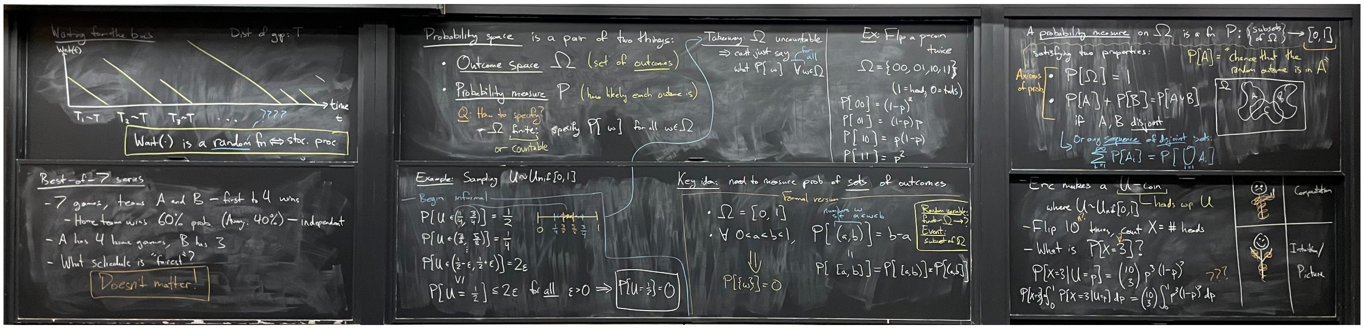
# Lecture 2



## Lecture 2 learning outcomes

- Define the following terms
    - *Probability measure*
    - *Outcome space*
    - *Probability space*
    - *Random variable*
    - *Event*
- Describe a distribution that is neither discrete nor continuous
- Distinguish between distributions and random variables

# Announcements

- Textbook problem numbers and homework numbers won't necessarily line up!
- Office hours will (very likely) start next week
    - Some might start the week after
- Will post lecture notes, but they're just bullet points
- Study partner matching by Learning Strategies Center:

> Find study partners! Studying with peers is a great way to connect with other Cornell students and is a powerful tool for learning. Finding people to study with can be challenging, and Cornell's Learning Strategies Center (LSC) helps match you with study partners. To find out more about study groups and partners, and to sign-up for study partners for class you are in, visit the LSC's Studying Together webpage.

# Overview

- Motivating hard examples

- Eric's coin puzzle
- Best-of-seven sports series

# Motivating examples

- Goals of today's lesson
  - Learn how to clearly define random processes
  - Learn how to think about a random process from multiple different perspectives
- Start with two examples where changing perspective makes a problem way easier

## Eric's coin puzzle

- This is a puzzle my friend Eric tweeted once
- The setup
  - Eric generates a random number $U \sim \mathrm{Unif}[0,1]$
  - Then, Eric creates a biased coin with heads probability $U$
    - Call this a *$U$-coin*
  - Eric flips the $U$-coin $n$ times and records the number of heads $X$
    - For concreteness, imagine $n = 10$
- The question: what is the distribution of $X$?
  - That is, we want to find $\mathbf{P}[X = k]$ for each $k$
    - For concreteness, imagine $k = 3$
- Attempting the question with standard tools
  - If we knew that $U = p$, then we'd know that $X$ would have a binomial distribution, so

$$\mathbf{P}[X = k \mid U = p] = \binom{n}{k} p^k (1-p)^{n-k}$$

  - We know $U \sim \mathrm{Unif}[0,1]$, so its density function is $f_U(p) = 1$ for $p \in [0,1]$
  - From the law of total probability, we get

$$\mathbf{P}[X = k] = \int_0^1 \mathbf{P}[X = k \mid U = p]\, f_U(p)\, \mathrm{d}p = \binom{n}{k} \int_0^1 p^k (1-p)^{n-k}\, \mathrm{d}p$$

    - ... which is an integral that I straight-up don't know how to do, so I'm stuck
- A hint at what the answer might be
  - If $k = n$, then the integral is doable:

$$\mathbf{P}[X = n] = \binom{n}{n} \int_0^1 p^n\, \mathrm{d}p = \frac{1}{n+1}$$

  - Similar integral yields $\mathbf{P}[X = 0] = \frac{1}{n+1}$

    - ... so do we always get $\frac{1}{n+1}$?

## Best-of-seven sports series

- Let's model a best-of-seven series of sports games
- The setup
    - Two evenly-matched sports teams, A and B, play a best-of-seven series
    - Each game, one team is the home team, and the other is away
    - Suppose the home team wins 60% of the time due to home-field advantage
        - Independently of any past or future games
    - Team A gets up to 4 home games, while team B gets up to 3 home games
    - The first team to get to win 4 games wins the series
- The question
    - How does the schedule of which team is home when affect the probability A wins the series?
        - Is it better for A to do all of its home games first?
        - Is it better for A to save its home games for last?
    - Based on varying the schedule, what are the minimum and maximum probabilities we can get for A winning the series?
- How do we approach the question?
    - List out all $\binom{7}{3} = 35$ possible schedules and compute A's winning probability for each?

