Random Variables and Events

An *event* is a subset of the outcome space Ω

A *random variable* is a function $\Omega \to \mathbb{R}$ (or $\Omega \to$ something else)

Example: Flipping two p-coins. Let's formalize following:

- Event A is "first coin was heads"
- Random variable N is "number of heads"
- Event $\{N=1\}$ is "got exactly 1 heads"
- Random variable $(N-1)^2$ is, well, "number of heads, minus 1, squared"

Approach 1: Using outcome space $\Omega = \{\underbrace{00}_{\text{tails, tails}}, \overbrace{01}^{\text{tails, heads}}, \underbrace{10}_{\text{heads, tails}}, \overbrace{11}^{\text{heads, heads}}\}$

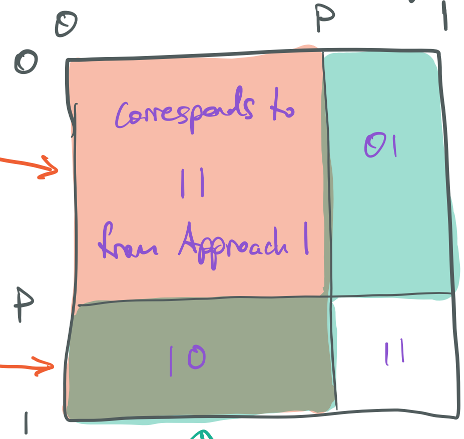
- $A = \{10, 11\}$
- $N(00) = 0, \quad N(01) = 1, \quad N(10) = 1, \quad N(11) = 2$
- $\{N=1\} = \{\omega \in \Omega \mid N(\omega) = 1\} = \{01, 10\}$

  ↳ Remember, want to express this as a subset of Ω

- $[(N-1)^2](\omega) = (N(\omega) - 1)^2$

  ↳ Remember, want to express this as a function $\Omega \to \mathbb{R}$

Approach 2: Using outcome space $\Omega = [0,1]^2$ w/ uniform probability measure



- $A = [0,p] \times [0,1]$
- $N((x,y)) = \begin{cases} 0 & \text{if } x > p \text{ and } y > p \\ 2 & \text{if } x \le p \text{ and } y \le p \\ 1 & \text{otherwise} \end{cases}$
- $\{N = 1\} = \underbrace{\{\omega \in \Omega \mid N(\omega) = 1\}}$
- $\left[(N-1)^2\right](\omega) = \underbrace{(N(\omega)-1)^2}$

Seem familiar?

Shorthand notation: Let X and Y be random variables $\Omega \to \mathbb{R}$

- The event shorthand: (add the green)

$$\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$$

$$\{X \le t\} = \{\omega \in \Omega \mid X(\omega) \le t\}$$

$$\underbrace{\{X > Y\}}_{\text{Shorthand}} = \underbrace{\{\omega \in \Omega \mid X(\omega) > Y(\omega)\}}_{\text{Full notation for subset of } \Omega} \qquad \text{etc.}$$

Works w/ any random variables and any predicate about them

- The random variable shorthand: (shuffle the orange)

Example: Turn "X+Y" into a function $\Omega \to \mathbb{R}$

$$\underbrace{[X+Y](\omega)}_{} = X(\omega) + Y(\omega)$$

That is, what is the value of the function $[X+Y]$ evaluated at some specific outcome $\omega \in \Omega$?

Whatever you get by evaluating each of X and Y at that outcome ω, then add the results

$$\left[aX^2 + bX + c\right](\omega) = a\left(X(\omega)\right)^2 + bX(\omega) + c$$

$$\left[f(X)\right](\omega) = f\left(X(\omega)\right)$$

Try it: Using either approach 1 or 2, define

$X =$ "1 if first is heads, 0 otherwise" $: \Omega \to \mathbb{R}$

$Y =$ "1 if second is heads, 0 otherwise" $: \Omega \to \mathbb{R}$

Check that $[X+Y]$ and N are the same function

Terminology: $X(\omega)$ is called the realization of X for outcome ω

Interpretation: when the random outcome is ω, $X(\omega)$ is the specific value the random variable X has. The fact that there are usually lots of $\omega \in \Omega$ is why random variables can have multiple possible values