MITA 数学手册

2025 秋

目录 Content

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初等函数导数表

幂函数 $\forall\, \mu \in \mathbf{R}$ ：

$$(x)' = 1 \quad , \quad (x^{\mu})' = \mu x^{\mu - 1} \ .$$

指数函数 & 对数函数 $\forall\, a \in (0 , +\infty) \setminus \{ 1 \}$ ：

$$(a^x)' = a^x \ln a \quad , \quad (\log_a x)' = \frac{1}{x \ln a} \ .$$

三角函数

$$(\sin x)' = \cos x \quad , \quad (\cos x)' = - \sin x \quad , \quad (\tan x)' = \sec^2 x \ ,$$

$$(\sec x)' = \tan x \sec x \quad , \quad (\csc x)' = - \cot x \csc x \quad , \quad (\cot x)' = - \csc^2 x \ .$$

反三角函数

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}} \quad , \quad (\arccos x)' = -\frac{1}{\sqrt{1 - x^2}} \quad , \quad (\arctan x)' = \frac{1}{x^2 + 1} \ ,$$

$$(\operatorname{arcsec}\ x)' = \frac{1}{| x | \sqrt{x^2 - 1}} \quad , \quad (\operatorname{arccsc}\ x)' = -\frac{1}{| x | \sqrt{x^2 - 1}} \quad , \quad (\operatorname{arccot}\ x)' = -\frac{1}{x^2 + 1} \ .$$

双曲函数

$$(\sinh x)' = \cosh x \quad , \quad (\cosh x)' = \sinh x \quad , \quad (\tanh x)' = \operatorname{sech}^2 x \ ,$$

$$(\operatorname{sech}\ x)' = - \tanh x \ \operatorname{sech}\ x \quad , \quad (\operatorname{csch}\ x)' = - \coth x \ \operatorname{csch}\ x \quad , \quad (\coth x)' = -\operatorname{csch}^2 x \ .$$

反双曲函数

$$(\operatorname{arsinh}\ x)' = \frac{1}{\sqrt{x^2 + 1}} \quad , \quad (\operatorname{arcosh}\ x)' = \frac{1}{\sqrt{x^2 - 1}} \quad , \quad (\operatorname{artanh}\ x)' = \frac{1}{1 - x^2} \ ,$$

$$(\operatorname{arsech}\ x)' = -\frac{1}{x \sqrt{1 - x^2}} \quad , \quad (\operatorname{arcsch}\ x)' = -\frac{1}{| x | \sqrt{x^2 + 1}} \quad , \quad (\operatorname{arcoth}\ x)' = \frac{1}{1 - x^2} \ .$$

裂项求和

裂项求和又叫作伸缩级数分解, 是一种通用求和技巧, 即通过初等变换使得求和项前后相互抵消, 从而极大地简化求和计算, 在下面的例子中, 裂项的部分将被 [ ] 包裹以示区分, 并且求和符号将被简化表示, 我们约定 n 为恒正求和指标.

分式裂项 $\forall\, m_1\,,\, m_2 \in \mathbf{Z}\ :\ m_1 \neq m_2 \quad \Longrightarrow$

$$\sum \frac{1}{\prod\limits_{m=m_1}^{m_2}(\,kn+m\,)} = \sum \frac{1}{m_2-m_1}\left[\,\frac{1}{\prod\limits_{m=m_1}^{m_2-1}(\,kn+m\,)} - \frac{1}{\prod\limits_{m=m_1+1}^{m_2}(\,kn+m\,)}\,\right].$$

作为推论, 我们有

$$\sum \frac{1}{n^2+kn} = \frac{1}{k}\sum\left[\,\frac{1}{n}-\frac{1}{n+k}\,\right]\,,\quad (\forall\, n : n \neq -k\,)$$
$$\sum \frac{1}{k^2n^2-1} = \frac{1}{2}\sum\left[\,\frac{1}{kn-1}-\frac{1}{kn+1}\,\right].\quad (\forall\, n : n \neq 1/k\,)$$

