1/29/2024

Recap: Divide and conquer principle

Consider

$$z^* = \min\{f(x) : x \in S\}$$

If a collection of disjoint sets $\{S_1, S_2, \ldots, S_k\}$ satisfy

$$S = S_1 \cup S_2 \cup \ldots \cup S_k$$

then $\{S_1, S_2, \ldots, S_k\}$ is called a partition of S

Let $z^i = \min\{f(x) : x \in S_i\}$, and $z^i \geq z^i_{LB}$ $\longleftarrow$ a lower bound

Observation 1:

$$z^* = \min\{z^1, z^2, z^3, \ldots, z^k\}$$

Observation 2:

$$z^* \leq \min\{z^1, z^3, z^8, \ldots\}$$ $\longleftarrow$ some z^is are missing here

Observation 3:

$$z^* \geq \min\{z^1_{LB}, z^2_{LB}, \ldots, z^k_{LB}\}.$$

(In branch and bound, we dynamically decide how to partition of S.)

Consider a generic IP: $z^* = \min\{c^T x : x \in S\}$ where $S = P \cap \mathbb{Z}^n$

Partitioning S:

$[0]$ S

$[1]$ $S = S_1 \cup S_2$

$S_1 = S_3 \cup S_4$

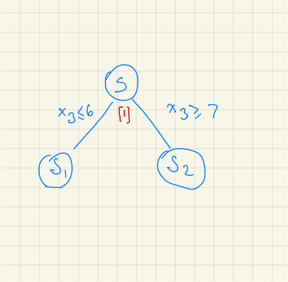
$[2]$ $S = S_3 \cup S_4 \cup S_2$

$S_2 = S_5 \cup S_6$

$[3]$ $S = S_3 \cup S_4 \cup S_5 \cup S_6$

$S_5 = S_7 \cup S_8$

$[4]$ $S = S_3 \cup S_4 \cup S_7 \cup \cancel{S_8} \cup S_6$



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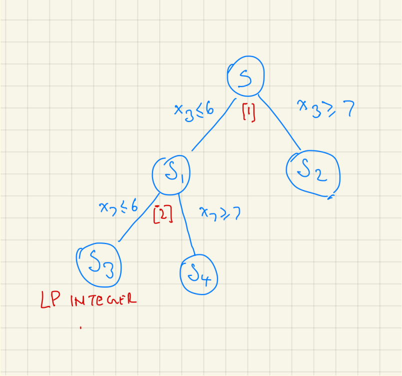

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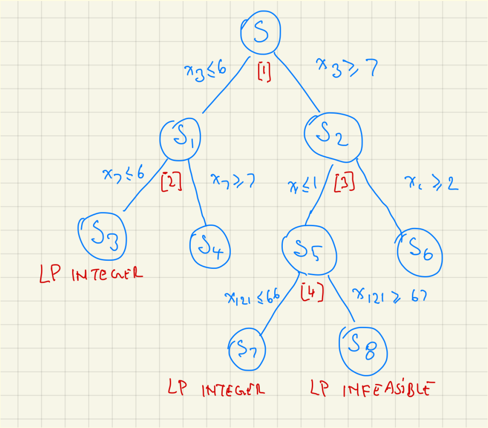

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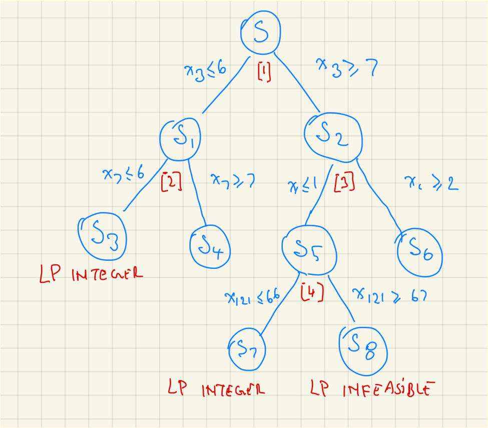

[1] $z^* = \min\{z^1, z^2\}$

[2] $z^* = \min\{z^2, z^3, z^4\} \longleftarrow z^* \leq \boldsymbol{U} = z^3$ (IP^3 solved)

[3] $z^* = \min\{z^3, z^4, z^5, z^6\}$

[4] $z^* = \min\{z^3, z^4, z^6, z^7, \cancel{z^8}\} \longleftarrow z^* \leq \boldsymbol{U} = \min\left\{z^3, z^7\right\}$

Consider a generic IP: $z^* \; = \; \min\left\{c^T x \; : \; x \in S\right\}$ where $S = P \cap \mathbb{Z}^n$

Partitioning S:

