Problem Set 7

Due: Wednesday, December 5, 2018

Instructions:

- Write your name and student ID at the top of the first page.
- Show all work and write neatly. Answers will be considered incorrect if supporting work is not shown, if the final number does not follow from the supporting logic (even if the final number turns out to be correct), or if your answer is illegible.
- The problem set is due at the beginning of class on the due date.
- This problem set consists of two parts: an individual assignment and a group assignment. Your write-up for individual assignment should be completed and submitted individually. Your write-up for the group assignment should be completed and submitted by your group separately (one copy for the whole group with group members' names on the first page). A group consists of one to four students.
- Each write-up should include two parts: (1) answers to the questions, including output tables and graphs, and (2) codes used for solving the numerical exercises. Please hand in your solutions in that order, with your answers and codes in separate sections. You may include tables and graphs in the main body of your written answers or append them at the end of part (1).

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1. This problem illustrates how the aggregate equity premium depends on the distribution of economic disasters and booms.

Consider a two-period equilibrium model with a representative agent. Agent's utility is given by

$$u(c_0) + E_0[u(c_1)],$$

where

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}.$$

Let the initial aggregate endowment be $e_0 = 1$. Suppose that the log of the aggregate endowment at time $t = 1$, $\ln e_1$, has the following distribution function $pdf(x)$:

$$pdf(x) = (1-p)\,\Phi(x; \mu - \sigma_1^2/2, \sigma_1) + p\,\Phi(x; \mu - \sigma_2^2/2, \sigma_2),$$

where

$$\Phi(x; a, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-a)^2/(2\sigma^2)}.$$

Thus, the distribution is a mixture of Gaussian distributions. Assume the following parameter values:

$$\gamma = 10;\ \mu = 0.01;\ \sigma_1 = 0.02;\ \sigma_2 = 0.15;\ p = 0.05$$

(a) Simulate 100,000 realization of the endowment process based on the above distribution. Estimate the probabilities of the log endowment $\ln c_1$ being below 0, -0.02, -0.06, -0.12 respectively. Fit a Gaussian distribution to the values of the lof endowment you've generated, compare the theoretical values of the CDF of the fitted distribution to your estimates. Comment on the ability of the Gaussian distribution to describe "crises" (extremely low values of the endowment) in your model.

(b) Compute and plot the SPD in this model, given by

$$SPD = \frac{u'(c_1)}{u'(c_0)}$$

(c) Compute numerically (use numerical integration or Monte Carlo simulations) the risk-free interest rate in this model and the price of a "stock market index," defined as the claim on the aggregate endowment. Change p to 0.02. How do the answers change? Comment on how the value of p affects the risk premium on the stock market index.

(d) Repeat the exercise in (c), using $p = 0.05$, but instead of the true model, use the Gaussian distribution you've fitted in part (a). Comment on how the risk premium on the market index depends on the shape of the endowment distribution.

2. This problem illustrates how an implied volatility smile may arise in an equilibrium model with tail risk in aggregate output.

Continue with the setting of problem 1. Use $p = 0.05$. Let P_0 denote the price of the stock market index at time 0. Consider a set of put options on the time-1 value of the index (use the cum-dividend value, which equals the value of the aggregate endowment). Let the strike prices of these options be 0.9, 0.91, 0.92, ..., 1.1.

(a) Compute prices of the put options above. Use Monte Carlo simulation. (Hint: to increase precision, use the same set of simulated draws from the endowment to compute prices of all options at once).

(b) Compute the implied volatilities of the options and argue that the Black-Scholes option pricing model does not apply in this setting. Try to explain intuitively the shape of the implied volatility curve.

3. This problem illustrates that junior and senior CDO tranches exhibit different degrees of aggregate risk exposure.

Continue with the setting of problem 1. Use $p = 0.05$. Consider a set of corporate bonds, indexed by $n = 1, ..., 100$. Suppose that each bond pays off according to the following model: bond n pays \$1 at time $t = 1$ if a draw of the random variable x_n exceeds zero. Assume that

$$x_n = a_0 + a_1(e_1 - 1) + \sigma_\epsilon \epsilon_n,$$

where $\epsilon_n \sim \mathcal{N}(0, 1)$, and all random variables ϵ_n are jointly independent and independent of e_1. Otherwise, if $x_n < 0$, bond n defaults and pays nothing.

Thus, all bonds are more likely to default when the aggregate endowment is low. Assume

$$a_0 = 3, \ a_1 = 6, \ \sigma_\epsilon = 1.5.$$

Consider a CDO based on the above 100 bonds, with three tranches. The "equity" tranche has a maximum time-1 payoff equal to \$3. The payoff of this tranche is reduced by \$1 with each default among the 100 bonds underlying the CDO. The lowest possible payoff of the tranche is zero.

The next tranche is a "mezzanine" tranche. This tranche promises to pay a maximum of \$4 and its payoff is reduced by \$1 with each default *after* the equity tranche payoff has reached zero. Thus, the equity tranche absorbs the first few defaults, and then the mezzanine tranche gets affected. The lowest possible payoff of the tranche is zero.

The last tranche, the "senior" tranche, pays off the entire value of the portfolio of 100 bonds minus the payoffs of the equity and mezzanine tranches. This tranche is the safest of the three.

Next, consider CDS contracts, one for each tranche. Each CDS contract offers protection against losses on the tranche it covers. Its time-1 payoff (from the perspective of the protection buyer) is equal to the losses on the tranche minus the CDS premium, with premium payable at time 1. The premium is determined so that the price is fair and no money needs to change hands at time 0.

(a) Using Monte Carlo simulation with 1,000,000 draws, compute equilibrium CDS premia for the three contracts described above. Using simulations, decompose each premium into default premium and risk premium.

(b) Compute expected losses on each tranche *conditionally* on the realized endowment value, e_1. Plot the results against e_1.

(c) Using the results from parts (a) and (b), explain intuitively the differences in the decomposition of the CDS premia into the default and the risk premium components. (Hint: appeal to consumption CAPM).