



MITA 数学手册

2025 秋

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初等函数导数表

幂函数 $\forall \mu \in \mathbf{R}$:

$$(x)' = 1 \quad , \quad (x^\mu)' = \mu x^{\mu-1} .$$

指数函数 & 对数函数 $\forall a \in (0, +\infty) \setminus \{1\}$:

$$(a^x)' = a^x \ln a \quad , \quad (\log_a x)' = \frac{1}{x \ln a} .$$

三角函数

$$\begin{aligned} (\sin x)' &= \cos x \quad , \quad (\cos x)' = -\sin x \quad , \quad (\tan x)' = \sec^2 x \quad , \\ (\sec x)' &= \tan x \sec x \quad , \quad (\csc x)' = -\cot x \csc x \quad , \quad (\cot x)' = -\csc^2 x . \end{aligned}$$

反三角函数

$$\begin{aligned} (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \quad , \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \quad , \quad (\arctan x)' = \frac{1}{x^2+1} \quad , \\ (\operatorname{arcsec} x)' &= \frac{1}{|x|\sqrt{x^2-1}} \quad , \quad (\operatorname{arccsc} x)' = -\frac{1}{|x|\sqrt{x^2-1}} \quad , \quad (\operatorname{arccot} x)' = -\frac{1}{x^2+1} . \end{aligned}$$

双曲函数

$$\begin{aligned} (\sinh x)' &= \cosh x \quad , \quad (\cosh x)' = \sinh x \quad , \quad (\tanh x)' = \operatorname{sech}^2 x \quad , \\ (\operatorname{sech} x)' &= -\tanh x \operatorname{sech} x \quad , \quad (\operatorname{csch} x)' = -\coth x \operatorname{csch} x \quad , \quad (\coth x)' = -\operatorname{csch}^2 x . \end{aligned}$$

反双曲函数

$$\begin{aligned} (\operatorname{arsinh} x)' &= \frac{1}{\sqrt{x^2+1}} \quad , \quad (\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}} \quad , \quad (\operatorname{artanh} x)' = \frac{1}{1-x^2} \quad , \\ (\operatorname{arsech} x)' &= -\frac{1}{x\sqrt{1-x^2}} \quad , \quad (\operatorname{arcsch} x)' = -\frac{1}{|x|\sqrt{x^2+1}} \quad , \quad (\operatorname{arcoth} x)' = \frac{1}{1-x^2} . \end{aligned}$$

裂项求和

裂项求和又叫作伸缩级数分解, 是一种通用求和技巧, 即通过初等变换使得求和项前后相互抵消, 从而极大地简化求和计算, 在下面的例子中, 裂项的部分将被 [] 包裹以示区分, 并且求和符号将被简化表示, 我们约定 n 为恒正求和指标.

分式裂项 $\forall m_1, m_2 \in \mathbf{Z} : m_1 \neq m_2 \implies$

$$\sum \frac{1}{\prod_{m=m_1}^{m_2} (kn+m)} = \sum \frac{1}{m_2-m_1} \left[\frac{1}{\prod_{m=m_1}^{m_2-1} (kn+m)} - \frac{1}{\prod_{m=m_1+1}^{m_2} (kn+m)} \right].$$

作为推论, 我们有

$$\begin{aligned} \sum \frac{1}{n^2+kn} &= \frac{1}{k} \sum \left[\frac{1}{n} - \frac{1}{n+k} \right], \quad (\forall n : n \neq -k) \\ \sum \frac{1}{k^2n^2-1} &= \frac{1}{2} \sum \left[\frac{1}{kn-1} - \frac{1}{kn+1} \right]. \quad (\forall n : n \neq 1/k) \end{aligned}$$

我们也可以结合部分分式来裂项求和, 例如

$$\begin{aligned} \sum \frac{n^2}{4n^2-1} &= \frac{1}{8} \sum \left(2 + \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right] \right), \\ \sum \frac{3n+1}{(n+1)(n+2)(n+3)} &= 4 \sum \left(\left[\frac{2}{n+2} - \frac{1}{n+1} - \frac{1}{n+3} \right] + \left[\frac{1}{n+2} - \frac{1}{n+1} \right] \right). \\ \sum \frac{a_{n+1}-a_n}{a_n a_{n+1}} &= \sum \left[\frac{1}{a_n} - \frac{1}{a_{n+1}} \right], \end{aligned}$$