我们也可以结合部分分式来裂项求和, 例如

$$\sum \frac{n^2}{4n^2-1} = \frac{1}{8}\sum\left(2+\left[\,\frac{1}{2n-1}-\frac{1}{2n+1}\,\right]\right)\,,$$
$$\sum \frac{3n+1}{(\,n+1\,)(\,n+2\,)(\,n+3\,)} = 4\sum\left(\left[\,\frac{2}{n+2}-\frac{1}{n+1}-\frac{1}{n+3}\,\right]+\left[\,\frac{1}{n+2}-\frac{1}{n+1}\,\right]\right).$$

$$\sum \frac{a_{n+1}-a_n}{a_na_{n+1}} = \sum\left[\,\frac{1}{a_n}-\frac{1}{a_{n+1}}\,\right]\,,$$

其中 $\{\,a_n\,\}$ 为非零复数列. 作为推论, 我们有

$$\sum \frac{a^n}{(\,a^n+k\,)(\,a^{n+1}+k\,)} = \frac{1}{a-1}\sum\left[\,\frac{1}{a^n+k}-\frac{1}{a^{n+1}+k}\,\right]\,,\quad (\,a\neq 1\,,\,\forall\, n : a^n \neq -k\,)$$
$$\sum \frac{qn-n+q}{(\,n^2+n\,)\,q^{n+1}} = \sum\left[\,\frac{1}{nq^n}-\frac{1}{(\,n+1\,)\,q^{n+1}}\,\right]\,,\quad (\,q\neq 0\,)$$
$$\sum \frac{a^{2^n}}{1-a^{2^{n+1}}} = \sum\left[\,\frac{1}{1-a^{2^n}}-\frac{1}{1-a^{2^{n+1}}}\,\right].\quad (\,a\neq 1\,)$$

对于含有 $(\,-1\,)^n$ 的通项, 需要在裂项时变更符号, 例如

$$\sum \frac{(\,2n+1\,)(\,-1\,)^n}{n\,(\,n+1\,)} = \sum\left(\frac{(\,-1\,)^n}{n}+\frac{(\,-1\,)^n}{n+1}\right) = \sum\left[\,\frac{(\,-1\,)^n}{n}-\frac{(\,-1\,)^{n+1}}{n+1}\,\right].$$

根式裂项

$$\sum \frac{1}{\sqrt{n}+\sqrt{n+k}} = \frac{1}{k}\sum\left[\sqrt{n+k}-\sqrt{n}\right] , \quad (k\neq 0,\forall n: n+k>0)$$

$$\sum \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}} = \frac{1}{2}\sum\left[\sqrt{2n+1}-\sqrt{2n-1}\right] ,$$

$$\sum \frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}} = \sum\left[\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right] ,$$

$$\sum \frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}} = \sum\left[\sqrt{\frac{n}{n+1}}-\sqrt{\frac{n-1}{n}}\right] .$$

阶乘裂项

$$\sum (n\cdot n!) = \sum\left[(n+1)!-n!\right] ,$$

$$\sum \frac{n}{(n+1)!} = \sum\left[\frac{1}{n!}-\frac{1}{(n+1)!}\right] ,$$

$$\sum \frac{1}{n!(n+2)} = \sum\left[\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\right] ,$$

$$\sum \frac{n+2}{n!+(n+1)!+(n+2)!} = \sum\left[\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\right] ,$$

$$\sum (n+1)n!! = \sum\left[(n+2)!!-n!!\right] ,$$

$$\sum \frac{n+1}{(n+2)!!} = \sum\left[\frac{1}{n!!}-\frac{1}{(n+2)!!}\right] ,$$

差比数列前 $\boldsymbol{n}$ 项和 $\forall\, k, b\in\mathbf{C}$, $q\in\mathbf{C}\setminus\{1\}$:

$$\sum (kn+b)q^{n-1} = \sum\left[\left(\frac{k}{q-1}(n+1)+t\right)q^n-\left(\frac{k}{q-1}n+t\right)q^{n-1}\right] ,$$

