The Gamma distributions

A Gamma random variable with parameters $\lambda > 0$ and $\alpha > 0$ has density function

$$f_X(x) = \frac{\lambda}{\Gamma(\alpha)} e^{-\lambda x} (\lambda x)^{\alpha-1} \mathbf{1}_{(0,+\infty)}(x) \text{ where } \Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds.$$

By integration by parts, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. Because $\Gamma(1) = 1$, this implies $\Gamma(n) = (n-1)!$. Note that an exponential distribution with parameter $\lambda > 0$ is a special case of a Gamma, with parameters λ and $\alpha = 1$.

One easily computes $E(X)$, $E(X^2)$ and $\mathrm{Var}(X)$ as follows.

$$E(X) = \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda s} (\lambda s)^\alpha ds = \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda},$$

$$E(X^2) = \frac{1}{\lambda^2\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda s} (\lambda s)^{\alpha+1} ds = \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{\lambda^2}, \text{ and } \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}.$$

The MGF of a Gamma with parameters $\lambda > 0$ and $\alpha > 0$ is

$$M(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \ \ t \in (-\infty, \lambda).$$

Using this, we immediately check that the sum of k independent Gammas with the same parameter λ and second parameters $\alpha_1, \ldots, \alpha_k$ is a Gamma distribution with parameters λ and $\alpha_1 + \cdots + \alpha_k$.