Dyson
Cornell
SC Johnson College of Business

Sample Problems - Solutions
AEM 2241 - Finance

1 Finance Matters

1.1. Some of these problems are more complex that the problems you may be expected to solve during the exam. Nevertheless, you should attempt to solve them as **sub-parts** of more complex questions may be posed as exam questions. You will benefit most if you work on these problems before you look at the solutions. The course staff can answer any questions that you may have in relation to these problems.

1.2. The problem set is issued before all the material testable on the exam was taught. It may happen that some topics may not have been covered (yet) in class before the exam. If so, just skip the relevant parts.

1.3. All assumptions, conventions, and notations that we normally use can be relied without further explanations. If you use non-standard notations, explain what they mean.

1.4. Unless stated otherwise, we ask that dollar amounts be rounded to two decimals, and interest rates be rounded to four decimals. For example: \$156,798.38, \$9.75, 0.0315 = 3.15%, 0.1425=14.25%.

1.5. Be careful to distinguish between per-period quantities, such as per-period coupon payments and interest rates, and their annualized versions, which are the ones that must be typically provided as results.

1.6. [When working on paper] Unless we tell you otherwise, you may use either formulas or financial calculators to solve a problem. Whether you are using formulas or calculators, you must show what you did (e.g. what formulas you used, what values you replaced, or what calculator buttons you pushed and what values you entered), and **you must briefly explain the logic of your solutions**.

1.7. [When working on paper] To eliminate any possible ambiguity, whenever using the calculator, you must indicate values for all 5 TVM buttons; except for the value that you compute. Indicate explicitly what buttons you pushed to get a result, and what the result is.

1.8. Whenever possible, interpret the meaning of the results in terms consistent with the problem.

Good Luck!

2 Present Value Calculations

Each row of the table below represents two cash flows, one being the present value of the other when viewed as an investment over a time horizon t, at an interest rate r. One or two numbers are missing in each row.

2.1. For rows (a) and (c) compute the value of the missing number using the formulas given in class. Write down the suitable general formulas first, then replace letter symbols with known numerical values; finally, compute the respective results.

2.2. For rows (b) and (d), use a financial calculator. Similar to what we did in class, show the values that you would set up for the TVM variables, then show what key combinations you would press to get to the solution; also, provide the calculator's answer.

2.3. Is there anything special about the problem in row (e)? If so, what is it? Explain, in no more than two sentences, how a situation like the one shown may arise. For this part you can use formulas or the calculator, but you must state what you did and show the steps that lead to the solution.

2.4. For part (f), where there are two missing numbers, determine, using formulas, **two** combinations of values for PV and r that would make the connection between the four variables in row (e) correct. Note: You need to provide two pairs of numbers, (PV_1, r_1) and (PV_2, r_2), which are both consistent with the data in row (f). You may be able to determine part of the answer by making a choice, as the problem is not fully determined.

Part	**PV [$]**	**t [years]**	**r [%]**	**FV [$]**
(a)	7,513	7	9	
(b)		29	13	48,318
(c)	48,000	15		185,000
(d)	18,400		9	289,715
(e)	200,000	5		175,415
(f)		8		89,980

Solution:

2.1. For row (a) we have: $FV = PV \cdot (1+r)^t = 7,513 \cdot (1+0.09)^7 = \$13,734.06$.

For row (c) we have: $r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1 = \left(\frac{185{,}000}{48{,}000}\right)^{\frac{1}{15}} - 1 = 0.0941 = 9.41\%$.

2.2. For row (b) we have: $N = 29$, $I/Y = 13$, $FV = 48,318$, $PMT = 0$. Now pressing $CPT\ PV$ yields the result: $-\$1,395.84$.

For row (d) we have: $I/Y = 9$, $PV = 18,400$, $PMT = 0$, $FV = -289,715$. Now pressing $CPT\,N$ yields the result: $31.99 \approx 40$ periods.

Observations:

Payment is set to 0 by default after resetting the calculator, or after clearing the TVM registers. If you did not indicate $PMT = 0$, but the solution is otherwise correct, we will accept it.

We set the future value to be negative, and the present value to be positive. With these signs, the PV can be interpreted as an inflow (say, loan), and the FV as an outflow (say, the loan repayment). Choosing opposite signs for FV and PV, respectively, would have also been acceptable. Is it not acceptable to set both cash flows to have the same signs, as the calculator would show an error.

Both these observations hold in all similar situations, but we will not repeat them every time.

2.3. In the case of row (e), the future value is **less** than the present value, which implies negative interest rates.

Using formulas, we have: $r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1 = \left(\frac{175{,}415}{200{,}000}\right)^{\frac{1}{5}} - 1 = -0.0259 = -2.59\%$.

Using a calculator, we set $N = 5$, $PV = 200,000$, $FV = -175,415$, then press $CPT\,I/Y$ to get -2.59%.

2.4. There are four variables that appear in the formula that connects future values to present values. Given any three of these, we can determine the fourth one. Given only two of them, however, the formula on its own does not determine the two missing quantities. The easiest way to handle this problem is to choose one quantity arbitrarily, and then compute the other one using either the formula, or the calculator. We illustrate both methods below:

Using a formula: We choose $r = 10\%$ and compute $PV = \frac{FV}{(1+r)^t} = \frac{89{,}980}{(1+0.10)^8} = \$41,976.33$.

Using the calculator: We choose, say, $PV = \$30,000$. We proceed as usual and set $N = 8$, $PV = 30,000$, $PMT = 0$, $FV = -89,900$. We get the answer of $r = 14.70\%$ after pressing $CPT\,I/Y$.

The two pairs that we generated are $(PV_1, r_1) = (\$41,976.33, 10\%)$ and (PV_2, r_2)=$(\$30,000, 14.70\%)$.

Any combination of methods to generate the two pairs is acceptable. Signs are needed for calculator computations, if you performed them, but were not required in the answers. If you provided signs when you specified the pairs of values, they are acceptable.

3 Simple vs. Compound Interest Rates

Consider an investment of \$10,000 that you make at the end of year 0. You are guaranteed an interest rate of 7.5% per annum for 10 years, compounded annually. We say that two interest rates are equivalent over a given time horizon t if the total interest earned by time t is the same in both cases.

3.1. Given a time horizon of $t = 5$ years, what is the simple interest rate that is equivalent to the compound interest rate described above?

3.2. Given a time horizon of $t = 10$ years, what is the simple interest rate that is equivalent to the compound interest rate described above?

3.3. Do the answers to the two questions above depend on your initial investment, assuming that the terms of the investment remain otherwise the same? Why?

3.4. Is it possible to have the same simple interest rate be equivalent to a given positive compound interest rate **simultaneously** over **several** investment periods longer than one year?

Solution:

The total interest TI earned using compound interest is $TI = FV - PV = PV \cdot \left[(1+r)^t - 1\right]$. We recall that "interest does not earn interest" when we use simple interest rates. Thus if the simple interest rate is r_s, the annual interest earned is always $PV \cdot r_s$. Over t time periods, we earn $PV \cdot r_s \cdot t$ total interest. We equate the two quantities: $TI = PV \cdot \left[(1+r)^t - 1\right] = PV \cdot r_s \cdot t$. We immediately get the simple interest rate: $r_s = \frac{(1+r)^t - 1}{t}$.

3.1. $r_s = \frac{(1+r)^t-1}{t} = \frac{(1+0.075)^5-1}{5} = 0.0871 = 8.71\%$.

3.2. $r_s = \frac{(1+r)^t-1}{t} = \frac{(1+0.075)^{10}-1}{10} = 0.1061 = 10.61\%$.

3.3. Given the assumptions that we use in class, and given the context of our problem, future values are strictly proportional to present values irrespective of the interest rate convention that we use. This implies that total interest earned is also proportional to present values. Because of this proportionality, when we equate the total interest earned using the two interest rate conventions, the present value will appear as a factor on both sides. We can simplify by dividing both sides with the present value, thus eliminating the potential dependency of the answer on the size of the initial investment.

3.4. If the compound interest rate is 0, then we immediately get that $r_s = 0$, and the answer does not depend on the period t. It is thus possible for the simple interest rate to be equivalent to the compound interest over multiple (in fact: any) time periods. This case was, however, excluded by the problem's text.

The solutions to the first two parts of this problem seem to suggest that the answer may be negative. A full analysis is mathematically more involved than the typical material in this course, so we do not provide it here. We will accept any reasonable, less than fully formal argument, such as the one below.

One can, for example, use Excel to plot the value of $r_s = \frac{(1+r)^t - 1}{t}$ for $t = 1, 2, 3, ...$, for several values of $r > 0$. For all positive values of r, all lines are upward sloping, which means that no matter what r was chosen, the equivalent r_s is always increasing as t increases. For fixed t, the slope will be greater the greater r is. To satisfy the condition of the problem, a horizontal line would have to cross the line corresponding to a given r more than once. However, this is not possible. A similar analysis can be made for $r < 0$, though this was not required.

It would have been acceptable not to use a general formula, like above, but a step-by-step calculation. For part (3.2), say, we compute the total interest TI earned using compound interest as $TI = FV - PV = PV \cdot \left[(1+r)^t - 1\right] = 10,000 \cdot \left[(1+0.075)^{10} - 1\right] = \$10,610.32$. We then use equation $TI = PV \cdot r_s \cdot t = \$10,610.32$ to determine $r_s = \frac{10{,}610.32}{10{,}000 \cdot 10} = 10.61\%$. We get the same answer, but it is harder to see that the answer does not depend on the initial investment. We could check that by computing r_s for several PV values, then generalizing our empirical observations. This is not the approach we recommend, though we would be flexible if you submitted a solution along these lines.

4 Discounted Cash Flow Valuation

The present value (at time 0) of the cash flow stream below is \$7,500 when discounted at 9% annually. What is the value of the missing cash flow? What would be the missing cash flow if the present value were \$4,500? Beside magnitude (the amount of dollars), is there anything different between the unknown cash flows in these two situations? Interpret this difference. Use formulas.

Year	Cash Flow [$]
1	1,700
2	x
3	2,450
4	2,980

Solution:

We compute the present value (at time 0) of each cash flow, and then we set up an equation by equating the known total present value to the sum of present values:

Year	Cash Flow [$]	Present Value [$]
1	$1,700$	$\frac{1{,}700}{1+0.09} = 1,559.63$
2	x	$\frac{x}{(1+0.09)^2}$
3	$2,450$	$\frac{2{,}450}{(1+0.09)^3} = 1,891.85$
4	$2,980$	$\frac{2{,}980}{(1+0.09)^4} = 2,111.11$
	Total	$\frac{x}{1.09^2} + 5,562.59$

We now write equation $\frac{x}{1.09^2} + 5,562.59 = 7,500$, from which we determine $x = \$2,301.84$. If the present value is $\$4,500$, we get that $x = -\$1,262.46$.

The signs of the two unknown cash flows are different in the two cases considered. If we follow the usual sign convention, we can say that future cash flows are all inflows when their total present value is $\$7,500$; however, there is a cash outflow in year 2 in the case when the total present value of the cash flows is $\$4,500$.

Observations:

Using the calculator to solve this problem would not be acceptable, since the text specified "use formulas." However, you could have used a calculator to check your results.

It is possible to write more general formulas that allow for the calculation of the unknown cash flow: $PV = \sum_{i=1}^{4} \frac{CF_i}{(1+r)^i}$, where CF_i is the cash flow received/paid at time i. Thus $CF_2 = (1+r)^2 PV - (1+r) CF_1 - \frac{CF_3}{1+r} - \frac{CF_4}{(1+r)^2}$. Of course, by replacing variables with their corresponding values in this formula, we get the same answers as before.

5 Time Value of Money

Use formulas to solve this problem, except, possibly, for the first and last sub-part.

The 6-month interest rate is 3.5%. Consider two successive six-month periods.

5.1. Fill out the statement below relying on our usual conventions, so that it is consistent with the problem text.

"The interest rate is __**7**__% per year, compounded semi-annually."

Solution:

If the yearly rate is r, and it compounds m times a year, then the per-period rate is $r' = \frac{r}{m}$. Conversely, if the per-period rate is r', then the yearly rate must be $r = m \cdot r'$. In this case $r' = 3.5\%$, $m = 2$, and thus $r = 2 \cdot 3.5\% = 7\%$.

5.2. Compute the EAR.

Solution:

$$EAR = \left(1 + \frac{r}{2}\right)^2 - 1 = \left(1 + r'\right)^2 - 1 = (1 + 0.035)^2 - 1 = 7.12\%.$$

5.3. What is the present value at time 0 of \$500 to be paid at the end of the second 6-month period?

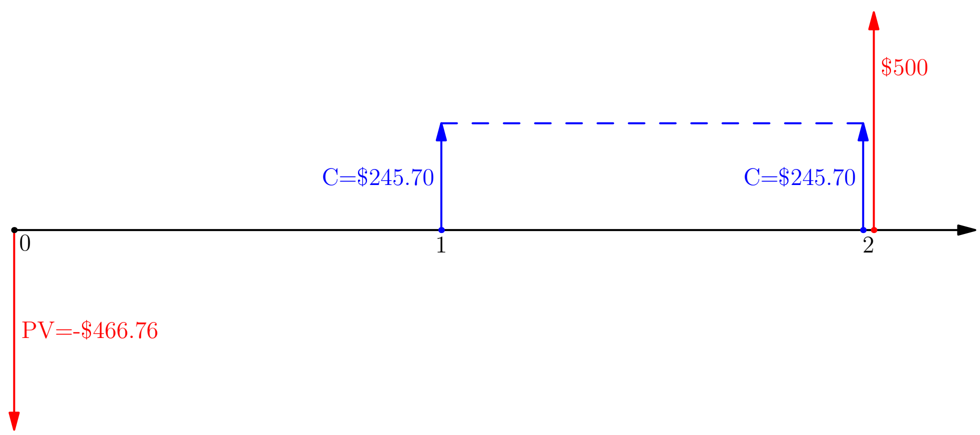
Solution:

$$PV = \frac{FV}{(1+r)^2} = \frac{500}{(1+0.035)^2} = \$466.76.$$

5.4. Now consider two equal payments made at the end of the first and second six-month periods, respectively. How big should these payments be so that their total present value at time 0 is equal to the value obtained in part (5.3) above?

