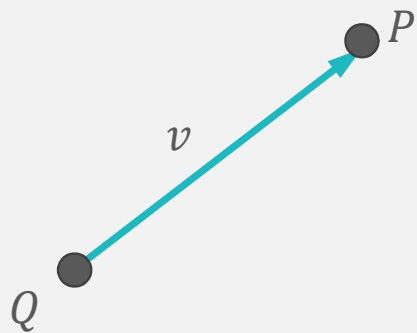
- Basic geometric objects and relationships among them can be described using scalars, points, and vectors
 - A **point** is a location in space
 - A **scalar** is used to specify quantities such as distance between two points.
 - Scalars obey a set of rules that are abstractions of the operations of ordinary arithmetic
 - Addition/subtraction, multiplication/division
 - A **vector** is a quantity with two attributes: direction and magnitude
 - A vector does not have a fixed location in space



Points

■ Operations between point and vector:

- Point-point subtraction yields a vector.
- Equivalent to point-vector addition.



$$v = P - Q$$

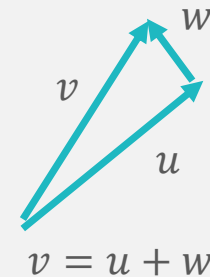
$$P = Q + v$$

Linear Vector Space

■ Mathematical system for manipulating vectors:

□ Scalar-vector multiplication: $v' = \alpha v$

□ Vector-vector addition: $v = u + w$



Head-to-tail rule

Vector Space

- A **linear combination** of n vectors u_1, u_2, \dots, u_n is a vector of the form:

- $u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$

- If $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$ only when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, then u_1, u_2, \dots, u_n are linear independent and form the **basis**

- $\alpha_1, \alpha_2, \dots, \alpha_n$ give the **representation** of u

- If u_1', u_2', \dots, u_n' is some other basis and $u = \alpha_1' u_1' + \alpha_2' u_2' + \dots + \alpha_n' u_n'$

- $\alpha_1', \alpha_2', \dots, \alpha_n'$ give another **representation** of u

- There exist an $n \times n$ matrix T such that
$$\begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_n' \end{bmatrix} = T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

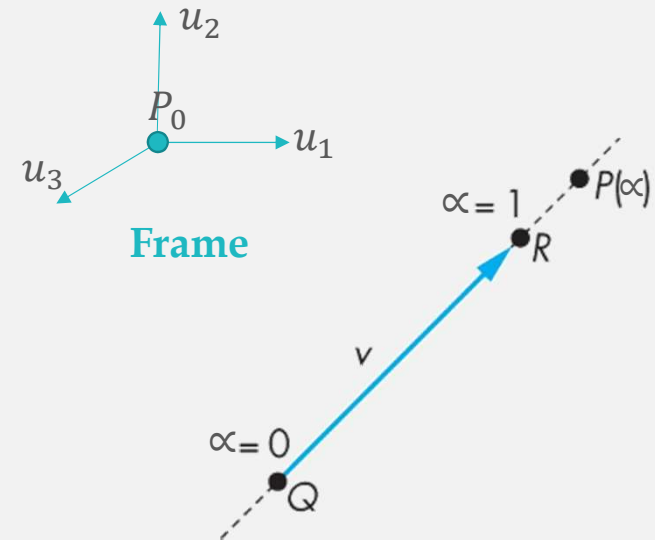
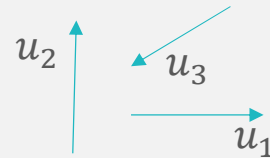
Affine Space

Affine Space:

- **Point** + a **vector** space
- Any arbitrary vector can be written uniquely as

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$
- Any arbitrary point can be written uniquely as

$$P = P_0 + \beta_1 u_1 + \beta_2 u_2 + \cdots + \alpha_n u_n$$



Operations:

- Vector-vector addition
- Scalar-vector multiplication
- Point-vector addition
- **Affine addition** (addition of two points)

$$P = Q + \alpha v \quad v = R - Q$$

$$\Rightarrow P = Q + \alpha (R - Q) = \alpha R + (1 - \alpha) Q$$

$$P = \alpha_1 R + \alpha_2 Q \quad \text{where } \alpha_1 + \alpha_2 = 1$$

Lines

- A line is a set of all points that pass through P_0 in the direction of the vector d .

- Parametric form: $P(\alpha) = P_0 + \alpha d$

- Two-dimension forms:

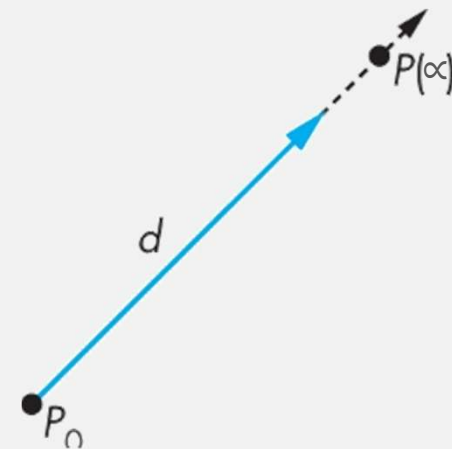
- Explicit: $y = mx + h$

- General/Implicit form: $ax + by + c = 0$

- Parametric form:

$$x(\alpha) = \alpha x_0 + (1 - \alpha)x_1$$

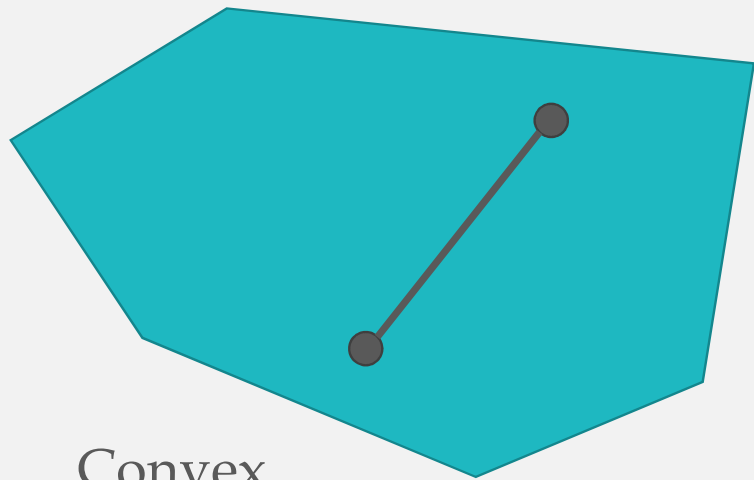
$$y(\alpha) = \alpha y_0 + (1 - \alpha)y_1$$



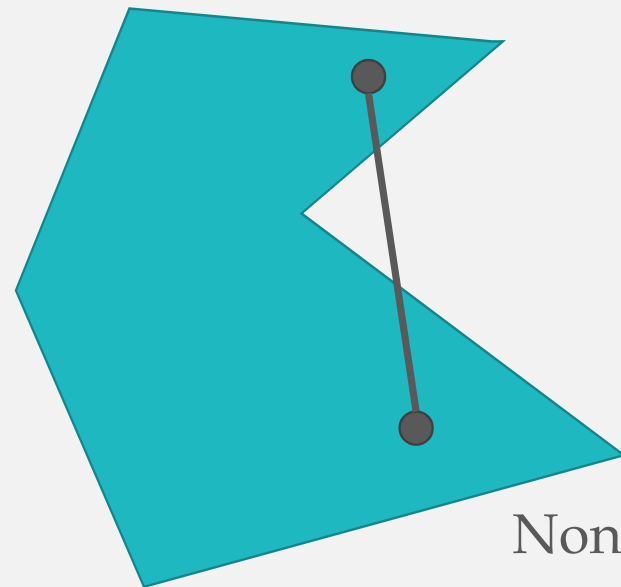
Convexity

■ A convex object :

- For any two points in the object, all points on the line segment between these two points are also in the object.