## Summary

- Saw two questions that are difficult to attack by direct computation
- We're going to learn to model the questions formally
- The same formal-modeling skill is also going to help us model the questions from a different perspective, which will make them easier

# Formally defining probability

## Probability measures

> ### Definition
>
> A **probability measure on set $\mathbb{S}$**, also called a **distribution** on $\mathbb{S}$, is a function $\mathcal{D}$ that takes as input subsets $A \subseteq \mathbb{S}$, outputs values $\mathcal{D}(A) \in [0, 1]$, and satisfies the following properties:
>
> - $\mathcal{D}(\mathbb{S}) = 1$.

- **Countable additivity:** If $A_1, A_2, \ldots$ is a sequence of *disjoint* subsets of $\mathbb{S}$, then

$$\mathcal{D}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathcal{D}(A_i).$$

  - Finite additivity, where there are finitely many disjoint sets $A_i$, is a special case.

- Often we call the probability measure $\mathbf{P}$
  - That is, formally speaking, the "$\mathbf{P}$" that we write when talking about probabilities is a probability measure
- What does this meaning of "distribution" have to do with classic distributions you've learned about in previous classes?
  - Will see next time!

## Probability spaces and outcomes

### Definition

A **probability space** is a set $\Omega$ together with a distribution $\mathbf{P}$ on $\Omega$. The set $\Omega$ is usually called the **outcome space** or **sample space**, and $\mathbf{P}$ is usually called the **probability measure**. The probability space is then the pair $(\Omega, \mathbf{P})$.

- We often call elements $\omega \in \Omega$ *outcomes* or *samples*
  - Or sometimes *sample paths*, especially when $\omega$ describes how something evolves over time
- Example: Eric's coin
  - Outcome space is $\Omega = [0,1] \times \{0,1\}^n$
    - There's the probability $U \in [0,1]$
    - There's the result of each coin flip, which we can model as an element of $\{0,1\}$
      - **Notation:** For most of this course,

$$\begin{aligned} \text{heads} &= 1 \\ \text{tails} &= 0 \end{aligned}$$

    - Example outcome $\omega \in \Omega$ and $n = 10$:

$$\omega = \big(0.42, (1,0,0,1,0,0,0,1,0,0)\big)$$

  - Probability measure
    - Honestly, kind of a pain to write down
    - Here's my best attempt:

$$\mathbf{P}\left[\left\{\big(\omega_0, (\omega_1, \ldots, \omega_n)\big) \in \Omega \,\middle|\, \omega_0 \in [p, p+\delta)\right\}\right] \approx \delta \times p^{\sum_{i=1}^n \omega_i} \times (1-p)^{\sum_{i=1}^n (1-\omega_i)}$$

- ... blergh
- Example: best-of-seven series
  - Outcomes will be sequences of $0$s and $1$s
    - $1$ means team A wins, $0$ means team B wins
  - So outcome space is a set of sequences of $0$s and $1$s that end when they get either a fourth $1$ or fourth $0$
    - So an outcome looks like $\omega = (\omega_1, \dots, \omega_n)$ for some $n \leq 7$
    - Example: $(0, 1, 1, 0, 1, 1)$
    - **Notation:** When it's clear enough, we'll write tuples as just a string of characters, e.g.

$$(0, 1, 1, 0, 1, 1) \rightsquigarrow 011011$$

  - Full outcome space:

$$\Omega = \{0000, 00010, 000110, \dots, 011101, 01111, \dots 1110000, 111001, \dots, 1111\}$$

    - *Optional exercise:* how many outcomes are in $\Omega$?
  - Probability measure is... well, you could write it out, but it's kind of annoying
    - It depends on the exact schedule of which games are home and which games are away
    - Example:

$$\mathbf{P}[\{0000\}] = \left(\tfrac{2}{5}\right)^4$$
$$\mathbf{P}[\{1111\}] = \left(\tfrac{3}{5}\right)^4$$

  - **Question:** If there were no home-field advantage, i.e. if team A had a 50% chance of winning every game whether home or away, would the probability measure be uniform on $\Omega$?
    - No!
    - Example:

$$\mathbf{P}\left[\{0000\}\right] = \left(\tfrac{1}{2}\right)^4$$
$$\mathbf{P}\left[\{1111\}\right] = \left(\tfrac{1}{2}\right)^4$$
$$\mathbf{P}\left[\{00010\}\right] = \left(\tfrac{1}{2}\right)^5 \neq \left(\tfrac{1}{2}\right)^4$$

## Random variables and events

- With Eric's coin, what we actually want to know about is the number of heads $X$
  - Specifically, the distribution of $X$

- What type of thing is $X$, and how does it fit into our definitions of distributions and probability spaces?

> **Definition**
>
> - A **random variable** is a function $X$ from an outcome space $\Omega$ to some other set $\mathbb{S}$. We write $X : \Omega \to \mathbb{X}$.
> - An **event** is a subset $A$ of an outcome space $\Omega$. We write $A \subseteq \Omega$.

## Formalizing can be subtle

- Your friend flips two coins, then tells you "I got at least one heads"
- **Activity:** What event is your friend saying happened?
- Two ways to formalize this
  - Decide exactly which event your friend's statement corresponds to
  - Expand the outcome space to include your friend's statement, then decide exactly what the probabilities should be
- Both depend on carefully thinking through what *would have happened* even if your friend had gotten two tails
  - More in discussion and on the homework!