Note: If you need the result of part (5.3), but you cannot calculate it, then use \$400 for the needed total present value.

Solution:

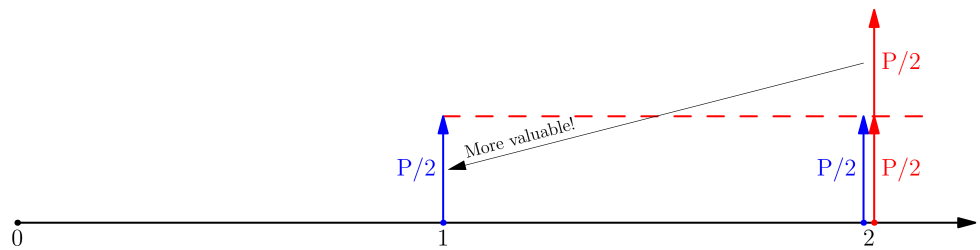


We can see the two cash flows as representing a regular annuity with a maturity of two six-month time periods.

$$PV = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^2}{r} \Rightarrow C = \frac{r}{1-\left(\frac{1}{1+r}\right)^2} \cdot PV = \frac{0.035}{1-\left(\frac{1}{1+0.035}\right)^2} \cdot 466.76 = \$245.70.$$

5.5. More abstractly, consider a payment P made at the end of the second 6-month period and its time-0 present value PV_0. Separately, consider two payments of size P/2 made at the end of the first and second six-month period, respectively, as well as their total present value $PV_0^{'}$. Can you determine, relying either on a qualitative argument **or** on formulas, whether PV_0 or $PV_0^{'}$ is bigger under ordinary circumstances, when interest rates are positive?

Solution:



The one-payment and the two-payment case overlap in that both pay (at least) $\frac{P}{2}$ at the end of the year. These cash flows have the same present value, since they have the same size and timing. In both cases an extra cash of size $\frac{P}{2}$ also occurs. In the two-payment case, the payment occurs after one period, while in the other case it occurs after two periods. When rates are positive, a cash flow closer in time to time 0 will have a higher present value than a cash flow of the same size that is farther in time. Hence, the total present value of the two cash flows must be greater than the present value of the one cash flow, even though the total cash flows, expressed in nominal dollars, are the same in both cases.

6 Financial Product

You purchased (invested in) a financial product that will pay you \$500,000 every six months for 4 years, starting exactly six months from today. The relevant interest rate is 12% per year, compounded quarterly.

6.1. Given the frequency of payments and the quarterly compounding interest, which of the following interest rates is closest to the rate you must use to discount the first cash flow you will receive?

- ✗ 3%
- ✗ 4%
- ✓ 6%
- ✗ 12%

Answer:

The quarterly interest rate is $r_3 = \frac{0.12}{4} = 3.00\%$. The first payment will be made in exactly six months, so the relevant rate is $r_6 = (1 + r_3)^2 - 1 = 6.09\%$.

While it was not requested, we compute the present value of the first \$500,000, which is $PV = \frac{500{,}000}{1+r_6} = \frac{500{,}000}{1+0.0609} = \$471,297.95$.

6.2. The financial product that you purchased will be phased out (eliminated) and replaced with a similar product which has only 4 annual payments of \$1,000,000, with the first payment arriving exactly one year from now. From your perspective, is the new contract more, or less valuable than the old contract?

- ✗ The new contract is more valuable.
- ✗ The two contracts are equally valuable.
- ✓ The new contract is less valuable.

Answer:

From the perspective of the person who receives the payments ("you"), this new contract is less valuable.

Solution 1: The answer is obvious even in the absence of any calculations, as the net effect of the change is to move some payments later than their time of payment under the original contract. For example, the payment due in 6 months will be made in 12 months, and will be paid out jointly with the \$500,000 originally due at that time. So half of the 8 original \$500,000 payments, specifically those due at 6, 18, 30, and 42 months will be pushed back to 12, 24, 36, and 48 months, respectively, and thus will be less valuable. The \$500,000 payments that would have been made at 12, 24, 36, and 48 months originally are still paid out at the original dates (but they are merged with the payments that have been delayed).

Solution 2: A quantitative argument is not necessary, but can easily be provided. Both contracts are simple annuities that we can value using standard formulas. Let $C_{original}$ be the size of the original \$500,000 payments, C_{new} the size of the new \$1,000,000 payments, $r_6 = \left(1 + \frac{0.12}{4}\right)^2 - 1 = 6.09\%$ the per-6-month period interest rate, and $r_{12} = \left(1 + \frac{0.12}{4}\right)^4 - 1 = 12.55\%$ the per-12-month period interest rate. We have:

$$PV_{original} = C_{original} \cdot \frac{1 - \left(\frac{1}{1+r_6}\right)^8}{r_6} = \$500,000 \cdot \frac{1 - \left(\frac{1}{1+0.0609}\right)^8}{0.0609} = \$3,093,067.49$$

$$PV_{new} = C_{new} \cdot \frac{1 - \left(\frac{1}{1+r_{12}}\right)^4}{r_{12}} = \$1,000,000 \cdot \frac{1 - \left(\frac{1}{1+0.1255}\right)^4}{0.1255} = \$3,002,443.10$$

As can be seen, $PV_{original} > PV_{new}$.

6.3. In a different scenario, the financial product you invested in initially will be eliminated, and will be replaced with a similar contract. This time, however, you will receive four \$1,000,000 payments, with the first one due in exactly 9 months. The second, third, and fourth payments will each be made exactly one, two, and three years after the first payment, respectively (i.e., payments following the first one will be made at the ends of successive one-year intervals following the first payment).

To make the exchange fair, if the value of the new contract to you is greater than that of the old contract, you will have to pay the difference in contract values to your counterparty. Should the value of the new contract be less than the value of the old contract, you will receive the difference in contract values from your counterparty. If the old and new contract have the same value, no money exchanges hands.

What will happen?

- ✗ The answer cannot be determined.
- ✗ No money will exchange hands.
- ✓ You will receive a payment from your counterparty.
- ✗ You will make a payment to your counterparty.

Answer:

Solution 1: For many students, the most straightforward approach is to compute the numerical value of the two contracts. In fact, we did almost all the work when answering the question above. The original contract has a time-0 value of $PV_{original}$. The new contract introduced in this part will have a time-0 value of PV_{new2}. However, if we valued the new contract at a time 3 months into the past (i.e., 3 months before time-0), then the present value of the new contract in this part would be equal to PV_{new} from above (think about it: four payments of \$1,000,000 are paid out at at yearly intervals versus the reference time "3 months ago"). To get PV_{new2} we must compute the future value at time 0 of the new contract's value at time "3 months ago": $PV_{new2} = PV_{new} \cdot (1 + r_3) = \$3,002,443.10 \cdot \left(1 + \frac{0.12}{4}\right) = \$3,092,516.39$. We conclude that the new contract is less valuable than the original contract from the perspective of the person receiving the money, and that this person ("you") must receive a payment from the counterparty.

Solution 2: The answer can also be determined without numerical calculations, though it is perhaps not as obvious as that for the question above. The explanation below is long because we would like you to understand how such an argument can be developed. With practice, you will be able to almost instantaneously determine the answer in situations like this.

Consider the first two payments of the original contract, the one made at 6-months, and the one made at 12-months. These two payments are equal. The first payment is pushed back by 3 months (from 6 months to 9 months), while the second payment is brought forward by 3 months (from 12 months to 9 months). The original 6 month payment loses value when pushed back to 9 months, the original 12 month payment gains value when brought forward to 9 months. The question we need to answer is what is the net effect of the value loss and of the value gain.

Let $r_3 = \frac{0.12}{4} = 3\%$ be the quarterly interest rate and let $f_N = \frac{1}{(1+r_3)^N}$ be the discount factor resulting from a cash flow that is paid N quarters into the future (N is equal to 2, 3, and 4 for payments made at 6, 9, and 12 months, respectively). We know from observations and remarks made in the class about discount factors, but also from the general properties of function $h(x) = \frac{1}{a^x}$, where $a > 1$, and $x \geqslant 0$, that this function decreases steeply, but then flattens out as the values of x grow. We immediately infer that $f_2 - f_3 > f_3 - f_4$; this is because there is the same distance in months (3 months) between 6 and 9 months, and 9 and 12 months, respectively, but the decrease in discount factors over 3 months is steeper to the "left" of 9 months (i.e., toward time 0), than to the right of this new payout time. We can rewrite the inequality by moving two discount factors from one side to the other side: $f_3 - f_4 - (f_2 - f_3) < 0$, which then becomes $2 \cdot f_3 - f_2 - f_4 < 0$.

For the first two payments of the old contract, the change in value from moving them both to 9 months is $C_{original} \cdot (f_3 - f_2) + C_{original} \cdot (f_3 - f_4)$, which is equal to $C_{original} \cdot (f_3 - f_2 + f_3 - f_4) = C_{original} \cdot (2 \cdot f_3 - f_2 - f_4) < 0$.

Now, the same intuition holds for the pair of payments originally scheduled for payout at 18 and 24 months, respectively, which are both moved to a payout date of 21 months. Again, this leads to a net decrease in value. The same holds for pairs of payments scheduled to be paid out at 30 and 36 months, and 42 and 48 months, respectively.

The new contract is thus less valuable from the perspective of the person receiving the payments ("you"). You must receive money.

This argument has been developed in great detail to illustrate that you can decide many problems of this type without having to perform numerical calculations, or even any calculations, assuming that you have practiced enough to build intuition in the area of NPV computations. Indeed, this line of reasoning can be pursued as a mental exercise, while the numerical solution given earlier is not amenable to such an approach.

Many seasoned businesspersons stress the importance of being able to think rapidly and in intuitive terms about time-value-of-money and general NPV matters. Social conventions and other practicalities often prevent negotiating parties from using calculators, computers, or phones during meetings - these intuitions, however, can be deployed rapidly, and they can guide one's negotiating strategy.

Solution 3: We can be more mathematical with our approach to the second solution; for example, we can show algebraically that the original first two payments lose value when they are moved. Let Δ be the change in value of these two payments only:

$$\begin{aligned}
\Delta &= C_{original} \cdot (2 \cdot f_3 - f_2 - f_4) \\
&= C_{original} \cdot \left[2 \cdot \frac{1}{(1+r_3)^3} - \frac{1}{(1+r_3)^2} - \frac{1}{(1+r_3)^4}\right] \\
&= C_{original} \cdot \frac{2 \cdot (1+r_3) - (1+r_3)^2 - 1}{(1+r_3)^4} \\
&= C_{original} \cdot \frac{2 + 2r_3 - 1 - 2r_3 - r_3^2 - 1}{(1+r_3)^4} \\
&= -C_{original} \cdot \frac{r_3^2}{(1+r_3)^4} < 0.
\end{aligned}$$

There are three other pairs of payments that can also be treated analytically, of course. The analytic solution makes it particularly easy to notice that the decrease in value would hold for any payment size and interest rates, and it is not specific to the combination of specific values chosen.

7 You Won the Lottery!

You have just won the lottery and will receive 10 yearly payments, as follows: you get $1,500,000 in the first year, after which yearly payments will increase by 2.7% per year. A company specializing in purchasing annuities (yes, they do exist!) offers you instant $14,000,000 in cash to purchase the right to receive your winnings. The relevant interest rate is 3% per year. Will you take the offer?

Solution:

This is a growth annuity, its value can be computed using the formulas given in class:

$$PV = C \cdot \frac{1 - \left(\frac{1+g}{1+r}\right)^t}{r-g} = 1,500,000 \cdot \frac{1 - \left(\frac{1+0.027}{1+0.030}\right)^{10}}{0.030 - 0.027} = \$14,373,706.12.$$

Since the present value of the annuity exceeds that of the offer by almost $375,000, you should refuse the offer.

Annuities are purchased routinely by specialized firms. Because of factors that we do not consider here, such as taxes, estate planning, desire to access wealth early, and others, an offer may imply a steep discount to the present value of the annuity, yet still be acceptable to an annuity seller.

Calculator Solution:

A calculator solution requires some preparation, and it is more complicated than the direct evaluation of the formula above. Let $\frac{1+g}{1+r} = \frac{1}{1+u}$; this implies that $u = \frac{r-g}{1+g} = \frac{0.030-0.027}{1+0.027} = 0.29\%$. Also, note that $r - g = u\,(1+g)$. We rewrite the general formula: $PV = C \cdot \frac{1-\left(\frac{1+g}{1+r}\right)^t}{r-g} = \frac{C}{1+g} \cdot \frac{1-\left(\frac{1}{1+u}\right)^t}{u} = C' \cdot \frac{1-\left(\frac{1}{1+u}\right)^t}{u}$, where $C' = \frac{C}{1+g} = \frac{1,500,000}{1+0.027} = \$1,460,564.75$. Now we can price this as a regular annuity: N=10; I/Y=0.29; PMT=1,460,564.75; FV=0. Press CPT PV to get -14,375,364.50. Thus $PV = \$14,375,364.50$. Note the small difference due to rounding the per-period interest rate to 0.29%; use the more precise value of 0.292113% to get a better match!

8 Annuity, But You Have to Wait

Consider an annuity with a yearly payment of $\$75,000$ that makes its first payment at the end of year 5, and consists of 10 payments. Assume that the appropriate interest rate is 11% per year.

8.1. Using formulas for annuities, value this annuity (compute its present value) as of the end of year 4 (i.e. as if time had passed, and "now" were at the end of year 4); name this value PV_4.

8.2. Using formulas for annuities, value this annuity as of the end of year 5; name this value PV_5. What kind of annuity is this when viewed from the end of year 5?

8.3. Compute the present value of this annuity at time 0 in two different ways, starting separately from PV_4 and PV_5, respectively. Compare and comment very briefly on the two answers that you get.

