
Name :

NetID:

Instructions When more than one solution is given, figure out which you attempted and try to understand the different solutions.

(1) There are 3 independent problems. The third problem uses Calculus (integrals). The point total is 25. You have 75 minutes.

(2) Write your answers on this booklet. Write your name and Cornell NetID on the top of this page before you begin.

(3) Write clearly using a black or blue pen or pencil (your exam will be scanned).

(4) Always provide reasons for your answers and explain your computations. For numerical answers, give either a simplified fraction or a decimal answer, which ever comes more easily. For instance, $3 \times 7^2/(15)^9$ is very acceptable (much better than the expanded version) but $8/12$ is not very good.

(5) No books, notes, or documents are allowed during this prelim. Do not use (do not touch) electronic devices during the exam, for any reasons.

(6) Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty.

Problem 1: (9 pts) In this problem, it is preferable to solve the 3 questions in order and to make sure the earlier questions are done correctly.

(a-3pts) What is the probability mass function of the random variable X which record the maximum of the two numbers shown in two independent rolls of a fair die?

The maximum seen after two rolls of a die is a number in $\{1, 2, 3, 4, 5, 6\}$. The probability mass function is $p_1 = 1/36$, $p_2 = 3/36 = 1/12$, $p_3 = 5/36$, $p_4 = 7/36$, $p_5 = 9/36 = 1/4$ and $p_6 = 11/36$.
This is because a maximum of k results from rolls (k, k) or (i, k) or (k, i) with $i \in \{1, \dots, k-1\}$ which gives $2(k-1)+1$ different ways to get it.

(b-3pts). A person flips a biased coin that comes Heads with probability $1/4$. If it comes Heads, they roll a fair die twice and record the maximum of the two numbers shown. If the coin shows Tails, they roll the die once and record the number shown. Call Y the recorded number. What are the possible values taken by Y and what is the probability that $Y = 6$?

The possible values of Y are $\{1, 2, 3, 4, 5, 6\}$. To compute the probability that $Y = y$ we call H, T the events that the coin comes-up Heads/Tails, M the maximum seen in two consecutive rolls, R the result of the roll of one die. Then

$$P(Y = y) = P(H)P(Y = y|H) + P(T)P(Y = y|T) = \frac{1}{4}P(M = y) + \frac{3}{4}P(R = y) = \frac{1}{4}p_y + \frac{3}{24} = \frac{1}{4}p_y + \frac{1}{8}$$

where p_y is as in question a. It follows that

$$P(Y = 6) = \frac{1}{8} + \frac{11}{4 \times 36} = \frac{29}{144}.$$

(c-3pts) Referring to the experiment in (b), if the person doing the experiment tells us that $Y = 6$, what is the probability that the coin came up Heads? (give the answer as a reduced fraction)

$$P(H|Y = 6) = \frac{P(Y = 6|H)P(H)}{P(Y = 6)} = \frac{11}{36} \times \frac{1}{4} \times \frac{144}{29} = \frac{11}{29}.$$

Problem 2: (8 pts) In this problem, each question refers to the same setup but can be answered independently of the other questions. A person decide to paint their left-hand fingernails using the colors yellow, red and blue. For each finger, they pick a color uniformly at random.

(a-2pts) What is the probability that the little finger is not painted yellow?
For each given finger, the color is picked uniformly at random in $\{y, r, b\}$. It follows that the probability of the little finger is not painted yellow is $2/3$.

(b-2pts) Let y, r, b be the number of fingers painted yellow, red and blue. What is the set Ω of possible triplets (y, r, b)? What is the name and formula for the probability distribution of (y, r, b)?

The set of possible is $\{(y, r, b) : y, r, b \in \{10, 1, 2, \dots, 5\}, y + r + b = 5\}$. The probability distribution is multinomial with parameters $n = 5, k = 3, p_1 = p_2 = p_3 = 1/3$. The formula is $P(y, r, b) = \left(\frac{1}{3}\right)^5 \binom{5}{y,r,b} = \left(\frac{1}{3}\right)^5 \frac{5!}{y!r!b!}$.

(c-2pts) What is the probability that all three colors be used? (for this probability, give an expression, do not try to compute the decimal value)

Solution 1: The probability that all three colors are used is 1 minus the probability that at least one color is not used. Let A_x, $x = y, r, b$, be the event that color x is not used. We want to compute $P(A_y \cup A_r \cup A_b) = P(A_y) + P(A_r) + P(A_b) - (P(A_y \cap A_r) + P(A_y \cap A_b) + P(A_r \cap A_b) + P(A_y \cap A_r \cap A_b)$. Obviously, $P(A_y \cap A_r \cap A_b) = 0$ and $P(A_y \cap A_r) = P(A_y \cap A_b) = P(A_r \cap A_b) = (1/3)^5$ (in each case, we need to pick the remaining color for every finger). Finally, $P(A_y) = P(A_r) = P(A_b) = (2/3)^5$ (for each finger, we need to pick one of the two allowed colors). So, the probability that all three colors be used is $1 - 3((2/3)^5 - (1/3)^5)$. This is about .62.

