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Disproportionality and Malapportionment: Measuring Electoral Inequity

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Political and social scientists often wish to measure inequity—the deviation of some actual distribution from a comparative standard distribution based on merit, need, or some other expectation. Measurements of income inequality have the longest history, but others can be identified, including two of direct interest in electoral studies: measures of disproportionality and malapportionment. While many economists agree that inequality measures must meet certain statistical axioms, there has been no recognition of these requirements for the wider case of inequity. This article presents a method for developing simple and easily interpreted measures of inequity that meet the desired axioms. Disproportionality and malapportionment applications are illustrated and contrasting results obtained with alternative indices highlighted.

There are many applications in political and social science in which distributions of goods must be compared with some external standard. The most obvious example is *inequality* measurement, in which shares of, say, income or wealth are compared with the standard of equal shares. The search for an adequate measure of such economic inequality has deep historical roots (Pigou, 1920; Dalton, 1920; Gini, 1912; Shutz, 1951; Atkinson, 1970; Sen, 1973; Cowell, 1977; Coutler, 1980). A second example is the *disproportionality* of election results, the measurement of which has sparked a growing literature, including an extended debate in this journal (Rae, 1971; Loosemore and Hanby, 1971; Rose, 1983; Lijphart, 1984; Taagepera and Shugart, 1989; Strom, 1989; Bartolini and Mair, 1990; Mackie and Rose, 1991; Mair, 1989; Gallagher, 1991; Fry and McLean, 1991; Cox and Shugart, 1991; Lijphart, 1993). A disproportionality index compares the distribution of seats in a parliamentary assembly to that which might be expected from the vote distribution. *Malapportionment*—the discrepancy between the shares of legislative representation and the shares of population held by geographical units—is, at least

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mathematically, an equivalent concept. Coulter (1989) refers to such deviations from a distributional standard as *inequity*. The clumsier, but less normative, phrases *distributional deviation* and *distributional variance* are also used in this article.

If the history of these measures is so extensive, what is the need for another offering? First, while economists concentrate on the standards by which inequality indices can be evaluated, there has been no recognition of such standards in similar attempts by political scientists. Second, those few current indices that do pass all or most of the statistical tests have other problems associated with a lack of simplicity and interpretability. Third, there appears to be no recognition that on a technical level these measures attack similar problems and that a unified approach might be possible. Fourth, the argument about inequity has been garbled by confusion about its normative and non-normative features. The methodology in this article is provided in an attempt to solve these problems, suggesting a simple tool of use to electoral scholars and, it is hoped, to a larger audience of political and social scientists as well.

Axioms for Inequality Measurement

Inequity measurement is a generalization of equality measurement. To have confidence in an inequity index we must at least be sure that it has acceptable properties when applied to the special case of equality. Shorrocks (1984) shows that if we accept certain statistical axioms as necessary for an inequality index, then only a limited class of indices will suffice. This finding is well-documented (Jenkins, 1989; Moulin, 1988), so only a brief summary is needed here. Potential axioms that might be applied include the following:

Anonymity (or *Symmetry*): The amount of inequality depends only on the shares of the distributed good held by individuals, not on the identity of those individuals.

Principle of Transfers (Pigou, 1920; Dalton, 1920): If one distribution can be obtained from a second distribution through the transfer of shares from one individual to a poorer one, then inequality is lower in the latter distribution than in the former. If an inequality index always reflects this latter situation with a strictly lower level of inequality then the *strong principle of transfers* is satisfied. If the index merely never takes on a lower value after such a transfer (i.e. sometimes remains the same) the *weak principle of transfers* is satisfied.

Constant Relative Inequality Aversion (or *Scale-Invariance; Mean-Independence*) (Atkinson, 1970): The value of an inequality index should not change if all shares are multiplied by the same constant.

Increasing Absolute Inequality Aversion (Atkinson, 1970): If a constant is added to all shares, inequality should decrease. To give an extreme example, if two people have \$100 and \$1,000 respectively—high inequality—and each is then given \$1,000,000, inequality has decreased.

Separability (or *Decomposability*) (Cowell, 1977; Shorrocks, 1984): If a distribution has a given level of inequality and is partitioned into subgroups, a reduction (or increase) in the inequality of one subgroup should, *ceteris paribus*, reduce (or increase) the inequality of the group as a whole. A stronger principle—*Additive Separability*—states that the index value for a supergroup must be an additive function of the intragroup inequalities and an intergroup inequality term. Even stronger, ***Linear Separability*** requires that index values be a linear combination of intragroup inequity terms and intergroup inequity.

Population Symmetry (or *Population Homogeneity*): If two distributions, each with the same mean and with the same inequality value, are combined, the index value for the combined population should be the same as it was for the groups separately. Although superficially appealing, the inconsistency of this axiom with our intuition can be illustrated with a simple example. Imagine a pair of individuals with \$100 split very unequally: [0 100]. Inequality is obviously very high for such a pair, at a maximum in one sense. If they are grouped with another similar pair, the resulting group has a distribution of [0 0 100 100]. While inequality still might be considered high, it is clearly not as high before. Neither the whole of the good nor the whole of its absence is held by a single individual. Many have expressed doubts about the usefulness of this axiom (Cowell, 1977; Coulter, 1989; Jenkins, 1989).

Lorenz Dominance (Lorenz, 1905): Next to the principle of transfers, this is probably the most well-known axiom, but somewhat more complex. A description of the Lorenz curve and Lorenz Dominance would be of little utility here, because of its broad treatment elsewhere (Atkinson, 1970; Sen, 1973; Cowell, 1977; Coulter, 1989; Jenkins, 1989; Moulin, 1988). Also, Lorenz Dominance is equivalent to the simultaneous application of the Strong Principle of Transfers, Scale-Invariance, and Population Homogeneity. Since the last of these is questionable, so is Lorenz Dominance.