Solution:

8.1. As we look forward in time "standing" at the end of year 4, we see a regular annuity with a maturity of 10 years; we can price it immediately: $PV_4 = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r} = 75,000 \cdot \frac{1-\left(\frac{1}{1+0.11}\right)^{10}}{0.11} = \$441,692.40$.

8.2. The annuity is an annuity due when viewed from time 5. To compute PV_5, we can just roll forward PV_4 and compute its future value one period later: $PV_5 = PV_4 \cdot (1+r) = 441,692.40 \cdot (1+0.11) = \$490,278.56$. Alternatively, we can compute PV_5 as the value of an annuity due with a maturity of 10 years, valued at time 5: $PV_5 = C \cdot \frac{1+r}{r} \cdot \left[1 - \left(\frac{1}{1+r}\right)^t\right] = 75,000 \cdot \frac{1+0.11}{0.11} \cdot \left[1 - \left(\frac{1}{1+0.11}\right)^{10}\right] = \$490,278.56$.

8.3. We compute the present value of PV_4 at time 0 first: $PV = \frac{PV_4}{(1+r)^4} = \frac{441,692.40}{(1+0.11)^4} = \$290,956.46$. The present value of PV_5 at time 0 is $PV' = \frac{PV_5}{(1+r)^5} = \frac{490,278.56}{(1+0.11)^5} = \$290,956.46$. The two present values are equal: $PV = PV'$, which shows that our calculations have been consistent. If computing present values "passing" through different intermediate moments of time gave different results, our methodology would not be correct.

The equality of present values can be seen directly from the general formulas:

$$PV = \frac{PV_4}{(1+r)^4} = \frac{C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r}}{(1+r)^4} = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^{10}}{r \cdot (1+r)^4},$$

$$PV' = \frac{PV_5}{(1+r)^5} = \frac{C \cdot \frac{1+r}{r} \cdot \left[1-\left(\frac{1}{1+r}\right)^t\right]}{(1+r)^5} \cdot = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^t}{r \cdot (1+r)^4}.$$

We are used to thinking about present values as being computed at time 0. This problem makes it clear that we can compute present values using any reference times that may be convenient.

Calculator Solution:

We use the same notations as above; also, we do not repeat the explanations and reasoning given above. For brevity, we do not interpret or explain the choices of cash flow signs.

8.1. N=10; I/Y=11; PMT=75,000; FV=0. Press CPT PV to get -441,692.40. Thus $PV_4 = \$441,692.40$.

8.2. First, set the calculator so that cash flows arrive at the start of time periods: 2ND BGN; 2ND SET.
N=10; I/Y=11; PMT=75,000; FV=0. Press CPT PV to get -490,278.56. Thus $PV_5 = \$490,278.56$.
We pushed the same buttons, but we obtained different results. This is because of the change in the timing of cash flows.

8.3. Now you must reset or otherwise change back the setting about the timing of the cash flow: 2ND RESET, or 2ND BGN, 2ND SET.
N=4; I/Y=11; PMT=0; FV=-441,692.40. Press CPT PV to get 290,956.46. Thus $PV = \$290,956.46$.
N=5; I/Y=11; PMT=0; FV=-490,278.56. Press CPT PV to get 290,956.46. Thus $PV' = \$290,956.46$.

9 One or Several?

Consider an annuity payment that will pay 75,000 every six months for 5 full years. Afterwards, the payment will increase to 100,000 every six months, for 5 more years. The relevant interest rate is 7.5% per annum, compounded semi-annually.

9.1. Explain how this annuity is equivalent to the difference of **two** regular annuities, each starting at time 0, one with a maturity of 10 years, one with a maturity of 5 years. Specify in full the details of the two component annuities.

9.2. Using the observation made in the previous part, value the component annuities, and combine these values to get the value of the composite annuity. If you could not identify the two annuities into which the initial annuity can be decomposed, feel free to use any other method to value the initial annuity.

Note: Many complex financial instruments can be seen as collections of simpler instruments that can be understood and/or valued in isolation. In such cases the value of the complex financial instrument emerges by the suitable aggregation of the values of its respective components.

Solution:

9.1. Consider a regular annuity A_1, which has a maturity of 10 years, starts at time 0, and has constant payments of \$100,000. Now consider a regular annuity A_2, which has a maturity of 5 years, starts at time 0, and has constant payments of $\$25,000 = 100,000 - 75,000$. Consider a person who **receives** annuity A_1 and **pays** annuity A_2; the net effect will be to produce the cash flow pattern described in the problem's text.

9.2. Annuity A_1 has $C_1 = \$100,000$, $t_1 = 10$ ($2 \cdot 10 = 20$ semi-annual intervals), per-period interest rate $\frac{r}{2} = \frac{7.5\%}{2} = 3.75\%$: $PV_1 = C_1 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_1}}{\frac{r}{2}} = 100,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{20}}{\frac{0.075}{2}} = \$1,389,620.42$. Annuity A_2 has $C_2 = \$25,000$, $t_2 = 5$ ($2 \cdot 5 = 10$ semi-annual intervals), same per-period interest rate: $PV_2 = C_2 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_2}}{\frac{r}{2}} = 25,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{10}}{\frac{0.075}{2}} = \$205,319.68$. The present value of the cash flows actually received is $PV = PV_1 - PV_2 =$\$1,184,300.74.
It is possible to solve the problem in a different way, without necessarily being able to answer the prior question. We can see the cash flows received as consisting of two regular annuities, one, A_3, paying a constant semi-annual $C_3 = \$75,000$, starting at time 0, with a maturity of $t_3 = 5$ years, the other, A_4, paying a constant semi-annual $C_4 = 25,000$, starting at the end of the fifth year, with a maturity of $t_4 = 5$ years. The present value (at time 0) of A_3 is easy to calculate: $PV_3 = C_3 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_3}}{\frac{r}{2}} = 75,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{20}}{\frac{0.075}{2}} = \$1,042,215.32$. The present value of A_4 **at the end of year 5**, which we know how to compute, must be discounted back to time 0. We get $PV_4 = \frac{1}{\left(1+\frac{r}{2}\right)^{10}} \cdot C_4 \cdot \frac{1-\left(\frac{1}{1+\frac{r}{2}}\right)^{2t_4}}{\frac{r}{2}} = \frac{1}{\left(1+\frac{0.075}{2}\right)^{10}} \cdot 25,000 \cdot \frac{1-\left(\frac{1}{1+\frac{0.075}{2}}\right)^{10}}{\frac{0.075}{2}} = \$142,085.42$. This time the present value of the cash flows to be received is the **sum** of the present values of A_3 and $A_4 : PV = PV_3 + PV_4 = 1,042,215.32 + 142,085.42 = \$1,184,300.74$. As before, the two answers computed by using different approaches are identical.

Calculator Solution:

We use the same notations as above; also, we do not repeat the explanations and reasoning given above. For brevity, we do not interpret or explain the choices of cash flow signs.

9.1. Same as above.

9.2. N=20; I/Y=3.75; PMT=100,000; FV=0. Press CPT PV to get -1,389,620.42. Thus $PV_1 = \$1,389,620.42$.

N=10; I/Y=3.75; PMT=25,000; FV=0. Press CPT PV to get -205,319.68. Thus $PV_2 = \$205,319.68$.
As above, the final answer is the difference $PV = PV_1 - PV_2$.
PV_3 and PV_4 can be computed similarly, but then PV_4 must be discounted back to time 0, similar to the approach we took in solution 8.3.

10 Amortizing loans with variable payments

Consider an amortizing loan with **variable** payments. The initial loan amount is \$10,000,000, while the initial loan maturity is 30 years. The borrower must make monthly payments, with payments due at the end of each month. The borrower's total payment on this loan that is due at the end of the 73^{rd} month equals \$87,777.78.

10.1. What is the monthly principal repayment on this loan?

Solution:

Each month the same amount of principal is repaid (in addition to the entire interest due at that time). Let the initial loan amount be L and the number of months over the whole duration of the loan be N=360.

$$\text{monthly repayment=p=}\frac{L}{N} = \frac{10,000,000}{360} = \$27,777.78.$$

10.2. What is the balance outstanding on this loan at the beginning of the 73^{rd} month?

Solution:

At the beginning of the 73^{rd} months 72 full months, i.e., 6 years, have already passed. This represents $\frac{6}{30} = \frac{1}{5}$ of the loan's original maturity. Since the loan principal is repaid in constant chunks each month, one fifth of the principal was paid back, while four-fifths are not yet repaid.

$$\text{beginning balance} = BB = \frac{4}{5} \cdot L = \frac{4}{5} \cdot 10,000,000 = \$8,000,000.$$

10.3. What is the interest rate that would have been quoted on this loan when it was issued?
Note: If you need to use results from either parts (10.1) or (10.2) above, or from both, but you

were not able to determine the respective values, you may use \$50,000 for the answer to part (10.1), and \$5,000,000 for the answer to part (10.2), respectively.

Solution:

The monthly payment (pymt) consists of the (constant-size) principal repayment (p, from above) plus the one-period interest on the beginning balance. Let the monthly (i.e., per-period) interest be r.

$$pymt = p + BB \cdot r \Rightarrow r = \frac{pymt - p}{BB} = \frac{87,777.78 - 27,777.78}{8,000,000} = 0.75\%.$$

This, however, is not the rate that would be quoted on the loan. The usual conventions imply that the rate would be quoted on an annual basis as “9% per annum, compounded monthly” $(9\% = 12 \cdot 0.75)$.

11 Balloons

You are an important local real estate investor; you just got a \$10,000,000 balloon loan to buy a new office building in your home town. The nominal maturity of the loan is 30 years, but the loan has a 10-year balloon payment. In other words, the loan will end at the end of the 10th year, and the outstanding balance will be paid off in a lump sum at that time. The interest on the loan is 6.7% per annum, compounded monthly.

11.1. Assume that the loan has fixed payments.

(11.1.1) What is the monthly fixed payment that you have to make?

(11.1.2) Let PV be the present value at time 0 of the fixed payments made over the 10-year life of the loan. What is PV?

(11.1.3) Can you find a connection between the \$10,000,000 principal, PV, and the balloon payment at the end of year 10? Compute, using this connection, the size of the balloon payment at the end of year 10. If you cannot find the connection, a more work-intensive approach is to use an amortization table to compute the answer (use Excel!).

11.2. Assume now that this loan has variable payments. At the end of each month fixed, equal portions of the loan's principal are paid down, such that the principal would be fully amortized at the end of the hypothetical 30-year loan period. What is the size of the balloon payment at the end of year 10 in this case?

Solution:

11.1. The loan can be seen as a regular annuity with a maturity of 30 years. The monthly interest rate is $\frac{r}{12} = \frac{6.7\%}{12} = 0.56\%$. The number of periods is $12 \cdot 30 = 360$. Let us denote the principal of the loan by $PV_{loan} = \$10,000,000$.

(11.1.1) The monthly payment is $C = \frac{r}{1-\left(\frac{1}{1+r}\right)^t} \cdot PV_{loan} = \frac{0.0056}{1-\left(\frac{1}{1+0.0056}\right)^{360}} \cdot 10,000,000 = \$64,660.52$.

(11.1.2) The present value of the payments is the value of a regular annuity with the payment equal to the loan's monthly payment, i.e. \$64,660.52. The interest rate is the monthly rate computed earlier. This time, however, there are t'=$12 \cdot 10 = 120$ periods (payments). $PV = C \cdot \frac{1-\left(\frac{1}{1+r}\right)^{t'}}{r} = 64,660.52 \cdot \frac{1-\left(\frac{1}{1+0.0056}\right)^{120}}{0.0056} = \$5,638,808.22$.

(11.1.3) The difference between the loan's principal and the present value of the payments over the first 10 years is the present value of the balance outstanding at the end of 10 years. The same amount is also the present value of the all the payments that would be made after 10 years, if the loan reached its maturity.

We can turn this reasoning around: the future value of $PV_{loan} - PV$ must be the loan balance outstanding at 10 years in the future. We have: $Balance_{10} = (1+r)^{t'} \cdot (PV_{loan} - PV) = (1+0.0056)^{120} \cdot (10,000,000 - 5,638,808.22) = \$8,523,872.53$.

If may seem surprising that after "paying off" \$5.6 million in the first 10 years, the borrower still owes \$8.5 million. Note, however, that the majority of payments made in the early years represented interest, not the repayment of the principal. You can see this best if you look at the slides made available to you earlier, illustrating the evolution of the structure of payments for fixed-payment loans.

11.2. Each month the borrower must repay a fixed share of the principal. The amortization schedule is built as if the loan had a maturity of 30 years; thus the monthly repaid principal is $P_{repaid} = \frac{10,000,000}{360} = \$27,777.78$. After 10 years, or 120 periods, the principal **not** repaid, i.e. the outstanding balance of the loan is $P_{loan} - 120 \cdot P_{repaid} = 10,000,000 - 120 \cdot 27,777.78 = \$6,666,666.40$. An alternative way of thinking is even simpler: Each month the same amount of principal is repaid. If 10 out of the 30 years of the loan's hypothetical maturity passed, then one third of the principal was paid off, and two thirds of the principal are still left. These two thirds of principal are the balance of the loan at the end of year 10: $\frac{2}{3} \cdot P_{loan} = \frac{2}{3} \cdot 10,000,000 = \$6,666,666.66$. Note the slight difference in the two answers - this is due to rounding errors; they are not significant.

Calculator Solution:

11.1. We use the same notations as above; also, we do not repeat the explanations and reasoning given above. For brevity, we do not interpret or explain the choices of cash flow signs.

Note that in the calculations below, in order to be consistent with the formula-based solutions, we used a per-period interest rate of 0.56%. If you used the calculator to compute the per-period interest and you directly stored the result in the the I/Y register, then you set your interest rate to $\frac{6.7\%}{12}$ =0.558333... If you did this, then the results below would be different; for example, $C = \$64,527.80$. Strictly speaking, this latter result is more accurate, as it used more decimals in its input data. Our solution emulates one possible way in which a student may solve an exam problem by first computing and writing down some intermediate results (e.g., the per-period interest rate), and then performing next steps with the rounded prior results. From our perspective, these differences are not significant. In a real case, however, working with more decimals and minimizing numerical errors due to rounding and truncation would be important.

