Risk, Return, and the Security Market Line

AEM 2241 - Finance

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Expected Returns Risk Portfolios News and Returns Diversification SML and CAPM

Outline

Expected Return

- In this part of the course, "expected" and "expectation" are technical terms that have precisely formulated mathematical meanings.
- We will avoid jargon, but we will be careful to provide enough precision, as well as examples, to make the meaning of these (and other) terms intuitive.
- "Expectation" is a generalization of the informal term "average."
- Consider a stock whose return over a one-period period could be r_i with probability p_i, for $i = 1, \ldots, N$. The **expected return** over the year considered will be $Er = \Sigma_{i=1}^{N} r_i \times p_i$.
- Example: a stock will have a return -20% or of $+70\%$, both with a probability of 50%. The expected return of the stock is $Er = -0.20 \times 0.50 + 0.70 \times 0.50 = 25\%$.

Expected Return: Intuition

- Consider a stock whose return over a one-period period could be r_i with probability p_i, for $i = 1, \ldots, N$.
- Imagine that you watch many (say, M) one-year periods and record each time return r_i has been seen; let this number be m_i. Obviously $m_1 + m_2 + \cdots + m_N = M$.
- The **average return** that we will experience is

$$Average(r) = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_N r_N}{M} = \frac{m_1}{M} r_1 + \frac{m_2}{M} r_2 + \cdots + \frac{m_N}{M} r_N.$$

- If M is very large, then $\frac{m_i}{M} \approx p_i$. We say that the relative frequency of an event is approaching the probability of that event if M grows very large.

Expected Return: Intuition (2)

- Assuming that M is indeed very large, let us rewrite the formula for the average return:

$$Average(r) \approx p_1 r_1 + p_2 r_2 + \cdots + p_N r_N = \sum_{i=1}^{N} p_i r_i = Er.$$

- If we **sampled** the stock return many times (i.e., if M is large), then the average return would be approximately equal to the expectation of the stock return.
- In the limit, when $M \to \infty$, we have equality:

$$Average(r) = Er.$$

Expectation: Examples

Stock 1		Stock 2		Stock 3	
Prob.	Return	Prob.	Return	Prob.	Return
50%	-20%	50%	30%	30%	-30%
50%	70%	50%	10%	20%	-10%
				20%	10%
				30%	30%

- Stock 1:

$$Er = -0.2 \cdot 0.5 + 0.7 \cdot 0.5 = 25\%.$$

- Stock 2:

$$Er = 0.3 \cdot 0.5 + 0.1 \cdot 0.5 = 20\%.$$

- Stock 3:

$$Er = -.30 \cdot 0.3 - 0.20 \cdot 0.1 + 0.20 \cdot 0.1 + .30 \cdot 0.3 = 0\%.$$

Expectation: Some Properties

- The expected value may **not** be a value that can occur (see prior slide).
- Different combinations of probabilities and returns may yield the same expected return.
- Return distributions with the same expected value are not equivalent investments (more on this later).
- We will only consider return distributions that have a finite number of possible returns (outcomes). Each of this outcomes will have a non-zero probability to occur. These probabilities add up to exactly 100%.
- If it possible to consider return distributions with an infinite (even continuous) number of possible outcomes. This is what is often done in advanced mathematical finance.

Risk

- What is investment risk? For many, it is the possibility that the investment will lose a significant portion of its value.
- Usually, people think of **risk** as being only the risk of a loss. This is not the view that mathematical finance adopts.
- More generally, but still informally, we say that an investment may be risky if its outcome is highly uncertain.
- In this sense, an investment that goes up by either 10% with probability 50%, or 40% with probability 50% is still risky, since the outcomes are vastly different.
- We say that an investment whose value (thus, return) may fluctuate a lot is a **volatile** investment.
- Volatile investments may suffer large losses, but may also have large gains. One does not really find assets that are both volatile and have sustained price movements much larger in one direction and not the other.

