Networks: Fall 2023	Homework 3
David Easley and Yian Yin	Due 3:30pm, Thursday, September 21, 2023

Homework solutions should be submitted by upload to Gradescope, which can be accessed via Canvas. The file you upload **must be typed and submitted in PDF format**. Handwritten assignments will not be graded. However, you can draw graphs and insert them into your pdf. You can create a separate file with the solutions (you don't need to repeat the questions); it is fine to create the homework in any format provided it's typed and handed in as a single PDF file. When you upload your pdf to Gradescope be sure to assign your answers to the correct question.

To be eligible for full credit, your homework must come in by 3:30pm Thursday. We will also accept late homeworks after 3:30pm Thursday until 3:30pm Friday for a deduction of 10% of the total number of points available. Gradescope will stop accepting homework uploads after 3:30pm Friday; after this point we can only accept late homework when it is accompanied by a University-approved reason that is conveyed to and **approved by the TA in charge of this homework prior to the due date of the homework**. (These include illness, family emergencies, and travel associated with university activities.)

The TA in charge of this homework is Jonathan Aimuyo (oja7@cornell.edu).

Reading: The questions below are primarily based on the materials in Chapters 9 and 10.

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(1) [6 points] You work for an online auction site that sells items using ascending-bid auctions. In this question we will consider only one auction for one item and we will assume that each user knows how much the item is worth to them and does not care about how others value it (the bidders have independent private values).

The site has set up an automatic bidding agent and is considering a proposed announcement to explain the bidding agent to users. Here is proposed, still rough, draft of the announcement:

If you choose to use our new automatic bidding agent, it will participate in the ascending-bid auction on your behalf, as follows.

- At the beginning of the auction, you provide a number b to the bidding agent. If you use our bidding agent and provide a number b you cannot change b or or stop using the bidding agent once the auction begins.
- At any point while the auction is taking place, if the current highest bid x has been submitted by a bidder other than you, and x is strictly less than your number b, then in the next time step of the auction the bidding agent will bid x + \$0.01 on your behalf. (Note that one cent is being used as the minimum increment by which the bid is increased in the ascending-bid auction, in order to have the new bid be strictly greater than the current bid x.)

- The bidding agent stops submitting bids on your behalf if the current bid is greater than or equal to your number b or if you are the only person left in the auction.

Note also that although the bidding agent uses your number b to guide its behavior, it will not directly reveal the value of b that you give to anyone.

This automatic bidding agent will make it easier for you to participate in our site as you only need to give the agent the number b and then you can ignore the auction until you find out whether you won. So all you need to do is to chose b. The number you want to give to the bidding agent should depend on your value for the item; lets suppose for example that your value is \$100. It may also depend on how many other bidders participate in the auction. Lets suppose that the number of bidders is N; we know N even if you don't. Our recommendation is that you select b such that $b = 100(N-1)/N$. That way the bidding agent will always bid less than your value, but if there are many bidders and thus more competition for the object, then the agent will bid up to an amount that's close to your value.

Do you agree with the recommended number b in this announcement? Explain. If you agree explain why $b = 100(N-1)/N$ is optimal. If you think that $b = 100(N-1)/N$ is not correct, then what number would you recommend (to a bidder with value \$100) and why? [Although this auction has a minimum increment of \$0.01 you may ignore any complications that this small discrete value introduces.]

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(2) [6 points] A seller has one item that he would like to sell. He can at any time sell it to a dealer who will pay \$50 for the item. He is also considering two other ways to try to sell the item: posting a price of \$100 or using a second price auction. He knows that there are two people interested in buying or bidding on the item. He also knows that these two people have independent, private values for the item and that the value for any person is either \$100 or \$80 with equal probability. [These values are independent, so the probability of each of the four possible pairs of values $(100, 100), (100, 80), (80, 100), (80, 80)$ is $1/4$.]

If the seller posts a price of \$100 then the item will be sold \$100 if at least one of the people interested in the item has a value for it of \$100; otherwise it will not be purchased at the posted price and our seller will have to sell it to the dealer for \$50. If the seller runs a second price auction the two people interested in the item will bid optimally in the auction.

(a) Suppose the seller attempts to sell the item at a posted price of \$100.

  (i) What is the probability that someone will buy the item at the price of \$100?

