1.

$X_n \sim Ge(\frac{\lambda}{n})$

$Y_n = \frac{1}{n} X_n$

证明 $\{Y_n\}$ 依分布收敛到 Y，其中 $Y \sim Exp(\lambda)$

解：

pf: 已知 $X_n \sim Ge(\frac{\lambda}{n})$

则 $P(X_n = k) = (1-\frac{\lambda}{n})^{k-1} \frac{\lambda}{n}$

$Y_n = \frac{1}{n} X_n$

$P(\cancel{Y_n = k}) = \cancel{P(\frac{1}{n} X_n = k) = P(X_n = nk)}$

$P(Y_n = \frac{k}{n}) = P(X_n = k) = (1-\frac{\lambda}{n})^{k-1} \frac{\lambda}{n}$.

则对应分布函数 $F_n = \sum_{k'=1}^{k} (1-\frac{\lambda}{n})^{k'-1} \frac{\lambda}{n}$

$= \frac{\lambda}{n} \cdot \frac{1-(1-\frac{\lambda}{n})^{k}}{1-(1-\frac{\lambda}{n})} = 1-(1-\frac{\lambda}{n})^{\frac{k}{n} \cdot n}$

即 ~~$F_n(y)$ $F(Y_n = y_n) =$~~

$F_n(y) = 1-(1-\frac{\lambda}{n})^{y \cdot n}$

当 $n \to \infty$ $F_n(y)$ 依分布收敛到 $F(y) = 1-e^{-\lambda y}$

即 $Y \sim exp(\lambda)$

2.

$X_n \xrightarrow{L} X \quad Y_n \xrightarrow{P} c$

(a) $X_n+Y_n \xrightarrow{L} X+c$

(b) $X_nY_n \xrightarrow{L} cX$

(c) $X_n/Y_n \xrightarrow{L} X/c$，这里c非零

解：pf:

(a) 设 x_n 的~~实~~密度函数为 $f_n(x)$

Y_n的密度函数为 $g_n(y)$.

$$f_{X_n+Y_n}(z)=\int_{-\infty}^{+\infty} f_n(x)g_n(z-x)dx$$

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f_n(x)g_n(z-x)\,dx=\lim_{n\to\infty} f_{X_n+Y_n}(z).$$

$$=\int_{-\infty}^{+\infty} f(x)g(z-x)\,dx=f_{X+c}(z).$$

(b).

$$\cancel{f_{X_nY_n}(z)=\int_{-\infty}^{+\infty} f_n(x) f_{Y_n} f_{X_n}}$$

$$\lim_{n\to\infty} f_{X_nY_n}(z)=\lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{f_{X_n}(t)}{|t|}f_{Y_n}\left(\frac{z}{t}\right)dt$$

$$=\int_{-\infty}^{+\infty}\frac{f_X(t)}{|t|}f_Y\left(\frac{z}{t}\right)dt=f_{cX}(z)$$

(c)、同理 $\lim_{n\to\infty} f_{X_n/Y_n}(z)=\lim_{n\to\infty}\int_{-\infty}^{+\infty} t f_{X_n}(zt) f_{Y_n}(t)\,dt$

$$=\int_{-\infty}^{+\infty} t f_X(zt) f_Y(t)\,dt=f_{X/c}(z)$$

3.

求 $\sqrt{\frac{n}{n+1}}(X_{n+1}-\overline{X})/S_n$ 的分布。

$X_{n+1}-\overline{X}\sim N(0,\sigma^2+\frac{\sigma^2}{n})=N(0,\sigma^2(1+\frac{1}{n}))$

$\frac{X_{n+1}-\overline{X}}{\sigma\sqrt{1+\frac{1}{n}}}\sim N(0,1)$ $\frac{S_n}{\sigma}\sqrt{n-1}\sim\chi^2_{n-1}$

解:

$$\frac{\sqrt{n}}{\sqrt{n+1}}\frac{X_{n+1}-\overline{X}}{S_n}=\frac{\sqrt{\frac{n}{n+1}}\frac{X_{n+1}-\overline{X}}{\sigma\sqrt{1+\frac{1}{n}}}}{\sqrt{\frac{(n-1)S_n^2}{\sigma^2}/n-1}}$$

分子服从 $N(0,1)$ 分母服从 χ^2_{n-1}，且二者独立.

则

上式服从 t_{n-1} 分布

4. $\dfrac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{\sigma\sqrt{\frac{\alpha^2}{m}+\frac{\beta^2}{n}}} \sim N(0,1)$ (由正态分布的再生性可知)

$$\frac{(m-1)S_{1m}^2+(n-1)S_{2n}^2}{\sigma^2} \sim \chi^2_{m+n-2}$$

$$T=\frac{\dfrac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{\sigma\sqrt{\frac{\alpha^2}{m}+\frac{\beta^2}{n}}}}{\sqrt{\dfrac{(m-1)S_{1m}^2+(n-1)S_{2n}^2}{\sigma^2(m+n-2)}}} \sim t_{m+n-2}$$

1. (1) 已知 $Y\sim\chi^2_k$ 则 $E(\sqrt{Y})=\sqrt{2}\dfrac{\Gamma(k+1/2)}{\Gamma(k/2)}$

$$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1}$$

$$E\left(\frac{\sqrt{n-1}\,S}{\sigma}\right)=\frac{\sqrt{2}\,\Gamma(n/2)}{\Gamma(n-1/2)}$$

$$E(S)=\frac{\sqrt{2}\,\sigma\,\Gamma(n/2)}{\sqrt{n-1}\,\Gamma(n-1/2)}$$

(2) 已知 $\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1}$

则 $Var\left(\dfrac{(n-1)S^2}{\sigma^2}\right)=2\cdot(n-1)$

$$\Rightarrow Var\,S^2=\frac{2(n-1)}{(n-1)^2}\sigma^4=\frac{2\sigma^4}{n-1}$$

6.

$E(\hat{\theta}_2-\sigma^2)^2-E(\hat{\theta}_1-\sigma^2)^2$

$=E(\hat{\theta}_2^2-\cancel{2\hat{\theta}_2\sigma^2}\,2\hat{\theta}_2\sigma^2+\sigma^4)-E(\hat{\theta}_1^2-2\hat{\theta}_1\sigma^2+\sigma^4)$

$=\cancel{E(\hat{\theta}_2^2-\hat{\theta}_1}\,E(\hat{\theta}_2^2)-2\sigma^2E(\hat{\theta}_2)-E(\hat{\theta}_1^2)-2\sigma^2E(\hat{\theta}_1)$

$=Var(\hat{\theta}_2)+[E(\hat{\theta}_2)]^2-2\sigma^2E(\hat{\theta}_2)-E(\hat{\theta}_1)^2-Var(\hat{\theta}_2)-2\sigma^2E(\hat{\theta}_1)$

$=\left[\frac{(n-1)^2}{(n+1)^2}-1\right]\left(Var(\hat{\theta}_1)+E^2(\hat{\theta}_1)\right)-2\sigma^2\left(\frac{n-1}{n+1}E(\hat{\theta}_1)-E(\hat{\theta}_1)\right)$

$Var(\hat{\theta}_1)=\frac{2\sigma^4}{n-1}$ (上题结果)

$E(\hat{\theta}_1)=\sigma^2$ (无偏估计的定义)

代入上式

~~得~~ 上式最终得：

$$E(\hat{\theta}_2-\sigma^2)^2-E(\hat{\theta}_1-\sigma^2)^2=\frac{-4\sigma^4}{n^2-1}<0$$

恒成立

因此 $\hat{\theta}_2$ 的均方误差更小。

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