Functions of multivariate random variables

Function of one real variable: Given a continuous random variable X with density function f_X and a function g, how to find the density function of $Y = g(X)$?

Method 1: use the cumulative distribution of X, \(F_X(x)=P(X\le x)=\int_{\{t\le x\}}f_X(t)dt\) and us it to compute $F_Y(y) = P(Y \le y) = P(g(X) \le y)$. Then take derivative because $f_Y(y) = F'_Y(y)$.

Method 2 (assuming more on g): Assume that g is differentiable, one-to-one and its derivative is zero at almost finitely many points. Then

$$f_Y(y) = f_X(g^{-1}(y))\frac{1}{|g'(g^{-1}(y))|}$$

at point y such that $g^{-1}(y)$ exists and $g'(g^{-1}(y)) \neq 0$. At any other points, $f_Y(y) = 0$.

For a more general case, see Fact 5.27 in the book, page 193.

Example: Let X be a uniform random variable on the interval $[0, 1]$ and set $Y = X(1 - X)$. Find the density of Y.

Solution: In any exercise of this type, it is a good idea to figure out first what is the range of values taken by Y. Here this means figuring out what is the range of the function $x \mapsto x(1 - x)$ on the interval $[0, 1]$. Because the function is continuous and the interval $[0, 1]$ bounded and closed, we know that the image of the interval is also a bounded closed interval. The smallest possible value is obviously 0, taken at 0 and 1. Using calculus, we find that the function $x \mapsto x(1 - x)$ is maximum at $x = 1/2$ where it takes the value $1/4$. So the random variable Y takes values in $[0, 1/4]$. We compute its cumulative distribution function
F_Y. If $y < 0$, $F_Y(y) = 0$; if $y > 1/4$, $F_Y(y) = 1$ and if $0 \le y \le 1/4$,
$F_Y(y) = P\left(X(1 - X) < y\right)$. We need to express the condition $X(1 - X) < y$ in a a more transparent way in terms of X. For fixed y, the quantity $X(1 - X) - y$ is a quadratic polynomial in X. To study its sign, we write it as $-X^2 + X - y$ and complete the square

$$X(1 - X) - y = -X^2 + X - y = -(X - 1/2)^2 + (1/4) - y.$$

From this expression it is clear that, for $0 \le y \le 1/4$,

$$-(X-1/2)^2+(1/4)-y<0 \iff (X-1/2)^2>(1/4)-y$$
$$\iff \begin{cases} X-(1/2)>\sqrt{(1/4)-y} \\ \quad\text{or} \\ (1/2)-X>\sqrt{(1/4)-y}. \end{cases}$$

So, we can compute

$$P(Y<y)=P(X(1-X)<y)=P(-X^2+X-y<0)=P\left(X>(1/2)+\sqrt{(1/4)-y}\right)$$

Because X is uniform on $[0,1]$, $F_X(x)=P(X<x)=x,\ \ x\in[0,1]$. This gives

$$P(Y<y)=1-\frac{1}{2}-\sqrt{\frac{1}{4}-y}+\frac{1}{2}-\sqrt{\frac{1}{4}-y}=1-2\sqrt{\frac{1}{4}-y}.$$

We can check that $F_Y(0)=0$ and $F_Y(1/4)=1$ as expected. Finally, we take the derivative in y to find

$$f_Y(y)=\frac{1}{\sqrt{\frac{1}{4}-y}}=\left(\frac{1}{4}-y\right)^{-1/2},\ \ y\in[0,1/4].$$

Find this same result using Fact 5.27 Page 193.

The multivariate case: Change of variables and computation of densities

Assume that we have a change of variables formula $u=g(x,y), v=h(x,y)$ from a region K to a region L so that $(x,y)\mapsto(u,v)=(g(x,y),h(x,y))$ is one to one and onto between K and L. Let (q,r) be the inverse map $(u,v)\mapsto(x,y)=(q(u,v),r(u,v))$.

Assume that g,h are continuous with continuous partial derivatives over K. Define the Jacobian determinants

$$J_K(x,y)=\begin{vmatrix} \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \\ \frac{\partial h}{\partial x}(x,y) & \frac{\partial h}{\partial y}(x,y) \end{vmatrix}$$

and

$$J_L(u,v) = \begin{vmatrix} \frac{\partial q}{\partial u}(u,v) & \frac{\partial q}{\partial v}(u,v) \\ \frac{\partial r}{\partial u}(u,v) & \frac{\partial r}{\partial v}(u,v) \end{vmatrix}.$$

By construction, it is always the case that $J_K((q(u,v),r(u,v))J_L(u,v) = 1$ (as long as these Jacobian exists), because the composition of the two maps gives the identity map (either on K or on L depending of the order of composition). We assume that one of the two Jacobian exists and never vanishes on the relevant domain (this implies that the other exists and never vanishes either).