Risk-Free Investments

- Given our definition of risk, a risk-free investment will be an asset that has a predictable (certain) return: $r_F = Er_F$. It should also be an investment that is also accessible (you can purchase it). If not, it is as if it did not exist.
- Such investments are rare.
- It has been conventionally assumed that short-term US Treasuries are the assets that most closely approximate the ideal of a risk-free investment. They are also available in large volumes, given heavy US borrowing.
- Bonds of other developed, financially stable countries may approximate this ideal.
- It is assumed that the rates paid on short-term Treasuries closely correspond to the risk-free rate of the (global) economy. There are caveats to this statement.

Quantifying Risk

- Risk = uncertain outcomes (returns). How do we quantify it?
- A good measure of risk would be one that measures the spread of possible outcomes around the central (average) value.
- Such a measure exists: it is the variance.
- The return variance is the average squared distance of each possible return from the average stock return. Alternatively: **the return variance is the expected squared difference of each possible return from the expected stock return**.

$$Var(r) = \sigma^2(r) = E(r - Er)^2 = Er^2 - (Er)^2 = \sum_{i=1}^{N} p_i(r_i - Er)^2.$$

- The **standard deviation** or the **volatility** of an investment is the square root of its variance:

$$\sigma(r) = \sqrt{Var(r)} = \sqrt{E(r - Er)^2} = \sqrt{Er^2 - (Er)^2} = \sqrt{\sum_{i=1}^{N} p_i(r_i - Er)^2}.$$

Variance and Volatility of Returns

Stock 1, Er=25%		Stock 2, Er=20%		Stock 3, Er=0%	
Prob.	Return	Prob.	Return	Prob.	Return
50%	-20%	50%	30%	30%	-30%
50%	70%	50%	10%	20%	-10%
				20%	10%
				30%	30%

- Stock 1:

$$\sigma^2(r) = 0.5 \cdot (-0.20 - 0.25)^2 + 0.5 \cdot (0.70 - 0.25)^2 = 20.25\%.$$

- Stock 2:

$$\sigma^2(r) = 0.5 \cdot (0.30 - 0.20)^2 + 0.5 \cdot (0.10 - 0.20)^2 = 1\%.$$

- Stock 3:

$$\sigma^2(r) = .30 \cdot (-0.30 - 0)^2 + 0.2 \cdot (-0.10 - 0)^2 + 0.2 \cdot (0.10 - 0)^2 + .30 \cdot (0.30 - 0)^2 = 5.8\%$$

Portfolios

- If you invest your wealth in a group of financial instruments (say, stocks) you have created a portfolio of investments.
- We can describe a portfolio by specifying what instruments it contains, and how much money is invested in each instrument.
- More often, we specify the instruments, the total amount, and the weight of each instrument within the portfolio.
- The **weight** of an instrument within the portfolio is the amount of money invested in that instrument, expressed as a percentage of the total value of the portfolio.
- The sum of a portfolio's weights must add up to 100%.
- Portfolio weights are often denoted using the letter w. The textbook uses x for the same purpose. We stick with the usual convention and not follow the textbook in this respect. With respect to your work, either approach is acceptable.

Portfolio Example

- Consider a portfolio that consists of \$50 invested in stock A and \$150 invested in stock B.
- The total value of the portfolio is \$50+\$150 = \$200.
- The weights of the stocks in the portfolio are $w_A = \frac{50}{200} = 25\%$, and $w_B = \frac{150}{200} = 75\%$, respectively.
- Note that $w_A + w_B = 100\%$.
- If we know the value of the portfolio P and the weights of its components, we can determine the value of the investments in individual instruments. For example, the amount invested in stock A is $w_A \cdot P = 25\% \cdot 200 = \$50$.

Portfolio Return and Variance

- Portfolio returns r_p are determined by the portfolio's weights and the components' returns:

$$r_p = w_1 \cdot r_1 + w_2 \cdot r_2 + \cdots + w_N \cdot r_N = \sum_{i=1}^{N} w_i \cdot r_i.$$

  The return of the portfolio is equal to the weighted sum of its component instruments. **The same relationship holds for return expectations.**
- The variance of a portfolio can be computed using the weights of its components, as well as their variances and covariances. We will not explore this avenue.
- We'll compute the return of the portfolio for each possible case. We'll then use these returns and their probabilities to directly compute the variance of the portfolio return.

