MATH 4710 **FINAL** **FALL 2021**

Instructions

(1) There are 4 independent problems. The point total is 35. The exam is due on gradescope on Sunday December 12 at 2:00pm (grace period until 6:00pm)

(2) You can write your answers on any reasonable media that is convenient to you as long as you can produce a clean pdf to upload on gradescope. Write your name and Cornell NetID on the top of the first page before you begin.

(3) Write clearly using a black or blue pen or pencil. When you are done, create a pdf file of your work and upload it on gradescope. Make sure to assign problems to pages on gradescope.

(4) **Always provide logical reasons for your answers and explain your computations**. Finding answers is not enough, you need to provide detailed justifications. Being able to explain how you obtain the answer is as important as obtaining the answer. Do not refer to homework problems to justify your solutions. You will receive partial credit for well explained steps in the right direction even if you are not able to provide a complete solution. For numerical answers, give either a simplified fraction (first choice when possible) or a decimal answer, which ever comes more easily. For instance, $3 \times 7^2/(15)^9$ is very acceptable (much better than the expanded version) but $8/12$ is not very good. You can use a scientific calculator to compute decimal answers but only after writing explicitly what it is that you compute and when there is no simple simple rational answer.

(5) You can use your notes, our canvas website including all documents provided there and the book. Do not use other websites or the internet (except for Desmos or a simple electronic calculator). Do not discuss exam problems with other students. Do not discuss exam problems with anyone except Pr. Saloff-Coste (ask Professor Saloff-Coste privately, by email, or in office hours if you have questions. It is OK to do so).

(6) Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty.

Problem 1: (10 pts) In this problem, a deck of N cards, marked 1 to N, has been well shuffled. You are trying to guess the value of each card in the deck from top to bottom. Someone handles the deck and gives you some information as you proceed.

(a-3pts) You make a guess (in whatever way you want) for the card at the top of the deck. The deck handler checks if you are right or wrong and disregards the card. You do not see the card and you are NOT told whether you were right or wrong nor given any information about the disregarded card. This is repeated until the deck is exhausted. Let X be the number of correct guesses you make during the entire process through the deck. What is $E(X)$? Propose a strategy that makes the variance of X smallest.

For each i between 1 and N, let $X_i = 1$ if you guess correctly at step i, and $X_i = 0$ otherwise. We have $X = \sum_1^N X_i$, hence $E(X) = \sum_{i=1}^N E(X_i)$ by linearity of the expectation. At step i, whatever the strategy you use, your guess as probability $1/N$ to be correct because the deck is well shuffled. So $E(X_1) = P(X_i = 1) = 1/N$ and $E(X) = 1$. If you decide to always guess "card 1," at every step, then you are certain to have exactly one guess right and the variance of X is 0 in this case (same if you decide to always guess one particular card instead of card 1).

(b-4pts) This time, after each of your guesses, the deck handler shows you the card on top before it is disregarded. You decide to choose the following strategy: you always name the card with the lowest mark among the cards that have not been disregarded yet. For example, if cards 7, 2, 1 and 5 have been disregarded in the first four steps, you will guess 3 for the card on top because 3 is the lowest of the cards not yet disregarded. Let Y be the number of correct guesses you make during the entire process through the deck. What is $E(Y)$? What is a good approximation of it? What is $\lim_{N\to\infty} P(Y \geq N^{1/4})$?

We define X_i as above and we again have $Y = \sum_1^N X_i$ and $E(X) = \sum_1^N E(X_i)$. This time, at step i, we know the $i-1$ that have already been disregarded and you guess the lowest of the remaining values. The probability that you are right is $1/(N-i+1)$. It follows that $E(Y) = \sum_{i=1}^N 1/(N-i+1) = \sum_1^N 1/i \sim \log N$.. By Markov inequality, $P(Y \geq N^{1/4}) \leq N^{-1/4}E(Y) \sim N^{-1/4}\log N$. The limit as N goes to infinity is 0.

(c-3pts) This time, at each step, you are told if you are right or wrong but you are not told what the disregarded card is. You decide to name the cards in order, naming each card repeatedly until it appears and is disregarded. So, at the beginning, you always say "card 1" until you are correct, that is, card 1 appears and is disregarded; after that, you say card 2 until it appears, if ever, and so on. Let Z be the number of correct guesses you make during the entire process through the deck. What is $E(Z)$? What is a good approximation of it?

