MATH 4710 **PRELIM 1** **FALL 2021**

Name and Cornell NetID:

Instructions

(1) There are 3 independent problems. The point total is 25. You have 50 minutes.

(2) Write your answers on these booklet. Write your name and Cornell NetID on the top of this page before you begin.

(3) Write clearly using a black or blue pen or pencil.

(4) Always provide reasons for your answers and explain your computations. For numerical answers, give either a simplified fraction or a decimal answer, which ever comes more easily. For instance, $3 \times 7^2/(15)^9$ is very acceptable (much better than the expanded version) but $8/12$ is not very good.

(5) No books, notes, or documents are allowed during this prelim. Do not use (do not touch) electronic devices during the exam, for any reasons.

(6) Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty.

Problem 1: (8 pts) An urn contains 4 red balls, 7 green balls, and 4 yellow balls. Each of question (a) and (b) can be solved without solving the other.

(a-4pts) We draw balls uniformly with replacement repeatedly and independently until we get two red in succession and stop. What is the probability that we draw exactly 4 balls?

The probability space for 5 draws with replacement is $\{r, g, y\}^4$ and the probability of a given outcome with a reds, b green and c yellow is $(4/15)^a(7/15)^b(4/15)^c$. Call E the event that we stop at draw four. This means that the last two draws (draws 3 and 4) were red and the second draw was not red. The first draw can be any color. So $P(E) = P(\text{1st draw is any color}) \times P(\text{second draw is not red}) \times P(\text{draws 3 and 4 are red}) = 1 \times (11/15) \times (4/15)^2 = 11 \times 4^2/(15)^3$.

(b-4pts) We draw balls uniformly with replacement, repeatedly and independently. What is the probability that we will draw a green ball before we draw any red ball?

To answer this question, we first compute the probability p_k that we get the first red or green a the k-th draw and it is green. For $k = 1, 2, \ldots$, $p_k = (4/15)^{k-1}(7/15)$. So the probability that we draw a green ball before drawing a red ball is $(7/15)\sum_{k=1}^{\infty}(4/15)^{k-1} = \frac{7}{15}\frac{1}{1-4/15} = \frac{7}{11}$.

Other solution: We condition on the result of the first draw: Call A the event we will draw a green ball before we draw any red ball and call R,G,Y the event that the first draw is R, G, Y.

$$\begin{aligned} P(A) &= P(AR) + P(AG) + P(AY) = P(R)P(A|R) + P(G)P(A|G) + P(Y)P(A|Y) \\ &= (4/15) \times 0 + (7/15) \times 1 + (4/15) \times P(A). \end{aligned}$$

So $P(A)(1 - 4/15) = 7/15$ which gives $P(A) = 7/11$.

Problem 2: (10 pts) There are two dice. Die A has 4 red and 2 blue faces. Die B has 1 red and 5 blue faces.

(a-3pts) We roll both dice. What is the probability that the faces shown are of different colors?

Assume as we may that the faces are also numbered with the appropriate number of low number faces being red. Use $\{1, \ldots, 6\}^2$ with the uniform distribution as our probability space. Each individual outcome as probability $1/36$ to appear. Of these individual outcomes, ordered pairs in $\{1, 2, 3, 4\} \times \{2, 3, 4, 5, 6\}$ and in $\{5, 6\} \times \{1\}$ have two colors. So the probability of a bi-color outcome is $22/36 = 11/18$.

(b-3pts). We now consider the following experiment: Someone flip a fair coin. If it comes Heads, they pick die A. If it comes Tails, they pick die B. Then they throw the die repeatedly and report the results of the rolls, red or blue. At any given throw, say the k-th throw, what is the probability that the result is red ?

Let H be the event that the coin flips give Heads. Let R be the result that the k-th roll yields red. It does not matter what k is so we can think of $k = 1$. We have

$$P(R) = P(R \cap H) + P(R \cap H^c) = P(R|H)P(H) + P(R|H^c)P(H^c)$$

and $P(H) = P(H^c) = 1/2$, $P(R|H) = 2/3$ and $P(R|H^c) = 1/6$. This gives

$$P(R) = 1/3 + 1/12 = 5/12.$$

(c-4pts) We continue with the set-up of (b). If the first throw yields red, what is the probability that the result of the tenth throw will be red?

Let E the the event that the tenth throw yields red. Write

$$\begin{aligned} P(E|R) &= P(E \cap H|R) + P(E \cap H^c|R) = \frac{P(E \cap H \cap R)}{P(R)} + \frac{P(E \cap H^c \cap R)}{P(R)} \\ &= \frac{P(E \cap R|H)P(H)}{P(R)} + \frac{P(E \cap R|H^c)P(H^c)}{P(R)} \\ &= \frac{(2/3)^2(1/2)}{5/12} + \frac{(1/6)^2(1/2)}{5/12} = \frac{8}{15} + \frac{1}{30} = \frac{17}{30}. \end{aligned}$$

Problem 3: (7 pts) A scrabble bag contains 5 tiles: E,E E,R,S. We pick a pair of tiles uniformly at random in the bag. If we draw two different letters, we call the draw successful.

(a-3pts) What is the probability to get two different letters?

We compute the probability that we get two different letters (for this, we imagine the letters being E1,E2,E3,R,S). There are 7 pairs of letters containing different letters and there are $\binom{5}{2} = 10$ pairs of any kind. So, the probability to get a pair with different letters is $7/10$.

(b-4pts) We repeat the same experiment, replacing all letters in the bag after each draw, until the third successful draw. Consider the events

$$T = \{\text{The third successful draw happens at the tenth draw}\}$$

and

$$E_k = \{\text{the first successful draw happens at the } k\text{-th draw}\}.$$

For each $k = 1, \ldots, 8$, what is the probability that we drew our first pair of distinct letters at the k-th draw given that that the third successful draw happens at the 10th draw?

We are asked to compute

$$P(E_k|T) = \frac{P(E_k \cap T)}{P(T)}.$$

Reasoning as in (a), we have $P(T) = (3/10)^7(7/10)^3\binom{9}{2}$.

Now, we compute $P(E_k \cap T) = (3/10)^7(7/10)^3(9-k)$ where $9-k$ stands for the number of positions that can produce the second successful draw between the first at k and the third at the tenth draw. This gives

$$P(E_k|T) = \frac{9-k}{\binom{9}{2}} = \frac{1}{4}\left(1 - \frac{k}{9}\right).$$