Example: Portfolio Return and Variance

Probability	Stock A	Stock B	Stock C	Portfolio
40%	10%	15%	20%	$0.50 \cdot 0.10 + 0.25 \cdot 0.15 + 0.25 \cdot 0.20 = 13.75\%$
60%	8%	4%	0%	$0.50 \cdot 0.08 + 0.25 \cdot 0.04 + 0.25 \cdot 0 = 5.00\%$

- We'll assume that $w_A = 50\%$ and $w_B = w_C = 25\%$.
- $Er_A = 0.40 \cdot 0.10 + 0.60 \cdot 0.08 = 8.8\%$;
  $Er_B = 0.40 \cdot 0.15 + 0.60 \cdot 0.04 = 8.4\%$;
  $Er_C = 0.40 \cdot 0.20 + 0.60 \cdot 0 = 8\%$.
- $Er_p = 0.40 \cdot 0.1375 + 0.60 \cdot 0.05 = 8.5\%$.
- $Var(r_p) = \sigma^2(r_p) =$
  $0.40 \cdot (0.1375 - 0.085)^2 + 0.60 \cdot (0.05 - 0.085)^2 = 0.0018375$.
- $Vol(r_p) = \sigma(r_p) = \sqrt{0.0018375} \approx 4.2866\% \approx 4.29\%$.

News and Returns

- Assume a simple world in which we enumerated each possible future return and its associated probability, as well as the events that must occur so that the respective returns realize.
- Under these circumstances, we can compute the expected return. The future **realized return will generally not be equal to the expected return**.
- The deviation from expected return can be called **unexpected return**. This is due to unexpected events, i.e., **surprises**.

  $$realized\ return = r = Er + (unexpected\ return)$$

  The deviation from expected return is also called **innovation**.
- These are surprises not because we did not imagine they could have occurred, but because we did not know which of the possible future events will actually occur.

News and Returns (2)

- Innovations in the return process are caused by genuine news.
- Given the speed with which information is disseminated, anything that is known or can be anticipated has already been reflected in asset prices (thus, also in returns).
- What matters is what is neither known or anticipated, i.e. the surprise part (if it exists) of any new event:

$$\textit{announcement} = (\textit{expected part}) + \textit{surprise}$$

- News is not uniformly good or bad: even events of major impact, such as the current pandemic, wars, or economic crises may have positive impact on some firms.

Risk: Systematic and Idiosyncratic

- Return innovations (surprises) come in many forms, and they affect various companies in different ways.
- At one extreme, surprises may affect every company/entity, and thus every investment/financial instrument. At the other extreme, they may affect only one company/entity/financial instrument.
- The risk due to events that affect (substantially) every market participant is called **systematic risk** or **market risk**.
- The risk due to events that affect individual market participants is called **idiosyncratic risk**, also known as unsystematic, unique, or asset-specific risk.
- Few risks truly affect everything, or just one firm. These concepts are idealizations.
- With these insights we can decompose the total surprise:

$$\textit{realized return} = r = Er + (\textit{systematic portion}) + (\textit{idiosyncratic portion})$$

Risk Decomposition

- With these insights we can decompose realized returns for any particular asset or financial instrument as follows:

$$\textit{realized return} = r = Er + m + \epsilon,$$

  where m is the surprise in the market return, while ϵ is the surprise in the idiosyncratic return.
- Note that the expected return is a constant in this framework.
- Now, if we build a portfolio that combines suitable stocks, it may be possible to obtain a return where idiosyncratic returns cancel each other.

Expected Returns | Risk | Portfolios | News and Returns | Diversification | SML and CAPM

Diversification and Risk

- If one invests in a single asset (stock), the one bears all the associated risk, including the idiosyncratic risk.
- Risks are hard to estimate. In addition most investors are risk-averse. Given a choice, all else being equal, they typically prefer lower risk.
- The process of spreading investments across a portfolio of assets is called diversification.
- One can prove both empirically and theoretically that diversification reduces risk. **Diversification, however, cannot fully eliminate risk**.
- Well-diversified portfolios can be designed using mathematical techniques, but also by making random choices (“monkeys throwing darts”).