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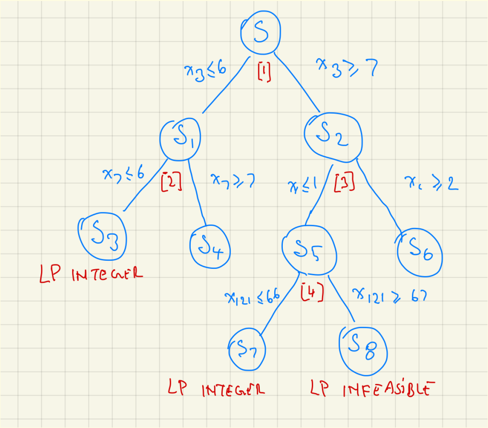

$$z^* \;\ge\; \boldsymbol{L} \;=\; \min\{\cancel{z^3_{LB}}, z^4_{LB},\; z^6_{LB},\; \cancel{z^7_{LB}},\; \cancel{z^8_{LB}}\}$$

$$z^* \;\le\; \boldsymbol{U} \;=\; \min\{z^3, z^7\} \longleftarrow$$ because we solved IP^3 and IP^7

When $\boldsymbol{L} = \boldsymbol{U}$ we have solved the IP.

When we solve the LP relaxation of a subproblem

- When we solve the LP relaxation of IP_1 (call it LP_1)
  - If LP_1 is infeasible, then IP_1 is also infeasible.
  - If the optimal solution to LP_1 is integral then IP_1 is solved to optimality and $z^{\mathsf{IP}_1} = z^{\mathsf{LP}_1}$

    [As we have a feasible integer solution at hand, z^{IP_1} is an upper bound on IP (minimization)]
  - If the LP solution is **not** integral, then we can again divide IP_1 into two new subproblems IP_3 and IP_4.

- Repeating this process, we create subproblems IP_k for $k = 1, \ldots$
- We have to further divide any subproblem IP_k unless its LP relaxation LP_k returns an integral solution or it is infeasible.
- Notice that at each step we might replace one IP/LP with two!
- This might lead to exponential blow up!

  (However, this is still the best way to solve general IPs.)

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Pruning subprobems

- After solving its LP relaxation, we do not partition a subproblem IP_k into 2 new subproblems in one of these cases:
  - If its LP relaxation is infeasible.
  - If the LP solution is integral (in this case we solved IP_k to optimality and found a new lower bound $\boldsymbol{L} = z^{\mathsf{IP}_k}$ for IP)

- Let's now assume that we have already found an integral solution earlier, giving us an upper bound $\boldsymbol{U}$.
  - What happens if $z^{\mathsf{LP}_k} \geq \boldsymbol{U}$?
  - IP_k cannot contain a better solution to the IP

$$\underbrace{z^{\mathsf{IP}_k} \geq z^{\mathsf{LP}_k}}_{\text{relaxation}} \geq \underbrace{\boldsymbol{U} \geq z^*}_{\text{upper bound}}$$

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Branch&Bound for minimization problems

(IP) $z^* = \min\left\{c^T x : x \in P \cap \mathbb{Z}^n\right\}$

- Set the list of problems to solve to $\mathcal{L} \leftarrow \{\text{IP}\}$ and set $\boldsymbol{U} = +\infty$.
- **While** $\mathcal{L} \neq \emptyset$
  - Pick a subproblem IP′ from $\mathcal{L}$ and set $\mathcal{L} = \mathcal{L} \setminus \{\text{IP}'\}$
  - Relax IP′ → solve LP′ → obtain solution $x^{\text{LP}'}$ with obj. $z^{\text{LP}'}$
  - **If** LP′ is infeasible, **break**
  - **If** $x^{\text{LP}'}$ is integral, update $\boldsymbol{U} \leftarrow \min\{\boldsymbol{U}, z^{\text{LP}'}\}$, **break**
  - **If** the optimal value $z^{\text{LP}'} \geq \boldsymbol{U}$, **break**
  - Choose a fractional x_j i.e., $x_j^{\text{LP}'} \notin \mathbb{Z}$

    Create IP″ with additional constraint: $x_j \leq \lfloor x_j^{\text{LP}'} \rfloor$

    Create IP‴ with additional constraint: $x_j \geq \lceil x_j^{\text{LP}'} \rceil$

    $\mathcal{L} \leftarrow \mathcal{L} \cup \{\text{IP}'', \text{IP}'''\}$

Branch&Bound for **maximization** problems

(IP) $z^* = \max\left\{c^T x : x \in P \cap \mathbb{Z}^n\right\}$

- Set the list of problems to solve to $\mathcal{L} \leftarrow \{\text{IP}\}$ and set $\boldsymbol{L} = -\infty$.
- **While** $\mathcal{L} \neq \emptyset$
  - Pick a subproblem IP′ from $\mathcal{L}$ and set $\mathcal{L} = \mathcal{L} \setminus \{\text{IP}'\}$
  - Relax IP′ → solve LP′ → obtain solution $x^{\text{LP}'}$ with obj. $z^{\text{LP}'}$
  - **If** LP′ is infeasible, **break**
  - **If** $x^{\text{LP}'}$ is integral, update $\boldsymbol{L} \leftarrow \max\{\boldsymbol{L}, z^{\text{LP}'}\}$, **break**
  - **If** the optimal value $z^{\text{LP}'} \leq \boldsymbol{L}$, **break**
  - Choose a fractional component $x_j^{\text{LP}'} \notin \mathbb{Z}$

