Important Formulas in Finance

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1 General Comments

Unless we state otherwise, all cash flows occur at the end of a period and all interest is compound interest. Typically $r > 0$, but interest rates could also be negative or zero. Periods could be days, weeks, months, years. We ignore calendar idiosyncrasies related to months and years of variable length.

Not all formulas, not all versions of formulas, and not all interpretations of formulas are given. This is just an aide-mémoire, not a full reference. Some observations may be beyond the scope of the course (e.g., root finding methods for determining the implied interest rate for an annuity).

2 Time-Value of Money

PV = present value at time 0; FV = future value at time t; r = interest rate; t = number of periods.

- One period case: $FV = PV \cdot (1 + r)$.
- Multi-period case: $FV = PV \cdot (1 + r)^t$; $PV = \frac{FV}{(1+r)^t}$; $t = \frac{\ln \frac{FV}{PV}}{\ln(1+r)}$; $r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1$.
- Doubling time of an investment: t=$\frac{\ln 2}{\ln(1+r)} \approx \frac{\ln 2}{r}$ (if r is small). If r is expressed in percents, then $t \approx \frac{72}{r}$ periods.
- Future value factor $= (1 + r)^t$; present value (or discount) factor $= \frac{1}{(1+r)^t}$.

3 Annuities

PV = present value of an annuity; FV = future value of an annuity; C = the constant payment of an ordinary annuity; r = interest rate; t = number of periods; g = growth rate of payouts.

- $PV = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r}$; $C = \frac{r}{1-\left(\frac{1}{1+r}\right)^t} \cdot PV$; $t = -\frac{\ln\left(1-\frac{PV}{C}\cdot r\right)}{\ln(1+r)}$. There is no general formula for determining r - you can use a calculator, trial and error, or a systematic root finding method.
- Present value interest factor for annuities: $PFIVA(r, t) = \frac{1-\left(\frac{1}{1+r}\right)^t}{r}$, thus $PV = C \cdot PFIVA(r, t)$.
- $FV = C \cdot \frac{(1+r)^t-1}{r}$; $C = \frac{r}{(1+r)^t-1} \cdot FV$; $t = \frac{\ln\left(1+\frac{FV}{C}\cdot r\right)}{\ln(1+r)}$. There is no general formula for determining r.
- Annuity FV factor $= \frac{(1+r)^t-1}{r}$, thus $FV = C \cdot (Annuity\, FV\, factor)$.
- Perpetuity: $PV = \frac{C}{r}$.
- Annuity due: $PV = C \cdot \frac{1+r}{r} \cdot \left[1 - \frac{1}{(1+r)^t}\right]$.
- Growth annuity: $PV = C \cdot \frac{1-\left(\frac{1+g}{1+r}\right)^t}{r-g}$, if $r > g$. C is the payment at the end of period 1 (i.e., the first payment).
- Growth perpetuity: PV=$\frac{C}{r-g}$, if $r > g$.

4 Compounding Conventions; EAR

- If the nominal annual interest rate is r, and it compounds m times per year, then $FV = PV \cdot \left(1 + \frac{r}{m}\right)^m$.
- The effective annual rate is $EAR = \left(1 + \frac{r}{m}\right)^m - 1$.
- In the limit, when m tends to infinity, $FV = PV \cdot e^r$, $EAR = e^r - 1$.

5 Bonds

P = face value; C = yearly coupon (in dollars); t or T = maturity in years; y = yield; B = price. Note that the formulas depend on per-compounding-period interest rates and coupons. Formulas below are for semi-annual compounding; they may need to be adjusted.

- Bond price: $B = \frac{C}{2} \cdot \frac{1-\left(\frac{1}{1+\frac{y}{2}}\right)^{2t}}{\frac{y}{2}} + P \cdot \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}$.
- Bond yields can be computed using the interval bisection method.

6 Inflation

R = nominal interest rate; h = inflation rate; r = real interest rate.

