Counting and Probability

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Reading: Devore 2.1–2.3

Probability Theory

Deals with the analysis and modeling of *random phenomena*.

For now, we'll focus on random phenomena where all outcomes are equally likely, e.g.:

- Flipping a fair coin (outcomes: heads, tails)
- Rolling a fair die (outcomes: 1, 2, 3, 4, 5, 6)

A **probability** is a number between 0 and 1 that measures how likely it is that something (an *event*) will occur.

Example: For a fair die, the probability of rolling a 7 is 0, the probability of rolling a number less than 4 is 0.5, and the probability of rolling something less than 7 is 1.

Computing Probabilities

An **event** is a *collection of outcomes* you're interested in.

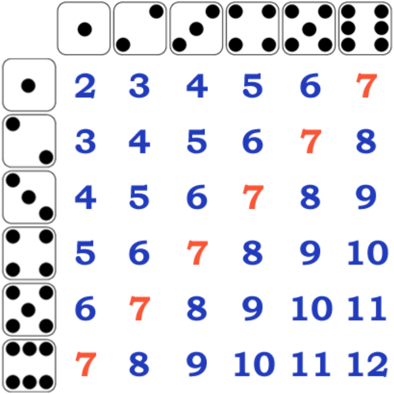
When all outcomes are equally likely, the **probability** that a given event occurs is:

$$P(\text{Event}) = \frac{\#\text{ of outcomes belonging to the event}}{\#\text{ of possible outcomes}}$$

Example:

- For a fair coin, $P(\text{getting heads}) = 1/2$.
- For a fair die, $P(\text{rolling } 1) = 1/6$ and $P(\text{rolling an even number}) = 3/6 = 1/2$.

Example: Rolling Two Fair Dice



Event	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Counting & Probability

When all outcomes are equally likely, computing probabilities requires that you *count* the number of "desired" and possible outcomes.

Counting is hard sometimes.

Example: In poker, the probability of getting a royal flush is

$$P(\text{royal flush}) = \frac{\#\text{ ways to get 10, J, Q, K, A of the same suit}}{\#\text{ possible 5-card hands}}$$

We'll need some *counting techniques*.

Counting: Some Examples

Example: To travel by bus from North Campus to The Commons and back, you can:

- take Route 10, 30, or 31 on the way down;
- take Route 10, 30, 31, 70, or 72 on the way up.

How many options do you have?

Example: You're ordering a burrito, for which you can choose from 4 types of protein, 2 types of beans, and 3 types of salsa. How many different burritos can you order?

Multiplication Rule

For the burrito example:

- for each type of protein, you can choose from two types of beans, and
- for each protein-bean combo, you can choose from three types of salsa.

So, there're $4 \times 2 \times 3 = 24$ possible burritos. (Draw a "tree"!)

Multiplication Rule: If you have K types of things, and for type $i = 1, 2 \ldots, K$ there're N_i items, then there're

$$N_1 \times N_2 \cdots \times N_K$$

ways to select one item of each type.

Example: License Plates

New York license plate numbers consist of 3 letters and 4 digits.

What's the probability that a randomly-generated plate contains four repeating digits (e.g., RWX-3333)?

A. $10/(26^3 \times 10^4)$
B. $10/(3^{26} \times 4^{10})$
C. $(26^3 \times 10)/(26^3 \times 10^4)$
D. $(26^3 \times 10)/(3^{26} \times 4^{10})$

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Example: Selecting Subsets

You're packing for a trip, and have 5 different shirts.

Question: How many ways can you select shirts to pack? (You might pack none, or all 5 shirts.)

(Ordered) Sampling With Replacement

Task: Draw a sample of size k *with replacement* from a set of n distinct items.

Question: How many *ordered samples* are possible?

A. nk
B. k^n
C. n^k

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Example: Eating Lunch

The ORIE faculty go out to lunch every day. There're only 3 restaurants in Collegetown that they all like.

Question: How many weeks can they go without repeating the same Monday-Friday sequence of restaurants?

A. 15
B. 5^3
C. 3^5
D. π

(Ordered) Sampling Without Replacement

Example: You have 1,000 songs on your computer, and you want to make a playlist with 10 songs. How many different playlists can you make?

Hint: There're 1,000 possible songs to put first, 999 possible songs to put second, ... (Draw a "tree"!)

(Ordered) Sampling Without Replacement

Task: Draw a sample of size k *without replacement* from a set of n distinct items.

