Discounted Cash Flows Valuation

AEM 2241 - Finance

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Goals for this Section

- Determine the future and present value of investments with multiple cash flows.
- Define and study several kinds of annuities.
- Explain how loan payments are calculated and how to find the interest rate on a loan.
- Describe how loans are amortized or paid off.
- Show how interest rates are quoted (and misquoted).

Some of these goals will be achieved in the second lecture on discounted cash flows valuation.

How to Handle Multiple Cash Flows

- Until now, we considered two cash flows (values, cash amounts) that occurred at different times, which were made "equivalent" across time by the concept of interest rates.
- One can have more than two cash flows that are connected this way; in principle, any sequence of cash flows, irrespective of timing and/or the sign of these cash flows, can be analyzed.
- When we solve problems of this type we keep in mind that a dollar paid/received at a given moment in time is **not equivalent** to a dollar paid/received at a different moment in time. Informally, we can say that "dollars received at different times are different dollars." We use the concepts of present value and future value to convert amounts received/paid at different times to equivalent ("contemporaneous") dollars.

Future Values: Periodic Deposits/Investments

- You have \$5,000 in your bank account.
- You can deposit \$2,000 at the **end** of each year for the five years.
- You earn 10% per year. Interest compounds yearly.
- How much will you have at the **end** of the fifth year?

Overview | FV With Multiple CFs | PV With Multiple CFs | Annuities | Perpetuities | Special Annuities

Future Values: Alternate Solutions

- We can approach the problem in one of two ways:
  - We "roll forward" each cash flow from the date it occurs to the end date we are interested in. Each amount preserves its separate identity until all amounts are totaled.
  - We consolidate all the cash we have at the end of each year; and roll the total forward for one year at a time. All amounts are amalgamated together and lose their individual identity.
- Hidden assumption: Given two amounts A_1 and A_2, we assume that the interest earned separately on these two amounts is the same as the interest earned jointly on $A_1 + A_2$: $\mathit{interest}(A_1) + \mathit{interest}(A_2) = \mathit{interest}(A_1 + A_2)$. This is not necessarily the case, as financial transactions have both fixed and variable costs.

Solution 1: Periodic Deposits

- Roll forward each cash flow independently:

	Year 0	Year 1	Year 2	Year 3	Year 4	Year 5
New Deposit	$5,000$					$5,000x1.1^5 = 8,052.55$
		$2,000$				$2,000x1.1^4 = 2,928.20$
			$2,000$			$2,000x1.1^3 = 2,662.00$
				$2,000$		$2,000x1.1^2 = 2,420.00$
					$2,000$	$2,000x1.1 = 2,200.00$
						$2,000$
Total						$20,262.75$

- The final amount is equal to $\$5,000x1.1^5 + \$2,000x\left(1.1^4 + 1.1^3 + 1.1^2 + 1.1 + 1\right)$. Note the geometric progression whose sum multiplies the per-period deposit of $\$2,000$.

Solution 2: Periodic Deposits

- We amalgamate cash flows at the end of each year, then roll forward the total one year at a time.
- The two solutions lead to the same final result; however, the second solution also allows us to track the changes in the yearly total value of the deposit.
- Other solutions are possible; however, they would not offer any significant additional insights. Can you think of any such solutions?

	Year 0	Year 1	Year 2	Year 3	Year 4	Year 5
Existing cash		$5,000.00x1.1$	$7,500.00x1.1$	$10,250.00x1.1$	$13,275.00x1.1$	$16,602.50x1.1$
New deposit	$5,000$	$2,000$	$2,000$	$2,000$	$2,000$	$2,000$
Total	$5,000$	$7,500.00$	$10,250.00$	$13,275.00$	$16,602.50$	$20,262.75$

Present Value - Multiple Cash Flows

- Suppose you will face an outlay of cash of \$1,000 in one year, and \$2,000 in two years. Payments occur at the end of each year.
- You have access to an investment opportunity that earns 10% per year.
- How much do you need to invest at the beginning of the first year so that these payments can be made?
- Like before, we have two solutions:
  - Discount each future cash flow back to the start of year 1 (start of the investment period; same as the end of year 0); or
  - Discount each future cash flow back only one year at a time and amalgamate (add up) cash flows.

