MATH 4710 **PRELIM** **SPRING 2019**

Name :________________________________ Cornell NetID:________________________

Instructions

(1) Write clearly using a black or blue pen or pencil. Provide reasons for your answers and explain your computations. For numerical answers, give either a simplified fraction or a decimal answer, which ever comes more easily.

(2) There are three completely independent problems and you have fifty minutes. The first two questions of each problem are the simplest questions.

(3) Keep your cell phone away. Do not use electronic devices. No books or notes are permitted. No communications with anyone during the prelim.

(4) Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty.

Problem 1: A bin contains three different types of disposable flashlights. The following table gives, for each type, the probability that a flashlight of this type lasts over 100 hours and the percentage of flashlights of this type in the bin.

	Type 1	Type 2	Type 3
Probability to last > 100 hours	.8	.6	.2
Percentage of that type in the bin	25%	50 %	25 %

(a) What is the probability that a flashlight chosen uniformly at random in the bin lasts over 100 hours?

Let A the event that the chosen flashlight last more than 100 hours and B_i the event that the chosen flashlight is of type i, $1 \leq i \leq 3$.

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3).$$

Using the table, we get

$$P(A) = .8 \times .25 + .6 \times .5 + .2 \times .25 = .55.$$

(b) We have chosen a flashlight uniformly at random in the bin and it has lasted more than 100 hours. What is the probability that this flashlight is of type 3?

We are asked to compute $P(B_3|A)$ which is

$$P(B_3|A) = \frac{P(B_3 \cap A)}{P(A)} = \frac{P(A|B_3)P(B_3)}{P(A)} = \frac{.05}{.55} = 1/11.$$

(c) For any of the flashlight in the bin, the probability that it lasts over 200 hours given that already lasted 100 hours is 1/2. What is the probability that a flashlight chosen uniformly in the bin will last over 200 hours?

Call A' that event (lasting over 200 ours). Obviously $A \cap A' = A'$ and it follows that

$$P(A') = P(A' \cap A) = P(A'|A)P(A) = .55 \times .5 = .275.$$

Problem 2: A regular die is rolled N times, the rolls being independent. The number N is fixed and larger than 7. It is reported to us that exactly 7 of the N rolls produced a six, call A this event. Let B_i be the event that roll number i produced a six.

(a) What is the probability that the fifth roll produced a six given that exactly 7 of the N rolls produced a six, that is, what is $P(B_5|A)$?

$$P(B_5|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{(1/6)^7(5/6)^{N-7}\binom{N-1}{6}}{(1/6)^7(5/6)^{N-7}\binom{N}{7}} = \frac{7}{N}.$$

(b) Are the events B_3 and B_5 conditionally independent given A?

The previous computation makes it clear that $P(B_3|A) = P(B_5|A) = 7/N$ and we need to compute

$$P(B_3 \cap B_5|A) = \frac{(1/6)^7(5/6)^{N-7}\binom{N-2}{5}}{(1/6)^7(5/6)^{N-7}\binom{N}{7}} = \frac{7 \times 6}{N(N-1)} \neq (7/N)^2.$$

The events B_3, B_5 are not independent given A.

(c) True or False (explain your answer): When U and V have positive probability and $P(U|V) > P(V|U)$ then $U \cap V$ is non empty.

True:

$$P(U|V) = \frac{P(U \cap V)}{P(V)}, \quad P(V|U) = \frac{P(U \cap V)}{P(U)}.$$

If these two quantities are different, it must be that $P(U \cap V) \neq 0$ (otherwise they would both equal 0). An event with positive probability cannot be empty.

Problem 3: A certain type of radio tubes (a technology from the past) have a life time in hours that can be described as a continuous random variable X having probability density function

$$f_X(x) = \begin{cases} 0 & \text{for } x \leq 100, \\ \frac{100}{x^2} & \text{for } x > 100. \end{cases}$$

(a) What is the probability that one of these tubes lasts no more than 150 hours?

Call E that event. We have $P(E) = \int_{100}^{150} \frac{100}{x^2} dx = 1 - 10/15 = 1/3$.

(b) My radio set contains 5 such tubes. What is the probability that exactly two of these tubes will have to be replaced within the first 150 hours of operation? (Explain first what natural hypothesis you are making in order to compute this probability, then compute it).

We can think of using each tube as a sort of experiment. In each of these experiments, the probability of the even F (failure of the tube) is $1/3$. We assume that the behavior (failure or not) of the different tubes are independent. Then we know that the number of failures is a binomial random variable with parameter 5 and $1/3$. So

$$P(\text{exactly two tubes fail before with the first 150 hours}) = (1/3)^2(2/3)^3\binom{5}{2} = 80/243.$$

(c) If a fair coin is flipped $2N$ times (independently), show that the probability that the number of Heads is even is equal to $1/2$.

The number X of Heads is a random variable which is a Binomial $2N, 1/2$. It follows that

$$P(X \text{ is even}) = 2^{-2N} \sum_{i=0}^{N} \binom{2N}{2i}.$$

By the binomial theorem

$$2^{2N} = (1+1)^{2N} = \sum_{j=0}^{2N} \binom{2N}{j} \text{ and } 0 = (1-1)^{2N} = \sum_{j=0}^{2N} (-1)^i \binom{2N}{j}.$$

These two formulas show that the sum of the terms with $j = 2i$ (even number of Heads), $\sum_{i=0}^{N} \binom{2N}{2i}$, is equal to the sum of the odd terms, $\sum_{i=0}^{N-1} \binom{2N}{2i+1}$, and thus equal to half the sum of all terms, that is, $\sum_{i=0}^{N} \binom{2N}{2i} = 2^{2N-1}$. We conclude that $P(X \text{ is even}) = 2^{-2N} \sum_{i=0}^{N} \binom{2N}{2i} = \frac{1}{2}$.

<u>Here is a very different solution.</u> Proceed by induction on N. Call $\mathcal{P}_N$ the property "The probability that $2N$ flips of a fair coin result in an even number of Heads is $1/2$". (a) We verify that $\mathcal{P}_1$ is true: After two flips, an even number of Heads means 0 Heads or 2 Heads. Each has probability $1/4$ so the total probability of that event is $1/2$. (b) Let us prove that $\mathcal{P}_N$ implies $\mathcal{P}_{N+1}$: An even number of Heads in $2(N+1)$ flips is obtained in two disjoint ways: (1) An even number of Heads in the first $2N$ flips and either 0 or 2 Heads in the last two or (2) an odd number of Heads in the first $2N$ flips and 1 Heads in the last two. By the induction hypothesis, the probability of an even number of Heads in the first $2N$ flips is $1/2$ and, also, the probability of an odd number of Heads in the first $2N$ flips is $1/2$ (Why?). Using independence and the fact that, in each of the two cases, the probability that the last two flips produce the needed result is $1/2$, we find that

$$P(\text{even number of Heads in } 2(N+1) \text{ flips}) = 1/2 \times 1/2 + 1/2 \times 1/2 = 1/2.$$

We conclude that property $\mathcal{P}_N$ is true for all N.

The key point for you is to distinguish between making a valid argument or not. No guesses, no hasty conclusion that something $= 1/2$ just because you were told the result should be $1/2$. Much better to realize you do not know how to proceed and stop than to keep going without a clue.