Define Z_i to be equal to 1 if, during the entire process, you guess the card with value i correctly. Otherwise $Z_i = 0$. We have $E(Z) = \sum_1^N E(Z_i)$ (same reasoning as before even so the Z_is are different from the X_is). Note that you can be absolutely certain that you will guess the card with value 1 correctly because you will name it until it appears on top. So $P(Z_1 = 1) = 1$. What is $P(Z_2 = 1)$? You will guess the card with value 2 correctly exactly if it appears after card 1 in the shuffled deck. This happens with probability 1/2. What about the value 3 card? You will guess it correctly exactly if the cards with values $1, 2, 3$ appear in this order in the deck. The chance of that is $1/3!$ (there are 3! different possible orders for 3 cards and each is equally likely. More formally, there are $\binom{n}{3}$ different ways to pick the 3 positions to place $1, 2, 3$ in order and $(n-3)!$ ways to place the others cards, and $\binom{n}{3} \times (n-3)!/n! = 1/3!$. By the same token, $P(Z_i = 1) = 1/i!$ and thus $E(Z) = \sum_{i=1}^N 1/i! \sim e - 1$.

Problem 2: (6 pts) 10 professors go to a formal dinner. They are named $1, 2, \ldots, 10$. Each of the $\binom{10}{2}$ pairs of professors are friends with probability p, and the events that various pairs are pairs of friends are mutually independent. So the events A= 1 and 2 are friends, and B=1 and 3 are friends, and C=2 and 3, are friends are mutually independent. The large dinner room has ten round tables and each table can seat ten people.

The professors arrive one by one, in order. When professor j arrives, if one or more of their friends is already sitting at a table, they join one of their sitting friends at a table. If none of the earlier arrivals are friends of professor j, professor j sits at a new table.

(a-2pts) Let X be the number of tables used during this dinner. What is $E(X)$? What is a good approximation for it if $p > 1/2$?

Let $X_i = 1$ if professor i starts a new table, and 0 otherwise. The probability that $X_i = 1$ is the probability that professor i has no friends among professors $1, \ldots, i-1$. Since each is his friend with probability p, independently, $P(X_i = 1) = (1-p)^{i-1}$. Now, $X = \sum_{i=1}^{10} X_i$ and (if p is not too small for the estimate)

$$E(X) = \sum_{i=1}^{10} E(X_i) = \sum_{i=1}^{10} (1-p)^{i-1} = \frac{1-(1-p)^{10}}{p} \sim 1/p.$$

(b-2pts) What is the probability that $X = 1$? (do not try to simplify the answer)

We have $X = 1$ when all professors sit at the same table. Given that professors $1, 2, \ldots, i-1$ are sitting at the same table when professor $i \geq 2$ arrives, the probability that he joins them is $1-(1-p)^{i-1}$. Let A_i be the event that professors $1, \ldots, i$ sits at the same table. Then, $P(A_1) = 1$ and, for $i \geq 2$,

$$P(A_i) = P(A_1)P(A_2|A_1)\cdots P(A_i|A_{i-1}) = 1 \times \prod_{k=2}^{i}(1-(1-p)^{k-1}).$$

It follows that

$$P(X=1) = \prod_{i=2}^{10}(1-(1-p)^{i-1}) = \prod_{i=1}^{9}(1-(1-p)^{i}).$$

(c-2pts) What is the probability that $X = 10$ (gives the answer in the simplest form you can)?

Let B_i be the events that professors $1, \ldots, i$ all sit at different tables. Observe that $P(B_i|B_{i-1}) = (1-p)^{i-1}$ (given that professors $1, \ldots, i-1$ sits at different tables, the probability that professor i will seat at a new table on arrival is $(1-p)^{-i-1}$). Also, $P(B_1) = 1$. It follows that $P(B_i) = P(B_1)P(B_2|B_1)\cdots P(B_i|B_{i-1})$, and

$$P(X=10) = \prod_{i=2}^{10}(1-p)^{i-1} = (1-p)^{\sum_{i=2}^{10}(i-1)} = (1-p)^{45}.$$

We have used $\sum_{k=1}^{n} k = n(n+1)/2$.

Problem 3: (11 pts) The random variables X, Y are independent with respective densities

$$f_X(x) = 6x(1-x)\mathbf{1}_{(0,1)}(x), \quad f_Y(y) = 2y\mathbf{1}_{(0,1)}(y).$$

(a-3pts) Compute $P(X < Y)$.

