ORIE 3310/5310 : Optimization II

1. Linear and Integer Programming

Prof. Oktay Günlük
Cornell University, School of ORIE

Spring 2024

Linear Programming Review (22/1/2024)

Notation

- A vector $x \in \mathbb{R}^n$ is an array of real numbers:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{or} \qquad x^T = [x_1, x_2, \ldots, x_n].$$

- The inner (or, "dot") product of two vectors $x, y \in \mathbb{R}^n$ is:

$$x^T y = \sum_{i=1}^{n} x_i y_i. \qquad \leftarrow \text{this is a number}$$

- A $m \times n$ matrix A is an array of real numbers a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

We say $A \in \mathbb{R}^{m \times n}$ (number of rows first, columns second)

Notation

- Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $x \in \mathbb{R}^n$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- The matrix-vector product of A and x is:

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix}$$

- The number of columns of A must equal the number of rows of x.
- The number of rows of Ax equals the number of rows of A.

Notation

- Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $x \in \mathbb{R}^n$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- The matrix-vector product of A and x is:

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix}$$

- The number of columns of A must equal the number of rows of x.
- The number of rows of Ax equals the number of rows of A.

Notation

Given two matrices A and B, you can multiply them

$$AB$$

only if the number of columns of A is same as the number of rows of B:

$$A \in \mathbb{R}^{m \times k} \qquad \text{and} \qquad B \in \mathbb{R}^{k \times n}.$$

The resulting matrix $D = AB$ has m rows and n columns: $D \in \mathbb{R}^{m \times n}$

$$D = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1k} \\ \dots & \vdots & \vdots \\ a_{m1} & \dots & a_{mk} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & \dots & b_{1n} \\ b_{21} & \dots & b_{2n} \\ \dots & \vdots & \vdots \\ b_{k1} & \dots & b_{kn} \end{bmatrix}}_{B} = \begin{bmatrix} \cdots & d_{1j} = \sum_{i=1}^{k} a_{1i} b_{ij} & \cdots \\ \cdots & d_{2j} = \sum_{i=1}^{k} a_{2i} b_{ij} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \underbrace{d_{mj} = \sum_{i=1}^{k} a_{mi} b_{ij}}_{j\text{th column of } D} & \cdots \end{bmatrix}$$

Remember matrix multiplication is not commutative:

AB is not same as BA

A linear programming problem

$$
\begin{array}{lll}
\text{minimize} & 2x_1 - x_2 + 4x_3 & \text{objective function} \\
\text{subject to} & \left.\begin{array}{r} x_1 + x_2 + x_4 \le 2 \\ 3x_2 - x_3 = 5 \\ x_3 + x_4 \ge 3 \end{array}\right\} & \text{constraints} \\
& x_1, x_2, x_3, x_4 \ge 0 & \text{non-negativity}
\end{array}
$$

- $x^T = [x_1, x_2, x_3, x_4]$ are the decision variables.
- The objective function is linear and it can be written as $c^T x$, where

$$c^T = [2, -1, 4, 0]$$

- The constraints are linear equations and inequalities

  They have the form $a^T x = b$, or $a^T x \le b$, or $a^T x \ge b$.

  – Example: The first constraint is of the form $a^T x \le b$, with

$$a^T = [1, 1, 0, 1] \qquad \text{and} \qquad b = 2$$

A linear programming problem

$$
\begin{array}{lrl}
\text{minimize} & 2x_1 - x_2 + 4x_3 & \text{objective function} \\
\text{subject to} & & \\
& \left.\begin{array}{r} x_1 + x_2 + x_4 \leq 2 \\ 3x_2 - x_3 = 5 \\ x_3 + x_4 \geq 3 \end{array}\right\} & \text{constraints} \\
& x_1, x_2, x_3, x_4 \geq 0 & \text{non-negativity}
\end{array}
$$

- $x^T = [x_1, x_2, x_3, x_4]$ are the decision variables.
- The objective function is linear and it can be written as $c^T x$, where

$$c^T = [2, -1, 4, 0]$$

- The constraints are linear equations and inequalities

  They have the form $a^T x = b$, or $a^T x \leq b$, or $a^T x \geq b$.

  – Example: The first constraint is of the form $a^T x \leq b$, with

$$a^T = [1, 1, 0, 1] \qquad \text{and} \qquad b = 2$$

Writing LPs in " $\geq$ " form

minimize	$2x_1 - x_2 + 4x_3$	objective function
subject to	$x_1 + x_2 + x_4 \leq 2$	constraints
	$3x_2 - x_3 = 5$	
	$x_3 + x_4 \geq 3$	
	$x_1, x_2, x_3, x_4 \geq 0$	non-negativity

Notice that:

- The first constraint can also be rewritten as

$$-x_1 - x_2 - x_4 \geq -2$$

- The second constraint is equivalent to the two constraints:

$3x_2 - x_3 \geq 5$ and $3x_2 - x_3 \leq 5$ (i.e., $-3x_2 + x_3 \geq -5$).