Solution 2: We are using (y, r, b): All three color used means one color used 3 times and two color used once or two color used twice and one color used once. An outcome of the first type has probability $p_1 = 3^{-5}\binom{5}{3,1,1}$, an outcome of the second type have probability $p_2 = 3^{-5}\binom{5}{2,2,1}$. It remains to count how many outcomes of each type there are:

Type 1: 3 outcomes: We just need to choose which color gets the 3: $(3, 1, 1), (1, 3, 1), (1, 1, 3)$.

Type 2: 3 outcomes: We just need to to choose which color get 1: $(1, 2, 2), (21, 2), (2, 2, 1)$.

The answer is # type $1 \times p_1 +$ # type $2 \times p_2$).

Solution 3 Use the space $\{\text{yellow}, \text{red}, \text{blue}\}^5$ instead of the space of (y, r, b), $y+r+b = 5$. In that space, $\{\text{yellow}, \text{red}, \text{blue}\}^5$, all outcomes have the same probability 3^{-5}. It thus suffices to count all the elements of $\{\text{yellow}, \text{red}, \text{blue}\}^5$ which have at least one of each color. This is $3^5 - \#\{\text{at most 2 colors}\}$. Then count the cases with at most two colors: $3(2^5 - 2) + 3$ where $2^5 - 2$ counts the cases with exactly two given colors, and the last 3 is for one color paintings. See (d).

(d-2pts) In how many different ways can the person paint their nails using at most two of the three colors?

There are exactly three ways to paint all the fingers the same color. Painting with exactly two colors, we have to pick a pair of colors and there are 3 ways to do that: $(y, r), (y, b), (r, b)$. Then we have to pick the non-empty non-full subset of fingers that are painted with the first of the two colors. There are $2^5 - 2$ ways to do that. So the answer is $3 + 3(2^5 - 2) = 3 \times 2^5 - 3 = 93$.

Problem 3: (8 pts) The questions can be solved independently of each other.

(a-4pts) Find the value of the constant c so that the function $f(x) = c|x|e^{-3|x|}$, $x \in (-\infty, +\infty)$, is the probability density of a random variable X. Does $E(X)$ exists? If it does , what is its value? (think before you compute)

We must have $c\int_{-\infty}^{+\infty} |x|e^{-3|x|}dx = 1$ Using symmetry ($f(-x) = f(x)$), and integration by parts (the boundary terms are 0), $\int_{-\infty}^{+\infty} |x|e^{-3|x|}dx = 2\int_0^{+\infty} xe^{-3x}dx = 2(1/3)\int_0^{+\infty} e^{-3x}dx = 2/9$ because the derivative of e^{-3x} is $-3e^{-3x}$ and its antiderivative is $(-1/3)e^{-3x}$. This implies that $c = (9/2)$.

The expectation $E(X)$ exists because the function $e^{-3|x|}$ decays much faster than any power of $|x|$ at $\pm\infty$. Specifically, we need to show that $|x|^2e^{-3|x|}$ is integrable at infinity. We can find a constant C such that $|x|^2|e^{-3|x|/2} \le C$ for all $x \in \mathbb{R}$ (just find the maximum of this function using calculus) and this implies $|x|^2e^{-3|x|} \le Ce^{-3|x|/2}$. This upper bound is integrable at infinity as desired.

Without calculation, $E(X) = 0$ because of the symmetry $f(x) = f(-x)$: use the change of variable $x \to -x$ to see that

$$\int_{-\infty}^{+\infty} xf(x)dx = \int_{+\infty}^{-\infty} -xf(-x)d(-x) = -\int_{-\infty}^{+\infty} xf(x)dx.$$

(The only real a such that $a = -a$ is 0).

(b-4pts) The random variable U has density $(9/2)|u|e^{-3|u|}$, $u \in \mathbb{R}$. What is the cumulative distribution of $V = |U|$? What is the density of V?)

The random variable V takes only non-negative values. cumulative distribution of V is ($v \ge 0$)

$$\begin{aligned} F_V(v) &= P(V < v) = P(-v < U < v) = 2(9/2)\int_0^v ue^{-3u}du \\ &= 3\int_0^v u \times (-e^{-3u})'du = 3\int_0^v e^{-3u}du - 3ve^{-3v} = 1 - e^{-3v} - 3ve^{-3v}. \end{aligned}$$

Taking derivative, we find that probability density of V is $f_V(v) = 3^{-3v} - 3e^{-3v} + 9ve^{-3v} = 9ve^{-3v}$ for $v \ge 0$ and 0 otherwise.