The gambler's ruin problem

The Gambler's ruin problem

Two players, A and B, have respective initial fortunes $n, N-n$. They play a game that consists in a referee flipping a coin. If Heads, A receives a dollar from B; if Tails, B receives a dollar from A. The game goes on until one of them runs out of money. We assume the different coin flips are independent and the coin is such that it comes up Heads with probability p (we will also use the notation $q = 1 - p$ which is the probability the coin comes up Tails).

What is the probability that A will end up with all the money?

This is solve by using a simple useful idea. We think of the total amount of money, N, as fixed and we call P_n the probability that player A ends up with all the money when he or she starts with n dollars. We are going to think of n as a parameter that may vary. Let H be the event that the result of the first flip is Heads. Let E the event that player A ends up with all the money. Now, if player A starts with n dollars,

$$P_n = P(E) = P(E|H)P(H) + P(E|H^c)P(H^c) = pP(E|H) + qP(E|H^c).$$

But observe that it must be the case that

$$P(E|H) = P_{n+1} \text{ and } P(E|H^c) = P_{n-1}.$$

This is because we can think of what happens after the first flip as a new equivalent game (the total money in play remains equal to N) starting fresh under a new initial condition (depending on the result of the first flip).

This gives the equation

$$P_n = pP_{n+1} + qP_{n-1}$$

where n is an integer which varies between 0 and N. In linear algebra, this is called a linear difference equation and even if you do not remember HOW to solve it, it would be good to remember that it can be solved! You should also remember that you should expect to have to use some initial or boundary conditions.

In our problem, it is obvious that $P_0 = 0$ (if A has no money, then B has all the money and wins) and also $P_N = 1$. Now, to solve this particular problem,

we can observe that $p + q = 1$ and write the equation in the form

$$pP_n + qP_n = pP_{n+1} + qP_{n-1}$$

and rearrange this into

$$P_{n+1} - P_n = \frac{q}{p}(P_n - P_{n-1})\,.$$

Using induction (or, simply, inspection), we see that

$$P_{n+1} - P_n = \left(\frac{q}{p}\right)^n (P_1 - P_0)$$

and

$$\begin{aligned} P_n - P_0 &= (P_n - P_{n-1}) + (P_{n-1} - P_{n-2}) + (P_{n-2} - P_{n-2}) + \cdots + (P_1 - P_0) \\ &= \left(\sum_0^{n-1} (q/p)^i\right)(P_1 - P_0). \end{aligned}$$

Remember that $P_0 = 0$ and $\sum_0^{n-1} a^i = (1-a^n)/(1-a)$ (partial sum of a geometric series). The equation above becomes

$$P_n = P_1 \times \begin{cases} \frac{1-(q/p)^n}{1-(q/p)} & \text{if } q/p \neq 1 \\ n & \text{if } q = p. \end{cases}$$

We still need to use the boundary condition $P_N = 1$ to compute P_1

$$P_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} & \text{if } q/p \neq 1 \\ 1/N & \text{if } q = p. \end{cases}$$

This gives

$$P_n = \begin{cases} \frac{1-(q/p)^n}{1-(q/p)^N} & \text{if } q/p \neq 1 \\ n/N & \text{if } q = p. \end{cases}$$

Now, it is not obvious , a priori, that this game always ends after a finite number of flips! But, the argument above tells that if A starts with n dollars, the probability that B ends up with all the money is

$$Q_n = \begin{cases} \frac{1-(p/q)^{N-n}}{1-(p/q)^N} & \text{if } q/p \neq 1 \\ (N-n)/N & \text{if } q = p. \end{cases}$$

We simply switch p and q and replaced n with $N-n$ (why?). Now, with a little patience, you can check that $P_n + Q_n = 1$ which means that one of the two players always ends with all the money! With probability 1, the game does not last for ever.