    Create IP″ with additional constraint $x_j \leq \lfloor x_j^{\text{LP}'} \rfloor$

    Create IP‴ with additional constraint $x_j \geq \lceil x_j^{\text{LP}'} \rceil$

    $\mathcal{L} \leftarrow \mathcal{L} \cup \{\text{IP}'', \text{IP}'''\}$

For maximization problems

(IP) $z^* = \max\left\{c^T x : x \in P \cap \mathbb{Z}^n\right\}$

B&B summary:

- $\mathcal{L} = \{IP^2, IP^5, IP^7, IP^8, IP^9, \ldots\}$ contains the subproblems that have the potential to contain the optimal solution to the IP.
- The best LP objective value of the subproblems in $\mathcal{L}$ give an upper bound on z^*

$$z^* \ \leq \ \max\{z^{LP^2}, z^{LP^5}, z^{LP^7}, z^{LP^8}, z^{LP^9}, \ldots\} \quad \longleftarrow \ \boldsymbol{U}$$

- Whenever we encounter an integer solution we update

$$\boldsymbol{L} \leftarrow \max\{\boldsymbol{L}, z^{\mathsf{LP}'}\}$$

  (we also remember the best integral solution so far and call it the incumbent)

- We stop when $\mathcal{L} = \emptyset$. The incumbent is the optimal solution to IP with value $\boldsymbol{L}$.
- If we terminate early (i.e. $\mathcal{L} \neq \emptyset$) then we know: $\boldsymbol{U} \geq z^* \geq \boldsymbol{L}$.

One last example

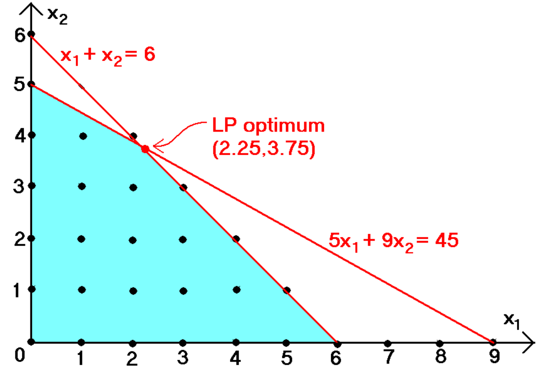
Example (maximization)

.

(IP)

$$
\begin{array}{lccccccc}
\max & 5x_1 & + & 8x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 6 \\
 & 5x_1 & + & 9x_2 & \leq & 45
\end{array}
$$

$x_1, x_2 \geq 0,$ and integer



- The optimal solution of the LP relaxation is $x^* = [2.25, 3.75]^T$,
- The optimal LP objective value is $z^{LP} = 41.25$.
- Therefore we know that optimal IP value is at most $U = 41.25$

  (Note: This actually means that $U = 41 = \lfloor 41.25 \rfloor$, why?)
- We branch on x_2 : $x_2 \leq 3$ or $x_2 \geq 4$

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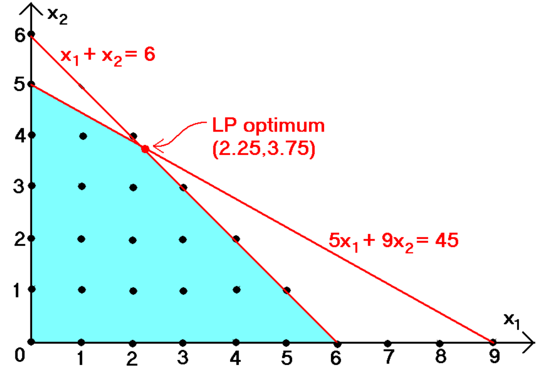

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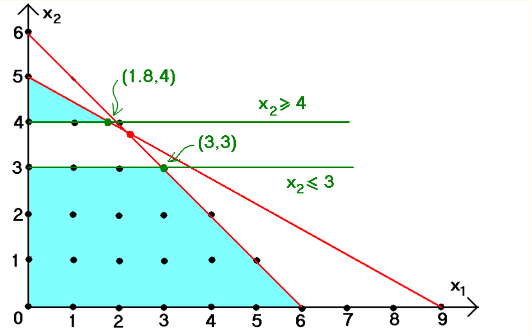
- We branch on x_2 : $\quad x_2 \leq 3$ or $x_2 \geq 4$

Create 2 subproblems



(IP1)

$$\begin{array}{lrcrcl} \max & 5x_1 & + & 8x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & 5x_1 & + & 9x_2 & \leq & 45 \\ & & & x_2 & \leq & 3 \end{array}$$

$x_1, x_2 \geq 0,$ and integer

(IP2)

$$\begin{array}{lrcrcl} \max & 5x_1 & + & 8x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & 5x_1 & + & 9x_2 & \leq & 45 \\ & & & x_2 & \geq & 4 \end{array}$$

