Week 5 Recap

Random variables

One can think of "random variables" as a fancy way to think about "events." We will use the notation RV for short.

Given a probability space $(\Omega, \mathcal{F}, P)$, a (real valued) random variable on Ω is a function $X : \Omega\mathbb{R}$ such that for any interval $I \subseteq \mathbb{R}$), $X^{-1}(I) \in \mathcal{F}$. This property allows us to compute $P(\{\omega : X(\omega) \in I\}) = P(X^{-1}(I))$. One often write $P(\{\omega : X(\omega) \in I\}) = P(X \in I)$ for short.

Random variables can also take values in other sets (colors, letters, vector spaces).

An event A can always be obtained as $A = \{\omega : \mathbf{1}_A(\omega) = 1\}$ where $\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$ This $\mathbf{1}_A$ is a random variable.

RV can take values in a finite set, a countable set, or a continuous (uncountable space).

RV X taking only finitely many or countably many values $\{x_1, x_2, \ldots\}$. In this case, any probability question regarding X can be answered based on the knowledge of $P(\{\omega : X(\omega) = x_i\}) = p_{x_i} = p_X(x_i)$ for all I (a finite or countable number of values of x_i. This is called the probability mass function of X.

Examples: In a sequence of repeated independent identical experiments with probability of success p, what is the probability mass function of (a) the number of successes X in the first N experiments? (b) the rank Y of the the occurrence of the first success

***Continuous RV* X taking values in $\mathbb{R}$.** A random variable $X : \Omega \to \mathbb{R}$ is said to be a *continuous random variable* if there exists an integrable function $f_X : \mathbb{R} \to [0, +\infty)$ such that, for any interval I,

$$P(X \in I) = \int_I f_X(x)dx.$$

The function f_X is called the probability density function of X and X is a continuous RV with density function f_X.

Example: Verify that, for any $\lambda > 0$, the function $f(y) = \lambda e^{-\lambda y} \mathbf{1}_{(0,+\infty)}(y)$ is the probability density function of a random variable X such that, for any $0 \leq a \leq b < +\infty$, $P(X \in (a, b)) = e^{-\lambda a} - e^{-\lambda b}$.

Cumulative distribution function In general the cumulative distribution function F_X of a random variable X taking values in an ordered set (e.g., $\mathbb{R}$ and $\mathbb{N}$, etc), is defined by

$$F_X(s) = P(X \leq s).$$

If one knows the function F_X, it is easy to compute the probability that $a < X \leq b$ because it is equal to

$$P(a < X \leq b) = F(b) - F(a).$$

The cumulative probability function is very helpful in dealing with random variables, especially continuous random variable. Read in the book about the properties shared by all cumulative probability functions.

Fact: Let X be a continuous RV with probability density function f_X. At any real x_0 where f_X is continuous, we can compute $f_X(x_0)$ by taking the derivative of the cumulative probability distribution F_X at x_0, that is, $f_X(x_0) = F'_X(x_0)$.

Expectation and moments By definition, when it exists, the expectation $E(X)$ of a random variable X defined on $\Omega, \mathcal{F}, P)$ is the "weighted average" of the values taken by X weighted according to P. One could write $E(X) = \int X(\omega)P(d\omega)$ except that we do not know what the right hand-side means in general.

- For RV taking a finite or countable number of values x_i with probability mass function $p_X(x_i)$, the definition is (assuming the sum converges when it is a countable sum)

$$E(X) = \sum x_i p_X(x_i).$$

- For continuous RV with probability density function f_X, the definition is

$$E(X) = \int x f(x) dx$$

  assuming that this integral exists.

Important properties of expectation:

- If X is non-negative an its expectation exists the $E(X) \geq 0$.
- If the expectation of X exists and Y=aX+b then $E(Y) = aE(X) + b$.
- If the RV variables $X_1, \dots, X_n$ are all defined on the same probability space and their expectations exists then the expectation of $Y = X_1 + \cdots + X_n$ exists and $E(Y) = \sum_{i=1}^n E(X_i)$.

Exercise: In a well shuffled deck of N cards marked 1 to N, what is the expectation of the number of cards whose marking coincides with their position in the deck?

One way to obtain a new RV Y from an old one X is to use a function g, defined on a set containing all the values of X and set$Y = g(X)$.

- For a RV X taking a finite or countable number of values x_i with probability mass function $p_X(x_i)$, and a function g, the expectation of $Y = g(X)$ when it exists is

$$E(Y) = E(g(X) = \sum g(x_i)p_X(x_i).$$

- For a continuous RV with probability density function f_X, and a function g, the expectation of $Y = g(X)$ is

$$E(Y) = E(g(X)) = \int g(x)f(x)dx.$$

Special case (the moment of X): Given a random variable X, the "n-moment" of X equals $E(X^n)$ if that expectation make sense.

At the end of the week, you should know how to compute the expectation of of anyone of the following RVs:

Bernoulli p: E(X)=p. Binomial n, o: E(X)=np. Geometric p: $E(X) = 1/p$. Negative binomial r, p: $E(X) = r/p$. Poisson λ: E(X)=\lambda\).

Exponential λ: $E(X) = 1/\lambda$. Normal μ, σ^2: $E(X) = \mu$.