A Theory of Political Districting*

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Fair districting requires more than compact, contiguous equal-sized districts; namely, sets of districts *should also possess certain features. Specifically, they should be neutral (treat all parties alike) and responsive to changes in votes. In order to establish the extent to which these goals can be achieved, we give a precise definition to the concept of neutrality and expand the notion of responsiveness into three characteristics: the range in actual votes over which a districting plan is responsive; the degree of responsiveness in the vicinity of the "normal vote" (i.e., competitiveness); and the constancy of the swing ratio (i.e., the rate at which vote changes yield seat changes) over a range of votes. We show that while all possible values for these features are readily attainable when considered individually, certain combinations of values cannot be achieved. Finally, we identify the nature of the compromises required and the properties that the compromises possess, and show the kinds of trade-offs that result in reasonably fair districting plans.*

In this article we consider the task of dividing a political unit into legislative districts. The most frequently considered features of size, compactness, and contiguity have been extensively dealt with elsewhere and are not our concern here. Rather, we are concerned with four features of districting plans which provide a useful characterization of the way in which the partisan allocation of legislative seats responds to changing vote totals. After defining the features and showing how they can be achieved individually, we show that certain combinations of these characteristics—in fact those which are probably the most desirable combinations—cannot be achieved. Having shown that trade-offs are a virtual necessity, we then turn to the more difficult task of prescribing standards for fair districting.

Four Characteristics of Districting Plans

Our ultimate goal is the normative one of proposing standards against which the quality of districting plans for political units can be judged. Even apart from the widely accepted standards of compactness, contiguity, and equal size, a large number of other standards could be established. For example, "accessibility" within districts could be maximized (Taylor, 1973, p. 948), or the degree of homogeneity of the districts could be maximized or minimized (Mayhew, 1971, pp. 270–73). Here we limit our investigation to characteristics which indicate the responsiveness of an entire districting scheme to changing vote totals. As Tufte (1973) has pointed out, many districting schemes can be devised which satisfy criteria such as compactness, equality of size, and so on, but which many observers would still regard as unfair because large shifts in votes result in few or no seats changing hands. If we accept the premise that seats in a representative body should change in some specified way as vote totals change, then it is apparent that there is a need to incorporate the partisan division of the vote into criteria for fair districting.[1]

We suggest that four parameters (or characteristics) can be used for this purpose:

(1) *Neutrality.* A districting plan which treats all parties alike in allocating seats per given vote totals is said to be neutral. Stated more formally, a districting plan is neutral when v percent of the popular vote results in s percent of the seats, and this holds for all parties and all vote percentages. Although it would be useful to have a simple measure of the degree to which a given districting scheme departs from neutrality, no such measure is easily defined. Therefore, for present purposes this is a 0–1 attribute: A districting plan is neutral (1) or it is not (0).

(2) *Range of Responsiveness.* The range of responsiveness (RR) of a districting plan is defined as the percentage range of the total popular vote (for the entire political unit) over

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[1]There is a serious question as to whether the courts would enforce the use of the partisan division of the vote as a districting criterion. (Legislators already rely on the party vote, but not in an attempt to devise fair districting schemes.) Although the answer is not at all clear, legal scholars do not reject the possibility (Dixon, 1968, p. vi, Ch. 13; Baker, 1971).

which seats change from one party to the other. Specifically, the low end of the RR is the minimum percentage of the total vote required for a party to win at least one seat (or one-half seat as interpreted in n. 9), while the upper end is the minimum percentage of the total vote required to win all of the seats. Obviously the RR can be any part of the 0–100 percent vote range.

(3) *Constant Swing Ratio.* The swing ratio (SR) of a districting plan is defined as the rate at which a party gains seats per unit increment in votes. When this rate is identical for all vote percentage points over a specified range, the SR is said to be constant (over that range). As with the neutrality parameter, it would be useful to have some measure of degree of departure from a constant SR. However, here too we use a 0–1 parameter, with the SR constant (1) or not (0).

(4) *Competitiveness.* The competitiveness of a districting plan is defined as the percentage of districts in which the "normal" vote (defined below) is within some fixed distance of 50 percent.[2]

All four of these parameters are important considerations in evaluating the adequacy of a given districting plan. The neutrality condition is a simple criterion of fairness since it merely requires that parties be treated the same way in relating seat totals to vote totals. In addition, it is closely related to the concept of "manufactured majorities," in which a party wins a majority of the seats with less than half of the votes (Rae, 1971, pp. 74–76). The range of responsiveness is important because it helps determine whether or not a party survives a landslide election or a series of such elections (Mayhew, 1971, p. 266; Campbell, et al., 1960, p. 553). Under some circumstances (especially in proportional representation systems) the lower end of the RR is particularly controversial since it represents the "threshold of representation" (Rokkan, 1968, pp. 12–25; Rae, 1971, passim). The swing ratio over a range of votes is important because it describes the rate at which changes in the vote are translated into changes in legislative seats. Moreover, districting plans may be sufficiently insensitive to vote changes that the responsiveness of legislators (i.e., constituency accountability) becomes problematic (Tufte, 1973). Finally, the competitiveness of individual districts and of sets of districts is important because it is said to be related to a variety of political phenomena such as congressional voting behavior (Fiorina, 1974; Mayhew, 1971, pp. 255–69) and the policy output of state legislatures (e.g., Cnudde and McCrone, 1969).

In addition to the fact that each of these four characteristics is important in its own right, they also possess important interdependencies. To see an illustration of this point, consider the competitiveness parameter. A districting plan might be judged acceptable in terms of competition if, for example, all districts had a "normal" vote of 55 percent favoring one party. If, however, one imposed a wide range of responsiveness, or required a constant SR over the votes interval 40 percent to 60 percent, a very different kind of districting plan would be required. Indeed, the changes brought about by considering several characteristics simultaneously highlights one of the major points we make; namely, that *all four* characteristics must be considered in evaluating the fairness of a districting plan.[3]

In order to understand better the properties of each of these four characteristics, as well as the ramifications of their interdependencies, we need to work with a well-defined model. The model we use is described in the next section.

The Model

Let us consider the task of dividing a political unit (e.g., a state) into electoral districts. We assume the following:

1. There are two political parties (which we call Republican and Democrat).
2. Districts are of equal size.
3. Clustering of the population (such as most Democrats living in one geographical area) does not prevent achievement of desired districting.
4. A "normal" vote for the political unit of interest (symbolized TNV, for total normal vote) describes the expected partisan division of the vote in future elections for the office in question. Each district with-

[2]None of the currently available definitions of competitiveness, including the one used here, is entirely satisfactory. As we will shortly make clear, however, the problems associated with these definitions are largely overcome by considering all four of the characteristics we have defined.

[3]Implicit throughout our discussions are the parameter values we tentatively favor–neutrality (=1), a wide RR, a constant (or nearly constant) SR over relatively wide ranges, and a fairly high degree of competitiveness. Even if one favors widely different values, however, we believe that these parameters are important features by which to judge the quality of a districting plan.

in the political unit is also characterized by a normal vote (symbolized DNV_i, where i is an index referencing a particular district).

5. There are no constituency effects in the vote.

Assumption 1 is a simplification for convenience and can be eliminated by developing models for multiparty systems. Assumption 2 is a relatively innocuous assumption in that minor variations from equal size will not substantially change the results of our analysis. Moreover, the essential results presented below hold for virtually all district sizes, although the specific details do differ. Assumption 3 is a simplification that we feel must be incorporated in our model because it is almost impossible to deal abstractly and yet systematically with real-world population clustering. Furthermore, part of our objective is to show that even when real-world problems such as this are assumed away, it is still impossible to formulate districting plans with certain combinations of features that many would likely view as desirable. If one were in fact attempting to implement our proposed compromises, one would simply have to attempt to do the best possible in the face of population clustering.

Assumption 4 states that we know, or can estimate, a "normal" or expected vote in the total political unit and within each electoral district. Specifically, if DNV_i represents the proportion of voters in district i who "normally" vote Democratic, then the TNV is simply a weighted mean of the individual DNV's, where the weights employed depend on the size of each district. Here, since we assume that all districts are equally sized, $TNV = \frac{1}{n} \Sigma\ DNV_i$, where n is the number of districts. For future reference it should be noted that, unless otherwise stated, all references to proportions (or percentages) will be with respect to votes and seats won by the Democratic party.