其中 $\{a_n\}$ 为非零复数列. 作为推论, 我们有

$$\begin{aligned} \sum \frac{a^n}{(a^n+k)(a^{n+1}+k)} &= \frac{1}{a-1} \sum \left[\frac{1}{a^n+k} - \frac{1}{a^{n+1}+k} \right], \quad (a \neq 1, \forall n : a^n \neq -k) \\ \sum \frac{qn-n+q}{(n^2+n)q^{n+1}} &= \sum \left[\frac{1}{nq^n} - \frac{1}{(n+1)q^{n+1}} \right], \quad (q \neq 0) \\ \sum \frac{a^{2^n}}{1-a^{2^{n+1}}} &= \sum \left[\frac{1}{1-a^{2^n}} - \frac{1}{1-a^{2^{n+1}}} \right]. \quad (a \neq 1) \end{aligned}$$

对于含有 $(-1)^n$ 的通项, 需要在裂项时变更符号, 例如

$$\sum \frac{(2n+1)(-1)^n}{n(n+1)} = \sum \left(\frac{(-1)^n}{n} + \frac{(-1)^n}{n+1} \right) = \sum \left[\frac{(-1)^n}{n} - \frac{(-1)^{n+1}}{n+1} \right].$$

根式裂项

$$\begin{aligned}\sum \frac{1}{\sqrt{n} + \sqrt{n+k}} &= \frac{1}{k} \sum \left[\sqrt{n+k} - \sqrt{n} \right], \quad (k \neq 0, \forall n: n+k > 0) \\ \sum \frac{1}{\sqrt{2n-1} + \sqrt{2n+1}} &= \frac{1}{2} \sum \left[\sqrt{2n+1} - \sqrt{2n-1} \right], \\ \sum \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}} &= \sum \left[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right], \\ \sum \frac{n - \sqrt{n^2-1}}{\sqrt{n(n+1)}} &= \sum \left[\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n-1}{n}} \right].\end{aligned}$$

阶乘裂项

$$\begin{aligned}\sum (n \cdot n!) &= \sum [(n+1)! - n!], \\ \sum \frac{n}{(n+1)!} &= \sum \left[\frac{1}{n!} - \frac{1}{(n+1)!} \right], \\ \sum \frac{1}{n!(n+2)} &= \sum \left[\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right], \\ \sum \frac{n+2}{n! + (n+1)! + (n+2)!} &= \sum \left[\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right], \\ \sum (n+1)n!! &= \sum [(n+2)!! - n!!], \\ \sum \frac{n+1}{(n+2)!!} &= \sum \left[\frac{1}{n!!} - \frac{1}{(n+2)!!} \right],\end{aligned}$$

差比数列前 n 项和 $\forall k, b \in \mathbf{C}, q \in \mathbf{C} \setminus \{1\}$:

$$\sum (kn+b)q^{n-1} = \sum \left[\left(\frac{k}{q-1}(n+1) + t \right) q^n - \left(\frac{k}{q-1}n + t \right) q^{n-1} \right],$$

其中

$$t = \frac{b}{q-1} - \frac{kq}{(q-1)^2}.$$

作为推论, 我们有等比数列前 n 项和

$$\sum a \cdot q^{n-1} = \frac{a}{q-1} \sum [q^n - q^{n-1}].$$

三角裂项

$$\begin{aligned}
\sum \sin \theta \cos (2n\theta + \theta) &= \sum [\sin 2(n+1)\theta - \sin 2n\theta] , \\
\sum \sin \theta \sin (2n\theta + \theta) &= \frac{1}{2} \sum [\cos 2n\theta - \cos 2(n+1)\theta] , \\
\sum \frac{\sin \theta}{\cos n\theta \cdot \cos (n+1)\theta} &= \sum [\tan (n+1)\theta - \tan n\theta] , \quad (\forall n : \cos n\theta \neq 0) \\
\sum \frac{\sin \theta}{\sin n\theta \cdot \sin (n+1)\theta} &= \sum [\cot n\theta - \cot (n+1)\theta] , \quad (\forall n : \sin n\theta \neq 0) \\
\sum 3^{n-1} \sin^3 \frac{\theta}{3^n} &= \frac{1}{4} \sum \left[3^n \sin \frac{\theta}{3^n} - 3^{n-1} \sin \frac{\theta}{3^{n-1}} \right] , \\
\sum \frac{1}{2^n} \tan \frac{\theta}{2^n} &= \sum \left[\frac{1}{2^n} \cot \frac{\theta}{2^n} - \frac{1}{2^{n-1}} \cot \frac{\theta}{2^{n-1}} \right] , \\
\prod \cos \frac{\theta}{2^n} &= 2^{-n} \prod \left[\sin \frac{\theta}{2^{n-1}} / \sin \frac{\theta}{2^n} \right] .
\end{aligned}$$