$x_1, x_2 \geq 0,$ and integer

Solve the LP relaxation of the subproblems

$$(\mathsf{IP}_1) \quad \max \quad 5x_1 + 8x_2 \quad \text{s.t.} \quad x \in P, \quad x_2 \leq 3, \quad x_1, x_2 \text{ integer}$$

$$(\mathsf{IP}_2) \quad \max \quad 5x_1 + 8x_2 \quad \text{s.t.} \quad x \in P, \quad x_2 \geq 4, \quad x_1, x_2 \text{ integer}$$

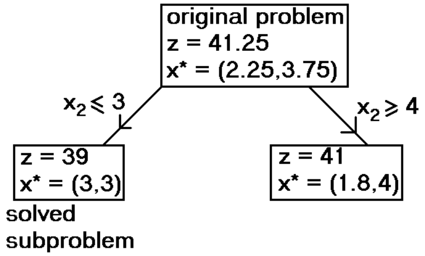
LP optimal solution:

$x^{LP1} = [3, 3]^T$, with $z^{LP1} = 39$

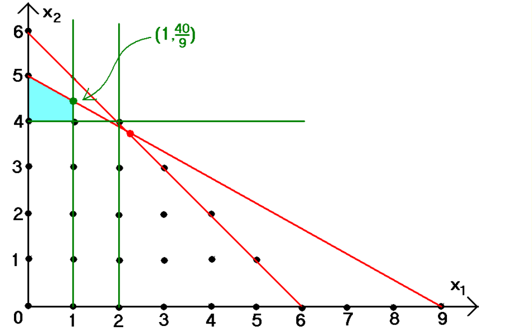
LP optimal solution:

$x^{LP2} = [1.8, 4]^T$, with $z^{LP2} = 41$

- IP_1 is solved to optimality. We have now a lower bound of $\boldsymbol{L} = 39$.
- We need to explore (divide) IP_2 further: branch on x_1.



Create 2 more subproblems



$$(\mathsf{IP}_3) \quad \max \quad 5x_1 + 8x_2$$
$$\text{s. t.} \quad x \in P,\ x_2 \geq 4 \quad x_1 \leq 1$$
$$x_1, x_2 \text{ integer}$$

LP optimal solution:

$x^{LP3} = [1, 4\frac{4}{9}]^T$ with $z^{LP1} = 40\frac{5}{9}$

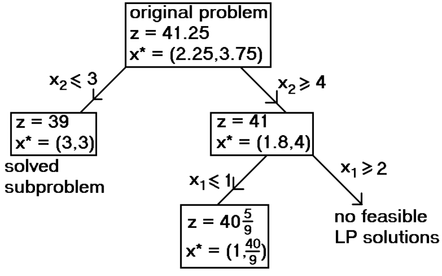
$$(\mathsf{IP}_4) \quad \max \quad 5x_1 + 8x_2$$
$$\text{s. t.} \quad x \in P,\ x_2 \geq 4 \quad x_1 \geq 2$$
$$x_1, x_2 \text{ integer}$$

LP infeasible:

Subproblem IP_4 is also infeasible.

After solving x^{LP3} and x^{LP4},
this is the current B&B tree $\rightarrow$.

- We have a lower bound of $\boldsymbol{L} = 39$ for the IP
- We have improved the IP upper bound to $\boldsymbol{U} = 40\frac{5}{9}$ $\rightarrow 40$



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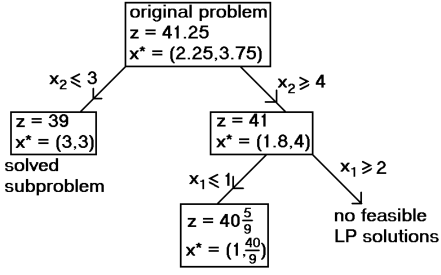


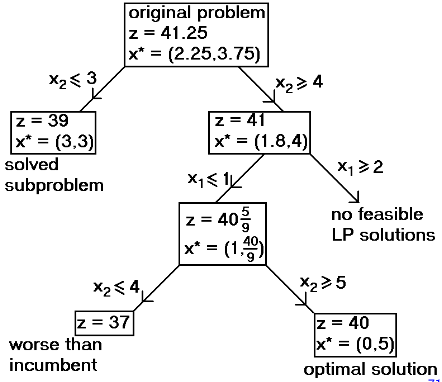
Exploring IP_3 further we create 2 new subproblems branching on x_2

- IP_5 has an additional constraint

$$x_2 \leq 4$$

- IP_6 has an additional constraint

$$x_2 \geq 5$$



Solving the knapsack problem

The 0-1 Knapsack problem

- You are going on a camping trip
- You have a knapsack that can carry a maximum weight $b > 0$.
- There are n different items that you could take.
- Each item of type i has weight $a_i > 0$.
- Each item of type i has value $c_i > 0$.
- You want to load the knapsack with items (possibly several items of the same type).
- Which items should you pack?
(Without exceeding the knapsack capacity)

The 0-1 knapsack problem

- Now assume that only one unit of each item type can be selected. In this case we use binary variables instead of general integer variables.

