Sample Problems - Solutions
AEM 2241 - Finance

1 Finance Matters

1.1. Some of these problems are more complex that the problems you may be expected to solve during the exam. Nevertheless, you should attempt to solve them as **sub-parts** of more complex questions may be posed as exam questions. You will benefit most if you work on these problems before you look at the solutions. The course staff can answer any questions that you may have in relation to these problems.

1.2. The problem set is issued before all the material testable on the exam was taught. It may happen that some topics may not have been covered (yet) in class before the exam. If so, just skip the relevant parts.

1.3. All assumptions, conventions, and notations that we normally use can be relied without further explanations. If you use non-standard notations, explain what they mean.

1.4. Unless stated otherwise, we ask that dollar amounts be rounded to two decimals, and interest rates be rounded to four decimals. For example: \$156,798.38, \$9.75, 0.0315 = 3.15%, 0.1425=14.25%.

1.5. Be careful to distinguish between per-period quantities, such as per-period coupon payments and interest rates, and their annualized versions, which are the ones that must be typically provided as results.

1.6. [When working on paper] Unless we tell you otherwise, you may use either formulas or financial calculators to solve a problem. Whether you are using formulas or calculators, you must show what you did (e.g. what formulas you used, what values you replaced, or what calculator buttons you pushed and what values you entered), and **you must briefly explain the logic of your solutions**.

1.7. [When working on paper] To eliminate any possible ambiguity, whenever using the calculator, you must indicate values for all 5 TVM buttons; except for the value that you compute. Indicate explicitly what buttons you pushed to get a result, and what the result is.

1.8. Whenever possible, interpret the meaning of the results in terms consistent with the problem.

Good Luck!

2 Present Value Calculations

Each row of the table below represents two cash flows, one being the present value of the other when viewed as an investment over a time horizon t, at an interest rate r. One or two numbers are missing in each row.

2.1. For rows (a) and (c) compute the value of the missing number using the formulas given in class. Write down the suitable general formulas first, then replace letter symbols with known numerical values; finally, compute the respective results.

2.2. For rows (b) and (d), use a financial calculator. Similar to what we did in class, show the values that you would set up for the TVM variables, then show what key combinations you would press to get to the solution; also, provide the calculator's answer.

2.3. Is there anything special about the problem in row (e)? If so, what is it? Explain, in no more than two sentences, how a situation like the one shown may arise. For this part you can use formulas or the calculator, but you must state what you did and show the steps that lead to the solution.

2.4. For part (f), where there are two missing numbers, determine, using formulas, **two** combinations of values for PV and r that would make the connection between the four variables in row (e) correct. Note: You need to provide two pairs of numbers, (PV_1, r_1) and (PV_2, r_2), which are both consistent with the data in row (f). You may be able to determine part of the answer by making a choice, as the problem is not fully determined.

Part	**PV [$]**	**t [years]**	**r [%]**	**FV [$]**
(a)	7,513	7	9	
(b)		29	13	48,318
(c)	48,000	15		185,000
(d)	18,400		9	289,715
(e)	200,000	5		175,415
(f)		8		89,980

Solution:

2.1. For row (a) we have: $FV = PV \cdot (1+r)^t = 7,513 \cdot (1+0.09)^7 = \$13,734.06$.

For row (c) we have: $r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1 = \left(\frac{185,000}{48,000}\right)^{\frac{1}{15}} - 1 = 0.0941 = 9.41\%$.

2.2. For row (b) we have: $N = 29$, $I/Y = 13$, $FV = 48,318$, $PMT = 0$. Now pressing $CPT\ PV$ yields the result: $-\$1,395.84$.

For row (d) we have: $I/Y = 9$, $PV = 18,400$, $PMT = 0$, $FV = -289,715$. Now pressing $CPT\,N$ yields the result: $31.99 \approx 40$ periods.

Observations:

Payment is set to 0 by default after resetting the calculator, or after clearing the TVM registers. If you did not indicate $PMT = 0$, but the solution is otherwise correct, we will accept it.

We set the future value to be negative, and the present value to be positive. With these signs, the PV can be interpreted as an inflow (say, loan), and the FV as an outflow (say, the loan repayment). Choosing opposite signs for FV and PV, respectively, would have also been acceptable. Is it not acceptable to set both cash flows to have the same signs, as the calculator would show an error.

Both these observations hold in all similar situations, but we will not repeat them every time.

2.3. In the case of row (e), the future value is **less** than the present value, which implies negative interest rates.

Using formulas, we have: $r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1 = \left(\frac{175,415}{200,000}\right)^{\frac{1}{5}} - 1 = -0.0259 = -2.59\%$.

Using a calculator, we set $N = 5$, $PV = 200,000$, $FV = -175,415$, then press $CPT\,I/Y$ to get -2.59%.

2.4. There are four variables that appear in the formula that connects future values to present values. Given any three of these, we can determine the fourth one. Given only two of them, however, the formula on its own does not determine the two missing quantities. The easiest way to handle this problem is to choose one quantity arbitrarily, and then compute the other one using either the formula, or the calculator. We illustrate both methods below:

Using a formula: We choose $r = 10\%$ and compute $PV = \frac{FV}{(1+r)^t} = \frac{89,980}{(1+0.10)^8} = \$41,976.33$.

Using the calculator: We choose, say, $PV = \$30,000$. We proceed as usual and set $N = 8$, $PV = 30,000$, $PMT = 0$, $FV = -89,900$. We get the answer of $r = 14.70\%$ after pressing $CPT\,I/Y$.

The two pairs that we generated are $(PV_1, r_1) = (\$41,976.33, 10\%)$ and (PV_2, r_2)=$(\$30,000, 14.70\%)$.

Any combination of methods to generate the two pairs is acceptable. Signs are needed for calculator computations, if you performed them, but were not required in the answers. If you provided signs when you specified the pairs of values, they are acceptable.

3 Simple vs. Compound Interest Rates

Consider an investment of $10,000 that you make at the end of year 0. You are guaranteed an interest rate of 7.5% per annum for 10 years, compounded annually. We say that two interest rates are equivalent over a given time horizon t if the total interest earned by time t is the same in both cases.

3.1. Given a time horizon of $t = 5$ years, what is the simple interest rate that is equivalent to the compound interest rate described above?

3.2. Given a time horizon of $t = 10$ years, what is the simple interest rate that is equivalent to the compound interest rate described above?

3.3. Do the answers to the two questions above depend on your initial investment, assuming that the terms of the investment remain otherwise the same? Why?

3.4. Is it possible to have the same simple interest rate be equivalent to a given positive compound interest rate **simultaneously** over **several** investment periods longer than one year?

Solution:

The total interest TI earned using compound interest is $TI = FV - PV = PV \cdot \left[(1+r)^t - 1\right]$. We recall that "interest does not earn interest" when we use simple interest rates. Thus if the simple interest rate is r_s, the annual interest earned is always $PV \cdot r_s$. Over t time periods, we earn $PV \cdot r_s \cdot t$ total interest. We equate the two quantities: $TI = PV \cdot \left[(1+r)^t - 1\right] = PV \cdot r_s \cdot t$. We immediately get the simple interest rate: $r_s = \frac{(1+r)^t - 1}{t}$.

3.1. $r_s = \frac{(1+r)^t-1}{t} = \frac{(1+0.075)^5-1}{5} = 0.0871 = 8.71\%$.

3.2. $r_s = \frac{(1+r)^t-1}{t} = \frac{(1+0.075)^{10}-1}{10} = 0.1061 = 10.61\%$.

3.3. Given the assumptions that we use in class, and given the context of our problem, future values are strictly proportional to present values irrespective of the interest rate convention that we use. This implies that total interest earned is also proportional to present values. Because of this proportionality, when we equate the total interest earned using the two interest rate conventions, the present value will appear as a factor on both sides. We can simplify by dividing both sides with the present value, thus eliminating the potential dependency of the answer on the size of the initial investment.

3.4. If the compound interest rate is 0, then we immediately get that $r_s = 0$, and the answer does not depend on the period t. It is thus possible for the simple interest rate to be equivalent to the compound interest over multiple (in fact: any) time periods. This case was, however, excluded by the problem's text.

The solutions to the first two parts of this problem seem to suggest that the answer may be negative. A full analysis is mathematically more involved than the typical material in this course, so we do not provide it here. We will accept any reasonable, less than fully formal argument, such as the one below.

One can, for example, use Excel to plot the value of $r_s = \frac{(1+r)^t - 1}{t}$ for $t = 1, 2, 3, ...$, for several values of $r > 0$. For all positive values of r, all lines are upward sloping, which means that no matter what r was chosen, the equivalent r_s is always increasing as t increases. For fixed t, the slope will be greater the greater r is. To satisfy the condition of the problem, a horizontal line would have to cross the line corresponding to a given r more than once. However, this is not possible. A similar analysis can be made for $r < 0$, though this was not required.

It would have been acceptable not to use a general formula, like above, but a step-by-step calculation. For part (3.2), say, we compute the total interest TI earned using compound interest as $TI = FV - PV = PV \cdot \left[(1+r)^t - 1\right] = 10,000 \cdot \left[(1+0.075)^{10} - 1\right] = \$10,610.32$. We then use equation $TI = PV \cdot r_s \cdot t = \$10,610.32$ to determine $r_s = \frac{10,610.32}{10,000 \cdot 10} = 10.61\%$. We get the same answer, but it is harder to see that the answer does not depend on the initial investment. We could check that by computing r_s for several PV values, then generalizing our empirical observations. This is not the approach we recommend, though we would be flexible if you submitted a solution along these lines.

4 Discounted Cash Flow Valuation

The present value (at time 0) of the cash flow stream below is \$7,500 when discounted at 9% annually. What is the value of the missing cash flow? What would be the missing cash flow if the present value were \$4,500? Beside magnitude (the amount of dollars), is there anything different between the unknown cash flows in these two situations? Interpret this difference. Use formulas.

Year	Cash Flow [$]
1	1,700
2	x
3	2,450
4	2,980

Solution:

We compute the present value (at time 0) of each cash flow, and then we set up an equation by equating the known total present value to the sum of present values:

Year	Cash Flow [$]	Present Value [$]
1	$1,700$	$\frac{1{,}700}{1+0.09} = 1,559.63$
2	x	$\frac{x}{(1+0.09)^2}$
3	$2,450$	$\frac{2{,}450}{(1+0.09)^3} = 1,891.85$
4	$2,980$	$\frac{2{,}980}{(1+0.09)^4} = 2,111.11$
	Total	$\frac{x}{1.09^2} + 5,562.59$

We now write equation $\frac{x}{1.09^2} + 5,562.59 = 7,500$, from which we determine $x = \$2,301.84$. If the present value is $\$4,500$, we get that $x = -\$1,262.46$.