Change of variable formula Let $Z = (X,Y)$ have joint density function $f_Z(x,y)$ with $f_Z = 0$ outside K. Let $W = (U,V)$ be defined by

$$U = g(X,Y), V = h(X,Y).$$

This makes sense because Z takes value in K and g, h are defined on K. Assume that the Jacobian $J_K(x,y)$ does not vanish over K. The joint density function of $W = (U,V)$, f_W is 0 outside L and given for $(u,v) \in L$ by

$$\begin{aligned} f_W(u,v) &= f_Z(q(u,v),r(u,v))\frac{1}{|J_K(q(u,v),r(u,v))|} \\ &= f_Z(q(u,v),r(u,v))|J_L(u,v)|. \end{aligned}$$

Example: If X,Y are independent normal $\mathcal{N}(0,1)$ random variables and $(X,Y) = Re^{i\Theta}$, $R > 0, \Theta \in [0,2\pi)$, what is joint distribution of (R,Θ)? Are R,Θ independent? What is the distribution of R?

Use the change of variable $K = \mathbb{R}^2 \setminus \{(0,0)\} \ni (x,y) \leftrightarrow (r,\theta) \in (0,\infty) \times [0,2\pi) = L$. Note how we where careful to avoid $(0,0)$ and observe that it is much easier to express this (famous) change of variables by writing (x=r \cos \theta,y=r\sin \theta\) than expressing r and θ in terms of x and y. This is because, although $r = \sqrt{x^2+y^2}$ is relatively simple, the formula $\theta = \tan^{-1} y/x$ is NOT really correct for the full range of x, y. However, we have a good change of variable and we can compute the Jacobian $J_L(r,\theta) = r$. Set $W = (R,\Theta)$. We obtain (on $(0,\infty) \times [0,2\pi)$)

$$f_W(r,\theta) = f_{(X,Y)}(r\cos\theta, r\sin\theta) \times r = \frac{1}{2\pi} re^{-r^2/2}.$$

This is the product of $\frac{1}{2\pi}\mathbf{1}_{[0,2\pi)}(\theta)$ which the density of a uniform on $[0,2\pi)$ and of
$re^{-r^2/2}\mathbf{1}_{(0,\infty)}(r)$
(also known as the Raleigh distribution). We can conclude that R and Θ are independent, Θ being uniform on $[0,2\pi)$ and R is Raleigh.

Two more examples as exercises:

Assume that X, Y are independent normal $\mathcal{N}(0,1)$ random variables. Repeat the above argument carefully to find the distribution of (T, Θ), $T = X^2 + Y^2$, $X = \sqrt{T}\cos\Theta, Y = \sqrt{T}\sin\Theta$.

Let X, Y be independent exponential λ random variables. What is the joint density of $Z = (X, Y)$. Find the density of $(U, V) = (X + Y, X/Y)$. Is it rue or not that $X + Y$ and X/Y are independent?

Independence : If the random variables $X_1, \ldots, X_k$ are mutually independent and we partition them into ℓ disjoint groups and define random variables $Y_1, \ldots, Y_\ell$ so that, for each $i \in \{1, \ldots, \ell\}$, Y_i is a function of the X_js in group i, then the variables $Y_1, \ldots, Y_\ell$ are mutually independent.

Functions of many random variables: Let $Y = g(X_1, \ldots, X_k)$. To compute $E(Y)$ it often bets to write

$$E(Y) = \int_{\mathbb{R}^k} g(x_1, \ldots, x_k) f_{X_1,\ldots,X_k}(x_1, \ldots, x_k) dx_1 \ldots dx_k.$$

Finding the density a function of many continuous random variables: Let X, Y be uniform in the unit square with lower-left corner at $(0,0)$, i.e., $(0,1)^2$. What is the density function of $Z = XY$? For this type of question, it is often best to start with computing the cumulative function of $Z, F_Z(z) = P(Z \le z)$. Also, it is important to figure out what is the important range of z. For instance, here, it is clear that $0 < Z < 1$ so $F_Z(z) = 0$ if $z \le 0$ and $F_Z(z) = 1$ if $z \ge 1$. For $z \in (0,1)$, we write

$$F_Z(z) = P(XY \le z) = \iint_{(0,1)^2; xy \le z} dxdy = \int_0^1 \left(\int_0^{\min\{1, z/x} dy \right) dx.$$

this is because, for any given $x \in (0,1)$, we near to integrate over all $y \in (0,1)$ such that $y \le z/x$. This means integrating over all $0 < y < \min\{1, z/x\}$ (the discrepancy between $<$ and $\le$ does not matter here). Because of the $\min$ it helpful to spelt the integral offer x into the integral between 0 and z and the integral between z and 1:

$$F_Z(z) = P(XY \le z) = \int_0^1 \left(\int_0^{\min\{1, z/x} dy \right) dx = \int_0^z \int_0^1 dydx + \int_z^1 \int_0^{z/x} dydx = z + z$$

Not that this function is 0 at 0 and 1 at 1. Its derivative, $F_Z'(z) = 1 + \ln(1/z) - 1 = \ln(1/z)$ is the density of the random variable Z.