The joint density of the pair (X, Y) is $f(x,y) = 12xy(1-x)\mathbf{1}_{(0,1)}(x)\mathbf{1}_{(0,1)}(y)$. Let T the triangle

$$T = \{(x,y) : 0 < x < y < 1\}.$$

We have

$$P(X < Y) = 12 \int\int_T xy(1-x)dxdy = 12\int_0^1 x(1-x)\left\{\int_x^1 ydy\right\}dx = 6\int_0^1 x(1-x)(1-x^2)dx.$$

We expand and compute $x(1-x)(1-x^2) = x - x^2 - x^3 + x^4$ and

$$\int_0^1 x(1-x)(1-x^2)dx = \int_0^1 (x - x^2 - x^3 + x^4)dx = \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{7}{60}.$$

This gives $P(X < Y) = 7/10$.

(b-2pts) For $(x,y) \in (0,1)^2$, consider the map $(x,y) \mapsto (u,v), u = -\log x, v = y/x$. What is the image R of $K = (0,1)^2$ under this map? (hint: it is a subset of the first quadrant in the (u,v)-plane). Prove that the map in question is a bijection between $(0,1)^2$ and R.

By definition, u and v are positive and $ve^{-u} = y < 1$ so that R is contained in the region

$$\{(u,v) : 0 < u < +\infty, 0 < v < e^u\},$$

that is, the region under the graph of the function $v = e^u$ in the first quadrant. Moreover, given a point in that region, $\{(u,v) : 0 < u < +\infty, 0 < v < e^u\}$, the pair (x,y) with $x = e^{-u}$ and $y = ve^{-u}$ lies in $(0,1) \times (0,1)$. It follows that $R = \{(u,v) : 0 < u < +\infty, 0 < v < e^u\}$ and the given map from $(0,1)^2$ to R is a bijection with inverse $(u,v) \mapsto (e^{-u}, ve^{-u})$, going from R to $(0,1)^2$.

(c-3pts) Consider the random variables $U = -\log X, V = Y/X$. Find the joint density function of the pair (U,V).

The joint density function of (U,V), $f_{(U,V)}(u,v)$, is obtained by applying the change of variable formula (6.32) – Fact 6.41, in the book, that is,

$$f_{(U,V)}(u,v) = f_{(X,Y)}(x(u,v), y(u,v)) \left|\det\begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v}\end{pmatrix}\right|, \text{ if } (u,v) \in R,$$

and 0 otherwise. Here, $f_{(X,Y)}(x,y) = 12xy(1-x)\mathbf{1}_{(0,1)}(x)\mathbf{1}_{(0,1)}(y)$ and $x(u,v) = e^{-u}, y(u,v) = ve^{-u}$. It follows that

$$\left|\det\begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v}\end{pmatrix}\right| = \left|\det\begin{pmatrix} -e^{-u} & 0 \\ -ve^{-u} & e^{-u}\end{pmatrix}\right| = e^{-2u},$$

and

$$f_{(U,V)}(u,v) = 12ve^{-4u}(1-e^{-u})\mathbf{1}_R((u,v)).$$

(d-3pts) Are U and V independent? What is the density function of U?

The density of the pair (U,V) does not split as a product because of the term $\mathbf{1}_R$. For instance,

$$f((1/2,1/2)) > 0, f((3/4, e^{-3/4}/2)) > 0$$

because $1/2 < e^{-1/2} \approx .6$ but $f((3/4,1/2) = 0$ because $e^{-3/4} < 1/2$. This shows that $f((u,v)) \neq f_1(u)f_2(v)$ because if that was the case, we would have both $f_1(3/4)f_2(1/2) = 0$ and $f_1(3/4) > 0, f_2(1/2) > 0$ which is a contradiction.

Let us compute the first marginal, f_U, ($f_U(u) = 0$ is u is not positive) using the formula

$$f_U(u) = 12e^{-4u}(1-e^{-u})\int_{-\infty}^{+\infty} v\mathbf{1}_K((u,v))dv = 12e^{-4u}(1-e^{-u})\int_0^{e^u} vdv = 6e^{-2u}(1-e^{-u}), u > 0.$$

Problem 4: (8 pts)

(a-3pts) The pair (X, Y) has joint density $f(x, y) = 2$ if $0 < x < y < 1$ and 0 otherwise. What is the density function of X? What is density function of $Z = X + Y$?