The signs of the two unknown cash flows are different in the two cases considered. If we follow the usual sign convention, we can say that future cash flows are all inflows when their total present value is $\$7,500$; however, there is a cash outflow in year 2 in the case when the total present value of the cash flows is $\$4,500$.

Observations:

Using the calculator to solve this problem would not be acceptable, since the text specified "use formulas." However, you could have used a calculator to check your results.

It is possible to write more general formulas that allow for the calculation of the unknown cash flow: $PV = \sum_{i=1}^{4} \frac{CF_i}{(1+r)^i}$, where CF_i is the cash flow received/paid at time i. Thus $CF_2 = (1+r)^2 PV - (1+r) CF_1 - \frac{CF_3}{1+r} - \frac{CF_4}{(1+r)^2}$. Of course, by replacing variables with their corresponding values in this formula, we get the same answers as before.

5 Time Value of Money

Use formulas to solve this problem, except, possibly, for the first and last sub-part.

The 6-month interest rate is 3.5%. Consider two successive six-month periods.

5.1. Fill out the statement below relying on our usual conventions, so that it is consistent with the problem text.

"The interest rate is __**7**__% per year, compounded semi-annually."

Solution:

If the yearly rate is r, and it compounds m times a year, then the per-period rate is $r^{'} = \frac{r}{m}$. Conversely, if the per-period rate is $r^{'}$, then the yearly rate must be $r = m \cdot r^{'}$. In this case $r^{'} = 3.5\%$, $m = 2$, and thus $r = 2 \cdot 3.5\% = 7\%$.

5.2. Compute the EAR.

Solution:

$$EAR = \left(1 + \frac{r}{2}\right)^2 - 1 = \left(1 + r^{'}\right)^2 - 1 = (1 + 0.035)^2 - 1 = 7.12\%.$$

5.3. What is the present value at time 0 of $500 to be paid at the end of the second 6-month period?

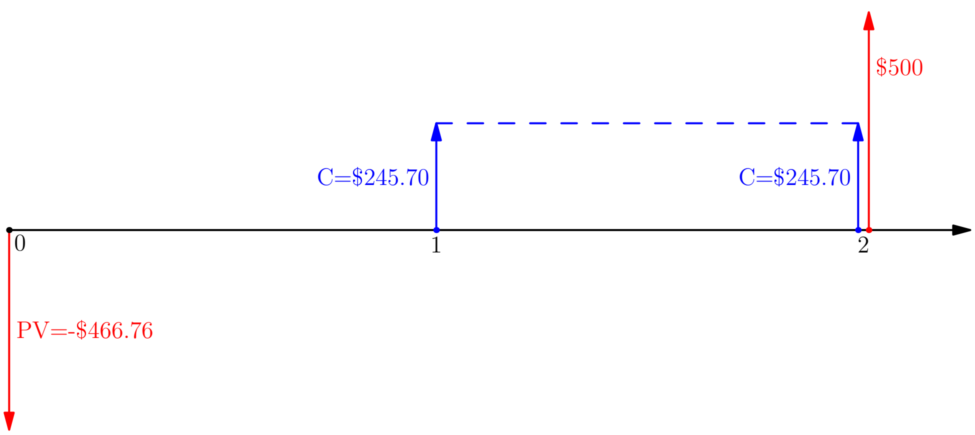
Solution:

$$PV = \frac{FV}{(1+r)^2} = \frac{500}{(1+0.035)^2} = \$466.76.$$

5.4. Now consider two equal payments made at the end of the first and second six-month periods, respectively. How big should these payments be so that their total present value at time 0 is equal to the value obtained in part (5.3) above?

Note: If you need the result of part (5.3), but you cannot calculate it, then use \$400 for the needed total present value.

Solution:

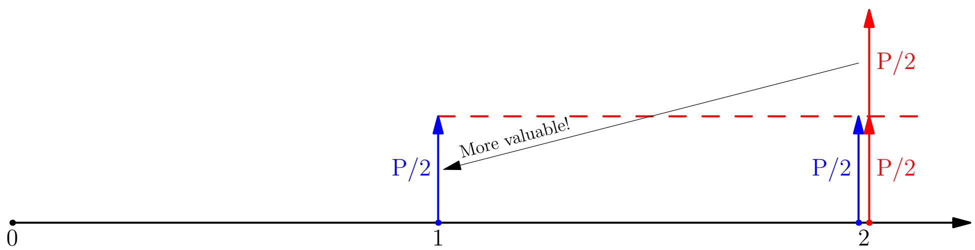


We can see the two cash flows as representing a regular annuity with a maturity of two six-month time periods.

$$PV = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^2}{r} \Rightarrow C = \frac{r}{1-\left(\frac{1}{1+r}\right)^2} \cdot PV = \frac{0.035}{1-\left(\frac{1}{1+0.035}\right)^2} \cdot 466.76 = \$245.70.$$

5.5. More abstractly, consider a payment P made at the end of the second 6-month period and its time-0 present value PV_0. Separately, consider two payments of size P/2 made at the end of the first and second six-month period, respectively, as well as their total present value $PV_0^{'}$. Can you determine, relying either on a qualitative argument **or** on formulas, whether PV_0 or $PV_0^{'}$ is bigger under ordinary circumstances, when interest rates are positive?

Solution:



The one-payment and the two-payment case overlap in that both pay (at least) $\frac{P}{2}$ at the end of the year. These cash flows have the same present value, since they have the same size and timing. In both cases an extra cash of size $\frac{P}{2}$ also occurs. In the two-payment case, the payment occurs after one period, while in the other case it occurs after two periods. When rates are positive, a cash flow closer in time to time 0 will have a higher present value than a cash flow of the same size that is farther in time. Hence, the total present value of the two cash flows must be greater than the present value of the one cash flow, even though the total cash flows, expressed in nominal dollars, are the same in both cases.

6 Financial Product

You purchased (invested in) a financial product that will pay you \$500,000 every six months for 4 years, starting exactly six months from today. The relevant interest rate is 12% per year, compounded quarterly.

6.1. Given the frequency of payments and the quarterly compounding interest, which of the following interest rates is closest to the rate you must use to discount the first cash flow you will receive?

- ✗ 3%
- ✗ 4%
- ✓ 6%
- ✗ 12%

Answer:

The quarterly interest rate is $r_3 = \frac{0.12}{4} = 3.00\%$. The first payment will be made in exactly six months, so the relevant rate is $r_6 = (1 + r_3)^2 - 1 = 6.09\%$.

While it was not requested, we compute the present value of the first \$500,000, which is $PV = \frac{500{,}000}{1+r_6} = \frac{500{,}000}{1+0.0609} = \$471,297.95$.

6.2. The financial product that you purchased will be phased out (eliminated) and replaced with a similar product which has only 4 annual payments of \$1,000,000, with the first payment arriving exactly one year from now. From your perspective, is the new contract more, or less valuable than the old contract?

✗ The new contract is more valuable.

✗ The two contracts are equally valuable.

✓ The new contract is less valuable.

Answer:

From the perspective of the person who receives the payments ("you"), this new contract is less valuable.

Solution 1: The answer is obvious even in the absence of any calculations, as the net effect of the change is to move some payments later than their time of payment under the original contract. For example, the payment due in 6 months will be made in 12 months, and will be paid out jointly with the \$500,000 originally due at that time. So half of the 8 original \$500,000 payments, specifically those due at 6, 18, 30, and 42 months will be pushed back to 12, 24, 36, and 48 months, respectively, and thus will be less valuable. The \$500,000 payments that would have been made at 12, 24, 36, and 48 months originally are still paid out at the original dates (but they are merged with the payments that have been delayed).

Solution 2: A quantitative argument is not necessary, but can easily be provided. Both contracts are simple annuities that we can value using standard formulas. Let $C_{original}$ be the size of the original \$500,000 payments, C_{new} the size of the new \$1,000,000 payments, $r_6 = \left(1 + \frac{0.12}{4}\right)^2 - 1 = 6.09\%$ the per-6-month period interest rate, and $r_{12} = \left(1 + \frac{0.12}{4}\right)^4 - 1 = 12.55\%$ the per-12-month period interest rate. We have:

$$PV_{original} = C_{original} \cdot \frac{1 - \left(\frac{1}{1+r_6}\right)^8}{r_6} = \$500,000 \cdot \frac{1 - \left(\frac{1}{1+0.0609}\right)^8}{0.0609} = \$3,093,067.49$$

$$PV_{new} = C_{new} \cdot \frac{1 - \left(\frac{1}{1+r_{12}}\right)^4}{r_{12}} = \$1,000,000 \cdot \frac{1 - \left(\frac{1}{1+0.1255}\right)^4}{0.1255} = \$3,002,443.10$$

As can be seen, $PV_{original} > PV_{new}$.

6.3. In a different scenario, the financial product you invested in initially will be eliminated, and will be replaced with a similar contract. This time, however, you will receive four \$1,000,000 payments, with the first one due in exactly 9 months. The second, third, and fourth payments will each be made exactly one, two, and three years after the first payment, respectively (i.e., payments following the first one will be made at the ends of successive one-year intervals following the first payment).

To make the exchange fair, if the value of the new contract to you is greater than that of the old contract, you will have to pay the difference in contract values to your counterparty. Should the value of the new contract be less than the value of the old contract, you will receive the difference in contract values from your counterparty. If the old and new contract have the same value, no money exchanges hands.

What will happen?

- ✗ The answer cannot be determined.
- ✗ No money will exchange hands.
- ✓ You will receive a payment from your counterparty.
- ✗ You will make a payment to your counterparty.

Answer:

Solution 1: For many students, the most straightforward approach is to compute the numerical value of the two contracts. In fact, we did almost all the work when answering the question above. The original contract has a time-0 value of $PV_{original}$. The new contract introduced in this part will have a time-0 value of PV_{new2}. However, if we valued the new contract at a time 3 months into the past (i.e., 3 months before time-0), then the present value of the new contract in this part would be equal to PV_{new} from above (think about it: four payments of \$1,000,000 are paid out at at yearly intervals versus the reference time "3 months ago"). To get PV_{new2} we must compute the future value at time 0 of the new contract's value at time "3 months ago": $PV_{new2} = PV_{new} \cdot (1 + r_3) = \$3,002,443.10 \cdot \left(1 + \frac{0.12}{4}\right) = \$3,092,516.39$. We conclude that the new contract is less valuable than the original contract from the perspective of the person receiving the money, and that this person ("you") must receive a payment from the counterparty.