Convex



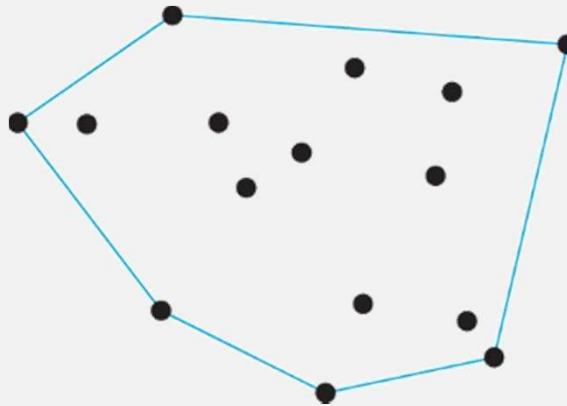
Non-Convex

Convex Hull

- Given a set of points $\{P_1, P_2, \dots, P_n\}$, we can extend the **affine sum** to find the set of points forming a **convex hull**, where each point is in the form of:

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n \text{ where } \alpha_1 + \alpha_2 + \dots + \alpha_n = 1, \alpha_i \geq 0$$

- The convex hull is formed by stretching a tight fitting surface over the given set of points, i.e. **shrink-wrapping** the points .



Dot and Cross Products

■ Dot (inner) product: $u \cdot v$

■ $u \cdot v = 0$ iif. u and v are orthogonal

■ In the Euclidean space, the magnitude of a vector is:

$$|u|^2 = u \cdot u$$

■ The cosine of the angle between two vectors:

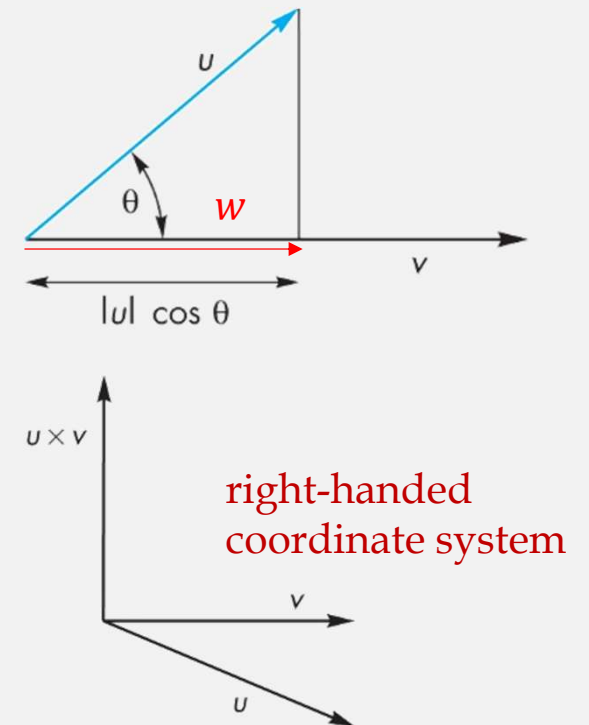
$$\cos \theta = \frac{u \cdot v}{|u||v|}$$

■ Cross (outer) product: $u \times v$

■ Given two non-parallel vectors u and v , we can determine a third vector that is orthogonal to them:

■ The magnitude of the cross product gives the magnitude of the sine of the angle between u and v :

$$|\sin \theta| = \frac{|u \times v|}{|u||v|}$$



Planes

- A plane is an affine space defined by a direct extension of the parametric line.

- Each point on the plane expanded by **three points** P, Q , and R can be defined by

$$T(\alpha, \beta) = \beta[\alpha P + (1 - \alpha)Q] + (1 - \beta)R$$

$$= \beta\alpha P + \beta(1 - \alpha)Q + (1 - \beta)R$$

$$= \alpha'P + \beta'Q + \gamma'R \text{ where } \alpha' + \beta' + \gamma' = 1$$

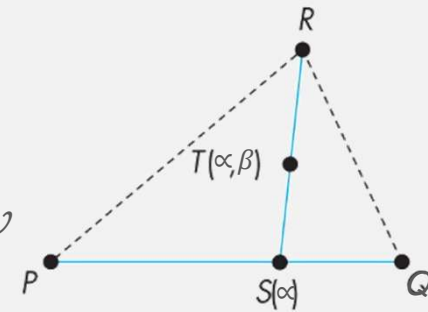
$(\alpha', \beta', \gamma')$ is the **barycentric coordinate** representation of T

- A plane can be expressed by **a point P_0 and two nonparallel vectors u and v**

$$\rightarrow T(\alpha, \beta) = P + \beta(1 - \alpha)(Q - P) + (1 - \beta)(R - P)$$

$$\rightarrow T(\alpha, \beta) = P + \alpha'u + \beta'v$$

$$\rightarrow \text{If a point } T \text{ lies in the plane: } P - T = \alpha'u + \beta'v$$



$$S(\alpha) = \alpha P + (1 - \alpha)Q, \quad 0 \leq \alpha \leq 1$$

$$T(\beta) = \beta S + (1 - \beta)R, \quad 0 \leq \beta \leq 1$$

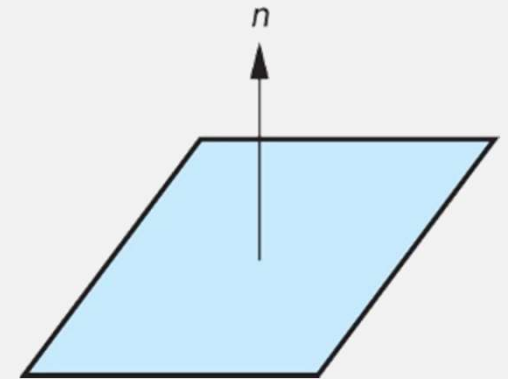
Normal

- Every plane has a vector n normal/perpendicular/orthogonal to it.

- Obtained by $n = u \times v$

- The plane equation can be also expressed by:

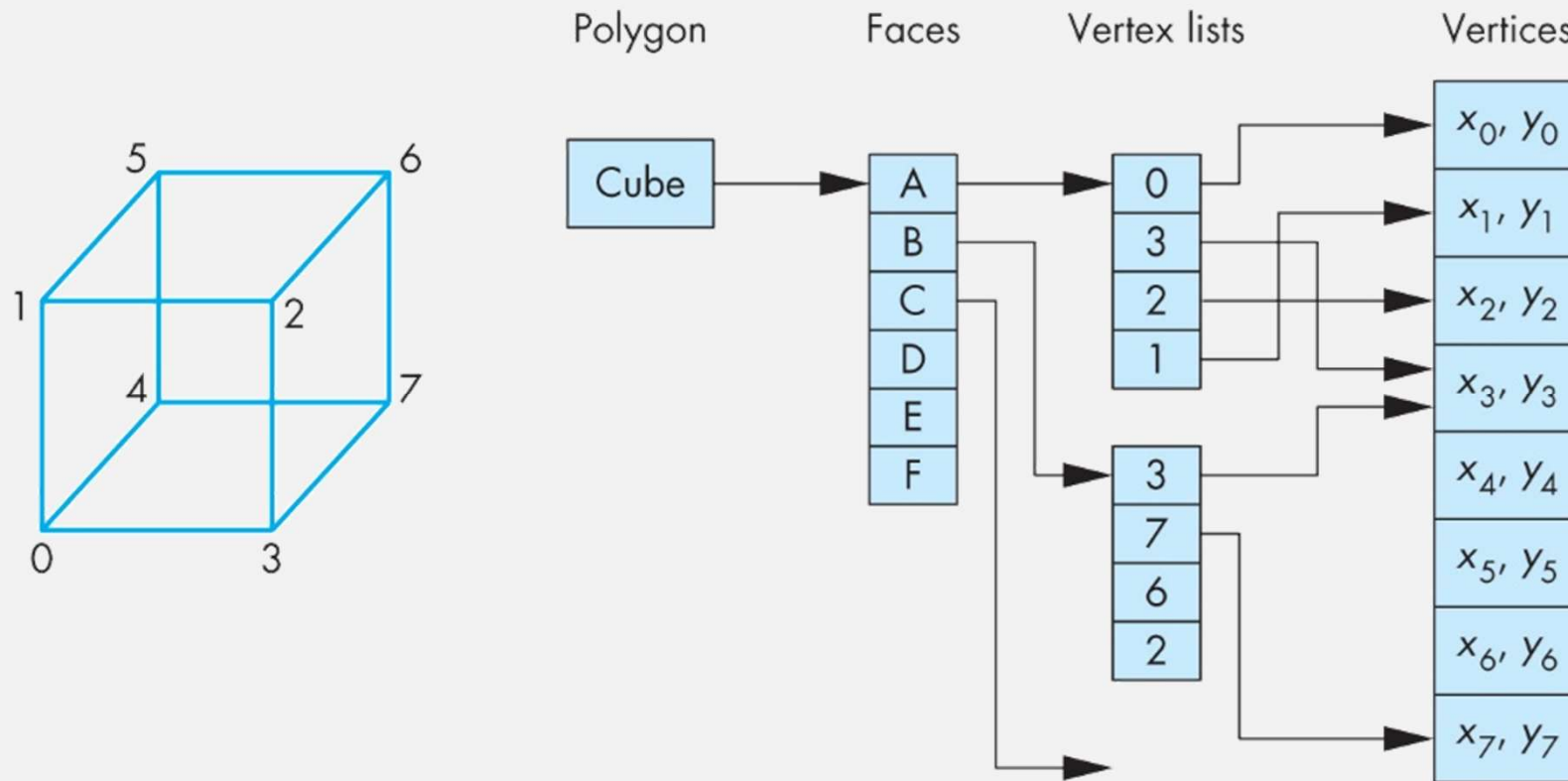
$$n \cdot (P - P_0) = 0$$



3-D Objects

- Three features characterize 3-D object that fit well with existing graphics hardware and software:
 - The objects are described by their **2D surfaces** and can be thought of as being hollow.
 - The objects can be specified through **a set of vertices** in 3-D.
 - The objects either are composed of or can be approximated by **flat, convex polygons**.

Vertex-list Representation of a Cube

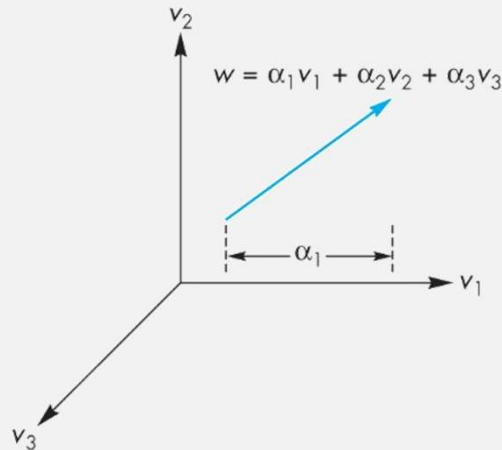


Coordinate Systems and Frames

- Any vector w can be represented uniquely in terms of three linearly independent vectors/bases, v_1 , v_2 , and v_3 :

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

α_1, α_2 , and α_3 are the components of w with respect to the three basis.



$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$w = \mathbf{a}^T \mathbf{v}$$

Coordinate Systems and Frames (Cont.)



- With a particular reference point (the origin), we can represent all points unambiguously in the affine space.
- The origin and the basis vectors determine a **frame**.

Coordinate Systems and Frames (Cont.)

- Within a given frame

- Every vector can be written uniquely as:

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{a}^T \mathbf{v}$$

- Every point can be written uniquely as:

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = P_0 + \mathbf{b}^T \mathbf{v}$$

Change of Coordinate Systems

■ Object/model Frame \rightarrow World Frame \rightarrow Camera/eye Frame

■ can be done by the models-view matrix

$$\begin{array}{l} \text{2 basis: } \{v_1, v_2, v_3\} \\ \{u_1, u_2, u_3\} \end{array} \quad \begin{array}{l} u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3 \\ u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3 \\ u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3 \end{array} \quad \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

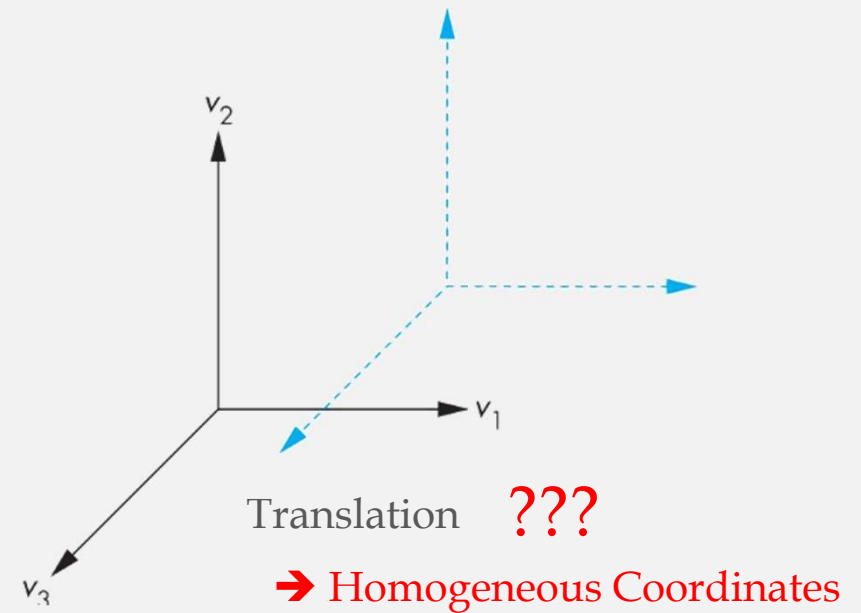
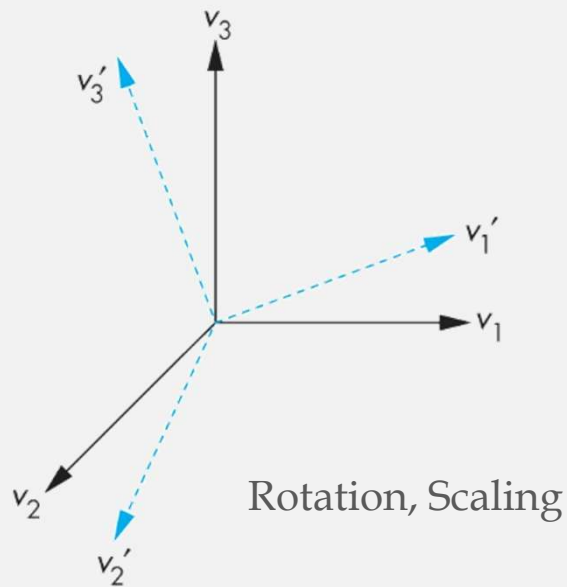
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{M}\mathbf{v}$$

■ Consider a vector $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{a}^T \mathbf{v}$ and $w = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 = \mathbf{b}^T \mathbf{u}$

$$w = \mathbf{b}^T \mathbf{u} = \mathbf{b}^T \mathbf{M}\mathbf{v} = \mathbf{a}^T \mathbf{v} \quad \rightarrow \quad \mathbf{a} = \mathbf{M}^T \mathbf{b} \quad \rightarrow \quad \mathbf{b} = \mathbf{M}^{T^{-1}} \mathbf{a} = \mathbf{T}\mathbf{a}$$

Change of Coordinate Systems (Cont.)

$$\blacksquare \mathbf{b} = \mathbf{M}^{T^{-1}} \mathbf{a} = \mathbf{T} \mathbf{a}$$



Homogeneous Coordinates

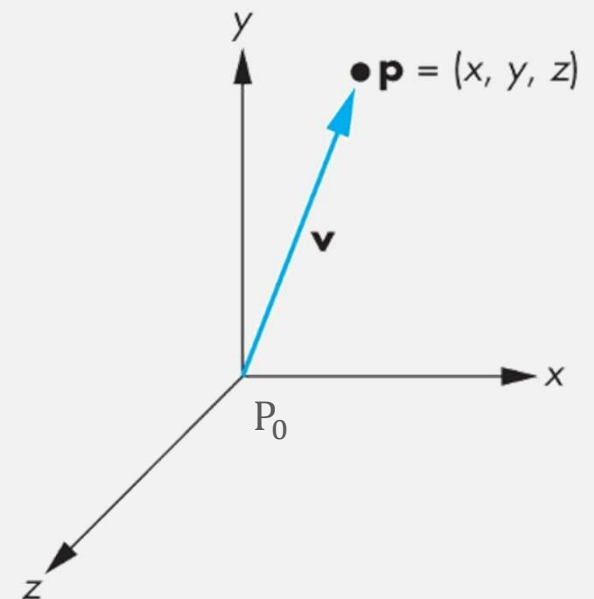
- Any point P can be represented as

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = xv_1 + yv_2 + zv_3 + P_0$$

$$\rightarrow P = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

- Any vector w can be represented as

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$



Homogeneous Coordinates (Cont.)