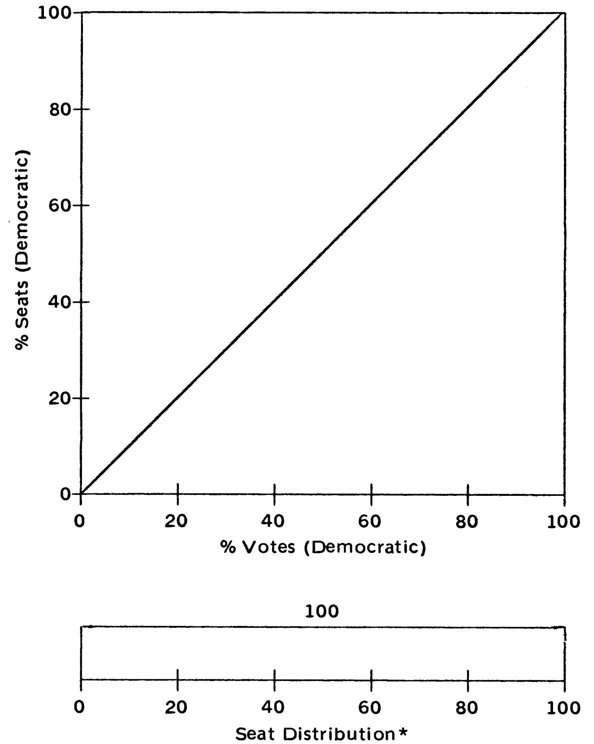
For our purposes, it is immaterial how one might go about measuring or determining a normal vote. Our only concern is that it can be used as a base-line for predicting the expected division of the party vote in subsequent elections. Past voting results are probably the most frequently used mechanism for estimating a normal vote, but registration figures, survey results, or some other means (such as forecasting future election results on the basis of projected demographic changes in the electoral districts) could also be used. We would argue that incumbent legislators frequently make assessments of this sort in an attempt to anticipate the partisan consequences of proposed districting plans. Thus, our assumption simply extends what we view as established practice by requiring that a specific estimate of future votes in a district be available rather than a general characterization of the district as, for example, "safe Democratic."

Assumption 5 states that there are no constituency effects in the vote. We ignore constituency effects simply because there is little one can do to anticipate them and take them into account. For example, it may be thought that a particular district is safely Democratic. But, if an especially attractive Republican candidate were to seek election, or if a particularly unattractive Democratic candidate were to be nominated, or if an unexpected issue arose which was favorable to the Republicans, or if the constituency changed rapidly because of migration, the seat might be closely contested. Events of this sort will obviously influence the actual competitiveness of electoral districts and, of course, will also affect the responsiveness of a given districting plan to changes in the vote total. Yet for the most part constituency-specific factors can be either roughly taken into account when estimating the normal vote or are effects which simply cannot be foreseen at the time of districting. In either case, it seems safe to ignore them in our model.

Given these assumptions, our model can be graphically represented as a distribution of seats (characterized by their district normal votes) and a related seats-votes curve. An example is shown in Figure 1, where a political unit with TNV = 50% is divided into 100 districts with DNV's uniformly distributed between zero and 100 percent.[4] The seats-votes curve depicted shows the expected proportion of the seats the Democrats will win for each percentage of the total vote they receive. Figure 1 represents an extreme instance of proportional representation: at each point on the curve the Democrats expect to win exactly as many seats as they do votes.

Before proceeding, we need to clarify the way in which deviations in the total *actual* vote (TAV) from the total *normal* vote (TNV) are

[4]See the Appendix for the precise way in which the DNV's are distributed. Except for Figure 8, all of our figures assume 100 districts. Our analysis is best illustrated with a fairly large number of districts, and 100 is convenient since each district represents exactly one percent of the seats. Small numbers of districts do not alter the basic results of our analysis, but they do make it more difficult to achieve satisfactory results. This is taken up below.



*Districts (i.e., DNV's) are uniformly distributed across the entire 0—100% Votes interval. The Appendix contains the exact specification of all seat distributions displayed in the figures.

Figure 1. Hypothetical Seat Distribution and Related Seats-Votes Graph

distributed among the districts. Consider the situation in Figure 1. In accordance with assumption 5, the district actual vote (DAV) deviates uniformly from the DNV in all districts. Therefore, since it is assumed that no constituency effects are present, if the Democratic vote percentage were to drop, say, 10 percent below normal in a given election, it must be 10 percent below normal in each and every district. Now suppose that the TAV is 40 percent Democrat. Since districts cannot obtain less than zero percent of the vote, there are districts in which the DAV cannot be 10 percent below the DNV (i.e., all districts in which DNV $<$ 10%). Consequently (since by supposition the TAV = 40%), the vote in other districts must necessarily decline by more than 10 percent. It is possible, but unnecessarily complicated, to figure out what that decline must be. It turns out that for a TAV of 40 percent, the 89 most Democratic districts would get 10.61 percent less than their normal vote, so that the DAV's would be .39 percent, 1.39 percent, and so on.

A simpler procedure, and the one we use, is to interpret a difference between the TNV and TAV as meaning that this same difference exists between the DNV and DAV in all districts, subject to the limits of zero and 100 percent. Thus, for example, if TNV–TAV = 10%, the vote in each district is DNV – 10% if the DNV $\geqslant$ 10%, and 0% if the DNV $<$ 10%. This simplification permits us to speak of the TAV as some percentage, say 40 percent, while our interpretation is that the actual vote in each district is (TNV–TAV)% below the DNV, subject to the limits of zero and 100 percent. (Algebraically, DAV = DNV – (TNV – TAV), with DAV constrained to be between zero and 100 percent, inclusive.) In practice, this interpretation is likely to cause few problems since electoral districts are not usually designed to have extremely large or extremely small DNV's, and since actual votes do not usually deviate too far from the normal vote.

Having shown how the actual vote is distributed among the districts, we now define the swing ratio and show how it is calculated. The swing ratio (SR) is defined as the percent change in seats corresponding to a one percent change in votes. Let us show how to determine this rate of change with a specific example. Consider once again Figure 1, and suppose that the TAV = 40% (meaning, as noted, that the actual vote in each district is either 10 percent less than the district normal vote, or zero). Given this situation, the Democrats would win all seats having a DNV > 60% (i.e., 40 seats), since they would receive over 50 percent of the popular vote in each of these districts. If the Democratic vote were to increase by one percent to a TAV = 41%, the Democrats would win all seats with a DNV > 59%, for a one-seat gain.[5] Thus, in going from 40 to 41 percent, a one-percent increase in the vote total leads to a one-percent increase in seats, so the swing ratio (SR) equals 1.0. Note that this same rate of seat changes to vote changes is true for all percentages in Figure 1, so SR = 1.0 for the entire 0–100 percent range.

Since seats cannot be split fractionally between the parties,[6] it will sometimes be useful to calculate an average SR over a range of votes. For example, if 25 districts are uniformly spread over 15 percentage points, one-third of the time a gain of one percent in votes will yield a gain of only one seat while two-thirds of the time a one-percent gain in votes will yield two seats. What is important, however, is that the average gain over that range is 25/15 = 1.67 seats per one-percent gain in votes, and that the uniform distribution of seats yields the closest possible approximation of that seats-votes relationship. (That is, the average gain is equivalent to the linear regression line over that range.) In such situations, the average gain is the value reported for the SR.[7]

Achieving Individual Parameter Values

Using the model just described, it is possible to design districting plans in which the four responsiveness parameters take on specified values. We show how this can be done for each parameter in turn.

Neutrality. A districting plan is neutral when the distribution of district normal votes around the total normal vote is symmetric.[8] This can be easily demonstrated. First, for any arbitrary TNV in the interval zero to 100 percent, consider the case where TAV = 50% Democratic. By definition there exists an x such that TNV $- x =$ TAV $= 50\%$, where $-50\% \leqslant x \leqslant 50\%$. It follows that the Democrats would win all seats for which DNV $> (50 + x)\%$ since in these districts the actual vote (DAV) would be more than 50 percent Democratic. Since $(50 + x)\% =$ TNV, and since the DNV's are distributed symmetrically about the TNV, the Democrats would win exactly one-half of the seats.[9] Hence the system is neutral when the actual vote is 50 percent Democratic. Now suppose that the Democratic vote *increased* from 50 percent by some number y, where $0\% < y \leqslant 50\%$. The Democrats would take from the Republicans the following seats (in addition to maintaining those for which the DNV > TNV):

$$(1)\quad \begin{cases} \text{half of the seats for which DNV = TNV (having already won half of these seats);} \\ \text{half of the seats for which DNV = TNV} - y\text{, and} \\ \text{all seats for which TNV} - y < \text{DNV} < \text{TNV.} \end{cases}$$

Conversely, if the Democratic vote *decreased*

[5]As noted in the Appendix, we give DNV's values which are ϵ above or below an integer value. Thus there is a DNV = $(60 + \epsilon)\%$ Democratic and another with DNV = $(59 + \epsilon)\%$ Democratic. When TAV = 40% Democratic, the former district will be won by the Democrats (with a vote of $(50 + \epsilon)\%$) but the latter will be lost (with a vote of $(49 + \epsilon)\%$). When TAV = 41%, the latter district will also be won by the Democrats. Hence the one seat gain.

[6]A minor exception to this can occur in our model because we have to deal with the possibility that a vote exactly equals 50 percent. See n. 9 below.

[7]This feature is not peculiar to our model. No matter how calculated, any swing ratio that is not integer-valued involves some smoothing since a one-percent increase in votes cannot bring about a fractional gain in seats. Most calculations of the SR involve regressing seats on votes over some range of votes. If desired, the precise SR's at each percentage of the vote can be determined from the distributions given in the Appendix.

[8]Symmetry is actually a necessary and sufficient condition for neutrality. We only prove the sufficiency condition.

[9]We assume that when the actual vote in a district is exactly 50 percent Democratic, the seat(s) is (are) split equally between the parties. This is unrealistic, of course, as it results in parties winning "half seats," but it seems much more appropriate than alternative rules (e.g., arbitrarily assigning the seat(s) to one party). One can think of a half seat as a .50 probability that the seat would be won.

from 50 percent by y, the Democrats would lose to the Republicans the following seats (in addition to those seats already lost for which the DNV $<$ TNV):

(2) $\begin{cases} \text{half of the seats for which DNV = TNV (having already lost half of these seats);} \\ \text{half of the seats for which DNV = TNV} + y\text{, and} \\ \text{all seats for which TNV} < \text{DNV} < \text{TNV} + y. \end{cases}$

Since by stipulation the distribution of DNV's is symmetric about the TNV, the same number of seats satisfies (1) and (2). Hence, the system is neutral throughout the entire 0–100 percent range.

