Finite versus countable additivity

By definition, a probability space $\Omega, \mathcal{F}, P)$ has countably additive properties: $\mathcal{F}$ is stable under countable union,

- for a finite of countable family $A_i, i \in I$ of disjoint elements of $\mathcal{F},$ (these are subsets of Ω), we have $P(\cup_{i\in I} A_i) = \sum_{I \in I} P(A_i)$.

Let Ω be an infinite countable set. Let $\mathcal{F}$ be the family of all finite sets and their complements. Show that this family is stable under finite union but not countable union. For $A \in \mathcal{F}$, set $P(A) = 0$ if A is finite and $P(A) = 1$ otherwise. Show that this is a finitely additive P. Then show that this finitely additive P cannot be extended to a countably additive measure on all subsets of Ω. What is the problem?

A set function m on all subsets of $\mathbb{Z}$ is called a mean if it is finitely additive, non-negative and $m(\mathbb{Z}) = 1$. It is an invariant means if it gives the same value to a set A and any of its translates $A + r = \{a + r : a \in A\}$ where $r \in \mathbb{Z}$.

- Prove that an invariant mean, if it exists, cannot be countably additive.
- It is a fact that there are many many invariant means on $\mathbb{Z}$. A simple one is constructed by "taking limits" of

$$P_k(A) = \frac{\#A \cap \{-k, \ldots, k\}}{k}$$

  when k tends to infinity (too complicated to explain here).

This is an entry into the world of "Amenability" and the "Banach--Tarski paradox." IT IS NOT PART OF THE COURSE.