(1/24/2024)

Upper bounds and lower for minimization problems

Given an optimization problem:

$$z^* = \min_{x \in X} \ f(x) \qquad \text{where} \quad X \subseteq \mathbb{R}^n$$

- An upper bound $U \in \mathbb{R}$ is a number that satisfies:

  $$\min_{x \in X} \ f(x) \leq U$$

  It is perfectly OK if there are solutions $x' \in X$ such that $f(x') > U$

  We only need the condition to hold for the optimal solution $x^* \in X$.

  Any feasible point $\bar{x} \in X$ gives an upper bound $U = f(\bar{x})$

- A lower bound $L \in \mathbb{R}$ is a number that satisfies:

  $$\min_{x \in X} \ f(x) \geq L$$

  Which means that

  $$f(x) \geq L \quad \text{for all} \quad x \in X$$

  Finding lower bounds is usually complicated.

- Small upper bounds and large lower bounds are better.

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Lower bounds via duality

- For any LP, there is a corresponding dual LP.

Primal LP

$$
\begin{aligned}
\text{minimize} \quad & 4x_1 + 3x_2 \\
\text{subject to} \quad & 1x_1 + 1x_2 \geq 2 \\
& 0x_1 + 1x_2 \geq 1 \\
& 1x_1 - 1x_2 \geq 3, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Its Dual LP

$$
\begin{aligned}
\text{maximize} \quad & 2p_1 + 1p_2 + 3p_3 \\
\text{subject to} \quad & 1p_1 + 0p_2 + 1p_3 \leq 4 \\
& 1p_1 + 1p_2 - 1p_3 \leq 3 \\
& p_1, p_2, p_3 \geq 0.
\end{aligned}
$$

- More generally:

Primal

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax \geq b \\
& x \geq 0
\end{aligned}
$$

Dual

$$
\begin{aligned}
\text{maximize} \quad & b^T p \\
\text{subject to} \quad & A^T p \leq c \\
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Weak Duality Thm: If $\bar{x}$ is primal feasible and $\bar{p}$ is dual feasible:

(min. objective) $\quad c^T \bar{x} \geq b^T \bar{p} \quad$ (max. objective)

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$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p \leq c \\ & p \geq 0. \end{aligned}$$

Strong Duality Thm: If x^* is primal optimal and p^* is dual optimal:

(min. objective) $\quad c^T x^* = b^T p^* \quad$ (max. objective)

LP notation example: Production planning

- A firm produces n different products using m different materials.
- Let $b_i \geq 0$, $i = 1, \ldots, m$, be the available amount of the ith material.
- The jth product, $j = 1, \ldots, n$, requires $a_{ij} \geq 0$ units of the ith material and results in a revenue of $p_j \geq 0$ per unit produced.
- Decide how much of each product to produce to maximize revenue.

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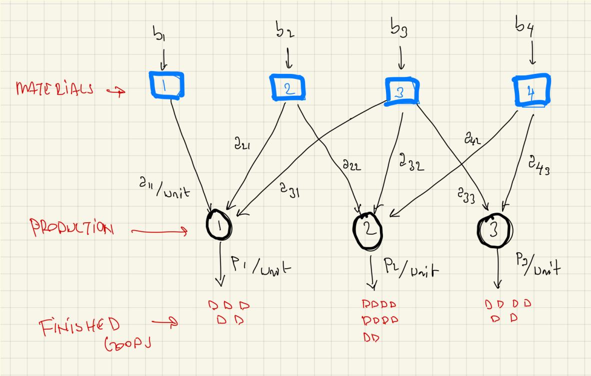

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- The decision variables x_j, denote how much jth product is produced.
- The problem can now be formulated as follows:

$$
\begin{array}{ll}
\text{maximize} & p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \\
\text{subject to} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\
& a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \\
& \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m \\
& x_1, x_2, \cdots, x_n, \geq 0
\end{array}
$$

LP notation example: Production planning

- A firm produces n different products using m different raw materials.
- The decision variables x_j, denote how much jth product is produced.