Shorrocks (1984) shows that for an index to meet all of these axioms, it must be in the *Generalization Entropy Family* (GEF) of indices. These indices take the form:

$$E_\alpha = \frac{1}{(\alpha^2 - \alpha)}\left(\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{X_i}{\mu}\right)\alpha\right) - 1\right) \qquad \text{for } \alpha \neq 0,1$$

$$E_0 = \frac{1}{N}\sum_{i=1}^{N}\log\left(\frac{\mu}{X_i}\right)$$

$$E_1 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{X_i}{\mu}\right)\log\left(\frac{X_i}{\mu}\right) \tag{1}$$

where N = the number of components receiving distributive shares,
X_i = the share received by component i,
S = the total supply of the good to be distributed ($= \sum_i X_i$), and
μ = the mean share ($= S/N$).

Jenkins (1989) shows that the Population Homogeneity axiom, and therefore also Lorenz Dominance, are unnecessary for this result.

All that remains for the inequality researcher who accepts these axioms is the choice of α. We can note that, prior to the identification of the GEF, several special cases had been identified. Theil's index is a transformation of E_1 (Theil, 1967); variance is a transformation of E_2; and Atkinson (1970) defined a family of indices that all have GEF equivalents (although the converse is not true). Several other popular inequality indices are not in the GEF, most notably the Shutz index (1951),[1] which violates the principle of transfers among others, and the Gini index (1912),[2] which is not separable. Extensive examinations of these indices and their properties are given by Atkinson (1970), Sen (1973), Cowell (1977), Coulter (1989), and Moulin (1988).

How then are we to choose from among those indices that are otherwise acceptable? While the accepted statistical axioms are the most important—a violation of one indicates some lack of logic or consistency—there are other intuitive 'quasi-axioms' that can help one choose:

Definition: This criterion simply requires that the index be well defined for all possible distributions it might be required to evaluate.

Simplicity: First, computation of the index should be simple, requiring as few calculations as possible, *ceteris paribus*. Second, any choices to be made by the index user, about the value of parameters, for instance, should be clear and understandable. This second feature might arguably be the inverse of another virtue: flexibility.

Interpretability: Index values should have as clear and unambiguous an interpretation as possible. It is helpful, for instance, if an index is *properly bounded*, ranging from zero (maximum equity) to one (maximum inequity). Second, it is helpful if an index has a clear and intuitive *theoretical justification*.

Information Use: An index that makes use of all available information is to be preferred to one that ignores some data, *ceteris paribus*.

Stability: The stability of an index is measured by the effects of the addition of small or null components to a distribution. An index that varies continuously with such additions is to be preferred to one that does not. This is more important for the general case of inequity, where components may have small or zero expectations.

The application of Definition allows us to eliminate the possibilities of $\alpha \leq 0$, since all of these indices are undefined when any component receives a zero share. The Simplicity criterion would seem to indicate a preference for a small integral value of α. The Interpretation criterion is particularly useful. There is little theoretical or intuitive reason for choosing an α of 17, 8.46, or even 3. On the other hand, E_1 (Theil's entropy) has a theoretical basis in information theory and E_2 (variance) has supporting arguments from both statistics and physics.

In the case of the latter, variance and standard deviation have long been established as measures of dispersion within statistical samples, being fundamental parameters of many probabilistic distributions. The concept of 'squared distances' is integral to the oft-used least-squares regression method and chi-squared significance tests.[3] In physics, moment of inertia and radius of gyration are variance-type measures that are used to indicate force away from the centre. Also, the variance is the only one of the entropy measures that will measure the deviation of shares from the mean equivalently to deviations from each other. Finally, standard deviation is useful because of its clear interpretation as the average distance from the mean, measured in units of the distributive good itself. For these reasons, the inequity indices below are derived from the concepts of variance and standard deviation.

The decisions to choose an index from the GEF and, further, to choose a variance-based measure, are arguably normative ones. Measures not based on these principles will evaluate distributions differently, and the principles thus have potential political implications. Nevertheless, the statistical arguments for these choices are compelling. There are yet other choices to be made with more clearly normative implications.[4]

Measuring Inequity

The generalization from inequality to inequity is relatively straightforward. With equality, the comparative standard for each individual is the mean share of the distributive good: S/N. With equity, each individual may have a different expectation, E_i, unconstrained except it must not be negative and the sum of expectations must equal the sum of the distributive good, S. The inequity equivalent of standard deviation, for instance, is:

$$\textit{Deviation from Equity} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(X_i - E_i\right)^2} \tag{2}$$

Variance from Equity is the square of this expression. As an index, this measure suffers from a problem shared by variance and all of the GEF—improper bounding. With all of these measures the upper bound is not one, but varies considerably with the number of components and the total good supplied. For situations in which we are comparing different distributions to the same set of expectations or even the same set of components, this is unimportant and there is little or no reason to use a more complicated index. For the more general case of direct comparisons between distributions with different numbers of components, however, the problem is genuine. The simplest solution is to define maximum deviation such that the index can be normalized to vary from zero to one. Inequity is then:

$$\textit{Inequity} = \frac{\textit{Deviation}}{\textit{Maximum Deviation}} \tag{3}$$

Two different inequity indices that adopt such a structure have been proposed. Each has its own underlying definition of maximum deviation. The first is that proposed by Coulter (1980). Coulter's index, like the GEF, is a set of measures with an infinitely variable parameter to be chosen by the analyst. The only point of equivalence between the GEF and Coulter's measure is the specific measures related to variance. This Coulter index is:[5]

$$I_c = \sqrt{\frac{\sum_{i=1}^{N}\left(x_i - e_i\right)^2}{1 - 2\left[\min\left(e_i\right)\right] + \sum_{i=1}^{N} e_i^2}} \tag{4}$$

where x_i = the percentage share received by component i, and
e_i = the percentage share expected by component i.