- The $0,1$ **knapsack set** K **is:**

$$K := \Big\{ x \in \{0,1\}^n : \sum_{i=1}^{n} a_i x_i \leq b \Big\}.$$

- The $0,1$ **knapsack problem:**

$$\max\{c^T x : x \in K\}.$$

In other words,

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_i x_i \leq b \\ & x_i \in \{0,1\} \ \text{ for all } \ i = 1, \ldots, n. \end{aligned}$$

Why don't we just enumerate possible solutions?

- In the 0-1 Knapsack Problem, what we want is to :
  - Choose a subset $S \subseteq I$ of possible items $I = \{1, \ldots, n\}$
  - Make sure they fit: $\sum_{i \in S} a_i \leq b$
  - Maximize: reward $= \sum_{i \in S} c_i$

Why don't we simply numerate all possible subsets of I, consider the ones that weigh at most b, pick the best among them.

How long will it take to do this using the fastest supercomputer:

n	Solutions to check	Time
3	8	0
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50	2^{50}	2 sec
110		

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110	2^{110}	Take a guess

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110	2^{110}	69 billion years*

*That's four times the age of the universe as we know it!

(remember, I has 2^n distinct subsets.)

Solving The Knapsack Problem

- We will solve the following knapsack problem using branch-and-bound:

$$
\begin{array}{rrrrrrl}
z^{\mathsf{IP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\
\text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\}
\end{array}
$$

- We start with solving its LP relaxation

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} a_i x_i \leq b \\
& 1 \geq x_i \geq 0 \quad \cancel{x_i \in \{0,1\}} \quad \text{for all} \quad i = 1, \ldots, n.
\end{aligned}
$$

- The LP relaxation is solved using the greedy algorithm:

$$z^{\mathsf{LP}} = \max \quad 2x_1 \quad +1x_2 \quad +4x_3 \quad +5x_4 \quad +10x_5$$
$$\text{s. t.} \quad 8x_1 \quad +2x_2 \quad +5x_3 \quad +4x_4 \quad +2x_5 \quad \leq 9, \qquad 1 \geq x_i \geq 0 \ \forall i$$

- We will next look at properties of an optimal solution of this LP.
- Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$ be the optimal solution to LP
- Can we have $\bar{x}_4 > \bar{x}_5$?
  - Notice that x_5 gives *more bang for the buck:*

$$\frac{c_5}{a_5} = \frac{10}{2} > \frac{c_4}{a_4} = \frac{5}{4}$$

  - If $\bar{x}_4 > \bar{x}_5$, using $1 \geq x_i \geq 0$ for all x_i, we know that

$$1 \geq \bar{x}_4 > \bar{x}_5 \geq 0 \implies (i)\ 1 > \bar{x}_5, \ \text{and} \qquad (ii)\ \bar{x}_4 > 0$$

  - Now consider a new solution x' obtained by by decreasing $\bar{x}_4$ by a tiny number $\delta > 0$ and increasing $\bar{x}_5$ twice as much:

$$x' = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 - \delta, \bar{x}_5 + 2\delta)$$

$$
\begin{array}{rrrrrrll}
z^{\mathsf{LP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & & \\
\text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9, & 1 \geq x_i \geq 0\ \forall i
\end{array}
$$

- We will next look at properties of an optimal solution of this LP.
- Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$ be the optimal solution to LP
- Can we have $\bar{x}_4 > \bar{x}_5$?
  - Notice that x_5 gives *more bang for the buck:*

$$\frac{c_5}{a_5} \;=\; \frac{10}{2} \;>\; \frac{c_4}{a_4} \;=\; \frac{5}{4}$$

  - If $\bar{x}_4 > \bar{x}_5$, using $1 \geq x_i \geq 0$ for all x_i, we know that

$$1 \geq \bar{x}_4 > \bar{x}_5 \geq 0 \implies (i)\ 1 > \bar{x}_5, \quad \text{and} \qquad (ii)\ \bar{x}_4 > 0$$

  - Now consider a new solution x' obtained by by decreasing $\bar{x}_4$ by a tiny number $\delta > 0$ and increasing $\bar{x}_5$ twice as much:

$$x' = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 - \delta, \bar{x}_5 + 2\delta)$$

$$
\begin{array}{rllllll}
z^{\mathsf{LP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\
\text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9, \quad 1 \geq x_i \geq 0\ \forall i
\end{array}
$$

- If $\bar{x}_4 > \bar{x}_5$, then for some small $\delta > 0$, consider

$$x' = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 - \delta, \bar{x}_5 + 2\delta)$$

- As $a_4/a_5 = 4/2 = 2$, the constraint is still satisfied:

$$\sum a_i x'_i = \sum a_i \bar{x}_i - 4\delta + 2 \cdot 2\delta = \sum a_i \bar{x}_i \leq b$$