$$
\begin{aligned}
\text{maximize} \quad & p_1 x_1 + \cdots + p_n x_n \\
\text{subject to} \quad & a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i \quad & i = 1, \ldots, m \\
& x_j \geq 0 \quad & j = 1, \ldots, n.
\end{aligned}
$$

LP problem using summations:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} p_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots m \\
& x_j \geq 0 \quad j = 1, \ldots, n.
\end{aligned}
$$

Written in matrix form:

$$
\begin{aligned}
\max \quad & p^T x \\
\text{s.t.} \quad & Ax \leq b, \\
& x \geq 0
\end{aligned}
$$

$$p^T = [p_1, \ p_2, \ \cdots \ , p_n \ ]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(In this example we allow x variables to take fractional value, ex: $x_3 = 2.7$)

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Integer Programming

IP Example: Labor scheduling

- A hospital wants to make a weekly night shift (10pm-6am) schedule for its nurses.
- The demand for nurses for the night shift on day j is an integer d_j, for all $j \in \{Su, Mo, Tu, We, Th, Fr, Sa\}$.
- Every nurse works 5 consecutive days and takes the next 2 days off.
- Find the minimum number of nurses the hospital needs to hire.

Question: What are the decision variables?

IP Example: Labor scheduling

What are the decision variables?

- We could try using a decision variable y_j equal to the number of nurses that work on day j.
- However, with this definition we would not be able to capture the constraint that every nurse works 5 days in a row.
- We need to know the starting day of a nurse to model the problem correctly.
- We define x_j as the number of nurses starting their week on day $j \in \{Mo, Tu, We, Th, Fr, Sa, Su\}$.
- We can now write a constraint for every day of the week to make sure that the demand is satisfied.

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IP Example: Labor scheduling

We then have the following problem formulation:

$$
\begin{array}{rllllllll}
\text{minimize} & x_{Su} & +x_{Mo}+x_{Tu} & +x_{We}+x_{Th} & +x_{Fr}+x_{Sa} & \\
\\
\text{subject to} & x_{Su} & & +x_{We}+x_{Th} & +x_{Fr}+x_{Sa} & \geq d_{Su} \\
& x_{Su} & +x_{Mo} & +x_{Th} & +x_{Fr}+x_{Sa} & \geq d_{Mo} \\
& x_{Su} & +x_{Mo}+x_{Tu} & & +x_{Fr}+x_{Sa} & \geq d_{Tu} \\
& x_{Su} & +x_{Mo}+x_{Tu} & +x_{We} & +x_{Sa} & \geq d_{We} \\
& x_{Su} & +x_{Mo}+x_{Tu} & +x_{We}+x_{Th} & & \geq d_{Th} \\
& & +x_{Mo}+x_{Tu} & +x_{We}+x_{Th} & +x_{Fr} & \geq d_{Fr} \\
& & +x_{Tu} & +x_{We}+x_{Th} & +x_{Fr}+x_{Sa} & \geq d_{Sa}
\end{array}
$$

$$x_j \geq 0 \ \text{ and } \ \underbrace{x_j \in \mathbb{Z}}_{integer}, \quad j \in \{Mo,Tu,We,Th,Fr,Sa,Su\}$$

- This would be an LP except for the integrality constraints.

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$$
\begin{array}{rlllllllll}
\text{minimize} & x_{Su} & +x_{Mo} + x_{Tu} & +x_{We} + x_{Th} & +x_{Fr} + x_{Sa} & \\
\\
\text{subject to} & x_{Su} & & +x_{We} + x_{Th} & +x_{Fr} + x_{Sa} & \geq d_{Su} \\
& x_{Su} & +x_{Mo} & \quad\;\; + x_{Th} & +x_{Fr} + x_{Sa} & \geq d_{Mo} \\
& x_{Su} & +x_{Mo} + x_{Tu} & & +x_{Fr} + x_{Sa} & \geq d_{Tu} \\
& x_{Su} & +x_{Mo} + x_{Tu} & +x_{We} & \quad\;\; + x_{Sa} & \geq d_{We} \\
& x_{Su} & +x_{Mo} + x_{Tu} & +x_{We} + x_{Th} & & \geq d_{Th} \\
& & +x_{Mo} + x_{Tu} & +x_{We} + x_{Th} & +x_{Fr} & \geq d_{Fr} \\
& & \quad\;\; + x_{Tu} & +x_{We} + x_{Th} & +x_{Fr} + x_{Sa} & \geq d_{Sa}
\end{array}
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$$x_j \geq 0 \;\; \text{and} \;\; \underbrace{x_j \in \mathbb{Z}}_{integer}, \quad j \in \{Mo, Tu, We, Th, Fr, Sa, Su\}$$

