Week 1 Recap

Random experiments: Flipping a coin, rolling a die, rolling a pair of dice to record the sum, buying a mystery box containing a Smiski, all of those can be thought of as basic random experiments. In each case, we a have a clear idea of what the different possible outcomes of the experiment are. What is much less clear is what is the source of what we call randomness. For instance, if you flip a coin, there is no randomness if you can describe all the physical parameters of the flip. The laws of physics say that the coin will react again exactly in the same way if you repeat the flip in exactly the same fashion. Let us describe in details several experiments that represent key examples we would like to understand

Jade flips a coin We would like to bet on whether the result is Heads or Tails.

Peter rolls a pair of dice If the sum is 10 or more, the dragon Peter is fighting will be defeated!

The birthday problem (fancy version) How many students need to take this course in order for the probability that at least three of them share the same birthday to be at least $1/2$?

The matching problem Professor S hands back homework to the students in a completely random way. How many students will receive their own homework?

The coupon collector problem Claire wants to collect all the Smiskis that exist in the bath collection (say there are 10 of them). What is the chance that she will have to buy more than nineteen mystery box to succeed?

A **probability space** is a triplet $(\Omega, \mathcal{F}, P)$ where:

- Ω is an abstract set whose elements, ω, represents potential individual outcomes of an experiment;
- $\mathcal{F}$ is a subset of the set $\mathcal{P}(\Omega)$ of all subsets of Ω. Subsets contained in $\mathcal{F}$ represent potential *events* resulting from the experiment.
- P is a function defined on $\mathcal{F}$ which takes values in $[0,1]$. If $A \in \mathcal{F}$ is an event then $P(A)$ is the probability of that event.

Moreover, the triplet $(\Omega, \mathcal{F}, P)$ has the following properties:

- $\Omega \in \mathcal{F}$, and $\mathcal{F}$ is stable under taking complement and countable unions.
- $P(\Omega) = 1$;
- If a set A in $\mathcal{F}$ is the disjoint union of a finite or countable collection of sets $A_i, i \in I$, that is, $A = \cup_{i\in I} A_i$, $A_i \cap A_j = \emptyset$ for all $i, j,\ i \neq j$, then $P(A) = \sum_{i\in I} P(A_i)$.

The set-function $P : \mathcal{F} \to [0,1]$ is called a probability distribution on $(\Omega, \mathcal{F})$.

A **random variable,** X, is a function. $X : \Omega \to R, \omega \mapsto X(\omega)$ where R is the set in which the function X takes value (often the set of natural integers or the set of reals or a vector space, or $\{0, 1\}$, or the set of colors yellow, red, blue, green, and magenta.

Exercise: Name and describe the random variables that are relevant for the experiments mentioned above.

Simple consequences of the axioms of probability On any probability space $(\Omega, \mathcal{F}, P)$, it is always true that:

- For any subset $A \in \mathcal{F}, 0 \leq P(A) \leq 1$.
- $P(\Omega) = 1, P(\emptyset) = 0$.

For any subset $A \in \mathcal{F}$ with complement $A^c = \Omega \setminus A$, $P(A^c) = 1 - P(A)$.

- For any two subsets $A, B \in \mathcal{F}$,

$$A \subseteq B \Longrightarrow P(A) \leq P(B).$$

Inclusion-Exclusion Principle On any probability space $(\Omega, \mathcal{F}, P)$ and for arbitrary events A, B, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

More generally, for arbitrary events A, B, C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B$$

Moreover, for arbitrary events $A_i, 1 \leq i \leq k$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} P(A_{i_1} \cap \cdots \cap A_{i_j}).$$

Exercise: Why does the second equality also give the first? Apply the last equality to obtain the second equality. Write down what this says for 4 events.