Solution 2: The answer can also be determined without numerical calculations, though it is perhaps not as obvious as that for the question above. The explanation below is long because we would like you to understand how such an argument can be developed. With practice, you will be able to almost instantaneously determine the answer in situations like this.

Consider the first two payments of the original contract, the one made at 6-months, and the one made at 12-months. These two payments are equal. The first payment is pushed back by 3 months (from 6 months to 9 months), while the second payment is brought forward by 3 months (from 12 months to 9 months). The original 6 month payment loses value when pushed back to 9 months, the original 12 month payment gains value when brought forward to 9 months. The question we need to answer is what is the net effect of the value loss and of the value gain.

Let $r_3 = \frac{0.12}{4} = 3\%$ be the quarterly interest rate and let $f_N = \frac{1}{(1+r_3)^N}$ be the discount factor resulting from a cash flow that is paid N quarters into the future (N is equal to 2, 3, and 4 for payments made at 6, 9, and 12 months, respectively). We know from observations and remarks made in the class about discount factors, but also from the general properties of function $h(x) = \frac{1}{a^x}$, where $a > 1$, and $x \geqslant 0$, that this function decreases steeply, but then flattens out as the values of x grow. We immediately infer that $f_2 - f_3 > f_3 - f_4$; this is because there is the same distance in months (3 months) between 6 and 9 months, and 9 and 12 months, respectively, but the decrease in discount factors over 3 months is steeper to the "left" of 9 months (i.e., toward time 0), than to the right of this new payout time. We can rewrite the inequality by moving two discount factors from one side to the other side: $f_3 - f_4 - (f_2 - f_3) < 0$, which then becomes $2 \cdot f_3 - f_2 - f_4 < 0$.

For the first two payments of the old contract, the change in value from moving them both to 9 months is $C_{original} \cdot (f_3 - f_2) + C_{original} \cdot (f_3 - f_4)$, which is equal to $C_{original} \cdot (f_3 - f_2 + f_3 - f_4) = C_{original} \cdot (2 \cdot f_3 - f_2 - f_4) < 0$.

Now, the same intuition holds for the pair of payments originally scheduled for payout at 18 and 24 months, respectively, which are both moved to a payout date of 21 months. Again, this leads to a net decrease in value. The same holds for pairs of payments scheduled to be paid out at 30 and 36 months, and 42 and 48 months, respectively.

The new contract is thus less valuable from the perspective of the person receiving the payments ("you"). You must receive money.

This argument has been developed in great detail to illustrate that you can decide many problems of this type without having to perform numerical calculations, or even any calculations, assuming that you have practiced enough to build intuition in the area of NPV computations. Indeed, this line of reasoning can be pursued as a mental exercise, while the numerical solution given earlier is not amenable to such an approach.

Many seasoned businesspersons stress the importance of being able to think rapidly and in intuitive terms about time-value-of-money and general NPV matters. Social conventions and other practicalities often prevent negotiating parties from using calculators, computers, or phones during meetings - these intuitions, however, can be deployed rapidly, and they can guide one's negotiating strategy.

Solution 3: We can be more mathematical with our approach to the second solution; for example, we can show algebraically that the original first two payments lose value when they are moved. Let Δ be the change in value of these two payments only:

$$\begin{aligned}
\Delta &= C_{original} \cdot (2 \cdot f_3 - f_2 - f_4) \\
&= C_{original} \cdot \left[2 \cdot \frac{1}{(1+r_3)^3} - \frac{1}{(1+r_3)^2} - \frac{1}{(1+r_3)^4} \right] \\
&= C_{original} \cdot \frac{2 \cdot (1+r_3) - (1+r_3)^2 - 1}{(1+r_3)^4} \\
&= C_{original} \cdot \frac{2 + 2r_3 - 1 - 2r_3 - r_3^2 - 1}{(1+r_3)^4} \\
&= -C_{original} \cdot \frac{r_3^2}{(1+r_3)^4} < 0.
\end{aligned}$$

There are three other pairs of payments that can also be treated analytically, of course. The analytic solution makes it particularly easy to notice that the decrease in value would hold for any payment size and interest rates, and it is not specific to the combination of specific values chosen.

7 You Won the Lottery!

You have just won the lottery and will receive 10 yearly payments, as follows: you get $1,500,000 in the first year, after which yearly payments will increase by 2.7% per year. A company specializing in purchasing annuities (yes, they do exist!) offers you instant $14,000,000 in cash to purchase the right to receive your winnings. The relevant interest rate is 3% per year. Will you take the offer?

Solution:

This is a growth annuity, its value can be computed using the formulas given in class:

$$PV = C \cdot \frac{1 - \left(\frac{1+g}{1+r}\right)^t}{r-g} = 1,500,000 \cdot \frac{1 - \left(\frac{1+0.027}{1+0.030}\right)^{10}}{0.030 - 0.027} = \$14,373,706.12.$$

Since the present value of the annuity exceeds that of the offer by almost $375,000, you should refuse the offer.

Annuities are purchased routinely by specialized firms. Because of factors that we do not consider here, such as taxes, estate planning, desire to access wealth early, and others, an offer may imply a steep discount to the present value of the annuity, yet still be acceptable to an annuity seller.

Calculator Solution:

A calculator solution requires some preparation, and it is more complicated than the direct evaluation of the formula above. Let $\frac{1+g}{1+r} = \frac{1}{1+u}$; this implies that $u = \frac{r-g}{1+g} = \frac{0.030-0.027}{1+0.027} = 0.29\%$. Also, note that $r - g = u\,(1+g)$. We rewrite the general formula: $PV = C \cdot \frac{1-\left(\frac{1+g}{1+r}\right)^t}{r-g} = \frac{C}{1+g} \cdot \frac{1-\left(\frac{1}{1+u}\right)^t}{u} = C' \cdot \frac{1-\left(\frac{1}{1+u}\right)^t}{u}$, where $C' = \frac{C}{1+g} = \frac{1,500,000}{1+0.027} = \$1,460,564.75$. Now we can price this as a regular annuity: N=10; I/Y=0.29; PMT=1,460,564.75; FV=0. Press CPT PV to get -14,375,364.50. Thus $PV = \$14,375,364.50$. Note the small difference due to rounding the per-period interest rate to 0.29%; use the more precise value of 0.292113% to get a better match!

8 Annuity, But You Have to Wait

Consider an annuity with a yearly payment of $\$75,000$ that makes its first payment at the end of year 5, and consists of 10 payments. Assume that the appropriate interest rate is 11% per year.

8.1. Using formulas for annuities, value this annuity (compute its present value) as of the end of year 4 (i.e. as if time had passed, and "now" were at the end of year 4); name this value PV_4.

8.2. Using formulas for annuities, value this annuity as of the end of year 5; name this value PV_5. What kind of annuity is this when viewed from the end of year 5?

8.3. Compute the present value of this annuity at time 0 in two different ways, starting separately from PV_4 and PV_5, respectively. Compare and comment very briefly on the two answers that you get.

Solution:

8.1. As we look forward in time "standing" at the end of year 4, we see a regular annuity with a maturity of 10 years; we can price it immediately: $PV_4 = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r} = 75,000 \cdot \frac{1-\left(\frac{1}{1+0.11}\right)^{10}}{0.11} = \$441,692.40$.

8.2. The annuity is an annuity due when viewed from time 5. To compute PV_5, we can just roll forward PV_4 and compute its future value one period later: $PV_5 = PV_4 \cdot (1+r) = 441,692.40 \cdot (1+0.11) = \$490,278.56$. Alternatively, we can compute PV_5 as the value of an annuity due with a maturity of 10 years, valued at time 5: $PV_5 = C \cdot \frac{1+r}{r} \cdot \left[1 - \left(\frac{1}{1+r}\right)^t\right] = 75,000 \cdot \frac{1+0.11}{0.11} \cdot \left[1 - \left(\frac{1}{1+0.11}\right)^{10}\right] = \$490,278.56$.

8.3. We compute the present value of PV_4 at time 0 first: $PV = \frac{PV_4}{(1+r)^4} = \frac{441,692.40}{(1+0.11)^4} = \$290,956.46$. The present value of PV_5 at time 0 is $PV^{'} = \frac{PV_5}{(1+r)^5} = \frac{490,278.56}{(1+0.11)^5} = \$290,956.46$. The two present values are equal: $PV = PV^{'}$, which shows that our calculations have been consistent. If computing present values "passing" through different intermediate moments of time gave different results, our methodology would not be correct.

The equality of present values can be seen directly from the general formulas:

$$PV = \frac{PV_4}{(1+r)^4} = \frac{C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r}}{(1+r)^4} = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^{10}}{r \cdot (1+r)^4},$$
$$PV^{'} = \frac{PV_5}{(1+r)^5} = \frac{C \cdot \frac{1+r}{r} \cdot \left[1 - \left(\frac{1}{1+r}\right)^t\right]}{(1+r)^5} \cdot = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r \cdot (1+r)^4}.$$

We are used to thinking about present values as being computed at time 0. This problem makes it clear that we can compute present values using any reference times that may be convenient.

Calculator Solution:

We use the same notations as above; also, we do not repeat the explanations and reasoning given above. For brevity, we do not interpret or explain the choices of cash flow signs.

8.1. N=10; I/Y=11; PMT=75,000; FV=0. Press CPT PV to get -441,692.40. Thus $PV_4 = \$441,692.40$.

8.2. First, set the calculator so that cash flows arrive at the start of time periods: 2ND BGN; 2ND SET.
N=10; I/Y=11; PMT=75,000; FV=0. Press CPT PV to get -490,278.56. Thus $PV_5 = \$490,278.56$.
We pushed the same buttons, but we obtained different results. This is because of the change in the timing of cash flows.

8.3. Now you must reset or otherwise change back the setting about the timing of the cash flow: 2ND RESET, or 2ND BGN, 2ND SET.
N=4; I/Y=11; PMT=0; FV=-441,692.40. Press CPT PV to get 290,956.46. Thus $PV = \$290,956.46$.
N=5; I/Y=11; PMT=0; FV=-490,278.56. Press CPT PV to get 290,956.46. Thus $PV' = \$290,956.46$.

9 One or Several?

Consider an annuity payment that will pay 75,000 every six months for 5 full years. Afterwards, the payment will increase to 100,000 every six months, for 5 more years. The relevant interest rate is 7.5% per annum, compounded semi-annually.