Writing LPs in “ $\geq$ ” form

$$
\begin{array}{lll}
\text{minimize} & 2x_1 - x_2 + 4x_3 & \text{objective function} \\
\text{subject to} & \left.\begin{array}{r} x_1 + x_2 + x_4 \leq 2 \\ 3x_2 - x_3 = 5 \\ x_3 + x_4 \geq 3 \end{array}\right\} & \text{constraints} \\
 & x_1, x_2, x_3, x_4 \geq 0 & \text{non-negativity}
\end{array}
$$

Notice that:

- The first constraint can also be rewritten as

$$-x_1 - x_2 - x_4 \geq -2$$

- The second constraint is equivalent to the two constraints:

$$3x_2 - x_3 \geq 5 \quad \text{and} \quad 3x_2 - x_3 \leq 5 \quad (\text{i.e., } -3x_2 + x_3 \geq -5).$$

Writing LPs in " $\geq$ " form

$$
\begin{array}{lll}
\text{minimize} & 2x_1 - x_2 + 4x_3 & \text{objective function} \\
\text{subject to} & \left.\begin{aligned} x_1 + x_2 + x_4 &\leq 2 \\ 3x_2 - x_3 &= 5 \\ x_3 + x_4 &\geq 3 \end{aligned}\right\} & \text{constraints} \\
 & x_1, x_2, x_3, x_4 \geq 0 & \text{non-negativity}
\end{array}
$$

Notice that:

- The first constraint can also be rewritten as

$$-x_1 - x_2 - x_4 \geq -2$$

- The second constraint is equivalent to the two constraints:

$$3x_2 - x_3 \geq 5 \quad \text{and} \quad 3x_2 - x_3 \leq 5 \quad (\text{i.e., } -3x_2 + x_3 \geq -5).$$

Therefore any linear programming problem can then be written as:

$$\text{minimize} \quad c^T x \quad \text{subject to} \quad Ax \geq b, \quad x \geq 0.$$

Example LP in matrix notation

- We can write any linear programming problem in "$\geq$" form.

$$
\begin{array}{lrcl}
\min & 2x_1 - x_2 + 4x_3 & & \\
\text{s.t.} & x_1 + x_2 + x_4 & \leq & 2 \\
 & 3x_2 - x_3 & = & 5 \\
 & x_3 + x_4 & \geq & 3 \\
 & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

Example LP in matrix notation

- We can write any linear programming problem in "$\geq$" form.

$$
\begin{array}{lrcl}
\min & 2x_1 - x_2 + 4x_3 & & \\
\text{s.t.} & x_1 + x_2 + x_4 & \leq & 2 \\
 & 3x_2 - x_3 & = & 5 \\
 & x_3 + x_4 & \geq & 3 \\
 & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
\qquad
\begin{array}{lrcr}
\min & 2x_1 - x_2 + 4x_3 & & \\
\text{s.t.} & -x_1 - x_2 - x_4 & \geq & -2 \\
 & 3x_2 - x_3 & \geq & 5 \\
 & -3x_2 + x_3 & \geq & -5 \\
 & x_3 + x_4 & \geq & 3 \\
 & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

- A vector x is feasible to the first LP if and only if it is feasible to the second LP.

Example LP in matrix notation

- We can write any linear programming problem in "$\geq$" form.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b, \; x \geq 0 \end{aligned}$$

where

$$c^T = (2, \; -1, \; 4, \; 0),$$

$$A = \begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and, $b^T = (-2, \; 5, \; -5, \; 3)$.

$$\begin{array}{lrcr} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & -x_1 - x_2 - x_4 & \geq & -2 \\ & 3x_2 - x_3 & \geq & 5 \\ & -3x_2 + x_3 & \geq & -5 \\ & x_3 + x_4 & \geq & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Example LP in matrix notation

- We can write any linear programming problem in "$\geq$" form.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b,\ x \geq 0 \end{array}$$

where

$$c^T = (2,\ -1,\ 4,\ 0),$$

$$A = \begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and, $b^T = (-2,\ 5,\ -5,\ 3)$.

$$\begin{array}{lrcr} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & -x_1 - x_2 - x_4 & \geq & -2 \\ & 3x_2 - x_3 & \geq & 5 \\ & -3x_2 + x_3 & \geq & -5 \\ & x_3 + x_4 & \geq & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Data for this LP:

- n variables, $c \in \mathbb{R}^n$
- m constraints, $b \in \mathbb{R}^m$
- $A \in \mathbb{R}^{m \times n}$

Example LP in matrix notation cont.

- Sometimes it is more natural to write LPs in " $\leq$ " form.

$$\begin{array}{lrcr} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & -x_1 - x_2 - x_4 & \geq & -2 \\ & 3x_2 - x_3 & \geq & 5 \\ & -3x_2 + x_3 & \geq & -5 \\ & x_3 + x_4 & \geq & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array} \qquad \begin{array}{lrcr} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_4 & \leq & 2 \\ & -3x_2 + x_3 & \leq & -5 \\ & 3x_2 - x_3 & \leq & 5 \\ & -x_3 - x_4 & \leq & -3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \min & c^T x \\ \text{s.t.} & -Ax \leq -b \\ & x \geq 0 \end{array}$$

Example LP in matrix notation cont.

- Sometimes it is more natural to write LPs in "$\leq$" form.

$$
\begin{array}{lrcr}
\min & 2x_1 - x_2 + 4x_3 & & \\
\text{s. t.} & -x_1 - x_2 - x_4 & \geq & -2 \\
& 3x_2 - x_3 & \geq & 5 \\
& -3x_2 + x_3 & \geq & -5 \\
& x_3 + x_4 & \geq & 3 \\
& x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
\qquad
\begin{array}{lrcr}
\min & 2x_1 - x_2 + 4x_3 & & \\
\text{s. t.} & x_1 + x_2 + x_4 & \leq & 2 \\
& -3x_2 + x_3 & \leq & -5 \\
& 3x_2 - x_3 & \leq & 5 \\
& -x_3 - x_4 & \leq & -3 \\
& x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

$$
\begin{array}{ll}
\min & c^T x \\
\text{s. t.} & Ax \geq b \\
& x \geq 0
\end{array}
\quad \longrightarrow \quad
\begin{array}{ll}
\min & c^T x \\
\text{s. t.} & -Ax \leq -b \\
& x \geq 0
\end{array}
$$