(11.1.1) N=360; I/Y=0.56; PV=-10,000,000; FV=0. Press CPT PMT to get 64,660.52. Thus $C = \$64,660.52$.

(11.1.2) N=120; I/Y=0.56; PMT=-64,660.52; FV=0. Press CPT PV to get 5,638,808.22. Thus $PV = \$5,638,808.22$.

(11.1.3) N=120, I/Y=0.56; PV=-4,361,191.78 (this is equal to $PV_{loan} - PV$); PMT=0. Press CPT FV to get $End\,Balance_{10} = \$8,523,872.53$.

11.2. Same solution as above.

12 Better Late Than Never

You work for a bank. Exactly five years ago, you helped Al Kapon, a well-known local businessperson, to get a \$15,000,000, 20-year variable-payment amortizing loan in order to build a "soft drink bottling facility." The loan carries an interest of 7% per annum, has monthly payments, and is structured like similar loans discussed in class; in particular, Kapon is expected to pay down the same amount of principal every month for the duration of the loan, in addition to the interest due monthly.

12.1. What is the payment due at the end of the very first month of this loan?

12.2. On the fifth anniversary of the loan Kapon comes to your office unexpectedly, and states that his business is in trouble. However, he hopes that difficulties are temporary, and that his business and finances will recover within one year. After some back and forth, you agree on behalf of the bank to forsake principal payments due for the next 12 months. However, Kapon still must pay in full the interest due at the end of each month.

(12.2.1) What is the loan balance at the end of 5 years, when Kapon asks for the modification of the loan?

(12.2.2) What payments will be made at the end of each month for the duration of the year when principal payments are suspended?

12.3. Principal repayments resume after the 12 months elapse. For the remainder of the loan's original term, the same amount of principal will be repaid every month, so that by the loan's original maturity date the principal is fully paid off.

(12.3.1) What will be the monthly principal payments due after the end of the principal repayment suspension?

(12.3.2) Provide the row of the loan's updated amortization table corresponding to the first month in which after principal payments have resumed.

Solution:

12.1. This loan is amortized over $12 \cdot 20 = 240$ months, and each month an equal share of principal is repaid, until the principal (and thus the loan) is fully paid off. The monthly principal payment is $\frac{15{,}000{,}000}{240} = \$62,500$. For the first month of the loan we must pay interest for the full loan amount, since no part of it was paid back yet. The interest payment is equal to $15,000,000 \cdot \frac{0.07}{12} = \$87,500$. The total payment due at the end of the first month is thus $62,500 + 87,500 = \$150,000$.

12.2.

(12.2.1) The interest is always paid in full; the principal is paid down in constant-size chunks. After 5 years of the original loan term of 20 years have elapsed, a proportion $f = \frac{5}{20} = 25\%$ of the principal was paid off. Still outstanding is 75% of the original balance, that is $15,000,000 \cdot 0.75 = \$11,250,000$.

(12.2.2) There will be no repayment of the principal during the 12 months of suspension, so only interest (on the principal outstanding at the time) will be paid. Because the principal outstanding does not change, the monthly interest, and thus the monthly payments will be constant and equal to $11,250,000 \cdot \frac{0.07}{12} = \$65,625$.

(12.2.3) After the principal repayment suspension ends, 6 years out of the original 20 years will have elapsed. At this point the loan has a leftover maturity of 14 full years, i.e., of $14 \cdot 12 = 168$

months. The still-outstanding principal will be repaid in equal monthly parts over this period. Thus the monthly principal payment must be $\frac{11{,}250{,}000}{168} = \$66,964.29$.

(12.2.4) We already know the new monthly principal payment. We also know the total principal outstanding when the principal suspension ends, which allows for the computation of the interest for the first month after the suspension ends. We need no calculation, in fact, as this is the same interest payment that was paid monthly during the suspension period! Can you say why?

We can now fill out the requested row of the amortization table, shown in Table 2.

Month	**Beg. Balance**	**Payment**	**Interest**	**Principal**	**End Balance**
...	...	...	...	...	...
73	$\$11,250,000.00$	$132,589.29$	$\$65,625.00$	$\$66,964.29$	11,183,035.71
...	...	...	...	...	...

Table 2: Partial amortization of the modified loan discussed in problem 12.212.2.4.

13 Beginning Balance at the End...

Assume that you have a fixed-payment amortized loan with a principal of $8,000,000, a yearly interest rate of 9% compounded monthly, and a maturity of 10 years.

To the closest thousand, what is the beginning balance of the loan at the start of the last month (i.e., at the beginning of the month at the end of which the very last payment due on the loan is made)?

✗ $100,000

✓ $101,000

✗ $102,000

✗ None of the numbers above can be the answer.

Answer:

Let the monthly fixed payment on this loan be C. C must cover both the monthly interest payment on the beginning balance BB, and the respective month's decrease in loan balance (principal). If the monthly interest is $r = \frac{0.09}{12} = 0.75\%$, then this month's interest is $BB \cdot r$, and the repaid principal is BB itself (this is because the balance goes down to 0 at the end of the month; it is the last month, after all).

Thus we must have $C = BB \cdot r + BB$, from which we get $BB = \frac{C}{1+r}$.

The loan is just a regular annuity, so we can immediately determine its fixed payment to be $C = PV \cdot \frac{r}{1-\left(\frac{1}{1+r}\right)^{120}} = \$8,000,000 \cdot \frac{0.0075}{1-\left(\frac{1}{1+0.0075}\right)^{120}} = \$101,340.62$.

Finally, we have that $BB = \frac{101,340.62}{1+0.0075} = \$100,586.22$.

14 Collecting Coupons

Assume that a bond sells for \$948; it has semi-annual coupons, a maturity of 8 years, yields 5.1%, and has a face value of \$1,000. What is the coupon rate of this bond?

Solution:

We start by writing the general formula that connects bond prices to their (semi-annual) coupon, yield, face value, and maturity:

$$B = \frac{C}{2} \cdot \frac{1 - \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}}{\frac{y}{2}} + P \cdot \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}.$$

Note that in this formula the appropriate adjustments have been made to account for semi-annual coupons and the yield compounding twice a year. The coupon rate can be determined immediately:

$$C = y \cdot \frac{B - P \cdot \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}}{1 - \left(\frac{1}{1+\frac{y}{2}}\right)^{2t}} = 0.051 \cdot \frac{948 - 1,000 \cdot \left(\frac{1}{1+\frac{0.051}{2}}\right)^{16}}{1 - \left(\frac{1}{1+\frac{0.051}{2}}\right)^{16}} = \$43.00.$$

The result implies that this bond pays \$43 for each $\$1,000$ of face value. The annual coupon rate, which the problem is asking for, is $p = \frac{43}{1,000} = 4.3\%$.

15 But How Much Do I Make?

Assume that a bond with a maturity of 10 years, face value of \$1,000, coupon rate of 5%, with semi-annual coupons, has a market price of \$903.25. You have already determined that the yield of the bond is between 6% and 7%. Show yields as percentages with two decimals precision.

15.1. Assuming that the yield of the bond were 6.5%, show how you would set up your calculator to compute the implied bond value and provide this respective value.

15.2. Set up a table similar to that given in class to determine an approximate value for the yield of the bond. Use your calculator to compute bond values, but do not show the details of your calculator operations. Stop when the mid-yield produces a price within \$0.50 of the bond's true price.

Hints: You only need to compute the total value of the bond for various yields, not also the part attributable to coupons or principal. Also, since only the yield changes, you do not have to re-enter all the values into the TVM worksheet - just change the yield and recompute the value.

Solution:

15.1. $N = 20$ (the number of periods is $2 \cdot 10 = 20$); $I/Y = 3.25$ (the rate compounds twice a year, $y = \frac{6.5\%}{2} = 3.75\%$); $PMT = 25$ (the semi-annual coupon is $\frac{50}{2} = \$25$); $FV = 1,000$. Now we press $CPT\,PV$ to obtain the value of the bond as -890.95. Note that the value given in the problem text is not the value that we get (or that we should use) here.

15.2. We provide the table below. Please note that values that are simply repeated from above are not shown (i.e., and empty cells means that the value in the cell is identical to that in the first non-empty cell above it). Values taken over from the columns corresponding to the mid-yield in the previous row are shown on gray background. Note that each row requires the computation of a single new yield (the mid-yield), and of a single bond value, which is easy to do using the calculator. We show only annualized yields in this table; in actual calculations per-period yields would be used. Details were discussed extensively on slides, in class, and in the additional hand-out on computing bond yields.

As can be seen, the yield of this bond is approximately 6.32%. Depending on how you rounded and/or truncated your results, your table may be different, but the final result should not be significantly different.

Low Yield	Bond Price	High Yield	Bond Price	Mid-Yield	Bond Price
6.00%	925.61	7.00%	857.88	6.50%	890.95
		6.50%	890.95	6.25%	908.08
6.25%	908.08			6.38%	899.47
		6.38%	899.47	6.31%	903.76
6.31%	903.76			6.34%	901.61
		6.34%	901.61	6.33%	902.69
		6.33%	902.69	6.32%	903.22
		6.32%	903.22	6.32%	903.49
6.32%	903.49			6.32%	903.36

16 Treasuries

On February 28, 2020, the financial press announced that yields for US Treasury bonds with 10-year maturities were at record low levels, never before seen. Indeed, early in the day, 10-year Treasury yields were as low as 1.18%, corresponding to a price of $1,029.50 per $1,000 face value. Typical Treasury bonds pay semi-annual coupons.

16.1. Without performing any computations, can you provide a **lower bound** (lower limit) for the yearly coupon rate of this 10-year US Treasury bond? If yes, state what this lower bound is, and how you know it is correct. A trivial lower bound of 0 is not an acceptable answer.

Solution:

We know that when the yearly coupon is equal numerically to the bond's (annualized) yield, the bond trades at par, no matter what the compounding period is. To trade at par, the yearly coupon would have to be 1.18%. The bond trades above par. Given that the face value and yield are fixed, the increase of the price (payment present value) above par must come from coupons that are higher than 1.18% per year.

16.2. What is the implied annual coupon rate for this bond?

Solution:

The bond has a maturity of 20 6-month periods, and the 6-month yield is $\frac{1.18}{2} = 0.59\%$. We use the calculator to determine the answer: 20 N, 0.59 I/Y, -1,029.50 PV, 1,000 FV. We then press CPT PMT to get 7.47. This result means that the bond pays a coupon of \$7.47 per \$1,000 of face value each 6-month period. The yearly coupon is thus $2 \cdot 7.47 = \$14.94$. The yearly coupon rate is $c = \frac{14.94}{1{,}000} = 1.49\%$.

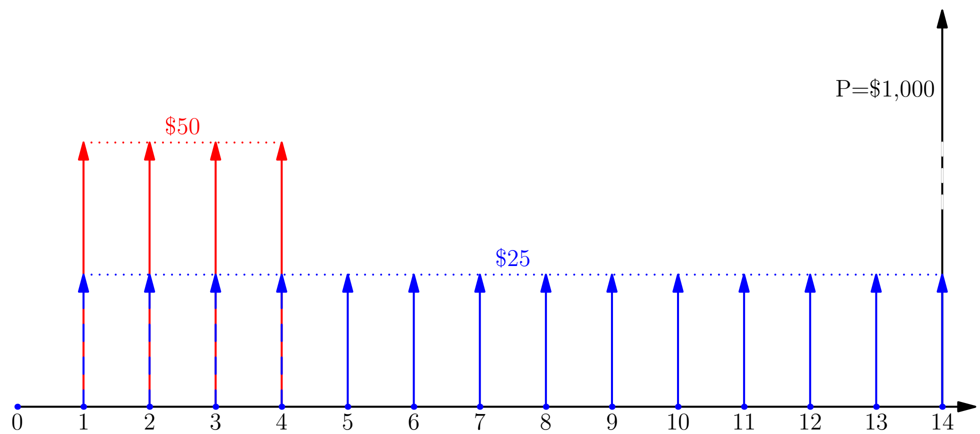
17 Bond Decomposition

You are studying a bond that has a leftover maturity of 7 years, has a face value of \$1,000, and a stated coupon rate of 5% per year, payable semiannually. The yield of the bond is 4.7% per annum; further, you may assume that per-period rates are constant (they do not depend on the time horizon).

The bond contract includes covenants (agreements) meant to protect bond investors' interest. In the recent past the company breached one of these covenants. Even though some improvements did occur since then, the breach is expected to persist for the next two years, after which it is expected to be cured (eliminated). While the breach persists, the coupon rate will be double the normal (stated) one. If default were to occur, such an accelerated payment schedule assures that investors get more of their money back; if default does not occur, then the increased coupons act as a penalty for the breach of the covenant.

17.1. Decompose this bond into simpler financial instruments that were studied in class. Specify what these simpler instruments are, what their parameters are, and how do you know your decomposition is correct.

Solution:



This bond has a coupon period of 6 months. The normal (stated) yearly coupon is $C = 5\% \cdot 1,000 = \$50$, while the increased yearly coupon is $2 \cdot C = \$100$. The actually paid 6-month coupons are of \$25 and \$50, respectively. There will 4 payments of \$50, covering 4 successive periods (2 full years), followed by 10 payments of \$25, covering the next 10 successive periods (5=7-2 years). At the end of the 7 years (14 successive periods) there will be a principal payment of \$1,000.

We can decompose our atypical bond into an annuity paying \$25 each period, maturing in 2 years, and a regular bond paying \$25 every period, maturing in 7 years. Putting together the cash flows of these two instruments we get the cash flows of the atypical bond, so the decomposition must be correct. This is not the only possible decomposition.

17.2. Determine the current price of the bond.

Solution:

The per-period yield of this bond is $\frac{4.7\%}{2} = 2.35\%$. We use the calculator to value the annuity: 4 $\boxed{\text{N}}$, 2.35 $\boxed{\text{I/Y}}$, 25 $\boxed{\text{PMT}}$, 0 $\boxed{\text{FV}}$. We then press $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$ to get -94.39. The result means that the annuity's value is A=\$94.39.