我们可以结合导数的线性性质来裂项求和, 例如

$$\sum \frac{1}{2^n} \tan \frac{\theta}{2^n} = -\frac{d}{d\theta} \ln \left| \prod \cos \frac{\theta}{2^n} \right| = -\frac{d}{d\theta} \ln \left| 2^{-n} \prod \left[\sin \frac{\theta}{2^{n-1}} / \sin \frac{\theta}{2^n} \right] \right| .$$

反三角裂项

$$\begin{aligned}
\sum \arctan \frac{1}{n^2 + n + 1} &= \sum [\arctan (n+1) - \arctan n] , \\
\sum \arctan \frac{1}{2n^2} &= \sum \left[\arctan \frac{n}{n+1} - \arctan \frac{n-1}{n} \right] .
\end{aligned}$$

其它裂项

$$\begin{aligned}
\sum \sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}} &= \sum \left[1 + \frac{1}{n} - \frac{1}{n+1} \right] , \\
\sum \frac{2}{\sqrt[3]{n^2 - 2n + 1} + \sqrt[3]{n^2 - 1} + \sqrt[3]{n^2 + 2n + 1}} &= \sum [\sqrt[3]{n+1} - \sqrt[3]{n}] , \\
\prod (1 + a^{2^n}) &= \prod \left[\frac{1 - a^{2^{n+1}}}{1 - a^{2^n}} \right] , \quad (a \neq 1) \\
\prod \frac{n^3 - 1}{n^3 + 1} &= \prod \left[\frac{n-1}{n+1} \right] \left[\frac{(n+1)^2 - (n+1) + 1}{n^2 - n + 1} \right] .
\end{aligned}$$

三角恒等式

在下面的一切恒等式中, 我们要求 α 和 β 使得对应的三角函数值有意义.

基本恒等式 $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin^2 \alpha + \cos^2 \alpha &= 1, & \tan \alpha \cdot \cot \alpha &= 1, \\ \tan^2 \alpha + 1 &= \sec^2 \alpha, & \cot^2 \alpha + 1 &= \csc^2 \alpha.\end{aligned}$$

诱导公式 I $\forall \alpha \in \mathbf{C}, k \in \mathbf{Z}$:

$$\begin{aligned}\sin(2k\pi + \alpha) &= \sin \alpha, & \cos(2k\pi + \alpha) &= \cos \alpha, \\ \tan(2k\pi + \alpha) &= \tan \alpha, & \cot(2k\pi + \alpha) &= \cot \alpha, \\ \sec(2k\pi + \alpha) &= \sec \alpha, & \csc(2k\pi + \alpha) &= \csc \alpha.\end{aligned}$$

诱导公式 II $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin(-\alpha) &= -\sin \alpha, & \cos(-\alpha) &= \cos \alpha, \\ \tan(-\alpha) &= -\tan \alpha, & \cot(-\alpha) &= -\cot \alpha, \\ \sec(-\alpha) &= -\sec \alpha, & \csc(-\alpha) &= \csc \alpha.\end{aligned}$$

诱导公式 III $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin(\pi + \alpha) &= -\sin \alpha, & \cos(\pi + \alpha) &= -\cos \alpha, \\ \tan(\pi + \alpha) &= \tan \alpha, & \cot(\pi + \alpha) &= \cot \alpha, \\ \sec(\pi + \alpha) &= -\sec \alpha, & \csc(\pi + \alpha) &= -\csc \alpha.\end{aligned}$$

诱导公式 IV $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin(\pi - \alpha) &= \sin \alpha, & \cos(\pi - \alpha) &= -\cos \alpha, \\ \tan(\pi - \alpha) &= -\tan \alpha, & \cot(\pi - \alpha) &= -\cot \alpha, \\ \sec(\pi - \alpha) &= \sec \alpha, & \csc(\pi - \alpha) &= -\csc \alpha.\end{aligned}$$

诱导公式 V $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin(\pi/2 + \alpha) &= \cos \alpha, & \cos(\pi/2 + \alpha) &= -\sin \alpha, \\ \tan(\pi/2 + \alpha) &= -\cot \alpha, & \cot(\pi/2 + \alpha) &= -\tan \alpha, \\ \sec(\pi/2 + \alpha) &= -\csc \alpha, & \csc(\pi/2 + \alpha) &= \sec \alpha.\end{aligned}$$

