MATH 4710 **PRELIM 1: SOLUTION** **FALL 2020**

Instructions

(1) There are 4 independent problems. The point total is 30. Start early so that you have time to come back to those questions that require you to work harder and read some documents. Take care to solve correctly those questions that are easiest for you. Even when it is not required, ask yourself: what probability space am I using? what are the elementary outcomes used to describe the problem?

(2) You can write your answers on any reasonable media you that is convenient to you as long as you can produce a clean pdf file to upload on gradescope. Write your name and Cornell NetID on the top of the first page before you begin.

(3) Make sure you clearly indicate the questions you are addressing and separate them neatly from each other. Write clearly using a black or blue pen or pencil if you write on paper.
When you upload your exam on gradescope, please assign problems to your pages.

(4) Provide reasons for your answers and explain your computations. For numerical answers, give either a simplified fraction or a decimal answer, which ever comes more easily. You can use a basic calculator (e.g., Desmos) if needed.

(5) You can use your notes, our canvas website including all documents provided there and the book. Do not use other websites or the internet (except for Desmos or a simple electronic calculator). Do not discuss prelim problems with other students. Do not discuss prelim problems with anyone except Pr. Saloff-Coste (ask Professor Saloff-Coste privately on Piazza, by email, or in office hours if you have questions. It is Ok to do so).

(6) Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty.

Problem 1: (6 pts) A hat contains five cards. One of the cards is red on both sides, two are black on both sides and the last two cards have one red side and one black side. A card is picked from the hat uniformly at random and put on the table so that only the color of the top side is revealed (the side shown up is equally likely to be either side of the chosen card). The side shown is red. What is the probability that the other side is black?

Let A be the event that the side shown on the table is red and B the event that the card on the table is a red/black card. We are asked to compute $P(B|A)$, the conditional probability that the card picked is a red/black card given that we know it has a red side. It is easy to compute $P(A|B) = 1/2$ and $P(B) = 2/5$ so $P(A \cap B) = 1/5$. We need to compute $P(A)$. Let C the event the red/red card is chosen, and D the event that a black/black card is chosen. We have

$$P(A) = P(A \cap B) + P(A \cap C) + P(A \cap D) = P(A|B)P(B) + P(A|C)P(C) + P(A|D)P(D).$$

This gives

$$P(A) = \frac{2}{5} \times \frac{1}{2} + \frac{1}{5} \times 1 + \frac{2}{5} \times 0 = \frac{2}{5}.$$

Finally, we can compute

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{5} \times \frac{5}{2} = \frac{1}{2}.$$

Equivalently, you can also use the Bayes formula

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|C)P(C) + P(A|D)P(D)}$$

directly. In this case, you should quote the book of the Notes or at least name the Bayes formula.

Problem 2: (10 pts) An electrician buys components in lots of size ten. When receiving a lot of components, the electrician inspects three components and rejects the lot unless all three are non-defective. Assume that 30 percent of the lots have 4 defective components and 70 percent have 1 defective component.

(a-6 pts) What is the probability that the electrician will reject the next lot sent to him?

Let A be the event that the electrician rejects the next lot sent to him. We will compute $P(A^c)$ as this require the sample of three to contain no defective component. Let B the event that the lot contain 4 defective component and C the event that the lot contains 1 defective component. We write

$$P(A^c) = P(A^c|B)P(B) + P(A^c|C)P(C) = \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} \frac{3}{10} + \frac{\binom{1}{0}\binom{9}{3}}{\binom{10}{3}} \frac{7}{10}.$$

This gives

$$P(A^c) = \frac{6 \times 5 \times 4}{3 \times 2} \frac{3 \times 2}{10 \times 9 \times 8} \frac{3}{10} + \frac{9 \times 8 \times 7}{3 \times 2} \frac{3 \times 2}{10 \times 9 \times 8} \frac{7}{10} = \frac{1}{20} + \frac{49}{100} = .54.$$

So, $P(A) = 1 - P(A^c) = .46$.

(b-4 pts) What do you think about the hypothesis made in this exercise about the distribution of defective components in a lot? Assuming the components are all produced through the same process and that this process as probability $0.1 = 1/10$ to produce a defective component, propose a probability distribution for the number X of defective components in a lot of ten components. Justify your proposal.

The assumption is strange. Why only 4 or 1 defective components? It is not very realistic. If we assume that the same fabrication process is repeated independently for each component and that the probability to obtain a defective component is $1/10$ then the number X of defective components in a lot should follow a binomial distribution with parameters $N = 10$ and $p = 1/10$. This means that for each $k \in \{0, 1, \ldots, 10\}$, $P(X = k) = (1/10)^k (9/10)^{10-k} \binom{10}{k}$.

Problem 3: (7 pts) A gambler has two coins in his or her pocket, one fair coin and one two-headed coin. One coin is chosen uniformly at random and flipped, showing Heads.

(a-3 pts) What is the probability that the coin being flipped is the fair coin?

Let A be the event that the coin being flipped is the fair coin. Let B the event that the flipped coin shows Heads. We want to compute

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{(1/2) \times (1/2)}{(1/2) \times (1/2) + 1 \times (1/2)} = \frac{1}{3}.$$

(b- 2pts) The same coin is flipped 2 more times, showing Heads each time. Now, what is the probability it is a fair coin?

Let A still be the event that the coin being flipped is the fair coin. Let C be the event that the first 3 flips give Heads. Again,

$$P(A|C) = \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|A^c)P(A^c)} = \frac{(1/8) \times (1/2)}{(1/8) \times (1/2) + 1 \times (1/2)} = \frac{1}{9}.$$

(c-2 pts) The same coin is flipped once more and shows Tails. What is the probability that coin being flipped is the fair coin?

The answer is we are certain that it is the fair coin that is being flipped because we have seen both faces and they carry Heads and Tails! The probability in question is 1. (You can use the same type of argument as in the two earlier question to see that these computations lead to the same conclusion!)

Problem 4: (7 pts). A committee consisting of six members has to be formed to represent a group of seven men and eight women. The committee is formed by choosing uniformly at random a subset of six persons from the group.

(a-4 pts) What is the probability that the committee will include at least three women and at least two men.

The probability space Ω is the set of all subset of exactly six people from the group of 15 so that cardinality of Ω is $\binom{15}{6} = 13 \times 11 \times 7 \times 5$. The subsets of 6 containing at least 3 women and at least 2 men have either 4 women and 2 men or 3 women and 3 men. There are $\binom{8}{4} \times \binom{7}{2} = 7^2 \times 5 \times 3 \times 2$ subsets of the first kind and $\binom{8}{3} \times \binom{7}{3} = 7^2 \times 5 \times 2^3$ subsets of the second kind. So, the desired probability is

$$\frac{7^2 \times 5 \times 2 \times (3+4)}{13 \times 11 \times 7 \times 5} = \frac{7^2 \times 2}{13 \times 11} = \frac{98}{143}.$$

(b-3 pts) If X is the number of women in the committee, what is $P(X = k)$?

There are $\binom{15}{6}$ distinct subsets of 6 people from the group of 15 people. The number of subsets comprising exactly k women, $0 \le k \le 6$, is $\binom{8}{k}\binom{7}{6-k}$. The probability that the committee includes exactly k women is

$$\frac{\binom{8}{k}\binom{7}{6-k}}{\binom{15}{6}}.$$

This is the hypergeometric distribution with parameters $N = 15$, $m = 8$ and $n = 6$.