We can use the calculator to also value the (typical) bond: 14 $\boxed{\text{N}}$, 2.35 $\boxed{\text{I/Y}}$, 25 $\boxed{\text{PMT}}$, 1,000 $\boxed{\text{FV}}$. We then press $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$ to get -1,017.72. The result means that the bond's value is B=\$1,017.72. The value of the atypical bond is $A + B = 94.39 + 1,017.72 = \$1,112.11$.

18 NPV Mystery I

The NPV of a regular coupon bond's cash flows, when the discount rate is equal to the bond's yield, is equal to the bond's price (value). This statement is...

✓ True

✗ False

Answer:

This statement is true; in fact, it is a restatement of the definition of the yield.

19 NPV Mystery II

For the types of bonds discussed in lectures, the NPV of the bond's principal (face value) always exceeds half of the bond's current price (value). This statement is...

✗ True

✓ False

Answer:

This statement is false. One can get intuition about it by thinking of a coupon bond with, say, a 30-year maturity and a face value of \$1,000. If the yearly coupon was a very reasonable 3.5%, say, then the bond would make 60 payments of $\frac{\$35}{2} = \$17.5$, for a total coupon payment of \$1,050 (these are nominal, non-discounted dollars). These coupon payments exceed the face value of the bond, and, but for the last one, all of them are paid earlier than the face value. Thus the cumulative net present value of the coupons must exceed the net present value of the face value, which means that more than half of the bond's value comes from coupons. This conclusion is even stronger if coupon rates are higher. In other words, the longer the maturity of the bond, and the higher the coupons, the more likely it is that the statement above is false.

There are bonds for which the majority of the value comes from the principal payment, an obvious example being zero-coupon bonds. Short-maturity bonds paying low coupon rates also derive most of their value from their respective principals.

20 Congratulations, You're an Analyst

You have just been hired as a junior analyst working for a bond trader. Your first assignment is to value a corporate bond paying semi-annual coupons at an annual rate of 9.5%, with a maturity of exactly 2 years. The bond has a face value of $1,000. A senior analyst has already processed the current Treasury price data and provided you with an up-to-date term structure chart, shown in Figure 1. You are told to treat this corporate bond similarly to government bonds; i.e. you will ignore all default, liquidity, and similar risks, which will be analyzed by more experienced colleagues.

20.1. The term structure of interest rates chart has a horizontal axis labeled "Maturity," and a vertical axis labeled "[Annual[1]] Yield." What kind of government bonds have their yields and maturities plotted on this chart?

20.2. From earlier problems you solved while in college, you learned to decompose more complex financial instruments into sums - or differences - of simpler instruments. Explain how you can decompose this corporate bond into a collection of zero-coupon bonds of different maturities, perhaps having atypical face values. Provide a brief statement explaining the decomposition and show, in a table, what would be the bond's maturities and face values, respectively.

20.3. You learned that in realistic settings cash flows that arrive later must be discounted at (usually) higher per-period interest rates. For each zero-coupon bond listed in part 20.2 above, use the term-structure chart and look up its corresponding yield. Next, compute the present value of each zero-coupon bond. Using these zero-coupon bond prices, and also relying your earlier insights, provide a computed (theoretical) price for your corporate bond.

20.4. You now have a price for your corporate bond - what is its yield?

20.5. You provide the result computed in item 20.4 to one of your colleagues, who explains that in the practice of your firm, in order to adjust for the risks that a bond like yours bears in addition to government bonds, its yield must be changed by 0.50% per year. The colleague did not say explicitly whether the yield should be increased or decreased. State whether the yield must be increased or decreased, explain why, and then compute the new bond price. Determine what is the percentage change in the bond price when comparing prices before and after the risk adjustment, respectively.

[1] As you will note, the chart only uses "Yield" as a label for the vertical axis. The usual bond terminology expresses yields in annualized terms, and you should do the same in this class, as well as in other finance-related work that you do. We provide a reminder here, but you should **no**t assume that similar reminders will also be present when taking an exam.

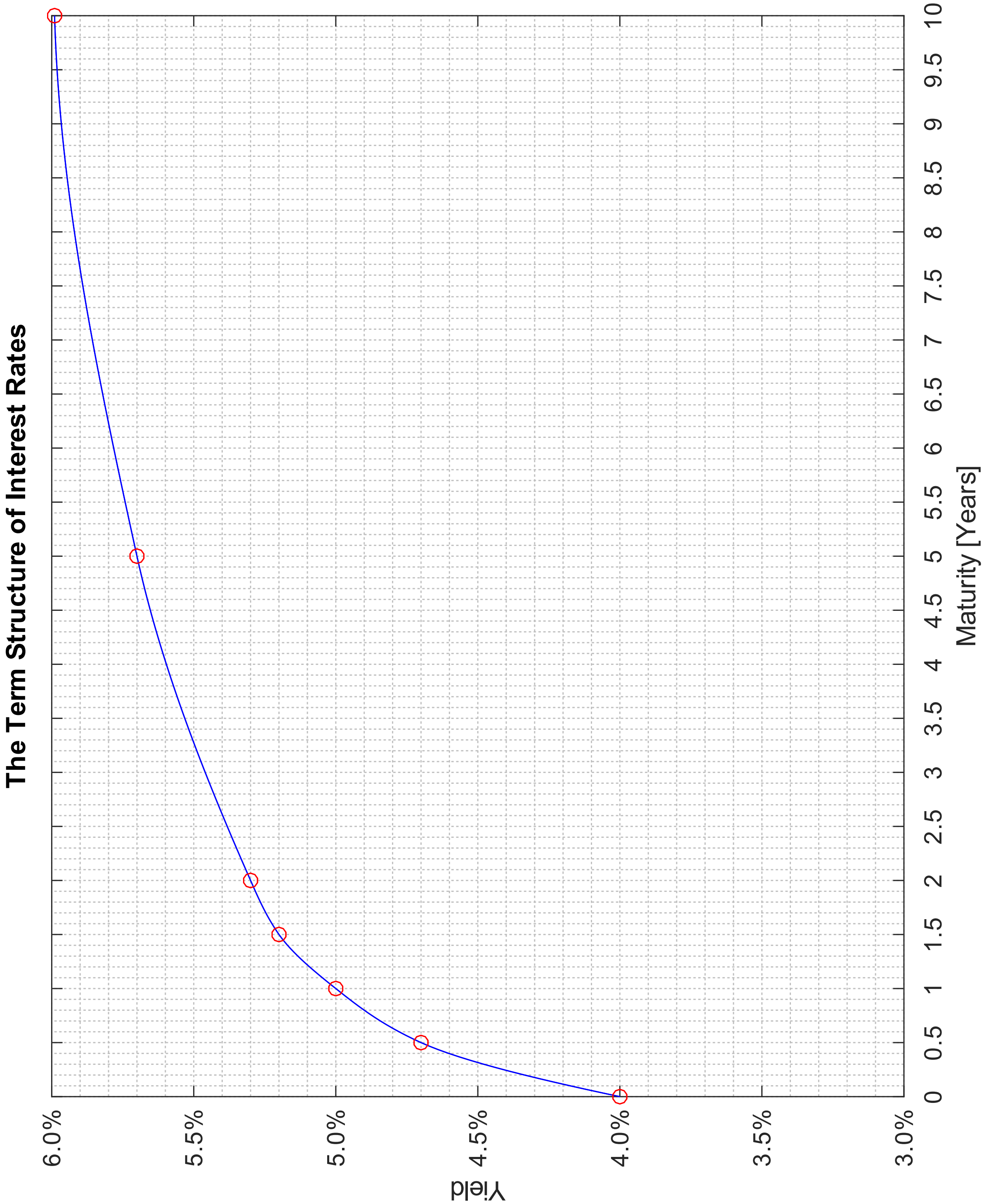


Figure 1: Term structure of interest rates on your first day as an analyst, as determined by a senior analyst using current government bond data.

20.6. Now adjust the yield in the **opposite direction** to that you decided was necessary in part 20.5 above. Compute the new price and the percentage price change when comparing the original (riskless) price and the price after the newest change in yield. Compare the percentage changes in the price of the corporate bond when the yield has been increased and decreased by the same amount, respectively. Which relative change is bigger? Could you have predicted which change is bigger by examining any of the slides discussed in class, without resorting to formulas?

Solution:

20.1. As discussed in class and in your textbook, the term structure of interest rates is the yield curve of **zero-coupon government bonds**. The problem already states that these are government bonds, so all you need to add is that these are also zero-coupon bonds.
Each point on the attached chart specifies yield of a zero-coupon bond of a certain maturity. In reality, there are no zero-coupon bonds of all possible maturities, but there exist techniques that use both regular and (the rare) zero-coupon bonds to infer this curve even for zero-coupon bonds that do not actually exist.

20.2. The corporate bond you were assigned makes 4 coupon payments 6, 12, 18, and 24 months into the future, respectively, and also repays its principal 24 months into the future. The individual coupon payments are of $\frac{0.095}{2} \cdot 1,000 = \$47.50$ each. Table 3 summarizes the cash flows that will be paid, as well as their timing.
If we ignore risks that government bonds do not bear, our corporate bond can be described as a package of zero-coupon government bonds with atypical face values; i.e., each cash flow in the table can be interpreted as the face value of a zero-coupon bond that matures at the time when the cash flow is due.

Time [Yrs]	**0.5**	**1.0**	**1.5**	**2.0**
Cash Flow	\$47.50	\$47.50	\$47.50	\$47.50+\$1,000=\$1,047.50

Table 3: Cash flows paid by the corporate bond discussed in problem 20.2.

20.3. We use the term structure of interest rates chart that we were given to read off the yields of the corresponding zero-coupon bonds. We note in passing that this is an upward sloping term structure curve, as is typical, because longer maturities usually also imply higher per-period interest rates. We extend Table 3 with the data read off the chart, as well as with present value calculations for each zero-coupon bond; the result is shown in Table 4.

If we add up the present values of all zero-coupon, and thus of the corporate bond's payments, we get a theoretical price for the corporate bond: $B = \$1,079.05$.
Corporate bonds bear more risk than Treasuries, because, for example, they are subject to default risk, while Treasuries are usually assumed not to carry default risk. Because rational investors need to be compensated for extra risks, the interest that they demand is higher for riskier bonds. This means that the discounting for the corporate bond should be more aggressive, which, in turn, means that the present value of the payments should be lower. The price we got is an upper bound - an overestimate - of the true price.

20.4. We can compute the yield of the bond using the bisection method. We presented that method extensively in earlier handouts, so in this case we'll just use the calculator to quickly get the result: 4 N, -1,079.05 PV, 47.50 PMT, 1,000 FV, and then hit CPT I/Y. The per-period yield is equal to 2.64%, while the annualized yield is $y \approx 2.64 \cdot 2 = 5.28\%$. Note that this value is lower than, and close, but not equal to the yield of the 2-year zero-coupon bond. Can you explain qualitatively why this is so?

20.5. The yield must increase; see the discussion related to interest rates (hence, also yields) and risk above. The new yield is thus y_{high} =5.28+0.50=5.78% (annualized). The price of the bond can be computed using the calculator: 4 N, 2.89 I/Y, 47.50 PMT, 1,000 FV, and then hit CPT PV to get $-1,069.32$. The price of the bond is thus \$1,069.32. As expected, the **increase** in yield led to a **decrease** in the computed bond price. The relative change in bond price is $\Delta_{increase} = \frac{B_{5.78}-B_{5.28}}{B_{5.28}} = \frac{B_{5.78}}{B_{5.28}} - 1 = \frac{1{,}069.32}{1{,}079.05} - 1 = 0.9017\%$.

20.6. Since we increased the bond yield in the prior part, we now have to decrease it. The new yield will be y_{low} =5.28-0.50=4.78% (annualized). The price of the bond can be computed using the calculator: 4 N, 2.39 I/Y, 47.50 PMT, 1,000 FV, and then hit CPT PV to get $-1,089.02$. The price of the bond is thus \$1,089.02. As expected, the **decrease** in yield led to an **increase** in the computed bond price. The relative change in bond price is $\Delta_{decrease} = \frac{B_{4.78}-B_{5.28}}{B_{5.28}} = \frac{B_{4.78}}{B_{5.28}} - 1 = \frac{1{,}089.02}{1{,}079.05} - 1 = 0.9240\%$.
$\Delta_{decrease}$ is positive, while $\Delta_{increase}$ is negative; if we ignore the sign and just focus on magnitudes, however, then $|\Delta_{decrease}| > |\Delta_{increase}|$. This means that the price increases faster if we decrease the yield, compared to the decrease in price when the increase the yield by the same amount. Speaking informally, the climb toward higher prices when we decrease the yield is steeper than the walk downhill when we increase the yield by the same amount.

Time [Yrs]	**0.5**	**1.0**	**1.5**	**2.0**
Cash Flow	\$47.50	\$47.50	\$47.50	\$47.50+\$1,000=\$1,047.50
“Zero” Yield	4.7%	5.0%	5.2%	5.3%
PV	$\frac{47.50}{\left(1+\frac{0.047}{2}\right)^1} = \$46.41$	$\frac{47.50}{\left(1+\frac{0.05}{2}\right)^2} = \$45.21$	$\frac{47.50}{\left(1+\frac{0.052}{2}\right)^3} = \$43.98$	$\frac{1,047.50}{\left(1+\frac{0.053}{2}\right)^4} = \$943.45.$

Table 4: Discounted cash flows used to provide a computed price for the corporate bond, un-adjusted for risk, as discussed in problem 20.3.

We have seen this effect, and we mentioned it in passing, when we looked at plots of bond values as a function of the bond's yield: the curves were steepest toward the left end, and their steepness decreased toward the right end. The quantitative effect that we notice here could have been qualitatively predicted from the plots we studied.

You may have considered that $\Delta_{decrease} \approx \Delta_{increase}$, given the closeness of their numerical values. This would not be unreasonable in the context of a homework, say. You could have then stated that small yield changes in opposite directions induce approximately equal relative changes in bond prices when yield changes are small. In other words, the climb toward higher prices is approximately as fast as the walk downhill, when the yield changes by a small amount. This statement, while approximately correct, is harder to reconcile with high-level qualitative observations that we made in class. However, if you took this approach, we would accept it.