- The new solution x' has a strictly better objective value for any $\delta > 0$:

$$\sum c_i x'_i = \sum c_i \bar{x}_i - 5\delta + 20\delta = \sum c_i \bar{x}_i + 15\delta > \sum c_i \bar{x}_i$$

- We also need to make sure that $1 \geq x' \geq 0$: How large can δ be?
  - $x'_4 = \bar{x}_4 - \delta \in [0,1] \implies \bar{x}_4 \geq \delta$
  - $x'_5 = \bar{x}_5 + 2\delta \in [0,1] \implies \bar{x}_5 + 2\delta \leq 1$

$\implies$ Pick $\delta = \min\left\{\bar{x}_4, \frac{1}{2}(1 - \bar{x}_5)\right\} > 0$ to obtain a better LP solution.

$$
\begin{array}{rllllll}
z^{\mathsf{LP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\
\text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9, \quad 1 \geq x_i \geq 0 \ \forall i
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\end{array}
$$

- We therefore established that if

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$$

  is an optimal solution to LP then we cannot have $\bar{x}_4 > \bar{x}_5$.
- Then, in any optimal solution we must have $\bar{x}_5 \geq \bar{x}_4$.

  [because $10/2 > 5/4$ meaning that x_5 gives more bang for the buck]
- How about $\bar{x}_4$ and $\bar{x}_3$?
- With the same reasoning optimal solution must have $\bar{x}_4 \geq \bar{x}_3$.

  [because $5/4 > 4/5$]
- And $\bar{x}_3 \geq \bar{x}_2$, and $\bar{x}_2 \geq \bar{x}_1$.

You will always prefer items that give *more bang for the buck* for the LP.

$$
\begin{array}{rlllllll}
z^{\mathsf{LP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & & \\
\text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9, & 1 \geq x_i \geq 0 \ \forall i
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  You will always prefer items that give *more bang for the buck* for the LP.

$$z^{\mathsf{LP}} = \max \quad 2x_1 \quad +1x_2 \quad +4x_3 \quad +5x_4 \quad +10x_5$$
$$\text{s.t.} \quad 8x_1 \quad +2x_2 \quad +5x_3 \quad +4x_4 \quad +2x_5 \quad \leq 9, \qquad 1 \geq x_i \geq 0 \ \forall i$$

- We therefore established that if

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- How about $\bar{x}_4$ and $\bar{x}_3$?

- With the same reasoning optimal solution must have $\bar{x}_4 \geq \bar{x}_3$.

  [because $5/4 > 4/5$]

- And $\bar{x}_3 \geq \bar{x}_2$, and $\bar{x}_2 \geq \bar{x}_1$.

  You will always prefer items that give *more bang for the buck* for the LP.

Solving the 0-1 knapsack problem with branch and bound

- To solve the LP relaxation of the Knapsack problem:

$$\begin{array}{rlllllll} z^{\mathsf{IP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\ \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\} \end{array}$$

- Look at the reward/weight ratio (c_i/a_i) of each item and sort the items:

$$\text{least desirable} \rightarrow \frac{2}{8} \leq \frac{1}{2} \leq \frac{4}{5} \leq \frac{5}{4} \leq \frac{10}{2} \leftarrow \text{most desirable}$$

- Therefore, in the optimal solution to LP, we have:

$$1 \geq x_5^{LP} \geq x_4^{LP} \geq x_3^{LP} \geq x_2^{LP} \geq x_1^{LP} \geq 0$$

Solving the 0-1 knapsack problem with branch and bound

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$$\begin{array}{rrrrrrrl} z^{\mathsf{IP}} = & \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\ & \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9 \\ & & x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\} \end{array}$$

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- Therefore, in the optimal solution to LP, we have:

$$1 \geq x_5^{LP} \geq x_4^{LP} \geq x_3^{LP} \geq x_2^{LP} \geq x_1^{LP} \geq 0$$

- Moreover if $x_4^{LP} = x_5^{LP}$ then they must either both 0, or 1.

  [otherwise, you can again decrease x_4 by δ and increase x_5 by 2δ to improve the objective.]

- Therefore, you will fill your knapsack (fractionally) with more profitable (meaning, larger c_i/a_i) items first.

Solving the 0-1 knapsack problem with branch and bound

- To solve the LP relaxation of the Knapsack problem:

$$\begin{array}{rrrrrrrl} z^{\mathsf{IP}} = & \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\ & \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9 \\ & & x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\} \end{array}$$

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$$1 \geq x_5^{LP} \geq x_4^{LP} \geq x_3^{LP} \geq x_2^{LP} \geq x_1^{LP} \geq 0$$

- The LP relaxation is solved using the greedy algorithm:
  - Set the remaining budget $B \leftarrow 9$
  - **For** $j = 5, 4, 3, 2, 1$ (Sorted from best to worst reward/weight ratio.)

    if $a_i \leq B$, **then** set $x_i^{LP} \leftarrow 1$ and $B \leftarrow B - a_i$

    if $a_i \leq B$, **else** set $x_i^{LP} \leftarrow B/a_i$; **stop**

Solve the LP relaxation

Solve the LP relaxation using the greedy algorithm:

$$\begin{array}{rrrrrrrl} z^{\mathsf{LP}} = & \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\ & \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9 \\ & & x_1, & x_2, & x_3, & x_4, & x_5 & \in [0,1] \end{array}$$

- Sort the items in decreasing order of reward/weight ratio:

$$5, 4, 3, 2, 1$$

- Set the current budget $B = 9$
- **for** $i = 5, 4, 3, 2, 1$ (In order of best to worst reward/weight ratio.)

