Coordinates and Transformations

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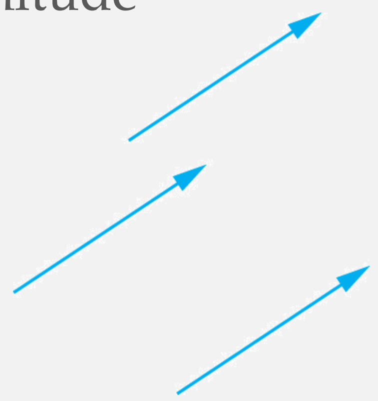
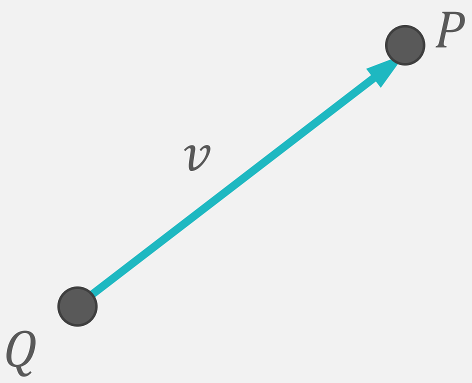
Scalars, Pints, & Vectors

- Basic geometric objects and relationships among them can be described using scalars, points, and vectors
  - A point is a location in space
  - A scalar is used to specify quantities such as distance between two points.
    - Scalars obey a set of rules that are abstractions of the operations of ordinary arithmetic
      - Addition/subtraction, multiplication/division
  - A vector is a quantity with two attributes: direction and magnitude
    - A vector does not have a fixed location in space



Points

- Operations between point and vector:
  - Point-point subtraction yields a vector.
  - Equivalent to point-vector addition.

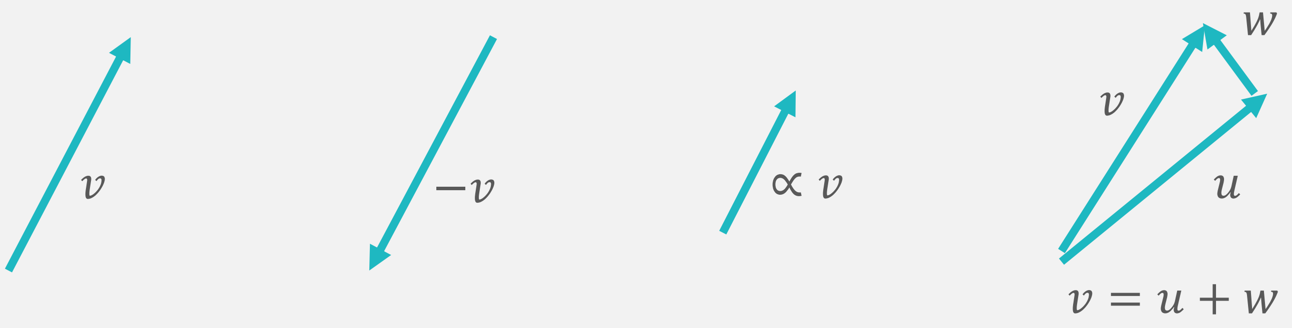


$$v = P - Q$$

$$P = Q + v$$

Linear Vector Space

- Mathematical system for manipulating vectors:
  - Scalar-vector multiplication: $v' = \propto v$
  - Vector-vector addition: $v = u + w$



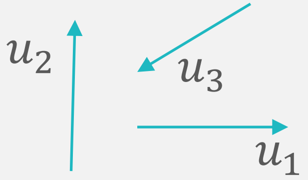
Head-to-tail rule

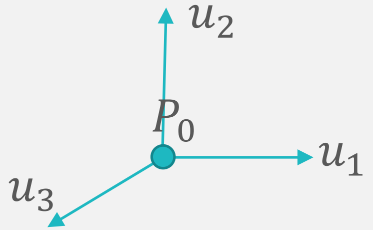
Vector Space

- A linear combination of n vectors $u_1, u_2, \ldots u_n$ is a vector of the form:
  - $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$
- If $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$ only when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then $u_1, u_2, \ldots u_n$ are linear independent and form the basis
  - $\alpha_1, \alpha_2 \ldots \alpha_n$ give the representation of u
- If $u_1', u_2', \ldots u_n'$ is some other basis and $u = \alpha_1' u_1' + \alpha_2' u_2' + \ldots + \alpha_n' u_n'$
  - $\alpha_1', \alpha_2' \ldots \alpha_n'$ give another representation of u
- There exist an nxn matrix **T** such that $\begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_3' \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_3 \end{bmatrix}$

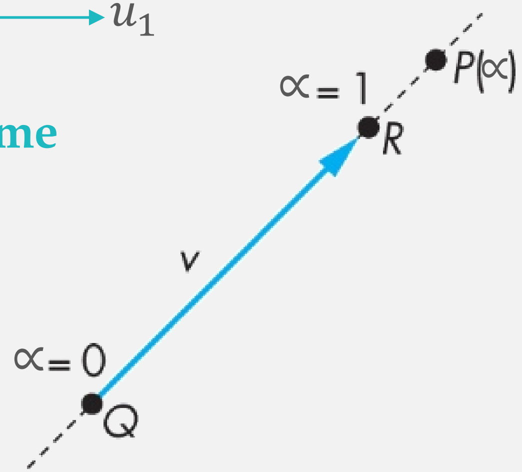
Affine Space

- Affine Space:
  - Point + a vector space
  - Any arbitrary vector can be written uniquely as
    $v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$
  - Any arbitrary point can be written uniquely as
    $P = P_0 + \beta_1 u_1 + \beta_2 u_2 + \cdots + \alpha_n u_n$





Frame



- Operations:
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Affine addition (addition of two points)

$P = Q + \propto v \qquad v = R - Q$

$\Rightarrow P = Q + \propto (R - Q) = \propto R + (1 - \propto)\, Q$

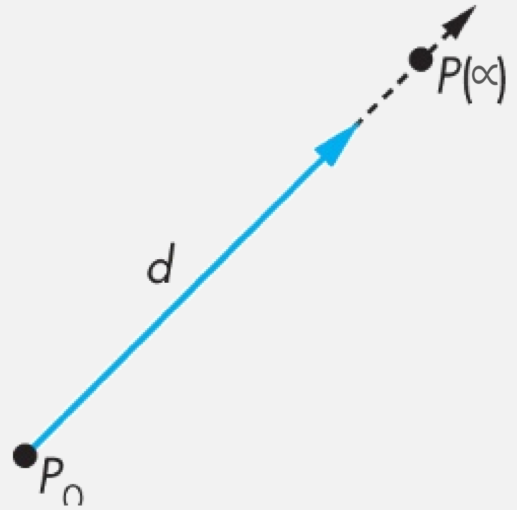
$P = \propto_1 R + \propto_2 Q$ where $\propto_1 + \propto_2 = 1$

Lines

- A line is a set of all points that pass through P_0 in the direction of the vector d.
  - Parametric form: $P(\propto) = P_0 + \propto d$
- Two-dimension forms:
  - Explicit: $y = mx + \mathrm{h}$
  - General/Implicit form: $ax + by + c = 0$
  - Parametric form:

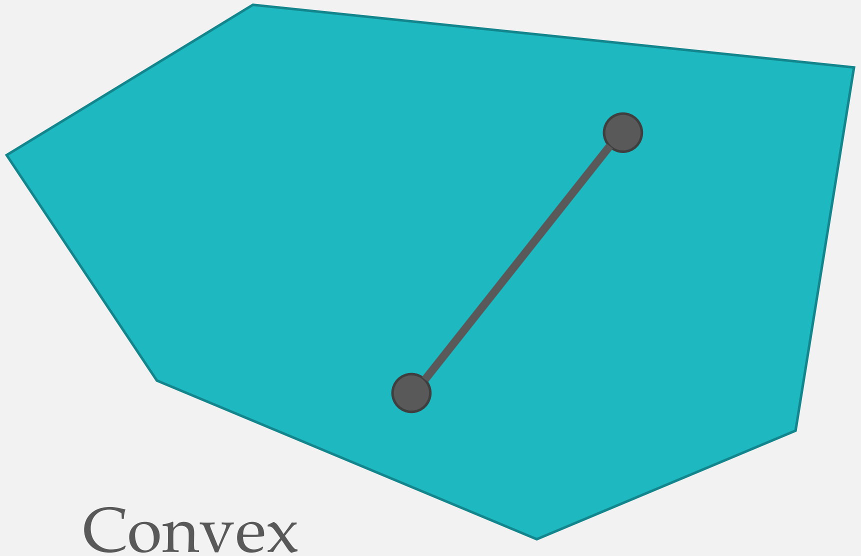
$$x(\propto) = \propto x_0 + (1 - \propto)x_1$$

$$\mathrm{y}(\propto) = \propto y_0 + (1 - \propto)y_1$$

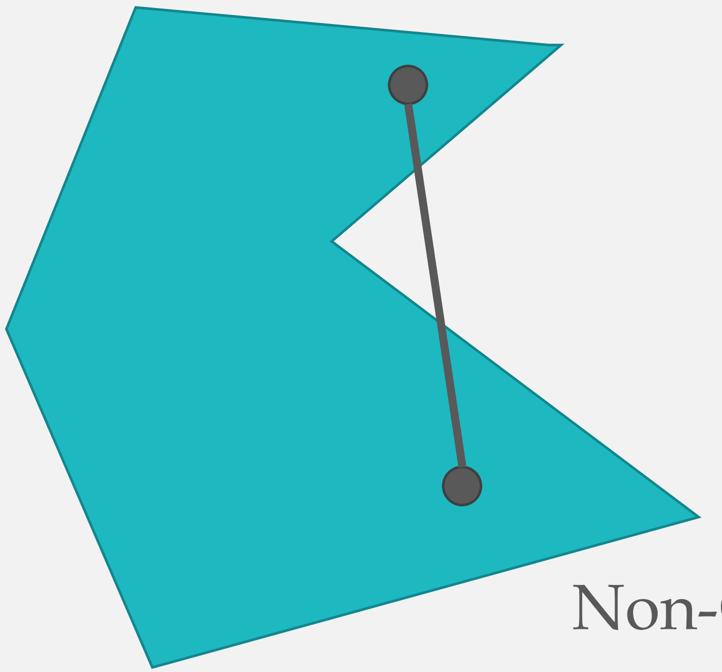


Convexity

- A convex object :
  - For any two points in the object, all points on the line segment between these two points are also in the object.