Example LP in matrix notation cont. again

- We can also write LPs in (standard) " $=$ " form.

$$\begin{array}{lrcl} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_4 & \leq & 2 \\ & 3x_2 - x_3 & = & 5 \\ & x_3 + x_4 & \geq & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_4 + s_1 & = & 2 \\ & 3x_2 - x_3 & = & 5 \\ & x_3 + x_4 - s_3 & = & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \\ & s_1, s_3 & \geq & 0 \end{array}$$

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax + Is = b \\ & x \geq 0, \quad s \geq 0 \end{array}$$

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax - Is = b \\ & x \geq 0, \quad s \geq 0 \end{array}$$

Example LP in matrix notation cont. again

- We can also write LPs in (standard) "$=$" form.

$$\begin{array}{lrcl} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_4 & \leq & 2 \\ & 3x_2 - x_3 & = & 5 \\ & x_3 + x_4 & \geq & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \min & 2x_1 - x_2 + 4x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_4 + s_1 & = & 2 \\ & 3x_2 - x_3 & = & 5 \\ & x_3 + x_4 - s_3 & = & 3 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \\ & s_1, s_3 & \geq & 0 \end{array}$$

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \quad \longrightarrow \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax + Is = b \\ & \quad x \geq 0, \quad s \geq 0 \end{array}$$

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \quad \longrightarrow \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax - Is = b \\ & \quad x \geq 0, \quad s \geq 0 \end{array}$$

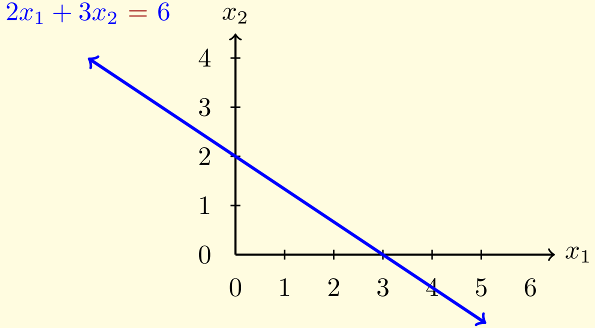
Halfspaces

- A hyperplane in $\mathbb{R}^n$ is the set points $x \in \mathbb{R}^n$ that satisfy the equation

$$a^T x \;=\; b \iff a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

This set of points divide the space $\mathbb{R}^n$ into two halfspaces.

Example: Consider the line $2x_1 + 3x_2 = 6$ in $\mathbb{R}^2$



Halfspaces

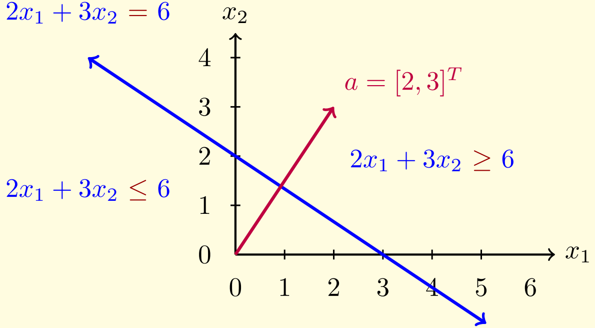
- All points in one of the halfspaces satisfy the inequality

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \;\geq\; b$$

- Points in the other satisfy

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \;\leq\; b.$$

- The vector $a = (a_1, a_2, \ldots, a_n)^T$ points in the direction where the points satisfying $a_1x_1 + a_2x_2 + \ldots + a_nx_n \;\geq\; b$ are.

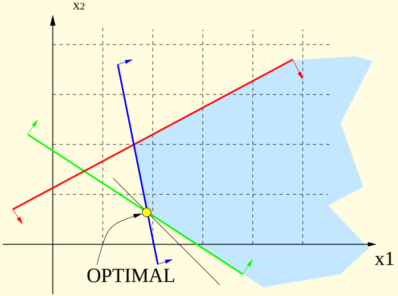


Geometrically an LP looks like

Intersecting all constraints (halfspaces) gives the feasible set P (polyhedron)

Solving the LP $\equiv$ find a point $x \in P$ with minimum $c^T x$.

$$\begin{aligned} \min \quad & 3x_1 + 3x_2 \\ & 2x_1 + 3x_2 \geq 6 \\ & -x_1 + 2x_2 \leq 2 \\ & 5x_1 + 1x_2 \geq 10 \end{aligned}$$



If there is an opt. solution, then there is an extreme point opt. solution

$$\underbrace{2x_1 + 3x_2 = 6,\ 5x_1 + x_2 = 10}_{\text{both constraints satisfied at equality (tight)}} \implies x = \left(\frac{24}{13}, \frac{10}{13}\right)$$

Optimal solutions of LPs

- Given a generic LP with n variables:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- There are four possible (mutually exclusive) outcomes:

(a) The feasible set P might be empty (problem is infeasible)

$$P = \left\{ x \in \mathbb{R}^n \, : \, Ax \geq b, \; x \geq 0 \right\} = \emptyset$$

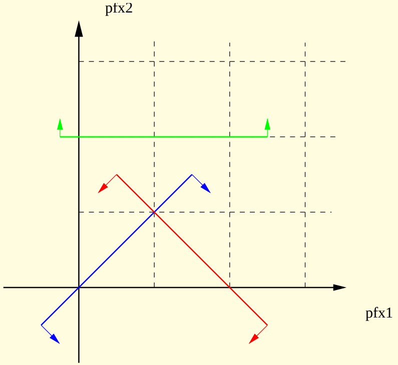
(b) There might be a unique optimal solution (one and only one).