诱导公式 VI $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin(\pi/2 - \alpha) &= \cos \alpha, & \cos(\pi/2 - \alpha) &= \sin \alpha, \\ \tan(\pi/2 - \alpha) &= \cot \alpha, & \cot(\pi/2 - \alpha) &= \tan \alpha, \\ \sec(\pi/2 - \alpha) &= \csc \alpha, & \csc(\pi/2 - \alpha) &= \sec \alpha.\end{aligned}$$

和差公式 $\forall \alpha, \beta \in \mathbf{C}$:

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, & \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, \\ \tan(\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}, & \cot(\alpha \pm \beta) &= \frac{\cot \alpha \cot \beta \mp 1}{\cot \alpha \pm \cot \beta}, \\ \sec(\alpha \pm \beta) &= \frac{\sec \alpha \sec \beta}{1 \mp \tan \alpha \tan \beta}, & \csc(\alpha \pm \beta) &= \frac{\sec \alpha \sec \beta}{\tan \alpha \pm \tan \beta}.\end{aligned}$$

辅助角公式 $\forall \alpha \in \mathbf{C}, A, B, C \in \mathbf{C}$:

$$\begin{aligned}A \sin \alpha + B \cos \alpha &= \sqrt{A^2 + B^2} \sin(\alpha + \varphi), \\ A \sin^2 \alpha + B \cos^2 \alpha + C \sin \alpha \cos \alpha &= \frac{\sqrt{A^2 + B^2 + C^2 - 2AB}}{2} \sin(2\alpha + \psi) + \frac{A+B}{2},\end{aligned}$$

其中辅助角 φ 和 ψ 满足

$$(\cos \varphi, \sin \varphi) = (A, B) \quad \text{和} \quad (\cos \psi, \sin \psi) = (C, B - A).$$

半角公式 $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\tan \frac{\alpha}{2} &= \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}, \\ \cot \frac{\alpha}{2} &= \frac{\sin \alpha}{1 - \cos \alpha} = \frac{1 + \cos \alpha}{\sin \alpha}.\end{aligned}$$

倍角公式 $\forall \alpha \in \mathbf{C}$:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad , \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \quad ,$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \quad , \quad \cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha} \quad ,$$

$$\sin 3\alpha = -4 \sin^3 \alpha + 3 \sin \alpha \quad , \quad \cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha \quad ,$$

$$\tan 3\alpha = \frac{\tan^3 \alpha - 3 \tan \alpha}{3 \tan^2 \alpha - 1} \quad , \quad \cot 3\alpha = \frac{\cot^3 \alpha - 3 \cot \alpha}{3 \cot^2 \alpha - 1} \quad ,$$

$$\sin 4\alpha = \cos \alpha (-8 \sin^3 \alpha + 4 \sin \alpha) \quad , \quad \cos 4\alpha = 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1 \quad ,$$

$$\tan 4\alpha = \frac{-4 \tan^3 \alpha + 4 \tan \alpha}{\tan^4 \alpha - 6 \tan^2 \alpha + 1} \quad , \quad \cot 4\alpha = \frac{\cot^4 \alpha - 6 \cot^2 \alpha + 1}{4 \cot^3 \alpha - 4 \cot \alpha} \quad ,$$

$$\sin 5\alpha = 16 \sin^5 \alpha - 20 \sin^3 \alpha + 5 \sin \alpha \quad , \quad \cos 5\alpha = 16 \cos^5 \alpha - 20 \cos^3 \alpha + 5 \cos \alpha \quad ,$$

$$\tan 5\alpha = \frac{\tan^5 \alpha - 10 \tan^3 \alpha + 5 \tan \alpha}{5 \tan^4 \alpha - 10 \tan^2 \alpha + 1} \quad , \quad \cot 5\alpha = \frac{\cot^5 \alpha - 10 \cot^3 \alpha + 5 \cot \alpha}{5 \cot^4 \alpha - 10 \cot^2 \alpha + 1} \quad .$$