9.1. Explain how this annuity is equivalent to the difference of **two** regular annuities, each starting at time 0, one with a maturity of 10 years, one with a maturity of 5 years. Specify in full the details of the two component annuities.

9.2. Using the observation made in the previous part, value the component annuities, and combine these values to get the value of the composite annuity. If you could not identify the two annuities into which the initial annuity can be decomposed, feel free to use any other method to value the initial annuity.

Note: Many complex financial instruments can be seen as collections of simpler instruments that can be understood and/or valued in isolation. In such cases the value of the complex financial instrument emerges by the suitable aggregation of the values of its respective components.

Solution:

9.1. Consider a regular annuity A_1, which has a maturity of 10 years, starts at time 0, and has constant payments of \$100,000. Now consider a regular annuity A_2, which has a maturity of 5 years, starts at time 0, and has constant payments of $\$25,000 = 100,000 - 75,000$. Consider a person who **receives** annuity A_1 and **pays** annuity A_2; the net effect will be to produce the cash flow pattern described in the problem's text.

9.2. Annuity A_1 has $C_1 = \$100,000$, $t_1 = 10$ ($2 \cdot 10 = 20$ semi-annual intervals), per-period interest rate $\frac{r}{2} = \frac{7.5\%}{2} = 3.75\%$: $PV_1 = C_1 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_1}}{\frac{r}{2}} = 100,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{20}}{\frac{0.075}{2}} = \$1,389,620.42$. Annuity A_2 has $C_2 = \$25,000$, $t_2 = 5$ ($2 \cdot 5 = 10$ semi-annual intervals), same per-period interest rate: $PV_2 = C_2 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_2}}{\frac{r}{2}} = 25,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{10}}{\frac{0.075}{2}} = \$205,319.68$. The present value of the cash flows actually received is $PV = PV_1 - PV_2 =$\$1,184,300.74.

It is possible to solve the problem in a different way, without necessarily being able to answer the prior question. We can see the cash flows received as consisting of two regular annuities, one, A_3, paying a constant semi-annual $C_3 = \$75,000$, starting at time 0, with a maturity of $t_3 = 5$ years, the other, A_4, paying a constant semi-annual $C_4 = 25,000$, starting at the end of the fifth year, with a maturity of $t_4 = 5$ years. The present value (at time 0) of A_3 is easy to calculate: $PV_3 = C_3 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_3}}{\frac{r}{2}} = 75,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{20}}{\frac{0.075}{2}} = \$1,042,215.32$. The present value of A_4 **at the end of year 5**, which we know how to compute, must be discounted back to time 0. We get $PV_4 = \frac{1}{\left(1+\frac{r}{2}\right)^{10}} \cdot C_4 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_4}}{\frac{r}{2}} = \frac{1}{\left(1+\frac{0.075}{2}\right)^{10}} \cdot 25,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{10}}{\frac{0.075}{2}} = \$142,085.42$. This time the present value of the cash flows to be received is the **sum** of the present values of A_3 and $A_4 : PV = PV_3 + PV_4 = 1,042,215.32 + 142,085.42 = \$1,184,300.74$. As before, the two answers computed by using different approaches are identical.

Calculator Solution:

We use the same notations as above; also, we do not repeat the explanations and reasoning given above. For brevity, we do not interpret or explain the choices of cash flow signs.

9.1. Same as above.

9.2. N=20; I/Y=3.75; PMT=100,000; FV=0. Press CPT PV to get -1,389,620.42. Thus $PV_1 = \$1,389,620.42$.

N=10; I/Y=3.75; PMT=25,000; FV=0. Press CPT PV to get -205,319.68. Thus $PV_2 = \$205,319.68$.
As above, the final answer is the difference $PV = PV_1 - PV_2$.
PV_3 and PV_4 can be computed similarly, but then PV_4 must be discounted back to time 0, similar to the approach we took in solution 8.3.

10 Amortizing loans with variable payments

Consider an amortizing loan with **variable** payments. The initial loan amount is \$10,000,000, while the initial loan maturity is 30 years. The borrower must make monthly payments, with payments due at the end of each month. The borrower's total payment on this loan that is due at the end of the 73$^{\text{rd}}$ month equals \$87,777.78.

10.1. What is the monthly principal repayment on this loan?

Solution:

Each month the same amount of principal is repaid (in addition to the entire interest due at that time). Let the initial loan amount be L and the number of months over the whole duration of the loan be N=360.

$$\text{monthly repayment=p=}\frac{L}{N} = \frac{10,000,000}{360} = \$27,777.78.$$

10.2. What is the balance outstanding on this loan at the beginning of the 73$^{\text{rd}}$ month?

Solution:

At the beginning of the 73$^{\text{rd}}$ months 72 full months, i.e., 6 years, have already passed. This represents $\frac{6}{30} = \frac{1}{5}$ of the loan's original maturity. Since the loan principal is repaid in constant chunks each month, one fifth of the principal was paid back, while four-fifths are not yet repaid.

$$\text{beginning balance} = BB = \frac{4}{5} \cdot L = \frac{4}{5} \cdot 10,000,000 = \$8,000,000.$$

10.3. What is the interest rate that would have been quoted on this loan when it was issued?
Note: If you need to use results from either parts (10.1) or (10.2) above, or from both, but you

were not able to determine the respective values, you may use \$50,000 for the answer to part (10.1), and \$5,000,000 for the answer to part (10.2), respectively.

Solution:

The monthly payment (pymt) consists of the (constant-size) principal repayment (p, from above) plus the one-period interest on the beginning balance. Let the monthly (i.e., per-period) interest be r.

$$pymt = p + BB \cdot r \Rightarrow r = \frac{pymt - p}{BB} = \frac{87,777.78 - 27,777.78}{8,000,000} = 0.75\%.$$

This, however, is not the rate that would be quoted on the loan. The usual conventions imply that the rate would be quoted on an annual basis as "9% per annum, compounded monthly" $(9\% = 12 \cdot 0.75)$.

11 Balloons

You are an important local real estate investor; you just got a \$10,000,000 balloon loan to buy a new office building in your home town. The nominal maturity of the loan is 30 years, but the loan has a 10-year balloon payment. In other words, the loan will end at the end of the 10th year, and the outstanding balance will be paid off in a lump sum at that time. The interest on the loan is 6.7% per annum, compounded monthly.

11.1. Assume that the loan has fixed payments.

(11.1.1) What is the monthly fixed payment that you have to make?

(11.1.2) Let PV be the present value at time 0 of the fixed payments made over the 10-year life of the loan. What is PV?

(11.1.3) Can you find a connection between the \$10,000,000 principal, PV, and the balloon payment at the end of year 10? Compute, using this connection, the size of the balloon payment at the end of year 10. If you cannot find the connection, a more work-intensive approach is to use an amortization table to compute the answer (use Excel!).

11.2. Assume now that this loan has variable payments. At the end of each month fixed, equal portions of the loan's principal are paid down, such that the principal would be fully amortized at the end of the hypothetical 30-year loan period. What is the size of the balloon payment at the end of year 10 in this case?

Solution:

11.1. The loan can be seen as a regular annuity with a maturity of 30 years. The monthly interest rate is $\frac{r}{12} = \frac{6.7\%}{12} = 0.56\%$. The number of periods is $12 \cdot 30 = 360$. Let us denote the principal of the loan by $PV_{loan} = \$10,000,000$.

(11.1.1) The monthly payment is $C = \frac{r}{1-\left(\frac{1}{1+r}\right)^t} \cdot PV_{loan} = \frac{0.0056}{1-\left(\frac{1}{1+0.0056}\right)^{360}} \cdot 10,000,000 = \$64,660.52.$

(11.1.2) The present value of the payments is the value of a regular annuity with the payment equal to the loan's monthly payment, i.e. \$64,660.52. The interest rate is the monthly rate computed earlier. This time, however, there are $t^{'}{=}12 \cdot 10 = 120$ periods (payments). $PV = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^{t^{'}}}{r} = 64,660.52 \cdot \frac{1-\left(\frac{1}{1+0.0056}\right)^{120}}{0.0056} = \$5,638,808.22.$

(11.1.3) The difference between the loan's principal and the present value of the payments over the first 10 years is the present value of the balance outstanding at the end of 10 years. The same amount is also the present value of the all the payments that would be made after 10 years, if the loan reached its maturity.

We can turn this reasoning around: the future value of $PV_{loan} - PV$ must be the loan balance outstanding at 10 years in the future. We have: $Balance_{10} = (1+r)^{t^{'}} \cdot (PV_{loan} - PV) = (1+0.0056)^{120} \cdot (10,000,000 - 5,638,808.22) = \$8,523,872.53.$

If may seem surprising that after "paying off" \$5.6 million in the first 10 years, the borrower still owes \$8.5 million. Note, however, that the majority of payments made in the early years represented interest, not the repayment of the principal. You can see this best if you look at the slides made available to you earlier, illustrating the evolution of the structure of payments for fixed-payment loans.

11.2. Each month the borrower must repay a fixed share of the principal. The amortization schedule is built as if the loan had a maturity of 30 years; thus the monthly repaid principal is $P_{repaid} = \frac{10,000,000}{360} = \$27,777.78$. After 10 years, or 120 periods, the principal **not** repaid, i.e. the outstanding balance of the loan is $P_{loan} - 120 \cdot P_{repaid} = 10,000,000 - 120 \cdot 27,777.78 = \$6,666,666.40$.

An alternative way of thinking is even simpler: Each month the same amount of principal is repaid. If 10 out of the 30 years of the loan's hypothetical maturity passed, then one third of the principal was paid off, and two thirds of the principal are still left. These two thirds of principal are the balance of the loan at the end of year 10: $\frac{2}{3} \cdot P_{loan} = \frac{2}{3} \cdot 10,000,000 = \$6,666,666.66$. Note the slight difference in the two answers - this is due to rounding errors; they are not significant.

Calculator Solution:

11.1. We use the same notations as above; also, we do not repeat the explanations and reasoning given above. For brevity, we do not interpret or explain the choices of cash flow signs.

Note that in the calculations below, in order to be consistent with the formula-based solutions, we used a per-period interest rate of 0.56%. If you used the calculator to compute the per-period interest and you directly stored the result in the the I/Y register, then you set your interest rate to $\frac{6.7\%}{12}$ =0.558333... If you did this, then the results below would be different; for example, $C = \$64,527.80$. Strictly speaking, this latter result is more accurate, as it used more decimals in its input data. Our solution emulates one possible way in which a student may solve an exam problem by first computing and writing down some intermediate results (e.g., the per-period interest rate), and then performing next steps with the rounded prior results. From our perspective, these differences are not significant. In a real case, however, working with more decimals and minimizing numerical errors due to rounding and truncation would be important.