This is an inequity index like that of Equation (3) where deviation is based on standard deviation and maximum deviation is defined to occur when the component expecting the smallest share receives all of the distributive good and all others receive nothing. The primary problem with this index is violation of the Stability axiom. The index can vary wildly with the addition or removal of a small or null component. For instance, consider a system where the expectation vector e, is [0.5 0.5] and the distribution vector x, is [0.6 0.4]. In this system, $I_c = 0.20$. If we add a null component so that e = [0.5 0.5 0] and x = [0.6 0.4 0], I_c drops 40 per cent to 0.12. This instability is undesirable in many applications.

The second inequity index is that given by Nagel (1984). His derivation is based on variance:

$$I_n = \frac{\sum_{i=1}^{N}\left(x_i - e_i\right)^2}{\sum_{i=1}^{N} e_i^2} \tag{5}$$

Maximum deviation is assumed to occur when all components receive nothing—when none of the good is distributed. The main problem with this index is that it is not properly bounded, although Nagel claims it to be. Consider a three component system with e = [0.2 0.4 0.4] and x = [1 0 0]. The value of I_n is 2.67! Indeed, I_n can take on any positive value, with sufficiently large N. This error occurs because there is improper accounting of the (potentially) undistributed good.

There is a definition of maximum deviation, however, which is not vulnerable to either of the above criticisms. Under this definition, maximum deviation occurs when a component with zero-expectation receives the entire distribution (and all others receive nothing). The difference from Coulter is that a zero-expecting component is implicitly assumed in every system, whether or not one is explicitly recognized. The difference from Nagel is that the 'undistributed' good is counted in the denominator. In this way, we can avoid both the bounding problems of I_n and the stability problems of I_c. The assumption is one of mathematical convenience that may or may not have an obvious intuitive correspondence with the realities of the situation at hand. In the case of electoral disproportionality, it is easy to imagine a small party or write-in candidate that was least deserving of receiving all of the parliamentary seats and that might not appear in the electoral statistics for use in a measure of the type Coulter describes. More generally, many distributive situations require a comparison of outcomes or procedures with varying levels of waste or inefficiency, a contingency allowed by this definition (assume waste is the zero-expecting component) and not by others. In the case of dividing a pie equally between three persons, this definition of maximum deviation might be less easy to accept, since it seems to require that the pie fall on the floor and be eaten by the zero-expecting dog. It is perhaps more intuitively appealing to imagine each distributional problem as having an infinite number of components, with some recognized components expecting non-zero shares and some unrecognized components expecting nothing.

Using this assumption the inequity index is:

$$I(X_i, E_i) = \sqrt{\frac{\sum_{i=1}^{N}\left(X_i - E_i\right)^2}{S^2 + \sum_{i=1}^{N} E_i^2}} = \sqrt{\frac{\sum_{i=1}^{N}\left(x_i - e_i\right)^2}{1 + \sum_{i=1}^{N} e_i^2}} \tag{6}$$

Call this quantity ***distributional deviation***. The square of this expression, $I^2(X_i,E_i)$, is ***distributional variance***.

One important special case is the *inequality index* implied by Equation (6):

$$I_E(X_i, S/N) = \sqrt{\frac{N}{S^2(N+1)}\sum_{i=1}^{N}\left(X_i - (S/N)\right)^2} = \sqrt{\frac{N}{N+1}\sum_{i=1}^{N}\left(x_i - 1/N\right)^2} \tag{7}$$

As expected, the index is essentially a normalized version of standard deviation.[6] Measurements of the deviation from more complex equity standards can be developed from the general index in a similar fashion. To apply the index, however, there still remain the surprisingly nontrivial problems of identifying the relevant components, their expectations, and the actual shares that they have received.

Disproportionality and Malapportionment

Disproportionality and malapportionment are two politically significant examples of distributional deviation, each of interest in electoral studies. *Disproportionality* occurs when political parties receive shares of legislative seats that are not equal to their shares of votes. *Malapportionment* occurs when geographical units have shares of legislative seats that are not equal to their shares of population. The two concepts are equivalent mathematically, but not politically. The intellectual histories of the two concepts have been very similar but largely independent, the former studied in proportional representation systems of Western Europe and the latter studied in such countries as the United States (Balinski and Young, 1982), the United Kingdom (McLean and Mortimore, 1992), and Japan (Hata, 1990; Hickman and Kim, 1992). I concentrate in this section on the concept of disproportionality, commenting on malapportionment only where its treatment must be different. Disproportionality is an example of inequity and should therefore be measurable using the inequity index given in this article. Before discussing how this is to be done, it is instructive to examine the various disproportionality indices that already exist.

Two of the most commonly used indices of disproportionality were proposed in 1971 (Loosemore and Hanby, 1971; Rae, 1971). Several others have been proposed in the interim (Lijphart, 1984; Gallagher, 1991; Fry and McLean, 1991). All but the most recent of these are variants of the Shutz inequality index. Even within the field of electoral studies, the problems associated with these indices are well-known (Lijphart, 1985; Coulter, 1989; Taagepera and Shugart, 1989). For lack of a better alternative, however, the problems have been tolerated in empirical studies that attempt to explore quantitatively the relationships between disproportionality and other political phenomena such as electoral volatility or the number of parties (Taagepera and Shugart, 1989; Strom, 1989; Mair, 1989; Bartolini and Mair, 1990; Lijphart, 1993).