$[i = 5]$ $2 = a_5 \leq B = 9$, therefore we set $x_5^{LP} = 1$ and $B = 9 - 2 = 7$

$[i = 4]$ $4 = a_4 \leq B = 7$, therefore we set $x_4^{LP} = 1$ and $B = 7 - 4 = 3$

$[i = 3]$ $5 = a_3 \not\leq B = 3$, therefore we set $x_3^{LP} = B/a_3 = 3/5$; **stop**

$$x^{\mathsf{LP}} = (0, 0, 3/5, 1, 1)$$

Solve the LP relaxation

- Solve the LP relaxation using the greedy algorithm:

$$\begin{array}{rrrrrrrl} z^{\mathsf{LP}} = & \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\ & \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 & \leq 9 \\ & & x_1, & x_2, & x_3, & x_4, & x_5 & \in [0,1] \end{array}$$

- Solution: $x^{\mathsf{LP}} = (0, 0, 3/5, 1, 1) \quad z^{\mathsf{LP}} = 17\frac{2}{5}$
- z^{LP} gives an upper bound for IP: $\boldsymbol{U} = 17\frac{2}{5}$.
- We do not have a lower bound (need an integral solution) (do we have one?)
- Split the problem into two sub-problems:
- $x_3^{\mathsf{LP}} = 3/5$ is the only fractional variable
  - IP_1: $x_3 \leq 0 \implies x_3 = 0$ (because $1 \geq x_3 \geq 0$)
  - IP_2: $x_3 \geq 1 \implies x_3 = 1$ (because $1 \geq x_3 \geq 0$)

Solve the LP relaxation

- Solve the LP relaxation using the greedy algorithm:

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Consider subproblem IP_1

Solve LP_1 with $x_3 = 0$ (we will consider IP_2 later)

$$\begin{array}{rrrrrrrl} z^{\text{LP1}} = & \max & 2x_1 & +1x_2 & +\,0 & +5x_4 & +10x_5 & \\ & \text{s.t.} & 8x_1 & +2x_2 & +\,0 & +4x_4 & +2x_5 & \leq 9 \\ & & x_1, & x_2, & & x_4, & x_5 & \in [0,1] \end{array}$$

- Solve LP_1 using the greedy algorithm.
- Solution: $x^{\text{LP1}} = (1/8, 1, 0, 1, 1)$ $z^{\text{LP1}} = 16\frac{1}{4}$
- x_1^{LP1} is fractional
- We will need to split the problem IP_1 into two sub-problems:
  - IP_3 would have an additional constraint $x_1 = 0$
  - IP_4 would have an additional constraint: $x_1 = 1$

Consider subproblem IP_1

Solve LP_1 with $x_3 = 0$ (we will consider IP_2 later)

$$\begin{array}{rlllllll} z^{\mathsf{LP1}} = & \max & 2x_1 & +1x_2 & +\,0 & +5x_4 & +10x_5 & \\ & \text{s.t.} & 8x_1 & +2x_2 & +\,0 & +4x_4 & +2x_5 & \leq 9 \\ & & x_1, & x_2, & & x_4, & x_5 & \in [0,1] \end{array}$$

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  - IP_3 would have an additional constraint $x_1 = 0$
  - IP_4 would have an additional constraint: $x_1 = 1$

Consider subproblem IP_2

Solve LP_2 problem with $x_3 = 1$

$$\begin{array}{rclcccccc}
z_2^{\text{LP}} & = & \max & 2x_1 & +1x_2 & +\,4 & +5x_4 & +10x_5 & \\
& & \text{s.t.} & 8x_1 & +2x_2 & +\,5 & +4x_4 & +2x_5 & \leq 9 \\
& & & x_1, & x_2, & & x_4, & x_5 & \in [0,1]
\end{array}$$

- Solve LP_2 (with budget $9 - 5 = 4$) using the greedy algorithm.
- Solution: $x^{\text{LP2}} = (0, 0, 1, 1/2, 1) \quad z^{\text{LP2}} = 16\frac{1}{2}$
- x_4^{LP2} is fractional
- We need to split the problem IP_2 into two sub-problems:
  - IP_5: $x_4 = 0$
  - IP_6: $x_4 = 1$

Consider subproblem IP_2

Solve LP_2 problem with $x_3 = 1$

$$
\begin{array}{rrrrrrrl}
z_2^{\mathsf{LP}} = \max & 2x_1 & +1x_2 & +\,4 & +5x_4 & +10x_5 & \\
\text{s.t.} & 8x_1 & +2x_2 & +\,5 & +4x_4 & +2x_5 & \leq 9 \\
& x_1, & x_2, & & x_4, & x_5 & \in [0,1]
\end{array}
$$

