## Partitions of n with k parts

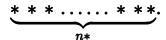
## Partition of n with k parts

A partition of n with k parts is a vector  $\mathbf{x}=(x_1,\ldots,x_k)$  with non-negative integer coordinates that sum to n. Observe that there are only finitely many such objects because each coordinate must be a number in  $\{0,1,\ldots,n\}$  and so there are at most  $n^k$  partitions of n with k parts. But how many are there exactly?

To solve this question, we start with an easier one which is very similar. How many partitions of n with k parts are there with no zero entries? More formally, how many elements are there in the set

$$\mathbf{V} = \mathbf{V}(n,k) = \left\{\mathbf{x} = (x_1,\ldots,x_k) : x_i \in \{1,2,\ldots\} ext{ and } \sum_1^k x_i = n
ight\}.$$

To figure this out, we use a famous argument called **stars and bars argument**. Imagine we have n stars arranged in a raw



Now, suppose we have k-1 bars | which can be inserted in the n-1 empty spaces between the n stars with at most one bar in one slot. Inserting the bars give configurations of stars and bars which looks as follows (for the sake of the illustration we use n=9 and k=5)

or (an other example)

How many such distinct configurations are there? That is not too hard to figure out because, in order to create a configuration, we just need to choose the k-1 slots in which we insert the k-1 bars that define the configuration. As already noted, there are n-1 slots between the stars. It follows that the total number of such configurations in  $\binom{n-1}{k-1}$ .

Now, observe that any one of these configurations is associated with a unique element in  ${f V}$  and vice-versa.

This is simply because the inserted bars decompose the n stars into k parts. For instance

$$*|***|*|***|*$$
 corresponds to  $(1,3,1,3,1)$ 

and

$$*|*|*|*|**|***$$
 corresponds to  $(1,1,1,2,4)$ .

This gives a one -to-one and onto map from configurations of stars and bars" and the set  $\mathbf{V}$  and it follows that they both have the same number of elements,  $\binom{n-1}{k-1}$ .

Now, our original question was to count the number of elements of (compare to the definition of  $\mathbf{V}$ )

$$\mathbf{U} = \mathbf{U}(n,k) = \left\{\mathbf{x} = (x_1,\ldots,x_k) : x_i \in \{0,1,2,\ldots\} ext{ and } \sum_1^k x_i = n
ight\}.$$

To do that it suffices to observe that adding 1 to each coordinate of an element  $\mathbf{x}=(x_1,\ldots,x_k)$  of  $\mathbf{U}(n,k)$  automatically give us an element of  $\mathbf{V}(n+k,k)$ , that is a k-vector with integer coordinates all greater or equal to 1 and summing to n+k. The map

$$\mathbf{U}(n,k) 
ightarrow \mathbf{V}(n,k), \;\; \mathbf{x} = (x_1,\ldots,x_k) \mapsto \mathbf{y} = (x_1+1,\ldots,x_k+1)$$

is on-to-one and onto. It follows that  $|\mathbf{U}(n,k)|$ , the number of elements in  $\mathbf{U}(n,k)$ , is equal to

$$|\mathbf{U}(n,k)| = |\mathbf{V}(n+k,k)| = inom{n+k-1}{k-1}.$$

The number of partitions of n with k non-zero parts, that is, the number of vectors

$$\mathbf{x}=(x_1,\ldots,x_k)$$
,  $x_i\in\{1,2,\ldots\}$ ,  $1\leq i\leq k$  and  $\sum_{i=0}^k x_i=n$ , is  $\binom{n-1}{k-1}$  .

The number of partitions of n with k parts, that is, the number of vectors

$$\mathbf{x}=(x_1,\ldots,x_k)$$
 ,  $x_i\in\{0,1,2,\ldots\}$  ,  $1\leq i\leq k$  and  $\sum_{i=0}^k x_i=n$  , is  $\binom{n+k-1}{k-1}$  .

Exercise: We pick a k-tuple in  $\{0,1,\ldots,n\}^k$  uniformly at random . What is the probability that the picked k-tuple is a partition of n with k parts?

**Set partitions** A partition of a set A is a set of disjoint non-empty subsets of A whose union is A. Let us restrict ourselves to the case of a finite set A, say with n elements. Any given set partition is made of a certain number, say k, of non-empty disjoint subsets  $A_i \subseteq A$ ,  $1 \le i \le k$ . So, for instance, there is only one set partition of A in one subset:  $A_1 = A$ . And there is n set partitions of A into n subsets one of which is a singleton, namely, if n is a set partition of n with two subsets one of which is

each of  $A_1=\{a_i\}, A_2=A_1^c, 1\leq i\leq n$ , is a set partition of A with two subsets one of which is a singleton. How many set partition of A have exactly n-1 subsets?

Call  $B_n$  the number of set partitions of a set A with n elements. Why does  $B_n$  depend only on n and not on what kind of elements the set A is made of? Compute  $B_1$ ,  $B_2$ ,  $B_3$ ,  $B_4$ . Find a counting proof that

$$B_{n+1} = \sum_{k=0}^n inom{n}{k} B_k.$$