The Loosemore-Hanby index (which Rose (1983) appears to have described independently) is an inequity version of the Shutz index, I_s. Let S_i and V_i be the assembly seats and popular vote, respectively, of party i; let S and V be the totals of each; let s_i and v_i be the percentage shares of each. The index of disproportionality is:

$$D_{lh} = \frac{1}{2}\sum_{i=1}^{N} |s_i - v_i| \tag{8}$$

The primary problem is insensitivity to transfers. Define the *advantage ratio*, A_i, of a party to be the ratio of votes to seats (expected share to actual share). D_{lh} recognizes transfer between a component with $A_i > 1$ and a component with $A_i < 1$, but fails to recognize transfers between two parties both on the same side of the mean ($A = 1$). This index, or a transformation, has been used widely in electoral studies (Taagepera and Shugart, 1989; Mair, 1989; Mackie and Rose, 1991).

D_{lh} is also perceived by some to be overly sensitive to the number of parties. Rae's suggested solution to this problem is to look at the *average* absolute deviation from proportionality:

$$D_r = \frac{1}{N}\sum_{i=1}^{N} |s_i - v_i| \tag{9}$$

This index is probably worse—it still does not reflect transfers properly, it is now extremely sensitive to the addition of small parties to the system, and it is improperly bounded. Rae suggests that only parties that receive 0.5 per cent of the vote be counted to avoid this problem. This arbitrary limit now introduces some information loss into the index. One important study to use the Rae index is that of Bartolini and Mair (1990).

Lijphart (1984) recommends a compromise between these two, in which the seat-share/vote-share differences of the two largest parties are averaged. Thus:

$$D_l = \frac{1}{2}\left(|s_1 - v_1| + |s_2 - v_2|\right) \tag{10}$$

where party 1 and party 2 are the two largest parties. The important axiomatic violations of the previous indices are not solved, however, and there is now a loss of information. Nevertheless, D_l has been used by other researchers (Strom, 1989).

Fry and McLean (1991), in addition to outlining a Gini-type index that they reject, suggest adapting the poverty measure given by Foster *et al.* (1984). Adjusting the polarity of the index to measure disproportionality gives:

$$D_{fm}(\alpha) = \sum_{A_i=0}^{1} (1 - A_i)^{\alpha} v_i, \quad \alpha \geq 0 \tag{11}$$

$D_{fm}(0)$ gives the total vote share of those underrepresented; $D_{fm}(1)$ gives a Shutz transformation; $D_{fm}(2)$ gives a measure closely related to the variance-based measure given in this article but with some additional assumptions. In fact, $D_{fm}(2)$ is a special case of Equation (6) and recommended in certain circumstances, as described below.

Gallagher (1991) suggests that every proportional representation system contains an implicit definition, and thus an index, of disproportionality. This leads him to suggest some new possibilities based on existing proportional representations (PR) schemes. The first, based on the *largest remainders method* of seat distribution, is a 'least-squares index':

$$D_{ls} = \sqrt{\frac{1}{2}\sum_{i=1}^{N}(s_i - v_i)^2} \tag{12}$$

Gallagher's second suggestion is derived from the *d'Hondt* or *greatest divisor method*:

$$D_{dh} = \max(A_i) \tag{13}$$

Thus, disproportionality is simply the largest advantage ratio in the election. Gallagher suggests that to reflect the threshold used in some systems, this might be the largest A_i among parties with a v_i of greater than 0.5 per cent.[7] Finally, Gallagher states that the *Sainte-Laguë* or *major fractions method* provides the following index:

$$D_{sl} = \sum_{i=1}^{N}\frac{(s_i - v_i)^2}{v_i} \tag{14}$$

Gallagher's primary point, that each method of PR contains its own definition of disproportionality, is only partly right. Each PR method contains its own *concept* of disproportionality, but actually minimizes an infinite number of *indices*, any of which might be chosen. The apparent lack of relationship between the Gallagher indices occurs because there is no consistent mapping from method to index. For instance, in a given election, largest remainders is the only procedure that will minimize D_{ls}, but D_{ls} is one of many functions (including D_{lh}) that will be minimized by the largest remainders method.

There are two unrecognized conflicts in the discussion to date. The first is about the *statistical form of the index*. This accounts for the difference between D_r and D_{lh} as well as for differences among the various inequality indices. The resolution suggested in this paper is a normalized version of variance or standard deviation—Equation (6). This decision, while having normative implications, is primarily a statistical one. The second conflict is about *what is being distributed and to whom*. This conflict, while guided by the logic of the distributive situation, is almost purely a normative one. It is also maddeningly nontrivial.

Returning to disproportionality, it seems natural to address this question by suggesting that S seats are distributed to P parties, p_i, each of which expects a share, s_i, equivalent to its vote share, v_i. Putting these values into Equation (6) provides:

$$D_p = I(S_i, V_i) = \sqrt{\frac{\sum_{i=1}^{P}(s_i - v_i)^2}{1 + \sum_{i=1}^{P} v_i^2}} \tag{15}$$

Such an index, like D_{ls} and D_{lh}, would judge the largest remainders method to generate the least disproportionality for any given vote distribution. Consider, however, the possibility that it is the *voters* of party p_i, each of whom is given a portion of a seat, S_i/V_i, and each of whom has an expected portion, S/V. With some algebra:

$$D_v = I\left(S_i/V_i, S/V\right) = \sqrt{\frac{1}{(V+1)}\sum_{i=1}^{P}\frac{\left(s_i - v_i\right)^2}{v_i}} \qquad (16)$$

Note the resemblance between this and D_{st} as given by Gallagher. This is not coincidence—Sainte-Laguë designed his PR system on just this assumption of voter equality. The analogous US apportionment system was designed by Daniel Webster using the same assumption. Note, however, that Equation (16) has a normalization term that, for large-scale democracies, will drive the index towards zero. With such an index, the differences in proportionality among systems in which most voters are represented will be small in absolute (but not percentage) terms in order to allow differences between representative democracies and, say, dictatorships to be measured on the same properly bounded scale.