- Solve LP_2 (with budget $9 - 5 = 4$) using the greedy algorithm.
- Solution: $x^{\mathsf{LP2}} = (0, 0, 1, 1/2, 1) \quad z^{\mathsf{LP2}} = 16\frac{1}{2}$
- x_4^{LP2} is fractional
- We need to split the problem IP_2 into two sub-problems:
  - IP_5: $\quad x_4 = 0$
  - IP_6: $\quad x_4 = 1$

Knapsack B&B

$$
\text{(IP)} \qquad \begin{array}{rcrrrrrl}
z^{\mathsf{IP}} & = & \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 \\
& & \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 \quad \leq 9 \\
& & & x_1, & x_2, & x_3, & x_4, & x_5 \quad \in \{0,1\}
\end{array}
$$

LP solution: $x^{\mathsf{LP}} = (0, 0, 3/5, 1, 1), \quad z^{\mathsf{LP}} = 17\frac{2}{5}$

(IP_1) $[x_3 = 0]$ $\quad x^{\mathsf{LP1}} = (1/8, 1, 0, 1, 1) \quad z^{\mathsf{LP1}} = 16\frac{1}{4}$

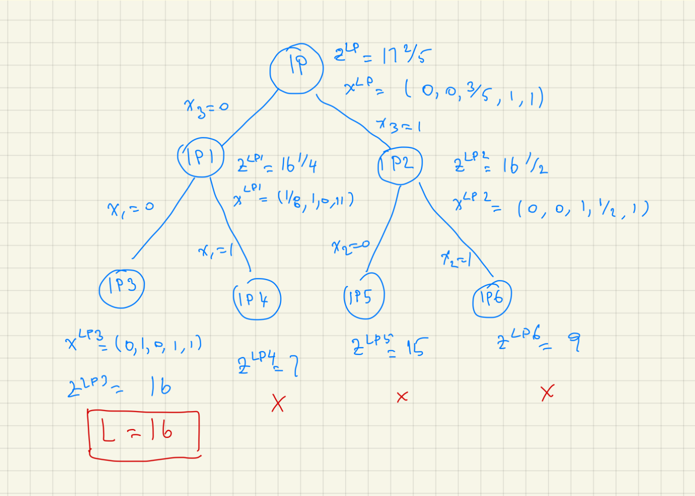
- (IP_3) $[x_3 = 0, x_1 = 0]$ $\quad x^{\mathsf{LP3}} = (0, 1, 0, 1, 1),\ z^{\mathsf{LP3}} = z^{\mathsf{IP3}} = 16 \ \leftarrow \ \boldsymbol{L}$
- (IP_4) $[x_3 = 0, x_1 = 1]$ $\quad x^{\mathsf{LP4}} = (1, 0, 0, 0, 1/2), \quad z^{\mathsf{LP4}} = 7 < \boldsymbol{L}$

(IP_2) $x_3 = 1$ $\quad x^{\mathsf{LP2}} = (0, 0, 1, 1/2, 1) \quad w^2 = 16\frac{1}{2}$

- (IP_5) $[x_3 = 1, x_4 = 0]$ $\quad x^{\mathsf{LP5}} = (0, 1, 1, 0, 1), \quad z^{\mathsf{LP5}} = z^{\mathsf{IP5}} = 15 < \boldsymbol{L}$
- (IP_6) $[x_3 = 1, x_4 = 1]$ $\quad x^{\mathsf{LP6}} = (0, 0, 1, 1, 0), \quad z^{\mathsf{LP6}} = z^{\mathsf{IP6}} = 9 < \boldsymbol{L}$

The B&B tree

$$(\mathrm{IP}) \quad \begin{array}{rrrrrrl} z^{\mathrm{IP}} = \max & 2x_1 & +1x_2 & +4x_3 & +5x_4 & +10x_5 & \\ \text{s.t.} & 8x_1 & +2x_2 & +5x_3 & +4x_4 & +2x_5 \le 9 & \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\} \end{array}$$



A small detour: What is the "knap" in knapsack?

- In English knap means
  - Crest of a hill (summit), if it is a noun
  - To break with a quick blow, if it is a verb.
- But then, why do we call a knapsack a knapsack?
- Possible answers from different languages:
  (this part is not completely factual)
  - It comes from Arabic, where it means treasure.
  - It comes from German, where it means to bite
  - It comes from Dutch, where it means laundry.

  Question: Which one is it?

A small detour: What is the "knap" in knapsack?

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  - It comes from Arabic, where it means treasure.
  - It comes from German, where it means to bite
  - It comes from Dutch, where it means laundry.

  Question: Which one is it?

  Answer: Knapsack comes from the German *knappen*, "to bite," and some experts believe that the name evolved from the fact that soldiers carried food in their knapsacks.