Time Value of Money

AEM 2241 - Finance

Tibor Jánosi

Cornell University - Fall 2023

Outline

Goals for Today

- Determine the future value of an investment made today.
- Determine the value today of cash to be received at a future date.
- Compute the return of an investment.
- Compute how long it takes a certain investment to reach a particular value.

Note: In a more general setting "today" can be interpreted simply as "earlier than the future date marking the end of the investment."

Time Value of Money | Future Value | Present Value | Discount Rate | Number of Periods | Remarks and Conclusions

Definitions

- Present value - amount of money at "earlier" times; often, but not always, at time=0, or "now."
- Future value - amount of money at "later" times; often, but not always, at time>0.
- Interest rate - "exchange rate" between present value (earlier money) and future value (later money). Interest rates can be expressed in many ways, and they may be known under different names in different situations: discount rate; cost of capital; opportunity cost of capital; required return.
- Interest rates do not exist physically, unlike, say, currency. Interest rates are conceptual tools that express connections between various currency (value) amounts.

Future Value - Example 1

- Suppose you invest \$1,000 for one year at 5% per year. What is the future value in one year?
- Interest $= 1,000(.05) =$ \$50. Do not confuse **interest** (a currency amount) with **interest rate** (a number). Note: For simplicity, we will often not show the dollar sign.
- Value in one year $=$ principal $+$ interest $= 1,000 + 50 = 1,050$;
- Future Value (FV) $= 1,000(1 + .05) = 1,000 \cdot 1.05 = 1,050$.
- What would be the answer if we invested \$100 or \$1? Why? Is this assumption realistic?

Compound Interest

- In the context of the previous example, assume that at the end of year 1 we invest for another year under the same terms. How much money will we have at the end of two years?
- At the end of the first year we have \$1,050. In the second year, we will earn interest on both the original \$1,000 and the \$50 of interest already received. This exemplifies the concept of **compound interest**.
- $FV = 1,050(1.05) = 1,000(1.05)(1.05) = 1,000(1.05)^2 = 1,102.50$.
- **Without** compound interest, we would earn the same \$50 in interest in the second year as well. This situation exemplifies simple interest. $FV = 1,000 + 50 + 50 = 1,100$.

Compound Interest (Cont'd)

- We earn **extra interest** due to compounding = $FV_{compound} - FV_{simple} = 1,102.50 - 1,100 = 2.50$.
- When **nominal** interest rates are the same, compound interest is **higher** than simple interest (only if $r > 0$ and $t > 1$).
- It is possible, however, to choose simple interest rates that exactly match a given compound interest rate (and vice versa).
- The effect of compounding is small if either the nominal interest rate is small, or the investment period is short. The higher the **nominal** interest rate, and the longer the investment period, the bigger the excess of compound interest over simple interest.

Compounding Is Very General

- Compounding is a general phenomenon that can be applied in contexts that transcend a purely financial framework.
- Suppose your company expects to increase unit sales of widgets by 15% per year for the next 5 years. If you sell 3 million widgets in the current year, how many widgets do you expect to sell in the fifth year?
- Suppose that the world's population grows at 2% per year. Assuming that the growth trend stays constant, how many times bigger will the world's population be in 200 years?
- Over very long time horizons compounding formulas will give results that are valid mathematically, but are unrealistic. This applies to both financial and non-financial situations.

Future Value - General Case

- Suppose we invest PV dollars at the beginning of year 1. We earn an interest of $r\%$ each year; we keep the money invested for t years. How much money do we have at the end of t years?

- $FV = PV(1+r)^t$
  - FV = future value;
  - PV = present value;
  - r = **period** interest rate, expressed as a decimal;
  - t = number of **periods** (often, but not necessarily, years).

- **Total** interest earned during the period $= FV - PV$.
- Future value interest factor $= (1+r)^t$.

