Networks: Fall 2023	Homework 7
David Easley and Yian Yin	Due 3:30pm, Thursday, November 9, 2023

Homework solutions should be submitted by upload to Gradescope, which can be accessed via Canvas. The file you upload **must be typed and submitted in PDF format**. Handwritten assignments will not be graded. However, you can draw graphs and insert them into your pdf. You can create a separate file with the solutions (you don't need to repeat the questions); it is fine to create the homework in any format provided it's typed and handed in as a single PDF file. When you upload your pdf to Gradescope be sure to assign your answers to the correct question.

To be eligible for full credit, your homework must come in by 3:30pm Thursday. We will also accept late homeworks after 3:30pm Thursday until 3:30pm Friday for a deduction of 10% of the total number of points available. Gradescope will stop accepting homework uploads after 3:30pm Friday; after this point we can only accept late homework when it is accompanied by a University-approved reason that is conveyed to and **approved by the TA in charge of this homework prior to the due date of the homework**. (These include illness, family emergencies, SDS accommodations and travel associated with university activities.)

The TA in charge of this homework is Utku Umur Acikalin (ua45@cornell.edu).

Reading: The questions below are primarily based on the material in Chapters 17 and 18.

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(1) [6 points]

In this question we will consider a good with a network effect. Consumers are named using real numbers between 0 and 1. The reservation price for consumer x when a z fraction of the population uses the product is given by the formula $r(x)f(z)$, where $r(x) = 9 - 9x$ and $f(z) = z$. Suppose that the good is sold at a price of 2.

(a) What are the possible equilibrium fractions of the population purchasing the good?

(b) Which of the equilibria you found in part (a) are stable? Explain.

Solution: (a) The equilibrium fractions are 0 and solutions to $9x^2 - 9x + 2 = 0$. These are $\frac{1}{3}$ and $\frac{2}{3}$. No explanation is necessary. (b) 0 and $\frac{2}{3}$ are stable and $\frac{1}{3}$ is unstable. 0 is stable as the amount the critical person is willing to pay if that person is x with $0 < x < 1/3$ is less than the price, so the fraction purchasing the good will fall. A similar argument shows that the amount the critical person is willing to pay if that person x is $1/3 < x < 2/3$ is greater than the price. These facts also explain instability of $\frac{1}{3}$. Finally, for $x > 2/3$ the amount the critical person is willing to pay is less than the price. These facts also explain stabiity of $\frac{2}{3}$. An alternative explanation can be seen by graphing the quadratic rf and the price.

(2) [6 points]

Consider a good that has network effects in the sense of our model from Chapter 17. Consumers are named using real numbers between 0 and 1. The reservation price for consumer x when a z fraction of the population uses the product is given by the formula $r(x)f(z)$, where $r(x) = 6(1 - x)$ and $f(z) = 2z$. Suppose that the good is sold at price of 3/2.

(a) What are the possible equilibrium fractions of the population purchasing the good?

(b) Suppose that the fraction of potential users who are actually using the product is 1/2. What will happen to the fraction of users over time? What will this fraction converge to?

(c) Suppose that the fraction of users has converged as in part (b). Now the company selling this good is considering a price increase. What is the maximum price, call it p^*, the company could charge so that there would be an equilibrium fraction of users that is positive? Do you think that it would be a good idea for the company to increase the price to p^*? Briefly explain why or why not?

Solution: (a) 0, $(2 - \sqrt{2})/4 \approx 0.146$ and $(2 + \sqrt{2})/4 \approx 0.854$. (b) It will converge to 0.854. (c) $p^* = 3$. This is not a good idea. The only equilibrium fractions of users for a price of 3 are 0 and 1/2. The fraction will decline from 0.854 toward 1/2, but if anything happens to push it below 1/2 it will fall to 0.

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(3) [7 points] In the 1990s, sociologists formalized the idea of 'friendship paradox' in networks: on average an individual's friends have more friends than that individual themselves. We can consider a version of this observation in the directed network: on average the pages you link to (the people you follow) have more in-links (followers) than you. In this question, we will look at a specific case of this paradox. For simplicity we will consider a network of 6 webpages, where each page creates just one outbound link (Fig. 1). x is a page randomly selected from the network and y is the page x links to.

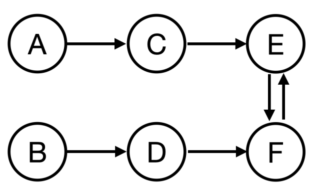

Figure 1: Q3: a network of 6 webpages.

(a) What is the probability for node x to have exactly k in-links in the network, for $k = 0, 1, 2$ respectively? Give an explanation for your answer.

(b) What is the probability for node y to have exactly k in-links in the network, for $k = 0, 1, 2$ respectively? Give an explanation for your answer.

(c) Does y have more or fewer average in-links than x? Give an explanation for your answer.

(d) Do you think your result in (c) will hold for other networks? Give an intuition for why (rigorous mathematical proof is not necessary).

Solution: (a) $f(0) = f(1) = f(2) = 1/3$. (b) $f(0) = 0, f(1) = 1/3, f(2) = 2/3$. (c): y has 1.667 in-links on average, which is higher than x which has 1 in-link on average. (d) Yes, since pages with high in-degrees are more likely to be sampled as y.

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(4) [7 points] You've been talking with the data scientists at a small Web site that hosts news articles, and they've been analyzing the number of views different news articles receive. They change the articles each day, and so the view counts they're looking at are for the articles posted in a given day. (As a simplifying assumption to make this question more tractable, we'll assume that even though the articles change from day to day, the behavior of the views is basically the same each day.) They believe that the views follow a power law, but they are not sure about the exponent of this power law. In particular they think that for some constant c, the number of articles each day that receive k views is c/k^α.

(a) They share with you the fact that each day, there are 8000 articles that receive 25 views, and 2000 articles that receive 50 views. With their belief that the number of articles each day that receive k views is c/k^α, can you estimate the exponent α? Give an explanation for your answer.

(b) Can you estimate the number of articles that receive 200 views? Give an explanation for your answer.

(c) Power law distribution has a nice property that it is scale-free – if we upscale or downscale a power law distribution by some magnitude, we will still observe a similar distribution. As an illustrative example, let us consider a new index – the *popularity index* (p) of news article, which is approximately half of the raw view counts v. The exact relationship between popularity p and view count v is defined as

$$p = \begin{cases} v/2, & v = 0, 2, 4, \cdots \\ (v-1)/2, & v = 1, 3, 5, \cdots \end{cases} \tag{1}$$

Now let us consider $h(p)$, which is defined as the fraction of articles with a popularity index value p. Given that the number of articles each day that receive k views is c/k^α, can you write $h(p)$? You may express your answer in terms of c and α.

(d) Given that $h(p)$ approximately follows another power law d/p^{β}, can you determine the parameters d and β? You may express your answer in terms of c and α.
Hint: For question (d) we are primarily interested in popular articles with high k, where you can use the approximation $x^{-\alpha} \approx (x+1)^{-\alpha}$.

Solution: (a) $\alpha = 2$. (b) 125 articles. (c) Note that an article has popularity p if and only if it has view count $2p$ or $2p+1$. Hence $h(p) = f(2p) + f(2p+1) = c[(2p)^{-\alpha} + (2p+1)^{-\alpha}]$. (d) $h(p) \approx c[(2p)^{-\alpha} + (2p)^{-\alpha}] = 2c(2p)^{-\alpha} = 2^{1-\alpha}cp^{-\alpha}$, hence $d = 2^{1-\alpha}c$ and $\beta = \alpha$.