  (ii) What will be the seller's expected revenue from trying to sell the item at a posted price of \$100?

(b) Suppose that instead of using the posted price the seller runs a second price auction.

  (i) For each of the possible pairs of values for the individuals how much will they each bid in the second price auction?

(ii) What will be the seller's expected revenue from running a second price auction?

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(3) [6 points] This question explores how one may use bipartite graphs to model and solve problems across different domains. For each given example, first reformulate the problem as a bipartite perfect matching problem. Then apply what you learned about perfect matching to answer the question at hand.

(a) The concept of perfect matching is widely applied in scheduling and assignment tasks. For example, in the Networks class we need to allocate four different office hour sessions – a Monday Zoom session, a Monday in-person session, a Tuesday in-person session, and a Wednesday Zoom session – to four TAs: Adam, Brian, Jemma, and Karen. The assignment must satisfy the following constraints: (i) Adam can only attend in-person sessions; (ii) Brian can only attend sessions on Monday; (iii) Jemma can only attend sessions on Wednesday; (iv) Karen can only attend Zoom sessions.

Determine whether you can find such an assignment. To answer this question, first reformulate the task as a perfect matching problem in a bipartite graph. Clearly define the nodes and edges, and provide a graph to illustrate. Then, identify whether a perfect matching (assignment) exists. If yes, provide a specific solution. If no, explain why.

(b) Let us consider another problem, where you are using a set of dominoes (tiles consisting of two adjacent squares) to cover a subset of a chessboard such that each square is covered exactly once. For example, Fig. 1 shows how to tile a given subset of 8 unit squares, where we use four dominoes (AC, BF, DE, and GH respectively).

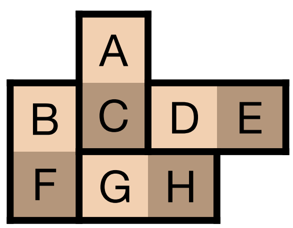

Figure 1: Q3: a subset of the chessboard. Dark edges represent the four dominoes.

Determine whether the subset in Fig. 2 can be tiled. To answer this question, first reformulate the task as a perfect matching problem in a bipartite graph. Clearly define the nodes and edges, and provide a graph to illustrate. Then, identify whether a perfect matching (tiling) exists. If yes, provide a specific solution. If no, explain why.

(c) Determine whether the subset in Fig. 3 can be tiled. To answer this question, first reformulate the task as a perfect matching problem in a bipartite graph. Clearly define the nodes and edges, and provide a graph to illustrate. Then, identify whether a perfect matching (tiling) exists. If yes, provide a specific solution. If no, explain why.

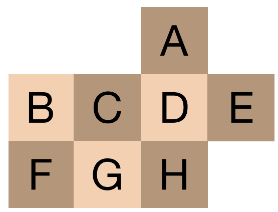

Figure 2: Q3b: a subset of the chessboard

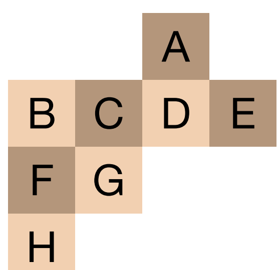

Figure 3: Q3c: a subset of the chessboard

(4) [5 points] Consider the following set of valuations that a set of buyers have for the items being offered by a set of sellers.

Buyer	Value for a's item	Value for b's item	Value for c's item
x	6	4	6
y	2	10	12
z	6	2	16

(a) Describe what happens if we run the bipartite graph auction procedure from Chapter 10 to find market-clearing prices on the matching market defined by this given set of valuations, by saying what the prices are at the end of each round of the auction, including what the final market-clearing prices are when the auction comes to an end.

You can either draw the preferred-seller graph from each round, or it is also fine to simply list the prices for each seller at the end of each round of the auction.

(Note: If in any round you notice that there are multiple choices for the constricted set of buyers, then under the rules of the auction, you may choose any such constricted set.)

(b) Suppose seller a, b, and c charges 2, 2, and 6 for their items respectively. Verify this is a market clearing price. Now suppose seller a has just added an additional warranty period of $t \geq 0$ months. Each additional month will increase buyer y's valuation of a by 2 and z's valuation of a by 1. x's valuation of a is not affected. Provide an updated valuation table (you can include t in the table). Calculate the range of t where the price (2,2,6) remains market clearing.