Similarly we can define a parallel index that assumes that the distribution is of voters to seats or representatives, D_s. In the case of apportionment, this is equivalent to minimizing differences in district size, and is the system in use for the US House of Representatives today. Other indices can be identified by attempting to minimize underrepresentation or overrepresentation of voters. The minimization of overrepresentation leads to the d'Hondt system and an index that, like D_v, will generally judge most election results to be nearly perfectly proportional. The minimization of underrepresentation leads to a system that awards each party with at least one seat and the Fry-McLean index, $D_{fm}(2)$. Note that this index is likely to evaluate many election results as highly disproportional, because there are nearly always microparties that are given no seats. These last two indices are qualitatively different from others in this family. While, like D_p, the concentration is on parties, there is a substantive change in the equity standard. All of these indices are given in Table 1 along with the assumptions of each. I have also identified, where applicable, the PR (and apportionment) schemes that each recommends.

The choice of index obviously depends on the purpose for which it is intended. For many cases, D_p is recommended—it contains natural assumptions about the distributional problem, remains well defined in all circumstances, and is reasonably sensitive to differences in election results. Also, most analyses that have used indices of disproportionality have been trying to link this feature of electoral systems to aspects of party behaviour, such as Duverger's Law (Duverger, 1954) or the 'freezing hypothesis' (Lipset and Rokkan, 1967). For such analyses, the inequity as observed by parties is most relevant. This should not be interpreted, however, as a recommendation for the largest remainder distribution scheme, which has several undesirable features (Balinski and Young, 1982).

Examples for Electoral Studies

Transfer Sensitivity and British Elections

Perhaps the most compelling alternative to D_p for disproportionality measurement is D_{lh}. It is simple and intuitive. It is certainly the most popular of the measures

TABLE 1. Disproportionality and malapportionment indices

Recommended distributive scheme • for PR • for apportionment	Components	I	X_i	E_i	Index
Largest remainder • Hare quota • Hamilton	Parties	D_p	S_i	$v_i S$	$\sqrt{\dfrac{\sum_{i=1}^{P}(s_i - v_i)^2}{1+\sum_{i=1}^{N} v_i^2}}$
Major fractions • Sainte-Laguë • Webster	Voters	D_v	$\dfrac{S_i}{V_i}$	$\dfrac{S}{V}$	$\sqrt{\dfrac{1}{V+1}\sum_{i=1}^{P}\dfrac{(s_i - v_i)^2}{v_i}}$
Equal proportions • (No name) • Hill/Huntington	Seats or representatives	D_s	$\dfrac{V_i}{S_i}$	$\dfrac{V}{S}$	$\sqrt{\dfrac{1}{S+1}\sum_{i=1}^{P}\dfrac{(s_i - v_i)^2}{v_i}}$
Greatest divisor • d'Hondt • Jefferson	Overrepresented voters	O_v	$\dfrac{S_i}{V_i}$	$\dfrac{S}{V}$	$\sqrt{\dfrac{1}{V}\sum_{A_i>1}\dfrac{(s_i - v_i)^2}{v_i}}$
Smallest divisor • (Unworkable) • Adams	Underrepresented voters	U_v $= D_{fm}(2)$	$\dfrac{S_i}{V_i}$	$\dfrac{S}{V}$	$\sqrt{\sum_{A_i<1}\dfrac{(s_i - v_i)^2}{v_i}}$

currently in use. The primary complaint I have given is that it does not satisfy the transfers axiom, but how important is this?

In recent British elections, most votes have gone to one of three parties: the Conservatives, Labour, and a 'Liberal' party.[8] The Liberals have generally been the smallest vote-getters of the three, but received as much as 25 per cent of the general election vote (in 1983). Even at this high point, however, the plurality system in combination with their relatively uniform geographic support limited them to less than 4 per cent of the seats. Both the Conservatives and Labour tend to receive a share of seats much larger than their share of votes—it is true for both that $A > 1$.

For the sake of illustration, simplify the system and assume v = [0.4 0.4 0.2] and s = [(x) (1-x) 0]. As x varies from 0 to 1 (as we change assumptions about the geographical dispersion of party support, perhaps), how does disproportionality change? The principle of transfers states that it *must* decline as x moves from 0 to 0.5 and increase again from 0.5 to 1. Figure 1 shows the values of D_p and D_{lh} as a function of x. The behaviour on the interval $0.4 \le x \le 0.6$ is of particular interest. While D_p is capable of recognizing [0.5 0.5 0] as the closest to [0.4 0.4 0.2] (because no shift in x brings the advantage ratios closer together), D_{lh} evaluates all of these distributions equally.