(11.1.1) N=360; I/Y=0.56; PV=-10,000,000; FV=0. Press CPT PMT to get 64,660.52. Thus $C = \$64,660.52$.

(11.1.2) N=120; I/Y=0.56; PMT=-64,660.52; FV=0. Press CPT PV to get 5,638,808.22. Thus $PV = \$5,638,808.22$.

(11.1.3) N=120, I/Y=0.56; PV=-4,361,191.78 (this is equal to $PV_{loan} - PV$); PMT=0. Press CPT FV to get $End\,Balance_{10} = \$8,523,872.53$.

11.2. Same solution as above.

12 Better Late Than Never

You work for a bank. Exactly five years ago, you helped Al Kapon, a well-known local businessperson, to get a \$15,000,000, 20-year variable-payment amortizing loan in order to build a "soft drink bottling facility." The loan carries an interest of 7% per annum, has monthly payments, and is structured like similar loans discussed in class; in particular, Kapon is expected to pay down the same amount of principal every month for the duration of the loan, in addition to the interest due monthly.

12.1. What is the payment due at the end of the very first month of this loan?

12.2. On the fifth anniversary of the loan Kapon comes to your office unexpectedly, and states that his business is in trouble. However, he hopes that difficulties are temporary, and that his business and finances will recover within one year. After some back and forth, you agree on behalf of the bank to forsake principal payments due for the next 12 months. However, Kapon still must pay in full the interest due at the end of each month.

(12.2.1) What is the loan balance at the end of 5 years, when Kapon asks for the modification of the loan?

(12.2.2) What payments will be made at the end of each month for the duration of the year when principal payments are suspended?

12.3. Principal repayments resume after the 12 months elapse. For the remainder of the loan's original term, the same amount of principal will be repaid every month, so that by the loan's original maturity date the principal is fully paid off.

(12.3.1) What will be the monthly principal payments due after the end of the principal repayment suspension?

(12.3.2) Provide the row of the loan's updated amortization table corresponding to the first month in which after principal payments have resumed.

Solution:

12.1. This loan is amortized over $12 \cdot 20 = 240$ months, and each month an equal share of principal is repaid, until the principal (and thus the loan) is fully paid off. The monthly principal payment is $\frac{15{,}000{,}000}{240} = \$62,500$. For the first month of the loan we must pay interest for the full loan amount, since no part of it was paid back yet. The interest payment is equal to $15,000,000 \cdot \frac{0.07}{12} = \$87,500$. The total payment due at the end of the first month is thus $62,500 + 87,500 = \$150,000$.

12.2.

(12.2.1) The interest is always paid in full; the principal is paid down in constant-size chunks. After 5 years of the original loan term of 20 years have elapsed, a proportion $f = \frac{5}{20} = 25\%$ of the principal was paid off. Still outstanding is 75% of the original balance, that is $15,000,000 \cdot 0.75 = \$11,250,000$.

(12.2.2) There will be no repayment of the principal during the 12 months of suspension, so only interest (on the principal outstanding at the time) will be paid. Because the principal outstanding does not change, the monthly interest, and thus the monthly payments will be constant and equal to $11,250,000 \cdot \frac{0.07}{12} = \$65,625$.

(12.2.3) After the principal repayment suspension ends, 6 years out of the original 20 years will have elapsed. At this point the loan has a leftover maturity of 14 full years, i.e., of $14 \cdot 12 = 168$

months. The still-outstanding principal will be repaid in equal monthly parts over this period. Thus the monthly principal payment must be $\frac{11{,}250{,}000}{168} = \$66,964.29$.

(12.2.4) We already know the new monthly principal payment. We also know the total principal outstanding when the principal suspension ends, which allows for the computation of the interest for the first month after the suspension ends. We need no calculation, in fact, as this is the same interest payment that was paid monthly during the suspension period! Can you say why?

We can now fill out the requested row of the amortization table, shown in Table 2.

Month	**Beg. Balance**	**Payment**	**Interest**	**Principal**	**End Balance**
...	...	...	...	...	...
73	$\$11,250,000.00$	$132,589.29$	$\$65,625.00$	$\$66,964.29$	11,183,035.71
...	...	...	...	...	...

Table 2: Partial amortization of the modified loan discussed in problem 12.212.2.4.

13 Beginning Balance at the End...

Assume that you have a fixed-payment amortized loan with a principal of \$8,000,000, a yearly interest rate of 9% compounded monthly, and a maturity of 10 years.

To the closest thousand, what is the beginning balance of the loan at the start of the last month (i.e., at the beginning of the month at the end of which the very last payment due on the loan is made)?

✗ \$100,000

✓ \$101,000

✗ \$102,000

✗ None of the numbers above can be the answer.

Answer:

Let the monthly fixed payment on this loan be C. C must cover both the monthly interest payment on the beginning balance BB, and the respective month's decrease in loan balance (principal). If the monthly interest is $r = \frac{0.09}{12} = 0.75\%$, then this month's interest is $BB \cdot r$, and the repaid principal is BB itself (this is because the balance goes down to 0 at the end of the month; it is the last month, after all).

Thus we must have $C = BB \cdot r + BB$, from which we get $BB = \frac{C}{1+r}$.

The loan is just a regular annuity, so we can immediately determine its fixed payment to be $C = PV \cdot \frac{r}{1-\left(\frac{1}{1+r}\right)^{120}} = \$8,000,000 \cdot \frac{0.0075}{1-\left(\frac{1}{1+0.0075}\right)^{120}} = \$101,340.62$.

Finally, we have that $BB = \frac{101,340.62}{1+0.0075} = \$100,586.22$.

14 Collecting Coupons

Assume that a bond sells for \$948; it has semi-annual coupons, a maturity of 8 years, yields 5.1%, and has a face value of \$1,000. What is the coupon rate of this bond?

Solution:

We start by writing the general formula that connects bond prices to their (semi-annual) coupon, yield, face value, and maturity:

$$B = \frac{C}{2} \cdot \frac{1 - \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}}{\frac{y}{2}} + P \cdot \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}.$$

Note that in this formula the appropriate adjustments have been made to account for semi-annual coupons and the yield compounding twice a year. The coupon rate can be determined immediately:

$$C = y \cdot \frac{B - P \cdot \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}}{1 - \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}} = 0.051 \cdot \frac{948 - 1,000 \cdot \left(\frac{1}{1+\frac{0.051}{2}}\right)^{16}}{1 - \left(\frac{1}{1+\frac{0.051}{2}}\right)^{16}} = \$43.00.$$

The result implies that this bond pays \$43 for each $\$1,000$ of face value. The annual coupon rate, which the problem is asking for, is $p = \frac{43}{1,000} = 4.3\%$.

15 But How Much Do I Make?

Assume that a bond with a maturity of 10 years, face value of \$1,000, coupon rate of 5%, with semi-annual coupons, has a market price of \$903.25. You have already determined that the yield of the bond is between 6% and 7%. Show yields as percentages with two decimals precision.

15.1. Assuming that the yield of the bond were 6.5%, show how you would set up your calculator to compute the implied bond value and provide this respective value.

15.2. Set up a table similar to that given in class to determine an approximate value for the yield of the bond. Use your calculator to compute bond values, but do not show the details of your calculator operations. Stop when the mid-yield produces a price within \$0.50 of the bond's true price.

Hints: You only need to compute the total value of the bond for various yields, not also the part attributable to coupons or principal. Also, since only the yield changes, you do not have to re-enter all the values into the TVM worksheet - just change the yield and recompute the value.

Solution:

15.1. $N = 20$ (the number of periods is $2 \cdot 10 = 20$); $I/Y = 3.25$ (the rate compounds twice a year, $y = \frac{6.5\%}{2} = 3.75\%$); $PMT = 25$ (the semi-annual coupon is $\frac{50}{2} = \$25$); $FV = 1,000$. Now we press $CPT\,PV$ to obtain the value of the bond as -890.95. Note that the value given in the problem text is not the value that we get (or that we should use) here.

15.2. We provide the table below. Please note that values that are simply repeated from above are not shown (i.e., and empty cells means that the value in the cell is identical to that in the first non-empty cell above it). Values taken over from the columns corresponding to the mid-yield in the previous row are shown on gray background. Note that each row requires the computation of a single new yield (the mid-yield), and of a single bond value, which is easy to do using the calculator. We show only annualized yields in this table; in actual calculations per-period yields would be used. Details were discussed extensively on slides, in class, and in the additional hand-out on computing bond yields.

As can be seen, the yield of this bond is approximately 6.32%. Depending on how you rounded and/or truncated your results, your table may be different, but the final result should not be significantly different.

Low Yield	Bond Price	High Yield	Bond Price	Mid-Yield	Bond Price
6.00%	925.61	7.00%	857.88	6.50%	890.95
		6.50%	890.95	6.25%	908.08
6.25%	908.08			6.38%	899.47
		6.38%	899.47	6.31%	903.76
6.31%	903.76			6.34%	901.61
		6.34%	901.61	6.33%	902.69
		6.33%	902.69	6.32%	903.22
		6.32%	903.22	6.32%	903.49
6.32%	903.49			6.32%	903.36

16 Treasuries

On February 28, 2020, the financial press announced that yields for US Treasury bonds with 10-year maturities were at record low levels, never before seen. Indeed, early in the day, 10-year Treasury yields were as low as 1.18%, corresponding to a price of $1,029.50 per $1,000 face value. Typical Treasury bonds pay semi-annual coupons.

16.1. Without performing any computations, can you provide a **lower bound** (lower limit) for the yearly coupon rate of this 10-year US Treasury bond? If yes, state what this lower bound is, and how you know it is correct. A trivial lower bound of 0 is not an acceptable answer.

Solution:

We know that when the yearly coupon is equal numerically to the bond's (annualized) yield, the bond trades at par, no matter what the compounding period is. To trade at par, the yearly coupon would have to be 1.18%. The bond trades above par. Given that the face value and yield are fixed, the increase of the price (payment present value) above par must come from coupons that are higher than 1.18% per year.

16.2. What is the implied annual coupon rate for this bond?

Solution:

The bond has a maturity of 20 6-month periods, and the 6-month yield is $\frac{1.18}{2} = 0.59\%$. We use the calculator to determine the answer: 20 [N], 0.59 [I/Y], -1,029.50 [PV], 1,000 [FV]. We then press [CPT] [PMT] to get 7.47. This result means that the bond pays a coupon of \$7.47 per \$1,000 of face value each 6-month period. The yearly coupon is thus $2 \cdot 7.47 = \$14.94$. The yearly coupon rate is $c = \frac{14.94}{1{,}000} = 1.49\%$.

