Homework 1

ORIE 3510/5510 & STSCI 3510, Spring 2024
Due Thursday, February 1 at 10:00pm on Gradescope

Problem 1	40 points
Problem 2	30 points
Problem 3	40 points
Total	110 points

1 Who's that Pokémon?

Gym Leader Monty brings two Pokémon to his daily teatime battle. To keep things interesting, he has started randomly deciding how to build his team each day.

- At breakfast, Monty chooses a Pokémon uniformly at random between a Venusaur (denoted V), a Charizard (denoted C), and a Blastoise (denoted B). Let's call this Pokémon the *breakfast-chosen Pokémon*.
- At lunch, Monty again chooses a Pokémon uniformly at random between a Venusaur, a Charizard, and a Blastoise. Let's call this Pokémon the *lunch-chosen Pokémon*.

Monty generates these choices independently, so he might choose two Pokémon of the same species, e.g. two Venusaurs.

At this particular teatime, you are the trainer battling Monty. The battle begins, and Monty's *leading Pokémon*, i.e. the one he sends into battle first, is a Charizard—your team's biggest weakness! You therefore want to know the probability that Monty's non-leading Pokémon is also a Charizard. But you quickly realize that this probability depends on how Monty chooses his leading Pokémon.

For now, assume that Monty uses the following procedure to choose his leading Pokémon:

- If possible, lead with a Charizard.
- Otherwise, if possible, lead with a Blastoise.
- Otherwise, lead with a Venusaur.

To make sense of this situation, let's model it with a probability space.

- Let $\mathcal{S} = \{\mathrm{V}, \mathrm{C}, \mathrm{B}\}$ be the set of possible Pokémon species.
- Let the set outcomes be $\Omega = \mathcal{S}^2$. Each outcome is an ordered pair $\omega = (\omega_1, \omega_2)$, where ω_1 is the species of the breakfast-chosen Pokémon, and ω_2 is the species of the lunch-chosen Pokémon.
- Let $\mathbf{P}$ be the probability measure that assigns each outcome in Ω equal probability.
- Let random variable $X_1 : \Omega \to \mathcal{S}$ be the species of Monty's leading Pokémon.
- Let random variable $X_2 : \Omega \to \mathcal{S}$ be the species of Monty's non-leading Pokémon.

The probability we want to compute is $\mathbf{P}[X_2 = \mathrm{C} \mid X_1 = \mathrm{C}]$.

[10 points] **1.1** Determine the following realizations of random variables on particular outcomes:

(a) $X_1((\mathrm{B}, \mathrm{C}))$.

(b) $X_2((\mathrm{B}, \mathrm{V}))$.

[10 points] **1.2** Write the event $\{X_1 = \mathrm{C}\}$ in two different ways:

(a) As an explicit list of elements of Ω.

(b) As a set expression of the form $\{\omega \in \Omega \mid \boxed{}\}$, where you fill in the $\boxed{}$.

[20 points] **1.3** Compute each of the following:

(a) $\mathbf{P}[X_1 = \mathrm{C}]$.

(b) $\mathbf{P}[X_2 = \mathrm{C}]$.

(c) $\mathbf{P}[X_1 = X_2 = \mathrm{C}]$.

(d) $\mathbf{P}[X_2 = \mathrm{C} \mid X_1 = \mathrm{C}]$.

(Hint: draw a picture of the outcome space, perhaps as a 3×3 *grid.)*

2 What is Monty thinking?

Let's reconsider the scenario from Problem 1 under different assumptions about how Monty chooses his leading Pokémon. This will change how X_1 and X_2 are defined, but the definitions of $\mathcal{S}$, Ω, and $\mathbf{P}$ remain the same.

Throughout, for the purposes of showing your work, it suffices to provide an appropriately annotated picture of the outcome space (e.g. as a 3×3 grid).

[15 points] **2.1** Assume that Monty leads with the breakfast-chosen Pokémon.

(a) Write the definition of $X_1((\omega_1, \omega_2))$.

(b) Compute $\mathbf{P}[X_2 = \mathrm{C} \mid X_1 = \mathrm{C}]$.

[15 points] **2.2** Assume that Monty uses the following procedure: if he chose at least one Venusaur, then he leads with the breakfast-chosen Pokémon; and otherwise, he leads with the lunch-chosen Pokémon.

(a) Write the definition of $X_1((\omega_1, \omega_2))$.

(b) Compute $\mathbf{P}[X_2 = \mathrm{C} \mid X_1 = \mathrm{C}]$.

3 Monty's feeling lucky

I actually had tea with Monty last week, and he told me his secret battle strategy: to choose his leading Pokémon, he *flips a fair coin*. If the coin is heads (denoted H), he leads with the breakfast-chosen Pokémon; and if the coins is tails (denoted T), he leads with the lunch-chosen Pokémon. The coin flip is independent of the species choices.

Our probability model from Problems 1 and 2 with Ω and $\mathbf{P}$ accounts for the breakfast and lunch Pokémon species choices. But now we need to account for an additional source of randomness: the coin flip. As such, we need to construct a new probability model.

[10 points] **3.1** Define a finite outcome space Ω' to model this scenario. *(Hint: one way is to have Ω' be a set of triples, analogous to how Ω is a set of pairs. But there are lots of valid approaches.)*

[10 points] **3.2** Define a probability measure $\mathbf{P}'$ on the outcome space Ω' you chose above. It suffices to specify the probability of each individual outcome, i.e. specify $\mathbf{P}'[\{\omega'\}]$ for all $\omega' \in \Omega'$.

[10 points] **3.3** Define the following two random variables on the outcome space Ω' you chose above:

(a) $X_1' : \Omega' \to \mathcal{S}$, the species of Monty's leading Pokémon.

(b) $X_2' : \Omega' \to \mathcal{S}$, the species of Monty's non-leading Pokémon.

[10 points] **3.4** Compute $\mathbf{P}'[X_2' = \mathrm{C} \mid X_1' = \mathrm{C}]$.