Diversification and Risk (2)

- **Unsystematic risk is virtually eliminated by diversification**, so a well-constructed portfolio with many assets has almost no systematic risk.
- This is because if portfolio assets were chosen well, then idiosyncratic events that affect them will be uncorrelated, and they will "wash out."
- **What is left is systematic risk.**
- Terminology:

Systematic Risk	Unsystematic Risk
Non-Diversifiable Risk	Unique Risk
Market Risk	Asset-Specific Risk
	Idiosyncratic Risk

Systematic Risk Principle

- Investors are rewarded, on average, for holding risky assets. Indeed, if there was no reward for investing in such assets, nobody would purchase them.
- Investors' reward is the return their investments earn. Riskier assets have, on average, a higher return than comparatively less risky assets.
- The **systematic risk principle** states that investors are only rewarded for the systematic risk that they bear, and not for the unsystematic risk.
- Because idiosyncratic risk can be diversified away cheaply, there is no reward, on average, for holding it.

Measuring Systematic Risk

- Not all assets have the same amount of systematic risk.
- Systematic risk is measured relative to the average asset (or, relative to "the market").
- The traditional notation for the amount of market risk an asset has is the Greek letter beta (β).

Asset	β
Risk-free asset	0.00
Johnson & Johnson	0.67
Pfizer	0.99
Average asset	1.00
Apple	1.44
CBS Corp.	1.71

Portfolio β

- Similar to the returns on portfolios, the β of a portfolio is equal the the weighted sum of the components' βs: $\beta_p = \Sigma_{i=1}^{N} w_i \cdot \beta_i$.
- Consider the following portfolio:

Stock	Amount	Weights	Er	β
A	\$1,000	$\frac{1{,}000}{10{,}000} = 10\%$	8%	0.80
B	\$2,000	$\frac{2{,}000}{10{,}000} = 20\%$	12%	0.95
C	\$3,000	$\frac{3{,}000}{10{,}000} = 30\%$	15%	1.10
D	\$4,000	$\frac{4{,}000}{10{,}000} = 40\%$	18%	1.40

$$Er_p = 0.10 \cdot 0.08 + 0.20 \cdot 0.12 + 0.30 \cdot 0.15 + 0.40 \cdot 0.18 = 14.9\%$$

$$\beta_p = 0.10 \cdot 0.80 + 0.20 \cdot 0.95 + 0.30 \cdot 1.10 + 0.40 \cdot 1.40 = 1.16$$

Portfolio Expected Returns and Betas

- Consider asset A with $Er_A = 20\%$, $\beta_A = 1.6$,and an asset B with $Er_B = 16\%$, $\beta = 1.2$. Further, consider the risk-free asset F with $r_F = 8\%$, $\beta = 0$.
- Consider the following portfolios:

w_A	w_F	Er_{P1}	β_{P1}
0%	100%	8%	0.00
25%	75%	11%	0.40
50%	50%	14%	0.80
75%	25%	17%	1.20
100%	0%	20%	1.60
125%	**-25%**	23%	2.00

w_B	w_F	Er_{P2}	β_{P2}
0%	100%	8%	0.00
25%	75%	10%	0.30
50%	50%	12%	0.60
75%	25%	14%	0.90
100%	0%	16%	1.20
125%	**-25%**	18%	1.50

As the table above shows, it is possible to have portfolio weights exceeding 100%, as long as the weights still sum to 100%. This is feasible if one can borrow freely.

Portfolio Expected Return as a Function of Beta

- Consider a combination of asset A and the risk-free asset F. With the usual notation, we have:

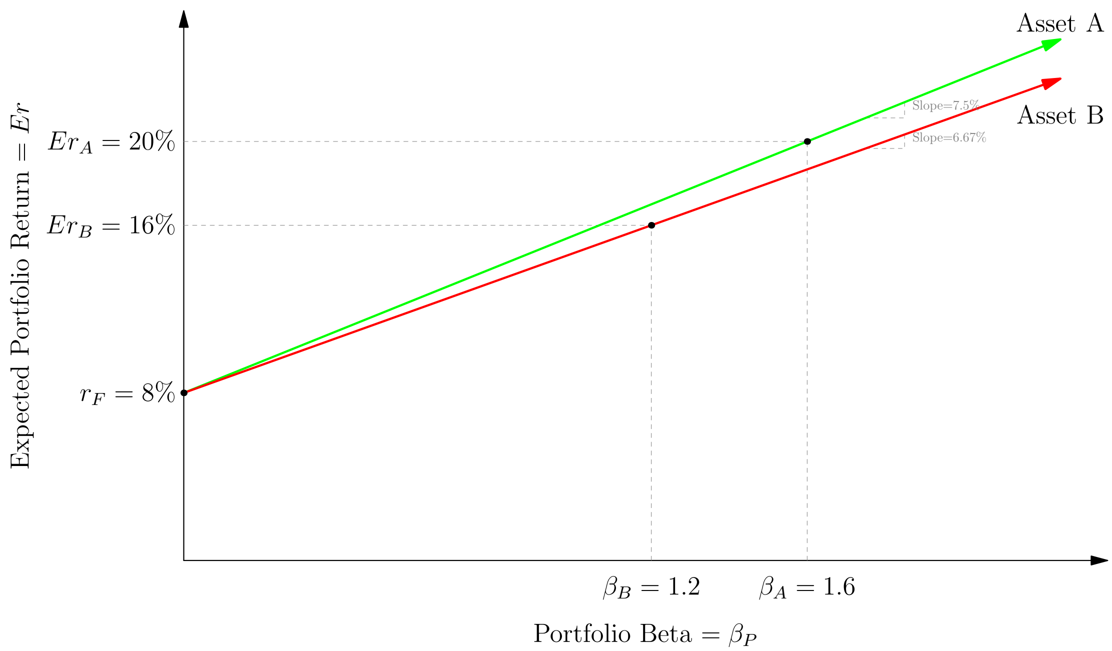
$$Er_P = w_A \cdot Er_A + w_F \cdot Er_F = w_A \cdot Er_A + (1 - w_A) \cdot r_F$$
$$\beta_P = w_A \cdot \beta_A + w_F \cdot \beta_F = w_A \cdot \beta_A + (1 - w_A) \cdot 0 = w_A \cdot \beta_A$$

- We express w_A from the second equation and we substitute it into the first. We get:

$$Er_P = \frac{Er_A - r_F}{\beta_A} \beta_P + r_F$$

- If we represent Er_P as a function of β_P, i.e., if we represent the portfolio's expected return as a function of the portfolio's systematic risk, we get a line with slope $\frac{Er_A - r_F}{\beta_A}$.

Different Assets, Different Slopes



Can Slopes Be Different?

- In the figure above, asset A clearly offers a better reward relative to its riskiness (per unit of risk) relative to asset B.
- When one asset is clearly better, investors, whom, we assume, all have access to the same information, will want to sell asset B and buy asset A.
- Market forces will inevitably lead to the increase of asset A's price, and to the decrease of asset B's price. The former effect decreases the return of asset A, and the latter effect increases the return of asset B.
- When the risk-to-reward ratio of asset A and asset B equalizes, this process stops.
- At this point, the slopes of the two lines will be identical:

$$\text{reward-to-risk ratio} = \frac{Er_A - r_F}{\beta_A} = \frac{Er_B - r_F}{\beta_B}.$$

The Security Market Line

- We conclude that **the reward-to-risk ratio must be the same for all the assets in the market**.
- Consider an asset A and the average asset M ("the market"). Recalling that $\beta_M = 1$, we have:

$$\frac{Er_A - r_F}{\beta_A} = \frac{Er_M - r_F}{\beta_M} = Er_M - r_F.$$

- Rearranging, we get the formula that expresses the Capital Asset Pricing Model (CAPM):

$$Er_A = r_F + (\mathrm{E}r_M\text{-}r_F)\cdot\beta_A.$$

- We call the term $Er_M - r_F$ the **market risk premium** - it is the reward (in return terms) of carrying the average amount of risk.

