Public Key Cryptography

- Two keys, one to encrypt, another to decrypt
  - Alice uses Bob's **public key** to encrypt
  - Only Bob's **private key** decrypts the message
- Based on "trap door, one way function"
  - "One way" means easy to compute in one direction, but hard to compute in other direction
  - Example: Given p and q, product $N = pq$ easy to compute, but hard to find p and q from N
  - "Trap door" is used when creating key pairs

Knapsack Problem

- Given a set of n weights $W_0, W_1, \ldots, W_{n-1}$ and a sum S, find $a_i \in \{0,1\}$ so that

  $$S = a_0W_0 + a_1W_1 + \ldots + a_{n-1}W_{n-1}$$

  (technically, this is the *subset sum* problem)
- Example
  - Weights (62,93,26,52,166,48,91,141)
  - Problem: Find a subset that sums to $S = 302$
  - Answer: $62 + 26 + 166 + 48 = 302$
- The (general) knapsack is NP-complete

Knapsack Problem

- **General knapsack** (GK) is hard to solve
- But **superincreasing knapsack** (SIK) is easy
- SIK — each weight greater than the *sum of all previous weights*
- **Example**
  - Weights $(2,3,7,14,30,57,120,251)$
  - Problem: Find subset that sums to $S = 186$
  - Work from largest to smallest weight
  - Answer: $120 + 57 + 7 + 2 = 186$

Knapsack Cryptosystem

1. Generate superincreasing knapsack (SIK)
2. Convert SIK to “general” knapsack (GK)
3. **Public Key:** GK
4. **Private Key:** SIK and conversion factor

- Goal…
  - Easy to encrypt with GK
  - With private key, easy to decrypt (solve SIK)
  - Without private key, Trudy has no choice but to try to solve GK

Knapsack Weakness

- Trapdoor: Convert SIK into "general" knapsack using modular arithmetic
- One-way: General knapsack easy to encrypt, hard to solve; SIK easy to solve
- This knapsack cryptosystem is **insecure**
  - Broken in 1983 with Apple II computer
  - The attack uses lattice reduction
- "General knapsack" is not general enough!
  - This special case of knapsack is easy to break

RSA

- Invented by Clifford Cocks (GCHQ) and **R**ivest, **S**hamir, and **A**dleman (MIT)
  - RSA is the *gold standard* in public key crypto
- Let p and q be two large prime numbers
- Let $N = pq$ be the **modulus**
- Choose e relatively prime to $(p-1)(q-1)$
- Find d such that $ed = 1 \bmod (p-1)(q-1)$
- **Public key** is (N,e)
- **Private key** is d

RSA

- Message M is treated as a number
- To encrypt M we compute

  $$C = M^e \bmod N$$
- To decrypt ciphertext C, we compute

  $$M = C^d \bmod N$$
- Recall that e and N are public
- If Trudy can factor $N = pq$, she can use e to easily find d since $ed = 1 \bmod (p-1)(q-1)$
- So, **factoring the modulus breaks RSA**
  - Is factoring the only way to break RSA?

Simple RSA Example

- Example of ***textbook*** RSA
  - Select "large" primes $p = 11$, $q = 3$
  - Then $N = pq = 33$ and $(p - 1)(q - 1) = 20$
  - Choose $e = 3$ (relatively prime to 20)
  - Find d such that $ed = 1 \bmod 20$
    - We find that $d = 7$ works
- **Public key:** $(N, e) = (33, 3)$
- **Private key:** $d = 7$

Simple RSA Example

- Public key: $(N, e) = (33, 3)$
- Private key: $d = 7$
- Suppose message to encrypt is $M = 8$
- Ciphertext C is computed as

  $C = M^e \bmod N = 8^3 = 512 = 17 \bmod 33$
- Decrypt C to recover the message M by

  $M = C^d \bmod N = 17^7 = 410{,}338{,}673$

  $= 12{,}434{,}505 * 33 + 8 = 8 \bmod 33$

More Efficient RSA (1)

