Taking limits to compute a probability

Limits of sequences of subsets (events)

A sequence of subsets A_k of Ω is monotone-increasing if $A_k \subseteq A_{k+1}$ and monotone-decreasing if $A_k \supseteq A_{k+1}$. In both case there is a simple way to define the "limit" of the sequence of subsets (events). Namely,

- the limit of an increasing sequence is the union $\bigcup_1^\infty A_k$. Note that for any integers k_1, k_2, $\bigcup_{k \geq k_1} A_k = \bigcup_{k \geq k_2} A_k$ in this case
- the limit of a decreasing sequence is the intersection $\bigcap_1^\infty A_k$. Note that for any integers k_1, k_2, $\bigcap_{k \geq k_1} A_k = \bigcap_{k \geq k_2} A_k$ in this case.

Consider one or the other case and call A the limit (the union in the first case, the intersection in the second case). We assume that each A_k is in $\mathcal{F}$ so that we can compute its probability $P(A_k)$. Because A is either a countable union or a countable intersection, it is in $\mathcal{F}$ (that is a key property of $\mathcal{F}$). This means we can ask about P(A). In both cases, it is true that $P(A) = \lim_{k \to \infty} P(A_k)$.

An example where this does no apply directly: Every morning, Jade rolls a die until she gets a six. What the chance that she never gets a six?

We can consider the event B_k that Jade gets the first six at the k-th roll. The probability of this event is $\frac{1}{6}(5/6)^{k-1}$. It is tempting to say that B_∞ (a notation for the event that Jade never gets a six) is the limit of B_k BUT that is incorrect. We can only take limits of monotone sequences of events and the sequence B_k is not monotone. In fact, let A_k be the event that Jade gets the first six no earlier than the k-th roll. Now, this is a monotone-decreasing sequence and its limit is indeed the event that Jade never gets a six. So we can compute $P(B_\infty)$ as the limit of $P(A_k)$.

It remains to compute $P(A_k)$. This is essentially the same computation than the one done in class involving a geometric series. In class we proved that $P(B_\infty^c) = 1$ and thus that $P(B_\infty) = 0$. If you compute $P(A_k)$ correctly, you will see that its limits when k tends to infinity is indeed 0.