The density function of X is obtained by computing $f_X(x) = \int f(x,y)dy = 2\int_x^1 dy = 2(1-x)$ if $0 \le x \le 1$ and $f_X(x) = 0$ otherwise. To find the density function of $Z = X + Y$, we first compute the distribution function of Z for $z \in [0, 2]$ ($F_Z(z) = 0$ if $z \le 0$ and $F_Z(z) = 1$ if $z \ge 2$). For $z \in [0, 2]$,

$$\begin{aligned}
F_Z(z) &= P(X + Y \le z) = 2\int\int_{\{0<x<y;x+y\le z\}} dxdy \\
&= 2\int_0^{\min\{1,z\}} \left\{\int_0^{\min\{z-y,y\}} dx\right\} dy = 2\int_0^{z/2} ydy + 2\int_{z/2}^{\min\{z,1\}} (z-y)dy \\
&= \frac{1}{4}z^2 + 2z(\min\{1,z\} - z/2) - ((\min\{1,z\})^2 - (z/2)^2) \\
&= \begin{cases} z^2/2 & \text{if } 0 \le z \le 1, \\ 1 - \frac{1}{2}(z-2)^2 & \text{if } 1 \le z \le 2. \end{cases}
\end{aligned}$$

Taking derivative, we find the density function of Z,

$$f_Z(z) = \begin{cases} z & \text{if } 0 \le z \le 1, \\ 2 - z & \text{if } 1 \le z \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(b-2pts) The random variables X and Y are independent with respective density $f_X(x) = xe^{-x}\mathbf{1}_{(0,+\infty)}(x)$ and $f_Y(y) = e^{-y}\mathbf{1}_{(0,+\infty)}(y)$. What is the density of $Z = X + Y$?

Solution 1: We recognize that X is a Gamma RV with parameter $\alpha = 2$ and $\lambda = 1$ and Y is an exponential RV with $\lambda = 1$. This means they have moment generating functions $[1/(1-t)]^2$ and $1/(1-t)$, for $t < 1$, respectively. Because X and Y are independent, $M_Z(t) = M_X(t)M_Y(t) = [1/(1-t)]^3$, and this shows that Z is a Gamma RV with parameter $\alpha = 3$, $\lambda = 1$. The density of such a random variable is $f_Z(z) = \frac{1}{2}z^2e^{-z}$.

Solution 2: Write $f_Z(z) = \int_0^z xe^{-x}e^{-(z-x)}dx = e^{-z}\int_0^z xdx = \frac{1}{2}z^2e^{-z}$.

Solution 3: We recognize that X is a Gamma RV with parameter $\alpha = 2$ and $\lambda = 1$ and Y is an exponential RV with $\lambda = 1$. As they are independent, 5.2.5 in the Notes implies that $Z = X + Y$ is a Gamma with parameters $\alpha = 3$ and $\lambda = 1$.

(c-3pts) If X and Y are independent exponential λ random variables. What is the moment generating function M of $Z = X - Y$? Compute $E(Z^2)$ in two different ways.

The Moment generating function of Z is

$$M_Z(t) = E(e^{t(X-Y)}) = E(e^{tX}e^{-tY}) = M_X(t)M_Y(-t) = \frac{\lambda^2}{\lambda^2 - t^2}.$$

First computation: (Using MGF) We compute

$$M_Z'(t) = \frac{2\lambda^2 t}{(\lambda^2 - t^2)^2}, M_Z''(t) = \frac{8\lambda^2t^2(\lambda^2 - t^2) + 2\lambda^2(\lambda^2 - t^2)^2}{(\lambda^2 - t^2)^4}$$

It follows that $E(Z^2) = \frac{2\lambda^6}{\lambda^8} = \frac{2}{\lambda^2}$.

Second computation: (Using $E(X) = E(Y) = 1/\lambda$ and $E(X^2) = E(Y^2) = 2/\lambda^2$) We know that $E(X^2) = E(Y^2) = 2/\lambda^2$ and $E(XY) = E(X)E(Y) = 1/\lambda^2$. It follows that $E((X-Y)^2) = E(X^2) + E(Y^2) - 2E(XY) = 2/\lambda^2$.