- $1 + R = (1 + r) \cdot (1 + h)$; $r = \frac{R-h}{1+h} \approx R - h$.

7 Stock Valuation

P_0 = time-0 price of the stock; D_i= dividend that will be paid at the end of period i; R = per-period interest rate; g, g_1 , g_2 = constant, per-period dividend growth rates. Many stock valuation problems involve perpetuities or combinations of perpetuities, which is also reflected in the formulas below.

- General formula, with dividends up to time t: $P_0 = \sum_{i=1}^{t} \frac{D_i}{(1+R)^i} + \frac{P_t}{(1+R)^t}$.
- $P_0 = \sum_{i=1}^{\infty} \frac{D_i}{(1+R)^i}$. In reality, there are no infinite streams of dividends.
- Zero growth dividends: $P_0 = \frac{D}{R}$.
- Dividend growth model: $P_t = \frac{D_{t+1}}{R-g}$, if $R > g$.
- Non-constant growth: $P_0 = \sum_{i=1}^{t} \frac{D_i}{(1+R)^i} + \frac{P_t}{(1+R)^t}$; $P_t = \frac{D_t \cdot (1+g)}{R-g}$ **if** we assume constant growth after t.
- Two-stage growth: $P_0 = \frac{D_1}{R-g_1} \cdot \left[1 - \left(\frac{1+g_1}{1+R}\right)^t\right] + \frac{P_t}{(1+R)^t}$; $P_t = \frac{D_{t+1}}{R-g_2} = \frac{D_0 \cdot (1+g_1)^t \cdot (1+g_2)}{R-g_2}$.
- Required return: $R = \frac{D_1}{P_0} + g$.
- Multiples: P_t = (benchmark PE ratio) $\cdot\, EPS_t$.

8 NPV and IRR

- NPV = sum of discounted (project) cash flows. The discount (demanded, required) rate must be specified.
- NPV = 0 for discount rate r $\Longleftrightarrow IRR = r$. The IRR may not be unique.
- NPV investment criterion: undertake the project if $NPV > 0$.
- Given projects A and B, a crossover point occurs when $NPV_A = NPV_B$ for a given discount rate. The discount rate can be determined by subtracting the cash flows of project B from those of project A, and computing the IRR of the resulting cash flows.
- IRR investment criterion: undertake the project if required return $< IRR$. The required return does not have to be made explicit, as long as one can argue that the IRR is high enough for the inequality to hold. Exception: if we have financing-type cash flows; then we undertake the project if required return $> IRR$.
- When comparing exclusive projects, NPV and IRR may disagree.
- IRR values can be computed using the interval bisection method.

9 Project Cash Flows

- Earnings before interest and taxes = EBIT = sales - costs - depreciation.
- Net income = NI = EBIT - taxes.
- Taxes = EBIT × tax rate. Tax rates may be constant, or they may depend on the magnitude of the EBIT. Taxes may apply to negative EBIT, case in which they are negative (a benefit).
- Operating cash flow = OCF = EBIT + depreciation − taxes.
  - OCF = net income + depreciation;
  - OCF = sales - costs - taxes;
  - OCF = (sales - costs) × $(1 - T_c)$ + depreciation × T_c, where T_c is the tax rate.
- Project cash flow = OCF − (change in net working capital) − (project capital spending). With a slight abuse of terminology, cash flows related to the sale of assets at the end of projects are also part of "capital spending."

10 Costs and Break-Even Points

- TC = total cost; VC = total variable cost; v = variable cost per unit; FC = total fixed cost; AC = average cost; MC = marginal cost; q = production level (number of widgets produced, units of service delivered, etc); D = depreciation. Writing a quantity as a function of q, e.g., $VC(q)$, indicates that the respective quantity depends on the production level.
- $VC(q) = q \times v$.
- $TC(q) = FC + VC(q)$.
- $AC(q) = \frac{TC(q)}{q} = \frac{FC}{q} + v$.