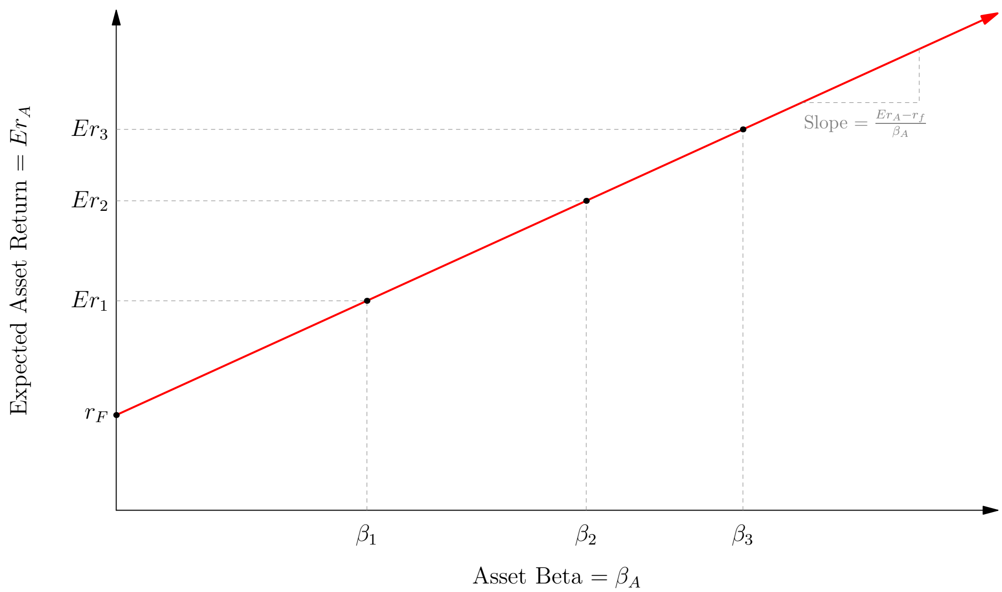
Capital Asset Pricing Model

- For any asset A, the CAPM predicts that

$$Er_A = r_F + (\mathrm{E}r_M\text{-}r_F)\cdot\beta_A.$$

- Thus the expected return for any particular asset depends on:
  - The pure time-value of money, as expressed by the risk-free rate r_F;
  - The reward for bearing (a unit) of systemic risk, as measured by the market premium $Er_M - r_F$;
  - The amount of systemic risk associated with the asset, as measured by the assets β_A.

Capital Asset Pricing Model (2)



SML & CAPM: Example

Consider the following information:

Security	Beta	Expected Return	Reward-to-Risk
SWMS Co	1.3	14%	$\frac{0.14-0.06}{1.3} = 6.15\%$
Insec Co	0.8	10%	$\frac{0.10-0.06}{0.8} = 5.00\%$

If $r_f = 6\%$ is any of these securities overvalues with respect to each other?

We compute the reward-to-risk ratio for the two stocks. These values are shown in the table above. The ratio for Insec is lower, i.e. we do not get paid enough for the risk we are assuming. If the price **now** were lower, the expected return would increase, and so would the ratio. We conclude that the price of Insec is too high relative to SWMS.

SML & CAPM: Example (2)

Now assume that only the information shown in the first three columns of the table above is known. What is the risk-free return r_F and the expected market return Er_M?

We assume that CAPM holds, and thus we can write the following system of equations:

$$Er_{SWMS} = r_f + \beta_{SWMS} \cdot (Er_M - r_f)$$
$$Er_{Insec} = r_f + \beta_{Insec} \cdot (Er_M - r_f)$$

Subtracting the second equation from the first, and replacing known numerical values, we get:

$$0.14 - 0.10 = (1.3 - 0.8) \cdot (Er_M - r_f)$$
$$Er_M - r_f = \frac{0.14 - 0.10}{1.3 - 0.8} = \frac{0.04}{0.5} = 8\%$$

SML & CAPM: Example (3)

We replace $Er_M - r_f = 0.08$ in the first equation to get $0.14 = r_f + 1.3 \cdot 0.08$; then $r_f = 0.14 - 1.3 \cdot 0.08 = 3.6\%$. Since $Er_M - r_f = 8\%$, we immediately get $Er_M = 0.08 + 0.036 = 11.6\%$.

If the CAPM holds for the two assets discussed here, the expected market return and the risk free rate have the following values:

$$Er_M = 11.6\%$$
$$r_F = 3.6\%$$