- This would be an LP except for the integrality constraints.

Question:

What symbols do we use for numbers?

- $\mathbb{R}$ Real numbers
- $\mathbb{C}$ Complex numbers
- $\mathbb{Q}$ Fractional numbers (Quotients)
- $\mathbb{N}$ Natural numbers
- $\mathbb{Z}$ Integers

Question: Why do we use $\mathbb{Z}$ for integers?

Answer: The use of the letter Z to denote the set of integers comes from the German word Zahlen ("numbers") and has been attributed to David Hilbert. [from Wikipedia]

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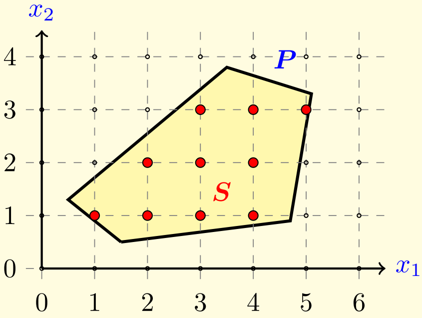
What is an Integer (Linear) Program?

A **pure integer program** (IP):

$$\begin{aligned} \min \quad & c^T x \\ \text{s. t.} \quad & Ax \geq b \\ & \;\; x \geq 0 \\ & \;\; x \;\; \text{integral (i.e., } x \in \mathbb{Z}^n). \end{aligned}$$

Feasible set: $S \;=\; \left\{ x \in \mathbb{Z}^n : Ax \geq b, x \geq 0 \right\} = \;$ *Polyhedron* $\cap\, \mathbb{Z}^n$.

When $n = 2$



Remember the labor scheduling problem

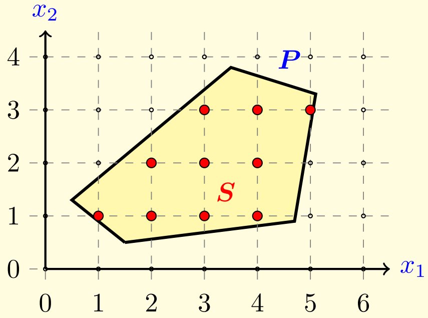
- If we ignore ("relax") the integrality constraints, we obtain the so-called LP relaxation of this problem.

$$
\begin{array}{rlllllllr}
\text{minimize} & x_{Su} & +x_{Mo} & +x_{Tu} & +x_{We} & +x_{Th} & +x_{Fr} & +x_{Sa} & \\
\text{subject to} & x_{Su} & & & +x_{We} & +x_{Th} & +x_{Fr} & +x_{Sa} & \geq d_{Su} \\
& x_{Su} & +x_{Mo} & & & +x_{Th} & +x_{Fr} & +x_{Sa} & \geq d_{Mo} \\
& x_{Su} & +x_{Mo} & +x_{Tu} & & & +x_{Fr} & +x_{Sa} & \geq d_{Tu} \\
& x_{Su} & +x_{Mo} & +x_{Tu} & +x_{We} & & & +x_{Sa} & \geq d_{We} \\
& x_{Su} & +x_{Mo} & +x_{Tu} & +x_{We} & +x_{Th} & & & \geq d_{Th} \\
& & +x_{Mo} & +x_{Tu} & +x_{We} & +x_{Th} & +x_{Fr} & & \geq d_{Fr} \\
& & & +x_{Tu} & +x_{We} & +x_{Th} & +x_{Fr} & +x_{Sa} & \geq d_{Sa}
\end{array}
$$

$$x_j \geq 0 \ \cancel{\text{and } x_j \in \mathbb{Z}}, \qquad j \in \{Su, Mo, Tu, We, Th, Fr, Sa\}.$$

- All feasible solutions to the integer program are also feasible to the LP relaxation.