- Modular exponentiation example
  - $5^{20} = 95367431640625 = 25 \bmod 35$
- A better way: **repeated squaring**
  - $20 = 10100$ base 2
  - $(1, 10, 101, 1010, 10100) = (1, 2, 5, 10, 20)$
  - Note that $2 = 1 \cdot 2$, $5 = 2 \cdot 2 + 1$, $10 = 2 \cdot 5$, $20 = 2 \cdot 10$
  - $5^1 = 5 \bmod 35$
  - $5^2 = (5^1)^2 = 5^2 = 25 \bmod 35$
  - $5^5 = (5^2)^2 \cdot 5^1 = 25^2 \cdot 5 = 3125 = 10 \bmod 35$
  - $5^{10} = (5^5)^2 = 10^2 = 100 = 30 \bmod 35$
  - $5^{20} = (5^{10})^2 = 30^2 = 900 = 25 \bmod 35$
- No huge numbers and it's efficient!

More Efficient RSA (2)

- Use $e = 3$ for all users (but not same N or d)
  - \+ Public key operations only require 2 multiplies
  - o Private key operations remain expensive
  - \- If $M < N^{1/3}$ then $C = M^e = M^3$ and cube root attack
  - \- For any M, if C_1, C_2, C_3 sent to 3 users, cube root attack works (uses Chinese Remainder Theorem)
- Can prevent cube root attack by padding message with random bits
- Note: $e = 2^{16} + 1$ also used ("better" than $e = 3$)

Diffie-Hellman

Diffie-Hellman Key Exchange

- Invented by Williamson (GCHQ) and, independently, by D and H (Stanford)
- A "key exchange" algorithm
  - Used to establish a shared symmetric key
  - *Not* for encrypting or signing
- Based on **discrete log** problem
  - **Given:** g, p, and $g^k \bmod p$
  - **Find:** exponent k

Diffie-Hellman

- Let p be prime, let g be a **generator**
  - For any $x \in \{1,2,\ldots,p-1\}$ there is n s.t. $x = g^n \bmod p$
- Alice selects her private value a
- Bob selects his private value b
- Alice sends $g^a \bmod p$ to Bob
- Bob sends $g^b \bmod p$ to Alice
- Both compute shared secret, $g^{ab} \bmod p$
- Shared secret can be used as symmetric key

Diffie-Hellman

- **Public:** g and p
- **Private:** Alice's exponent a, Bob's exponent b

$$g^a \bmod p \longrightarrow$$

$$\longleftarrow g^b \bmod p$$

Alice, a — Bob, b

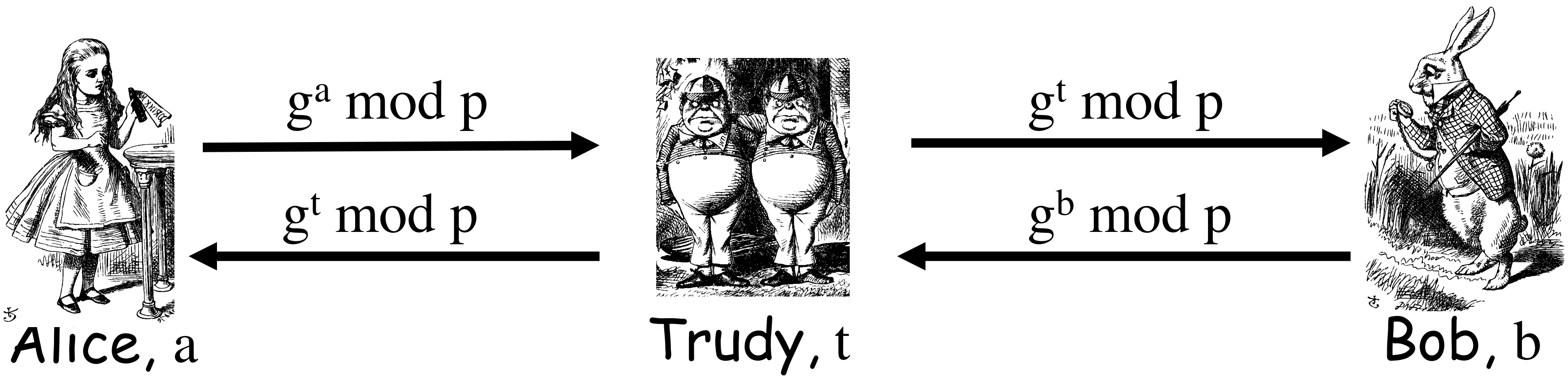
- Alice computes $(g^b)^a = g^{ba} = g^{ab} \bmod p$
- Bob computes $(g^a)^b = g^{ab} \bmod p$
- They can use $K = g^{ab} \bmod p$ as symmetric key