$\forall \alpha \in \mathbf{C} \quad , \quad n \in \mathbf{N}$:

$$\cos n\alpha = n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2n-2k-2)!!}{(2k)!!(n-2k)!} \cos^{n-2k} \alpha \quad ,$$

$$\sin 2n\alpha = \cos \alpha \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{(4n-2k-2)!!}{(2k)!!(2n-2k-1)!} \sin^{2n-2k-1} \alpha \quad ,$$

$$\sin (2n+1)\alpha = (2n+1) \sum_{k=0}^n (-1)^{n-k} \frac{(4n-2k)!!}{(2k)!!(2n-2k+1)!} \sin^{2n-2k+1} \alpha \quad ,$$

$$\begin{aligned} \tan n\alpha &= \frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k C_n^{2k+1} \tan^{2k+1} \alpha}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_n^{2k} \tan^{2k} \alpha} \quad , \\ \cot n\alpha &= \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_n^{2k} \cot^{n-2k} \alpha}{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k C_n^{2k+1} \cot^{n-2k-1} \alpha} \quad . \end{aligned}$$

降幂公式 $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin^2 \alpha &= \frac{1 - \cos 2\alpha}{2} , & \cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2} , \\ \tan^2 \alpha &= \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha} , & \cot^2 \alpha &= \frac{1 + \cos 2\alpha}{1 - \cos 2\alpha} , \\ \sec^2 \alpha &= \frac{2}{1 + \cos 2\alpha} , & \csc^2 \alpha &= \frac{2}{1 - \cos 2\alpha} .\end{aligned}$$

$\forall \alpha \in \mathbf{C}, n \in \mathbf{N}$:

$$\begin{aligned}\sin^{2n+1} \alpha &= \frac{1}{(-4)^n} \sum_{k=0}^n (-1)^k C_{2n+1}^k \sin(2n-2k+1)\alpha , \\ \cos^{2n+1} \alpha &= \frac{1}{4^n} \sum_{k=0}^n C_{2n+1}^k \cos(2n-2k+1)\alpha , \\ \sin^{2n} \alpha &= \frac{(2n)!}{4^n (n!)^2} + \frac{2}{(-4)^n} \sum_{k=0}^{n-1} (-1)^k C_{2n}^k \cos(2n-2k)\alpha , \\ \cos^{2n} \alpha &= \frac{(2n)!}{4^n (n!)^2} + \frac{2}{4^n} \sum_{k=0}^{n-1} C_{2n}^k \cos(2n-2k)\alpha .\end{aligned}$$

两元积化和差公式 $\forall \alpha, \beta \in \mathbf{C}$:

$$\begin{aligned}\sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] , \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] , \\ \sin \alpha \sin \beta &= -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)] .\end{aligned}$$

两元和差化积公式 $\forall \alpha, \beta \in \mathbf{C}$:

$$\begin{aligned}\sin \alpha \pm \sin \beta &= 2 \sin \frac{\alpha \pm \beta}{2} \cos \frac{\alpha \mp \beta}{2} , \\ \cos \alpha + \cos \beta &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} , \\ \cos \alpha - \cos \beta &= -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} .\end{aligned}$$

万能代换公式 $\forall \alpha \in \mathbf{C}$:

$$\begin{aligned}\sin \alpha &= \frac{2 \tan (\alpha / 2)}{1 + \tan^2 (\alpha / 2)} \quad , \quad \cos \alpha = \frac{1 - \tan^2 (\alpha / 2)}{1 + \tan^2 (\alpha / 2)} \quad , \quad \tan \alpha = \frac{2 \tan (\alpha / 2)}{1 - \tan^2 (\alpha / 2)} \quad , \\ \cot \alpha &= \frac{1 - \tan^2 (\alpha / 2)}{2 \tan (\alpha / 2)} \quad , \quad \sec \alpha = \frac{1 + \tan^2 (\alpha / 2)}{1 - \tan^2 (\alpha / 2)} \quad , \quad \csc \alpha = \frac{1 + \tan^2 (\alpha / 2)}{2 \tan (\alpha / 2)} \quad .\end{aligned}$$

三角平方差公式 $\forall \alpha, \beta \in \mathbf{C}$:

$$\begin{aligned}\sin^2 \alpha - \sin^2 \beta &= \sin (\alpha + \beta) \sin (\alpha - \beta) \quad , \\ \cos^2 \alpha - \sin^2 \beta &= \cos (\alpha + \beta) \cos (\alpha - \beta) \quad .\end{aligned}$$