- If (v_1, v_2, v_3, P_0) and (u_1, u_2, u_3, Q_0) are two frames:

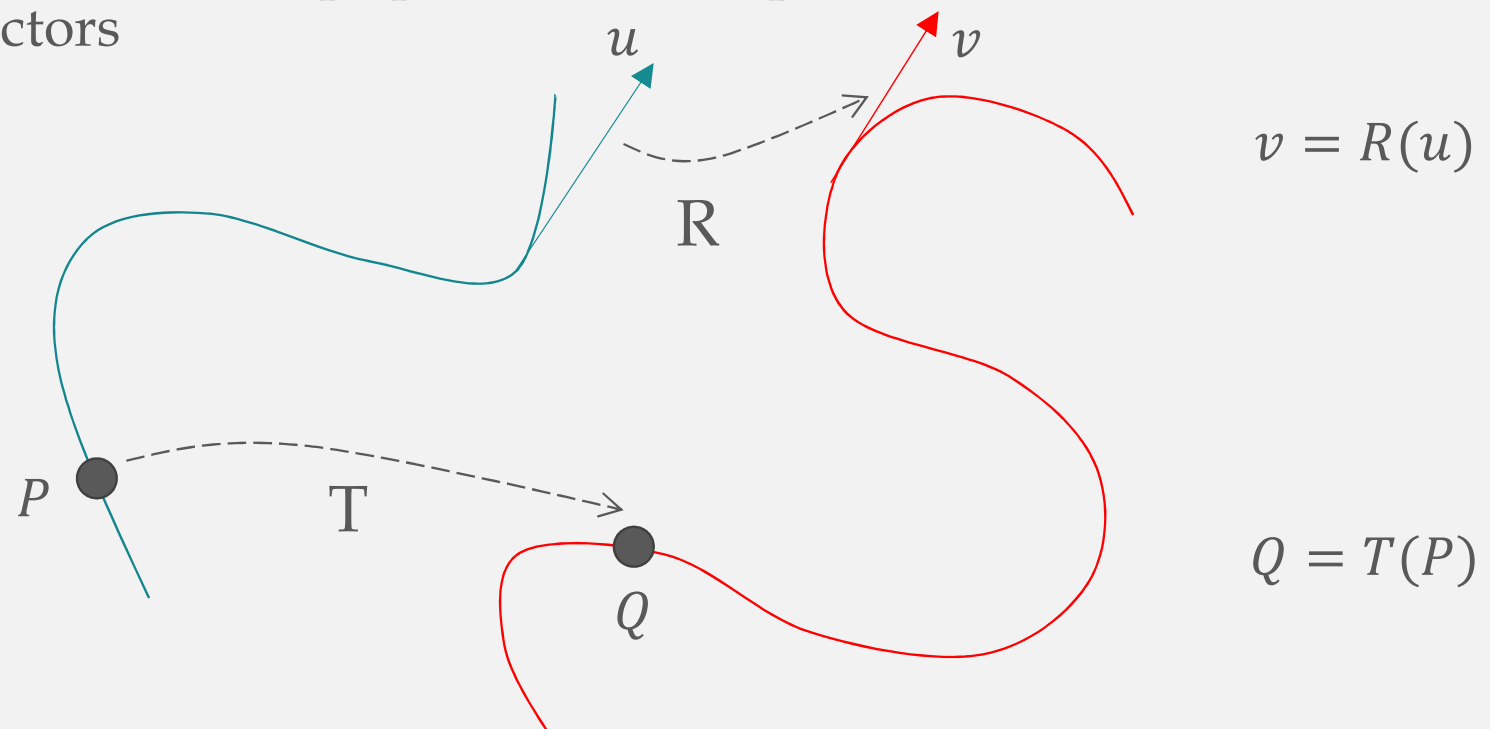
$$\begin{aligned}u_1 &= \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3 \\u_2 &= \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3 \\u_3 &= \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3 \\Q_0 &= \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + P_0\end{aligned}$$

$$\rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

Matrix Representation of the change of frames
(12 coefficients)

General Transformations

- A transformation maps points to other points and/or vectors to other vectors



Affine Transformations

- A transformation that preserves lines and parallelism
 - maps parallel lines to parallel lines
- If we know the transformations of p and q , we can obtain the transformations of linear combinations of p and q by taking linear combinations of their transforms:
 - $f(\alpha p + \beta q) = \alpha f(p) + \beta f(q)$
- Characteristic of many physically important transformations
 - Rigid body transformations: identity, translation, rotation, reflection
 - Non-rigid body transformation: scaling, shear

Translation

- Using the homogeneous coordinate representation in some frame

$$p = [x \quad y \quad z \quad 1]^T$$

$$p' = [x' \quad y' \quad z' \quad 1]^T$$

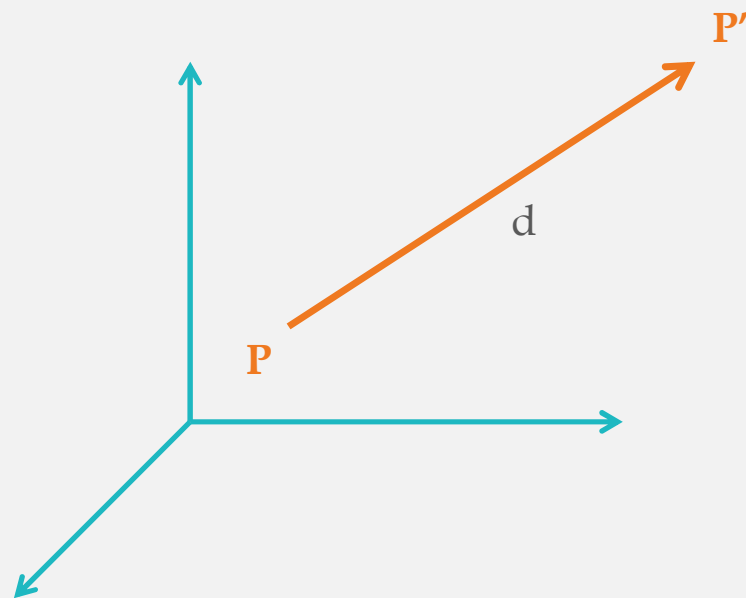
$$d = [d_x \quad d_y \quad d_z \quad 0]^T$$

Hence $p' = p + d$ or

$$x' = x + d_x$$

$$y' = y + d_y$$

$$z' = z + d_z$$



Translation Matrix

- We can also express translation using a 4 x 4 matrix T in homogeneous coordinates $p' = Tp$ where

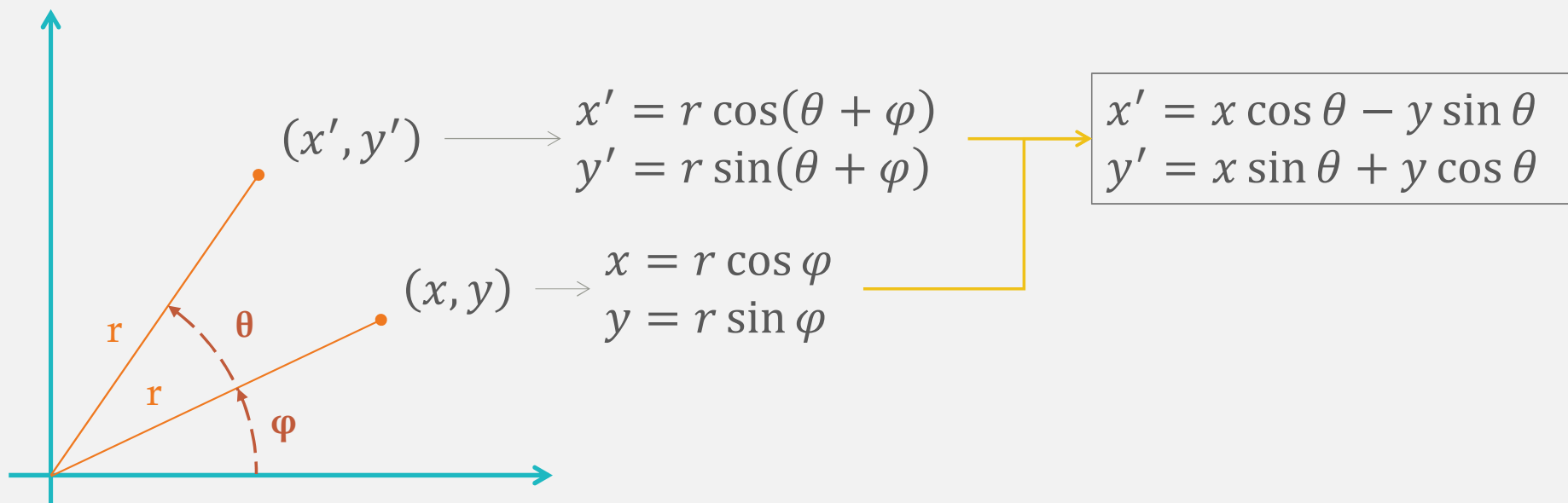
$$T = T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Why do we use a matrix form instead of vector addition?