LP relaxations – minimization objective



- P is the feasible set for the LP relaxation (ignore integrality)
- S is the feasible set for the integer program:

$$S = P \cap \mathbb{Z}^n$$

Question: How would

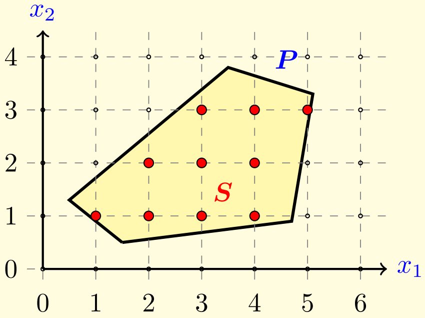
$$z_{IP} = \min c^T x \quad \text{s. t. } x \in S$$

compare to

$$z_{LP} = \min c^T x \quad \text{s. t. } x \in P \text{ ?}$$

Answer: $z_{IP} \geq z_{LP}$ $\leftarrow$ because $S \subseteq P$ and we are minimizing.

LP relaxations – minimization objective



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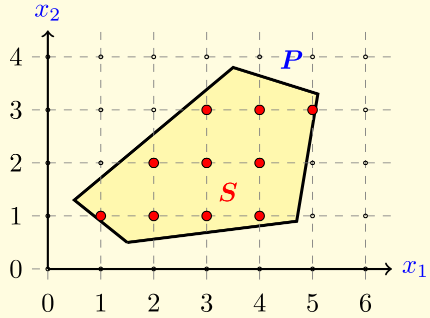

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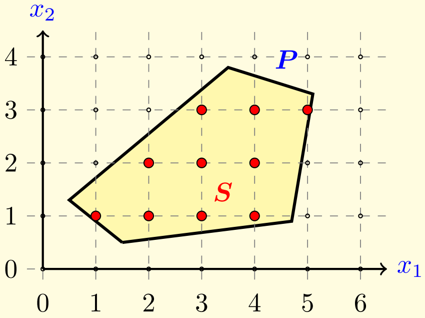

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Answer: $z_{IP} \leq z_{LP}$ $\leftarrow$ because $S \subseteq P$ and we are maximizing.

Relaxations – lower bounds (minimization objective)

- More generally, consider the following 2 optimization problems:

$$z_S = \min f(x) \quad \text{s.t. } x \in S \qquad \text{(small set)}$$

$$z_B = \min f(x) \quad \text{s.t. } x \in B \qquad \text{(big set)}$$

where we optimize the same objective function over 2 different feasible sets S and B such that

$$S \subseteq B.$$

(B is called a **relaxation** of S as it contains more solutions.)

- Assume that both problems are feasible and have an optimal solution.

Claim: $z_S \geq z_B$.

Proof. Let $x^S \in S$ be an optimal solution for the first problem.

As $S \subseteq B$ we know that that $x^S \in B$. Therefore $f(x^S)$ is an upper bound on z_B:

$$z_S = \underbrace{f(x^S) \geq \min_{x \in B} f(x)}_{\text{because } x^S \in B} = z_B$$

- In the LP/IP setting this means that the opt. LP value gives a lower bound on the opt IP value (for minimization).

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  where we optimize the same objective function over 2 different feasible sets S and B such that

$$S \subseteq B.$$

  (B is called a **relaxation** of S as it contains more solutions.)

- Assume that both problems are feasible and have an optimal solution.

  Claim: $z_S \geq z_B$.

  Proof. Let $x^S \in S$ be an optimal solution for the first problem.