21 We Have No Money, We're a Startup

Next Wonder, Inc., is a startup that cannot afford to pay dividends, since it has to finance its rapid growth, as it aims to take over the technology world. The company will pay no dividends for 5 years, but then it plans to pay \$2.5 per share, per year for the next 5 years. After these 10 years, the stock dividend will jump to \$5 in year 11, and will keep increasing by 3% per year indefinitely. The required return is 8% per year.

21.1. Assume that you are computing the price of the stock, P_{10}, at the end of year 10, just after the dividend due at the end of year 10 has been paid. What is P_{10}?

21.2. Assume that you are computing the price of the stock, P_5, at the end of year 5. What is P_5?

21.3. What is the price of the stock at time 0, P_0?

Solution:

21.1. Looking ahead from time t=10, we "see" a first dividend of \$5 to be paid in year 11; we also know that this dividend will increase by 3% per year. We can use the formula for constant dividend growth, when the price of the stock is given by the value of a growth perpetuity.

$$P_{10} = \frac{D_{11}}{R-g} = \frac{5.00}{0.08-0.03} = \$100.00.$$

We conclude that the time-10 price of the stock will be exactly \$100.

21.2. Price P_5 must include the present value (computed at time 5!) of the constant dividend $D = \$5$ paid in years 6, 7, 8, 9, and 10, respectively, plus the present value of the stock price at time 10. We note that the stream of constant dividend payments forms a regular annuity of maturity 5. We can immediately write:

$$\begin{aligned} P_5 &= D \cdot \frac{1 - \left(\frac{1}{1+R}\right)^5}{R} + \frac{P_{10}}{(1+R)^5} \\ &= 2.5 \cdot \frac{1 - \left(\frac{1}{1+0.08}\right)^5}{0.08} + \frac{100}{(1+0.08)^5} \\ &= 9.98 + 68.06 \\ &= \$78.04. \end{aligned}$$

21.3. There are no dividends paid in years 1 to 5, so their present value is obviously 0. Thus P_0 is the present value (at time 0) of P_5 :

$$P_0 = \frac{P_5}{(1+R)^5} = \frac{78.04}{(1+0.08)^5} = \$53.11.$$

22 Stock price

You are a new analyst following the stock of company Big Break, Inc., valued at \$15 per share on the morning of March 31, 2020. The company has been paying dividends of 35 cents on the very last day of each calendar quarter, and this was expected to continue indefinitely.

22.1. Assuming that the dividend model is reasonably accurate for this stock, what is the implied return rate demanded by investors in this stock, expressed in "per annum, compounded quarterly" terms?

Solution:

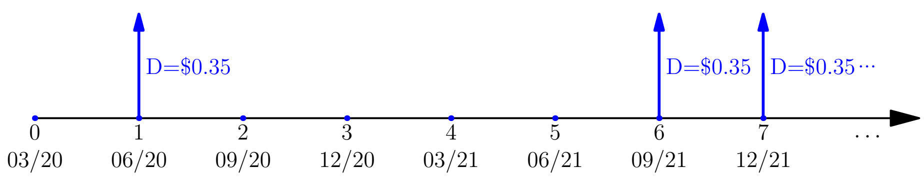
We use the notation $P_{mm/yy}$ to denote the price of the bond at the end of month mm in year 20yy. Before the announcement, the constant stream of dividends forms a perpetuity. Thus $P_{03/20} = \frac{D}{R}$, from which $R = \frac{D}{P_{03/20}} = \frac{0.35}{15} = 2.33\%$. This rate is a quarterly (per-period) rate, so we have to annualize it: $R_{annualized} = 4 \cdot 2.33 = 9.32\%$.

22.2. During a news conference late in the day on March 31, 2020, the company's CEO announces "temporary difficulties related to external funding needs" that will force the company to cancel the dividend for four successive quarters, "in order to conserve cash." The dividend due on June 30 will be paid, however, as it had already been declared, so the first missed payment will occur on September 30, 2020. The CEO, a highly credible, seasoned industry veteran, states that the dividend policy will be reinstated in its current form after the temporary suspension ends.
Note: If you need to use the value that was requested in part (22.1) above, but you were not able to compute it, you may use 5.5% for the "per annum, compounded quarterly" rate.

(22.2.1) Estimate the price of stock as of March 31, 2021.

Solution:

The diagram below illustrates the situation that prevails after the announcement:



The first dividend $D = \$0.35$ that will be paid after the suspension ends will occur at the end of September 2021. As of the prior quarter-end in June 2021, the dividend stream forms a regular perpetuity. We thus have that $P_{06/21} = \frac{D}{R} = P_{03/20} = \$15$. We can particularize our general stock price formula for end-March 2021, keeping in mind that there are no dividends paid in June 2021:

$$P_{03/21} = \frac{P_{06/21}}{1+R} = \frac{15}{1+0.0233} = \$14.66.$$

(22.2.2) Estimate the price of the stock as of March 31, 2020, just after the announcement.

Solution:

We already computed the price at the end of June 2021, five quarters into the future: $P_{06/21} = \$15$. We particularize our general stock price formula keeping in mind that there is a dividend in June 2020, one quarter into the future, but none during the subsequent four quarters:

$$P_{03/20} = \frac{D}{1+R} + \frac{P_{06/21}}{(1+R)^5} = \frac{0.35}{1+0.0233} + \frac{15}{(1+0.0233)^5} = 0.34 + 13.37 = \$13.71.$$

You can verify that the same result is obtained if you use $P_{03/21}$, which we have also computed:

$$P_{03/20} = \frac{D}{1+R} + \frac{P_{03/21}}{(1+R)^4}.$$

23 Dividends

Fly Over Airlines just paid an annual dividend of 80 cents, and a new analysis revealed that dividends will grow long term at a rate of 4% per year. The company's stock price is \$70. Estimate, to the closest percent, the rate of return demanded by investors who buy Fly Over's shares.

✓ 5%

✗ 6%

✗ 7%

✗ None of the above is the correct answer.

Answer:

We compute the required return using the model of constant-growth dividends discussed when we studied stock valuation. Thus the required return is $r = \frac{D_1}{P_0} + g$, where D_1 is the size of the next dividend, P_0 is the time-0 value of one share, and g is the constant dividend growth rate. Denoting the just-paid dividend by D_0, we get: $r = \frac{D_1}{P_0} + g = \frac{D_0 \cdot (1+g)}{P_0} + g = \frac{0.80 \cdot (1+0.04)}{70} + 0.04 = 5.19\%$.

24 What's an Option Worth?

Over the last few weeks we got into the habit of describing more complex financial instruments as collections of simpler instruments. For example, we can decompose a typical bond into an annuity (the coupon payments), and a separate single payment (the principal). Similarly, we have seen atypical annuities described as sums or differences of regular annuities. In this problem you will build a synthetic bond, by suitably combining fractional amounts of two other bonds.

Consider the following information about three Treasury bonds:

Maturity Date	Coupon Rate	Price	Bond Type
05/15/2024	6.500	106.31250	regular
05/15/2024	8.250	103.43750	callable
05/15/2024	12.000	134.78125	regular

All the bonds above have a face value of $1,000. Prices are expressed as percentages of face value. We do not know the precise date when these prices were quoted; as such, it is possible that we are not exactly at the beginning of a coupon period. We do know, however, that all coupon payment dates for all these bonds are the same; the coupons, of course, are not. The bond in the middle is a so-called callable bond, while the other two are regular Treasuries.

24.1. Read the textbook and/or research the web to understand what callable Treasuries are. Summarize your findings in a **brief** paragraph.

Solution:

A callable bond may be redeemed ("recalled") by its issuers under terms that were set before the bond was first sold. Often, the recalled bond will be redeemed for its face value. In principle, any type of bond can be made callable. Callable Treasury bonds have historically been very rare; some source may even indicate that they did not exist, but that is inaccurate.

24.2. Given what you learned about callable bonds, under what conditions would the option to call the callable Treasury be exercised and by whom (the bond holder, or the bond's issuer)? In other words, when would the callable Treasury be called and by whom?
Hint: Assume that the call decision is rational; consider changes in interest rate levels as time passes.

Solution:

If a bond issuer acts rationally, a callable bond would be recalled only if the issuer benefits from this action. This means that a bond will be retired only if the NPV of the bond payments still due are larger than the immediate redemption costs (face value, if we ignore administrative and other costs) of the bond. In effect, the issuer will take something of value from the bond-holder when redemption occurs (the holder loses the larger NPV of future payments and gets the smaller immediate redemption value). Since the issuer is the US Treasury, it would the Treasury that would initiate redemption.

24.3. From the **holder**'s perspective, would a callable Treasury bond be more, or less valuable than an otherwise identical non-callable, i.e., regular Treasury bond? Briefly state your opinion and justify it qualitatively.

Solution:

Holders and potential holders of bonds know at the time of their purchase that a bond may be recalled, and that they would lose money if a recall occurred. Hence, if they act rationally, they should be willing to pay **less** for a callable bond than for an otherwise identical non-callable (regular) bond.

24.4. Now consider the pricing information provided in the table above. Use the information on the regular bonds to construct a bond that is identical to the the callable bond in all respects, but for the callable feature. Use this artificial bond to compute the cost of the call option. From the perspective of the bond holder (the lender), is this cost positive or negative?

Hint: Let B_1, B_2, and B_3 be the three bonds shown in the table above, starting with the bond at the top. If we ignore the call feature, the listed Treasuries are characterized by a small number of parameters: their coupon sizes, coupon payment times, and face values. You should build a portfolio (combination) of the two regular bonds B_1 and B_3, such that their combination produces B_2 (again, except for the call feature). Let B_{13} be an artificial (often called "synthetic") bond that you can create from a combination of B_1 and B_3, and assume that B_{13} consists of a fraction f of B_1 and a fraction $1-f$ of bond B_2, where $0 \leq f \leq 1$. Write down equations so that the resulting coupon and face value of B_{13} (seen as a mix/combination/portfolio of B_1 and B_3) matches the respective parameters of B_2. Use these equations to determine f, and then use f to determine an appropriate price for B_{13}.

Solution:

We have encountered in the past financial instruments that could be decomposed into sums or differences of simpler instruments that we had studied. Pricing the more complex instrument can then be reduced to computing the equivalent sum or difference of prices of simpler instruments. All examples that we have encountered before, however, consisted of combinations of **integer** numbers of simpler instruments. As we will see in this case, a non-callable composite bond identical to the callable bond B_2 above, but for the call feature, can be constructed from **fractional** holdings of the two non-callable bonds B_1, and B_3.

If we ignore callability, a bond can be fully described by the size and time of its coupon payments, as well as its face value (the face value will be paid together with the last coupon, so there is no additional information needed with respect to the payment time of the face value). From the text of the problem, bonds B_1, B_2, and B_3 have the **same** coupon payment dates, and the same face value of \$1,000.

Let us take a fraction f, $0 \leq f \leq 1$, of bond B_1, and a fraction $1 - f$ of bond B_3; create a new bond by putting together the resulting coupon and face value payments. Let us call the resulting bond B_{13}, to indicate that it resulted from the combination of bonds B_1 and B_3.

With respect to face value, we have: $FV_{13} = f \cdot FV_1 + (1 - f) \cdot FV_3 = f \cdot 1,000 + (1 - f) \cdot 1,000 = (f + 1 - f) \cdot 1,000 = \$1,000$. It turns out that for any choice of f, the value of bond B_{13} is the same as that of the three other bonds, including bond B_2.

With respect to coupons, our goal is to recreate B_2's (yearly) coupon size: $C_{13} = f \cdot C_1 + (1 - f) \cdot C_3 = f \cdot 65 + (1 - f) \cdot 120 = 120 - 55 \cdot f$. We know that the (yearly) coupon of bond B_2 must be \$82.5, so we get the equation $C_{13} = C_2 \implies 120 - 55 \cdot f = 82.5$. From here, we immediately get $f = \frac{120-82.5}{55} = 0.682 = 68.2\%$.

The results above imply that if we took (approximately) 68.2% of bond B_1 and 100%-68.2%=31.8% of bond B_3, we would get bond B_{13}, which would be identical to bond B_2, but for callability (B_{13} is composed of two non-callable bonds, thus their "mix" is not callable either). The price of bond B_{13} is obtained from a similar combination of prices for the component bonds: $0.682 \cdot 106.31250 + 0.318 \cdot 134.78125 \approx 115.36556$.

So, in terms of face value percentages, the price of bond B_2 is 103.43750, while the price of (synthetic) bond B_{13} is 115.36556. These are identical bonds, but for their callability. We can easily establish that the callable bond does indeed trade at a discount (lower price). What is the difference? It is, in terms of percentages of face value, equal to $115.36556 - 103.43750 = 11.9281$.

For each \$1,000 of face value the existence of callability depresses the price of the callable bond by $\$1,000 \cdot 0.119281 = \$119.28$. Callability is an advantage to the issuer, but it comes at the price of a discount - buyers will pay less for these bonds. There is no free lunch in (rational) finance!

25 Computing IRRs

Consider a set of **signed** cash flows CF_i that arrive at times t_i, where $i = 1, 2, \cdots, N$, and $t_1 \leq t_2 \leq \cdots \leq t_N$. The (per-period) internal rate of return (IRR) is the constant interest rate that makes the net present value (NPV) of this series of cash flows to be equal to 0:

$$\sum_{i=1}^{N} \frac{CF_i}{(1+IRR)^{t_i}} = 0.$$

Clearly, this equality is only possible if some cash flows are positive, and some cash flows are negative. As mentioned in class, we have already encountered IRRs (in a hidden form) when we spoke about bond yields. Using, for now, **unsigned** cash flows, let B be the price of the bond, C the per-period coupon, and P the principal (face value) of the bond; further, assume that there are exactly N time periods to maturity. We can immediately write:

$$B = \sum_{i=1}^{N} \frac{C_i}{(1+y)^i} + \frac{P}{(1+y)^N},$$

or, equivalently,

$$-B + \sum_{i=1}^{N} \frac{C_i}{(1+y)^i} + \frac{P}{(1+y)^N} = 0.$$

Now, if we consider signed cash flows, and we assume that **B** is the purchase price of the bond, then **B** is a cash outflow, and thus **B** is a negative cash flow. Keeping this in mind, we can rewrite the equation above for **signed** cash flows:

$$\boldsymbol{B} + \sum_{i=1}^{N} \frac{C_i}{(1+y)^i} + \frac{P}{(1+y)^N} = 0.$$

To avoid confusion between the two interpretations of the bond price, we denoted the unsigned bond price by B, and the signed bond price by **B** (bold B). B and **B** are equal in magnitude, i.e., $B = |\boldsymbol{B}|$; only their signs are different (B>0, **B**<0).