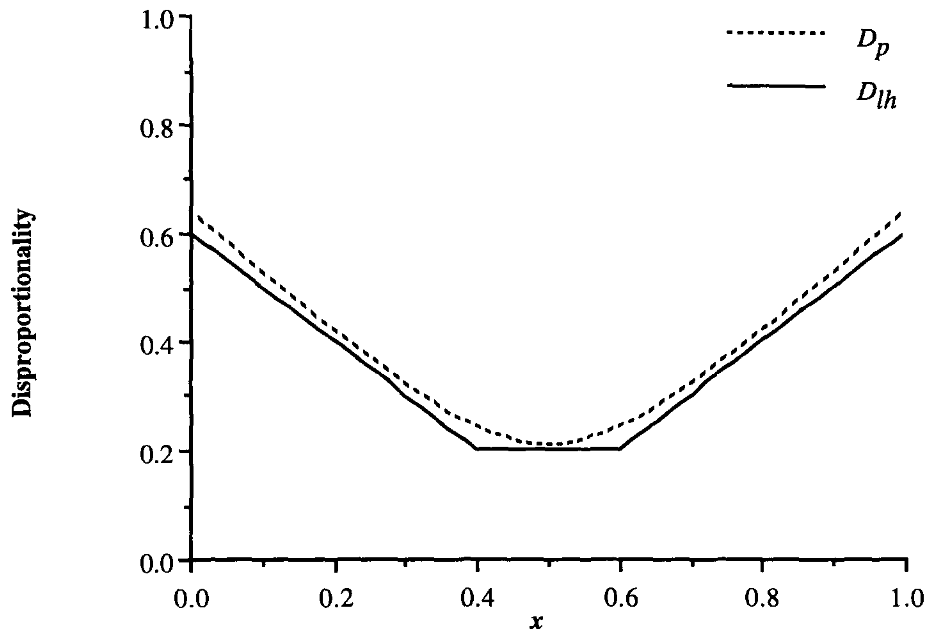

Fig. 1. Disproportionality in a simplified British election

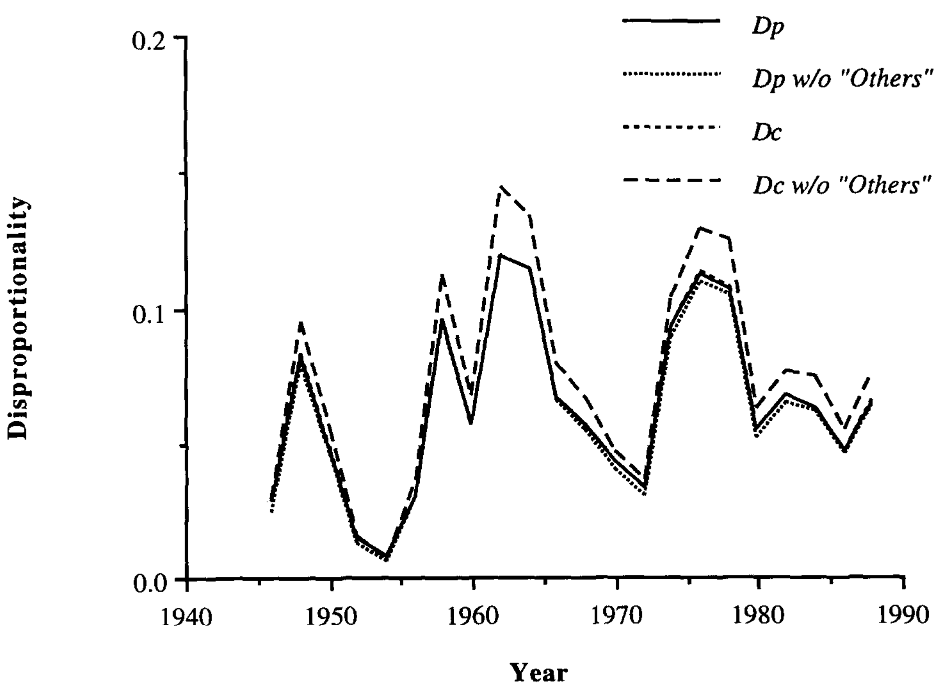

Fig. 2. Disproportionality in elections to the US House of Representatives. *Source*: Mackie and Rose (1991).

Stability and the House of Representatives

I have argued for the inequity index, I, defined in Equation (6) over the Coulter index, I_c, defined in Equation (4) on the basis of the stability axiom. The instability of I_c can be illustrated with a disproportionality study of the US House of Representatives. For this example I will use party-based disproportionality indices to analyse election data from Mackie and Rose (1991), a common source for electoral studies. The point of this example will be to illustrate the properties of D_p and $D_c = I_c(s_i, v_i)$ on data of varying quality.

Four sets of calculations have been made and shown in Figure 2. The first two are D_p and D_c based on the assumption that the 'Others' category in the data is another party. The second two are D_p and D_c where the 'Others' data is ignored (simulating, for instance, the case where our data source is less detailed). We can see in Figure 2 that with the 'Others' data, D_p and D_c have no perceptible differences. With removal of the 'Others' data, the new values of D_p are slightly smaller, but almost indistinguishable from the old. The values of D_c, however, are significantly larger than before. Coulter's index is highly sensitive to the data most likely to be of low quality or missing—those for the smallest parties.

Linear Separability and Malapportionment

Finally, the case of malapportionment of electoral districts for the US House of Representatives can be used to illustrate the utility of separability. (As a bonus, we also get a substantive result). Apportionment involves two stages. First, the algorithm is used to distribute a fixed number of legislative seats to the states. The House size has been fixed at 435 since the 1920s and the Hill/Huntington method has been used since 1941. The second stage is the drawing of districts by the state legislatures. Inequality will inevitably be introduced in the first stage. Until the 1964 Supreme Court decision in *Wesberry* v. *Sanders*,[9] inequality could also be introduced at the second stage. Since the decision, however, the Court has required approximate equality of district sizes within each state. How important was this decision for reducing overall malapportionment?

There are two interpretations of the distributional components of apportionment that might be adopted: voters who are given portions of Representatives, on the one hand, or Representatives who are given constituents. Although the former is strongly defensible, it is the latter interpretation I shall use since most of the discussion about malapportionment in the United States focuses on inequality of district sizes.[10] Either approach turns the problem into one of inequality measurement. Both distributional deviation and distributional variance are useful for this problem.