其中

$$t = \frac{b}{q-1}-\frac{kq}{(q-1)^2} .$$

作为推论, 我们有等比数列前 n 项和

$$\sum a\cdot q^{n-1} = \frac{a}{q-1}\sum\left[q^n-q^{n-1}\right] .$$

三角裂项

$$\sum \sin\theta\cos(2n\theta+\theta) = \sum\left[\,\sin 2(n+1)\theta - \sin 2n\theta\,\right]\ ,$$

$$\sum \sin\theta\sin(2n\theta+\theta) = \frac{1}{2}\sum\left[\,\cos 2n\theta - \cos 2(n+1)\theta\,\right]\ ,$$

$$\sum \frac{\sin\theta}{\cos n\theta\cdot\cos(n+1)\theta} = \sum\left[\,\tan(n+1)\theta - \tan n\theta\,\right]\ ,\quad (\forall n:\cos n\theta\neq 0)$$

$$\sum \frac{\sin\theta}{\sin n\theta\cdot\sin(n+1)\theta} = \sum\left[\,\cot n\theta - \cot(n+1)\theta\,\right]\ ,\quad (\forall n:\sin n\theta\neq 0)$$

$$\sum 3^{n-1}\sin^3\frac{\theta}{3^n} = \frac{1}{4}\sum\left[\,3^n\sin\frac{\theta}{3^n} - 3^{n-1}\sin\frac{\theta}{3^{n-1}}\,\right]\ ,$$

$$\sum \frac{1}{2^n}\tan\frac{\theta}{2^n} = \sum\left[\,\frac{1}{2^n}\cot\frac{\theta}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}\,\right]\ ,$$

$$\prod \cos\frac{\theta}{2^n} = 2^{-n}\prod\left[\,\sin\frac{\theta}{2^{n-1}}\Big/\sin\frac{\theta}{2^n}\,\right]\ .$$

我们可以结合导数的线性性质来裂项求和, 例如

$$\sum \frac{1}{2^n}\tan\frac{\theta}{2^n} = -\frac{\mathrm{d}}{\mathrm{d}\,\theta}\ln\left|\prod\cos\frac{\theta}{2^n}\right| = -\frac{\mathrm{d}}{\mathrm{d}\,\theta}\ln\left|2^{-n}\prod\left[\,\sin\frac{\theta}{2^{n-1}}\Big/\sin\frac{\theta}{2^n}\,\right]\right|\ .$$

反三角裂项

$$\sum \arctan\frac{1}{n^2+n+1} = \sum\left[\,\arctan(n+1) - \arctan n\,\right]\ ,$$

$$\sum \arctan\frac{1}{2n^2} = \sum\left[\,\arctan\frac{n}{n+1} - \arctan\frac{n-1}{n}\,\right]\ .$$

其它裂项

$$\sum \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} = \sum\left[\,1+\frac{1}{n}-\frac{1}{n+1}\,\right]\ ,$$

$$\sum \frac{2}{\sqrt[3]{n^2-2n+1}+\sqrt[3]{n^2-1}+\sqrt[3]{n^2+2n+1}} = \sum\left[\,\sqrt[3]{n+1}-\sqrt[3]{n}\,\right]\ ,$$

$$\prod\left(1+a^{2^n}\right) = \prod\left[\,\frac{1-a^{2^{n+1}}}{1-a^{2^n}}\,\right]\ ,\quad (a\neq 1)$$

$$\prod \frac{n^3-1}{n^3+1} = \prod\left[\,\frac{n-1}{n+1}\,\right]\left[\,\frac{(n+1)^2-(n+1)+1}{n^2-n+1}\,\right]\ .$$

三角恒等式

在下面的一切恒等式中, 我们要求 α 和 β 使得对应的三角函数值有意义.