Future Value as a Function of Interest Rates

- Future values are higher the higher the interest rate. **Increments** (increases) of the interest rate by the same amount have a bigger effect if the original interest rate was higher.
- Example: $t = 5$ years, PV=\$1,000. We use the formula $FV = PV\,(1+r)^t$.

r	FV_{base}	FV	$FV - FV_{base}$
5%	1,276.28		
6%=5%+1%		1,338.23	61.94
10%	1,610.51		
11%=10%+1%		1,685.06	74.55

- Each additional percent of interest will cost your more than the immediately prior percent. Each percent of discount you get on an interest rate will save you less than the immediately preceding percent obtained as discount.

Cash Flows Can Have Signs

- The mathematical formulas that we use assume that both present values and future values are positive amounts. We, as users of these formulas will keep track of which of these are outlays or inflows of cash.
- When using financial calculators, however, we must allow for outlays to be differentiated from inflows. To do this we use a **cash flow sign convention**.
- **Inflows** of cash (money you get) will be **positive**, since you have more money after an inflow.
- **Outflows** of cash (money you pay out or invest) will be **negative**, since you have less money after an outflow.
- You must use this convention consistently in all work that you do in this course, whenever the need for using signed cash flow arises (e.g., when using a financial calculator).

Using Financial Calculators

- The instructor will use the TI BA-II Plus (Professional).
- Use the following keys to enter known values:
  - FV = future value;
  - PV = present value;
  - I/Y = per-period interest rate, entered as a **percent**, not decimal;
  - P/Y = payments per period (year). For now, **set this to 1;**
  - N = number of periods.
- You set the values that are known, then you press CPT (compute) and the key corresponding to the value you do not know.
- The values entered above are set to be held in "registers." To be safe, you should clear registers after each problem (CLR TVM).

Present Value

- How much do I have to invest today to have some amount in the future?
- We already have the connection between present and future values. We treat this formula as an equation and solve for PV, given FV and the interest rate.
- $FV = PV \cdot (1+r)^t \Rightarrow PV = \frac{FV}{(1+r)^t}$;
  where $\frac{1}{(1+r)^t} =$ *present value* (*interest*) *factor*, also known as *discount factor*.
- **Discounting** means finding the present value of a certain future value, given the number of periods and the interest rate.
- Discounting allows us to talk about the value today of money received in the future.

Present Value - Example 1

- Suppose you need \$10,000 in one year for the down payment on a new car. If you can earn 7% annually, how much do you need to invest today?
- $PV = \frac{10{,}000}{(1.07)^1} = \frac{10{,}000}{1.07} = \$9,345.79$.
- Calculator:
- 1 N;
- 7 I/Y (remember: P/Y must be set to 1);
- 10,000 FV;
- CPT PV
- The result will be shown as $= -9,345.79$. This is a cash outflow "today." Why outflow? You invest!

Present Value as a Function of Investment Period

- Given a certain future value (FV) and an interest rate $r > 0$, **the present value is lower the longer the investment period** (the bigger t is).
- What is the present value of $500 to be received in 5 years? 10 years? The discount rate is 10%.
- 5 years: N = 5; I/Y = 10; FV = 500
  CPT PV = -310.46.
- 10 years: N = 10; I/Y = 10; FV = 500
  CPT PV = -192.77.
- Note that the signs of FV and PV differ!

Present Value as a Function of Interest Rate

- Given a certain future value (FV) and an investment period t, **the present value is lower the higher the interest rate** (the bigger r is).
- What is the present value of $500 received in 5 years if the interest rate is 10%? 15%?
- Rate = 10%: N = 5; I/Y = 10; FV = 500
  CPT PV = -310.46.
- Rate = 15%; N = 5; I/Y = 15; FV = 500
  CPT PV = -248.59.

Discount Rate

- In many situations, the future value and the present value are given (set), or they can be estimated. Assuming that the investment period is also known, what is the implied interest rate on the investment?

$$r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1$$

- When using financial calculators to directly compute this formula, you can also use both the y^x and $\frac{1}{x}$ keys. Use the specialized TVM functions as well as these calculator keys to verify that you get the same result for the implied interest rate!