In the last formula, it is easy to recognize that the yield of the bond is, in fact, the bond's IRR. You already know how to compute the yield of bonds - this problem will teach you how to compute IRRs. Given that we did not specify what the compounding period is in this problem, all interest rates will be expressed simply as "per-period" rates, consequently, you will not have to worry about annualizing the rates.

Consider the cash flows shown below:

Time	**0**	**1**	**2**	**3**	**4**
Cash Flow [$]	1,000,000	-1,525,000	947,000	-1,826,000	1,420,500

25.1. Using a suitable tool (such as Excel), compute the net present value of the cash flows shown above for interest rates ranging from 0% to 30%, by incrementing interest rates by 1% at each step. Complete a table like the one below:

Interest Rate	0%	1%	2%	...	30%
NPV	...	...	...	...	...

You may format the table vertically, or you may create multiple columns, if you find it necessary or useful.

Solution: Please see attached workbook.

25.2. Plot the values of the NPV as a function of the interest rate. Make sure that the plot axes are labeled, and that enough values are shown for the reader to understand your plot in detail. Make the plot big enough to be easily legible.

Solution: Please see attached workbook.

25.3. Study the plot and the table above, and answer the following questions:

(25.3.1) How many IRRs does this series of cash flows have?

Solution:

When we plot the NPV as a function of the discount rate, we get a curve that crosses the horizontal axis **twice**. In other words, there are two different discount rates for which the NPV is equal to 0. We conclude that this series of cash flows has two IRRs within the interval [0%, 30%].

(25.3.2) For each IRR whose existence you ascertained, provide an interest rate interval that contains the respective IRR. This step brackets each IRR.
Do not necessarily strive to provide very narrow intervals, as the procedure described below will quickly reduce their size. However, make sure that **each bracketing interval contains exactly one IRR**, and that **the sign of the NPV is different at the two ends of the interval,** i.e., to the left, and to the right of the IRR. In other words, each interval that you pick will contain exactly one IRR value, and the NPVs computed at the of the respective interval will have opposite signs.

Solution:

By examining the graph, we see that the first crossing of the horizontal axis occurs between 0% and 5%, and the second crossing occurs between 10% and 15%. These are the intervals that hold the two to-be-determined IRRs. Narrower or wider intervals are acceptable, as long as each interval contains only one IRR.

25.4. Consider now a presumptive IRR, let it be irr, and the corresponding bracketing interval $[low, high]$; i.e., $low < irr < high$. Like with bond yields, we must reduce the width of the interval so that we can find the approximate value of the IRR. We start each step by computing the midpoint of the interval: $mid = \frac{1}{2}\,(low+high)$. Now we compute the NPV of the given cash flows rates are equal to low, $high$, and mid, respectively. We must have one of the following situations:

NPV@low	NPV@mid	NPV@high	New Bracketing Interval
positive	**positive**	**negative**	[mid,high]
positive	**negative**	negative	[low,mid]
negative	**positive**	positive	[low,mid]
negative	**negative**	**positive**	[mid,high]

The rules[2] from the table above make sure that the updated, narrower bracketing interval for

[2]We note that the rules for computing bond yields were derived from these more general rules, and were particularized for the special case we were then examining. For example, for bonds, the cash flows represented by coupon payments and the principal payment are all positive, and thus a bond's NPV always decreases as yields increase. We commented

the IRR always contains the IRR we are chasing. This is because the NPV has different signs at the ends of the updated interval; this, combined with the fact that the bigger bracketing interval contains only one IRR, means that the NPV, seen as a function of the interest rate, will cross 0 somewhere within the interval, as it goes from positive to negative values, or from negative to positive values. This crossing identifies the location of the IRR.

For **each** IRR, and its corresponding initial bracketing interval, set up a table like the one below and fill out the rows by computing narrower and narrower intervals that contain the IRR. Stop when the computed NPV of the given cash flows is close to 0, e.g., it is within the interval [-$0.10,$0.10].

low		**high**		**mid**	
Per-Period	**NPV**	**Per-Period**	**NPV**	**Per-Period**	**NPV**
...	...	...	...	...	...

Analogously to the computation of bond yields, **after** you set up the first row, you only need to compute one new set of values for each table row, i.e., you must compute the new mid-point and its associated NPV. The other needed values are already available in the immediately prior row. For each table you construct, state what the best estimate of the corresponding IRR is.

Solution:

Please refer to the attached workbook. The estimates for the IRR values are 4.0580% and 14.2521%, respectively. You may have chosen different initial bracketing intervals, and thus your calculation may look slightly different. The essence of the calculation should be the same in all cases, however, and the results you got should be very close to ours.

on this effect when we plotted the bond price against bold yields earlier in the course. We can thus eliminate the last two rows from the more general IRR rule table, given that the NPV of a bond can never increase from negative to positive values as the rate increases from *low* to *high*. Can you show that the first two rows of the IRR rule table lead to the rules we provided for bond yield computations? Hint: You may wish to examine the bond formulas discussed at the beginning of this problem.

26 IRR

Consider the IRR calculation algorithm that uses the interval bisection method you learned in this course. You are given the table below, which shows part of such a calculation. Rely on our usual conventions to interpret the meaning of the various columns shown. Assume that there is exactly one IRR in the interval 18.00% to 21.50%.

Low		**High**		**Mid**	
Per-Year	**NPV**	**Per-Year**	**NPV**	**Per-Year**	**NPV**
...	...	...	...	...	...
18.00%	70,096.86	21.50%	**X**	19.75%	23,065.09
Y	23,065.09	**Z**	**Q**	20.63%	655.92
U	...	**V**	...	...	...
...	...	...	...	...	...

26.1. What is the value of Y in the table above?

- ✗ 18.00%
- ✓ 19.75%
- ✗ 20.63%
- ✗ 21.50%
- ✗ None of the values above is a correct value for Y.

Answer:

We note the the value in the NPV column next to Y is equal to the NPV value shown for the Mid section one row above. Based on the rules of the interval bisection method, we can infer that in the step that led to the row that holds Y the upper limit of the IRR-bracketing interval was preserved, while the lower limit of the bracketing interval was replaced. Thus Y must be equal to 19.75%.

26.2. What can we say for sure about Q in the table above?

- ✓ Q is negative.
- ✗ Q is very close to 0.
- ✗ Q is positive.
- ✗ None of the statements above are necessarily true.

Answer:

We know that the unique IRR is bracketed within the interval from 18.00% to 19.75%. This is only possible if the NPV at one end of the interval is positive, and the NPV at the other end of the interval is negative. In the line that holds X we see that the NPV is positive at the left (low) end of the interval, i.e., at 18.00%. This means that X, the NPV at the other end of the interval, must be negative. It cannot be zero, or very close to it, because then the algorithm would have stopped. Combining these insights with those from the prior point, we conclude that Q must be the equal to X, and thus Q must also be negative.

26.3. What is the value of V-U in the table above?

- ✗ V-U is greater than 2%.
- ✗ V-U is between 1% and 2%.
- ✓ V-U is between 0% and 1%.
- ✗ V-U is negative.

Answer:

As its name indicates, the length of the bracketing interval is halved at every step in the interval bisection method. The length of the interval in the row that holds X is 21.75%-18.00%=3.75%. U and V are two rows below the row that holds X, so the interval that U and V bracket has been halved twice compared to the interval two rows above. Thus $V - U = \frac{1}{4} \cdot 0.0375 = 0.9375\% \approx$ 0.94%.

27 Can You Trust Your Co-Worker?

After modeling, the cash flows for a project are estimated to be -$50M, $40M, and $15M; they will occur at the end of year 0, year 1, and year 2, respectively. You overheard a colleague stating that the IRR of these cash flows is approximately 7.82%. You suspect that this would be a rate that compounds annually. Is your colleague right?

✓ Yes

✗ No

Answer:

A simple NPV calculation, using the cash flows given, produces $0.00193 million as its answer. An examination of the NPV in the immediate vicinity of the 7.82% discount rate yields the following NPVs:

Rate [%]	7.81	7.82	7.83
NPV[$M]	0.007760	0.001925	-0.003908

These values indicate that there is an IRR value between 7.82% and 7.83%, with 7.82% being the closest approximation of this IRR when rounded to basis point precision, as the general rounding rules in this class demand. The problem itself stated that this may be an approximate IRR, so you should have answered YES.

28 Best or Worst?

You are creating a model for a project, and are currently working on scenarios, specifically, on developing a "best case" scenario. You have chosen (fixed) projected values for all but one input of your scenario. The only parameter of your scenario that is not yet fully determined by the choices you already made is the aggregate level of short-term credit that you may have to extend to your clients (by allowing them to pay for their purchases within 30 days from delivery). Given the remaining uncertainty, you believe that the total dollar value of this credit can range from a low value of L to a high value of H (L < H). The level of the credit will otherwise stay constant over the lifetime of the project.

Which value will you include in a best-case scenario?

✓ L would be included.

✗ H would be included.

✗ The answer cannot be determined based on the information given.

Answer:

The text states that "[y]ou have chosen (fixed) projected values for all but one input of your scenario." In other words, there are no more choices to be made, but for the choice between H and L. This means, inter alia, that the level of sales, for example, has been set, and that the impact of this decision on the NWC has already been accounted for. The choice between H and L is based on uncertainty that was impossible to eliminate **after** all these other possible choices were made. There is some otherwise unspecified source of uncertainty in the model (and, if the model is realistic, in the real world) that is captured by the difference between L and H.

Some may think that a high level of credit extended to clients could be a sign of success, and thus desirable, since more sales may require larger amounts of extended credit, even if credit extended grows slower than total sales. This may be the case in other situations/models, and in real life. Here the sales level, and all other parameters, have been fixed by hypothesis, so this line of analysis is not fruitful.

The question can thus be rephrased as follows: "**All else being equal**, would you prefer to extend high levels of credit to your clients, or low levels of credit?"

It is clear that L should be included in a best-case scenario. When the project is ramping up, this choice will produce a smaller negative cash flow as investment in working capital grows from 0 to L (as opposed to growing from 0 to H). When the working capital is recovered at the end, there will be a positive cash flow as credit is paid off. Even if the full working capital invested in credit is recovered, however, the net effect of credit will be to decrease the NPV of the project. This is because of the time-value of money (L is invested at time 0, and [at most] L is recovered at the end); the higher the level of extended credit, the bigger this negative contribution to the NPV will be in dollar terms. This conclusion stands even stronger if no outstanding credit is recovered, or if only a fraction of the outstanding credit is recovered. The fraction of credit recovered cannot be varied in comparing the impact of L and H, since this is a parameter that has already been fixed. All other realistic possibilities that you may consider yield the same conclusion: L is the choice for a best-case scenario under these circumstances.

29 I Have a Project For You...

Estimates for a new manufacturing project include the following: Initial investment in manufacturing equipment is $14,000. The equipment will be depreciated linearly over four years, which is also the lifetime of the project. The equipment has no salvage value. Fixed yearly costs are $11,000, variable costs per unit are $11, while the per-unit sale price is $25. The working capital needs of this project are negligible.

29.1. Given a production level of 2,000 units per year, what is the average manufacturing cost?

- ✗ $5.5
- ✗ $11.0
- ✓ $16.5
- ✗ None of the above.

Answer:

With the usual notations, $AC(q) = \frac{FC}{q} + MC = \frac{11,000}{2,000} + 11 = \$16.5$.

29.2. Given a constant per-year production level of 3,500 units and a tax rate of 25%, what number below is within $1,000 of the yearly OCF?

- ✗ $10,000
- ✗ $20,000
- ✓ $30,000
- ✗ No value above is within $1,000 of the yearly OCF.

Answer:

We compute the relevant quantities as follows: $\text{Total sales} = S = 3,500 \times 25 = \$87,500$; $\text{Costs} = C = FC + VC(q) = 11,000 + 3,500 \times 11 = \$49,500$; Depreciation =D=$\frac{14,000}{4} = \$3,500$ (per year). We then compute EBIT $= S - C - D = \$87,500 - \$49,500 + \$3,500 = \$34,500$; $\text{Taxes} = T = EBIT \times T_c = 34,500 \times 0.25 = \$8,625$; $\text{Net income} = NI = EBIT - T = 34,500 - 8,625 = \$25,875$.

Finally, $OCF = NI + D = 25,875 + 3,500 = \$29,375$. This OCF value is within $1,000 of $30,000.

29.3. Ignoring taxes, which of the numbers below is within 100 units of the yearly production level that is needed to achieve accounting break-even?

- ✓ 1,000
- ✗ 2,000
- ✗ 3,000
- ✗ None of the numbers above are within 100 of the accounting break-even production level.

Answer:

With the usual notations, we have $q = \frac{FC+D}{P-v} = \frac{11{,}000+3{,}500}{25-11} = 1,035.71 \approx 1,036$ units.

29.4. Assume that at a certain constant yearly production level L, the yearly OCF is $4,175 for each year over the lifetime of this project. If the required return is 14% per year, compounded annually, what can you say about the relationship of production level L and the financial break-even production level F?

- ✓ L is below F
- ✗ L is above F
- ✗ The relationship between L and F cannot be determined.

Answer:

To compute a financial break-even point, one needs to determine the NPV of the project's cash flows. In the context of our problem, we have operating cash flows in years 1 to 4, and we have capital spending in year 0 (when the project starts up). There are no sales of capital assets and there is no salvage value to these assets either. The working capital needs are negligible.