Solutions - PV with Multiple Cash Flows

- Solution 1: Discount each cash flow back directly to the start of the investment period:

	Year 0	**Year 1**	**Year 2**
Payments	$1,000/1.1 = 909.09$	$1,000$	
	$2,000/1.1^2 = 1,652.89$		$2,000$
Total	$2,561.98$		

- Solution 2: Discount each future cash flow back only one year at a time and add up cash flows:

	Year 0	**Year 1**	**Year 2**
Payments		$1,000$	$2,000$
Value of earlier outlays	$2,818.18/1.1 = 2,561.98$	$2,000/1.1 = 1,818.18$	
Total	$2,561.98$	$2,818.18$	$2,000.00$

Timing of Cash Flows

- The timing of cash flows is critical when computing present or future values.
- For problems in which cash flows occur at regular intervals, we must specify whether cash flows occur **at the beginning** or **at the end** of each period. For cash flows that occur irregularly, the time of each such cash flow must be specified.
- If cash flows occur regularly, but nothing is said about their timing, the implicit assumption is **that cash flows occur at the end of each period**.

Financial Calculators - Getting Familiar

- There are special formulas, to be discussed later, for valuing regular, level cash flows; if cash flows are different, or their timing is irregular, each cash flow can be handled separately, using the TVM functions of your calculator.
- Intermediate results can be stored in the memory of your calculator (using the **STO** and **RCL** buttons).
- Attention must be paid to the timing of regular cash flows. Use the **2ND** + **BGN** to check that when cash flows occur. Remember our convention, unless specified otherwise, they occur at the end of the period.
- You can change the timing of cash flows by pressing **2ND** + **BGN**, then **2ND** + **SET**. The default timing convention is also turned on if you reset the calculator by pressing **2ND** + **RESET**.

Financial Calculators - Examples

- You are offered an investment that will pay you \$200 in one year, \$400 the next year, \$600 in the third year, and \$800 in the fourth year. You can earn 12% per year on similar investments. What is the most you would pay for this opportunity?

- If you deposit \$100 in one year, \$200 in two years, and \$300 in three years, how much will you have in five years? Assume that you earn 7% on this account.
  Hint: When you roll forward an investment from the end of the second year to the end of the fifth year, you are investing over a period of 5-2=3 years.

Annuities and Perpetuities

- A series of constant (level) cash flows that occur at the end of regular intervals (periods) for some fixed (predetermined) number of periods is called an **ordinary annuity**. We say that the cash flows are in ordinary annuity form.

- If the cash flows occur indefinitely (forever), the annuity is called a **perpetuity**.

- Annuities occur frequently; for example, mortgages and leases can be interpreted as annuities. Remember, however, that in our world there is no risk/uncertainty. We would not worry about defaults affecting mortgages, say.

Valuing an Annuity

- Assume that you will receive an annuity of C dollars at the end of each period, for t periods. The per-period interest rate is r. What is the present value of the annuity?

$$PV_{annuity} = C \cdot \frac{1 - \left(\frac{1}{1+r}\right)^t}{r}.$$

Note that $\left(\frac{1}{1+r}\right)^t$ is the present-value (interest) factor (or discount factor) of the last annuity payment.

- $PVIFA(r,t) = \frac{1-\left(\frac{1}{1+r}\right)^t}{r}$ is the **present value interest factor for annuities**. Thus $PV_{annuity} = C \cdot PVIFA(r,t)$.

