Discounted Cash Flow Valuation II

AEM 2241 - Finance

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Outline

Effective Annual Rates (EAR)

- By convention, if the yearly stated interest rate is r, and if it compounds m times a year, the per-compounding period rate is $\frac{r}{m}$.
- If you invest PV at time 0, your investment grows to $FV = PV\left(1+\frac{r}{m}\right)^m$ by the end of the year (equivalently, by the end of the m compounding periods).
- The total interest earned is $FV - PV = PV\left(1+\frac{r}{m}\right)^m - PV = PV \cdot \left[\left(1+\frac{r}{m}\right)^m - 1\right]$.
- Now assume that you have a single investment period at rate EAR, and you end up with the same amount of money: $FV = PV \cdot (1 + EAR) = PV\left(1+\frac{r}{m}\right)^m$. Thus $EAR = \left(1+\frac{r}{m}\right)^m - 1$.
- Rate EAR is called the **effective annual rate**.

EAR vs. Stated Rates

- Assume that you invest $1,000 for one year, what is the EAR and how much total interest do you earn?
  - 10% per year, compounded semi-annually: $EAR = \left(1+\frac{0.10}{2}\right)^2 - 1 = 10.25\%$; $interest = 1,000 \cdot \left[\left(1+\frac{0.10}{2}\right)^2 - 1\right] = 1,000 \cdot 10.25\% =$ \$102.50;
  - 10% per year, compounded quarterly: $EAR = \left(1+\frac{0.10}{4}\right)^4 - 1 = 10.38\%$; $interest = 1,000 \cdot \left[\left(1+\frac{0.10}{4}\right)^4 - 1\right] = 1,000 \cdot 10.38\% =$ \$103.81;
  - 10% per year, compounded monthly: $EAR = \left(1+\frac{0.10}{12}\right)^{12} - 1 = 10.47\%$, $interest = 1,000 \cdot \left[\left(1+\frac{0.10}{12}\right)^{12} - 1\right] = \$104.71$.
- All else equal, the more compounding periods, the higher the future value and the total interest earned.

Making Rates Look Better

- You are a credit card issuer who wants to earn an *EAR* of 25% per year on all outstanding balances. Can you make this rate look smaller? How?
- You can try choosing a large number of compounding periods per year, and quoting the corresponding nominal annual, or just the per-period rate.
- We choose daily rates and assume a year has 365 days. $EAR = (1 + \frac{r}{m})^m - 1$, which we solve for r.
- $r = m \cdot \left[(1 + EAR)^{\frac{1}{m}} - 1\right] = 365 \cdot \left[(1 + 0.25)^{\frac{1}{365}} - 1\right] =$ 22.32%.
- The daily rate is equal to $\frac{r}{m} = \frac{0.2232}{365} = 0.0006115 = 0.06115\%$.

Continuous Compounding

- How high can the number of compounding periods go? In many transactions, interest is compounded daily.
- Mathematically, however, interest can compound infinitely often, over infinitely short time periods.

$$EAR_\infty = \lim_{m \to \infty} \left[\left(1 + \frac{r}{m}\right)^m - 1\right] = e^r - 1,$$

where e is the basis of the natural logarithms ($e \approx 2.7182$), and e^r is the exponential function.

- EAR_∞ is the highest yearly interest that can be earned using a stated (nominal) compounding interest of r.

Example

- Consider a stated annual interest rate of 12% per year. Let m be the number of compounding periods. We then have:

m	Stated rate	EAR
1	12%	$\left(1+\frac{0.12}{1}\right)^1 - 1 = 12.00\%$
2	12%	$\left(1+\frac{0.12}{2}\right)^2 - 1 = 12.36\%$
4	12%	$\left(1+\frac{0.12}{4}\right)^4 - 1 = 12.55\%$
12	12%	$\left(1+\frac{0.12}{12}\right)^{12} - 1 = 12.68\%$
365	12%	$\left(1+\frac{0.12}{365}\right)^{365} - 1 = 12.747\%$
∞	12%	$e^{0.12} - 1 = 12.750\%$

Pure Discount Loans

- The borrower gets a lump sum at time 0, and repays the loan at the end of the t periods agreed to.
- This is exactly the situation that we examined when talking about the time value of money.
- Example: How much can you borrow today, if you can pay \$25,000 in five years? The interest is 12% per year, compounded annually.

$$PV = \frac{FV}{(1+r)^5} = \frac{25,000}{(1+0.12)^5} = \$14,186.67.$$

- Treasury bills are issued by the US government and promise to pay \$1,000 in a given time period (3, 6, 9, and 12 months). They are pure discount loans.