  As $S \subseteq B$ we know that that $x^S \in B$. Therefore $f(x^S)$ is an upper bound on z_B:

$$z_S \; = \; \underbrace{f(x^S) \; \geq \min_{x \in B} f(x)}_{\text{because } x^S \in B} \; = z_B$$

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Relaxations – lower bounds (minimization objective)

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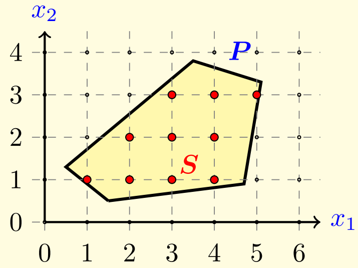
LP vs. IP (minimization objective)

Integer program v.s. its linear programming relaxation:



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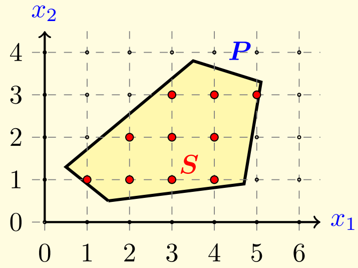

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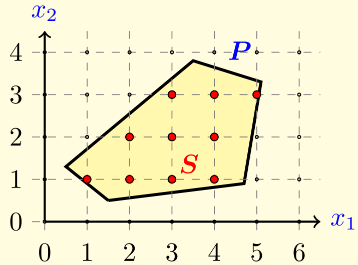

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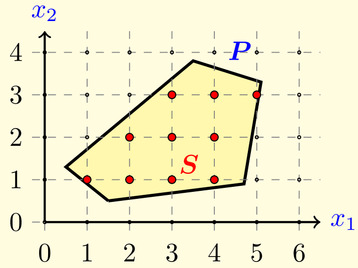

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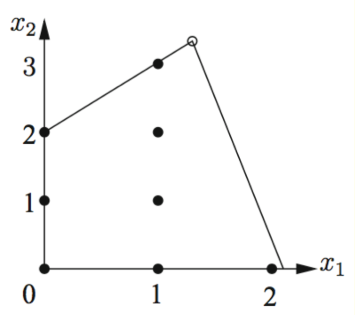
Example: Rounding LP solutions

$$\begin{aligned} \max \quad & 5.5x_1 + 2.1x_2 \\ \text{s. t.} \quad & -x_1 + x_2 \le 2 \\ & 8x_1 + 2x_2 \le 17 \\ & x_1, x_2 \ge 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$



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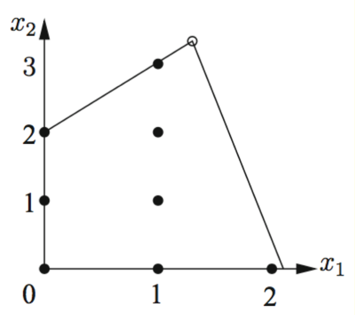

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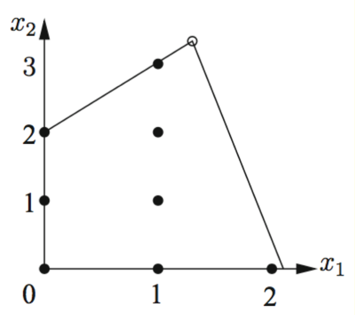

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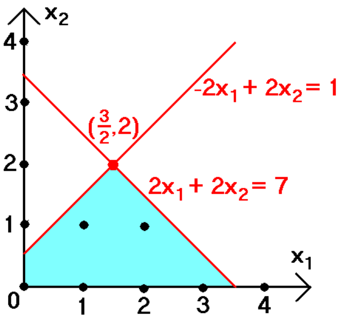
- It also looks like we can round $(1.3, 3.3)$ to the optimum solution $(1, 3)$, but...

Another example: Rounding LP solutions

$$\begin{aligned} \max \quad & 15x_2 \\ \text{s.t.} \quad & -2x_1 + 2x_2 \leq 1 \\ & 2x_1 + 2x_2 \leq 7 \\ & x_1, x_2 \geq 0 \\ & \cancel{x_1, x_2 \text{ integer}} \end{aligned}$$



- Optimal IP solutions ares $(1, 1)$ and $(2, 1)$ with obj. value 15.
- The opt. sol. of the LP relaxation is $(3/2, 2)$, with obj. value 30.
- Rounding the fractional component either up or down leads to an infeasible solution.