Discount Rate - Example 1

- You are looking at an investment that will pay $1,200 in 5 years if you invest $1,000 today. What is the implied rate of interest?

$$r = \left(\frac{1,200}{1,000}\right)^{\frac{1}{5}} - 1 = .03714 = 3.714\%.$$

- Calculator example:
  - $N = 5$;
  - $PV = -1,000$ (you pay [give out] $\$1,000$ today);
  - $FV = 1,200$ (you receive $\$1,200$ in 5 years);
  - $CPT\ I/Y = 3.714$.

Discount Rate - Example 2

- Suppose you are offered an investment that will allow you to double your money in 6 years. You have $\$10,000$ to invest. What is the implied rate of interest?
- Calculator example:
  - $N = 6$
  - $PV =$ -10,000
  - $FV =$ 20,000
  - $CPT\ I/Y = 12.25$.
- In the above we assume that PMT is set to 0, with is the default after a RESET or CLR TVM command.

Choosing Among Investments

- You are offered the following investments:
  - You can invest $500 today and receive $600 in 5 years. The investment is considered low risk.
  - You can invest the $500 in a bank account paying 4%. The bank is very reliable.
- What do you think we mean by **low risk** above? What is risk in Finance?
- Which investment should you choose?
- We have two ways to solve the problem. Let's do it!

Choosing Among Investments (Solution)

- **Solution 1**: We compute the future value at time 5 (years) for the two investments: $FV_1 = \$600$ (this is given); $FV_2 = 500 \cdot (1 + 0.04)^5 = \$608.33$. The second investment leads to a higher future value, so it should be preferred.
- **Solution 2**: We compute the implied interest rate for the two investments: $r_1 = \left(\frac{FV_1}{PV_1}\right)^{\frac{1}{5}} - 1 = 3.71\%$, while $r_2 = 4\%$ (this is given). Again, the second investment is preferable. We could have computed r_1 using the calculator: N=5, PV=-500; PMT=0; FV=600; press CPT I/Y to get 3.71%.
- Unsurprisingly, both methods agree in that the second investment is preferable.

Determining the Number of Periods

- Assume that the present value, the future value, as well as the interest rate are given. What is the corresponding number of investment periods?

$$t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$$

Note: **ln** is the natural logarithm. Any other logarithm base would work.

- The number of periods can be computed directly, using the formulas above, or using the calculator keys on the TVM worksheet.

Number of Periods - Example 1

- You want to purchase a new car, and you are willing to pay $\$20,000$.
- If you can invest at 10% per year and you currently have $\$15,000$, how long will it be before you have enough money to pay cash for the car?
- Calculator solution:
  - $I/Y = 10$;
  - $PV = -15,000$;
  - $FV = 20,000$;
  - $CPT\ N = 3.02$ years.
- Note the fractional number of years (periods) in the answer. Does this lead to any complications?

Time Value of Money | Future Value | Present Value | Discount Rate | Number of Periods | Remarks and Conclusions

Number of Periods - Example 2

- Suppose you want to buy a new house.
- You currently have $\$15,000$; you need to have a 10% down payment plus an additional 5% of the loan amount for closing costs.
- Assume the type of house you want will cost about $\$150,000$.
- Also assume that you can earn 7.5% per year for sure and **indefinitely** (as long as needed).
- How long will it be before you have enough money for the down payment and closing costs?

Number of Periods - Example 2 (Cont'd)

- How much do you need to have in the future?
  - Down payment $= 0.10 \cdot (150,000) = \$15,000$;
  - Closing costs $= 0.05 \cdot (150,000–15,000) = \$6,750$;
  - Total needed $= 15,000 + 6,750 = \$21,750$.
- Compute the number of periods.
- Using a financial calculator:
  - $PV = -15,000$;
  - $FV = 21,750$;
  - $I/Y = 7.5$;
  - $CPT\ N = 5.14$ years.
- Using a formula:

$$t = \ln(\frac{21,750}{15,000}) / \ln(1.075) = 5.14\, years.$$

Comprehensive Problem

- You have \$10,000 to invest for five years.
- How much additional interest will you earn if the investment provides a 5% annual return, when compared to a 4.5% annual return?
  The answer is the difference between the future values:
  $FV_1 - FV_2 = 10,000 \cdot \left[(1 + 0.050)^5 - (1 + 0.045)^5\right] \approx \$301.$
- How long will it take your \$10,000 to double in value if it earns 5% annually? Calculator solution: I/Y=5; PV=-10,000; PMT=0; FV=20,000. Press CPT N to get the number of years (periods) as 14.21.
- What annual rate has been earned if \$1,000 grows into \$4,000 in 20 years?
  $r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1 = \left(\frac{4,000}{1,000}\right)^{\frac{1}{20}} - 1 = 7.18\%.$

