# Week 2 Recap

**Tuples, ordered lists, subsets**

Product spaces are very common in basic probability. If $A$ and $B$ are two finite sets with $k$ and $\ell$ elements, respectively, then the product $A \times B$ is the set of all ordered pairs ($2$-tuples) $(a, b)$ with $a \in A$ and $b \in B$.
Similarly, the product $A_1 \times \cdots \times A_n$ is the set of all $n$-tuples $(a_1, \ldots, a_n)$ with $a_i \in A_i$, $1 \le i \le n$. If $|A_i| = k_i$, $1 \le i \le n$, then $|A_1 \times \cdots \times A_n| = k_1 \times \cdots \times k_n$.
A case of special importance is when all $A_i$ are the same set $A$. In this case, the product space

$$A^n = \underbrace{A \times \cdots \times A}_{n \text{\tiny{times}}}$$

is the set of all $n$-tuples $(a_1, \ldots, a_n)$ with $a_i \in A$, $1 \le i \le n$. It has $|A|^n$ elements.

It is very important to distinguish and understand the difference between tuples of elements in $A$ and subsets of $A$. Fix an integer $m$. The $m$-tuples of elements in $A$ are ordered lists of elements in $A$ and we denote such a list as
$(a_1, \ldots, a_m)$. The entries in the list can be repeated and the $3$-tuples $(a, a, b)$ and $(a, b, a)$ are two different $3$-tuples.

Now, assume that $m$ is no greater than the number $n$ of elements in $A$ and consider a subset of $A$ with $m$ elements. The notation used for such an object is $\{a_1, \ldots, a_m\}$
where all $a_i$ are distinct elements of $A$ and the subsets $\{a, b, c\}$ and $\{b, a, c\}$ are in fact the same subset. In other words, in a subset, the order in which we list the elements makes no difference. The same element cannot appear twice.

**Factorial and binomial coefficients** Now, it should be pretty clear to you that there are less subsets of $A$ with $m$ elements than there are $m$-tuples of elements in $A$. But how many subsets of $A$ with exactly $m$ elements are there?

For any set $A$ with $n$ elements, the number of subsets of $A$ with exactly $m$ elements is

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

where $0! = 1$ and $k! = 1 \times \cdots \times (k-1) \times k$ if $k \geq 1$.

For any set $A$ with $n$ elements, the number of ordered lists of exactly $m$ distinct elements is

$$\frac{n!}{(n-m)!} = n \times (n-1) \times \cdots \times (n-m+1).$$

The total number of ordered complete lists of $n$ distinct objects is $n!$. By definition, this is the same a the number of permutations of $n$ distinct objects, or the number of bijections from $\{1, \ldots, n\}$ onto itself, or the number of distinct arrangement of a deck of $n$ cards.

**Stirling formula**

$$\lim_{n \to +\infty} \frac{n!}{\sqrt{2\pi n}\, n^n\, e^{-n}} = 1$$

or, equivalently,

$$n! \sim \sqrt{2\pi n}\, n^n\, e^{-n}.$$

**Exercise:** Use Stirling formula to find the approximate size of $\binom{2n}{n}$.

**Read about binomial Theorem and binomial identities!**

Here is a short list of binomial coefficients identities which can be proved in several different ways. For each one, find a counting proof" (The last one of these identities is somewhat more difficult).

- $\binom{k}{2}\binom{n}{k} = \binom{n}{2}\binom{n-2}{k-2}$.
- $1 \times n + 2 \times (n-1) + \cdots + (n-1) \times 2 + n \times 1 = \binom{n+2}{3}$.
- $\binom{n+m}{2} - \binom{n}{2} - \binom{m}{2} = nm$.
- $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
- $\sum_{\ell=0}^{k} \binom{n}{\ell}\binom{m}{k-\ell} = \binom{n+m}{k}$.
- $\sum_{\ell=m}^{n} \binom{\ell}{m} = \binom{n+1}{m+1}$

As a clue, here is a list of counting problems" that may be helpful in proving these identities. You have to figure out which problem is helpful to establish which identity.

- How many different ways to choose k distinct balls in a set of balls marked $1, 2, \ldots, n+m$ of which $n$ are red and $m$ are blue.
- Pick a mixed team of $2$ in a class with $n$ girls and $m$ boys.
- Choosing $3$ distinct numbers $a < b < c$ in $\{0, 1, \ldots, n+1\}$ while paying attention to the value of $b$.
- Choosing $n$ numbers in $\{0, 1, \ldots, 2n-1\}$ while paying attention to how many are even.
- Choosing $m+1$ numbers in $\{0, 1, \ldots, n\}$ while paying attention to the largest of them.
- Picking a committee with a Chair and a Secretary.

**Read** the Canevas page about **partitions** of a set with $n$ elements in $k$ parts.