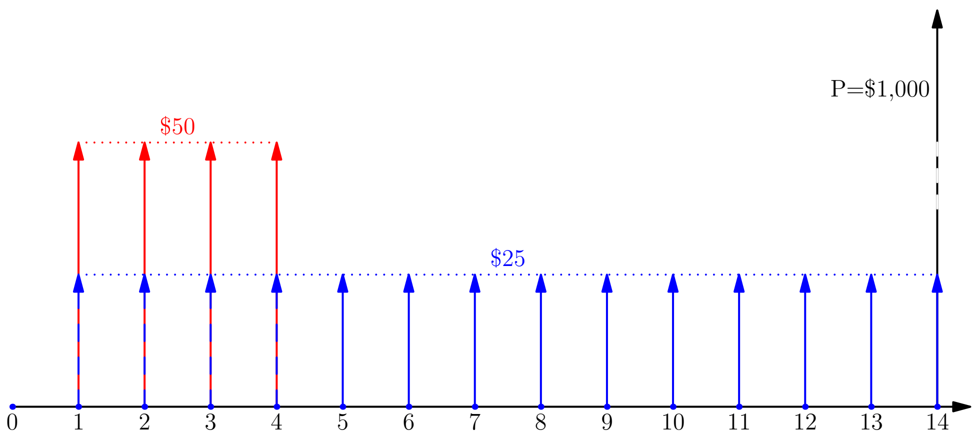
17 Bond Decomposition

You are studying a bond that has a leftover maturity of 7 years, has a face value of \$1,000, and a stated coupon rate of 5% per year, payable semiannually. The yield of the bond is 4.7% per annum; further, you may assume that per-period rates are constant (they do not depend on the time horizon).

The bond contract includes covenants (agreements) meant to protect bond investors' interest. In the recent past the company breached one of these covenants. Even though some improvements did occur since then, the breach is expected to persist for the next two years, after which it is expected to be cured (eliminated). While the breach persists, the coupon rate will be double the normal (stated) one. If default were to occur, such an accelerated payment schedule assures that investors get more of their money back; if default does not occur, then the increased coupons act as a penalty for the breach of the covenant.

17.1. Decompose this bond into simpler financial instruments that were studied in class. Specify what these simpler instruments are, what their parameters are, and how do you know your decomposition is correct.

Solution:



This bond has a coupon period of 6 months. The normal (stated) yearly coupon is $C = 5\% \cdot 1,000 = \$50$, while the increased yearly coupon is $2 \cdot C = \$100$. The actually paid 6-month coupons are of \$25 and \$50, respectively. There will 4 payments of \$50, covering 4 successive periods (2 full years), followed by 10 payments of \$25, covering the next 10 successive periods (5=7-2 years). At the end of the 7 years (14 successive periods) there will be a principal payment of \$1,000.

We can decompose our atypical bond into an annuity paying \$25 each period, maturing in 2 years, and a regular bond paying \$25 every period, maturing in 7 years. Putting together the cash flows of these two instruments we get the cash flows of the atypical bond, so the decomposition must be correct. This is not the only possible decomposition.

17.2. Determine the current price of the bond.

Solution:

The per-period yield of this bond is $\frac{4.7\%}{2} = 2.35\%$. We use the calculator to value the annuity: 4 $\boxed{\text{N}}$, 2.35 $\boxed{\text{I/Y}}$, 25 $\boxed{\text{PMT}}$, 0 $\boxed{\text{FV}}$. We then press $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$ to get -94.39. The result means that the annuity's value is A=\$94.39.

We can use the calculator to also value the (typical) bond: 14 $\boxed{\text{N}}$, 2.35 $\boxed{\text{I/Y}}$, 25 $\boxed{\text{PMT}}$, 1,000 $\boxed{\text{FV}}$. We then press $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$ to get -1,017.72. The result means that the bond's value is B=\$1,017.72.

The value of the atypical bond is $A + B = 94.39 + 1,017.72 = \$1,112.11$.

18 NPV Mystery I

The NPV of a regular coupon bond's cash flows, when the discount rate is equal to the bond's yield, is equal to the bond's price (value). This statement is...

✓ True

✗ False

Answer:

This statement is true; in fact, it is a restatement of the definition of the yield.

19 NPV Mystery II

For the types of bonds discussed in lectures, the NPV of the bond's principal (face value) always exceeds half of the bond's current price (value). This statement is...

✗ True

✓ False

Answer:

This statement is false. One can get intuition about it by thinking of a coupon bond with, say, a 30-year maturity and a face value of \$1,000. If the yearly coupon was a very reasonable 3.5%, say, then the bond would make 60 payments of $\frac{\$35}{2} = \$17.5$, for a total coupon payment of \$1,050 (these are nominal, non-discounted dollars). These coupon payments exceed the face value of the bond, and, but for the last one, all of them are paid earlier than the face value. Thus the cumulative net present value of the coupons must exceed the net present value of the face value, which means that more than half of the bond's value comes from coupons. This conclusion is even stronger if coupon rates are higher. In other words, the longer the maturity of the bond, and the higher the coupons, the more likely it is that the statement above is false.

There are bonds for which the majority of the value comes from the principal payment, an obvious example being zero-coupon bonds. Short-maturity bonds paying low coupon rates also derive most of their value from their respective principals.

20 Congratulations, You're an Analyst

You have just been hired as a junior analyst working for a bond trader. Your first assignment is to value a corporate bond paying semi-annual coupons at an annual rate of 9.5%, with a maturity of exactly 2 years. The bond has a face value of $1,000. A senior analyst has already processed the current Treasury price data and provided you with an up-to-date term structure chart, shown in Figure 1. You are told to treat this corporate bond similarly to government bonds; i.e. you will ignore all default, liquidity, and similar risks, which will be analyzed by more experienced colleagues.

20.1. The term structure of interest rates chart has a horizontal axis labeled "Maturity," and a vertical axis labeled "[Annual[1]] Yield." What kind of government bonds have their yields and maturities plotted on this chart?

20.2. From earlier problems you solved while in college, you learned to decompose more complex financial instruments into sums - or differences - of simpler instruments. Explain how you can decompose this corporate bond into a collection of zero-coupon bonds of different maturities, perhaps having atypical face values. Provide a brief statement explaining the decomposition and show, in a table, what would be the bond's maturities and face values, respectively.

20.3. You learned that in realistic settings cash flows that arrive later must be discounted at (usually) higher per-period interest rates. For each zero-coupon bond listed in part 20.2 above, use the term-structure chart and look up its corresponding yield. Next, compute the present value of each zero-coupon bond. Using these zero-coupon bond prices, and also relying your earlier insights, provide a computed (theoretical) price for your corporate bond.

20.4. You now have a price for your corporate bond - what is its yield?

20.5. You provide the result computed in item 20.4 to one of your colleagues, who explains that in the practice of your firm, in order to adjust for the risks that a bond like yours bears in addition to government bonds, its yield must be changed by 0.50% per year. The colleague did not say explicitly whether the yield should be increased or decreased. State whether the yield must be increased or decreased, explain why, and then compute the new bond price. Determine what is the percentage change in the bond price when comparing prices before and after the risk adjustment, respectively.

[1] As you will note, the chart only uses "Yield" as a label for the vertical axis. The usual bond terminology expresses yields in annualized terms, and you should do the same in this class, as well as in other finance-related work that you do. We provide a reminder here, but you should **no**t assume that similar reminders will also be present when taking an exam.

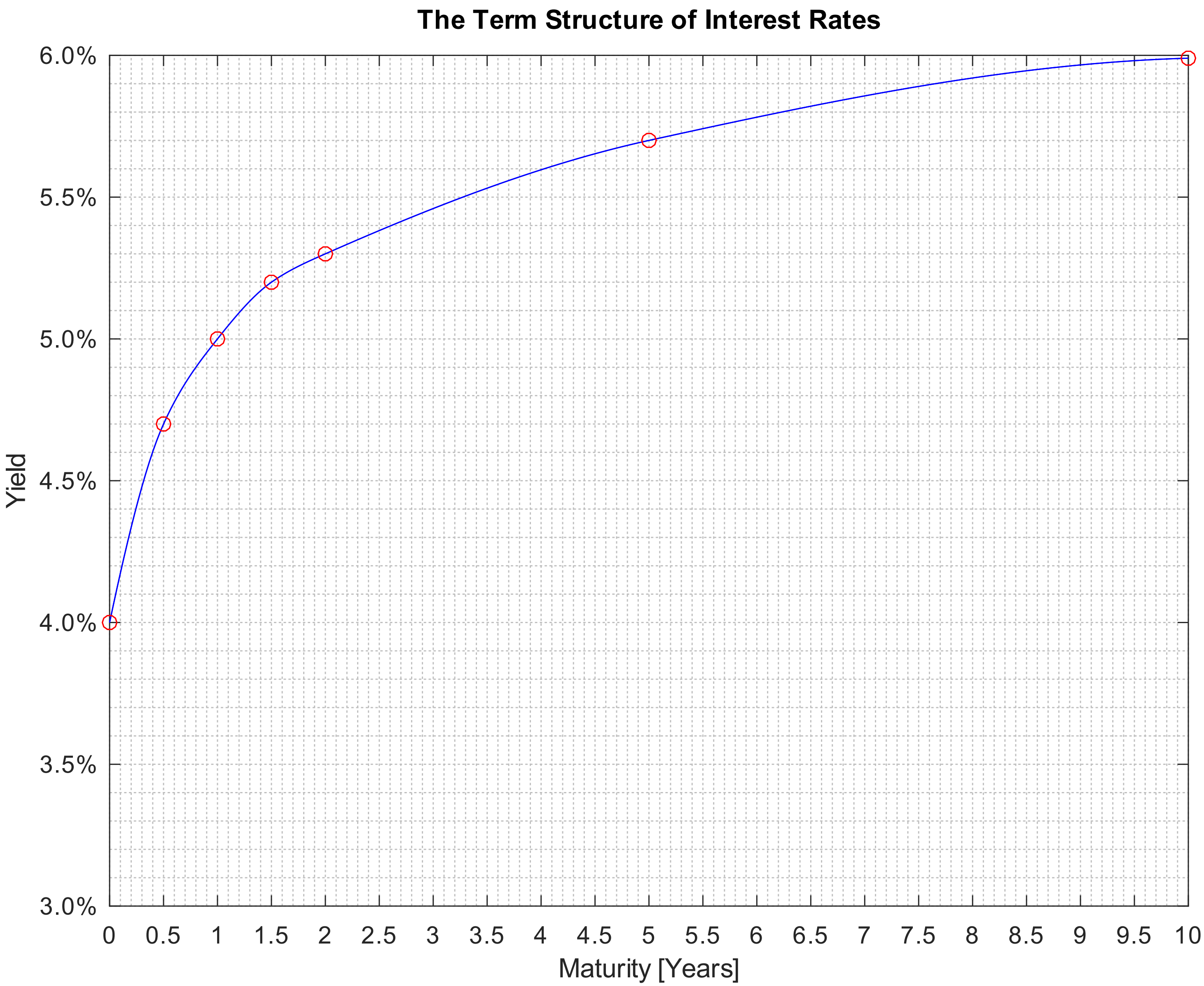


Figure 1: Term structure of interest rates on your first day as an analyst, as determined by a senior analyst using current government bond data.