切化弦公式 $\forall \alpha, \beta \in \mathbf{C}$:

$$\begin{aligned}\tan \alpha \pm \tan \beta &= \frac{\sin (\alpha \pm \beta)}{\cos \alpha \cos \beta} \quad , \\ \cot \alpha \pm \cot \beta &= \frac{\sin (\beta \pm \alpha)}{\sin \alpha \sin \beta} \quad , \\ \tan \alpha \pm \cot \beta &= \pm \frac{\cos (\beta \mp \alpha)}{\sin \beta \cos \alpha} \quad .\end{aligned}$$

内角恒等式 $\forall \alpha, \beta, \gamma \in \mathbf{C} : \alpha + \beta + \gamma = \pi \implies$

$$\begin{aligned}\sin \alpha + \sin \beta + \sin \gamma &= 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \quad , \\ \cos \alpha + \cos \beta + \cos \gamma &= 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + 1 \quad , \\ \sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 4 \sin \alpha \sin \beta \sin \gamma \quad , \\ \tan \alpha + \tan \beta + \tan \gamma &= \tan \alpha \tan \beta \tan \gamma \quad , \\ \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} &= \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \quad , \\ \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} &= 1 \quad , \\ \cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha &= 1 \quad , \\ \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma &= 2 + 2 \cos \alpha \cos \beta \cos \gamma \quad , \\ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= 1 - 2 \cos \alpha \cos \beta \cos \gamma \quad .\end{aligned}$$

Taylor 级数

Taylor 公式 任意阶可导函数 $f(x)$ 在 $x = x_0$ 处的 Taylor 展开式为

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x),$$

其中

$$R_n(x) = \int_{x_0}^x \frac{f^{(n+1)}(t)}{n!} (x - t)^n dt.$$

当余项 $R_n(x)$ 对一切定义域内的 x 均趋于零, 即

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

时, 我们有如下 Taylor 级数.

自然指数 $\forall x \in \mathbf{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

自然对数 $\forall x \in (-1, 1]$:

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

三角函数 $\forall x \in \mathbf{R}$:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots,$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots,$$

$\forall x \in (-\pi/2, \pi/2)$:

$$\tan x = \sum_{n=0}^{\infty} \frac{|B_{2n}| (16^n - 4^n) x^{2n-1}}{(2n)!} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots,$$

$$\sec x = \sum_{n=0}^{\infty} \frac{E_{2n} x^{2n}}{(2n)!} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots,$$

$\forall x \in (-\pi, \pi) :$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{4^n |B_{2n}| x^{2n-1}}{(2n)!} = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \cdots ,$$

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{|B_{2n}| (4^n - 2) x^{2n-1}}{(2n)!} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots .$$

双曲函数 $\forall x \in \mathbf{R} :$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots ,$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots ,$$

$\forall x \in (-\pi/2, \pi/2) :$

$$\tanh x = \sum_{n=0}^{\infty} \frac{B_{2n} (16^n - 4^n) x^{2n-1}}{(2n)!} = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots ,$$

$$\operatorname{sech} x = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n} x^{2n}}{(2n)!} = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \cdots ,$$

$\forall x \in (-\pi, \pi) :$

$$\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{4^n B_{2n} x^{2n-1}}{(2n)!} = \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \cdots ,$$

$$\operatorname{csch} x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{B_{2n} (4^n - 2) x^{2n-1}}{(2n)!} = \frac{1}{x} - \frac{1}{6}x + \frac{7}{360}x^3 - \frac{31}{15120}x^5 + \cdots .$$

反三角函数 $\forall x \in (-1, 1) :$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n-1)!! x^{2n+1}}{(2n)!! (2n+1)} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots ,$$

$\forall x \in [-1, 1] :$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

π 的 Leibniz 展开式

$$\pi = 4 \arctan 1 = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots .$$

反双曲函数 $\forall x \in (-1, 1)$:

$$\operatorname{arsinh} x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!! x^{2n+1}}{(2n)!! (2n+1)} = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots ,$$

$\forall x \in (-1, 1)$:

$$\operatorname{artanh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots .$$

根据反双曲函数的定义我们有

$$\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x} ,$$

其中用 x 代换 $(1+x)/(1-x)$ 可以得到下面这个应用更广泛的表达式:

快速对数展开式 $\forall x \in (0, +\infty)$:

$$\ln x = \sum_{n=0}^{\infty} \frac{2}{2n+1} \left(\frac{x-1}{x+1} \right)^{2n+1} = 2 \left(\frac{x-1}{x+1} \right) + \frac{2}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{2}{5} \left(\frac{x-1}{x+1} \right)^5 + \cdots .$$



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