Rotation (2D)

■ Consider rotation about the origin by θ degrees

■ radius stays the same, angle increases by θ



Rotation about the z axis

■ Rotation about z-axis in three dimensions

- leaves all points with the same z
- equivalent to rotation in two dimensions in planes of constant z

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$

$$z' = z$$

- or in homogeneous coordinates

$$p' = R_z(\theta)p$$

$$R = R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about x - and y -axes

■ Same argument as for rotation about z -axis

■ for rotation about x -axis, x is unchanged

■ for rotation about y -axis, y is unchanged

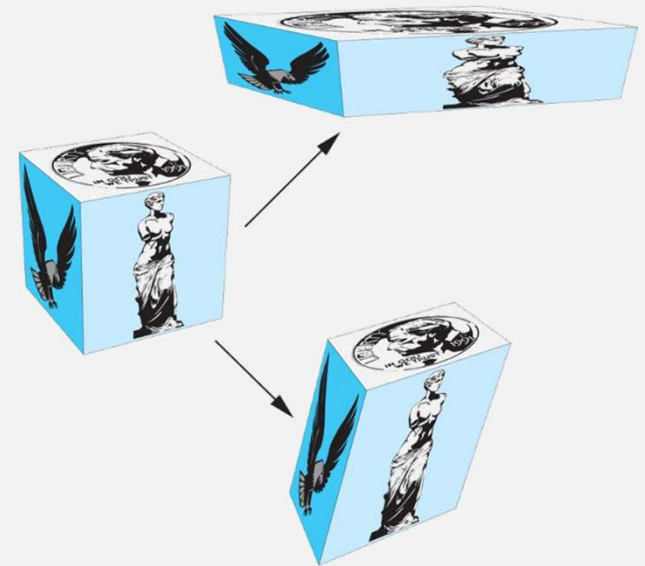
$$R = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- Expand or contract along each axis (fixed point of origin)

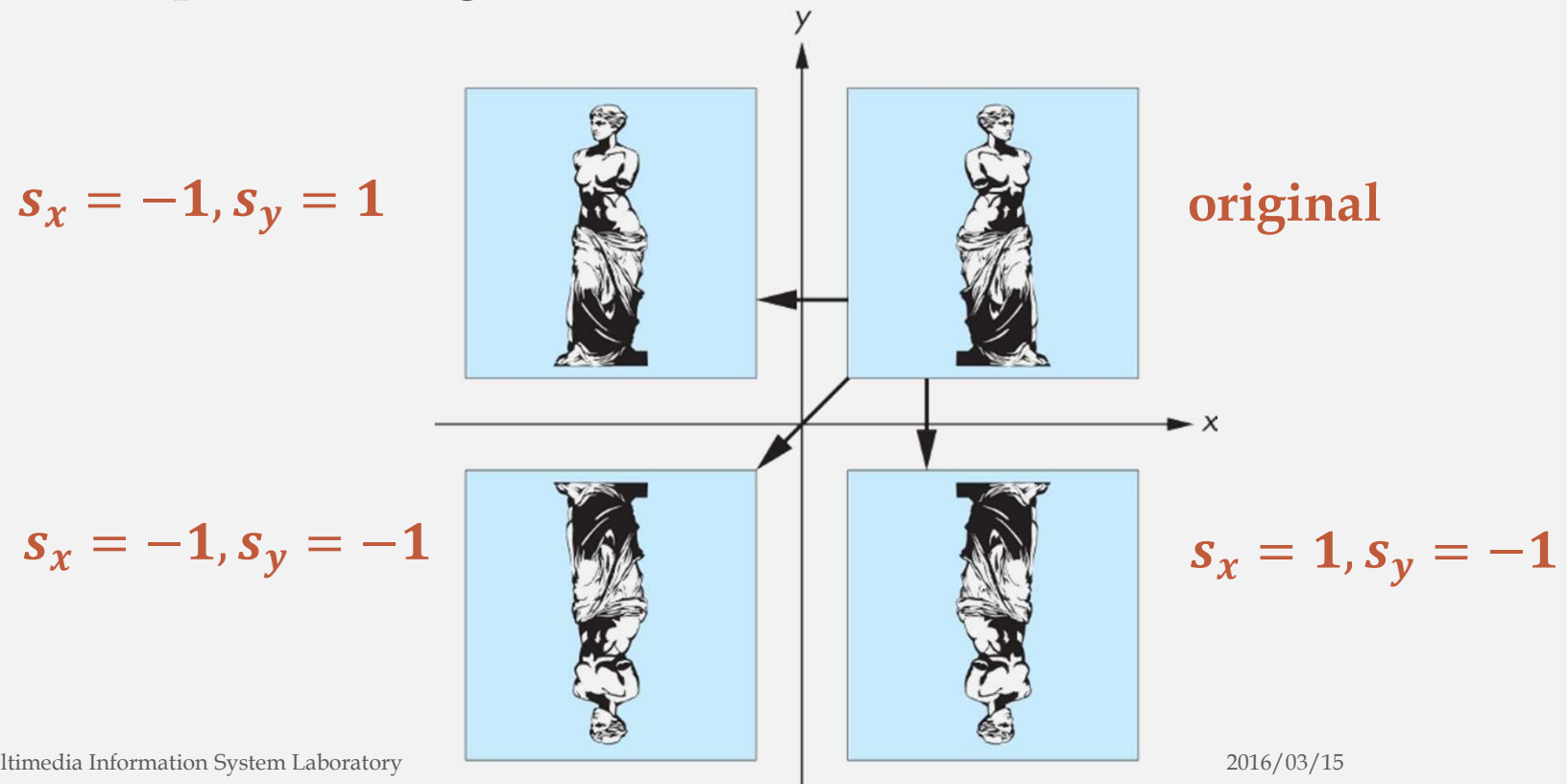
$$\begin{aligned}x' &= s_x x \\y' &= s_y y \\z' &= s_z z\end{aligned} \Rightarrow p' = Sp$$

$$S = S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Reflection

- Corresponds to negative scale factors



Inverses

- Compute inverse matrices by general formulas, or use simple geometric observations