Convex



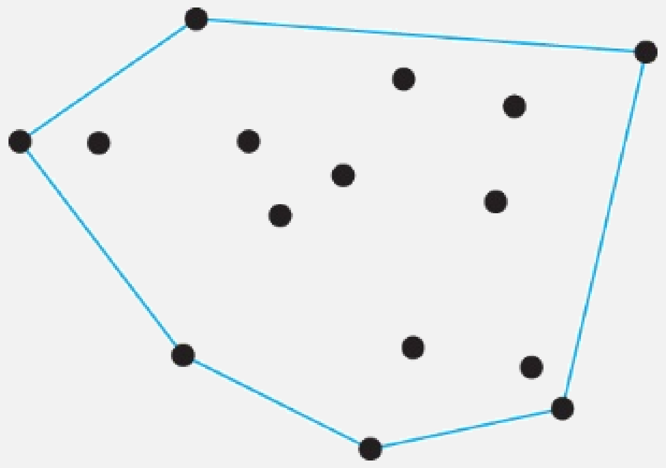
Non-Convex

Convex Hull

- Given a set of points $\{P_1, P_2, \ldots, P_n\}$, we can extend the affine sum to find the set of points forming a convex hull, where each point is in the form of:

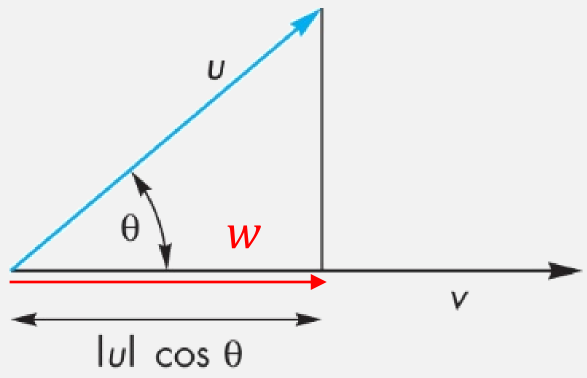
$$P = \propto_1 P_1 + \propto_2 P_2 + \cdots + \propto_n P_n \quad where \propto_1 + \propto_2 + \cdots + \propto_n = 1, \quad \propto_i \geq 0$$

- The convex hull is formed by stretching a tight fitting surface over the given set of points, i.e. shrink-wrapping the points .

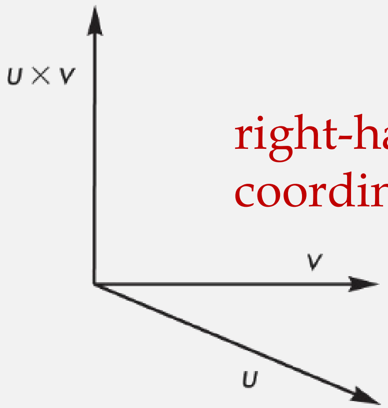


Dot and Cross Products

- Dot (inner) product: $u \cdot v$
  - $u \cdot v = 0$ iif. u and v are orthogonal
  - In the Euclidean space, the magnitude of a vector is:
    $$|u|^2 = u \cdot u$$
  - The cosine of the angle between two vectors:
    $$\cos\theta = \frac{u \cdot v}{|u||v|}$$



- Cross (outer) product: $u \times v$
  - Given two non-parallel vectors u and v, we can determine a third vector that is orthogonal to them:
  - The magnitude of the cross product gives the magnitude of the sine of the angle between u and v :
    $$|\sin\theta| = \frac{|u \times v|}{|u||v|}$$



right-handed coordinate system

Planes

- A plane is an affine space defined by a direct extension of the parametric line.
  - Each point on the plane expanded by three points P,Q, and R can be defined by

$$T(\propto,\beta) = \beta[\alpha P + (1-\alpha)Q] + (1-\beta)R$$

$$= \beta\alpha P + \beta(1-\alpha)Q + (1-\beta)R$$

$$= \alpha' P + \beta' Q + \gamma' R \ \text{ where } \alpha' + \beta' + \gamma' = 1$$

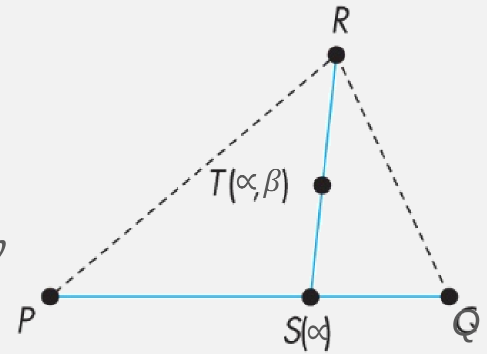
$(\alpha', \beta', \gamma')$ is the barycentric coordinate representation of T

  - A plane can be expressed by a point P_0 and two nonparallel vectors u and v

➔ $T(\propto,\beta) = P + \beta(1-\propto)(Q-P) + (1-\beta)(R-P)$

➔ $T(\propto,\beta) = P + \propto' u + \beta' \mathrm{v}$

➔ If a point T lies in the plane: $P - T = \propto' u + \beta' \mathrm{v}$



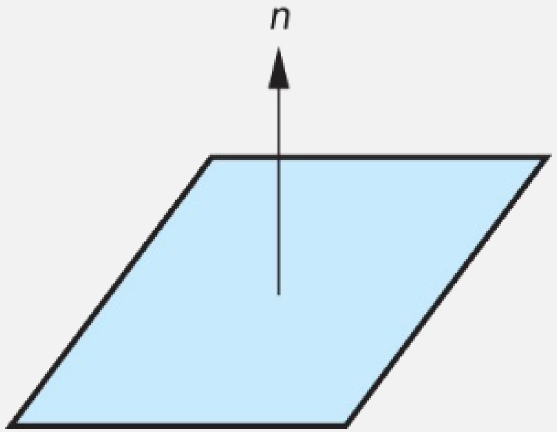
$$S(\alpha) = \alpha P + (1-\alpha)Q, \ 0 \le \alpha \le 1$$

$$T(\beta) = \beta S + (1-\beta)R, \ 0 \le \beta \le 1$$

Normal

- Every plane has a vector n normal/perpendicular/orthogonal to it.
  - Obtained by $n = u \times v$
- The plane equation can be also expressed by:

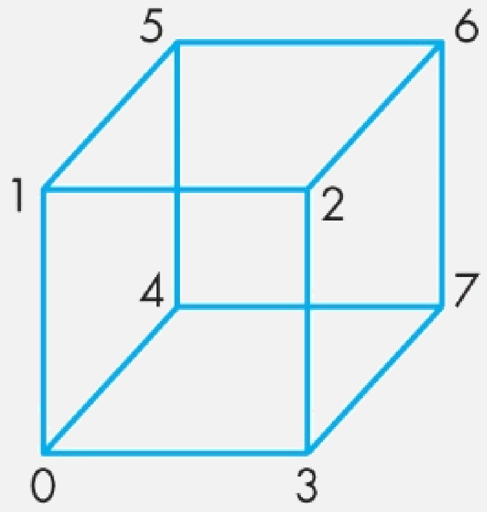
$$n \cdot (P - P_0) = 0$$

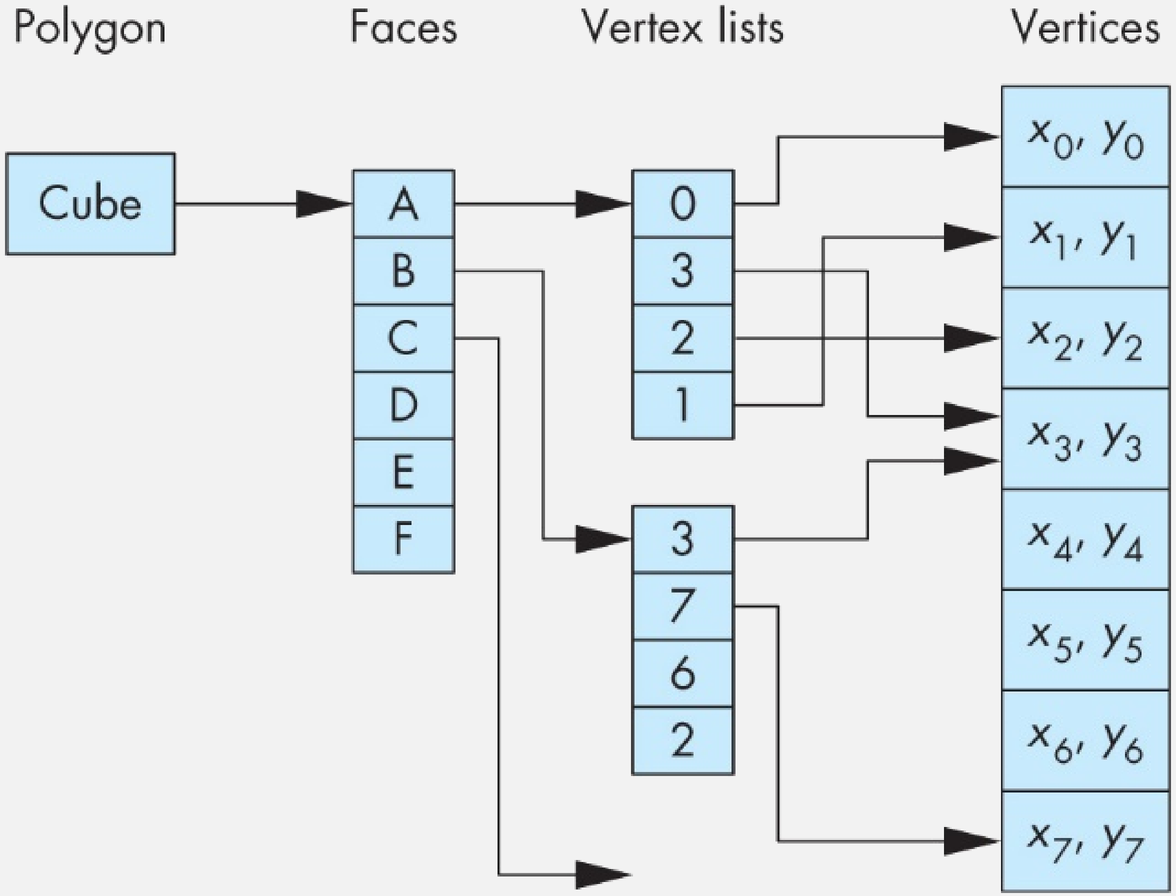


3-D Objects

- Three features characterize 3-D object that fit well with existing graphics hardware and software:
  - The objects are described by their 2D surfaces and can be thought of as being hollow.
  - The objects can be specified through a set of vertices in 3-D.
  - The objects either are composed of or can be approximated by flat, convex polygons.

Vertex-list Representation of a Cube



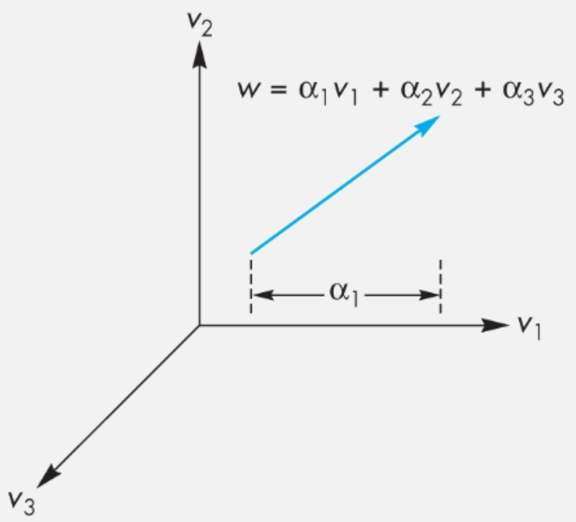


Coordinate Systems and Frames

- Any vector w can be represented uniquely in terms of three linearly independent vectors/bases, v_1, v_2, and v_3 :

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

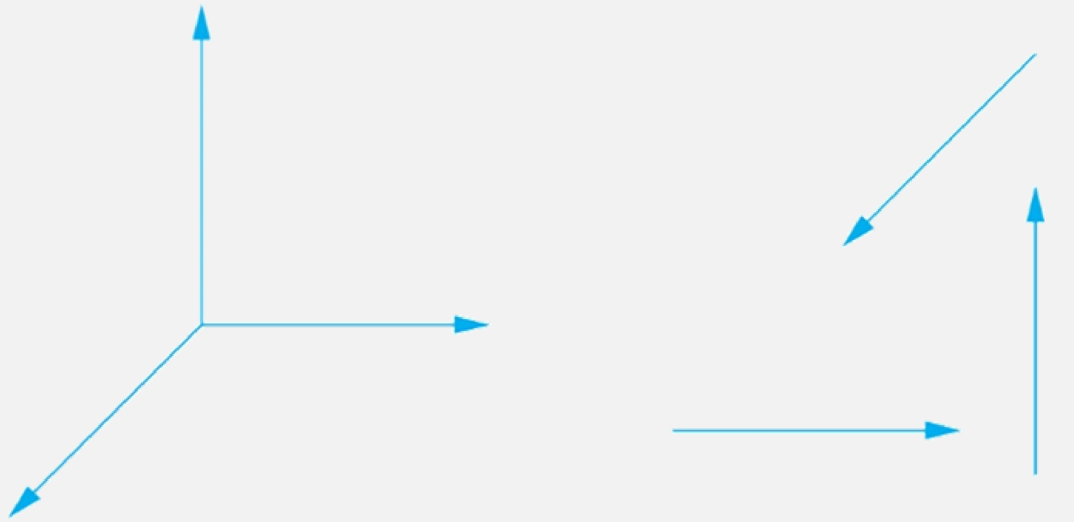
$\propto_1$,$\propto_2$, and $\propto_3$ are the components of w with respect to the three basis.



$$\mathbf{a} = \begin{bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathrm{w} = \mathbf{a}^T \mathbf{v}$$

Coordinate Systems and Frames (Cont.)



- With a particular reference point (the origin), we can represent all points unambiguously in the affine space.
- The origin and the basis vectors determine a frame.

Coordinate Systems and Frames (Cont.)

- Within a given frame
  - Every vector can be written uniquely as:

$$w = \alpha_1 v_1 + \ \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{a}^T \mathbf{v}$$

  - Every point can be written uniquely as:

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = P_0 + \boldsymbol{b}^T \mathbf{v}$$

Change of Coordinate Systems

- Object/model Frame → World Frame → Camera/eye Frame
  - can be done by the models-view matrix

2 basis: $\{v_1, v_2, v_3\}$ $\{u_1, u_2, u_3\}$

$$\begin{aligned} u_1 &= \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3 \\ u_2 &= \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3 \\ u_3 &= \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3 \end{aligned}$$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

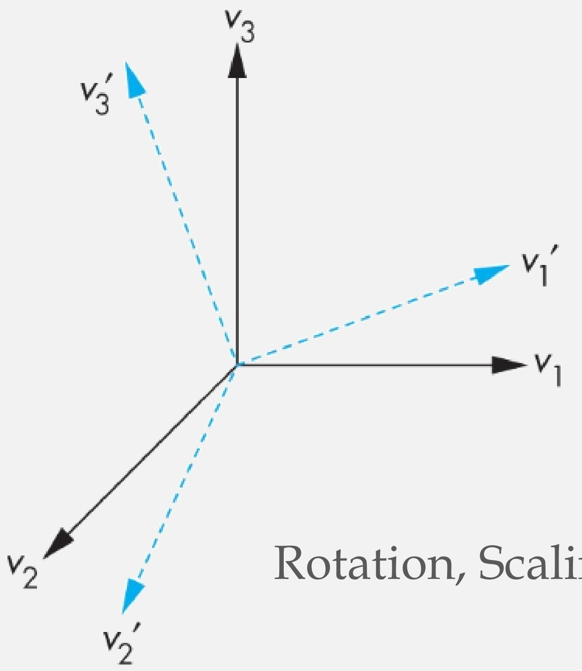
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{M}\mathbf{v}$$

  - Consider a vector $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{a}^T\mathbf{v}$ and $w = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 = \mathbf{b}^T\mathbf{u}$

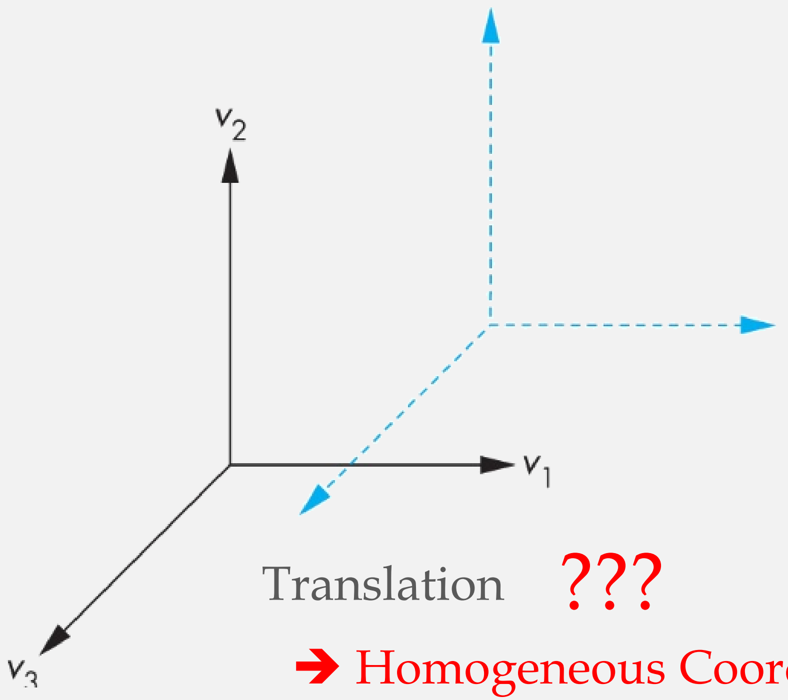
$$w = \mathbf{b}^T\mathbf{u} = \mathbf{b}^T\mathbf{M}\mathbf{v} = \mathbf{a}^T\mathbf{v} \quad \rightarrow \quad \mathbf{a} = \mathbf{M}^T\mathbf{b} \quad \rightarrow \quad \mathbf{b} = \mathbf{M}^{T^{-1}}\mathbf{a} = \mathbf{T}\mathbf{a}$$

Change of Coordinate Systems (Cont.)

