Basic Matching Model

INFO 4220 – Networks II: Market Design

Based on Haeringer Ch.9

Today

- Quickly look at the results of the game we played last lecture
- Traditional model: Gale-Shapley algorithm
- In the next few lectures we will study different settings that had some sort of market failure that was solved by using market design. For example:
  - National Residence Matching Program
  - The school Choice problem
  - Kidney exchanges
  - Dorm and course assignments

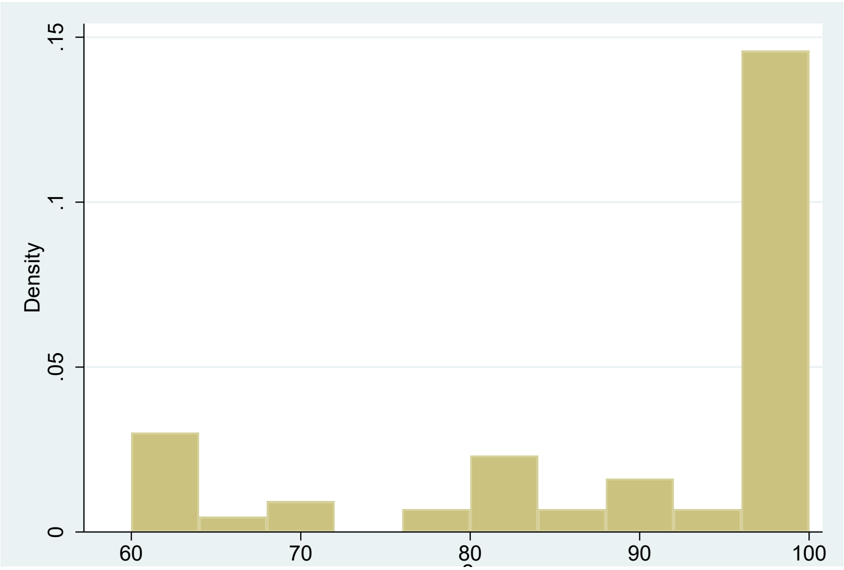
The Game we Played

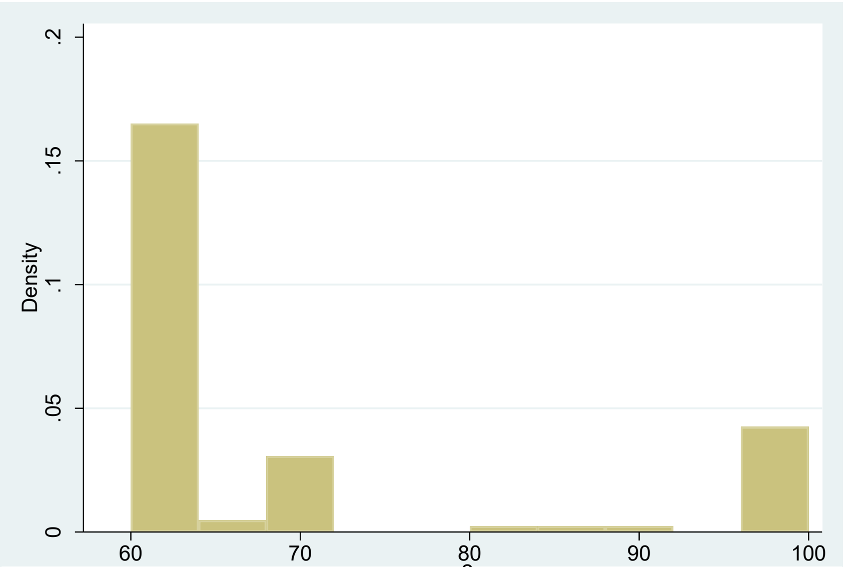
The 3 highest scores in this game will get a +1 bonus in the bonus counter

Game: Somehow, after grading PS1 your score and the score of your friend got lost in CANVAS (along with your submissions). Despite our best efforts, we (i.e. the teaching staff) cannot recover them. We know both of you had the exact same score and that it was in between 60 and 100, but we cannot recall the exact number. After a long meeting, we decided that we are going to ask each of you separately how much was your score. If you both say the same number, then we will give you both that score. If you say different numbers, we will know that the one of you saying the highest number is lying. Thus, we will give you both the lower score. But, to reward the honesty of the student reporting the lower score, we will give them a +4 bonus, and to the student reporting the higher score we will give a -4 penalty.