20.6. Now adjust the yield in the **opposite direction** to that you decided was necessary in part 20.5 above. Compute the new price and the percentage price change when comparing the original (riskless) price and the price after the newest change in yield. Compare the percentage changes in the price of the corporate bond when the yield has been increased and decreased by the same amount, respectively. Which relative change is bigger? Could you have predicted which change is bigger by examining any of the slides discussed in class, without resorting to formulas?

Solution:

20.1. As discussed in class and in your textbook, the term structure of interest rates is the yield curve of **zero-coupon government bonds**. The problem already states that these are government bonds, so all you need to add is that these are also zero-coupon bonds.
Each point on the attached chart specifies yield of a zero-coupon bond of a certain maturity. In reality, there are no zero-coupon bonds of all possible maturities, but there exist techniques that use both regular and (the rare) zero-coupon bonds to infer this curve even for zero-coupon bonds that do not actually exist.

20.2. The corporate bond you were assigned makes 4 coupon payments 6, 12, 18, and 24 months into the future, respectively, and also repays its principal 24 months into the future. The individual coupon payments are of $\frac{0.095}{2} \cdot 1,000 = \$47.50$ each. Table 3 summarizes the cash flows that will be paid, as well as their timing.
If we ignore risks that government bonds do not bear, our corporate bond can be described as a package of zero-coupon government bonds with atypical face values; i.e., each cash flow in the table can be interpreted as the face value of a zero-coupon bond that matures at the time when the cash flow is due.

Time [Yrs]	**0.5**	**1.0**	**1.5**	**2.0**
Cash Flow	\$47.50	\$47.50	\$47.50	\$47.50+\$1,000=\$1,047.50

Table 3: Cash flows paid by the corporate bond discussed in problem 20.2.

20.3. We use the term structure of interest rates chart that we were given to read off the yields of the corresponding zero-coupon bonds. We note in passing that this is an upward sloping term structure curve, as is typical, because longer maturities usually also imply higher per-period interest rates. We extend Table 3 with the data read off the chart, as well as with present value calculations for each zero-coupon bond; the result is shown in Table 4.

If we add up the present values of all zero-coupon, and thus of the corporate bond's payments, we get a theoretical price for the corporate bond: $B = \$1,079.05$.
Corporate bonds bear more risk than Treasuries, because, for example, they are subject to default risk, while Treasuries are usually assumed not to carry default risk. Because rational investors need to be compensated for extra risks, the interest that they demand is higher for riskier bonds. This means that the discounting for the corporate bond should be more aggressive, which, in turn, means that the present value of the payments should be lower. The price we got is an upper bound - an overestimate - of the true price.

20.4. We can compute the yield of the bond using the bisection method. We presented that method extensively in earlier handouts, so in this case we'll just use the calculator to quickly get the result: 4 $\boxed{\text{N}}$, -1,079.05 $\boxed{\text{PV}}$, 47.50 $\boxed{\text{PMT}}$, 1,000 $\boxed{\text{FV}}$, and then hit $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$. The per-period yield is equal to 2.64%, while the annualized yield is $y \approx 2.64 \cdot 2 = 5.28\%$. Note that this value is lower than, and close, but not equal to the yield of the 2-year zero-coupon bond. Can you explain qualitatively why this is so?

20.5. The yield must increase; see the discussion related to interest rates (hence, also yields) and risk above. The new yield is thus y_{high} =5.28+0.50=5.78% (annualized). The price of the bond can be computed using the calculator: 4 $\boxed{\text{N}}$, 2.89 $\boxed{\text{I/Y}}$, 47.50 $\boxed{\text{PMT}}$, 1,000 $\boxed{\text{FV}}$, and then hit $\boxed{\text{CPT}}\boxed{\text{PV}}$ to get $-1,069.32$. The price of the bond is thus $1,069.32. As expected, the **increase** in yield led to a **decrease** in the computed bond price. The relative change in bond price is $\Delta_{increase} = \frac{B_{5.78}-B_{5.28}}{B_{5.28}} = \frac{B_{5.78}}{B_{5.28}} - 1 = \frac{1{,}069.32}{1{,}079.05} - 1 = 0.9017\%$.

20.6. Since we increased the bond yield in the prior part, we now have to decrease it. The new yield will be y_{low} =5.28-0.50=4.78% (annualized). The price of the bond can be computed using the calculator: 4 $\boxed{\text{N}}$, 2.39 $\boxed{\text{I/Y}}$, 47.50 $\boxed{\text{PMT}}$, 1,000 $\boxed{\text{FV}}$, and then hit $\boxed{\text{CPT}}\boxed{\text{PV}}$ to get $-1,089.02$. The price of the bond is thus $1,089.02. As expected, the **decrease** in yield led to an **increase** in the computed bond price. The relative change in bond price is $\Delta_{decrease} = \frac{B_{4.78}-B_{5.28}}{B_{5.28}} = \frac{B_{4.78}}{B_{5.28}} - 1 = \frac{1{,}089.02}{1{,}079.05} - 1 = 0.9240\%$.
$\Delta_{decrease}$ is positive, while $\Delta_{increase}$ is negative; if we ignore the sign and just focus on magnitudes, however, then $|\Delta_{decrease}| > |\Delta_{increase}|$. This means that the price increases faster if we decrease the yield, compared to the decrease in price when the increase the yield by the same amount. Speaking informally, the climb toward higher prices when we decrease the yield is steeper than the walk downhill when we increase the yield by the same amount.

Time [Yrs]	**0.5**	**1.0**	**1.5**	**2.0**
Cash Flow	\$47.50	\$47.50	\$47.50	\$47.50+\$1,000=\$1,047.50
"Zero" Yield	4.7%	5.0%	5.2%	5.3%
PV	$\frac{47.50}{\left(1+\frac{0.047}{2}\right)^1} = \$46.41$	$\frac{47.50}{\left(1+\frac{0.05}{2}\right)^2} = \$45.21$	$\frac{47.50}{\left(1+\frac{0.052}{2}\right)^3} = \$43.98$	$\frac{1{,}047.50}{\left(1+\frac{0.053}{2}\right)^4} = \$943.45.$

Table 4: Discounted cash flows used to provide a computed price for the corporate bond, un-adjusted for risk, as discussed in problem 20.3.

We have seen this effect, and we mentioned it in passing, when we looked at plots of bond values as a function of the bond's yield: the curves were steepest toward the left end, and their steepness decreased toward the right end. The quantitative effect that we notice here could have been qualitatively predicted from the plots we studied.

You may have considered that $\Delta_{decrease} \approx \Delta_{increase}$, given the closeness of their numerical values. This would not be unreasonable in the context of a homework, say. You could have then stated that small yield changes in opposite directions induce approximately equal relative changes in bond prices when yield changes are small. In other words, the climb toward higher prices is approximately as fast as the walk downhill, when the yield changes by a small amount. This statement, while approximately correct, is harder to reconcile with high-level qualitative observations that we made in class. However, if you took this approach, we would accept it.

21 We Have No Money, We're a Startup

Next Wonder, Inc., is a startup that cannot afford to pay dividends, since it has to finance its rapid growth, as it aims to take over the technology world. The company will pay no dividends for 5 years, but then it plans to pay \$2.5 per share, per year for the next 5 years. After these 10 years, the stock dividend will jump to \$5 in year 11, and will keep increasing by 3% per year indefinitely. The required return is 8% per year.

21.1. Assume that you are computing the price of the stock, P_{10}, at the end of year 10, just after the dividend due at the end of year 10 has been paid. What is P_{10}?

21.2. Assume that you are computing the price of the stock, P_5, at the end of year 5. What is P_5?

21.3. What is the price of the stock at time 0, P_0?

Solution:

21.1. Looking ahead from time t=10, we "see" a first dividend of \$5 to be paid in year 11; we also know that this dividend will increase by 3% per year. We can use the formula for constant dividend growth, when the price of the stock is given by the value of a growth perpetuity.

$$P_{10} = \frac{D_{11}}{R-g} = \frac{5.00}{0.08-0.03} = \$100.00.$$

We conclude that the time-10 price of the stock will be exactly \$100.

21.2. Price P_5 must include the present value (computed at time 5!) of the constant dividend $D = \$5$ paid in years 6, 7, 8, 9, and 10, respectively, plus the present value of the stock price at time 10. We note that the stream of constant dividend payments forms a regular annuity of maturity 5. We can immediately write:

$$\begin{aligned} P_5 &= D \cdot \frac{1 - \left(\frac{1}{1+R}\right)^5}{R} + \frac{P_{10}}{(1+R)^5} \\ &= 2.5 \cdot \frac{1 - \left(\frac{1}{1+0.08}\right)^5}{0.08} + \frac{100}{(1+0.08)^5} \\ &= 9.98 + 68.06 \\ &= \$78.04. \end{aligned}$$

21.3. There are no dividends paid in years 1 to 5, so their present value is obviously 0. Thus P_0 is the present value (at time 0) of P_5 :

$$P_0 = \frac{P_5}{(1+R)^5} = \frac{78.04}{(1+0.08)^5} = \$53.11.$$

22 Stock price

You are a new analyst following the stock of company Big Break, Inc., valued at \$15 per share on the morning of March 31, 2020. The company has been paying dividends of 35 cents on the very last day of each calendar quarter, and this was expected to continue indefinitely.

22.1. Assuming that the dividend model is reasonably accurate for this stock, what is the implied return rate demanded by investors in this stock, expressed in "per annum, compounded quarterly" terms?