Amortized Loans

- Lenders often prefer several, equally spaced payments that, over time, repay both the original loan amount (the loan principal) and the interest earned on the loan.
- In this case, each payment contains a “principal” part, and an “interest” part.
- Because the principal decreases over time, we say that it is amortized. The loan itself is called an amortized loan.
- Why do typical lenders prefer amortized loans?
  - They provide steady income;
  - They reveal default early (if it occurs);
  - They make it more likely that at least part of the principal and due interest is repaid.

Amortized Loans: Example

- Consider a loan of $\$5,000$ that you receive today and that you will repay in 5 **equal** yearly payments, at an annually compounding interest rate of 9%. How much should your yearly payment be?
- The loan can be viewed as a regular annuity with $PV = \$5,000$. Thus $PV = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^5}{r}$. We must solve for C.
- $C = \frac{r}{1-\left(\frac{1}{1+r}\right)^5} \cdot PV = \frac{0.09}{1-\left(\frac{1}{1+0.09}\right)^5} \cdot 5,000 = \$1,285.46.$
- Over the 5 years the present value of the annuity (loan payments) is the same as the initial loan amount. But how much principal do we repay, and how much interest do we pay every year? We use an amortization schedule to find out.

Amortization Schedule

Year	Beg. Balance	Payment	Interest Paid	Principal Repaid	Ending Balance
0					$5,000.00$
1	$5,000.00$	$1,285.46$	$5,000.00 \cdot 0.09 = 450.00$	$1,285.46 - 450.00 = 835.46$	$5,000.00 - 835.46 = 4,164.54$
2	$4,164.54$	$1,285.46$	$4,164.54 \cdot 0.09 = 371.81$	$1,285.46 - 371.81 = 910.65$	$4,164.54 - 910.65 = 3,253.88$
3	$3,253.88$	$1,285.46$	$= 292.85$	$= 992.61$	$= 2,261.27$
4	$2,261.27$	$1,285.46$	$= 203.51$	$= 1,081.95$	$= 1,179.32$
5	$1,179.32$	$1,285.46$	$= 106.14$	$= 1,179.32$	$= 0.00$
Totals		**6,427.31**	**1,427.31**	**5,000.00**	

- Fill out the table by following the calculation patterns shown for year 1 and year 2!
- As time passes, each payment includes lower interest payments and higher principal (re)payments. This is always the case for fixed-payment amortized loans.
- Over the five years, the principal is repaid in full and an amount of $\$1,427.31$ is paid as interest. We can compute this directly: $\textit{interest paid} = 5 \cdot 1,285.46 - 5,000.00 = \$1,427.31$.

Amortized Loans: Variations

- Financial transactions exhibit many variations on the same fundamental idea. This is done to satisfy different target populations with different needs, in order to (de-)emphasize certain features (and thus to make these transactions more attractive), to simplify understanding of the terms, or for historical reasons.
- In some business loans yearly interest is paid on the beginning balance, and then the balance is decreased by **same amount** every year (e.g. by \$1,000 per year). In this case the yearly payment is always different. HW: Fill out an amortization table for such a loan if you borrow \$5,000 for 5 years at $r = 9\%$, compounded yearly, if you repay \$1,000 of the principal every year.
- In real estate it is common to use "bullet" loans. In this case the loan runs for, say, 5 years, but is calculated to amortize over, say, \$15 years (a longer period). At the end of the loan period (5 years), the borrower repays the then-outstanding loan balance, or, very commonly, refinances the loan.

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