Enter your score in the quiz in CANVAS. To determine your payoff, you will be paired with a random student in the course.

Game results





ROUND 1:

- Most people entered 100
  - If you are paired against 100 you get 100, if not, you get what they enter -4
- Highest payoff entered 99

ROUND 2:

- Most people entered 60
- Highest payoff entered 85
- 2 people entered 100 and were paired against each other

Game results

Round 1

- 42 entered 100
  - Average payoff 87.6
- 6 entered 99
  - Average payoff 96.5
- 4 entered 97
  - Average payoff 98.75
- 11 entered 96
  - Average payoff 89.45
- Most successful strategy was entering 97. Balances getting a high score if you win, with a high probability of being matched against a player that entered a higher score

Round 2

- 17 entered 100
  - Average payoff 40.35 (median 30)
- 1 entered 99 - Payoff 30
- 1 entered 90 - Payoff 39
- 70 entered 60
  - Average payoff 69.4
  - 11 had a payoff of 90
- Most successful strategy was entering 97. Balances getting a high score if you win, with a high probability of being matched against a player that entered a higher score

Matching Markets

- Gale & Shapley (1962) publish influential contribution in game theory that marks the start of market design
- Recently, market design has been increasingly employed in different settings with good outcomes
- In 2012 Shapley and Roth received the Nobel Memorial Prize in Economics Science for “the theory of stable allocations and the practice of market design” (David Gale died in 2008)
- According to A. Roth, this is one of the settings where economists “understand enough about how markets work so that we can help in that process”

The Basic Matching Model

- This model is based in Gale & Shapley’s model from 1962
- There are two sets of agents: Musicians and Singers
- The problem is that we want to match musicians to singers according to their preferences
- Each individual can be matched to at most one person
- A musician (or a singer) can be either be matched or unmatched to a singer (or a musician)

The Basic Matching Model

- This is what is known as a two-sided, one-to-one model
  - Two-sided: We are matching one side of the market (musicians) to other side of the market (singers)
  - One-to-one: Each agent can be matched to only one other agent
- There are other models:
  - Many-to-one matching (matching students to colleges)
  - Many-to-many matching (students and professors)

Preferences

- In the case of markets we have studied so far, prices were the coordination mechanism
- We are now assuming there is no exchange of money
  - There is only one way to be matched to someone else: You are either matched or you are not
  - Preferences over potential matches don't change
- **Each musician m has a preference ordering P_m over singers and the option to remain single**
- **Each singer s has a preference ordering P_s over musicians and the option to remain single**

Preferences

- Example: $P_{m_1=s_1,s_3,s_6,m_1.s_2,s_4\ldots}$
  - m_1's most preferred singer is $s_1$then s_3, etc.
  - We say that s_2 and s_4 are unacceptable for m_1
- Preferences are assumed to be strict: No one is indifferent between two different options
- There are other ways to represent preferences:
  - $s_1 P_{m_1} s_3$: Musician 1 (strictly) prefers s_1 to s_3
  - Read from top to bottom:

P_{m_1}
s_1
s_3
s_6

Preferences

The formal definition of preferences is as follows:

- The preferences of a singer s can be represented by a strict ordering over the set $M \cup \{s\}$ (i.e. all musicians and herself)
- The preferences of a musician m can be represented by a strict ordering over the set $S \cup \{m\}$ (i.e. all singers and himself)

Matching

A matching is a function μ that says who is matched to whom:

- $\mu(m)$ is the partner of m under the matching μ
- $\mu(s)$ is the partner of s under the matching μ

Any matching μ must satisfy the following properties:

- For each musician m, $\mu(m) \in S \cup \{m\}$
  - A musician is matched to a singer or himself
- For each singer s, $\mu(s) \in M \cup \{s\}$
  - A singer is matched to a musician or herself
- $\mu(m) = s \Leftrightarrow \mu(s) = m$
  - A musician is matched to a singer, if and only if, that singer is matched to that musician

Matching

- There may be many possible matching alternative, which we can denote: $\mu, \mu', \mu'', \ldots$
- We have said that musicians (singers) have preferences over singers (musicians). This will induce preferences over matchings
- Let's say that: $P_{s_1} = m_2, m_3, m_1, s_1, m_4$, and:
    - $\mu(s_1) = m_2$
    - $\mu'(s_1) = m_3$
    - Then s_1 prefers matching μ to matching μ'
- Note that while preferences over musicians (singers) are strict, preferences over matches don't need to. For example, consider the matching $\mu''(s_1) = m_2$
    - Singer s_1 is indifferent between matching μ and μ''

Stability

- As we are assuming there are no prices in matching, we cannot talk about equilibrium
- The "equivalent" concept is stability. It requires two conditions: Individual rationality and absence of blocking pairs
- A matching μ is individually rational if for each individual $v \in M \cup S$, v weakly prefers $\mu(v)$to v.
  - Notation: v weakly prefers $\mu(v)$to v = $\mu(v)R_v v$
  - A matching is individually rational if nobody would strictly prefer to be remain single rather than staying with the partner determined by the matching

Stability

- A pair (m, s) blocks a matching μ if:
  - $\mu(m) \neq s$ — m and s are not matched under μ
  - $sP_m\mu(m)$ — m prefers s to his match
  - $mP_s\mu(s)$ — s prefers m to her match
  - i.e. You can find a singer and a musician that are not currently matched, and that would prefer each other to their current match

- A matching μ is stable if
  - It is individually rational
  - There is no pair musician-singer that blocks μ.

Example

P_{m_1}	P_{m_2}
s_1	$\boxed{s_1}$
$\boxed{s_2}$	s_2
m_1	m_2

P_{s_1}	P_{s_2}
m_1	$\boxed{m_1}$
$\boxed{m_2}$	m_2
s_1	s_2

Consider the matching: $\mu(m_1) = s_2$ and $\mu(m_2) = s_1$

Why isn't this match stable?

Example

P_{m_1}	P_{m_2}
s_1	$\boxed{s_1}$
$\boxed{s_2}$	s_2
m_1	m_2

P_{s_1}	P_{s_2}
m_1	$\boxed{m_1}$
$\boxed{m_2}$	m_2
s_1	s_2

Consider the matching: $\mu(m_1) = s_2$ and $\mu(m_2) = s_1$

This match is not stable because: m_1 and s_1 block μ

- m_1 prefers to be matched to s_1 to his match
- s_1 prefers to be matched to m_1 to her match

Theorem

David Gale & Lloyd Shapley, 1962:

For any preferences, there always exist at least one stable matching

Ok, but how do we find stable matchings?

Deferred Acceptance Algorithm

- Musicians make offers to singers in the same order as in their preferences
  - Each musician start making offers to his favorite singer
  - If rejected, then the musician makes an offer to his second favorite singer
  - If rejected, then then musician makes an offer to his third favorite singer …
- Each singer always keeps the best musician’s offer (according to her preferences) among the musicians making offers to her (if any), and rejects all others
- Algorithm stops when there is no more rejection

Deferred Acceptance with Musicians Proposing

Step 1

Each musician makes an offer to his most preferred acceptable* singer

* If a musician finds all singers unacceptable, he remains single

Each singer who received at least one offer:

- Temporarily holds the offer from the most preferred musician among those who made an offer to her and are acceptable
- Rejects the other offers

Deferred Acceptance with Musicians Proposing

Step $k, k \geq 2$:

Each musician whose offer was rejected in the prior step makes an offer to his most preferred singer among the acceptable singers he has not yet made an offer to*

* If no such singer exists, he remains single

Each singer who received at least one offer in this step:

- Temporarily holds the offer from her most preferred musician among those that made an offer to her in this step and the offer she was holding (if any)
- Rejects all other offers

