Week 7 Recap

This week we consider many of the most important probability distributions, discrete or continuous, and learn how to compute their expectation, variance and moments when these quantities exist.

Monday October 2: Expectation, moments, and variance of Binomials, Geometric, Negative Binomials, Hypergeometric, and Poisson distributions.

Binomial (n,p): $E(X)=np,\ \mathrm{Var}(X)=np(1-p)$. We use the formulas

$$g_y(x)=(x+y)^n=\sum_0^n \binom{n}{k} x^k y^{n-k},$$

$$g_y'(x)=n(x+y)^{n-1}\sum_0^n k\binom{n}{k}x^{k-1}y^{n-k}=n(x+y)^{n-1}x^{-1}\sum_0^n k\binom{n}{k}x^k y^{n-k},$$

$$g_y''(x)=n(n-1)(x+y)^{n-2}\sum_0^n k(k-1)\binom{n}{k}x^{k-2}y^{n-k}=n(n-1)(x+y)^{n-2}x^{-2}\sum_0^n k(k$$

Setting $x=p, y=1-p$, this gives

$$E(X)=np \text{ and } E(X(X-1))=E(X^2)-E(X)=n(n-1)p^2.$$

It follows that

$$E(X^2)=n(n-1)p^2+np,\ \ \mathrm{Var}(X)n(n-1)p^2+np-(np)^2=np(1-p).$$

Geometric p: Use the same method with $g(x)=\sum_1^{+\infty}x^i=\frac{x}{x-1}$. This easily gives

$$E(X)=1/p, E(X(X-1)=2(1-p)/p^2, E(X^2)=(2-p)/p^2,\ \mathrm{Var}(X)=(1-p)/p^2.$$

Negative binomial r,p: We use $k\binom{k-1}{r-1}=r\binom{k}{r}$ and compute

$$
\begin{aligned}
E(X^n) &= \sum_{k=r}^{\infty} k^n \binom{k-1}{r-1} p^r (1-p)^{k-r} \\
&= \frac{r}{p} \sum_{k=r}^{\infty} k^{n-1} \binom{k}{r} p^{r+1} (1-p)^{k-r} \\
&= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{n-1} \binom{m-1}{r} p^{r+1} (1-p)^{m-(r+1)} \\
&= \frac{r}{p} E((Y-1)^{n-1})
\end{aligned}
$$

where Y is negative binomial with parameters $r+1, p$. For $n=1$, this gives $E(X) = r/p$. For $n=2$, $E(X^2) = (r/p)[E(Y) - 1]$ gives

$$E(X^2) = \frac{r}{p}\left(\frac{r+1}{p} - 1\right)$$

and

$$\mathrm{Var}(X) = \frac{r}{p}\left(\frac{r+1}{p} - 1\right) - \left(\frac{r}{p}\right)^2 = \frac{r(1-p)}{p^2}.$$

Poisson λ: $E(X) = \lambda, E(X(X-1)) = \lambda^2, E(X^2) = \lambda^2 + \lambda,\ \mathrm{Var}(X) = \lambda.$

Hypergeometric: Let X be a hypergeometric random variable with parameters N (population size), m (sub-population size), and n (sample size), so that

$$P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}.$$

Let Y be hypergeometric with parameters $N-1, m-1, n-1$. Use a binomial identity to show that

$$E(X^a) = \frac{nm}{N} E((Y+1)^{a-1})).$$

This gives

$$E(X) = \frac{nm}{N}, \quad E(X^2) = \frac{nm}{N}\left(\frac{(n-1)(m-1)}{N-1} + 1\right)$$

and

$$\mathrm{Var}(X) = \frac{nm}{N}\left(\frac{(n-1)(m-1)}{N-1} + 1 - \frac{nm}{N}\right).$$

Wednesday October 4: Computing with Exponential, Normal, Gamma distributions.

Exponential λ: Using integration by parts, for any $n \geq 1$ we have

$$E(X^n) = \lambda \int_0^{+\infty} x^n e^{-\lambda x} dx = n \int_0^{+\infty} x^{n-1} e^{-\lambda x} dx = \lambda^{-1} E(X^{n-1}).$$

This gives $E(X) = 1/\lambda, E(X^2) = 2/\lambda^2$ and $\mathrm{Var}(X) = 1/\lambda^2$.

Normal $N(0,1)$: The density of a normal $N(0,1)$ is $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Review why the integral of this function is equal to 1. By symmetry, $E(X) = 0$. To compute $\frac{1}{\sqrt{2\pi}} \int x^2 e^{-x^2/2} dx$, we write $x^2 e^{-x^2/2} = x \times x e^{-x^2/2}$ and integrate by parts. You will find that $E(X^2) = 1$. By definition, Y is $N(\mu, \sigma^2)$ exactly if $X = (Y - \mu)/\sigma$ is normal $N(0,1)$. This easily imply that $E(Y) = \mu$ and $\mathrm{Var}(Y) = \sigma^2$.

Gamma α, λ: This continuous probability distribution has density $f_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} \mathbf{1}_{(0,+\infty)}(x)$ where $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$. For $\alpha = 1$, this the exponential distribution with parameter λ. If X has distribution Gamma α, λ then $E(X) = \alpha/\lambda$ and $E(X^2) = \alpha(\alpha+1)/\lambda^2$ so that $\mathrm{Var}(X) = \alpha/\lambda^2$.