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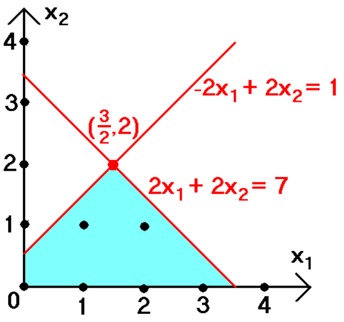

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One more example: Nurse scheduling

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\begin{array}{llllllllll}
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\\
\text{s.t.} & x_{Su} & & +x_{We} + x_{Th} & +x_{Fr} + x_{Sa} & \geq d_{Su} = 2 \\
& x_{Su} & +x_{Mo} & + x_{Th} & +x_{Fr} + x_{Sa} & \geq d_{Mo} = 2 \\
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& & + x_{Tu} & +x_{We} + x_{Th} & +x_{Fr} + x_{Sa} & \geq d_{Sa} = 2
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$$x_j \geq 0 \ \text{ and } \ \underbrace{x_j \in \mathbb{Z}}_{integer}, \quad j \in \{Su, Mo, Tu, We, Th, Fr, Sa\}$$

- Optimal LP solution is $x_j = 2/5 \ \forall j$ with cost $14/5 = 2.8$.
- Round-up solution $x_j = \lceil 2/5 \rceil = 1 \ \forall j$ is always feasible for IP (why?) with cost 7.
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Can we just round the fractional LP solution?

- We cannot always round the solution of the LP relaxation to the optimal solution of the IP:

  1. Rounding to it a feasible integer solution might be impossible
  2. Moreover, the IP can be infeasible even though the LP is feasible

     Example: $P = \{x \in \mathbb{R}^2 : 0.6 \geq x_1 \geq 0.2,\ 0.75 \geq x_2 \geq 0.41\}$
  3. Even if rounding to an integer solution is possible, the opt. IP sol. can be arbitrarily far (different) from the opt. LP sol.
  4. Even if there is an optimal integer solution "nearby", finding it might require checking **exponentially many** rounding options (up/down) as we might need to consider:

     $$\lfloor x_j \rfloor \quad \text{and} \quad \lceil x_j \rceil \qquad \text{for all fractional components } x_j$$

     There can be up to 2^n possibilities.

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Solving IPs: Branch and Bound

How do we solve an IP?

$$z^{IP} = \min \ c^T x \quad \text{s.t.} \quad Ax \geq b, \ x \geq 0, \ x \in \mathbb{Z}^n \qquad (IP)$$

- Relaxing integrality gives an LP problem (easy to solve)
- Solving LP gives an lower bound z^{LP} (for minimization)
- The optimal solution x^{LP}(to LP may have fractional components, say, $x_j^{LP} \notin \mathbb{Z}$, (eg., $x_1^{LP} = 3.31$) $\implies x^{LP}$ not feasible for (IP).

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Idea: Divide the solution set, and the IP into two new **subproblems** IP$_1$ and IP$_2$, with additional constraints

$$\text{IP}_1 : x_1 \leq \lfloor x_1^{LP} \rfloor = 3 \qquad\qquad \text{IP}_2 : x_1 \geq \lceil x_1^{LP} \rceil = 4$$

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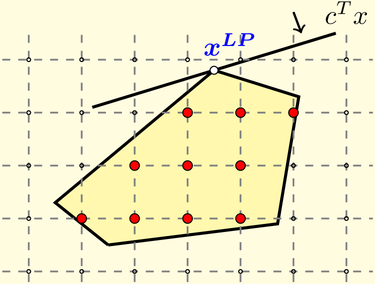
Note: **Any** feasible (optimal) solution of IP is either in IP_1 or in IP_2

Note: **No** feasible solution of LP_1 or LP_2 can have $x_1 = 3.31$

Partitioning step for integer programs



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$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & \underbrace{Ax \ge b,\ x \ge 0}_{P=\{x\in\mathbb{R}^n : Ax \ge b, x\ge 0\}} & x \in \mathbb{Z}^n \end{array}$$

- Let x^{LP} be the opt. sol. to the LP relaxation.