Deferred Acceptance with Musicians Proposing

END: Algorithm stops when no musician has an offer that is rejected

Final matching:

- Each singer is matched to the musician whose offer she was holding when the algorithm stopped (if any)
- Each musician is matched to the singer he was temporarily matched when the algorithm stopped (if any)
- That is why it is called deferred acceptance

Example

P_{m_1}	P_{m_2}
s_1	$\cancel{s_1}$
s_2	s_2
m_1	m_2

P_{s_1}	P_{s_2}
m_1	m_1
$\cancel{m_2}$	m_2
s_1	s_2

Step 1: Musician m_1 and m_2 both make an offer to s_1

- s_1 has offers from m_1 and m_2. Rejects m_2
- Singer s_2 does nothing (has no offers, nothing to reject)

Example

P_{m_1}	P_{m_2}
s_1	$\cancel{s_1}$
s_2	s_2
m_1	m_2

P_{s_1}	P_{s_2}
m_1	m_1
$\cancel{m_2}$	m_2
s_1	s_2

Step 1: Musician m_1 and m_2 both make an offer to s_1

- s_1 has offers from m_1 and m_2. Rejects s_2
- Singer s_2 does nothing (has no offers, nothing to reject)

Step 2: Musician m_1 is temporarily matched to s_1 (does nothing), and m_2 makes an offer to s_2

- Singer s_1 is still matched to m_1 and does nothing
- Singer s_2 does not reject m_2's offer

No more rejection, the algorithm stops: $\mu(m_1) = s_1, \mu(m_2) = s_2$

Which side makes offers?

In the deferred acceptance algorithm, it is important that:

- One side makes the offers, the other side accepts/rejects the offers
- No obligation to have musician proposing, it could be singers proposing
- (who proposes does matter for the outcome that is reached)

Example

P_{m_1}	P_{m_2}	P_{m_3}	P_{s_1}	P_{s_2}	P_{s_3}
s_2	s_1	s_1	m_1	m_3	m_1
s_1	s_2	s_2	m_3	m_2	m_3
s_3	s_3	s_3	m_2	m_1	m_2

Find the matching that would result from the DA algorithm with musicians proposing:

Example

P_{m_1}	P_{m_2}	P_{m_3}	P_{s_1}	P_{s_2}	P_{s_3}
s_2	s_1	s_1	m_1	m_3	m_1
s_1	s_2	s_2	m_3	m_2	m_3
s_3	s_3	s_3	m_2	m_1	m_2

Find the matching that would result from the DA algorithm with musicians proposing:

$$\mu(m_1) = s_1$$
$$\mu(m_2) = s_3$$
$$\mu(m_3) = s_2$$

Example

P_{m_1}	P_{m_2}	P_{m_3}
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3

P_{s_1}	P_{s_2}	P_{s_3}
m_1	m_3	m_1
m_3	m_2	m_3
m_2	m_1	m_2

Find the matching that would result from the DA algorithm with singers proposing

Gale and Shapley’s Theorem

- For any preferences, there always exists at least one stable matching
- The Deferred Acceptance Algorithm (with either musicians or singers proposing) always produces a matching that is stable according to the preferences used for the matching.

Proof: μ is a stable matching if:

1) μ is individually rational

2) There are no blocking pairs

Gale and Shapley’s Theorem

The outcome of the deferred acceptance algorithm is necessarily individually rational:

- No musician makes an offer to a singer that he considers unacceptable
- No singer accepts an offer from a musician that she considers unacceptable

Now let’s see if there are blocking pairs

Gale and Shapley's Theorem

Let's say that Bob is matched to Beth and Alice to Adam.

Let's assume (Bob, Alice) is a blocking pair:

1) Alice prefers Bob to Adam (her match in the D.A.)

2) Bob prefer Alice to Beth (his match in the D.A.)
   - As Bob was making the offers and he prefers Alice to Beth, he must have made an offer to Alice BEFORE making an offer to Beth
   - If Bob made offers AFTER proposing to Alice (eventually to Beth) then he must have been rejected by Alice

This is a contradiction because:

- If Alice accepted the offer from Adam, she must like Adam better than Bob

Which Stable Matching

- We have said that for all preferences there is going to be at least one stable matching
- There could be many stable matchings too
- How do we choose one of the possible stable matchings?
- Who is going to prefer which matching?