- Translation: $T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)$

- Rotation: $R^{-1}(\theta) = R(-\theta)$

- holds for any rotation matrix

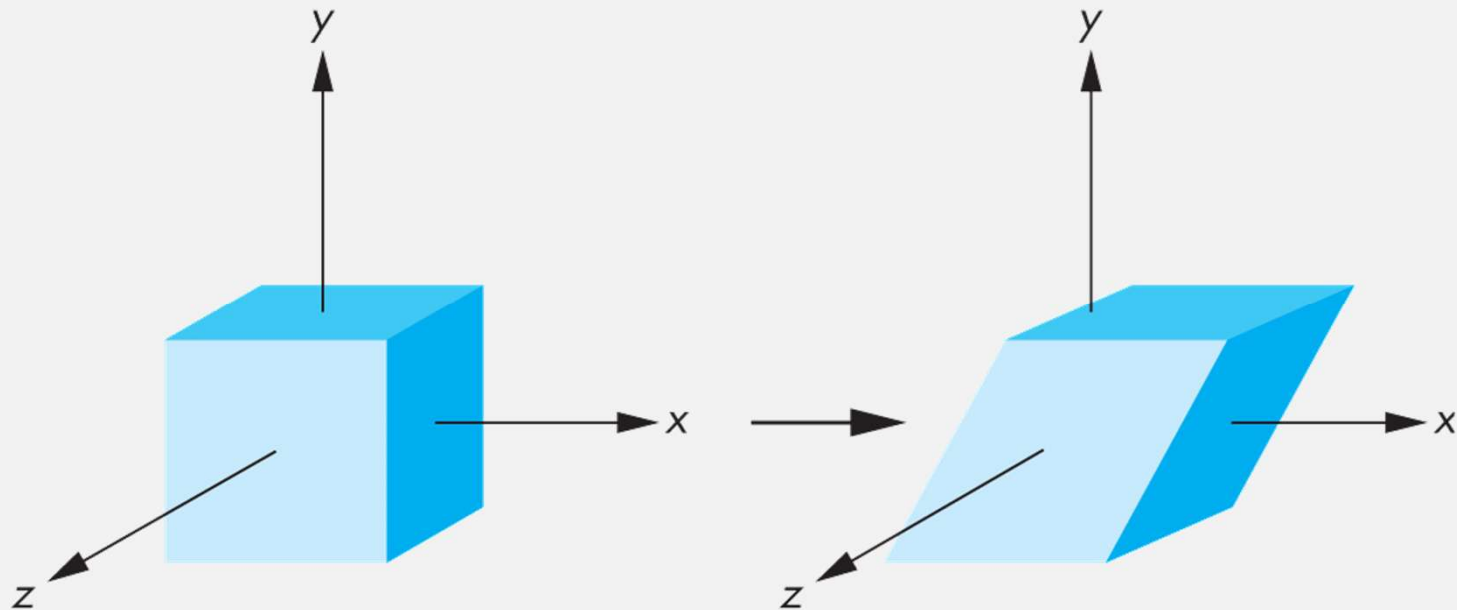
- $\cos -\theta = \cos \theta$ and $\sin -\theta = -\sin \theta \rightarrow R^{-1}(\theta) = R^T(\theta) \rightarrow R$ is an orthogonal matrix

- Normalized orthogonal matrices correspond to [rotation about the origin](#)

- Scaling: $S^{-1}(s_x, s_y, s_z) = S\left(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z}\right)$

Shear

- Equivalent to pulling faces in opposite directions



Shear Matrix

- Consider simple shear along x -axis

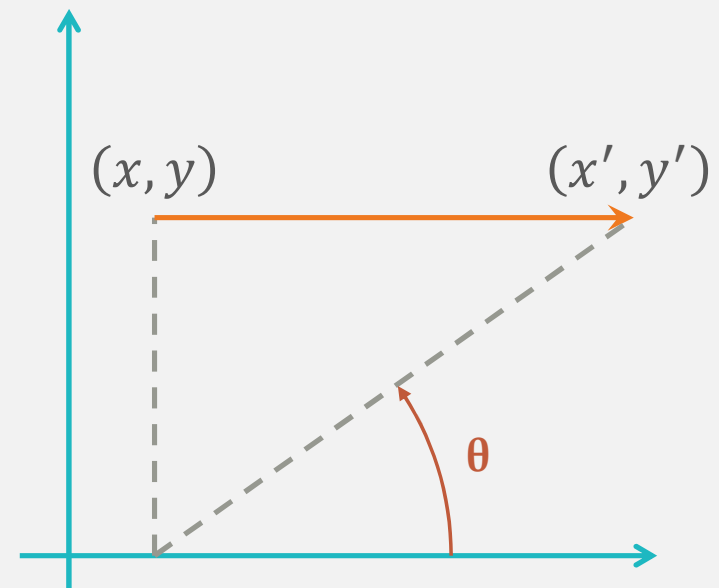
$$x' = x + y \cot \theta$$

$$y' = y$$

$$z' = z$$

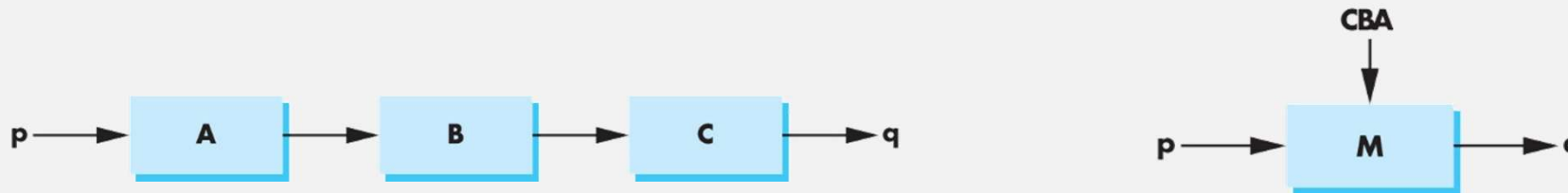
$$\Rightarrow p' = Hp$$

$$H(\theta) = \begin{bmatrix} 1 & \cot \theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Concatenation

- Form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices



*for each i
($C(B(Ap_i)))$)*

or

*$M=ABC,$
for each i
 Mp_i*

Order of Transformations

- Note that matrix on the right is the first applied

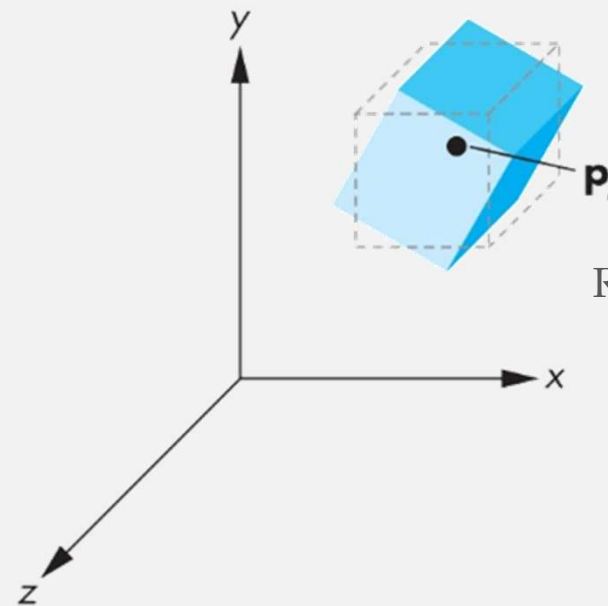
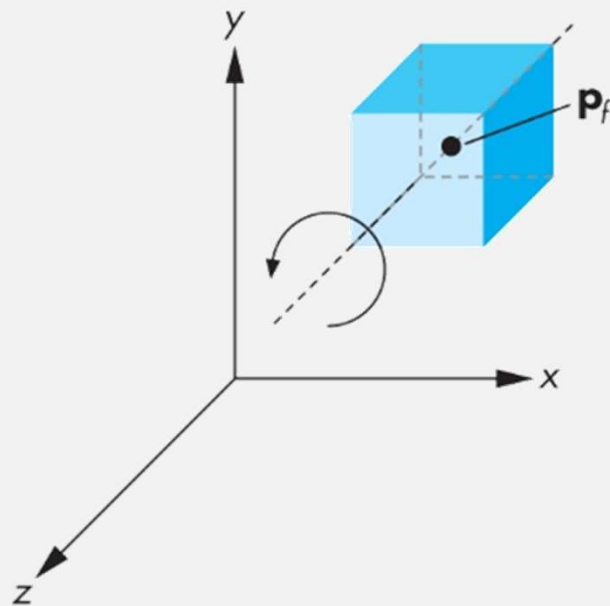
- Mathematically, the following are equivalent

$$p' = ABCp = A(B(Cp))$$

- Note many references use column matrices to represent points.
In terms of column matrices:

$$p'^T = p^T C^T B^T A^T$$

Rotation About a Fixed Point



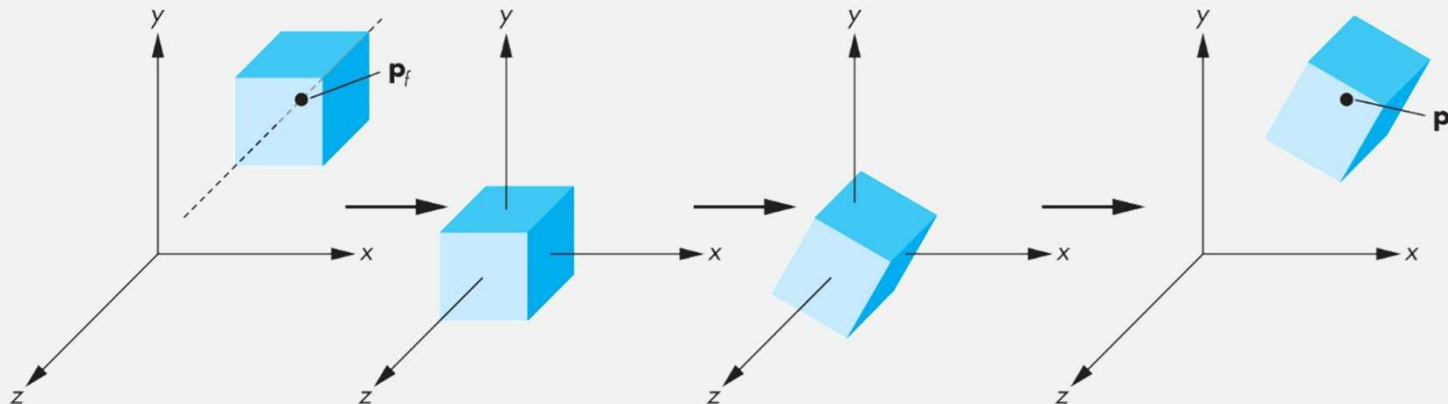
Rotation of cube about its center

Rotation About a Fixed Point other than the Origin

1. Move fixed point to origin
2. Rotate
3. Move fixed point back

$$M = T(p_f)R(\theta_z)T(-p_f)$$

$$M = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & x_f - x_f \cos \theta + y_f \sin \theta \\ \sin \theta & \cos \theta & 0 & y_f - x_f \sin \theta - y_f \cos \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

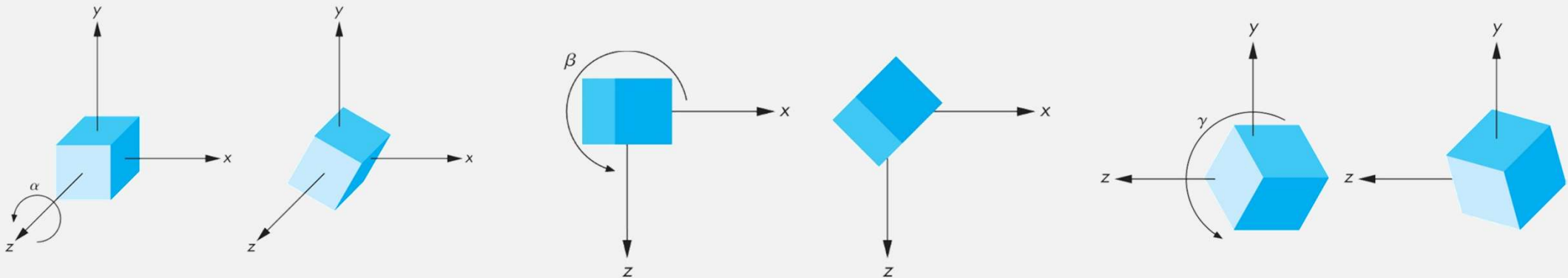


General Rotation About the Origin

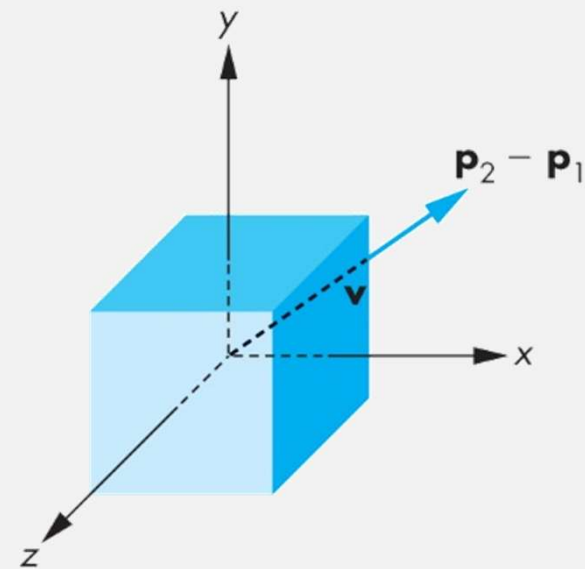
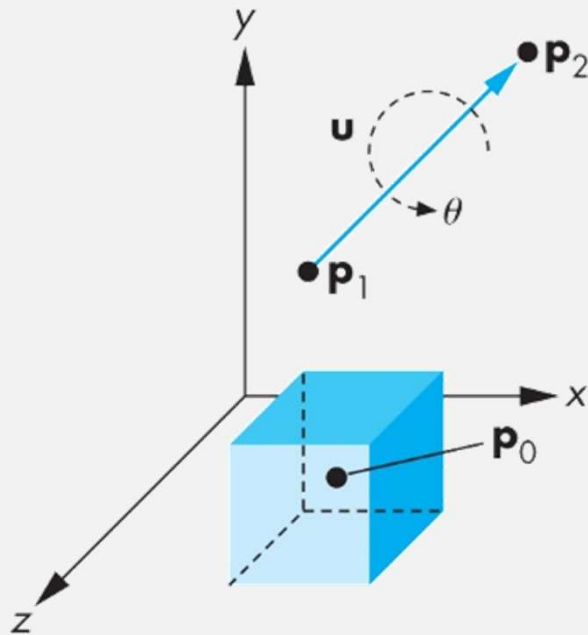
- Decompose into the concatenation of rotations about the x -, y -, and z -axes

$$R(\theta) = R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$$

$\theta_x, \theta_y, \theta_z$ are called the **Euler angles**.



Rotation About an Arbitrary Axis



Rotation About an Arbitrary Axis

- Rotate around an axis vector u

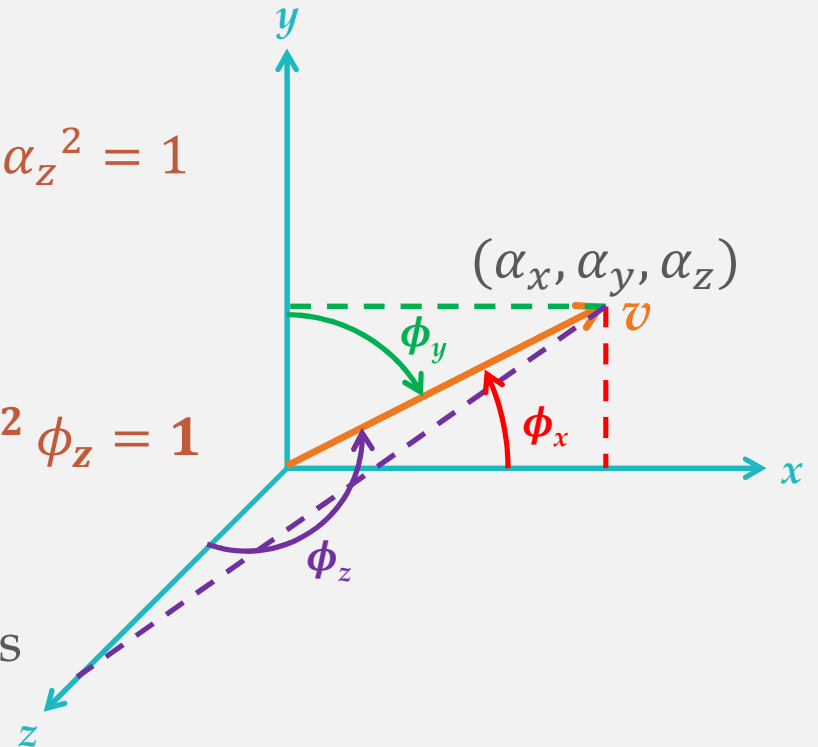
$$v = \frac{u}{|u|} = [\alpha_x \quad \alpha_y \quad \alpha_z]^T \quad \alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$$

$$\cos \phi_x = \alpha_x$$

$$\cos \phi_y = \alpha_y \quad \Rightarrow \quad \cos^2 \phi_x + \cos^2 \phi_y + \cos^2 \phi_z = 1$$

$$\cos \phi_z = \alpha_z$$

Hint: What we already have are rotations around x -, or y -, or z -axes.