(c) There might be multiple optimal solutions.
(all with the same objective value)

(d) The objective value might be unbounded ($-\infty$): No feasible solution is optimal.

Example: Infeasible problem

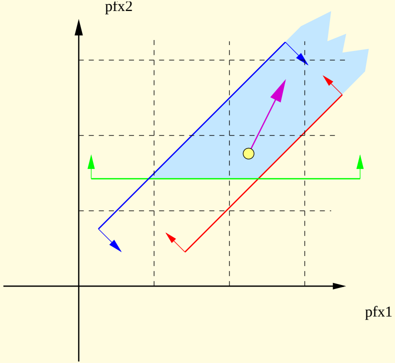
$$\begin{aligned} \min \quad & x_1 + x_2 \\ & x_1 - x_2 \geq 0 \\ & x_1 + x_2 \leq 2 \\ & x_2 \geq 2 \end{aligned}$$



$$P = \{x \in \mathbb{R}^2 \ : \ x_1 - x_2 \geq 0, \ x_1 + x_2 \leq 2, \ x_2 \geq 2\} = \emptyset$$

Example: Unbounded problem

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & 2x_1 - 2x_2 \geq -1 \\ & x_1 - x_2 \leq 1 \\ & x_2 \geq 3/2 \end{aligned}$$



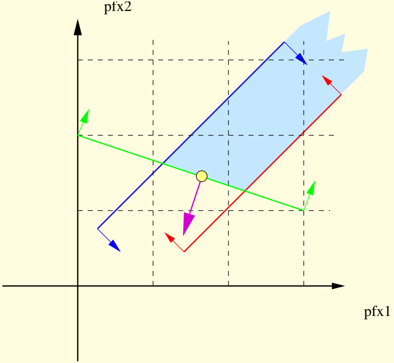
1. $P = \{x \in \mathbb{R}^2 \,:\, 2x_1 - 2x_2 \geq -1,\ x_1 - x_2 \leq 1,\ x_2 \geq 3/2\} \neq \emptyset$
2. For any $x \in P$, there exists a $x' \in P$ such that $c^T x' \geq c^T x + 1$

Note: If an LP is unbounded, its feasible region must be unbounded.

. But, an LP can be bounded while its feasible region is unbounded.

Example: Multiple optima

$$
\begin{aligned}
\min \quad & x_1 + 3x_2 \\
& 2x_1 - 2x_2 \geq -1 \\
& x_1 - x_2 \leq 1 \\
& x_1 + 3x_2 \geq 6
\end{aligned}
$$



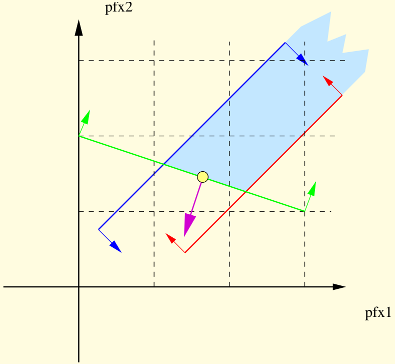
$$P = \{x \in \mathbb{R}^2 \ : \ 2x_1 - 2x_2 \geq -1, \ x_1 - x_2 \leq \ 1, \ x_1 + 3x_2 = \ 6\} \neq \emptyset$$

In conclusion an LP is either

- Feasible or infeasible.
- If feasible, the optimal value is either bounded or unbounded.
- If feasible and bounded, optimial solution may not be unique.

Example: Multiple optima

$$\begin{aligned} \min \quad & x_1 + 3x_2 \\ & 2x_1 - 2x_2 \geq -1 \\ & x_1 - x_2 \leq 1 \\ & x_1 + 3x_2 \geq 6 \end{aligned}$$



$$P = \{x \in \mathbb{R}^2 \; : \; 2x_1 - 2x_2 \geq -1, \; x_1 - x_2 \leq 1, \; x_1 + 3x_2 = 6\} \neq \emptyset$$

In conclusion an LP is either

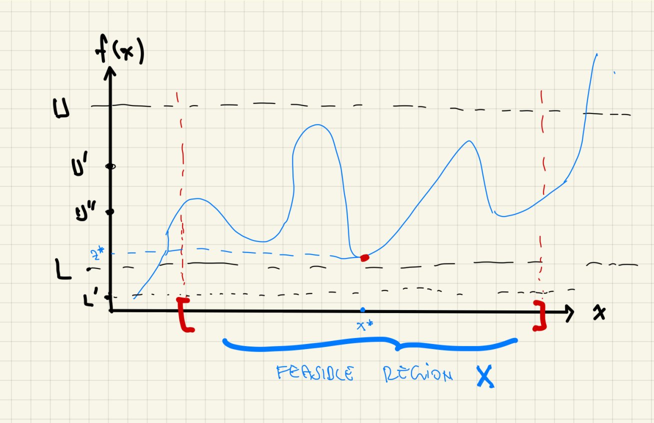
- Feasible or infeasible.
- If feasible, the optimal value is either bounded or unbounded.
- If feasible and bounded, optimial solution may not be unique.