- $\mathbf{b} = \mathbf{M}^{T^{-1}}\mathbf{a} = \mathbf{T}\mathbf{a}$



Rotation, Scaling



Translation ???

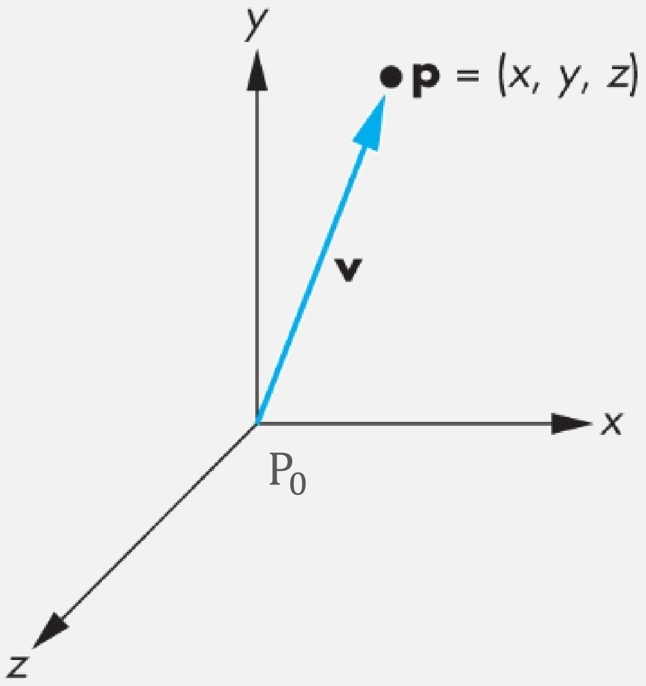
➔ Homogeneous Coordinates

Homogeneous Coordinates

- Any point P can be represented as

$$\mathrm{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = xv_1 + yv_2 + zv_3 + \mathrm{P}_0$$

$$\Rightarrow \mathrm{P} = [x \quad y \quad z \quad 1] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$



- Any vector w can be represented as

$$\mathrm{w} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 0] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

Homogeneous Coordinates (Cont.)

- If $(v_1, v_2, v_3, \mathrm{P}_0)$ and $(u_1, u_2, u_3, \mathrm{Q}_0)$ are two frames:

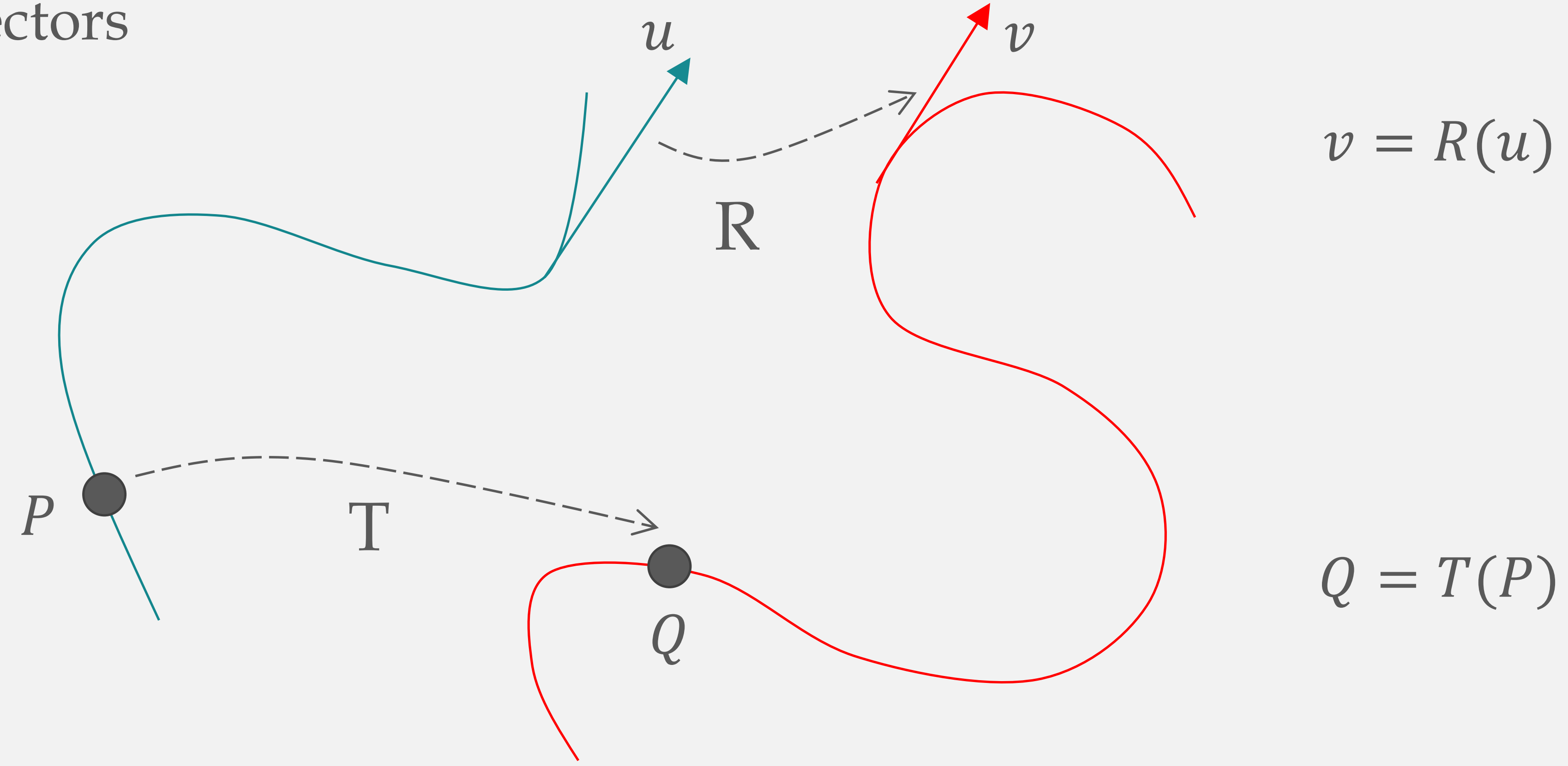
$$
\begin{aligned}
u_1 &= \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3 \\
u_2 &= \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3 \\
u_3 &= \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3 \\
Q_0 &= \gamma_{41} v_1 + \gamma_{42} v_2 + \gamma_{43} v_3 + P_0
\end{aligned}
$$

$$
\Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathrm{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \mathrm{P}_0 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}
$$

Matrix Representation of the change of frames
(12 coefficients)

General Transformations

- A transformation maps points to other points and/or vectors to other vectors



Affine Transformations

- A transformation that preserves lines and parallelism
  - maps parallel lines to parallel lines
- If we know the transformations of p and q, we can obtain the transformations of linear combinations of p and q by taking linear combinations of their transforms:
  - $f(\alpha p + \beta q) = \alpha f(p) + \beta f(q)$
- Characteristic of many physically important transformations
  - Rigid body transformations: identity, translation, rotation, reflection
  - Non-rigid body transformation: scaling, shear

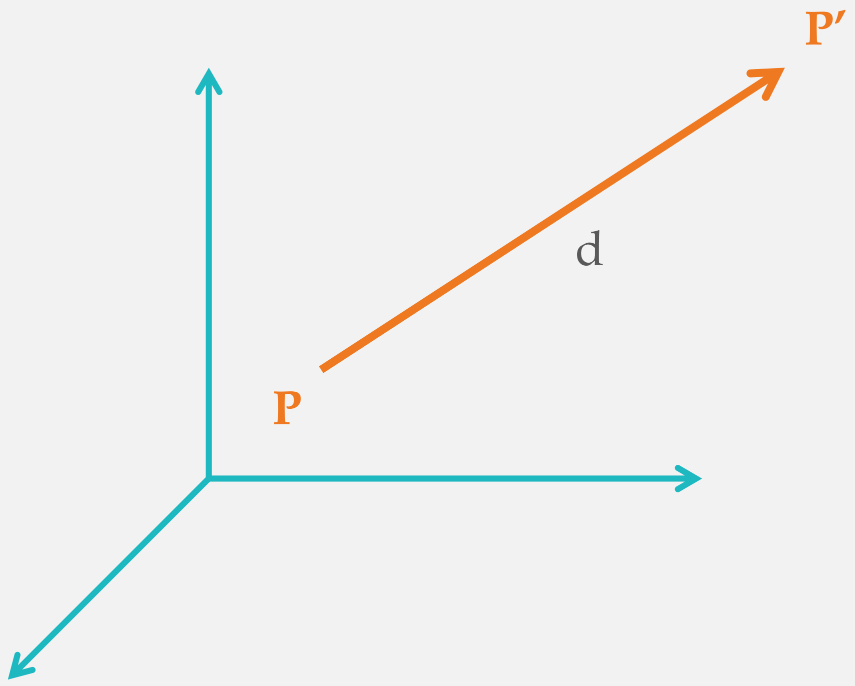
Translation

- Using the homogeneous coordinate representation in some frame

$$
\begin{aligned}
p &= [x \quad y \quad z \quad 1]^T \\
p' &= [x' \quad y' \quad z' \quad 1]^T \\
d &= [d_x \quad d_y \quad d_z \quad 0]^T
\end{aligned}
$$

Hence $p' = p + d$ or

$$
\begin{aligned}
x' &= x + d_x \\
y' &= y + d_y \\
z' &= z + d_z
\end{aligned}
$$