Using variants of Equation (7), Figures 3 and 4 show the inequity in district sizes over the last 100 years. Two elements are of interest. Note first the small levels of distributional deviation, which did not exceed 2 per cent in this period. This is due to the fact that in such a large distributional system even apparently large inequities are small relative to the theoretically possible maximum. It is true, however, that this theoretical range consists mostly of ridiculous distributions that could never have occurred (Bloomington, Indiana, gets 435 Representatives; everyone else gets zero). Second, we can note the dramatic drop in malapportionment from the 1970s brought about by the Supreme Court's reduction of interstate inequity. The linear

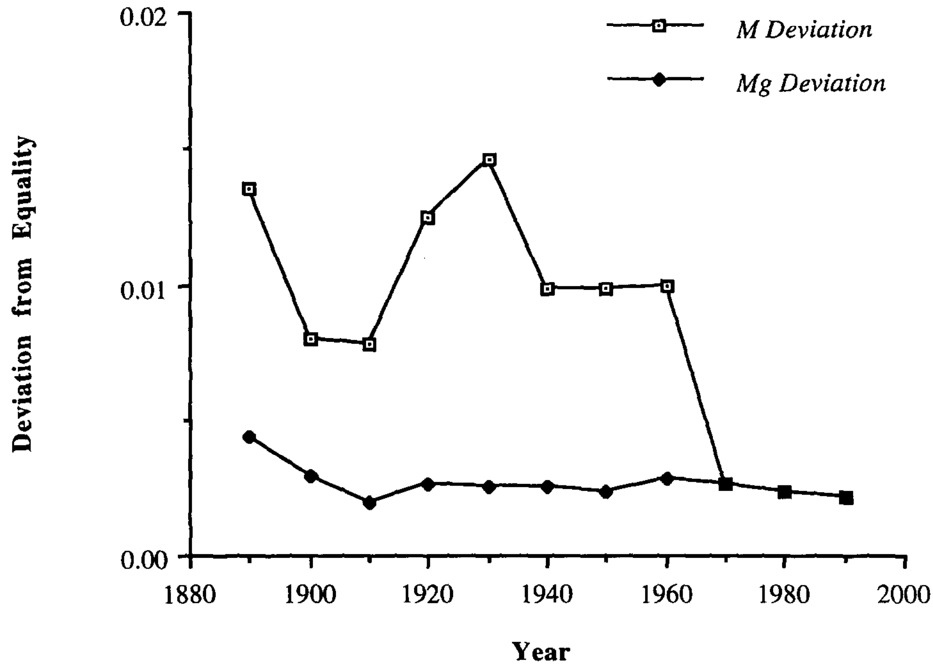

FIG. 3. Malapportionment of Congressional districts (distributional deviation).
Source: United States Congressional Directory, various dates.

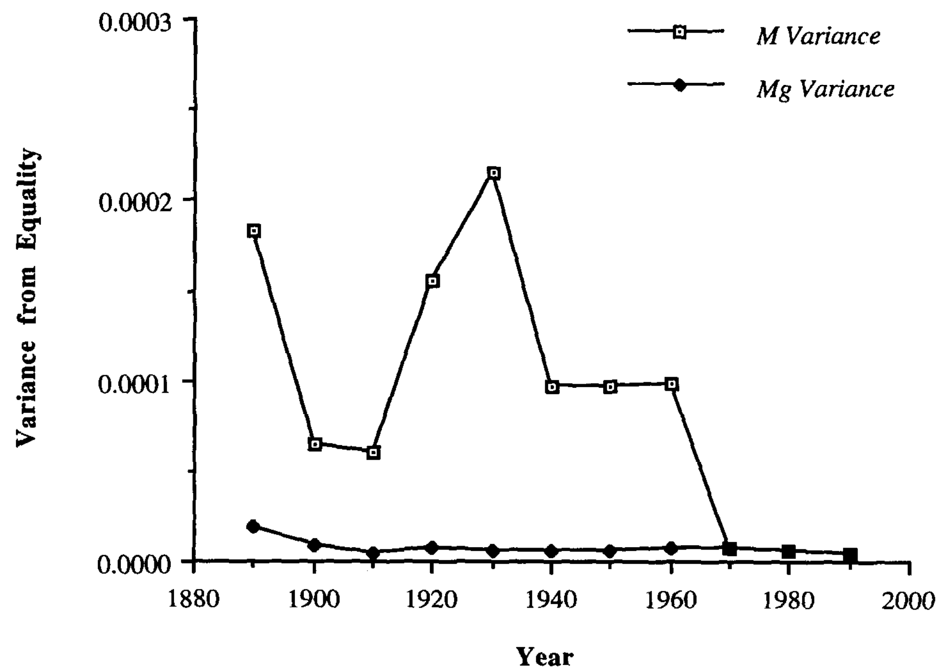

FIG. 4. Malapportionment of Congressional districts (distributional variance).
Source: United States Congressional Directory, various dates.

separability of distributional variance allows us to express total malapportionment as a sum of intrastate and interstate inequity terms:

$$M^2 = I^2\left(p_i, \frac{1}{D}\right) \approx \sum_{i=1}^{D}\left(p_i - \frac{1}{D}\right)^2 = M_g^{\,2} + \sum_{j=1}^{R}(d_j M_j)^2 \tag{17}$$

where D = the number of Congressional districts,
p_i = the fraction of total population in state i,
R = the number of states,
d_j = the fraction of Congressional districts in state j,
M_j = the intrastate malapportionment of district size in state j, and
M_g = interstate malapportionment.

