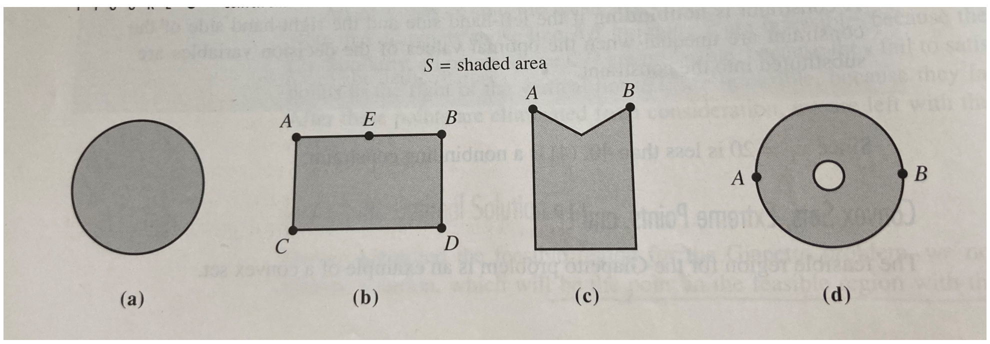
ORIE 3310: Optimization II – Spring 2024

Problem Set 1

Due: Tuesday, January 30, 9 PM

Q1: (a) A set $S \subset \mathbb{R}^n$ is called a convex set if it satisfies certain properties. Give a formal definition of a convex set.

(b) Consider the following 4 sets in $\mathbb{R}^2$ given by the shaded area. Which sets are convex and which sets are not? Explain your answers.



Q2: Consider the following linear program.

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) If we would like to write this LP in the following matrix form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, \quad x \geq 0 \end{aligned}$$

where $x = [x_1, x_2]^T$, what would A, b and c be?

(b) If we instead would like to write this LP in the following matrix form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b, \quad x \geq 0 \end{aligned}$$

what would A, b and c be?

(c) Draw the feasible region of the original LP and identify its extreme points.

(d) Calculate the objective function value of each of these extreme points and identify the optimal solution.

(e) Formulate an LP with a minimization objective that would give the optimal solution to the original (maximization) problem.

Q3: Consider a toy manufacturer that produces two types of wooden toys: soldiers and trains. A soldier sells for \$28 and uses \$9 worth of raw materials whereas a train sells for \$21 and uses \$8 worth of raw materials. In addition, each soldier increases the labor cost of the manufacturer by \$14 and each train by \$10.

The manufacturing process requires two types of skilled labor: carpentry and finishing. Each soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. Each train requires 1 hours of finishing labor and 1 hour of carpentry labor.

The manufacturer can purchase all the needed raw materials but has access to only 120 hours finishing labor and 90 hours of carpentry labor each week.

Weekly demand for trains is unlimited but at most 40 soldiers can be sold each week.

We would like to formulate an integer program to maximize the weekly profit of the manufacturer.

(a) Clearly explain your decision variables.

(b) Describe the objective function using these variables.

(c) Define all constraints necessary to define the feasible set of solutions.

Q4: Consider a school district with $I = \{1, \ldots, |I|\}$ neighborhoods, $J = \{1, \ldots, |J|\}$ schools and $G = \{1, \ldots, |G|\}$ grades at each school. Each school $j \in J$ has capacity C_{jg} for grade $g \in G$. In each neighborhood $i \in I$, the student population of grade $g \in G$ is S_{ig}. Finally, the distance from neighborhood $i \in I$ to school $j \in J$ is d_{ij}. Formulate a linear integer programming problem to assign all students to schools, while minimizing the total distance traveled by all students. Explain your variables, constraints and objective function clearly.

Q5: Consider the following linear programming problem written both in matrix form and in explicit form, where $c \in \mathbb{R}^n, a \in \mathbb{R}^n, b \in \mathbb{R}$. Here we denote a column vector of all 1s with $\mathbb{1}_n$ and a column vector of all 0s with $\mathbb{0}_n$. Both of these vectors $\mathbb{1}_n$ and $\mathbb{0}_n$ are n-dimensional.

$$
\begin{array}{lrcl}
\min & c^T x & & \\
& a^T x & \geq & b \\
& x & \leq & \mathbb{1}_n \\
& x & \geq & \mathbb{0}_n.
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{lrcll}
\min & \sum_{i=1}^n c_i x_i & & & \\
& \sum_{i=1}^n a_i x_i & \geq & b & \\
& x_i & \leq & 1 & \forall i \in \{1, \ldots, n\} \\
& x_i & \geq & 0 & \forall i \in \{1, \ldots, n\}.
\end{array}
$$

(a) Can this LP have an unbounded objective value? Explain why/why not.

(b) Construct an example when this LP is infeasible. (i.e. pick a vector a and a number b that would make this LP infeasible)

(c) What are the necessary and sufficient conditions in terms of a and b that would make this LP feasible?