Translation Matrix

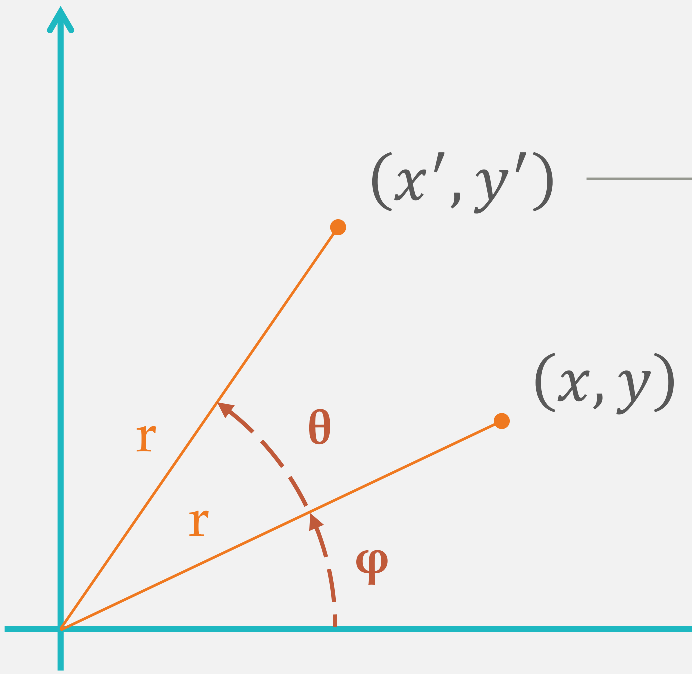
- We can also express translation using a 4 x 4 matrix **T** in homogeneous coordinates $p' = Tp$ where

$$T = T\left(d_x, d_y, d_z\right) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Why do we use a matrix form instead of vector addition?

Rotation (2D)

- Consider rotation about the origin by ***θ*** degrees
  - radius stays the same, angle increases by ***θ***



$$x' = r\cos(\theta + \varphi)$$
$$y' = r\sin(\theta + \varphi)$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$\boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

Rotation about the z axis

- Rotation about z-axis in three dimensions
  - leaves all points with the same z
  - equivalent to rotation in two dimensions in planes of constant z

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

  - or in homogeneous coordinates

$$p' = R_z(\theta)p$$

$$R = R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about *x*- and *y*-axes

- Same argument as for rotation about *z*-axis
  - for rotation about *x*-axis, *x* is unchanged
  - for rotation about *y*-axis, *y* is unchanged

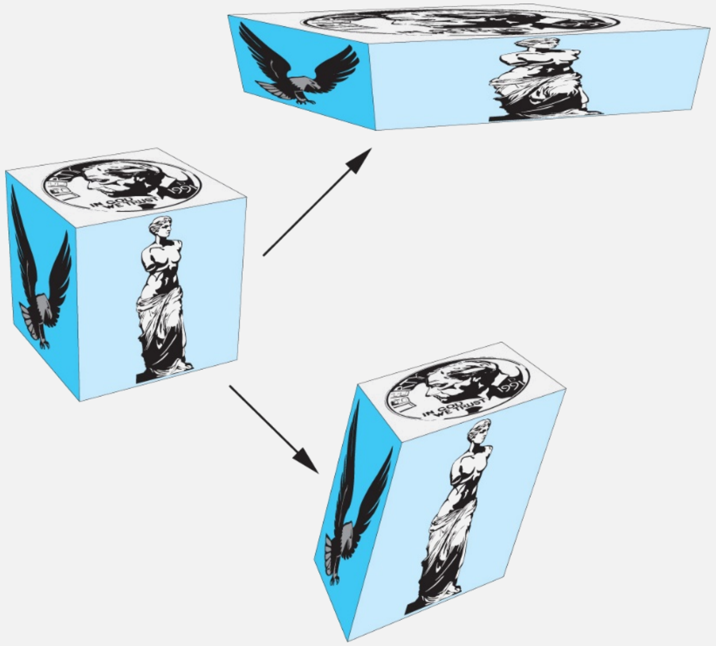
$$R = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- Expand or contract along each axis (fixed point of origin)