Note that neutrality is achievable regardless of the value of the TNV, the magnitude of the range of responsiveness, and the shape of the distribution of DNV's (as long as the distribution is symmetric about the TNV). Furthermore, neutrality is a property that is unaffected by small numbers of districts, non-constant swing ratios, and other factors that are considered below.

Range of Responsiveness. The range of responsiveness is a function of the greatest deviation of the district normal votes above and below the total normal vote. More precisely, let the least Democratic DNV be TNV$-x$, and the most Democratic DNV be TNV $+ y$. Then the RR is $(50 - y)\%$ Democratic (or 0% if $y > 50\%$) to $(50 + \epsilon + x)\%$ Democratic (or 100% if $x > (50 - \epsilon)\%$). To prove this, assume that in the most Democratic district DNV = TNV $+ y$. The difference between this DNV and 50 percent is TNV $+ y - 50\%$, which may be a negative value. Hence, the actual vote in the most Democratic district can be as low as TNV $+ y - 50\%$ below the TNV, and the district would still go (one-half) Democratic. Since $(\text{TNV} - (\text{TNV} + y - 50))\% = (50 - y)\%$, the system will be responsive when the total actual Democratic vote reaches $(50 - y)\%$ and is unresponsive below that value. A similar argument using the least Democratic district shows that the Democrats would win all the seats when that district's actual vote is $(50 + \epsilon - (\text{TNV} - x))\%$ above the TNV. Since $(50 + \epsilon - (\text{TNV} - x) + \text{TNV})\% = (50 + \epsilon + x)\%$, the system will be unresponsive after the total actual Democratic vote reaches $(50 + \epsilon + x)\%$.

Note that the RR is restricted to less than the 0–100 percent range if TNV $\neq$ 50%. For example, if TNV = 65%, the RR can at most be from 15 to 100 percent; if TNV = 80%, the RR is at most 30 to 100 percent.[10] If TNV $\neq$ 50% and the number of districts is small, the RR may be further restricted since it may not be possible to have districts with both zero percent and 100 percent DNV's. Nevertheless, within these constraints, any desired RR can be achieved simply by creating districts with DNV's at appropriate distances above and below the TNV.

Constant Swing Ratios. The swing ratio will be constant over the range $(50 - x)\%$ to $(50 + x)\%$ if the DNV's are distributed uniformly from (TNV $- x$) to (TNV $+ x$).[11] As noted above, if the most Democratic district has a DNV = TNV $+ x$ Democratic, the Democrats will begin winning seats when the TAV = $(50 - x)\%$. Then, because of the uniform distribution of DNV's, every one-percent increase in the vote will yield a constant increase in the number of seats (i.e., the average gain per one-percent increase in the vote will be constant) until all seats are won. Examples are given in Figure 2 where x = 10% and 25%. Note that a uniform swing ratio over the entire RR guarantees neutrality.

Competitiveness. Since it is often assumed that competitiveness is a desired property, we show that maximizing the number of competitive districts can be accomplished in a straightforward manner. If one instead wished to minimize the number of competitive districts, or fix the number at some specified level, this could be accomplished in a similar fashion. To begin with, if the TNV falls within what is considered the competitive range, competition can be maximized by simply making DNV_i = TNV for all districts. On the other hand, if the TNV is outside the competitive range, there exists an upper maximum percentage (less than 100 percent) on the number of districts that can be made competitive. It can be shown that this percentage is dependent on the TNV, the number of districts, and the level at which a DNV is regarded as competitive.

Rather than showing a general (and needlessly complex) solution, however, let us simply stipulate that competitive districts are those districts having a DNV of 45 to 55 percent, and suppose that the TNV is outside this range on

[10]Since ϵ can be arbitrarily small anyway, we will simply drop it where there is no danger of ambiguity.

[11]Constant swing ratios over other ranges (i.e., not symmetric about 50 percent) are constructed in a similar manner. Again we prove only a sufficiency condition.

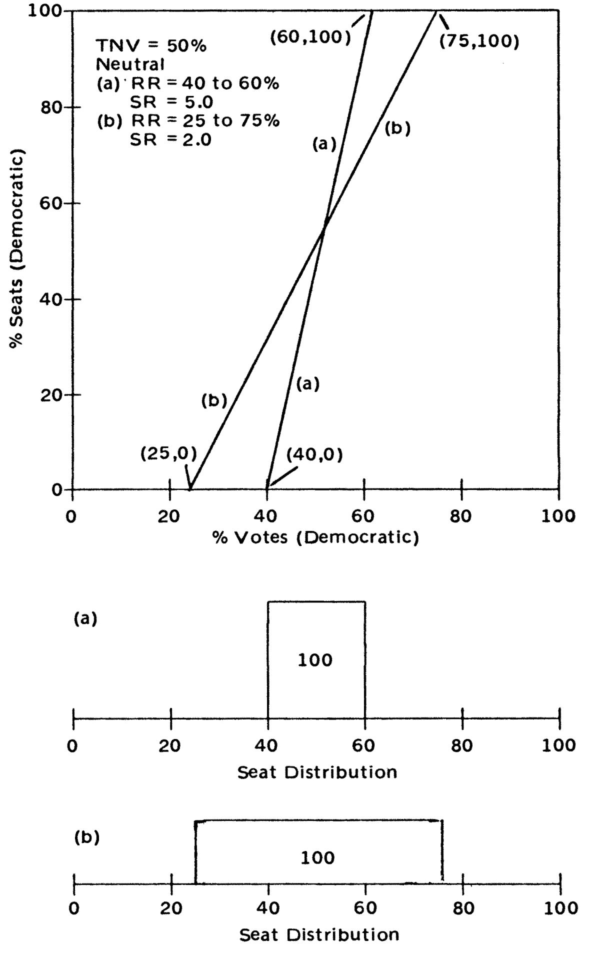


Figure 2. Examples of Uniform Swing Ratios with Different Ranges of Responsiveness

the Democratic side. The number of competitive districts can be maximized if we allow the competitive districts to be at the extreme pro-Democratic edge of the competitive range, i.e., at 55 percent Democratic. Therefore one solution to the problem of maximizing the number of competitive districts (or equivalently, minimizing the number of noncompetitive districts) is to solve the following constrained maximization (minimization) problem, viz., maximize a in:

$$55a + x[b] = n \times \text{TNV}, \text{ subject to} \tag{1}$$

$$a + [b] = n \text{ and} \tag{2}$$

$$\text{TNV} < x \leqslant 100\%, \tag{3}$$

where a = the number of competitive districts; $[b]$ = the smallest integer satisfying the equations (i.e., the number of noncompetitive districts), and x = DNV (in percent Democratic) of the noncompetitive districts.

To illustrate the solution suppose n = 10 and TNV = 60%. From (1),

$$55a + x[b] = 600. \tag{4}$$

Solving (2) for a and substituting into (4), yields

$$[b] = \frac{50}{x-55}. \tag{5}$$

Solving (5) subject to the constraint (3) yields

$[b] = 2$ and $x = 80\%$, and from constraint (2), $a = 8$. In other words, by making 2 districts 80 percent Democratic, 8 of the 10 districts can be made competitive (with DNV's = 55%).

It should be noted that this solution is not always unique, and may not be entirely satisfactory. Moreover, if the TNV is quite far from 50 percent, no amount of juggling (with equal-sized districts) can make a large proportion of the districts competitive. Nevertheless, by itself, maximizing the number of competitive districts is not a difficult task.

Impossibility Results

Thus far we have defined four parameters which describe the way in which sets of district seats respond to changing vote totals. Moreover, we have shown that certain values for these parameters can be easily achieved when considered individually: namely, neutrality, any (feasible) RR, a constant SR, and maximum competitiveness can all be obtained with ease. Simple as it is to achieve these values individually, however, it may be impossible to attain certain *combinations* of these parameter values in a given districting plan.

Whether or not a particular combination of parameter values can be achieved depends on the precise values specified for the parameters, on the TNV, and on the number of districts. If, for example, TNV = 50%, there is a large number of districts and one is satisfied with a districting plan that is neutral, has an RR from 25 to 75 percent, has a constant SR, and has at least 40 percent of the DNV's in the range (50 ± 10)%, then such a districting plan can be designed with all of these features. Figure 2(b) is such a plan for 100 districts. However, if it were instead required that half of the districts have DNV's in the range (50 ± 10)%, or required that the RR be 20 to 80 percent, or if the TNV = 67%, then no plan can be designed that satisfies all of these values.

It is possible, by means of a linear program, to express the interdependencies among the four parameters, for a given TNV and a fixed number of districts. Such a program would show, in a general way, the region of feasible solutions and the trade-offs among the parameter values for political units with a given number of districts and with a specified TNV. The problem with this approach is that there are really six variables that need to be taken into consideration (the four parameters plus the TNV and the number of districts), and lacking a quantitative measure of the degree to which neutrality and a constant SR are approached, this program would be extremely complex and of rather limited value. Nevertheless, at a less formal level, it is clear that if one values neutrality, a wide RR, a relatively constant SR, and a high degree of competitiveness, it may be impossible to design a completely satisfactory districting plan.