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(5) [7 points] Here's a "geometric" way of looking at the model of matching markets from Chapter 10. Since we've seen that a single market can have several different sets of market-clearing prices, it's natural to ask how we should think about the set of *all* possible market-clearing prices for a given market. It turns out that there is a natural geometric view of the market-clearing prices, which can be useful in developing a more global picture of them.

In this question, we give an illustration of how the geometric view works in a very simple setting. It is simple because we'll assume that there are only two sellers and two buyers, and we'll only consider prices that are lower than the valuation that any buyer has for any item (so any buyer would receive a positive payoff from buying anything).

Let's begin with the following example. Suppose we have sellers named a and b, and buyers named v and w. Each seller is offering a distinct item for sale, and the valuations of the buyers for the item are as follows.

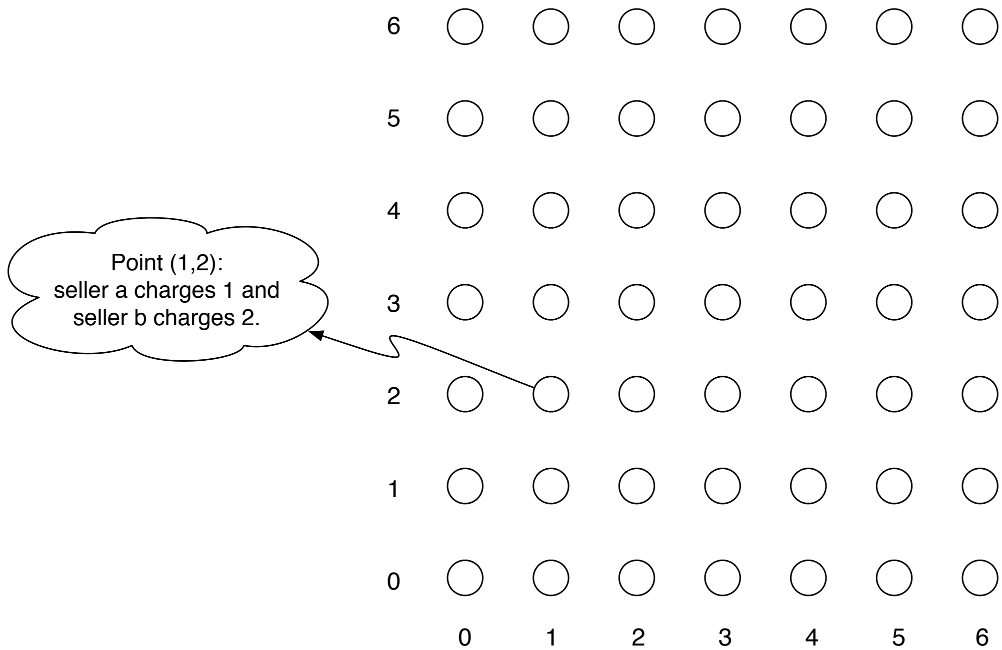

Figure 4: Q5: A grid of possible prices for two items.

Buyer	Value for a's item	Value for b's item
v	7	10
w	8	9

Now, let's say that a *pricing* of the two items is simply a choice of a price by each seller: a price p chosen by seller a, and a price q chosen by seller b. Consider the set of all possible pricings, where the price of each item is between 0 and 6 (and hence cheaper than any valuation). These pricings can be drawn as a grid of points, as shown in Figure 4, where the point (p, q) at x-coordinate p and y-coordinate q corresponds to the pricing in which seller a charges p and seller b charges q. So for instance, the point $(1, 2)$ indicated as an example in Figure 4 corresponds to the pricing in which seller a charges 1 for his item and seller b charges 2 for her item.

Given this grid of points, we can shade in all the points that correspond to pricings that are market-clearing. If we do this for our current example, we get the shading of points shown in Figure 5. This corresponds to the set of 15 market-clearing prices

$$(0,1),(0,2),(0,3),(1,2),(1,3),(1,4),(2,3),(2,4),(2,5),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6).$$

(a) Try doing the same process for the following different set of valuations. Again, suppose we have sellers named a and b, and buyers named v and w. Each seller is offering a distinct item for sale, and the valuations of the buyers for the item are as follows.