基本恒等式 $\forall\, \alpha \in \mathbf{C}$:

$$\sin^2\alpha + \cos^2\alpha = 1 \quad , \quad \tan\alpha \cdot \cot\alpha = 1 \; ,$$
$$\tan^2\alpha + 1 = \sec^2\alpha \quad , \quad \cot^2\alpha + 1 = \csc^2\alpha \; .$$

诱导公式 I $\forall\, \alpha \in \mathbf{C}$, $k \in \mathbf{Z}$:

$$\sin(2k\pi + \alpha) = \sin\alpha \quad , \quad \cos(2k\pi + \alpha) = \cos\alpha \; ,$$
$$\tan(2k\pi + \alpha) = \tan\alpha \quad , \quad \cot(2k\pi + \alpha) = \cot\alpha \; ,$$
$$\sec(2k\pi + \alpha) = \sec\alpha \quad , \quad \csc(2k\pi + \alpha) = \csc\alpha \; .$$

诱导公式 II $\forall\, \alpha \in \mathbf{C}$:

$$\sin(-\alpha) = -\sin\alpha \quad , \quad \cos(-\alpha) = \cos\alpha \; ,$$
$$\tan(-\alpha) = -\tan\alpha \quad , \quad \cot(-\alpha) = -\cot\alpha \; ,$$
$$\sec(-\alpha) = -\sec\alpha \quad , \quad \csc(-\alpha) = \csc\alpha \; .$$

诱导公式 III $\forall\, \alpha \in \mathbf{C}$:

$$\sin(\pi + \alpha) = -\sin\alpha \quad , \quad \cos(\pi + \alpha) = -\cos\alpha \; ,$$
$$\tan(\pi + \alpha) = \tan\alpha \quad , \quad \cot(\pi + \alpha) = \cot\alpha \; ,$$
$$\sec(\pi + \alpha) = -\sec\alpha \quad , \quad \csc(\pi + \alpha) = -\csc\alpha \; .$$

诱导公式 IV $\forall\, \alpha \in \mathbf{C}$:

$$\sin(\pi - \alpha) = \sin\alpha \quad , \quad \cos(\pi - \alpha) = -\cos\alpha \; ,$$
$$\tan(\pi - \alpha) = -\tan\alpha \quad , \quad \cot(\pi - \alpha) = -\cot\alpha \; ,$$
$$\sec(\pi - \alpha) = \sec\alpha \quad , \quad \csc(\pi - \alpha) = -\csc\alpha \; .$$

诱导公式 V $\forall\,\alpha\in\mathbf{C}$ ：

$$\sin(\pi/2+\alpha)=\cos\alpha \quad , \quad \cos(\pi/2+\alpha)=-\sin\alpha \ ,$$
$$\tan(\pi/2+\alpha)=-\cot\alpha \quad , \quad \cot(\pi/2+\alpha)=-\tan\alpha \ ,$$
$$\sec(\pi/2+\alpha)=-\csc\alpha \quad , \quad \csc(\pi/2+\alpha)=\sec\alpha \ .$$

诱导公式 VI $\forall\,\alpha\in\mathbf{C}$ ：

$$\sin(\pi/2-\alpha)=\cos\alpha \quad , \quad \cos(\pi/2-\alpha)=\sin\alpha \ ,$$
$$\tan(\pi/2-\alpha)=\cot\alpha \quad , \quad \cot(\pi/2-\alpha)=\tan\alpha \ ,$$
$$\sec(\pi/2-\alpha)=\csc\alpha \quad , \quad \csc(\pi/2-\alpha)=\sec\alpha \ .$$

和差公式 $\forall\,\alpha,\beta\in\mathbf{C}$ ：

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta \quad , \quad \cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta \ ,$$
$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta} \quad , \quad \cot(\alpha\pm\beta)=\frac{\cot\alpha\cot\beta\mp1}{\cot\alpha\pm\cot\beta} \ ,$$
$$\sec(\alpha\pm\beta)=\frac{\sec\alpha\sec\beta}{1\mp\tan\alpha\tan\beta} \quad , \quad \csc(\alpha\pm\beta)=\frac{\sec\alpha\sec\beta}{\tan\alpha\pm\tan\beta} \ .$$