Though a completely general approach does not seem useful at present, it is possible (by means of a few examples) to gain a more intuitive understanding of the incompatibilities among the parameter values and to understand the significance of these incompatibilities. Consider once again Figure 1. Figure 1 illustrates a districting plan that is neutral, has the maximum possible RR, and has a constant SR throughout that range. Relatively few seats, however, are competitive by commonly used standards (45–55 percent being the highly competitive range, 40–60 percent being somewhat competitive). Obviously more seats could be made competitive by simply creating a larger number of districts with DNV's close to 50 percent. If this were done "symmetrically," neutrality would be maintained. Moreover, if one district were left with a DNV = $(0 + \epsilon)\%$ and one with a DNV = $(100 - \epsilon)\%$, the RR would not be narrowed. It is readily apparent, however, that as more seats are made competitive, the SR will no longer be constant. At the extreme, the modified districting plan would still be neutral, have the maximum RR, and all but two of the districts would be extremely competitive. However, it would also have a SR of zero–i.e., be totally unresponsive to vote changes–throughout most of the range of votes.

In general, since DNV's must be spread out uniformly around the TNV to attain a constant SR, but bunched together in the competitive range to maximize competition, it can be seen that these two parameter values are basically incompatible (assuming an RR that is wide compared to the competitive range). This is clearest when TNV = 50%, as in Figure 1. However, the relevant considerations are the same, although the relationships among the specific values are more complex, when TNV $\neq$ 50%.

It is not merely the incompatibility between obtaining a constant SR and maximizing competitiveness that forms the root of the problem. If the TNV $\neq$ 50%, maintaining neutrality and maximizing competitiveness can also be mutually incompatible. Specifically, if the TNV is outside the competitive range, neutrality requires that at least half of the districts be safe since their DNV's must be at least as extreme as the TNV. In contrast, a non-neutral set of districts might include a larger number of districts that are competitive. As an example, consider a political unit in which the TNV = 60%, and assume for simplicity that there are only three districts, 1, 2, and 3. A neutral set of districts would require DNV's such as these:

	(a)	(b)	(c)	(d)	(e)	
1 ...	60	59 ...	55 ...	50 ...	45 ...	
2 ...	60	60 ...	60 ...	60 ...	60 ...	
3 ...	60	61 ...	65 ...	70 ...	75 ...	

At most, only one district is highly competitive by the standards mentioned above. In contrast, a non-neutral set of districts could contain two highly competitive seats, along with one very safe seat, such as:

	(a)	(b)	(c)
1	50	50	52
2	50	55	53
3	80	75	75

Thus, in some circumstances, attempting to maximize the number of competitive districts may require violating neutrality, and conversely, retaining neutrality may require less than the maximum number of competitive districts.

The significance of this incompatibility among parameter values becomes clear if we try to establish standards for fair districting. While no consensus may exist on what the values should be for the four parameters, it seems likely that even if there were consensus on individual parameter values, compromise would still be necessary because of conflicts among the values chosen. It makes little sense, therefore, to establish goals for each parameter alone. Rather, fair districting plans can be arrived at, if at all, only by considering all four parameters simultaneously. This we do in the next section.

Standards for Fair Districting

Prescribing standards for fair districting is a risky business, especially in light of the incompatibilities we have shown to exist among the four responsiveness parameters. Moreover, it would be foolish to propose establishment of exact standards such as a RR of 27 to 78 percent, or a SR of 2.37. This means that in much of the discussion that follows we will speak of a "wide" RR, a "quite high" SR, and so on. Despite these difficulties, we think the model developed here provides a useful framework with which to begin a discussion of standards for fair districting.

First, however, a word about our approach. It would be a relatively simple matter to construct figures showing some of the possible trade-offs among parameter values. For example, assuming neutrality and a constant SR throughout the RR, it is a simple matter to show the nature of the trade-offs between the RR and the number of competitive districts. At some point this kind of general figure may be useful. Currently, however, we feel that the presentation of several specific examples provides a better way of discussing the question of fairness and how to achieve it.

Since the districting task is easiest when the total normal vote is exactly 50 percent and there is a large number of districts, we begin with this case.

TNV = 50%, Large Number of Districts. In this case we think that the best districting plan is one that is neutral, has a fairly wide RR, in which the SR is constant over significant portions of that range, and in which there is a reasonably large number of competitive districts by virtue of a higher swing ratio near the TNV than farther away from it. In support of such a prescription, we note first of all that there is no a priori reason to violate neutrality (such as to increase competitiveness as illustrated earlier). Second, a "fairly" wide RR will virtually assure a major party of some representation without creating extremely one-sided districts (see examples of this below). Third, a constant SR insures that vote changes will lead to seat changes in a relatively well-specified manner; having a large number of competitive districts means that the rate at which seats respond to votes is relatively high. And fourth, having an SR lower near the ends of the RR than in the middle makes it possible to have relatively constant swing ratios both in the middle and near the ends of that range (but a lower swing ratio at the ends than in the middle) without creating a large number of safe seats for each party.

In Figure 3 we illustrate several districting plans with these prescribed features. Any line starting at 20 or 30 percent of the vote and ending at 70 or 80 percent represents the SR for an alternative districting plan. Thus, for example, one could arrange the DNV's so that the SR = 1.0 from 20% to 45%, SR = 5.0 from 45% to 50%, SR = 2.5 from 50% to 60%, and SR = 1.25 from 60% to 80%. However, since neutrality requires a symmetric distribution of DNV's around 50 percent, we will assume that this is the case in the following discussion.

All of the districting plans presented provide a reasonable approximation to the standards noted above. Neutrality, as noted, is easily attained. The RR (either 20 to 80 percent, or 30 to 70 percent) seems quite acceptable, assuming the TNV really is a meaningful predictor of future votes. With the wider RR, the actual vote could deviate almost 30 percent from normal before one party would be completely denied representation. The SR is constant across ranges of at least 10 percent and there are no "flat spots" (in the RR) where the SR is zero. And, the number of competitive districts, especially in 3(a), is quite high.

Of course, even in this ideal case incompatibilities among parameter values are apparent. Awareness of these incompatibilities, however, aids us in prescribing appropriate

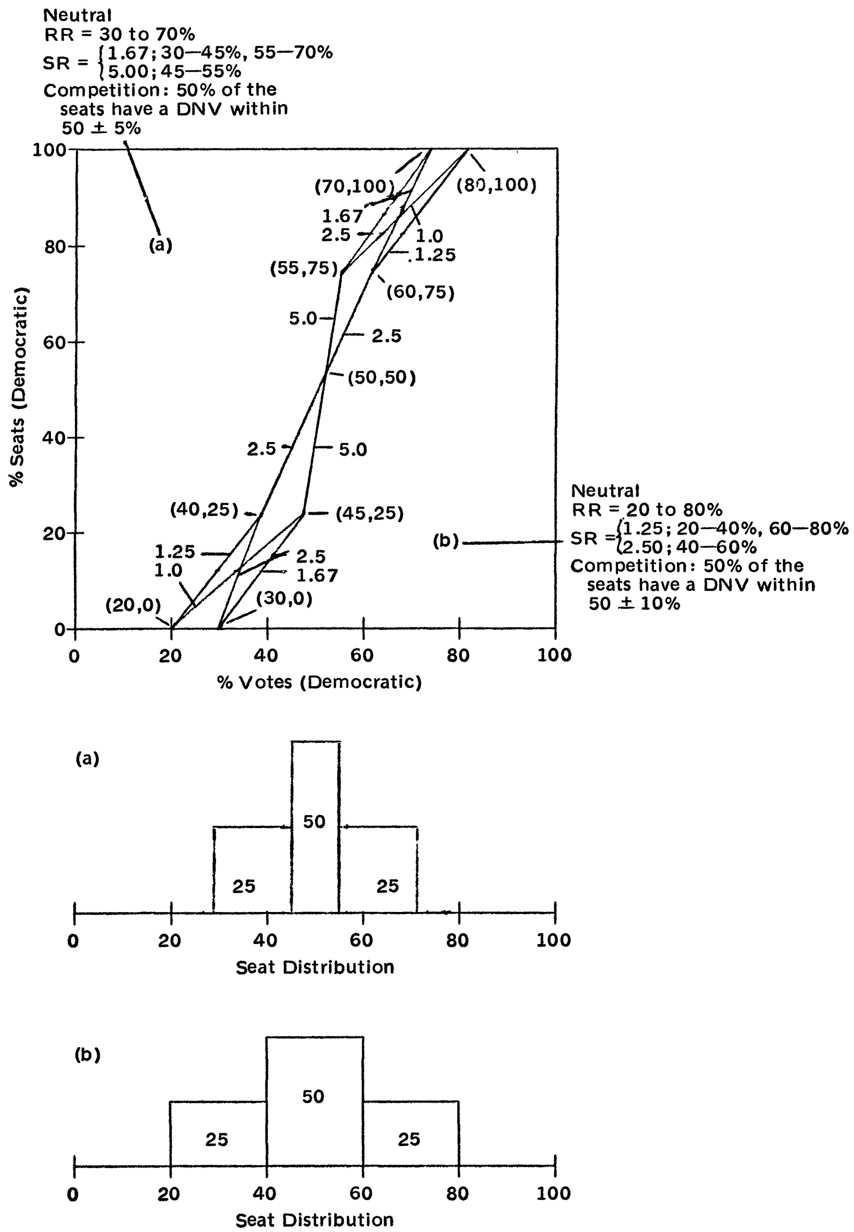


Figure 3. "Fair" Seats-Votes Curves for TNV = 50%

compromises to fit particular circumstances. Thus, if the TNV is regarded as a highly accurate predictor of what the vote in the constituency will be like over the life of the districting plan, then the narrower RR and the correspondingly higher SR's (as in Figure 3(a)) might be desirable. If the RR were fixed initially at 30 to 70 percent, however, we would still need to determine what the SR (or SR's) should be. To do this, consider what would happen if the actual vote dropped to 40 percent Democratic, only a 10 percent departure from the TNV. If the SR = 2.5 from 30% to 40%, the Democrats would win 25 percent of the seats, whereas an SR of 1.67 (from 30% to 45%) would give the Democrats only 17 percent of the seats. On these grounds the SR of 2.5 might be preferable. On the other hand, having the higher SR (2.5) near the ends of the RR necessarily lowers the number of competitive districts. Obviously, choosing between these (or other) SR's requires a subjective judgment; but at least the compromises involved are clear. Note that under these conditions, one might simply adopt a plan in which the SR is constant (2.5) throughout the entire RR.