- $MC = v < AC(q)$.
- Production levels derived from break-even calculations must typically be rounded up. Break-even calculations can be performed per-year, or over multiple years (project time horizon). If you are not given explicit tax information (such as the tax rate) and/or the problem does not explicitly refer to taxes, you can assume that no taxes are paid. If taxes are paid, they will paid at a flat rate T_c.
    - $OCF = (P - v) \times q - FC$. With taxes: $OCF = [(P - v)\, q - FC]\,(1 - T_c) + T_c D$.
    - Production level that corresponds to a given OCF: $q = \frac{FC+OCF}{P-v}$. With taxes: $q = \frac{FC}{P-v} + \frac{OCF-T_cD}{(1-T_c)(P-v)}$.
    - Accounting break-even: NI is zero; $q = \frac{FC+D}{P-v}$. With taxes: same.
    - Cash break-even: OCF is zero; $q = \frac{FC}{P-v}$. With taxes: $q = \frac{FC}{P-v} - \frac{T_cD}{(1-T_c)(P-v)}$.
    - Financial break-even is reached when the NPV of the operating cash flows is equal to the NPV of investments.

11 Risk, Return, and the SML

- Consider a stock whose return over a one-period period is r_i with probability p_i, for $i = 1, \ldots, N$. All possibilities must be included, so $\Sigma_{i=1}^{N} p_i = 1$.
- Expected return: $Er = \Sigma_{i=1}^{N} r_i \times p_i$.
- Variance of return: $Var(r) = \sigma^2(r) = E(r - Er)^2 = Er^2 - (Er)^2 = \sum_{i=1}^{N} p_i \times (r_i - Er)^2$.
- Volatility of return: $\sigma(r) = \sqrt{Var(r)} = \sqrt{E(r - Er)^2} = \sqrt{Er^2 - (Er)^2} = \sqrt{\sum_{i=1}^{N} p_i \times (r_i - Er)^2}$.
- $P =$ portfolio composed of N investments; $r_p =$ portfolio return; $V_i =$ total dollar value of the i^{th} investment; $w_i =$ (portfolio) weight of the i^{th} investment.
- Portfolio weights $w_i = \frac{V_i}{\Sigma_{j=1}^{N} V_i}$; $\Sigma_{i=1}^{N} w_i = 1$. If shorting is allowed, weights can be negative or can exceed 100%.
- Total portfolio value: $V_p = \Sigma_{j=1}^{N} V_i$; total dollar value of investment i: $V_i = w_i \times V_p$.
- $r_p = \sum_{i=1}^{N} w_i \times r_i$. Also, $Er_p = \sum_{i=1}^{N} w_i \times Er_i$.
- The variance and volatility of a portfolio can be computed as usual, using the probability of scenarios and the computed portfolio returns under each respective scenario.
- Risk = uncertainty of outcome, not probability of experiencing loss. We quantify risk using variance/volatility.
- Risk can be systematic or idiosyncratic. Diversification, if done correctly, can eliminate idiosyncratic risk.
- Realized return $= r = Er + m + \epsilon$, where m is the surprise (innovation) in the market return and ϵ is the surprise (innovation) in the idiosyncratic return.
- We assume that at least a risk-free asset exists: $r_F = Er_F$; $Var\,(r_F) = Vol\,(r_F) = 0$.
- Sytematic risk principle: Investors are only rewarded for systematic risk, and not for idiosyncratic risk.

- Systematic risk can be measured realtive to the average asset ("the market"). This measure of relative risk is the *beta* (β) of an asset. $\beta = \beta_M = 1$ by definition.
- The β of a portfolio is equal the the weighted sum of the components' βs: $\beta_p = \Sigma_{i=1}^{N} w_i \times \beta_i$.
- For any asset A, the slope of the SML is the reward-to-risk ratio, equal to $\frac{Er_A - r_F}{\beta_A}$. This ratio is the same for all assets, including for the market asset. The slope is equal to the market premium.
- Capital Asset Pricing Model (CAPM): $Er_A = r_F + (\mathrm{E}r_M\text{-}r_F)\times\beta_A$, where A is any asset; $Er_M - r_F$ is the market risk premium.