Value of an Annuity - Technical Note

- Consider an annuity with a per-period payment of C, a maturity of t periods, and a per-period interest rate of r. Then the value of the annuity is obtained by summing a geometric progression, representing the time-0 value of each future cash flow.

$$\begin{aligned} PV_{annuity} &= \sum_{i=1}^{t} \frac{C}{(1+r)^i} \\ &= \frac{C}{1+r} \cdot \sum_{i=0}^{t-1} \frac{1}{(1+r)^i} \\ &= \frac{C}{1+r} \cdot \frac{1 - \frac{1}{(1+r)^t}}{1 - \frac{1}{1+r}} \\ &= C \cdot \frac{1 - \frac{1}{(1+r)^t}}{r}. \end{aligned}$$

Value of an Annuity - Example

- You can pay \$632 dollars per months toward buying a car. Your bank quoted a 1% interest rate **per month** for 48-month loans. How much can you borrow to pay for the car, taxes, and other related expenses?
- $PV_{annuity} = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r} = 632 \cdot \frac{1-\left(\frac{1}{1+0.01}\right)^{48}}{0.01} \approx \$24,000.00$.
- We could have calculated the annuity factor explicitly as $PVIFA(r,t) = \frac{1-\left(\frac{1}{1+r}\right)^t}{r} = \frac{1-\left(\frac{1}{1+0.01}\right)^{48}}{0.01} = 37.9740$, then we multiply it with the monthly payment $PV_{annuity} = C \cdot PVIFA(r,t) = 632 \cdot 37.9740 \approx \$24,000.00$.
- So you can get a loan and buy the car if it does not cost more that \$24,000. Could you get a better car with a lease? How does a lease differ from outright buying, given the perspective we take in this class?

Value of an Annuity - Financial Calculators

- Consider the problem on the previous slide; let us solve it using a financial calculator!
- Set $N = 48$, $I/Y = 1$, $PMT = -632$, $FV = 0$, then hit **CPT + PV**.
- Note the following:
  - You may not have to set FV to 0, as this is its value by default. However, it is **safer** to always set values explicitly, especially when you perform many different calculations in sequence.
  - We set the monthly payment to a negative amount, consistent with our convention. The present value of the annuity, however, is positive (this is the loan that we will get).
  - In this example the period is a month, not a year. Still, the computations work as usual.

Annuity Problems - Financial Calculators

- An ordinary annuity has four parameters: C, r, t, and PV. When using calculators, it is possible to find the fourth parameter, no matter which three parameters are given.
- The annuity valuation formula can also be used to determine PV, C, or t, given the other three parameters. However, determining r without a calculator requires special methods, or a trial and error approach.
- Given: PV=\$100,000, r=18%, t=5 years. What is the yearly payment?
  Answer: Set **N=5**, **I/Y=18**, **PV=100,000**, **FV=0**. Press **CPT + PMT** to get -31,977.78.
  If you were to engage in this financial transaction, what would you call it?

Annuity Problems - Financial Calculators

- You charged \$1,000 on your credit card. You will only pay \$20 per month. Your monthly interest rate is 1.5%. How long will it take to pay off your debt?
  Answer: Set **I/Y=1.5**, **PV=1,000**, **PMT=-20**, **FV=0**. Press **CPT + N** to get 93.11.
  How do you interpret a fractional number of months?
- You received an offer for an annuity: If you pay \$6,710 upfront, you will get \$1,000 per year for 10 years. What is the interest rate implicit in this deal?
  Answer: Set **N=10**, **PV=-6,710**, **PMT=1000**, **FV=0**. Press **CPT + I/Y** to get 8.00%.

Annuities: Future Value

- We know the present value of an annuity already, we can compute its future value by multiplying with the future value factor $(1+r)^t$:

$$\begin{aligned} FV_{annuity} &= (1+r)^t \, PV_{annuity} \\ &= (1+r)^t \left[C \cdot \frac{1 - \frac{1}{(1+r)^t}}{r} \right] \\ &= C \cdot \frac{(1+r)^t - 1}{r}. \end{aligned}$$

- $Annuity\ \ FV\ \ factor = \frac{(1+r)^t - 1}{r}$; then

$$FV_{annuity} = (Annuity\ FV\ factor) \cdot C.$$

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Future Value Problem

- You save \$2,000 per year to a retirement account paying 8% per year. How much money will you accumulate at the end of 30 years?

- $Annuity\ \ FV\ \ factor = \frac{(1+r)^t - 1}{r} = \frac{(1+0.08)^{30} - 1}{0.08} = 113.28$; then $FV_{annuity} = (Annuity\ FV\ factor) \cdot C = 113.28 \cdot 2,000 = 226,566.42$ dollars.