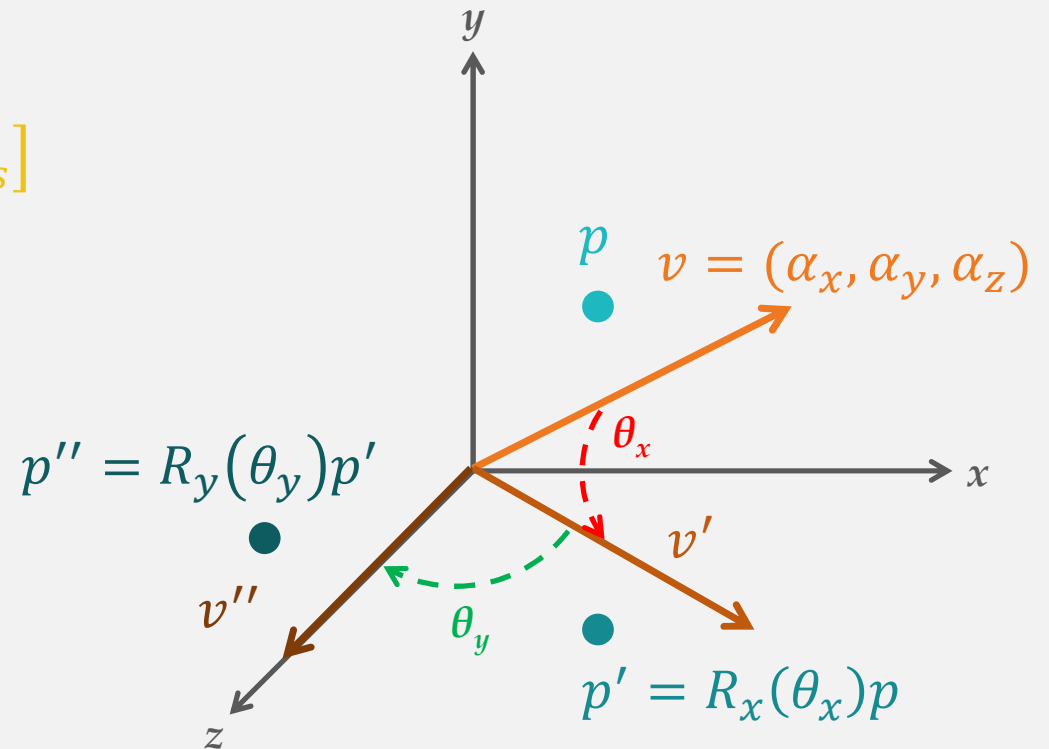


Rotation About an Arbitrary Axis (cont.)

1. Rotate the axis vector to match z- (x- or y-) axis $[R_{axis}]$
2. Rotate around z-axis $[R_z(\theta)]$
3. Rotate the axis vector back $[R_{axis}^{-1}]$

$$R_{axis} = R_y(\theta_y)R_x(\theta_x)$$

$$\begin{aligned} M &= R_{axis}^{-1}R(\theta)R_{axis} \\ &= R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x) \end{aligned}$$

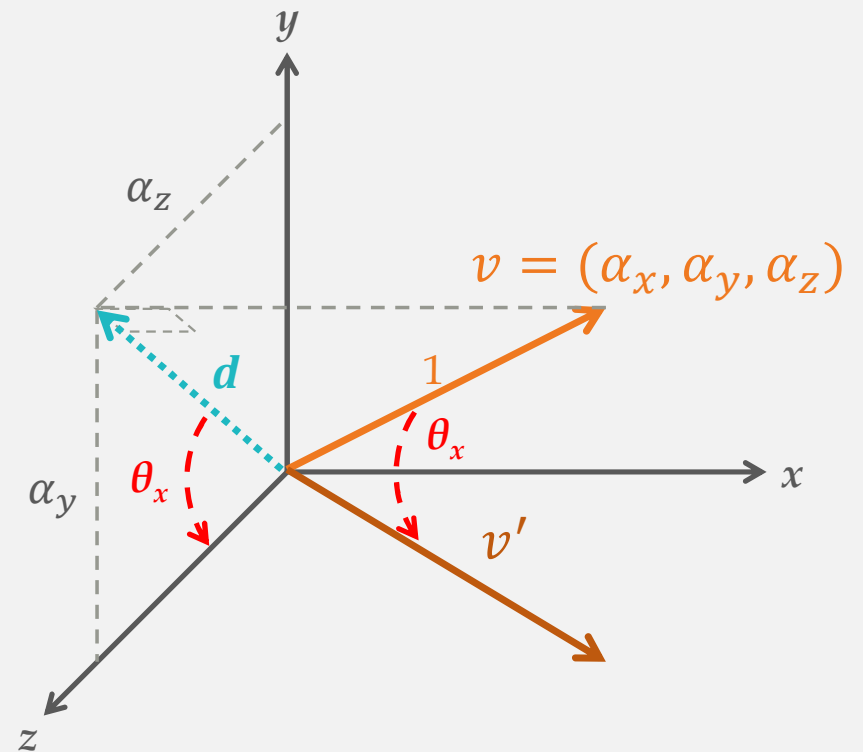


$$R_x(\theta_x)$$

$$R = R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x & 0 \\ 0 & \sin \theta_x & \cos \theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_z}{d} & -\frac{\alpha_y}{d} & 0 \\ 0 & \frac{\alpha_y}{d} & \frac{\alpha_z}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$

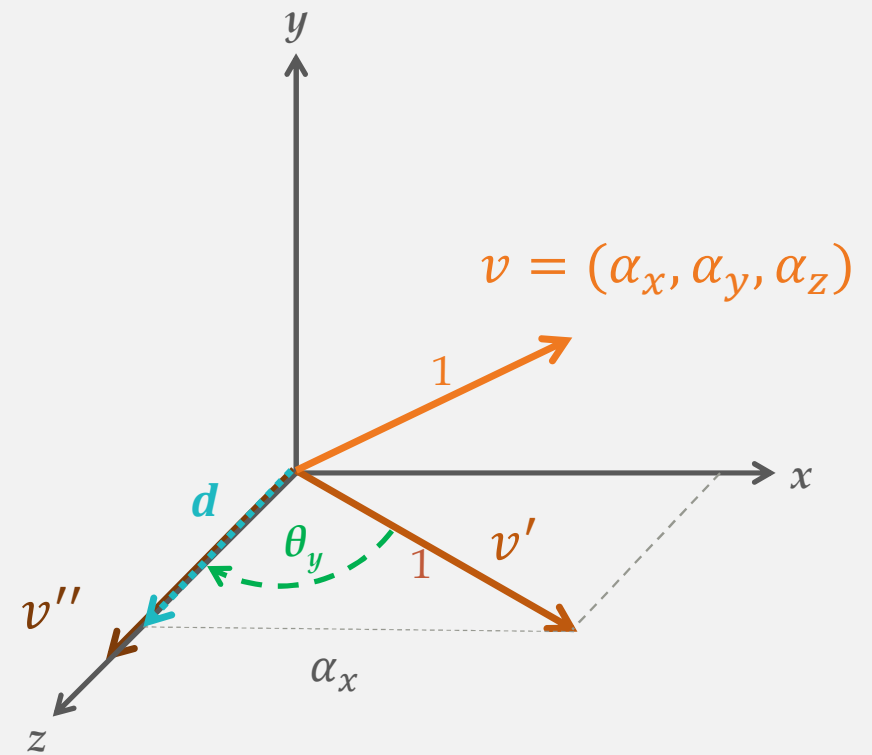


$$R_y(\theta_y)$$

$$R = R_y(\theta_y) = \begin{bmatrix} \cos \theta_y & 0 & \sin \theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & -d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = T(P_0)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-P_0)$$



Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size

- We apply an **instance transformation** to its vertices to

- Scale
- Orient
- Locate

