Week 10 Recap

Monday October 23 and Wednesday October 25: Moment Generating functions, functions of a random variable.

The {\em generating function}, $M_X : t \mapsto M_X(t)$ of a random variable X is the expectation of the random variable \(e^{tX\}\), that is, $M_X(t) = E(e^{tX})$, for any real t for which this expression is finite (because e^{tX} is positive, $E(e^{tX})$ is either finite or equal to $+\infty$).

Let X be a discrete random variable taking values $x_1, x_2, \dots, x_k, \dots$ with mass distribution function $p_{X,x_k} = P(X = x_k)$. The moment generating function of X is $M_X(t) = \sum p_{X,x_k} e^{tx_k}$. This is a finite sum or a series.

If X is a continuous random variable with density function f_X then $M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx$.

- If X is Bernoulli p then $M_X(t) = (1-p) + pe^t = 1 + p(e^t - 1)$.
- If X is binomial n, p then

$$M_X(t) = \sum_0^n e^{tk} p^k (1-p)^{n-k} \binom{n}{k} = \left(1 + p(e^t - 1)\right)^n.$$

  To see the last equality, group together e^{tk} and p^k in the form $e^{tk}p^k = (pe^t)^k$ and use the binomial theorem to express the resulting sum.
- If X is geometric p then

$$M_X(t) = \sum_1^\infty e^{tk} p(1-p)^{k-1} = pe^t \sum_1^\infty \left((1-p)e^t\right)^{k-1}.$$

  We recognize a geometric series which converges if and only if $0 \le (1-p)e^t < 1$, that is $t \in (-\infty, \log \frac{1}{1-p})$. For such t,

$$\sum_1^\infty \left((1-p)e^t\right)^{k-1} = \sum_0^\infty \left((1-p)e^t\right)^k = \frac{1}{1-(1-p)e^t}.$$

  It follows that $MX(t) = \frac{pe^t}{1-(1-p)e^t}$.
- If X is negative binomial r, p then

$$\begin{aligned} M_X(t) &= p^r \sum_r^\infty \binom{k-1}{r-1}(1-p)^{k-r}e^{tk} \\ &= p^r e^{tr} \sum_0^\infty \binom{k-1}{r-1}\left((1-p)e^t\right)^{k-r} \\ &= \frac{(pe^t)^r}{(1-(1-p)e^t)^r} \\ &= \left(\frac{pe^t}{1-(1-p)e^t}\right)^r. \end{aligned}$$

Here we use the fact that, for any $a \in (0,1)$, $\sum_r^\infty \binom{k-1}{r-1}(1-a)^{k-r} = \frac{1}{a^r}$. This identity is equivalent to $a^r \sum_r^\infty \binom{k-1}{r-1}(1-a)^{k-r} = 1$ which says that the negative binomial mass function (parameters r and a) is a probability mass function.

- If X is an exponential λ random variable, for all $t \in (-\infty, \lambda)$,

$$M_X(t) = \int_0^\infty \lambda e^{tx} e^{-x\lambda} dx = \lambda \int_0^\infty e^{-x(\lambda - t)} dx = \frac{\lambda}{\lambda - t}.$$

This integral converges if and only if $t \in (-\infty, \lambda)$.

Moments: If X is a random variable with the property that there exists a $\delta > 0$ such that the moment generating function M_X is well-defined on $(-\delta, \delta)$ then $E(X^n)$ equals the value at $t = 0$ of the n-th derivative of M_X, that is $E(X^n) = M_X^{(n)}(0)$.

Example: Using the definition, if X is normal $0, 1$, $\mathcal{N}(0,1)$, then

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tx} e^{-x^2/2} dx.$$

The first thing we do is look at this integral and observe that it converges for all real t because $-x^2/2$ beats tx whenever $|x| > 2|t|$. For instance, we can use that $\frac{x^2}{2} - tx \geq \frac{x^2}{4}$ whenever $|x| \geq 4|t|$. Next we compute

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tx} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2 - 2tx}{2}} dx.$$

The trick is to complete the square" and write $x^2 - 2tx = (x-t)^2 - t^2$. This is the same trick that is used to factorize degree 2 polynomials. Here, it gives

$$\begin{aligned} M_X(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(x-t)^2/2+t^2/2} dx \\ &= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}} dx = e^{t^2/2}. \end{aligned}$$

To see that last equality, use a simple change of variable in the last integral and remember that $\int_{-\infty}^{+\infty} e^{-y^2/2} dy = \sqrt{2\pi}$.

Now, computing moments is all about computing repeated derivatives of $M(t) = t \mapsto e^{t^2/2}$. We have $M'(t) = te^{t^2/2}$, $M''(t) = e^{t^2/2} + t^2 e^{t^2/2}$,
$M'''(t) = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2} = 3te^{t^2/2} + t^3 e^{t^2/2}$ and

$$\begin{aligned} M''''(t) &= 3e^{t^2/2} + 3t^2 e^{t^2/2} + 3t^2 e^{t^2/2} + t^4 e^{t^2/2} \\ &= 3e^{t^2/2} + 6t^2 e^{t^2/2} + t^4 e^{t^2/2}. \end{aligned}$$

All these computations give
$E(X^0) = M(0) = 1, \;\; E(X) = M'(0) = 0, \;\; E(X^2) = M''(0) = 1, E(X^3) = M'''(0) = 0,$

The key theorem about moment generating function is the following:

Theorem (Moment generating functions and equality in distribution) Assume there exists $\delta > 0$ such that the moment generating functions M_X, M_Y of two random variables X, Y are both finite on the interval $I = (-\delta, \delta)$ and that,

$$M_X = M_Y \text{ on } I, \quad \text{that is, } M_X(t) = M_Y(t) \text{ for all } t \in I.$$

Then X and Y are equal in distribution.

This means that you can "recognize" the distribution of a random variable just by knowing its moment generating function. For example, a random variable X with moment generating function $M_X(t) = \frac{4}{4-t}$, (t\in (-1/2,1/2)\), has to be an exponential random variable with parameter 4.

Read Section 5.2: Distribution of a function of a random variable.