辅助角公式 $\forall\,\alpha\in\mathbf{C}$, $A,B\in\mathbf{C}$ ：

$$A\sin\alpha+B\cos\alpha=\sqrt{A^2+B^2}\sin(\alpha+\varphi) \ ,$$
$$A\sin^2\alpha+B\cos^2\alpha+C\sin\alpha\cos\alpha=\frac{\sqrt{A^2+B^2+C^2-2AB}}{2}\sin(2\alpha+\psi)+\frac{A+B}{2} \ ,$$

其中辅助角 φ 和 ψ 满足

$$(\cos\varphi,\sin\varphi)=(A,B) \quad \text{和} \quad (\cos\psi,\sin\psi)=(C,B-A) \ .$$

半角公式 $\forall\,\alpha\in\mathbf{C}$ ：

$$\tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha} \ ,$$
$$\cot\frac{\alpha}{2}=\frac{\sin\alpha}{1-\cos\alpha}=\frac{1+\cos\alpha}{\sin\alpha} \ .$$

倍角公式　$\forall\, \alpha \in \mathbf{C}$ ：

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \quad , \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \ ,$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \quad , \quad \cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha} \ ,$$

$$\sin 3\alpha = -4\sin^3\alpha + 3\sin\alpha \quad , \quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha \ ,$$

$$\tan 3\alpha = \frac{\tan^3\alpha - 3\tan\alpha}{3\tan^2\alpha - 1} \quad , \quad \cot 3\alpha = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1} \ ,$$

$$\sin 4\alpha = \cos\alpha\left(-8\sin^3\alpha + 4\sin\alpha\right) \quad , \quad \cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1 \ ,$$

$$\tan 4\alpha = \frac{-4\tan^3\alpha + 4\tan\alpha}{\tan^4\alpha - 6\tan^2\alpha + 1} \quad , \quad \cot 4\alpha = \frac{\cot^4\alpha - 6\cot^2\alpha + 1}{4\cot^3\alpha - 4\cot\alpha} \ ,$$

$$\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha \quad , \quad \cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha \ ,$$

$$\tan 5\alpha = \frac{\tan^5\alpha - 10\tan^3\alpha + 5\tan\alpha}{5\tan^4\alpha - 10\tan^2\alpha + 1} \quad , \quad \cot 5\alpha = \frac{\cot^5\alpha - 10\cot^3\alpha + 5\cot\alpha}{5\cot^4\alpha - 10\cot^2\alpha + 1} \ .$$

$\forall\, \alpha \in \mathbf{C}$, $n \in \mathbf{N}$ ：

$$\cos n\alpha = n\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2n-2k-2)!!}{(2k)!!\,(n-2k)!} \cos^{n-2k}\alpha \ ,$$

$$\sin 2n\alpha = \cos\alpha \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{(4n-2k-2)!!}{(2k)!!\,(2n-2k-1)!} \sin^{2n-2k-1}\alpha \ ,$$

$$\sin(2n+1)\alpha = (2n+1)\sum_{k=0}^{n} (-1)^{n-k} \frac{(4n-2k)!!}{(2k)!!\,(2n-2k+1)!} \sin^{2n-2k+1}\alpha \ ,$$

$$\tan n\alpha = \frac{\displaystyle\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \mathrm{C}_n^{2k+1} \tan^{2k+1}\alpha}{\displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \mathrm{C}_n^{2k} \tan^{2k}\alpha} \ ,$$

$$\cot n\alpha = \frac{\displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \mathrm{C}_n^{2k} \cot^{n-2k}\alpha}{\displaystyle\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \mathrm{C}_n^{2k+1} \cot^{n-2k-1}\alpha} \ .$$

降幂公式 $\forall\, \alpha \in \mathbf{C}$ ：

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \quad , \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \ ,$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} \quad , \quad \cot^2\alpha = \frac{1+\cos 2\alpha}{1-\cos 2\alpha} \ ,$$

$$\sec^2\alpha = \frac{2}{1+\cos 2\alpha} \quad , \quad \csc^2\alpha = \frac{2}{1-\cos 2\alpha} \ .$$