Suppose, however, that it were considered unacceptable to have an electoral arrangement that would give no more than 25 percent of the seats to a party with 40 percent of the votes, and no seats at all to a party with up to 30 percent of the votes. These problems could be avoided by simply extending the RR. But if a constant SR were adopted, the number of competitive districts would be relatively low (approximately one-third), and the SR in the vicinity of the TNV (and throughout the RR) would be only 1.67. Here, then, it seems highly appropriate to extend the RR but to lower the SR near the ends of the RR.[12] The RR is thereby made sufficiently broad without unduly reducing the number of competitive districts.

It should be noted that an alternate but logically equivalent way of approaching the question of fairness is to ask whether the majority party has a great enough advantage rather than whether the minority party has a small enough disadvantage. Thus, for example, in reference to Figure 3 we previously asked whether it was fair for a party winning 40 percent of the votes to win only 25 percent, or perhaps only 16.7 percent, of the seats. Conversely, we could have asked: Is it fair for a party winning 60 percent of the votes to win "only" 75 percent or 83.3 percent of the seats? While this way of looking at things may suggest different political considerations,[13] we need not consider it further because of its mathematical equivalence to the approach above.

TNV ≠ 50%, Large Number of Districts. When the TNV differs from 50 percent, the mechanics of designing fair districting plans become somewhat more complicated. Moreover, when the TNV is far from 50 percent, the arguments in favor of neutrality may be much less powerful in light of the conflict between neutrality and competitiveness. Therefore it will be necessary to take another look at the question of neutrality. Again we think it most useful to proceed by the use of several examples.

First, suppose the TNV is close to 50 percent, say, 52 percent. In cases such as this the simplest approach is to construct the DNV's in a manner similar to that which was done for a TNV = 50%; that is, distribute the DNV's symmetrically about the TNV, bearing in mind the same considerations used earlier. Figure 4 shows one such plan. This figure is designed to be as similar as possible to the system in Figure 3(a). To accomplish this, the distribution of DNV's in Figure 4 is exactly comparable to that in Figure 3(a) except that it is centered at 52 percent rather than 50 percent.

Note that the SR's in Figure 4 are exactly comparable at every point to those in Figure 3(a). Despite the fact that the DNV's range from 32 to 72 percent, the RR is from 30 to 70 percent. The reason for this is easily seen. The most Democratic district has a DNV = $(72 - \epsilon)\%$. Therefore, when the TAV is $(22 - \epsilon)\%$ less than the TNV, this district and only this district will be won. (Technically the vote in this district will be exactly 50 percent, so one-half of the seat is won.) Since $(52 - (22-\epsilon))\% = (30 + \epsilon)\%$, this is equivalent to saying that the

[12]Such a compromise (not shown in Figure 3) might yield the following: SR = 1.0 from 20–30%, SR = 1.87 from 30–45%, SR = 2.4 from 45–55%, SR = 1.87 from 55–70%, SR = 1.0 from 70–80%. The RR = 20% to 80%, 42% of the seats have an expected vote of $(50 \pm 10)\%$, and the SR around the TNV is 2.4. A party winning 40 percent (30 percent) of the votes would win 29 percent (10 percent) of the seats.

[13]Most two-party systems in fact reward the winning party with more seats than votes (Rae, 1971, Ch. 4). This feature (though not necessarily the specific numbers used in our examples) can be defended on the grounds that it always yields a "working majority" even to a party winning a bare majority of the votes.

Democrats first begin to win seats when they win $(30 + \epsilon)\%$ of the total vote. Similar reasoning shows that the least Democratic seat will be won when the TAV is 18 percent above the TNV, or 70 percent.[14] In terms of the SR, then, this districting plan is identical to that in Figure 3(a). The Democrats are closer to winning most or all of the seats, but that is owing to a TNV $> 50\%$; the districting system itself is not biased against the Republicans.

Despite the similarities between the two figures, the districting plan in Figure 4 has fewer competitive districts than that in Figure 3(a). This characteristic occurs whenever a neutral districting plan is adopted and TNV $\neq$ 50%; that is, the number of competitive districts will be less than the number found in an otherwise identical districting plan for which the TNV = 50%.

Finally, observe that all considerations applying to Figure 3 also apply here as well. That is, the RR can be broadened or narrowed, the SR can be raised or lowered, and competition can be increased or decreased. But as before, the basic incompatibility of parameter values cannot be overcome.

A procedure such as the one outlined in Figure 4 could, of course, be used for any TNV. But it is obvious that as the TNV departs more and more from 50 percent, the number of competitive districts will decline to the vanishing point. Therefore, some other districting scheme must be developed for this situation. A clue as to what that scheme could be is contained in Figure 4. We noted that even though the TNV $\neq$ 50%, a symmetric distribution of DNV's *about* the TNV resulted in SR's centered *about* 50 percent (and, as we designed it, a relatively high value around the center and lower values away from 50 percent). But the actual vote is still expected to be near the TNV.

[14]This is in accord with our formula for the RR described on p. 0000 above. Let $x = y = 20 - \epsilon$. Then the RR is from $(50 - (20 - \epsilon))\% = (30 + \epsilon)\%$ to $(50 + \epsilon + (20 - \epsilon))\% = 70\%$.

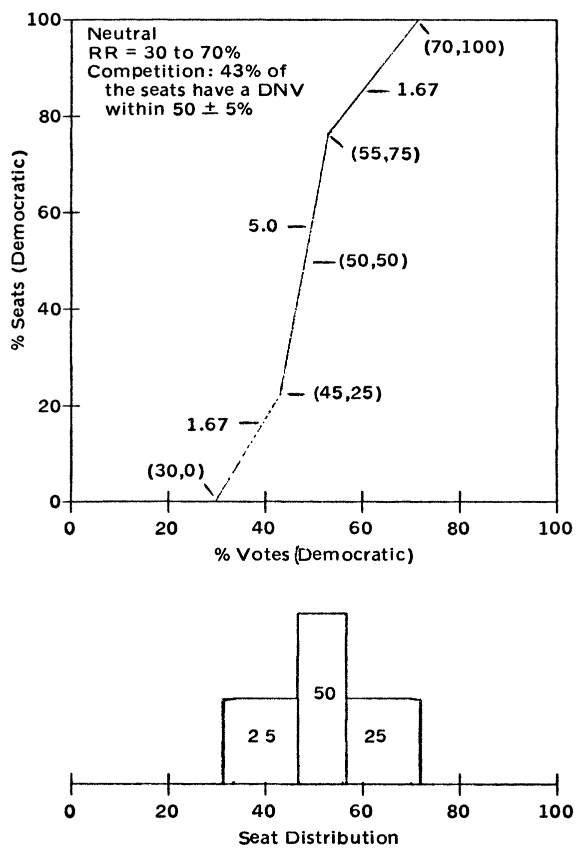


Figure 4. "Fair" Seats-Votes Curve for TNV = 52%

Hence, the SR is high in an area where votes are not expected (i.e., close to 50 percent) and, if the TNV is far from 50 percent, lower or even zero in the vicinity of the TNV. For example, a distribution of seats exactly like that in Figures 3(a) and 4, but centered around a TNV of 60 percent, would have only 16 percent of the seats in the highly competitive range; and, the SR would look exactly like those in Figures 3(a) and 4–i.e., it would be high around 50 percent but only 1.67 in the vicinity of the TNV.

Now, let us reverse the logic presented above. In order to increase the SR around the TNV, we need to distribute DNV's about 50 percent. The reason for this should now be apparent. As the actual vote fluctuates about the TNV, seats with DNV's near 50 percent are the ones that will move between the parties. The more seats there are with DNV's close to 50 percent, the more seats will change hands with a given vote change (i.e., the higher the SR will be around the TNV). However, it must also be remembered that the narrower the distribution of DNV's, the narrower the RR. Also, the more uniform the distribution of seats is around 50 percent, the more uniform the SR is (around the TNV).