Optimality

Denote by μ_M the matching we obtained with the DA algorithm with musicians making offers

- μ_M is called the musician-optimal matching because:
  - Each musician prefers μ_M to any other stable matching & Each singer prefers any stable matching to μ_M

- And similarly: μ_S is the singer-optimal matching:
  - Each singer prefers μ_S to any other stable matching & Each musician prefers any table matching to μ_S

Incentives in the DA algorithm

- We have been assuming that we know the true preferences of musicians and singers
  - Could they lie to obtain a better matching?
- Consider the following mechanism:
  - Musicians and singers (simultaneously) submit their preferences
  - The DA algorithm is used to create a matching using the submitted preferences
  - The matching is announced
- Will musicians and singers reveal their true preferences?

Strategyproofness of DA algorithm

Theorem:

A matching mechanism that uses the Deferred Acceptance algorithm is strategyproof for the proposing side

(strategyproof = it is a dominant strategy to reveal one's true preferences)

Musicians and singers don't have the same incentives

P_{m_1}	P_{m_2}
$\boxed{s_1}$	$\boxed{s_2}$
s_2	s_1

P_{s_1}	P_{s_2}	P'_{s_2}
m_2	m_1	m_1
$\boxed{m_1}$	$\boxed{m_2}$	

- The musician-optimal matching with true preferences: $\mu_M(m_1) = s_1, \mu_M(m_2) = s_2$.
- No suppose s_2 lies and says $P'_{s_2}: m_1$ (Singer 2 says only musician 1 is acceptable):
  - The musician-optimal matching now is:
    $$\mu'(m_1) = s_2, \mu'(m_2) = s_1$$
  - s_2 prefers her matching when lying than her match when being truthful

Incentives

- Deferred Acceptance is not strategyproof for both sides
- Is there another algorithm that would do the job?

Theorem:

There is no matching mechanism that satisfies, for any matching problem, the following two properties at the same time:

a) The matching is stable with respect to the submitted preference list
b) The mechanism is strategyproof for all individuals

Proof

P_{m_1}	P_{m_2}
s_1	s_2
s_2	s_1

P_{s_1}	P_{s_2}
m_2	m_1
m_1	m_2

Only 2 possible stable matchings:

$$\mu_M(m_1) = s_1 \qquad \mu_S(m_1) = s_2$$
$$\mu_M(m_2) = s_2 \qquad \mu_S(m_2) = s_1$$

- If the "magic algorithm" produces a stable matching. In this problem is must choose either μ_M or μ_S:
  - If we select μ_M: Then s_1 and s_2 could be better off lying
  - If we select μ_S: Then m_1 and m_2 could be better off lying
  - THUS: It is not strategyproof for all individuals

Take away – Matching & Stability

- A **matching function** says, for each individual, who is matched to whom
- If X prefers to remain single than being matched to Y, then we say Y is unacceptable to X
- A matching is **individually rational** if nobody is matched to an unacceptable partner
- A pair (m, s) is a **blocking pair** in matching μ if they both prefer each other to the match provided by μ
- A **matching is stable** if:
  - Individually rational
  - Not blocked by any pair

Take away – Deferred Acceptance

- The DA algorithm provides a stable matching:
  - The most preferred stable matching for the proposing side
  - The least preferred stable matching for the receiving side
- **DA is strategyproof for the proposing side**, but not for the receiving side
- In general, it is **impossible to have strategyproofness for both sides and stability**

Take away – Incentives

- DA with musicians proposing yield **Musician-optimal matching:**
  - All musicians prefer the musician-optimal matching to any other stable matching
  - All singers prefer any other stable matching than the musician-optimal matching
- DA with singers proposing: **Singer-optimal matching:**
  - All singers prefer the singer-optimal matching to any other stable matching
  - All musicians prefer any other stable matching than the singer-optimal matching

Basic Matching Model

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