$\forall\, \alpha \in \mathbf{C}$, $n \in \mathbf{N}$ ：

$$\sin^{2n+1}\alpha = \frac{1}{(\,-4\,)^n}\sum_{k=0}^{n}(\,-1\,)^k\,\mathrm{C}_{2n+1}^{k}\sin\,(\,2n-2k+1\,)\,\alpha \ ,$$

$$\cos^{2n+1}\alpha = \frac{1}{4^n}\sum_{k=0}^{n}\mathrm{C}_{2n+1}^{k}\cos\,(\,2n-2k+1\,)\,\alpha \ ,$$

$$\sin^{2n}\alpha = \frac{(\,2n\,)!}{4^n\,(\,n!\,)^2} + \frac{2}{(\,-4\,)^n}\sum_{k=0}^{n-1}(\,-1\,)^k\,\mathrm{C}_{2n}^{k}\cos\,(\,2n-2k\,)\,\alpha \ ,$$

$$\cos^{2n}\alpha = \frac{(\,2n\,)!}{4^n\,(\,n!\,)^2} + \frac{2}{4^n}\sum_{k=0}^{n-1}\mathrm{C}_{2n}^{k}\cos\,(2n-2k)\,\alpha \ .$$

两元积化和差公式 $\forall\, \alpha\,,\,\beta \in \mathbf{C}$ ：

$$\sin\alpha\cos\beta = \frac{1}{2}\,[\,\sin\,(\,\alpha+\beta\,)+\sin\,(\,\alpha-\beta\,)\,] \ ,$$

$$\cos\alpha\cos\beta = \frac{1}{2}\,[\,\cos\,(\,\alpha+\beta\,)+\cos\,(\,\alpha-\beta\,)\,] \ ,$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\,[\,\cos\,(\,\alpha+\beta\,)-\cos\,(\,\alpha-\beta\,)\,] \ .$$

两元和差化积公式 $\forall\, \alpha\,,\,\beta \in \mathbf{C}$ ：

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2} \ ,$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \ ,$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \ .$$

万能代换公式　$\forall\, \alpha \in \mathbf{C}$:

$$\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} \quad , \quad \cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} \quad , \quad \tan\alpha = \frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)} \ ,$$

$$\cot\alpha = \frac{1-\tan^2(\alpha/2)}{2\tan(\alpha/2)} \quad , \quad \sec\alpha = \frac{1+\tan^2(\alpha/2)}{1-\tan^2(\alpha/2)} \quad , \quad \csc\alpha = \frac{1+\tan^2(\alpha/2)}{2\tan(\alpha/2)} \ .$$

三角平方差公式　$\forall\, \alpha, \beta \in \mathbf{C}$:

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\sin(\alpha-\beta) \ ,$$

$$\cos^2\alpha - \sin^2\beta = \cos(\alpha+\beta)\cos(\alpha-\beta) \ .$$

切化弦公式　$\forall\, \alpha, \beta \in \mathbf{C}$:

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta} \ ,$$

$$\cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta} \ ,$$

$$\tan\alpha \pm \cot\beta = \pm\frac{\cos(\beta\mp\alpha)}{\sin\beta\cos\alpha} \ .$$

内角恒等式　$\forall\, \alpha, \beta, \gamma \in \mathbf{C}$: $\alpha+\beta+\gamma=\pi \quad \Longrightarrow$

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \ ,$$

$$\cos\alpha + \cos\beta + \cos\gamma = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1 \ ,$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma \ ,$$

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma \ ,$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2} \ ,$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1 \ ,$$

$$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1 \ ,$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 + 2\cos\alpha\cos\beta\cos\gamma \ ,$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 - 2\cos\alpha\cos\beta\cos\gamma \ .$$

Taylor 级数

Taylor 公式　任意阶可导函数 $f(x)$ 在 $x=x_0$ 处的 Taylor 展开式为

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+R_n(x)\ ,$$

其中

$$R_n(x)=\int_{x_0}^{x}\frac{f^{(n+1)}(t)}{n!}(x-t)^n\,\mathrm{d}t\ .$$

当余项 $R_n(x)$ 对一切定义域内的 x 均趋于零, 即

$$\lim_{n\to\infty}R_n(x)=0$$

时, 我们有如下 Taylor 级数.