$$x' = s_x x$$
$$y' = s_y y \quad \Longrightarrow \quad p' = Sp$$
$$z' = s_z z$$

$$S = S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

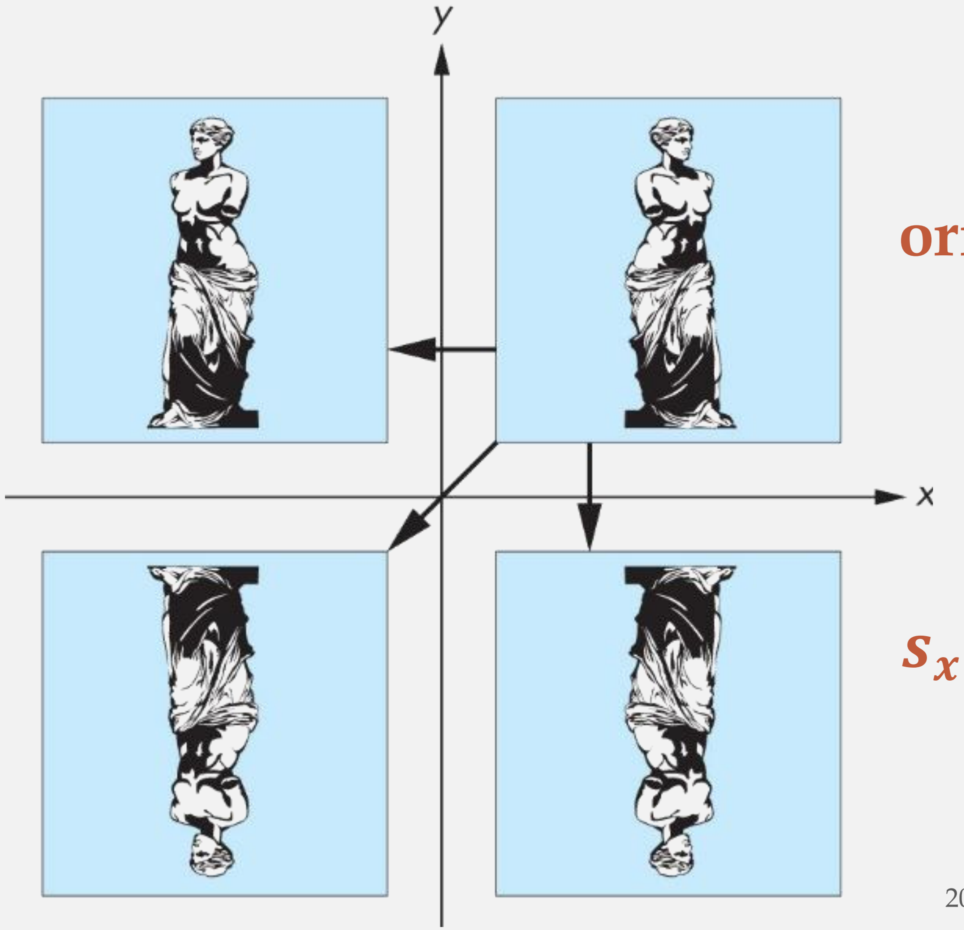


Reflection

- Corresponds to negative scale factors

$s_x = -1, s_y = 1$

original



$s_x = -1, s_y = -1$

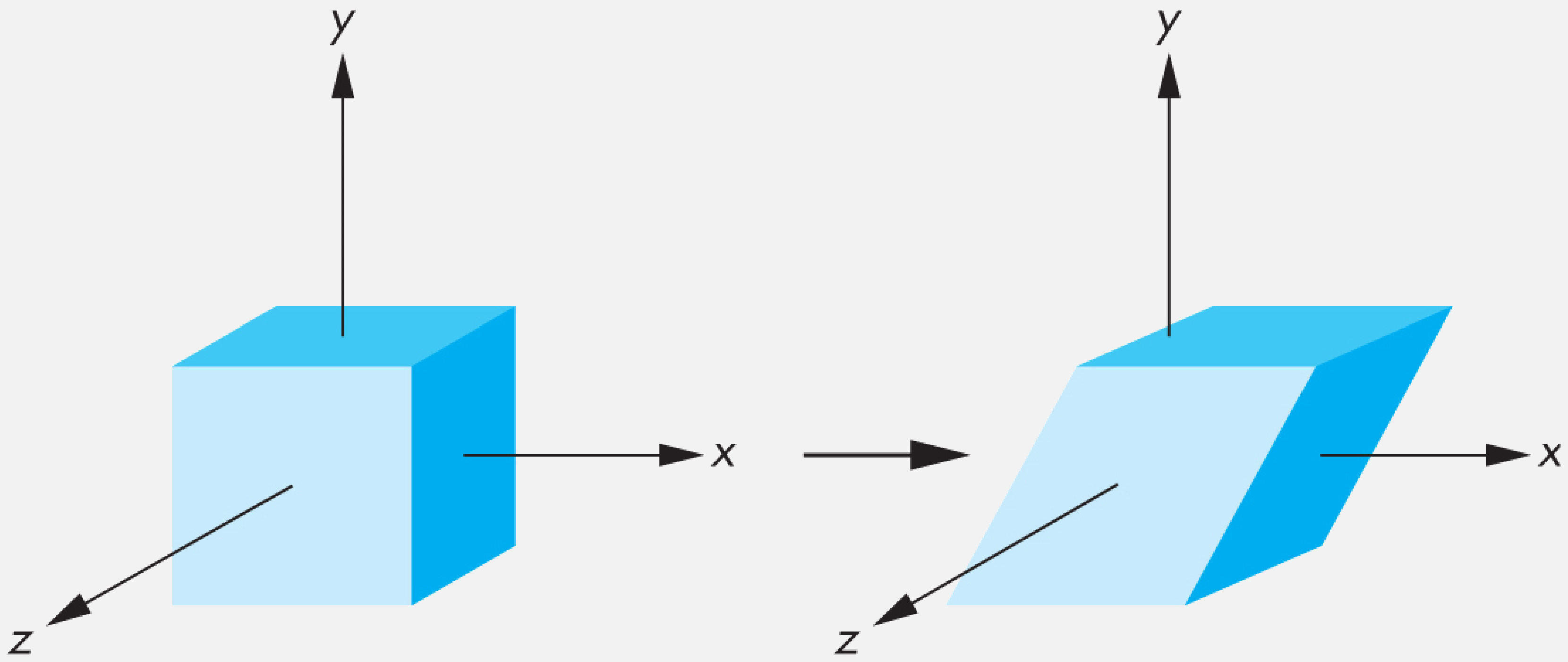
$s_x = 1, s_y = -1$

Inverses

- Compute inverse matrices by general formulas, or use simple geometric observations
  - Translation: $\boldsymbol{T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)}$
  - Rotation: $\boldsymbol{R^{-1}(\theta) = R(-\theta)}$
    - holds for any rotation matrix
    - $cos -\theta = cos\, \theta\; and\;\; sin -\theta = -\, sin\, \theta \rightarrow R^{-1}(\theta) = R^T(\theta) \rightarrow R$ is an orthogonal matrix
    - Normalized orthogonal matrices correspond to rotation about the origin
  - Scaling: $\boldsymbol{S^{-1}(S_x, S_y, S_z) = S\left(\frac{1}{S_x}, \frac{1}{S_y}, \frac{1}{S_z}\right)}$

Shear

- Equivalent to pulling faces in opposite directions



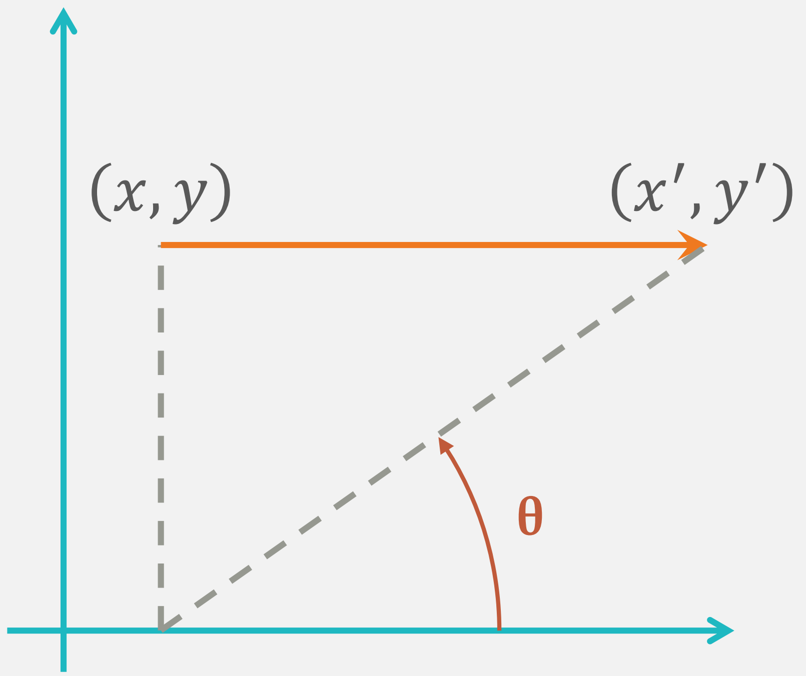
Shear Matrix

- Consider simple shear along x-axis

$$x' = x + y\cot\theta$$
$$y' = y$$
$$z' = z$$

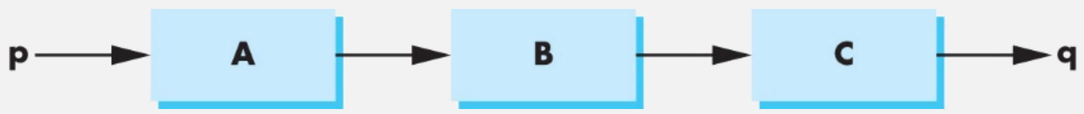
$$\Rightarrow \quad p' = Hp$$

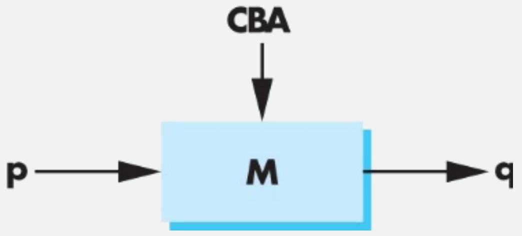
$$H(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Concatenation

- Form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices





$$\boxed{\begin{array}{l} \mathit{for\ each\ i} \\ \quad (C(B(Ap_i))) \end{array}}$$

or

$$\boxed{\begin{array}{l} M{=}ABC, \\ \mathit{for\ each\ i} \\ \quad Mp_i \end{array}}$$

Order of Transformations

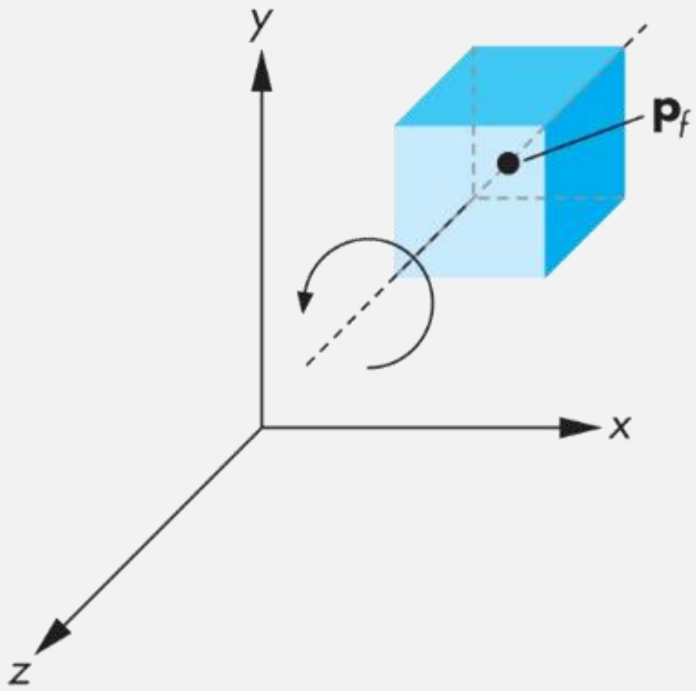
- Note that matrix on the right is the first applied
- Mathematically, the following are equivalent

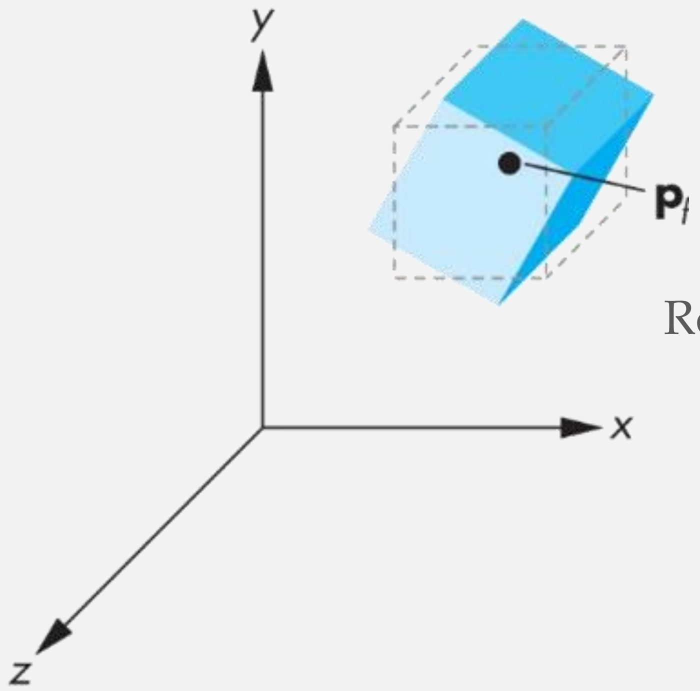
$$p' = ABCp = A\big(B(Cp)\big)$$

- Note many references use column matrices to represent points. In terms of column matrices:

$$p'^T = p^T C^T B^T A^T$$

Rotation About a Fixed Point





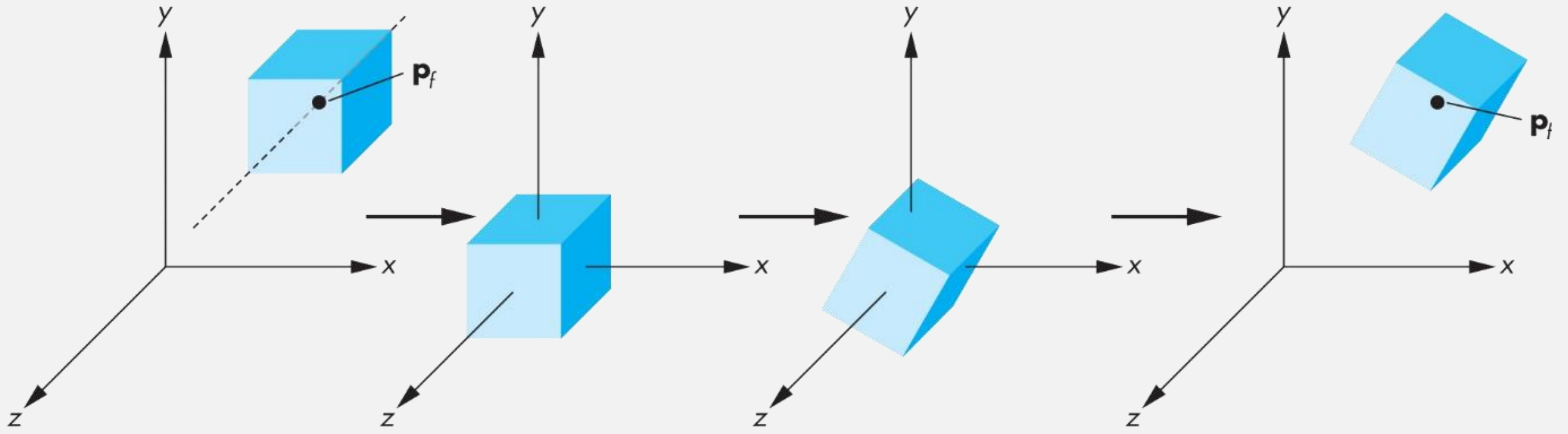
Rotation of cube about its center

Rotation About a Fixed Point other than the Origin

1. Move fixed point to origin
2. Rotate
3. Move fixed point back

$$\boldsymbol{M} = \boldsymbol{T}(\boldsymbol{p}_f)\boldsymbol{R}(\boldsymbol{\theta}_z)\boldsymbol{T}(-\boldsymbol{p}_f)$$

$$\boldsymbol{M} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & x_f - x_f\cos\theta + y_f\sin\theta \\ \sin\theta & \cos\theta & 0 & y_f - x_f\sin\theta - y_f\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

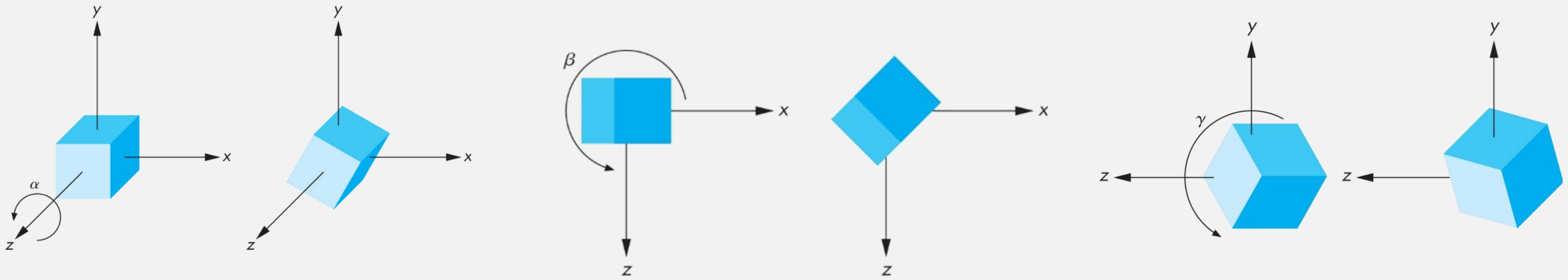


General Rotation About the Origin

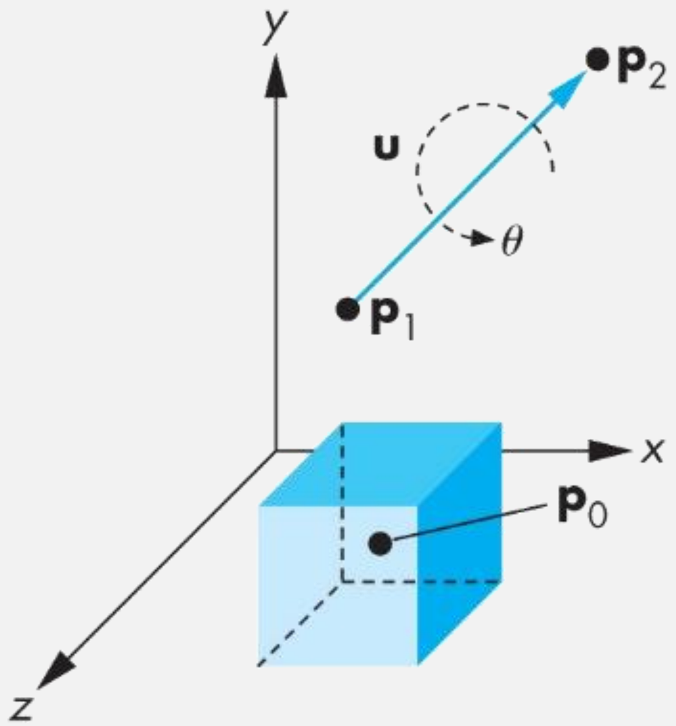
- Decompose into the concatenation of rotations about the *x*-, *y*-, and *z*-axes

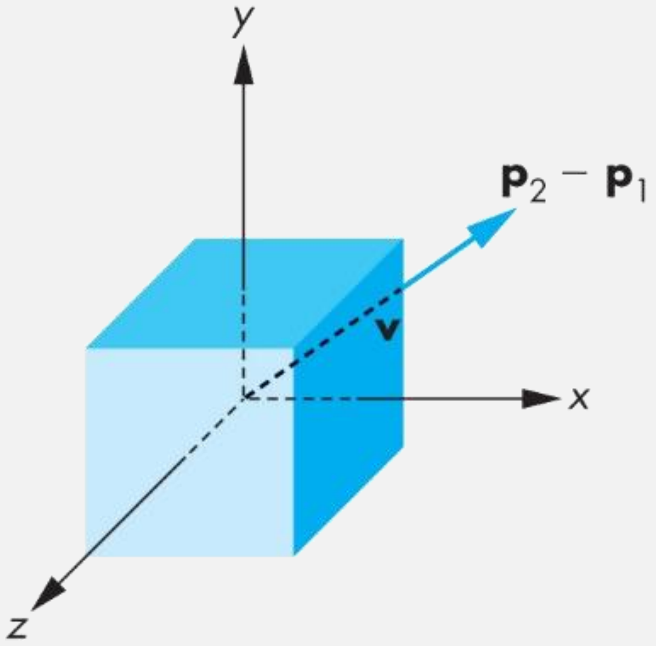
$$R(\theta) = R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$$

θ_x, θ_y, θ_z **are called the Euler angles.**



Rotation About an Arbitrary Axis



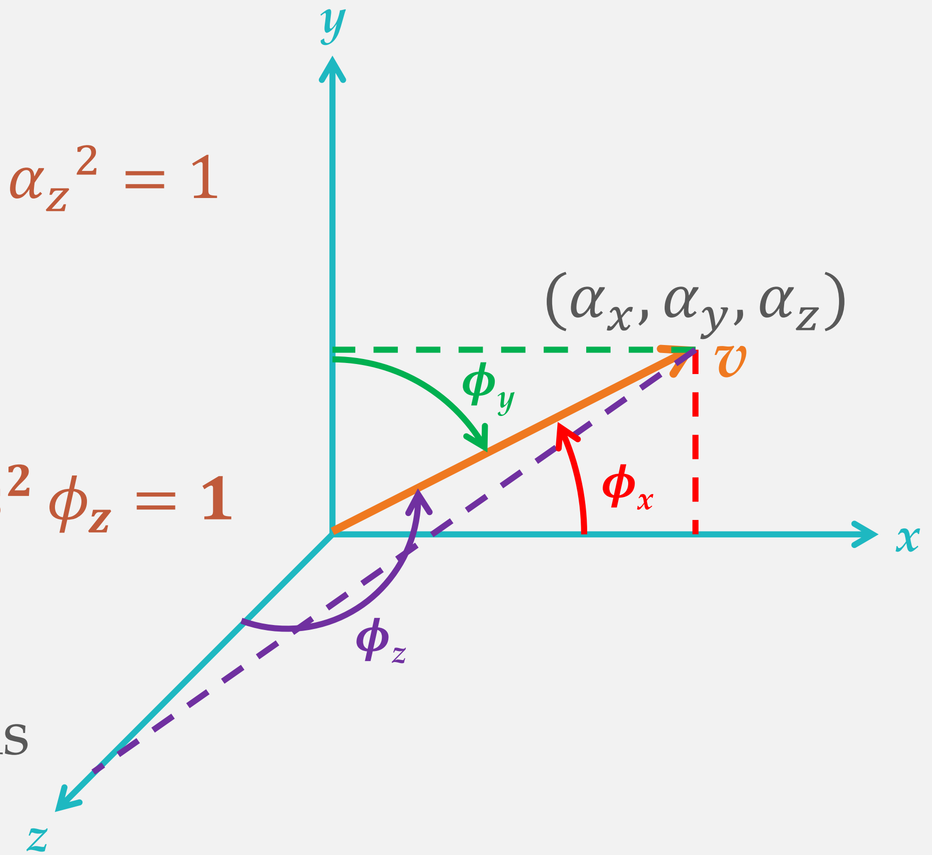


Rotation About an Arbitrary Axis

- Rotate around an axis vector ***u***

$$v = \frac{u}{|u|} = [\alpha_x \quad \alpha_y \quad \alpha_z]^T \qquad {\alpha_x}^2 + {\alpha_y}^2 + {\alpha_z}^2 = 1$$

$$\cos\phi_x = \alpha_x$$
$$\cos\phi_y = \alpha_y \quad \Rightarrow \mathbf{\cos^2\phi_x + \cos^2\phi_y + \cos^2\phi_z = 1}$$
$$\cos\phi_z = \alpha_z$$