Mental Calculations

- Even in an age of pervasive computing devices, mental calculations are important. Seasoned businesspeople emphasize the importance of being able to compute approximate present or future values, for example. A lot of money can be made or lost in a negotiation if you have a sense for numbers. Develop and practice your numerical skills!
- A good start is to learn this rule of thumb: "If your interest earns p% per year (period), then it will take approximately $\frac{72}{p}$ years (periods) to double your investment." This works reasonably well over a large range of interest rates; it will not work, however, for interest rates that are very high.
- With this rule of thumb, the interest rate must be expressed as a percentage, **not** as a decimal number.

Mental Calculations (Cont'd)

- PV = 1; FV = 2. We then get the time to doubling as being equal to $t_{doubling} = \frac{\ln 2}{\ln(1+r)}$.
- Fill out the following table by computing the appropriate time to doubling the initial investment:

r	Formula	Rule of Thumb
1%		
5%		
10%	7.27	7.20
15%		
20%		
25%		

For r=10%, the formula gives $t_{doubling} = \frac{\ln 2}{\ln(1+0.1)} = 7.27$years; the rule of thumb gives $t_{doubling} = \frac{72}{10} = 7.20$ years.

More on Interest Rates

- **Nominal** interest rates provide only partial information. One must also understand what kind of interest is used. For now, we know about simple interest and compound interest. We must always know the **interest rate convention** that is used.
- Interest rates can be **negative**. Such a situation arises when FV is lower than PV, e.g., when you lose money on an investment.
- We defined compounding on an integer number of periods; however, some of our calculations generate **fractional number of periods**. For now, we assume that our formulas work for fractional periods (i.e., when t is not an integer). This is not necessarily so in practice. For example, a bank may pay no interest on a time deposit if you withdraw your money before the end of the period.

More on Interest Rates (Cont'd)

- In all our discussions, we assumed that all stated facts are certain. For us, there is no doubt that we will receive the (promised) future cash flows, or that interest rates will not change in later years compared to what we expect them to be. In real life, **uncertainty** is a major source of risk, which greatly impacts considerations related to interest rates.
- Our formulas implicitly assume that present and future values scale linearly. For example if $PV_2 = k \cdot PV_1$, then the formulas predict that $FV_2 = k \cdot FV_1$. In other words, we assume that if we can invest \$1,000, we can also invest \$1, or \$1M in the **same** opportunity under the **same terms**. Typically, this is not the case: investments have size limits, taxes, laws, and other considerations make such situations impossible in real life.

Summary

- The crucial relationship is $FV = PV(1+r)^t$. All the information is embedded in this formula.
- The formula connects four quantities: PV, FV, r, t. If you know three of them, you can always compute the fourth.
- $PV = \frac{FV}{(1+r)^t}$ $\qquad t = \frac{ln\left(\frac{FV}{PV}\right)}{ln(1+r)}$ $\qquad r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1$
- In this course, we will use financial calculators to solve problems, unless explicitly stated otherwise. When using calculators, please remember to set **one** of FV and PV to be positive, and the other negative. If you do not, and you set both PV and FV to have the same sign, then you will get errors or nonsensical answers.

Conclusions

- Understanding the relationship between present values, future values, interest rates, and investment horizons is a fundamental skill in finance.
- There are many kinds of interest rate conventions, but they all ultimately express relationships between cash flows (present and future values).
- We used many simplifications in order to start understanding these relationships.
- There are many fascinating details related to interest rates that emerge when real-life complexity is properly accounted for.

Last updated on August 22, 2023.