自然指数　$\forall\, x\in\mathbf{R}$:

$$\mathrm{e}^x=\sum_{n=0}^{\infty}\frac{x^n}{n\,!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\ .$$

自然对数　$\forall\, x\in(-1\,,\,1]$:

$$\ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\ .$$

三角函数　$\forall\, x\in\mathbf{R}$:

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{(2n+1)\,!}=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots\ ,$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n}}{(2n)\,!}=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\ ,$$

$\forall\, x\in(-\pi/2\,,\,\pi/2)$:

$$\tan x=\sum_{n=0}^{\infty}\frac{|\,B_{2n}\,|\,(16^n-4^n)\,x^{2n-1}}{(2n)\,!}=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots\ ,$$

$$\sec x=\sum_{n=0}^{\infty}\frac{E_{2n}x^{2n}}{(2n)\,!}=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\cdots\ ,$$

$\forall\, x \in (\,-\pi\,,\,\pi\,)\ :$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{4^n\,|\,B_{2n}\,|\,x^{2n-1}}{(\,2n\,)\,!} = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \cdots\ ,$$

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{|\,B_{2n}\,|\,(\,4^n-2\,)\,x^{2n-1}}{(\,2n\,)\,!} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots\ .$$

双曲函数　$\forall\, x \in \mathbf{R}\ :$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(\,2n+1\,)\,!} = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\ ,$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(\,2n\,)\,!} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\ ,$$

$\forall\, x \in (\,-\pi/2\,,\,\pi/2\,)\ :$

$$\tanh x = \sum_{n=0}^{\infty} \frac{B_{2n}\,(\,16^n-4^n\,)\,x^{2n-1}}{(\,2n\,)\,!} = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots\ ,$$

$$\operatorname{sech}\,x = \sum_{n=0}^{\infty} (\,-1\,)^n\,\frac{E_{2n}x^{2n}}{(\,2n\,)\,!} = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \cdots\ ,$$

$\forall\, x \in (\,-\pi\,,\,\pi\,)\ :$

$$\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{4^n B_{2n} x^{2n-1}}{(\,2n\,)\,!} = \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \cdots\ ,$$

$$\operatorname{csch}\,x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{B_{2n}\,(\,4^n-2\,)\,x^{2n-1}}{(\,2n\,)\,!} = \frac{1}{x} - \frac{1}{6}x + \frac{7}{360}x^3 - \frac{31}{15120}x^5 + \cdots\ .$$

反三角函数　$\forall\, x \in (\,-1\,,\,1\,)\ :$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(\,2n-1\,)\,!!\,x^{2n+1}}{(\,2n\,)\,!!\,(\,2n+1\,)} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots\ ,$$

$\forall\, x \in [\,-1\,,\,1\,]\ :$

$$\arctan x = \sum_{n=0}^{\infty} (\,-1\,)^n\,\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\ .$$

$\boldsymbol{\pi}$ 的 Leibniz 展开式

$$\pi = 4\arctan 1 = \sum_{n=0}^{\infty} (\,-1\,)^n\,\frac{4}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots\ .$$

反双曲函数 $\forall\, x \in (-1\,,\,1)$:

$$\operatorname{arsinh}\, x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!\, x^{2n+1}}{(2n)!!\,(2n+1)} = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots\ ,$$

$\forall\, x \in (-1\,,\,1)$:

$$\operatorname{artanh}\, x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\ .$$

根据反双曲函数的定义我们有

$$\operatorname{artanh}\, x = \frac{1}{2}\ln\frac{1+x}{1-x}\ ,$$

其中用 x 代换 $(1+x)/(1-x)$ 可以得到下面这个应用更广泛的表达式:

快速对数展开式 $\forall\, x \in (0\,,\,+\infty)$:

$$\ln x = \sum_{n=0}^{\infty} \frac{2}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1} = 2\left(\frac{x-1}{x+1}\right) + \frac{2}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{2}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\ .$$

数协MITA2025新人群

群号：1062803942