Week 6 Recap

Monday September 25: Variance

Assume that a random variable X is such that $E(|X|^2)$ exists. Its variance, $\mathrm{Var}(X)$, is

$$\mathrm{Var}(X) = E(|X - E(X)|^2).$$

Expectation (the mean) is the simplest "approximation" of a random variable. Variance is a way to quantify the "dispersion" of a random variable, more precisely, it measures how much X differs from its expectation (or mean) $E(X)$.

Theorem Let X be a random variable with finite second moment ($E(|X|^2) < +\infty$). Then $E(|X|)$ exists and

$$\inf_{b\in\mathbb{R}}\{E(|X-b|^2)\} = E(X^2) - E(X)^2 = E(|X - E(X)|^2).$$

Proof: (a) Observe that, always, $|X| \le 1 + |X|^2$, so $E(|X|) \le 1 + E(X^2)$. **(b)** We have $|X-b|^2 = X^2 - 2bX + b^2$ and thus

$$E(|X-b|^2) = E(X^2) - 2bE(X) + b^2 = E(X^2) - E(X)^2 + E(X^2) - 2bE(X) + b^2 = E(X$$

This clearly show that the smallest possible value for $E(|X-b|^2$ is attained when b=E(X) and is equal to $E(X^2) - E(X)^2 = E(|X - E(X)|^2) = \mathrm{Var}(X)$.

In addition, we see that $E(|X|)^2 \le E(X^2)$, always.

It useful to note that $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$..

Jensen's inequality What is a convex (convex-up) function? One definition (equivalent to many others) is that a function $\phi : I \to \mathbb{R}$ is convex on the interval I if it is equal at any $x \in I$ to the supremum of the values of all affine functions below that always remain below it. More precisely, let

$$\mathcal{A}_\phi = \{f : I \to \mathbb{R}, f(x) = ax + b \text{ and } f \le \phi \text{ on } I\}.$$

This is exactly the set of all affine functions that always remain under ϕ. Now, a function is convex if, for any $x \in I$,

$$\phi(x) = \sup\{f(x) : f : x \to ax + b \in \mathcal{A}_\phi\}.$$

How would you relate this definition to the Calculus definition based on derivatives (a twice differentiable f is convex is $f'' \ge 0$)?

Jensen's inequality states that for any convex function defined on the range of X, we have ϕ,

$$\phi(E(X)) \le E(\phi(X)),$$

assuming that the right-hand side exists.

Proof of Jensen's Inequality: let X be a random variable. For any affine function $f : x \to ax + b$ in $\mathcal{A}_\phi$, it holds that $f(X) = aX + b \le \phi(X)$. Taking expectation, $aE(X) + b \le E(\phi(X))$ and this holds for any $f : x \to ax + b$ in $\mathcal{A}_\phi$. It follows that

$$\sup_{f \in \mathcal{A}_\phi} \{aE(X) + b\} \le E(\phi(X)).$$

The desired inequality then follows because the left-hand side is exactly $\phi(E(X))$.

For concave (convex-down) functions, the inequality is reversed: $E(\phi(X)) \le \phi(E(X))$.

Example: $x \to 1/x$ is convex on $(0, +\infty)$. Hence, for any positive random variable $\frac{1}{E(X)} \le E\left(\frac{1}{X}\right)$, assuming both quantities $E(1/X)$ and $E(X)$ exists.

Wednesday September 27: Prelim 1 in class