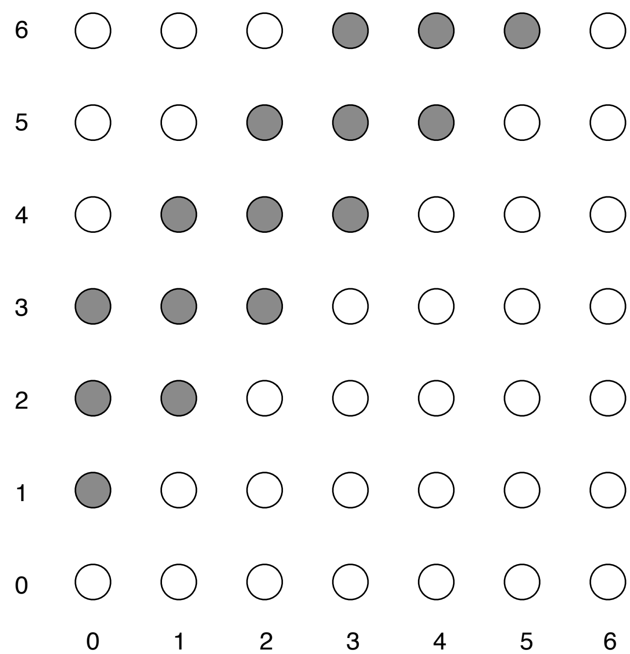

Figure 5: Q5: Shading in the market-clearing prices in the grid.

Buyer	Value for a's item	Value for b's item
v	7	9
w	11	10

Again, let's only consider prices between 0 and 6 (again, chosen so that the prices lie below all the valuations). Which grid points in this new example correspond to market-clearing prices?

(In your answer, you can either draw a shaded grid as in Figure 5, or if it's easier, you can write out the list of grid points as we did above for our original example.)

(b) If you did part (a) correctly, you should find that the set of shaded grid points — corresponding to market-clearing prices — consists of all the points that lie between two parallel diagonal lines. Our original example had this property too: the points corresponding to market-clearing prices lay neatly between two parallel diagonal lines, as shown in Figure 6.

In fact, this is a general phenomenon. Give an informal explanation for why the prices in these two examples lie between two diagonal lines. In particular, interpret the meaning of the upper diagonal line by saying why no pair of prices above this line can be market-clearing. Then, interpret the meaning of the lower diagonal line by saying why no pair of prices below this line can be market-clearing.

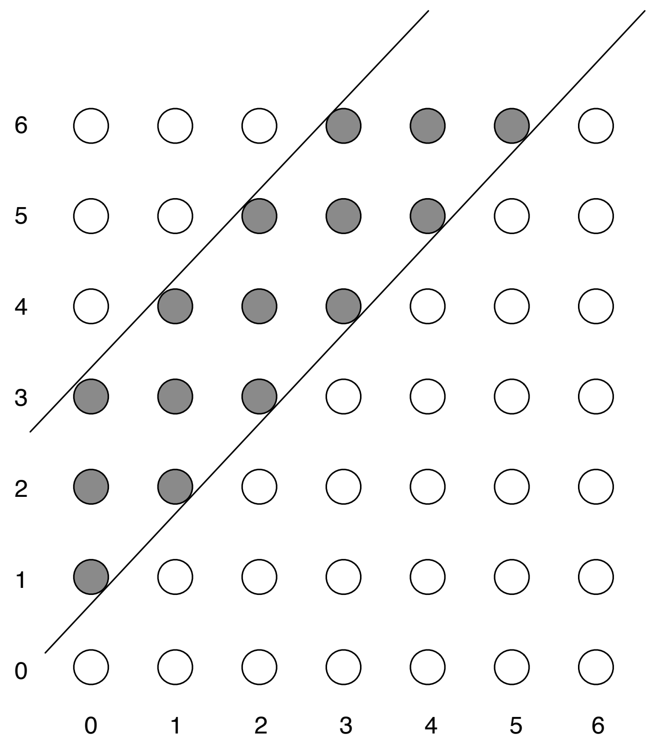

Figure 6: Q5: The market-clearing prices in the grid lie between two parallel diagonal lines.