Upper/lower bounds and LP duality

Upper and lower bounds for minimization problems

- Consider an optimization problem:

$$\text{the optimal value} \longrightarrow \quad z^* = \min_{x \in X} \; f(x) \qquad \text{where} \quad X \subseteq \mathbb{R}^n$$



- U is an upper bound for z^*, (so are U' and U'')
- L is a lower bound for z^* (L' is a lower bound as well)

Upper bounds for minimization problems

- Given an optimization problem:

$$z^* = \min_{x \in X} \ f(x) \qquad \text{where} \quad X \subseteq \mathbb{R}^n$$

- An upper bound $U \in \mathbb{R}$ is a number that can be certified to be greater than or equal to z^*

In other words: $$\min_{x \in X} \ f(x) \leq U$$

- This does not mean

$$f(x) \leq U \qquad \text{for all } x \in X$$

It is perfectly OK if there are solutions $x' \in X$ such that

$$f(x') > U$$

- We only need the condition to hold for the optimal solution $x^* \in X$

$$f(x^*) \leq U$$

- We can obtain upper bounds without solving the problem to optimality.
  - For example, **any** feasible point $\bar{x} \in X$ gives an upper bound

$$U = f(\bar{x})$$

Upper bounds for minimization problems

- Given an optimization problem:

$$z^* = \min_{x \in X} \; f(x) \qquad \text{where} \quad X \subseteq \mathbb{R}^n$$

- An upper bound $U \in \mathbb{R}$ is a number that can be certified to be greater than or equal to z^*

In other words: $$\min_{x \in X} \; f(x) \leq U$$

- This does not mean

$$f(x) \leq U \qquad \text{for all } x \in X$$

It is perfectly OK if there are solutions $x' \in X$ such that

$$f(x') > U$$

- We only need the condition to hold for the optimal solution $x^* \in X$

$$f(x^*) \leq U$$

- We can obtain upper bounds without solving the problem to optimality.
  - For example, **any** feasible point $\bar{x} \in X$ gives an upper bound

$$U = f(\bar{x})$$

Upper bounds for minimization problems

- Given an optimization problem:

$$z^* = \min_{x \in X} \; f(x) \qquad \text{where} \quad X \subseteq \mathbb{R}^n$$

- An upper bound $U \in \mathbb{R}$ is a number that can be certified to be greater than or equal to z^*

$$\text{In other words:} \qquad \min_{x \in X} \; f(x) \leq U$$

- This does not mean

$$f(x) \leq U \qquad\qquad \text{for all } x \in X$$

It is perfectly OK if there are solutions $x' \in X$ such that

$$f(x') > U$$

- We only need the condition to hold for the optimal solution $x^* \in X$

$$f(x^*) \leq U$$

- We can obtain upper bounds without solving the problem to optimality.
  - For example, **any** feasible point $\bar{x} \in X$ gives an upper bound

$$U = f(\bar{x})$$

Upper bounds for minimization problems: LP example

- Consider the LP with 4 variables:

$$
\begin{array}{rrrcl}
z^* & = & \min & 2x_1 - x_2 + 4x_3 & \\
 & & \text{s.t.} & x_1 + x_2 + x_4 & \leq & 6 \\
 & & & 3x_2 - x_3 & = & 5 \\
 & & & x_3 + x_4 & \geq & 2 \\
 & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

- And consider the point (or, solution): $\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \begin{array}{l} \leftarrow \bar{x}_1 \\ \leftarrow \bar{x}_2 \\ \leftarrow \bar{x}_3 \\ \leftarrow \bar{x}_4 \end{array}$

- The point $\bar{x}$ is feasible as it satisfies all constraints.
- It has an objective function value of : $2(0) - 1(2) + 4(1) + 0(2) = 2$
- Since we are minimizing, we conclude that z^* cannot be larger than 2.
- Therefore, $U = 2$ is **an** upper bound for this LP. $\longleftarrow \quad z^* \leq 2$

A slight detour

The LP:

$$
\begin{array}{rrrcl}
z^* = & \min & 2x_1 - x_2 + 4x_3 & & \\
& \text{s. t.} & x_1 + x_2 + x_4 & \leq & 6 \\
& & 3x_2 - x_3 & = & 5 \\
& & x_3 + x_4 & \geq & 2 \\
& & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

A feasible solution:

$$
\bar{x} = \begin{array}{cl} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} & \begin{array}{l} \leftarrow \bar{x}_1 \\ \leftarrow \bar{x}_2 \\ \leftarrow \bar{x}_3 \\ \leftarrow \bar{x}_4 \end{array} \end{array}
$$

with

$c^T\bar{x} = 2 \longleftarrow$ an UB on z^*

Question: Is this solution optimal? $\rightarrow$ Not sure: maybe, maybe not.

Question: Is this an extreme point solution? $\rightarrow$ NO!

- The feasible set lives in $\mathbb{R}^4$.
- An ext. pt. must satisfy 4 linearly indep. constraints as equality
- $\bar{x}$ only satisfies two: $3x_2 - x_3 = 5$ and $x_1 \geq 0$ constraint

Here is an ex. pt: $\hat{x}^T = [0, 5/3, 0, 2]$ with $c^T\hat{x} = -5/3 \leftarrow$ a better UB

A slight detour

The LP:

$$
\begin{array}{rlrcl}
z^* = & \min & 2x_1 - x_2 + 4x_3 & & \\
& \text{s.t.} & x_1 + x_2 + x_4 & \leq & 6 \\
& & 3x_2 - x_3 & = & 5 \\
& & x_3 + x_4 & \geq & 2 \\
& & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

A feasible solution:

$$
\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \begin{array}{l} \leftarrow \bar{x}_1 \\ \leftarrow \bar{x}_2 \\ \leftarrow \bar{x}_3 \\ \leftarrow \bar{x}_4 \end{array}
$$

with

$c^T \bar{x} = 2 \longleftarrow$ an UB on z^*

Question: Is this solution optimal? $\rightarrow$ Not sure: maybe, maybe not.