- Calculator: **N=30**, **I/Y=8**, **PV=0**, **PMT=-2,000**. Press **CPT + FV** to get \$226,566.42.

Perpetuities

- If the stream of payments continues forever, we have a perpetuity. The valuation formula becomes very simple in this case ($r > 0$):

$$PV_{perpetuity} = \lim_{t \to \infty} C \cdot \frac{1 - \frac{1}{(1+r)^t}}{r} = \frac{C}{r}.$$

- What this annuity's PVIFA?

- Does it make sense to talk about the future value of this annuity?

Perpetuity Problem

- **Preferred shares (stock)** are a special class of shares that (among others) promise a fixed cash dividend every period (usually every quarter) forever. The dividend must be paid before dividends on regular shares.
- Firm $\boldsymbol{F}$ wants to sell preferred shares at \$100 per share. Shares in a substantially identical company trade for \$40 and pay a \$1 dividend every quarter. What dividend must the firm offer to be competitive?
- The other firm sets the implicit interest rate (why?): $PV_{other\ firm} = \frac{(dividend)_{other\ firm}}{r} \Rightarrow \$40 = \$1\frac{1}{r} \Rightarrow r = \frac{1}{40} = 2.5\%.$
- $PV_F = \frac{(dividend)_F}{r} \Rightarrow (dividend)_F = r \cdot PV_F = 0.025 \cdot 100 =$ \$2.5.
- What would happen if firm F would pay, say, \$3 in quarterly dividends? What would happen if it paid only \$2?

Annuities Due

- In many cases annuity payments occur at the beginning of the period. In this case, annuities have the character of a prepayment. Such situations are common in leases. Such an annuity is called an **annuity due**.
- Note that viewed at time -1 (one period before time 0), the annuity due is equivalent to a regular annuity that starts at time -1 and has its first payment at time 0. We compute the present value at time 0 of the annuity due, PV_0, as the future value (as seen from time -1) of the ordinary annuity:

$$\begin{aligned} PV_0^{due} &= PV_{-1}^{ordinary} \cdot (1+r) = C \cdot \frac{1 - \frac{1}{(1+r)^t}}{r} \cdot (1+r) \\ &= C \cdot \frac{1+r}{r} \cdot \left[1 - \frac{1}{(1+r)^t}\right]. \end{aligned}$$

Growing Annuities and Perpetuities

- Higher interest rates, as well as other factors (e.g., inflation) reduce the present value of an annuity. To counteract this, annuity payments may be set to grow at a rate g, per period. The first payment will still be C, but the subsequent ones will be $C\,(1+g)$, $C\,(1+g)^2$, $C\,(1+g)^3$, ...

$$\textit{Growing } \textbf{\textit{annuity}} \textit{ present value} = C \cdot \frac{1 - \left(\frac{1+g}{1+r}\right)^t}{r-g}.$$

$$\textit{Growing } \textbf{\textit{perpetuity}} \textit{ present value} = \lim_{t\to\infty} C \frac{1 - \left(\frac{1+g}{1+r}\right)^t}{r-g} = \frac{C}{r-g}.$$

- Note that the growing perpetuity will have infinite present value if $g \geq r$.

Value of a Growing Annuity - Technical Note

- Consider an annuity with a per-period payment of C, a payment growth rate of g, a maturity of t periods, and a per-period interest rate r, $r > g$ (why?). We sum the geometric progression representing the time-0 value of each cash flow:

$$
\begin{aligned}
PV_{annuity} &= \sum_{i=1}^{t} \frac{C \cdot (1+g)^{i-1}}{(1+r)^i} = \frac{C}{1+r} \cdot \sum_{i=0}^{t-1} \frac{(1+g)^i}{(1+r)^i} \\
&= \frac{C}{1+r} \cdot \sum_{i=0}^{t-1} \left(\frac{1+g}{1+r}\right)^i = \frac{C}{1+r} \cdot \frac{1 - \left(\frac{1+g}{1+r}\right)^t}{1 - \frac{1+g}{1+r}} \\
&= C \cdot \frac{1 - \left(\frac{1+g}{1+r}\right)^t}{r-g}.
\end{aligned}
$$

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