If the TNV $\neq$ 50%, then, a fair districting plan requires construction of a set of districts centered at 50 percent. However, this prescription alone allows extremely wide latitude in the arrangement of districts, and some anomalous situations could arise. For example, ten districts with a TNV = 60% Democratic could be given DNV's of $(46 + \epsilon)$%, $(47 + \epsilon)$%, $(48 + \epsilon)$%, $(49 + \epsilon)$%, $(51 - \epsilon)$%, $(52 - \epsilon)$%, $(53 - \epsilon)$%, $(54 - \epsilon)$%, 100%, and 100% Democratic. But this would mean that (given our assumption of no constituency effects in the vote) if the TAV were 50 percent, or even as high as 56 percent Democratic, the Democrats would win only two seats! This kind of bizarre result can be prevented if it is required that when a party wins 50 percent of the votes it must win at least 50 percent of the seats. While conceivably there are circumstances in which this requirement might be relaxed (see n. 16 below), we think that it is generally appropriate since otherwise a party could win a majority of the votes but less than half of the seats. In imposing this condition, we are in effect requiring that the seats-votes curve pass through the (50,50) point, for if it did not, both parties would not meet the requirement. This in turn means that no more than half of the DNV's be greater than the TNV and no more than half be less than the TNV.[15]

At this point we still have a fair amount of latitude left in designing districting plans. In order to make some headway, let us proceed as follows. Let us first assume that proposed districting plans should be neutral; we will illustrate two districting plans under this assumption. Since it is our opinion that neutrality is generally a useful requirement, these plans illustrate our prescription for most cases in which TNV $\neq$ 50%. However, since our preference for neutrality is not an intense one, for reasons given below, we will shortly discard this requirement and illustrate a not-too-unreasonable, non-neutral districting plan.

Imposing neutrality means that the distribution of districts must be symmetric about the TNV. Let us assume that TNV $>$ 50% Democratic and concern ourselves only with the districts for which DNV $<$ TNV. Districts for which DNV $>$ TNV can then be distributed in such a way that the entire distribution of DNV's is symmetric about the TNV. As noted, we will increase the size of the SR *around* the TNV the more we cluster these DNV's *about* 50 percent. However, we also noted that the RR and the uniformity of the SR were dependent on the manner in which these DNV's are distributed. What seems like a reasonable compromise is to distribute these seats symmetrically about 50 percent, and, as in previous cases, to distribute them so that the SR is higher near the TNV than farther away and so that the SR's are uniform over fairly wide ranges.

Two such districting plans are illustrated in Figure 5. Comparison and evaluation of the plans shown, along with other possible plans, should now be a familiar exercise. Figure 5(a), for example, features a larger number of competitive districts than 5(b) and therefore a higher SR around the TNV. On the other hand, under 5(a), the Democrats would win 95 percent of the seats with 65 percent of the

[15]If the TAV = 50%, a party loses all seats for which

$$(1)\ \text{DNV} < \text{TNV},$$

and wins all seats for which

$$(2)\ \text{DNV} > \text{TNV}.$$

If more than half of the DNV's satisfy (1), then the party would win less than half of the seats. If more than half of the DNV's satisfied (2), it would win more than half of the seats; but then, of course, the other party would win less than half of the seats. Some seats may have DNV = TNV since they will be split evenly between the parties.

votes (only 5 percent above the TNV). Under 5(b) a 65 percent Democratic vote would yield a somewhat smaller Democratic majority (87.5 percent of the seats). Deciding which plan is preferable depends on how closely the actual vote is likely to be to the expected vote and on a judgment as to whether or not the Republicans should be given the chance (as in 5(a)) to win a lot of seats with the attendant risk of being virtually eliminated from the legislature.

Is the requirement of neutrality a good one? Consider the distribution of districts in Figure

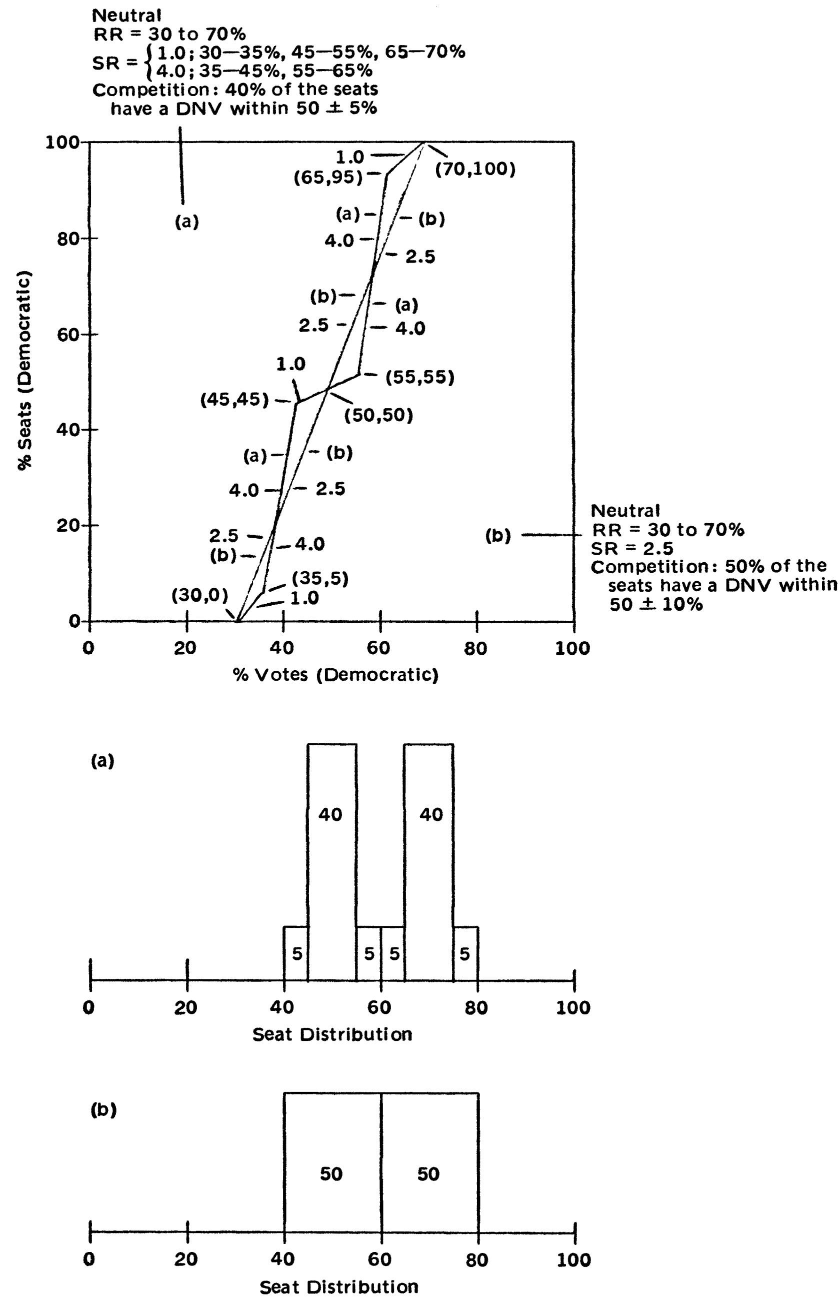


Figure 5. "Fair" Seats-Votes Curve for TNV = 60%

5(a). Half of the districts were distributed so that the mean DNV = 50%. Since TNV = 60%, the other half of the districts had to be distributed so that the mean DNV = 70%. Neutrality meant that they were in fact distributed symmetrically about 70 percent Democratic, with most of them being from 65 to 75 percent Democratic. If we violate neutrality, some of these districts could be made more competitive. The cost, however, would be that other districts would have to be made even less competitive. One such non-neutral districting plan is shown in Figure 6. The bias is readily apparent in the seats-votes curve. For example, if the Democrats won 55 percent of the votes, they would win 55 percent of the seats, whereas if the Republicans won 55 percent of the votes they would win 65 percent of the seats. We feel that the non-neutral plan gains very little at the expense of favoring one party. Admittedly, some seats are made more competitive. While they are still not very competitive in our example, they might be if the TNV were closer to 50 percent. But these competitive seats are gained at the expense of making some seats safer than they would otherwise be. For this reason, we do not feel the gain in competition is worth biasing the districting plan. Nonetheless, the gain in number of competitive seats is undeniable, and some may feel that this justifies the cost.[16] If so, Figure 6 serves as a model of how the districting plan can be

[16]Some may find this reasoning more persuasive when the number of districts is very small. In the