Question: Is this an extreme point solution? $\rightarrow$ NO!

- The feasible set lives in $\mathbb{R}^4$.
- An ext. pt. must satisfy 4 linearly indep. constraints as equality
- $\bar{x}$ only satisfies two: $3x_2 - x_3 = 5$ and $x_1 \geq 0$ constraint

Here is an ex. pt: $\hat{x}^T = [0, 5/3, 0, 2]$ with $c^T \hat{x} = -5/3 \leftarrow$ a better UB

A slight detour

The LP:

$$
\begin{aligned}
z^* = \min \quad & 2x_1 - x_2 + 4x_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_4 \le 6 \\
& 3x_2 - x_3 = 5 \\
& x_3 + x_4 \ge 2 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

A feasible solution:

$$
\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow \bar{x}_1 \\ \leftarrow \bar{x}_2 \\ \leftarrow \bar{x}_3 \\ \leftarrow \bar{x}_4 \end{matrix}
$$

with

$c^T \bar{x} = 2 \longleftarrow$ an UB on z^*

Question: Is this solution optimal? $\rightarrow$ Not sure: maybe, maybe not.

Question: Is this an extreme point solution? $\rightarrow$ NO!

- The feasible set lives in $\mathbb{R}^4$.
- An ext. pt. must satisfy 4 linearly indep. constraints as equality
- $\bar{x}$ only satisfies two: $3x_2 - x_3 = 5$ and $x_1 \ge 0$ constraint

Here is an ex. pt: $\hat{x}^T = [0, 5/3, 0, 2]$ with $c^T \hat{x} = -5/3 \leftarrow$ a better UB

A slight detour

The LP:

$$
\begin{aligned}
z^* = \min \quad & 2x_1 - x_2 + 4x_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_4 \leq 6 \\
& 3x_2 - x_3 = 5 \\
& x_3 + x_4 \geq 2 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

A feasible solution:

$$
\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \quad \begin{matrix} \leftarrow \bar{x}_1 \\ \leftarrow \bar{x}_2 \\ \leftarrow \bar{x}_3 \\ \leftarrow \bar{x}_4 \end{matrix}
$$

with

$c^T\bar{x} = 2 \longleftarrow$ an UB on z^*

Question: Is this solution optimal? $\rightarrow$ Not sure: maybe, maybe not.

Question: Is this an extreme point solution? $\rightarrow$ NO!

- The feasible set lives in $\mathbb{R}^4$.
- An ext. pt. must satisfy 4 linearly indep. constraints as equality
- $\bar{x}$ only satisfies two: $3x_2 - x_3 = 5$ and $x_1 \geq 0$ constraint

Here is an ex. pt: $\hat{x}^T = [0, 5/3, 0, 2]$ with $c^T\hat{x} = -5/3 \leftarrow$ a better UB

A slight detour

The LP:

$$\begin{array}{rrrrl} z^* & = & \min & 2x_1 - x_2 + 4x_3 & \\ & & \text{s.t.} & x_1 + x_2 + x_4 & \leq 6 \\ & & & 3x_2 - x_3 & = 5 \\ & & & x_3 + x_4 & \geq 2 \\ & & & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

A feasible solution:

$$\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \begin{array}{l} \leftarrow \bar{x}_1 \\ \leftarrow \bar{x}_2 \\ \leftarrow \bar{x}_3 \\ \leftarrow \bar{x}_4 \end{array}$$

with

$c^T\bar{x} = 2 \longleftarrow$ an UB on z^*

Question: Is this solution optimal? $\rightarrow$ Not sure: maybe, maybe not.

Question: Is this an extreme point solution? $\rightarrow$ NO!

- The feasible set lives in $\mathbb{R}^4$.
- An ext. pt. must satisfy 4 linearly indep. constraints as equality
- $\bar{x}$ only satisfies two: $3x_2 - x_3 = 5$ and $x_1 \geq 0$ constraint

Here is an ex. pt: $\hat{x}^T = [0, 5/3, 0, 2]$ with $c^T\hat{x} = -5/3 \leftarrow$ a better UB

Lower bounds for minimization problems

- Again consider the optimization problem:

$$z^* = \min_{x \in X} \; f(x) \qquad \text{where} \quad X \subseteq \mathbb{R}^n$$

- A lower bound $L \in \mathbb{R}$ is a number that can be certified to be less than or equal to z^*.

In other words: $$\min_{x \in X} \; f(x) \geq L$$

- We have to make sure that the best solution has to have an objective value larger than or equal to L.
- Which means that

$$f(x) \geq L \quad \text{for all} \;\; x \in X$$

- Finding lower bounds is usually complicated.
  - For LP problems, remember the weak duality theorem:
    (Feasible solutions to the dual give lower bounds for the primal LP.)