Diffie-Hellman

- Suppose Bob and Alice use Diffie-Hellman to determine symmetric key $K = g^{ab} \bmod p$
- Trudy can see $g^a \bmod p$ and $g^b \bmod p$
  - But… $g^a g^b \bmod p = g^{a+b} \bmod p \neq g^{ab} \bmod p$
- If Trudy can find a or b, she gets K
- If Trudy can solve **discrete log** problem, she can find a or b

Diffie-Hellman

- Subject to man-in-the-middle (MiM) attack



- Trudy shares secret $g^{at} \bmod p$ with Alice
- Trudy shares secret $g^{bt} \bmod p$ with Bob
- Alice and Bob don't know Trudy is MiM

Elliptic Curve Cryptography

Elliptic Curve Crypto (ECC)

- "Elliptic curve" is **not** a cryptosystem
- Elliptic curves provide different way to do the math in public key system
- Elliptic curve versions of DH, RSA, …
- Elliptic curves are more efficient
    - Fewer bits needed for same security
    - But the operations are more complex, yet it is a big "win" overall

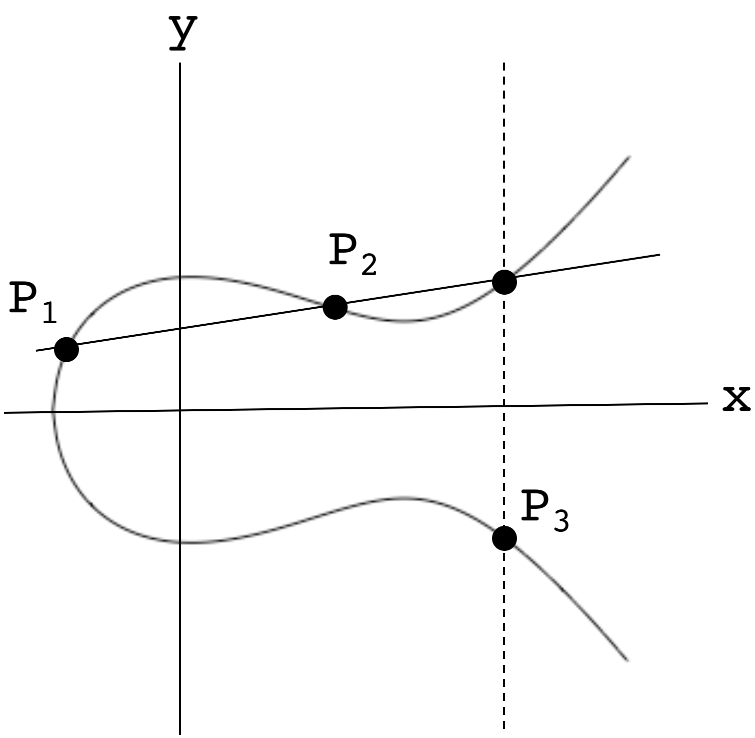
What is an Elliptic Curve?

- An elliptic curve E is the graph of an equation of the form

  $$y^2 = x^3 + ax + b$$

- Also includes a "point at infinity"
- What do elliptic curves look like?