The project cash flows are thus -$14,000 in year 0, and $4,175 in each of the four years from year 1 to year 4. At a discount rate of 14%, the NPV of these cash flows is -$1,835.25. Since this value is negative, production level L is below the level needed for financial break-even F.

One ambiguity that might have emerged in your mind was whether the OCF values shown were computed ignoring taxes or not. For the purposes of answering the problem, you were expected to use the OCF values as given; however, we point out that the answer did not depend on taxes.

$OCF = NI + D$, thus $NI = OCF - D = 4,175 - 3,500 = \$675$. Taxes, if applied, decreased net income. Let us assume that the project pays taxes: $NI = EBIT \times (1 - T_c) \implies EBIT = \frac{NI}{1-T_c} = \frac{675}{1-0.25} = \$900$, and taxes were $T = EBIT \times T_c = 900 \times 0.25 = \$225$. So **if** taxes were paid, and **if** we want to determine the answer on a pre-tax basis, we should add \$225 to each year's post-tax OCF, to get the yearly "no tax" OCF: $OCF_{no\,tax} = OCF_{tax} + \$225 = \$4,400$. Using "no tax" cash flows we determine the NPV to be -\$1,179.67; i.e., still negative. The answer is still that L is below F.

30 Equipment Sale

Your firm uses a complex manufacturing machine that, at the end of year 6 of its life, after the depreciation for the just-ending year has been applied, has a book value of \$50,000. The machine breaks at the very end of year 6, in a way that makes it useless for your business. However, the machine does not need to be immediately replaced, as you have spare manufacturing capacity, and there is no expectation that demand for your products will increase. Your tax rate is 25%.

Note: Do not look for hidden connections to MACRS. Take the depreciation rules as given and use them, when appropriate, to answer the questions below.

30.1. You can sell the machine at the very end of year 6 for \$60,000. If you sell the machine instantly for cash, which cash flows, if any, will change for year 6, compared to the case when you do not sell the machine at this time?

- ✗ OCF
- ✓ Cash flows related to capital spending.
- ✗ Both OCF and cash flows related to capital spending.
- ✗ Neither OCF, nor cash flows related to capital spending will change.

Answer:

We are right at the end of year 6. The firm had the machine in its possession throughout the year, and so the firm is entitled to take the whole yearly depreciation into account whether or not the machine breaks, and whether or not the machine is sold. Neither year-6 depreciation, nor year-6 OCF is impacted by what happens to the machine **at the very end** of year 6.

If the machine is sold at the end of year 6, however, there will be a cash flow related to capital spending. This cash flow will consist of the difference between the proceeds of the sale and the taxes paid on the excess of the sale price over the book value: $60,000 - (60,000 - 50,000) \times 0.25 = \$57,500$.

To answer this question, you only had to reason based on the rules we discussed in class; determining the size of the various cash flows was not needed.

30.2. Assume that you put off selling the machine at the end of year 6, but you sell it at the very end of year 7. By the end of this final year, the machine is fully depreciated. You sell the machine for $60,000. Which year 7 cash flows, if any, will change compared to the situation when the machine is sold at the end of year 6?

- ✗ OCF
- ✗ Cash flows related to capital spending.
- ✓ Both OCF and cash flows related to capital spending.
- ✗ Neither OCF, nor cash flows related to capital spending will change.

Answer:

If the machine is sold at the very end of year 7, then the full $50,000 of depreciation that was left at the end of year 6 is used in year 7. This depreciation will lead to year-7 tax savings of $50,000 \times 0.25 = \$12,500$. These tax savings will lead to increase in year-7 OCF compared to what OCF would have been had the machine been sold at the end of year 6.

Since the machine is fully written off by the end of year 7, the proceeds from the sale will be fully taxable, and thus there will be a cash flow related to capital spending of $60,000 - 60,000 \times 0.25 = \$45,000$. This cash flow would not exist if the machine had been sold at the end of year 6.

30.3. Assuming ordinary economic conditions, as well as the facts stated for this problem, including in parts (30.1) and (30.2) above, when should you sell the machine - at the end of year 6, or at the end of year 7? Assume that except for the timing of this sale, all other aspects of the business are the same under both scenarios.

Hint: With the numerical information given you may perform calculations, and then you can use basic financial knowledge to provide an answer. You may be able to avoid calculations altogether if you reason carefully about the impact of depreciation on the scenarios outlined here.

- ✓ You should sell at the end of year 6.
- ✗ The year of the sale is irrelevant, as it has no impact on the project's finances.
- ✗ You should sell at the end of year 7.
- ✗ The question cannot be decided.

Answer:

In a world of flat taxes (constant tax rates), this question is easy to answer. First, note that the book value of the machine at the end of year 6 is \$50,000; this is the same at the depreciation taken in year 7, if the machine was not sold at the end of year 6.

Now imagine that we are at the end of year 6. The machine still has a book value of \$50,000; if we sell it, we shield income equal to this book value from taxes. The value of the tax shield is $TS = 50,000 \times 0.25 = \$12,500$. You can imagine that you pay tax on the full \$60,000 sale proceeds, and then you "get back" TS.

Now imagine that you are at the end of year 7. You actually pay taxes on the full \$60,000 sale proceeds this time; however, you can shield an amount of income equal to the full year-7 depreciation (\$50,000) from taxes. This increases operating cash flow by the same amount TS as above.

Here is a summary of **of changes in relevant cash flows**:

Incremental cash flows	**If machine sold at year-6 end**	**If machine sold at year-7 end**
OCF	–	+TS
Capital spending	$\$60,000 \times (1 - 0.25) + TS$	$\$60,000 \times (1 - 0.25)$

As you can see, the total of the incremental cash flows (i.e. the total cash flows due only to the sale of the machine, **if** it occurred in a given year) is the same, and is also **positive** (i.e., the firm receives cash either way). The decision is then determined by time-value-of-money considerations. Under ordinary economic circumstances it is more advantageous to receive money earlier, as it has a higher NPV. We should sell the machine at the very end of year 6.

31 Cash Flows and NPVs

Quad Enterprises is considering a new three-year expansion project that requires an initial fixed asset investment of \$2.32 million. The fixed asset will be depreciated straight-line to zero over its three-year tax life, after which time it will be worthless. The project is estimated to generate \$1.735 million in annual sales, with costs of \$650,000. The tax rate is 21%.

31.1. What is the level of the OCF over the lifetime of this project?

31.2. What is the level of project cash flows over the lifetime of this project? If the required return (i.e., discount rate) is 12%, what is the project's NPV?

31.3. Assume, in addition to the above, that the project requires an initial investment in net working capital of \$250,000, and the fixed asset will have a market value of \$180,000 at the end of the project.

(31.3.1) What are the project's cash flows?

(31.3.2) What is the new NPV?

Solution:

31.1. Please refer to the attached workbook. We include a static image of the solution here; values are shown in thousands.

	Year 0	**Year 1**	**Year 2**	**Year 3**
Sales		1,735.00	1,735.00	1,735.00
Costs		650.00	650.00	650.00
Depreciation		773.33	773.33	773.33
EBIT		311.67	311.67	311.67
-Taxes		-65.45	-65.45	-65.45
+Depreciation		773.33	773.33	773.33
OCF		1,019.55	1,019.55	1,019.55

31.2. Please refer to the attached workbook. We include a static image of the solution here.

	Year 0	**Year 1**	**Year 2**	**Year 3**
OCF		1,019.55	1,019.55	1,019.55
Change in NWC		0.00	0.00	0.00
Capital Spending	-2,320.00			
Total Cash Flow	-2,320.00	1,019.55	1,019.55	1,019.55
Discount Rate	12%			
NPV	128.79			

31.3. Please refer to the attached workbook. We include a static image of the solution here. Asset was written down to $0; salvage costs are fully taxable.

	Year 0	**Year 1**	**Year 2**	**Year 3**
OCF		1,019.55	1,019.55	1,019.55
Change in NWC	-250.00			250.00
Capital Spending	-2,320.00			142.20
Total Cash Flow	-2,570.00	1,019.55	1,019.55	1,411.75
Discount Rate	12%			
NPV	157.95			

32 Cutting Costs

Masters Machine Shop is considering a four-year project to improve the efficiency of its production facilities. Buying a new machine press for \$385,000 is estimated to result in \$145,000 in annual pretax cost savings. The press falls in the five-year MACRS class, and it will have a salvage value of \$45,000 at the end of the project. The press also requires an initial investment in spare parts inventory of \$20,000, along with an additional \$3,100 in inventory at the end of each succeeding project year. At the end of the project all working capital is recovered.

If the shop's tax rate is 22% and its discount rate is 9%, should the company buy and install the machine press?

You may assume that the machine press is bought on the last day of Year 0. Further, assume that MACRS rules allow you to take a half-year depreciation **in the year of acquisition**. The MACRS schedule given in class and in the textbook already contain the half-year adjustment for the depreciation percentage shown for the first year.

You may find that in certain years a project generates losses. If EBIT is negative the "tax" computed on this amount is also negative. This should be interpreted as a form of tax shield. A business could use this tax shield to "cancel out" positive taxes in other parts of the business. It is also possible for losses to be "carried forward" or to be "carried backward" and for taxes to be reduced for past or future years. For this problem you should assume that this tax shield is usable, but you do not have to decide how. Mathematically, it is just the negative of a regular tax.

Solution:

Fixed Asset	385,000					
Salvage Value	45,000					
MACRS Year	**1**	**2**	**3**	**4**	**5**	**6**
MACRS Depr.	20.00%	32.00%	19.20%	11.52%	11.52%	5.76%
Tax Rate	22.00%					

Assume that salvage cost is SC, residual book value is BV, and tax rate is T_c. Further, assume $SC>BV$. Then you get SC, you pay tax on $SC - BV$, the excess over book value. The tax is $T_c \cdot (SC - BV)$. You are left with what you get minus the tax that you pay: $SC - T_c \cdot (SC - BV) = SC \cdot (1 - T_c) + T_c \cdot BV$. You get a tax benefit of $T_c \cdot BV$. If $BV = 0$, i.e., if the asset had been fully depreciated, then the after-tax cash flow would be $SC \cdot (1 - T_c)$.

	Year 0	Year 1	Year 2	Year 3	Year 4
Savings		145,000.00	145,000.00	145,000.00	145,000.00
Depreciation	77,000.00	123,200.00	73,920.00	44,352.00	44,352.00
EBIT	-77,000.00	21,800.00	71,080.00	100,648.00	100,648.00
-Taxes	16,940.00	-4,796.00	-15,637.60	-22,142.56	-22,142.56
+Depreciation	77,000.00	123,200.00	73,920.00	44,352.00	44,352.00
OCF	16,940.00	140,204.00	129,362.40	122,857.44	122,857.44

Also, note that all working capital is recovered in Year 4, and that at the sale of the capital asset everything above the book value of $22,176.00 is taxable. The impact of these observations is realized in the cell with red text and the cell with yellow background, respectively.

	Year 0	Year 1	Year 2	Year 3	Year 4
OCF	16,940.00	140,204.00	129,362.40	122,857.44	122,857.44
Change in NWC	-20,000.00	-3,100.00	-3,100.00	-3,100.00	29,300.00
Capital Spending	-385,000.00				39,978.72
Total Cash Flow	-388,060.00	137,104.00	126,262.40	119,757.44	192136.16
Discount Rate	9%				
NPV	72,584.84				

33 Scenarios

You are considering a new product launch. The project will cost \$1,950,000, have a four-year life, and have no salvage value; depreciation is straight-line to zero. Sales are projected to be 210 units per year; price per unit will be \$17,500, variable costs per unit will be \$10,600, and fixed costs will be \$560,000 per year. The required return on the project is 12%, and the relevant tax rate is 21%.

33.1. What is the base-case NPV?

33.2. Based on your experience, you think the unit sales, variable cost, and fixed cost projections above are within ±10%. What are the upper and lower bounds, respectively, for these projections? What are the best and worst case scenarios? You may assume that all these variables are independent. Note: Provide only the variables [inputs] that define these scenarios; do not set up the corresponding full models.

33.3. What is the cash break-even level of output for this project (ignoring taxes)?

33.4. What is the accounting break-even level of output for this project (ignoring taxes)?

Solution:

33.1. Please refer to the attached workbook. We include a static image of the solution here.

	Per Unit	**Total**
Sales	17,500.00	3,675,000.00
Variable Cost	10,600.00	2,226,000.00
Fixed Cost		560,000.00
Net Sales		889,000.00

	Year 0	Year 1	Year 2	Year 3	Year 4
Net Sales		889,000.00	889,000.00	889,000.00	889,000.00
Depreciation		487,500.00	487,500.00	487,500.00	487,500.00
EBIT		401,500.00	401,500.00	401,500.00	401,500.00
- Taxes		-84,315.00	-84,315.00	-84,315.00	-84,315.00
+ Depreciation		487,500.00	487,500.00	487,500.00	487,500.00
OCF		804,685.00	804,685.00	804,685.00	804,685.00

	Year 0	Year 1	Year 2	Year 3	Year 4
OCF		804,685.00	804,685.00	804,685.00	804,685.00
Change in NWC					
Capital Spending	-1,950,000.00				
Total Cash Flow	-1,950,000.00	804,685.00	804,685.00	804,685.00	804,685.00

Discount Rate	12%
NPV (base case)	494,109.46

33.2. Please refer to the attached workbook. We include a static image of the solution here.

Error Margin: 10%

	Base Case	Lower Bound	Upper Bound	Best Case	Worst Case
Unit Sales	210	189	231	231	189
Variable Cost	10,600.00	9,540.00	11,660.00	9,540.00	11,660.00
Fixed Cost	560,000.00	504,000.00	616,000.00	504,000.00	616,000.00

33.3. We must produce at least 82 units to become cash flow positive. We must produce at least 152 units to have non-negative net income. We used Solver with a goal of NPV of 0, by only varying the number of units. Solver's result was 180.1563. We round up.

	Units
Cash Break-Even	82
Accounting Break-Even	152
Financial Break-Even	181