Lower bounds for minimization problems: LP Example

- Consider the following simple (feasible) LP with 2 variables:

$$\begin{aligned} z^* \;=\; \min \quad & 2x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \geq 2. \end{aligned}$$

- We would like to find a lower bound L for z^*.
- L must be less than the objective value of **all** feasible solutions.
- Consider an arbitrary feasible point $\bar{x} = [\bar{x}_1, \bar{x}_2]$
- As $\bar{x}$ is feasible, we know that $\bar{x}_1 + 3\bar{x}_2 \geq 2$ and therefore

$$2\bar{x}_1 + 6\bar{x}_2 \geq 4$$

- As this holds for **all** feasible points, we have:

$$z^* \geq 4 \quad \leftarrow \quad L = 4 \text{ is a lower bound on } z^*$$

Lower bounds for minimization problems: LP Example

- Consider the following simple (feasible) LP with 2 variables:

$$\begin{aligned} z^* = \min \quad & 2x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \geq 2. \end{aligned}$$

- We would like to find a lower bound L for z^*.
- L must be less than the objective value of **all** feasible solutions.
- Consider an arbitrary feasible point $\bar{x} = [\bar{x}_1, \bar{x}_2]$
- As $\bar{x}$ is feasible, we know that $\bar{x}_1 + 3\bar{x}_2 \geq 2$ and therefore

$$2\bar{x}_1 + 6\bar{x}_2 \geq 4$$

- As this holds for **all** feasible points, we have:

$$z^* \geq 4 \quad \leftarrow \quad L = 4 \text{ is a lower bound on } z^*$$

Lower bounds for minimization: Another LP example

- Let's try a different example:

$$\begin{aligned} z^* \ = \ \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 4 \\ & \qquad\;\; x_2 \geq 1. \end{aligned}$$

- How can we express the objective function in terms of the constraints?
- We can do it by summing the constraints as follows:

$$\begin{array}{r} x_1 + x_2 \geq 4 \\ \underline{+ \ \ 2 \ \cdot \ (x_2 \geq 1)} \\ = \ x_1 + 3x_2 \geq 6 \end{array}$$

- As this holds for **all** feasible points, we have:

$$z^* \geq 6 \qquad \leftarrow \quad L = 6 \text{ is a lower bound on } z^*$$

Lower bounds for minimization: Another LP example

- Let's try a different example:

$$\begin{aligned} z^* \;=\; \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 4 \\ & \qquad\;\; x_2 \geq 1. \end{aligned}$$

- How can we express the objective function in terms of the constraints?
- We can do it by summing the constraints as follows:

$$\begin{array}{r} x_1 + x_2 \geq 4 \\ \underline{+\;\; 2 \;\cdot\; (x_2 \geq 1)} \\ = \; x_1 + 3x_2 \geq 6 \end{array}$$

- As this holds for **all** feasible points, we have:

$$z^* \geq 6 \qquad \leftarrow \quad L = 6 \text{ is a lower bound on } z^*$$

Lower bounds for minimization: One last LP example

- Here is a more interesting example:

$$
\begin{aligned}
z^* = \min \quad & x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& x_2 \geq 1 \\
& x_1 - x_2 \geq 3.
\end{aligned}
$$

- How can we express the objective function in terms of the constraints?
- For any choice of $p_1, p_2, p_3 \geq 0$ we can write

$$p_1 \cdot (x_1 + x_2 \geq 2) + p_2 \cdot (x_2 \geq 1) + p_3 \cdot (x_1 - x_2 \geq 3)$$

$$\Rightarrow \quad \underbrace{(p_1 + p_3)}_{\text{if } =1} x_1 + \underbrace{(p_1 + p_2 - p_3)}_{\text{and if } =3} x_2 \;\geq\; \underbrace{2p_1 + p_2 + 3p_3}_{\text{then you obtain } L}$$

- The lower bound L we obtain depends on the choice of p:
  - $p_1 = 1$, $p_2 = 2$, $p_3 = 0$ gives us the lower bound $L = 4$.
  - $p_1 = 0$, $p_2 = 4$, $p_3 = 1$ gives us the lower bound $L = 7$.

Lower bounds for minimization: One last LP example

- Here is a more interesting example:

$$\begin{aligned} z^* \; = \; \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & \phantom{x_1 + {}} x_2 \geq 1 \\ & x_1 - x_2 \geq 3. \end{aligned}$$

- How can we express the objective function in terms of the constraints?

- For any choice of $p_1, p_2, p_3 \geq 0$ we can write

$$p_1 \cdot (x_1 + x_2 \geq 2) + p_2 \cdot (x_2 \geq 1) + p_3 \cdot (x_1 - x_2 \geq 3)$$

$$\Rightarrow \quad \underbrace{(p_1 + p_3)}_{\text{if } \ =1} x_1 + \underbrace{(p_1 + p_2 - p_3)}_{\text{and if } \ =3} x_2 \;\geq\; \underbrace{2p_1 + p_2 + 3p_3}_{\text{then you obtain } \ L}$$

- The lower bound L we obtain depends on the choice of p:
  - $p_1 = 1$, $p_2 = 2$, $p_3 = 0$ gives us the lower bound $L = 4$.
  - $p_1 = 0$, $p_2 = 4$, $p_3 = 1$ gives us the lower bound $L = 7$.