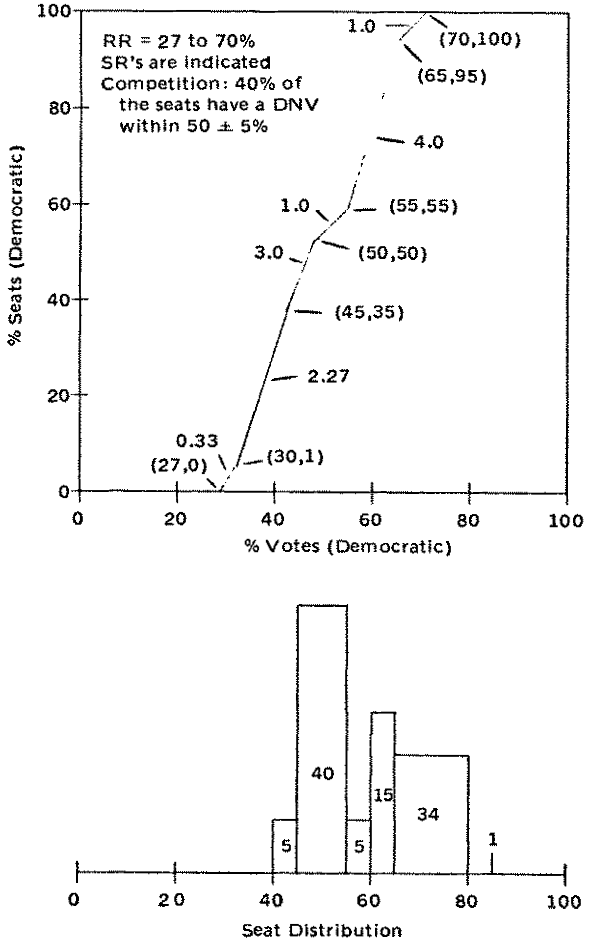


Figure 6. Non-Neutral Seats-Votes Curve for TNV = 60%

implemented.[17] Note that there is no change between Figures 5(a) and 6 for all DNV's < TNV.

examples on p. 1312, for example, there are only three districts. Consequently, neutral districting only allows one-third of the districts to be highly competitive, while a biased plan (biased even at the 50–50 point) raises that to two-thirds. The relative strength of the arguments may also depend on the structure of the political system. The argument in favor of neutrality is probably strongest when there is no intermediate structure such as states, some of which can be biased toward one party and some toward the other party. See our discussion of the "aggregation problem" in the conclusion.

[17]Other adjustments involving the neutrality question may have to be made if TNV >> 50%. First, it may be impossible to distribute half of the districts symmetrically about 50 percent. Second, while there exists a minimum percentage about which they can be distributed, a symmetric distribution about that percentage cannot be matched by the distribution of those districts for which DNV > TNV. For example, let there be four districts and TNV = 80%. Two districts can be centered around 60 percent–e.g., DNV_1 = 55% and DNV_2 = 65%–if DNV_3 = DNV_4 = 100% for the other two districts. But the DNV's of these latter districts cannot be altered. Either the non-neutral plan just noted, or a neutral plan in which DNV = 60% for two districts and DNV = 100% for two districts, seems best. The problem in this case lies not in our prescription as such, but in the fact that the fair districting criteria simply cannot be met to the same degree when the TNV is so far from 50 percent.

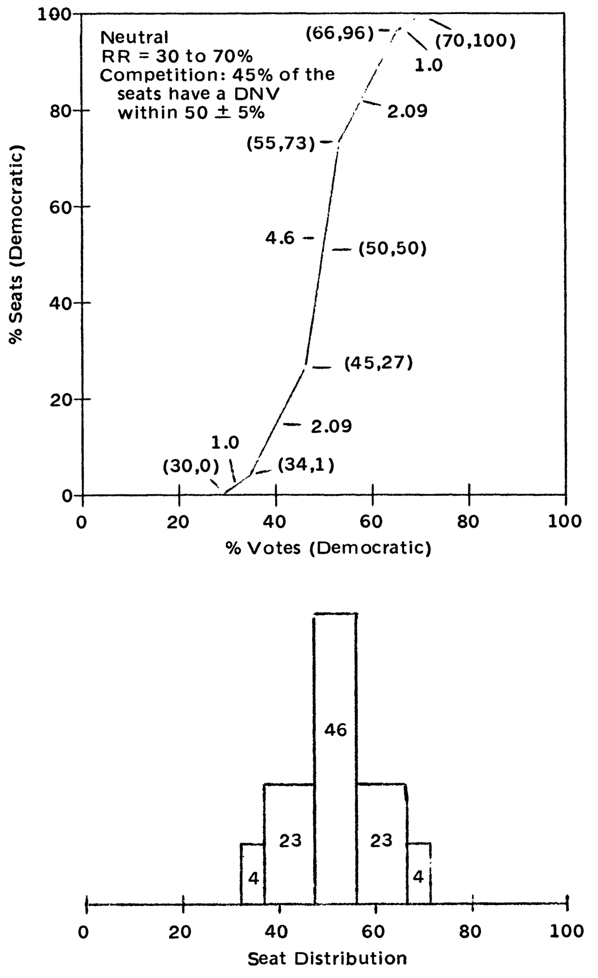


Figure 7. Alternative "Fair" Seats-Votes Curve for TNV = 52%

We now have explicated two schemes for constructing districting plans when the TNV $\neq$ 50%. The first method, illustrated in Figure 4, is rather obviously defective when the TNV is far from 50 percent, and the second approach was devised to deal with this situation. But the second method can just as well be utilized when the TNV is close to 50 percent. If it is utilized, one new consideration arises. Namely, in distributing (up to) half of the DNV's about 50 percent, there is no requirement that for all of these districts DNV $<$ TNV (assuming TNV $>$ 50%). However, if they do not meet this condition, the distribution of these DNV's will overlap the distribution of DNV's distributed about the point (TNV + (TNV − 50))%.

A specific example illustrates this point and also shows how application of this method gives slightly different results from the first method when the TNV is close to 50 percent. Suppose TNV = 52%. A symmetric distribution of seats about 52 percent yields the seat distribution and the seats-votes curve shown in Figure 4. If we instead distribute half of the DNV's symmetrically about 50 percent, and half about 54 percent, we can derive the results shown in Figure 7. The seat distribution was derived by uniformly distributing half of the DNV's as follows: 12 over the range 32–45 percent, 26 over 45–55 percent, and 12 over 55–68 percent. To make the entire distribution symmetric, the other half of the DNV's was distributed as follows: 12 over the range 36–49 percent, 26 over 49–59 percent, and 12 over 59–72 percent. Thus the distributions of the two halves overlap considerably. At the same time, since they do not overlap completely, there are ranges at the extremes of the RR in which there are fewer districts than in the comparable portion of Figure 4; this accounts for the lower SR at the extremes of the RR. The SR in Figure 7 is complementary about 50 percent, while that of Figure 4 is not. For this reason, there are slightly more competitive districts in Figure 7. On the other hand, the DNV's are slightly more highly clustered about the TNV in Figure 4 (50 percent of DNV's within 5 percent of 52 percent) than in Figure 7 (46 percent of the DNV's within 5 percent of 52 percent). Therefore the SR in the middle of the RR is slightly lower in the latter figure.

Despite the small differences between them, both figures represent what we think are fair seats-votes curves for TNV = 52%. As should be the case, either districting method outlined above is fair when TNV is close to 50 percent. (In fact, they are equivalent at TNV = 50%.) However, as already shown, the more the TNV departs from 50 percent, the less satisfactory the symmetric distribution of DNV's about the TNV becomes, and the more appropriate the two symmetric distributions about 50 percent and (TNV + (TNV − 50))% become.[18]

Small Number of Districts. If the number of districts is small, as is true of congressional districts in many states,[19] approximating seats-votes curves such as those shown above may become difficult. In fact, with a very small number of districts, there will necessarily be large flat spots in the seats-votes curve. Figure 8 illustrates this by showing two curves for a plan involving five districts. If the DNV's are distributed as in 8(a), the solid line is the seats-votes curve. If the most extreme DNV's are moved to $(40 + \epsilon)\%$ and $(60 - \epsilon)\%$, as in 8(b), the seats-votes curve from 40–46 percent and 54–60 percent is represented by the dotted line. The former has the advantage that all districts are in what is usually considered the "highly competitive" range, but has the disadvantage of a RR from only 45 to 55 percent (strictly $(45 + \epsilon)\%$ to 55%). If the Democratic proportion of the vote dropped as much as 5 percent below normal, the Democrats would have no representatives despite obtaining up to 45 percent of the vote. The latter version avoids the narrow RR, but 2 of the 5 districts are at the edge of what is often considered the "somewhat competitive" range. And in either case there are some changes in votes (within the RR), up to 7 percent, which result in no changes in seats.

Thus, when the number of districts is small the seats-votes curve will necessarily depart somewhat from those shown previously. Yet the basic considerations remain the same: neutrality can be simply maintained, the RR can be made large enough to ensure that neither party's representation is entirely eliminated by minor vote swings around the TNV, districts can be clustered near 50 percent in order to achieve a high degree of competitiveness, and yet districts can be spread out sufficiently to attain a reasonably constant SR. Fair districting may be harder to achieve, but the considera-

[18]Many of the schemes we have proposed as compromise solutions are reminiscent of piece-wise linear approximations to the nonlinear curve popularly referred to as the "Cube Law." This correspondence may be little more than coincidental since our formulations are theoretical prescriptions, while the Cube Law is proferred as an empirical description.

[19]Currently there are 35 states with fewer than 10 congressional districts.

tions involved and the operational procedures for establishing particular parameter values remain the same.

Conclusion

We have argued that the formation of an equitable districting plan requires attention to a number of factors not ordinarily considered. Specifically, an evaluation of the fairness of a districting plan must include judgments about the presence or absence of neutrality, the range of responsiveness, the constancy of the swing ratio, and the amount of competition present. We have demonstrated that simultaneous attainment of certain combinations of values for these parameters is impossible; but we have also suggested compromises that we think are reasonable and result in districting plans with desirable properties.