12 Cost of Capital

- Cost of equity r_E using the constant dividend growth model: $r_E = \frac{D_1}{P_0} + g$, where D_1 is the dividend at time 1, P_0 is the price of the equity at time 0, and g is the assumed-constant dividend growth rate.
- Given historical (yearly) growth rates $g_1, g_2, g_3, \ldots, g_N$, their arithmetic average is $g_A = \frac{1}{N}(g_1 + g_2 + \cdots + g_N)$, while the geometric average is $g_G = \sqrt[N]{(1+g_1)(1+g_2)\ldots(1+g_N)} - 1$.
- Cost of equity r_E using the SML: $r_E = r_F + \beta_E \times (r_M - r_F)$, where r_F is the risk-free rate, β_E is the equity β, and $r_M - r_F$ is the market risk premium.
- The cost of debt r_D can be identified with a corporation's bond yield (or another interest rate that characterizes loans taken out by the firm). These identifications are practical approximations.
- The cost of preferred debt is $r_P = \frac{D_P}{P_0}$, where D_P is the constant preferred dividend and P_0 is the price of one share of preferred stock at time 0.
- E=market value of the firm's equity; D=market value of a firm's debt; P=market value of the firm's preferred stock, V=market value of the firm. These are totals (aggregate values), not per-share values!
- Capital structure weights if only equity and debt exist: $V = E + D$; $w_E = \frac{E}{V} = \frac{E}{E+D}$; $w_D = \frac{D}{V} = \frac{D}{E+D}$; $w_E + w_D = 1$.
- Capital structure weights if equity, debt, and preferred stock exist: $V = E + P + D$; $w_E = \frac{E}{V} = \frac{E}{E+P+D}$; $w_P = \frac{P}{V} = \frac{P}{E+P+D}$; $w_D = \frac{D}{V} = \frac{D}{E+P+D}$; $w_E + w_P + w_D = 1$.
- If the interest on a firm's debt is I and the firm's tax rate is T_c, the after-tax cost of debt is $(1 - T_c) \times I$.
- WACC = weighted average cost of capital.
- WACC when only equity and debt exist: $WACC = w_E \times r_E + w_D \times r_D \times (1 - T_c)$.
- WACC when equity, preferred equity, and debt exist: $WACC = w_E \times r_E + w_P \times r_P + w_D \times r_D \times (1 - T_c)$.

13 WACC and Company Valuation

- A starred superscript (*) indicates quantities modified to remove the effect of interest payments. CFA = cash flow from assets (free cash flow); ΔNWC = change to the net working capital. The rest of the notations are defined above. Numerical subscripts indicate the time of valuation or the time of cash flows. Compare to section Project Cash Flows above; note the comment on capital spending there.

- $Taxes^{*} = EBIT \times T_c$.
- $CFA^{*} = EBIT + Depreciation - Taxes - \Delta NWC - (capital\, spending)$.
- $CFA^{*} = EBIT \times (1 - T_c) + Depreciation - \Delta NWC - (capital\, spending)$.
- Regular perpetuity: CFA^{*} stays constant. $Firm\, value_0 = \frac{CFA^{*}}{WACC}$.
- Growth perpetuity: CFA^{*} grows at a constant rate of g. $Firm\, value_0 = \frac{CFA^{*}_1}{WACC-g}$.
- Irregular growth can be handled like we did with equity valuation: we produce estimates up to time t, after which we estimate the value of the firm V_t at time t (typically using a perpetuity of some kind). $Firm\, value_0 = \frac{CFA^{*}_1}{1+WACC} + \frac{CFA^{*}_2}{(1+WACC)^2} + \frac{CFA^{*}_3}{(1+WACC)^3} + \cdots + \frac{CFA^{*}_t}{(1+WACC)^t} + \frac{V_t}{(1+WACC)^t}$.