LP Duality

- A natural question is: how can we obtain the best lower bound on z^*?

$$\begin{aligned} z^* \;=\; \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3. \end{aligned}$$

- Maximize $2p_1 + p_2 + 3p_3$ over the constraints that p should satisfy:

$$\begin{aligned} w^* \;=\; \max \quad & 2p_1 + p_2 + 3p_3 \\ \text{s.t.} \quad & p_1 + p_3 = 1 \\ & p_1 + p_2 - p_3 = 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

- This is the dual LP and w^* gives the best lower bound on z^*

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p = c \\ & p \geq 0. \end{aligned}$$

LP Duality

- A natural question is: how can we obtain the best lower bound on z^*?

$$\begin{aligned} z^* = \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & \phantom{x_1 +{}} x_2 \geq 1 \\ & x_1 - x_2 \geq 3. \end{aligned}$$

- Maximize $2p_1 + p_2 + 3p_3$ over the constraints that p should satisfy:

$$\begin{aligned} w^* = \max \quad & 2p_1 + p_2 + 3p_3 \\ \text{s.t.} \quad & p_1 + p_3 = 1 \\ & p_1 + p_2 - p_3 = 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

- This is the dual LP and w^* gives the best lower bound on z^*

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p = c \\ & p \geq 0. \end{aligned}$$

Non-negativity constraints

- Example:

$$\begin{aligned} z^* = \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

- For any choice of $p_1, p_2, p_3, p_4, p_5 \geq 0$ we can write

$$p_1 \cdot (x_1 + x_2 \geq 2) + p_2 \cdot (x_2 \geq 1) + p_3 \cdot (x_1 - x_2 \geq 3) + p_4 \cdot (x_1 \geq 0) + p_5 \cdot (x_2 \geq 0)$$

$$\Rightarrow \quad \underbrace{(p_1 + p_3 + p_4)}_{\text{if } \ =1} x_1 + \underbrace{(p_1 + p_2 - p_3 + p_5)}_{\text{and if } \ =3} x_2 \geq \underbrace{2p_1 + p_2 + 3p_3}_{\text{then you obtain } \ L}$$

- The lower bound L we obtain depends on the choice of p:
- Find the best p using linear programming

- To find the best lower bound on z^*:

$$\begin{aligned} z^* = \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

- Maximize $2p_1 + p_2 + 3p_3$ over the constraints that p should satisfy:

$$\begin{aligned} \max \quad & 2p_1 + p_2 + 3p_3 \\ \text{s.t.} \quad & p_1 + p_3 + p_4 = 1 \\ & p_1 + p_2 - p_3 + p_5 = 3 \\ & p_1, p_2, p_3, p_4, p_5 \geq 0. \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & 2p_1 + p_2 + 3p_3 \\ \text{s.t.} \quad & p_1 + p_3 \leq 1 \\ & p_1 + p_2 - p_3 \leq 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

Primal

$\min c^T x \quad \text{s.t. } Ax \geq b, \ x \geq 0$

Dual

$\max b^T p \quad \text{s.t. } A^T p \leq c, \ p \geq 0.$

- To find the best lower bound on z^*:

$$\begin{aligned} z^* = \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

- Maximize $2p_1 + p_2 + 3p_3$ over the constraints that p should satisfy:

$$\begin{aligned} \max \quad & 2p_1 + p_2 + 3p_3 \\ \text{s.t.} \quad & p_1 + p_3 + p_4 = 1 \\ & p_1 + p_2 - p_3 + p_5 = 3 \\ & p_1, p_2, p_3, p_4, p_5 \geq 0. \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & 2p_1 + p_2 + 3p_3 \\ \text{s.t.} \quad & p_1 + p_3 \leq 1 \\ & p_1 + p_2 - p_3 \leq 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

Primal	Dual
$\min c^T x \quad \text{s.t. } Ax \geq b,\ x \geq 0$	$\max b^T p \quad \text{s.t. } A^T p \leq c,\ p \geq 0.$

Lower bounds via duality

- For any LP, there is a corresponding dual LP.

Primal LP

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 3x_2 \\ \text{subject to} \quad & 1x_1 + 1x_2 \geq 2 \\ & 0x_1 + 1x_2 \geq 1 \\ & 1x_1 - 1x_2 \geq 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Its Dual LP

$$\begin{aligned} \text{maximize} \quad & 2p_1 + 1p_2 + 3p_3 \\ \text{subject to} \quad & 1p_1 + 0p_2 + 1p_3 \leq 4 \\ & 1p_1 + 1p_2 - 1p_3 \leq 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

- More generally:

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p \leq c \\ & p \geq 0. \end{aligned}$$

Lower bounds via duality

- For any LP, there is a corresponding dual LP.

Primal LP

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 3x_2 \\ \text{subject to} \quad & 1x_1 + 1x_2 \geq 2 \\ & 0x_1 + 1x_2 \geq 1 \\ & 1x_1 - 1x_2 \geq 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Its Dual LP

$$\begin{aligned} \text{maximize} \quad & 2p_1 + 1p_2 + 3p_3 \\ \text{subject to} \quad & 1p_1 + 0p_2 + 1p_3 \leq 4 \\ & 1p_1 + 1p_2 - 1p_3 \leq 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

- More generally:

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p \leq c \\ & p \geq 0. \end{aligned}$$

Weak Duality Thm: If $\bar{x}$ is primal feasible and $\bar{p}$ is dual feasible:

(min. objective) $\quad c^T \bar{x} \geq b^T \bar{p} \quad$ (max. objective)

Lower bounds via duality

- For any LP, there is a corresponding dual LP.

Primal LP

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 3x_2 \\ \text{subject to} \quad & 1x_1 + 1x_2 \geq 2 \\ & 0x_1 + 1x_2 \geq 1 \\ & 1x_1 - 1x_2 \geq 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Its Dual LP

$$\begin{aligned} \text{maximize} \quad & 2p_1 + 1p_2 + 3p_3 \\ \text{subject to} \quad & 1p_1 + 0p_2 + 1p_3 \leq 4 \\ & 1p_1 + 1p_2 - 1p_3 \leq 3 \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

- More generally:

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p \leq c \\ & p \geq 0. \end{aligned}$$

Strong Duality Thm: If x^* is primal optimal and p^* is dual optimal:

(min. objective) $\quad c^T x^* = b^T p^* \quad$ (max. objective)