Question: How many *ordered samples* are possible?

A. $n!/(n-k)! = n \times (n-1) \times \cdots \times (n-k+1)$
B. $n! = n \times (n-1) \times \ldots 2 \times 1$
C. $n!/k! = n \times (n-1) \times \cdots \times (k+1)$
D. $k! = k \times (k-1) \times \cdots \times 2 \times 1$

Permutations (Ordered Sampling Without Replacement)

(Ordered) sampling of k items without replacement from a set of n distinct items = a **permutation** (i.e., a re-ordering) of a list of k out of n possible items.

The number of ways of doing this is

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Example: How many ways can 5 people line up in front of the cash register?

Example: How many ways can 10 Olympians finish first, second, and third?

Combinations (Unordered Sampling Without Replacement)

For permutations the *order matters*.

- e.g., Klay, Steph, Kevin is different from Steph, Kevin, Klay.

Sometimes we don't care what the order is.

- e.g., if we only care whether Klay, Steph, Kevin participate.

Combinations: The number of ways to select a set of k items from a set of n distinct items (i.e., the number of *unordered samples without replacement*) is

$$C_{k,n} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Example: Poker

Question: What's the probability of getting a royal flush?

Answer:

$$
\begin{aligned}
P(\text{royal flush}) &= \frac{\#\text{ of ways to get 10, J, Q, K, A of the same suit}}{\#\text{ possible 5-card hands}} \\
&= \frac{4}{\binom{52}{5}} \\
&= \frac{4}{2,598,960} \approx 0.0000015
\end{aligned}
$$

Properties of Combinations

$$C_{k,n} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

$$= \text{“}n \text{ choose } k\text{”}$$

$$= \# \text{ subsets of size } k \text{ from a set of size } n$$

Useful Identity: $\binom{n}{k} = \binom{n}{n-k}$

Properties of Combination

Useful Identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

List the n items from 1 up to n. You either choose the first item, leaving $k-1$ items from the remaining $n-1$, or you don't choose the first item, leaving k items from the remaining $n-1$.

More Examples

You're hosting a dinner, and have a large supply of each of 10 varietals of wine. You wish to serve one bottle with the appetizer, one with the pasta course, and one with the meat course. You don't mind repeating the same varietal.

Question: How many ways can you do this?

A. 10^3
B. 3^{10}
C. $P_{3,10}$
D. $\binom{10}{3}$

More Examples

Quality Control: A factory makes 25 cars each day, and selects 5 at random to use for a quality check.

Question: How many ways can this selection be made?

A. 5^{25}

B. 25^5

C. $P_{5,25}$

D. $\binom{25}{5}$

Hypergeometric Probabilities: Example

You sample, without replacement, 3 balls from a bin containing 5 red balls and 3 blue balls.

Question: What's the probability of getting 1 red ball (and 2 blue balls)?

Hypergeometric Probabilities

Suppose we take a sample, without replacement, of size k from a population containing N_1 Type 1 members and N_2 Type 2 members. Then for $k_1 = 1, 2, \ldots, \min\{k, N_1\}$,

$$P(\text{get } k_1 \text{ Type 1 members}) = \frac{\binom{N_1}{k_1}\binom{N_2}{k-k_1}}{\binom{N_1+N_2}{k}}$$

Given N_1, N_2, and k, the probabilities that this assigns to the numbers k_1 is called the **hypergeometric distribution**.

Hypergeometric Probabilities

Example: A class consists of 210 students, 30 of whom are smokers and 180 of whom are non-smokers. If we select 12 students at random from this class, what is the probability that we get exactly 3 smokers?

Hypergeometric Probabilities

Example: A class consists of 210 students, 30 of whom are smokers and 180 of whom are non-smokers. If we select 12 students at random from this class, what is the probability that we get exactly 3 smokers?

We have $N_1 = 30, N_2 = 180, k = 12, k_1 = 3$. Plug into the previous formula we get

$$\frac{\binom{N_1}{k_1}\binom{N_2}{k-k_1}}{\binom{N_1+N_2}{k}} = \frac{\binom{30}{3}\binom{180}{9}}{\binom{210}{12}}$$

Summary

The number of ways to sample k things from a total of n things

1. with replacement, and order matters, is n^k;
2. without replacement, and order matters, is $P_{k,n}$
3. without replacement, and order doesn't matter, is $C_{k,n} = \binom{n}{k}$