We wish it were possible to conclude by proposing minimum standards which all districting plans should meet. Ultimately, we hope to do so. However, one problem keeps us from doing so at present. This is what we call "the aggregation problem." Assuming it is possible

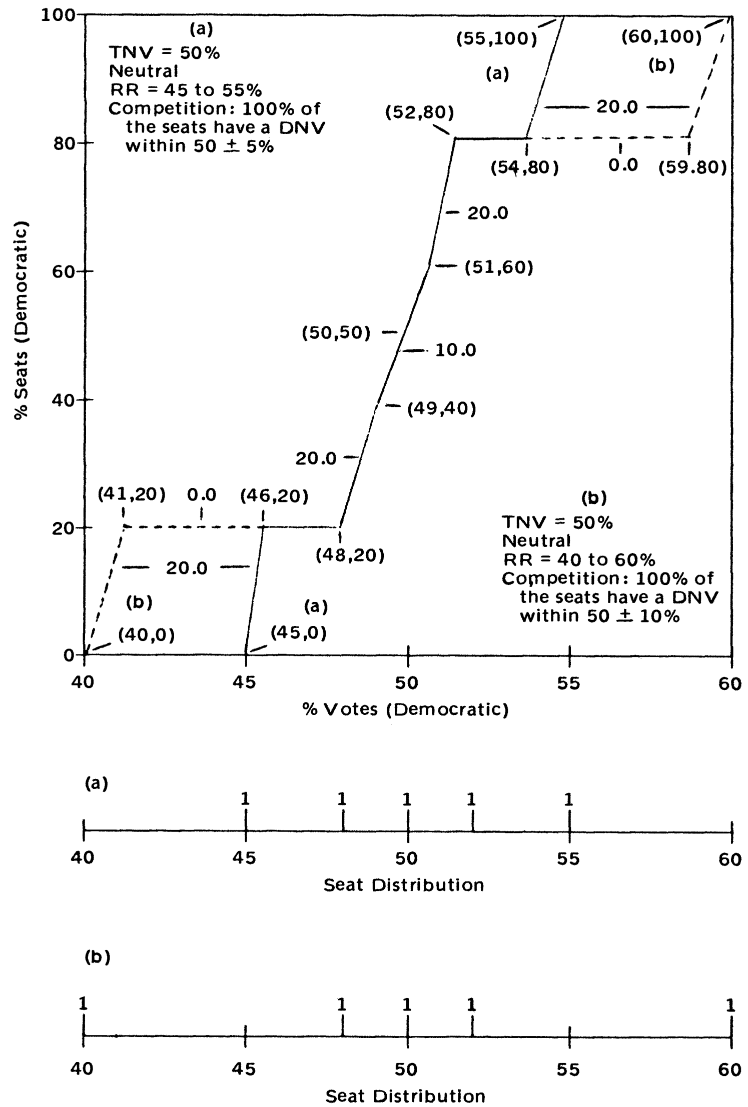


Figure 8. "Fair" Seats-Votes Curves for Five Districts

to arrive at agreed-upon standards for equitable districting plans within units of a federal system (e.g., American states), it is by no means obvious that these plans, when aggregated, will also result in an equitable arrangement at the aggregated level (e.g., at the national level). The presence of extremely one-sided districts, for example, may be viewed as inequitable and undesirable and can be easily avoided (except in extreme one-party states) when each state is considered in isolation. From a national perspective, however, such one-sided districts have historically meant that neither major party has been denied congressional representation, even when landslide elections have occurred. Thus, what might be thought of as inequitable at the state level, might also be viewed an extremely desirable feature from a national perspective.

Despite this remaining difficulty, as well as the possibility that additional complications can be brought into the model, we believe that the districting schemes derived here can serve as standards both for comparison and evaluation of existing districting plans and for the construction of equitable districting plans in the future.

Appendix

The relationship between the distribution of DNV's and the seats-votes curve is best shown by the kind of illustration in the figures. However, unless care is taken in determining precise distributions, minor problems can arise (such as a party winning all of the votes but winning slightly less than all of the seats). The use of ϵ values avoids these problems. Also, in many figures no district is created which has a DNV = TNV. We will speak of a uniform distribution even though this feature causes slight departures from uniformity.

For those who are interested, the exact distributions of DNV's for all figures are given below. All DNV's are given in percent Democratic.

Figure 1: 1 district each with a DNV of $0 + \epsilon$, $1 + \epsilon$, ..., $49 + \epsilon$, $51 - \epsilon$, $52 - \epsilon$, ..., $100 - \epsilon$.

Figure 2: (a) 5 districts each with DNV's of $40 + \epsilon$, $41 + \epsilon$, ..., $49 + \epsilon$, $51 - \epsilon$, $52 - \epsilon$, ..., $60 - \epsilon$; (b) 2 districts each with DNV's of $25 + \epsilon$, $26 + \epsilon$, ..., $49 + \epsilon$, $51 - \epsilon$, $52 - \epsilon$, ..., $75 - \epsilon$.

Figure 3: (a) 1 district each with DNV's of $30.0 + \epsilon$, $30.6 + \epsilon$, ..., $44.4 + \epsilon$, $55.6 - \epsilon$, $56.2 - \epsilon$, ..., $70.0 - \epsilon$; 5 districts each with DNV's of $45 + \epsilon$, $46 + \epsilon$, ..., $49 + \epsilon$, $51 - \epsilon$, $52 - \epsilon$, ..., $55 - \epsilon$; (b) 1 district each with DNV's of $20.0 + \epsilon$, $20.8 + \epsilon$, ..., $39.2 + \epsilon$, $60.8 - \epsilon$, $61.6 - \epsilon$, ..., $80.0 - \epsilon$; 1 district each with DNV's of $40.0 + \epsilon$, $40.4 + \epsilon$, ..., $49.6 + \epsilon$, $50.4 - \epsilon$, $50.8 - \epsilon$, ..., $60.0 - \epsilon$.

Figure 4: 1 district each with DNV's of $32.0 + \epsilon$, $32.6 + \epsilon$, ..., $46.4 + \epsilon$, $57.6 - \epsilon$, $58.2 - \epsilon$, ..., $72 - \epsilon$; 5 districts each with DNV's of $47 + \epsilon$, $48 + \epsilon$, ..., $51 + \epsilon$, $53 - \epsilon$, $54 - \epsilon$, ..., $57 - \epsilon$.

Figure 5: (a) 1 district each with DNV's of $40 + \epsilon$, $41 + \epsilon$, ..., $44 + \epsilon$, $56 - \epsilon$, $57 - \epsilon$, ..., $60 - \epsilon$, $60 + \epsilon$, $61 + \epsilon$, ..., $64 + \epsilon$, $76 - \epsilon$, $77 - \epsilon$, ..., $80 - \epsilon$; 4 districts each with DNV's of $45 + \epsilon$, $46 + \epsilon$, ..., $49 + \epsilon$, $51 - \epsilon$, $52 - \epsilon$, ..., $55 - \epsilon$, $65 + \epsilon$, $66 + \epsilon$, ..., $69 + \epsilon$, $71 - \epsilon$, $72 - \epsilon$, ..., $75 - \epsilon$; (b) 1 district each with DNV's of $40.0 + \epsilon$, $40.4 + \epsilon$, ..., $49.6 + \epsilon$, $50.4 - \epsilon$, $50.8 - \epsilon$, ..., $60 - \epsilon$, $60.0 + \epsilon$, $60.4 + \epsilon$, ..., $69.6 + \epsilon$, $70.4 - \epsilon$, $70.8 - \epsilon$, ..., $80 - \epsilon$.

Figure 6: 1 district each with DNV's of $40 + \epsilon$, $41 + \epsilon$, ..., $44 + \epsilon$, $56 - \epsilon$, $57 - \epsilon$, ..., $60 - \epsilon$, $65.45 - \epsilon$, $65.90 - \epsilon$, ..., $79.85 - \epsilon$, $80 - \epsilon$, $83 - \epsilon$; 4 districts each with DNV's of $45 + \epsilon$, $46 + \epsilon$, ..., $49 + \epsilon$, $51 - \epsilon$, $52 - \epsilon$, ..., $55 - \epsilon$; 3 districts each with DNV's of $61 - \epsilon$, $62 - \epsilon$, ..., $65 - \epsilon$.

Figure 7: 1 district each with DNV's of $32.0 + \epsilon$, $33.1 + \epsilon$, ..., $44.1 + \epsilon$, $55.9 - \epsilon$, $57.0 - \epsilon$, ..., $68.0 - \epsilon$, $45.0 + \epsilon$, $45.4 + \epsilon$, ..., $49.8 + \epsilon$, $50.2 - \epsilon$, $50.6 - \epsilon$, ..., $55.0 - \epsilon$, $36 + \epsilon$, $37.1 + \epsilon$, ..., $48.1 + \epsilon$, $59.9 - \epsilon$, $61.0 - \epsilon$, ..., $72.0 - \epsilon$, $49 + \epsilon$, $49.4 + \epsilon$, ..., $53.8 + \epsilon$, $54.2 - \epsilon$, $54.6 - \epsilon$, ..., $59.0 - \epsilon$.

Figure 8: (a) 1 district each with DNV's of $45 + \epsilon$, $48 + \epsilon$, 50, $52 - \epsilon$, $55 - \epsilon$; (b) 1 district each with DNV's of $40 + \epsilon$, $48 + \epsilon$, 50, $52 - \epsilon$, $60 - \epsilon$.

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