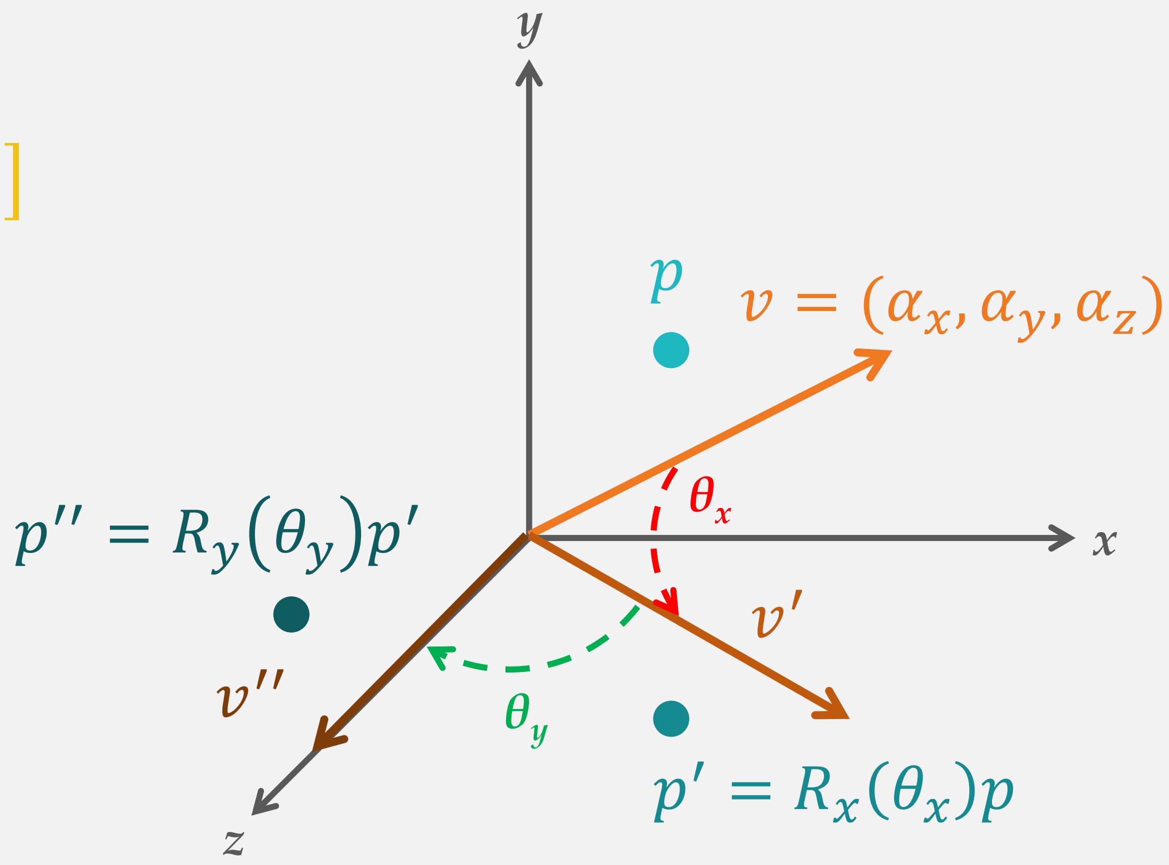
Hint: What we already have are rotations around ***x***-, or ***y***-, or ***z***-axes.

Rotation About an Arbitrary Axis (cont.)

1. Rotate the axis vector to match *z*- (*x*- or *y*-) axis $[R_{axis}]$
2. Rotate around *z*-axis $[R_z(\theta)]$
3. Rotate the axis vector back $[R_{axis}^{-1}]$

$$R_{axis} = R_y(\theta_y)R_x(\theta_x)$$

$$\begin{aligned} M &= R_{axis}^{-1}R(\theta)R_{axis} \\ &= R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x) \end{aligned}$$

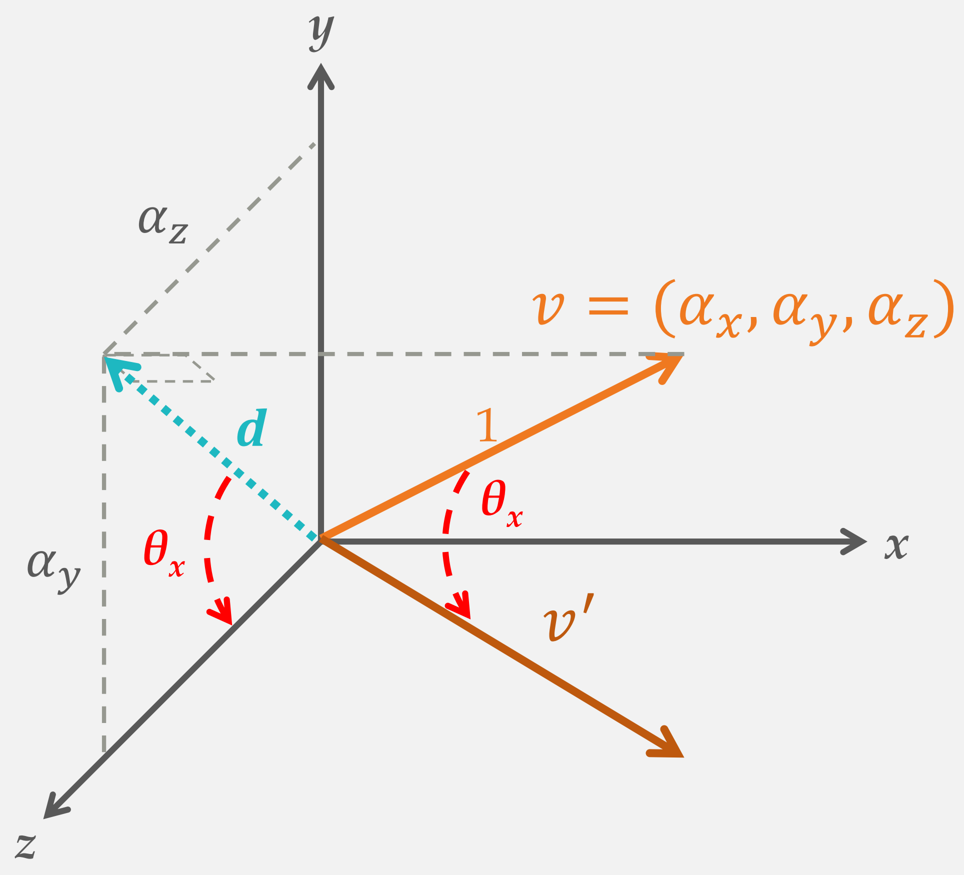


$R_x(\theta_x)$

$$R = R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_z}{d} & -\frac{\alpha_y}{d} & 0 \\ 0 & \frac{\alpha_y}{d} & \frac{\alpha_z}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

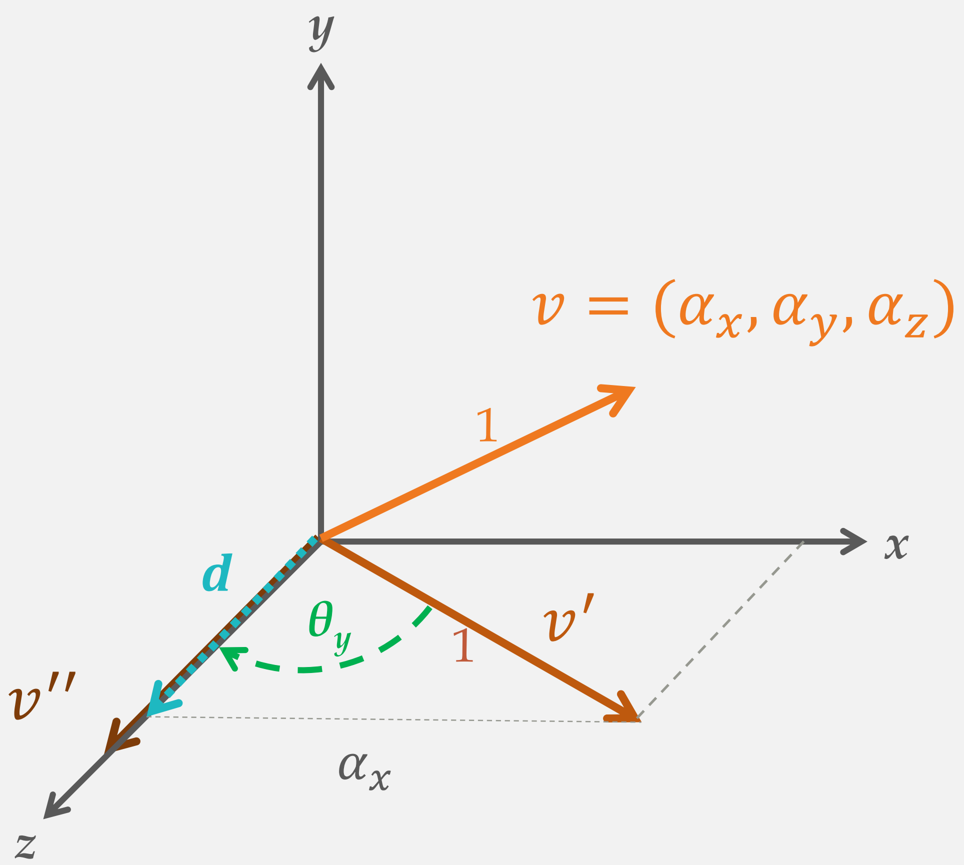
$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



$R_y(\theta_y)$

$$R = R_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & -d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$M = T(P_0)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-P_0)$$

Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size

- We apply an instance transformation to its vertices to
  - Scale
  - Orient
  - Locate