Elliptic Curve Picture



- Consider elliptic curve

  $$E: y^2 = x^3 - x + 1$$

- If P_1 and P_2 are on E, we can define addition,

  $$P_3 = P_1 + P_2$$

  as shown in picture
- Addition group (abelian group)

https://en.wikipedia.org/wiki/Elliptic_curve

Points on Elliptic Curve

- Consider $y^2 = x^3 + 2x + 3 \pmod 5$

  $x = 0 \Rightarrow y^2 = 3 \Rightarrow$ no solution (mod 5)

  $x = 1 \Rightarrow y^2 = 6 = 1 \Rightarrow y = 1,4 \pmod 5$

  $x = 2 \Rightarrow y^2 = 15 = 0 \Rightarrow y = 0 \pmod 5$

  $x = 3 \Rightarrow y^2 = 36 = 1 \Rightarrow y = 1,4 \pmod 5$

  $x = 4 \Rightarrow y^2 = 75 = 0 \Rightarrow y = 0 \pmod 5$

- Then points on the elliptic curve are

  (1,1) (1,4) (2,0) (3,1) (3,4) (4,0)

  and the point at infinity: ∞

Elliptic Curve Math

- Addition on: $y^2 = x^3 + ax + b \pmod{p}$

  $P_1=(x_1,y_1), P_2=(x_2,y_2)$

  $P_1 + P_2 = P_3 = (x_3,y_3)$ where

  $x_3 = m^2 - x_1 - x_2 \pmod{p}$

  $y_3 = m(x_1 - x_3) - y_1 \pmod{p}$

  And $m = (y_2-y_1)*(x_2-x_1)^{-1} \bmod p$, if $P_1 \neq P_2$

  $m = (3x_1{}^2+a)*(2y_1)^{-1} \bmod p$, if $P_1 = P_2$

Elliptic Curve Addition

- Consider $y^2 = x^3 + 2x + 3 \pmod 5$. Points on the curve are $(1,1)$ $(1,4)$ $(2,0)$ $(3,1)$ $(3,4)$ $(4,0)$ and ∞
- What is $(1,4) + (3,1) = P_3 = (x_3,y_3)$?

$$m = (1-4)*(3-1)^{-1} = -3*2^{-1}$$
$$= 2(3) = 6 = 1 \pmod 5$$
$$x_3 = 1 - 1 - 3 = 2 \pmod 5$$
$$y_3 = 1(1-2) - 4 = 0 \pmod 5$$

- On this curve, $(1,4) + (3,1) = (2,0)$

ECC Diffie-Hellman

- **Public:** Elliptic curve and point (x,y) on curve
- **Private:** Alice's A and Bob's B

$A(x,y)$ →

← $B(x,y)$

Alice, A

Bob, B

- Alice computes $A(B(x,y))$
- Bob computes $B(A(x,y))$
- These are the same since $AB(x,y) = BA(x,y)$

ECC Diffie-Hellman

- Public: Curve $y^2 = x^3 + 7x + b \pmod{37}$ and point $(2,5) \Rightarrow b = 3$
- Alice's private: $A = 4$
- Bob's private: $B = 7$
- Alice sends Bob: $4(2,5) = (7,32)$
- Bob sends Alice: $7(2,5) = (18,35)$
- Alice computes: $4(18,35) = (22,1)$
- Bob computes: $7(7,32) = (22,1)$

Larger ECC Example

- Example from Certicom ECCp-109
  - Challenge problem, solved in 2002
- Curve E: $y^2 = x^3 + ax + b \pmod{p}$
- Where

  p = 564538252084441556247016902735257

  a = 321094768129147601892514872825668

  b = 430782315140218274262276694323197

ECC Example

- The following point `P` is on the curve `E`

`(x,y) = (97339010987059066523156133908935, 149670372846169285760682371978898)`

- Let `k = 281183840311601949668207954530684`
- The `kP` is given by

`(x,y) = (44646769697405861057630861884284, 522968098895785888047540374779097)`

- And this point is also on the curve `E`

Really Big Numbers!

- Numbers are big, but not big enough
  - ECCp-109 bit (32 digit) solved in 2002
- Today, ECC DH needs bigger numbers
- But RSA needs way bigger numbers
  - Minimum RSA modulus today is 1024 bits
  - That is, more than 300 decimal digits
  - That's about 10x the size in ECC example
  - And 2048 bit RSA modulus is common…