(IP$_1$) $\min\ c^T x\ \text{s.t.}\ x \in P_1 \cap \mathbb{Z}^n$

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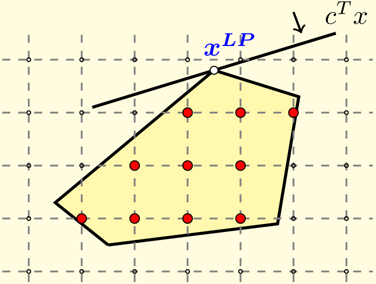

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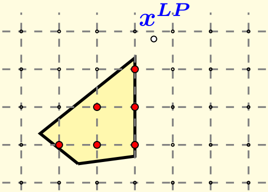

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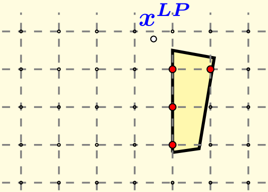

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Partitioning an IP problem into 2 subproblems

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- We have:

$$P \cap \mathbb{Z}^n = (P_1 \cap \mathbb{Z}^n) \cup (P_2 \cap \mathbb{Z}^n) \quad \longleftarrow \text{ integer points}$$

- Whereas:

$$P \supsetneq P_1 \cup P_2 \quad \longleftarrow \text{ fractional points}$$

Integer program (IP):

LP relaxation of IP:

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Integer program (IP):		LP relaxation of IP:
$z^{IP} = \min\{c^T x : x \in P \cap \mathbb{Z}^n\}$	$\geq$	$z^{LP} = \min\{c^T x : x \in P\}$
$z^{IP1} = \min\{c^T x : x \in P_1 \cap \mathbb{Z}^n\}$	$\geq$	$z^{LP1} = \min\{c^T x : x \in P_1\}$
$z^{IP2} = \min\{c^T x : x \in P_2 \cap \mathbb{Z}^n\}$	$\geq$	$z^{LP2} = \min\{c^T x : x \in P_2\}$
$z^{IP} \;=\; \min\{z^{IP1}, z^{IP2}\}$	$\geq$	$\min\{z^{LP1}, z^{LP2}\} \;\geq\; z^{LP}$

More generally: Divide and conquer principle

Consider

$$z^* = \min\{f(x) : x \in S\}$$

If a collection of disjoint sets $\{S_1, S_2, \ldots, S_k\}$ satisfy

$$S = S_1 \cup S_2 \cup \ldots \cup S_k$$

then $\{S_1, S_2, \ldots, S_k\}$ is called a partition of S

Let $z^i = \min\{f(x) : x \in S_i\}$, and $z^i \geq z^i_{LB}$ $\longleftarrow$ a lower bound

Observation 1:

$$z^* = \min\{z^1, z^2, z^3, \ldots, z^k\}$$

Observation 2:

$$z^* \leq \min\{z^1, z^3, z^8, \ldots\}$$ $\longleftarrow$ some z^is are missing here

Observation 3:

$$z^* \geq \min\{z^1_{LB}, z^2_{LB}, \ldots, z^k_{LB}\}.$$

(In branch and bound, we dynamically decide how to partition of S.)

More generally: Divide and conquer principle

Consider

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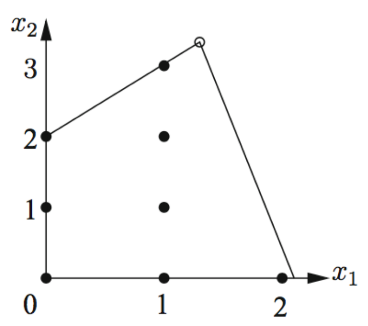
(In branch and bound, we dynamically decide how to partition of S.)