Solution:

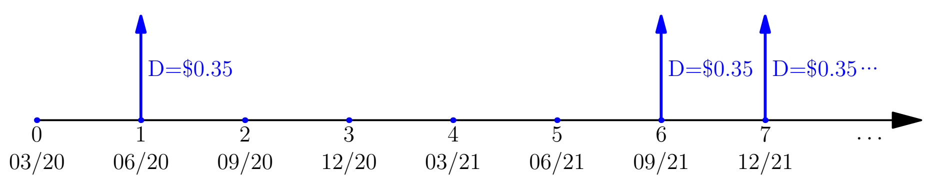
We use the notation $P_{mm/yy}$ to denote the price of the bond at the end of month mm in year 20yy. Before the announcement, the constant stream of dividends forms a perpetuity. Thus $P_{03/20} = \frac{D}{R}$, from which $R = \frac{D}{P_{03/20}} = \frac{0.35}{15} = 2.33\%$. This rate is a quarterly (per-period) rate, so we have to annualize it: $R_{annualized} = 4 \cdot 2.33 = 9.32\%$.

22.2. During a news conference late in the day on March 31, 2020, the company's CEO announces "temporary difficulties related to external funding needs" that will force the company to cancel the dividend for four successive quarters, "in order to conserve cash." The dividend due on June 30 will be paid, however, as it had already been declared, so the first missed payment will occur on September 30, 2020. The CEO, a highly credible, seasoned industry veteran, states that the dividend policy will be reinstated in its current form after the temporary suspension ends.
Note: If you need to use the value that was requested in part (22.1) above, but you were not able to compute it, you may use 5.5% for the "per annum, compounded quarterly" rate.

(22.2.1) Estimate the price of stock as of March 31, 2021.

Solution:

The diagram below illustrates the situation that prevails after the announcement:



The first dividend $D = \$0.35$ that will be paid after the suspension ends will occur at the end of September 2021. As of the prior quarter-end in June 2021, the dividend stream forms a regular perpetuity. We thus have that $P_{06/21} = \frac{D}{R} = P_{03/20} = \$15$. We can particularize our general stock price formula for end-March 2021, keeping in mind that there are no dividends paid in June 2021:

$$P_{03/21} = \frac{P_{06/21}}{1+R} = \frac{15}{1+0.0233} = \$14.66.$$

(22.2.2) Estimate the price of the stock as of March 31, 2020, just after the announcement.

Solution:

We already computed the price at the end of June 2021, five quarters into the future: $P_{06/21} = \$15$. We particularize our general stock price formula keeping in mind that there is a dividend in June 2020, one quarter into the future, but none during the subsequent four quarters:

$$P_{03/20} = \frac{D}{1+R} + \frac{P_{06/21}}{(1+R)^5} = \frac{0.35}{1+0.0233} + \frac{15}{(1+0.0233)^5} = 0.34 + 13.37 = \$13.71.$$

You can verify that the same result is obtained if you use $P_{03/21}$, which we have also computed:

$$P_{03/20} = \frac{D}{1+R} + \frac{P_{03/21}}{(1+R)^4}.$$

23 Dividends

Fly Over Airlines just paid an annual dividend of 80 cents, and a new analysis revealed that dividends will grow long term at a rate of 4% per year. The company's stock price is $70. Estimate, to the closest percent, the rate of return demanded by investors who buy Fly Over's shares.

✓ 5%

✗ 6%

✗ 7%

✗ None of the above is the correct answer.

Answer:

We compute the required return using the model of constant-growth dividends discussed when we studied stock valuation. Thus the required return is $r = \frac{D_1}{P_0} + g$, where D_1 is the size of the next dividend, P_0 is the time-0 value of one share, and g is the constant dividend growth rate. Denoting the just-paid dividend by D_0, we get: $r = \frac{D_1}{P_0} + g = \frac{D_0 \cdot (1+g)}{P_0} + g = \frac{0.80 \cdot (1+0.04)}{70} + 0.04 = 5.19\%$.

24 What's an Option Worth?

Over the last few weeks we got into the habit of describing more complex financial instruments as collections of simpler instruments. For example, we can decompose a typical bond into an annuity (the coupon payments), and a separate single payment (the principal). Similarly, we have seen atypical annuities described as sums or differences of regular annuities. In this problem you will build a synthetic bond, by suitably combining fractional amounts of two other bonds.

Consider the following information about three Treasury bonds:

Maturity Date	Coupon Rate	Price	Bond Type
05/15/2024	6.500	106.31250	regular
05/15/2024	8.250	103.43750	callable
05/15/2024	12.000	134.78125	regular

All the bonds above have a face value of $1,000. Prices are expressed as percentages of face value. We do not know the precise date when these prices were quoted; as such, it is possible that we are not exactly at the beginning of a coupon period. We do know, however, that all coupon payment dates for all these bonds are the same; the coupons, of course, are not. The bond in the middle is a so-called callable bond, while the other two are regular Treasuries.

24.1. Read the textbook and/or research the web to understand what callable Treasuries are. Summarize your findings in a **brief** paragraph.

Solution:

A callable bond may be redeemed ("recalled") by its issuers under terms that were set before the bond was first sold. Often, the recalled bond will be redeemed for its face value. In principle, any type of bond can be made callable. Callable Treasury bonds have historically been very rare; some source may even indicate that they did not exist, but that is inaccurate.

24.2. Given what you learned about callable bonds, under what conditions would the option to call the callable Treasury be exercised and by whom (the bond holder, or the bond's issuer)? In other words, when would the callable Treasury be called and by whom?
Hint: Assume that the call decision is rational; consider changes in interest rate levels as time passes.

Solution:

If a bond issuer acts rationally, a callable bond would be recalled only if the issuer benefits from this action. This means that a bond will be retired only if the NPV of the bond payments still due are larger than the immediate redemption costs (face value, if we ignore administrative and other costs) of the bond. In effect, the issuer will take something of value from the bond-holder when redemption occurs (the holder loses the larger NPV of future payments and gets the smaller immediate redemption value). Since the issuer is the US Treasury, it would the Treasury that would initiate redemption.

24.3. From the **holder**'s perspective, would a callable Treasury bond be more, or less valuable than an otherwise identical non-callable, i.e., regular Treasury bond? Briefly state your opinion and justify it qualitatively.

Solution:

Holders and potential holders of bonds know at the time of their purchase that a bond may be recalled, and that they would lose money if a recall occurred. Hence, if they act rationally, they should be willing to pay **less** for a callable bond than for an otherwise identical non-callable (regular) bond.

24.4. Now consider the pricing information provided in the table above. Use the information on the regular bonds to construct a bond that is identical to the the callable bond in all respects, but for the callable feature. Use this artificial bond to compute the cost of the call option. From the perspective of the bond holder (the lender), is this cost positive or negative?

Hint: Let B_1, B_2, and B_3 be the three bonds shown in the table above, starting with the bond at the top. If we ignore the call feature, the listed Treasuries are characterized by a small number of parameters: their coupon sizes, coupon payment times, and face values. You should build a portfolio (combination) of the two regular bonds B_1 and B_3, such that their combination produces B_2 (again, except for the call feature). Let B_{13} be an artificial (often called "synthetic") bond that you can create from a combination of B_1 and B_3, and assume that B_{13} consists of a fraction f of B_1 and a fraction $1-f$ of bond B_2, where $0 \leq f \leq 1$. Write down equations so that the resulting coupon and face value of B_{13} (seen as a mix/combination/portfolio of B_1 and B_3) matches the respective parameters of B_2. Use these equations to determine f, and then use f to determine an appropriate price for B_{13}.

Solution:

We have encountered in the past financial instruments that could be decomposed into sums or differences of simpler instruments that we had studied. Pricing the more complex instrument can then be reduced to computing the equivalent sum or difference of prices of simpler instruments. All examples that we have encountered before, however, consisted of combinations of **integer** numbers of simpler instruments. As we will see in this case, a non-callable composite bond identical to the callable bond B_2 above, but for the call feature, can be constructed from **fractional** holdings of the two non-callable bonds B_1, and B_3.

If we ignore callability, a bond can be fully described by the size and time of its coupon payments, as well as its face value (the face value will be paid together with the last coupon, so there is no additional information needed with respect to the payment time of the face value). From the text of the problem, bonds B_1, B_2, and B_3 have the **same** coupon payment dates, and the same face value of $1,000.

Let us take a fraction f, $0 \leq f \leq 1$, of bond B_1, and a fraction $1 - f$ of bond B_3; create a new bond by putting together the resulting coupon and face value payments. Let us call the resulting bond B_{13}, to indicate that it resulted from the combination of bonds B_1 and B_3.

With respect to face value, we have: $FV_{13} = f \cdot FV_1 + (1-f) \cdot FV_3 = f \cdot 1,000 + (1-f) \cdot 1,000 = (f + 1 - f) \cdot 1,000 = \$1,000$. It turns out that for any choice of f, the value of bond B_{13} is the same as that of the three other bonds, including bond B_2.

With respect to coupons, our goal is to recreate B_2's (yearly) coupon size: $C_{13} = f \cdot C_1 + (1-f) \cdot C_3 = f \cdot 65 + (1-f) \cdot 120 = 120 - 55 \cdot f$. We know that the (yearly) coupon of bond B_2 must be $82.5, so we get the equation $C_{13} = C_2 \implies 120 - 55 \cdot f = 82.5$. From here, we immediately get $f = \frac{120-82.5}{55} = 0.682 = 68.2\%$.

The results above imply that if we took (approximately) 68.2% of bond B_1 and 100%-68.2%=31.8% of bond B_3, we would get bond B_{13}, which would be identical to bond B_2, but for callability (B_{13} is composed of two non-callable bonds, thus their "mix" is not callable either). The price of bond B_{13} is obtained from a similar combination of prices for the component bonds: $0.682 \cdot 106.31250 + 0.318 \cdot 134.78125 \approx 115.36556$.

So, in terms of face value percentages, the price of bond B_2 is 103.43750, while the price of (synthetic) bond B_{13} is 115.36556. These are identical bonds, but for their callability. We can easily establish that the callable bond does indeed trade at a discount (lower price). What is the difference? It is, in terms of percentages of face value, equal to $115.36556 - 103.43750 = 11.9281$.

For each \$1,000 of face value the existence of callability depresses the price of the callable bond by $\$1,000 \cdot 0.119281 = \$119.28$. Callability is an advantage to the issuer, but it comes at the price of a discount - buyers will pay less for these bonds. There is no free lunch in (rational) finance!