The interstate term, M_g^2, is found by assuming each state is divided into equal districts and then calculating overall distributional variance. This is the inequity introduced in the first stage of apportionment. Each intrastate term, $(d_jM_j)^2$, is the distributional variance within the state, weighted by the percentage of districts held by the state. In this way, we can see the contribution of inequity within states (indeed, within particular states) compared directly with the inequity among states. Figure 4 shows that through the 1960s, intrastate malapportionment was responsible for as much as 97 per cent of overall variance from inequity. Today it is responsible for virtually none. Interpretation of the significance of this may vary, of course. The drop in inequity caused by *Wesberry* was certainly dramatic in percentage terms, but perhaps inequity was already so small that this was a misplaced concern. The latter interpretation is supported by the observation that the Court's insistence on equality of numbers has allowed other forms of gerrymandering to occur under the veil of equity.

These are matters for other methodological tools, however. The utility of a separable inequity measure should be clear. Many investigations are possible using the separability feature. We could, for instance, further examine how inequity among rural and urban districts has changed by grouping districts according to population density and calculating the intergroup inequity. More generally, the method can be used to evaluate many forms of inequity in and among components grouped by, say, income levels, race, or ethnicity, region, size, ideology, or indeed any identifiable feature.

Concluding Remarks

Income inequality, poverty, disproportionality, malapportionment, underrepresentation, and overrepresentation are all examples of inequity—deviation from distributional standards. Statistically, these problems are isomorphic and can be attacked with a standard methodology, one suggestion for which has been given in this article. The statistical choices about the form of the measurement should not be confused with the normative choices about the object of the measurement, however. Even when the distributional standard is clear, the choice of which deviations are important may not be.

Further, we must be careful not to expect more from an inequity index than measurement of inequity. For instance, an inequity index is not a social welfare function unless social welfare is assumed to be a function of deviation from the distributional standard. The economics literature has occasionally gone astray with the addition of this misleading and generally unnecessary normative consideration.

We must also be careful not to use inequity as an indicator of factors better illuminated by other tools. A prominent example in electoral studies is the use of disproportionality as a predictor of Duverger's 'psychological effect'. Under this effect, voters for small parties in plurality systems, becoming frustrated with their inability to win, switch their votes to large parties, leading to the demise of small parties. It is not, however, disproportionality as such that causes the effect, but *majoritarianism*—the tendency of large parties to be overrepresented and small parties underrepresented. In other words, the system must not only be disproportional, but the disproportionality must be consistently to the advantage of large parties. In a system where small parties had the advantage, Duverger's effect would not occur, even if (indeed, particularly if) disproportionality were quite high. Such studies require a different tool altogether.[11]

Also, I should note that I have not made any normative claims about what is *fair* in any distributive situation, electoral or otherwise. Only two central assertions have been made here. First, if we have a given standard for what is equitable and what counts as a deviation from that equity, we can develop an inequity measurement that meets a set of desired axioms. Second, the decisions about what is being distributed, to whom it is being distributed, and what is an equitable distribution, are difficult normative ones in every interesting case of inequity measurement. These choices must be justified by arguments beyond those of the current measurement framework. An inequity index can only measure inequity—it measures unfairness only when we accept that, for the case at hand, inequity is unfair. Since all electoral systems are disproportional to some extent, and all distributions of geographical representation are malapportioned to some extent, this does not appear empirically to be a generally accepted equivalence in either of the two cases of interest.

This article has given some brief illustrations of how specific forms of electoral inequity, such as disproportionality and malapportionment, might be measured. Many other electoral phenomena might be measured in this way as well. Electoral volatility (deviation of votes at one election from the standard of votes at the last election) and voter heterogeneity (deviation of the set of voters' preference orderings for candidates from the standard of equal support for each preference ordering) are just two possibilities. Beyond the context of electoral studies, many concepts of complex equality (Dunleavy, 1989; Rae, 1989) can be analysed within the inequity framework, and it is hoped that this discussion provides a solid foundation for such analyses.

Notes

1. $$I_s = \frac{1}{2}\sum_{i=1}^{N} | x_i - \frac{1}{N} |.$$

2. $$I_g = \frac{1}{2N}\sum_{i=1}^{j-1}\sum_{j=2}^{N}\left|x_i - x_j\right| = 1 - \frac{1}{N}\left(\sum_{i=1}^{N}(2(N-i)+1)x_i\right).$$

3. Further, sample variance is often the unbiased maximum-likelihood estimator for the dispersion in a sample from a known (or assumed) probability distribution.

4. It should be noted that this last step, the choice of $\alpha = 2$ over $\alpha = 1$ or $\alpha = 3$, 4, or higher, is still only minimally justified. It might be plausibly claimed that we should look at *all* of these moments and weigh their results somehow. If we are willing to do this, however, we should be willing to simply look at distributions and compare them. In building a summary measure we seek to reflect a salient characteristic of a distribution in one number. When we embark on this process, we accept that there must be some loss of information. The mean of a sample does not tell us everything about the sample, but it does tell us something useful about it. The choice of variance/standard deviation ($\alpha = 2$) is the most justified choice in this context.
5. Coulter's index with $\alpha = 2$.
6. The term $\frac{N}{N+1}$ can be ignored in large systems.
7. Cox and Shugart (1991) point out that this index is minimized by awarding all seats to parties with $v_i < 0.5$ per cent, a distributive scheme difficult to defend.
8. Here, 'Liberals' is used to refer to the major national third party: the Liberals, the Liberal/Social Democratic Alliance, or, currently, the Liberal Democrats.
9. 376 U.S. 1.
10. The Hill/Huntington method is based on minimizing this inequality, and most court cases on the apportionment issue are decided on such factors as 'maximum percent difference in district size'. The substantive results are not affected by a voter-based analysis.
11. King's methods present promise in this area (King, 1990).

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