Example

$$
\begin{aligned}
\text{(IP)} \quad \min \quad & x_1 - 2x_2 \\
\text{s. t.} \quad & -x_1 + x_2 \le 2 \\
& 8x_1 + 2x_2 \le 17 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ integer}
\end{aligned}
$$



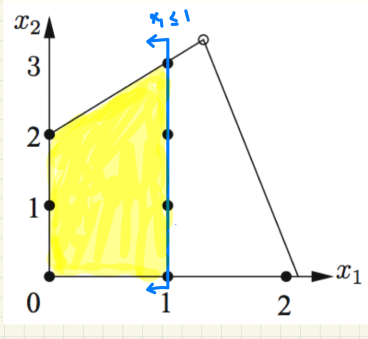
- Optimal solution to LP is $x^{\mathsf{LP}} = (1.3, 3.3)^T$ with objective $z^{\mathsf{LP}} = -5.3$
- We now have a lower bound for the IP: $\boldsymbol{L} = -5.3$
- We can branch on either x_1 or x_2
- Choose $x_1 \Rightarrow$ Create 2 subproblems by adding the constraints:

$$(i)\ x_1 \le 1 \ \text{ and } \ (ii)\ \ x_1 \ge 2$$

to the IP.

Example

$$
\begin{aligned}
\text{(IP1)} \quad \min \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \leq 2 \\
& 8x_1 + 2x_2 \leq 17 \\
& \color{darkred}{x_1 \leq 1} \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \text{ integer}
\end{aligned}
$$

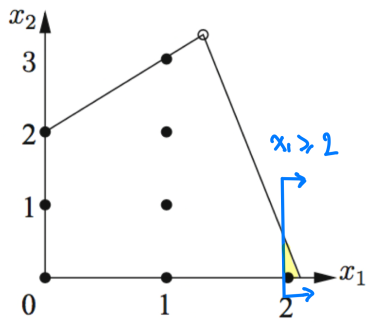


- Optimal solution to LP1 is $x^{\mathsf{LP1}} = (1, 3)^T$ with $z^{\mathsf{LP1}} = -5$
- As the LP solution is integral, we solved IP1 to optimality: $z^{\mathsf{IP1}} = -5$

  (We do not need to explore this subproblem anymore.)
- We now have an upper bound for the IP: $\boldsymbol{U} = -5$
- Let's go back to the other subproblem IP2.

Example

$$
\begin{aligned}
\text{(IP2)} \quad \min \quad & x_1 - 2x_2 \\
\text{s. t.} \quad & -x_1 + x_2 \le 2 \\
& 8x_1 + 2x_2 \le 17 \\
& x_1 \ge 2 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ integer}
\end{aligned}
$$



- Optimal solution to LP2 is $x^{\mathsf{LP2}} = (2, 0.5)^T$ with $z^{\mathsf{LP2}} = 1$
- Notice that

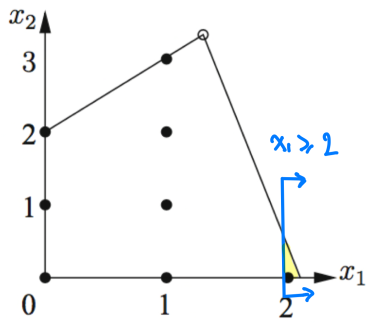
$$z^{\mathsf{IP2}} \ \ge \ z^{\mathsf{LP2}} \ = \ 1 \ > \ -5 \ = \ U$$

- There cannot be better integer solutions in this subproblem.
- We are done:

$$z^{\mathsf{IP}} = -5$$

Example

$$\begin{aligned} \text{(IP2)} \quad \min \quad & x_1 - 2x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 17 \\ & x_1 \geq 2 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$



- Optimal solution to LP2 is $x^{\text{LP2}} = (2, 0.5)^T$ with $z^{\text{LP2}} = 1$
- Notice that

$$z^{\text{IP2}} \geq z^{\text{LP2}} = 1 > -5 = \boldsymbol{U}$$

- There cannot be better integer solutions in this subproblem.
- We are done:

$$z^{\text{IP}} = -5$$