Conditional Probability

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Reading: Devore 2.4, 2.5

Conditional Probability

How can new information be used to update our beliefs about the likelihood of certain events?

Example: (Rolling a fair die twice) Let

$$A = \text{event that the sum of the rolls is at least 10}$$
$$B = \text{event that the first roll is a 5}$$

Then

$$P(A) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$$

Now I tell you that B happened. *Given this information*, what now is the probability that A happened?

$$P(\text{second roll is 5 or 6}) = \frac{1}{3}$$

Conditional Probability

Definition: Let A and B be events, and suppose $P(B) > 0$. The *conditional probability* that A occurs, given that B occurs ("probability of A given B") is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Note: $P(A|B)$ is not defined if $P(B) = 0$.

Example: Poker

A poker hand (5 cards) is dealt from a 52-card deck. Suppose you know that the hand contains at least 2 aces.

Question: What's the probability that the hand contains 3 aces?

Sample space = all 5-card hands

$A = \{3 \text{ aces}\}$, $B = \{\geq 2 \text{ aces}\}$

$A \subseteq B \implies P(A|B) = P(A \cap B)/P(B) = P(A)/P(B)$, where

$$P(A) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}},$$

$$P(B) = 1 - P(B^c) = 1 - \left[\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} + \frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} \right].$$

Multiplication Rule for Probabilities

According to the definition of conditional probabilities,

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Example: What's the probability that 2 cards drawn from a 52-card deck are both clubs?

Letting C_i be the event that the i^{th} card drawn is a club,

$$\begin{aligned} P(2 \text{ clubs}) &= P(C_1 \cap C_2) \\ &= P(C_1)P(C_2|C_1) = \frac{13}{52} \times \frac{12}{51} \end{aligned}$$

(Another way is $\binom{13}{2} / \binom{52}{2}$.)

Multiplication Rule for Probabilities

The multiplication rule can be extended to more events, e.g.,

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Example: The probability that 5 cards drawn from a 52-card deck are all clubs is

$$P(5 \text{ clubs}) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48}$$

Law of Total Probability

Can reduce the calculation of a probability into the calculation of simpler ones.

The events $A_1, \ldots, A_n$ form a *partition* of the sample space $\mathcal{S}$ if:

1. they're mutually exclusive (i.e., $A_i \cap A_j = \emptyset$ for all i, j);
2. $\cup_{i=1}^{n} A_i = \mathcal{S}$.

Law of Total Probability

Law of Total Probability: If B is an event, and $A_1, \ldots, A_n$ is a partition of $\mathcal{S}$, then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Why?

$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$

A_i's are disjoint $\implies$ $(B \cap A_i)$'s are disjoint

So

$$P(B) = P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) = \sum_{i=1}^{n} P(B \cap A_i).$$

Bayes' Rule

A useful tool for updating probabilities based on new information.

Bayes' Rule: Let $A_1, \ldots, A_n$ be a partition of $\mathcal{S}$, and let B be an event where $P(B) > 0$. If $P(A_j) > 0$ for all j, then

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$

Example: Drug Testing

The proportion of employees using illicit drugs is 0.001. A drug test correctly detects drug users 99% of the time, and gives a "false positive" for 2% of non-drug users.

Question: If an employee tests positive for drugs, what's the probability that they actually take drugs?

$D = \{\text{employee takes drugs}\}$
$P_o = \{\text{employee tests positive}\}$

$$
\begin{aligned}
P(D|P_o) &= \frac{P(D \cap P_o)}{P(P_o)} \\
&= \frac{P(P_o|D)P(D)}{P(P_o|D)P(D) + P(P_o|D^c)P(D^c)} \\
&= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.02 \times 0.999} \approx 0.047.
\end{aligned}
$$

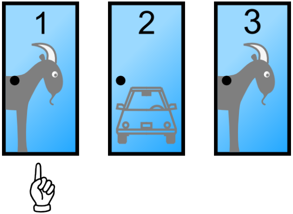
Example: Monty Hall Problem

A question from Marilyn vos Savant's *Ask Marilyn* column in *Parade Magazine* (1990):

You're on a game show, trying to select a door with a car behind it.

- 2 goats and 1 car are randomly distributed behind three doors.
- You pick a door (without opening it).
- The host (Monty Hall) picks a door and opens it.

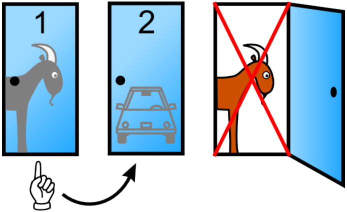
Question: Should you switch to the remaining door?



Example: Monty Hall Problem

Assumptions: The host must always:

1. open one of the 2 doors not picked by you
2. reveal a goat and never the car (if the two remaining doors both have goats, the host will randomly open one of them with equal probability)
3. offer the chance to switch between the originally chosen door and the remaining closed door



Example: Monty Hall Problem

Without loss of generality, suppose you select door 1. New information: the host opens door 3 with a goat.

$$A_i = \{\text{car is behind door } i\}$$
$$B = \{\text{host opens door 3 with a goat}\}$$

By Bayes' rule:

$$P(A_2|B) = \frac{P(B|A_2)P(A_2)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)}$$
$$= \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3}} = \frac{2}{3}.$$

$P(A_1|B) = \frac{1}{3}$, by a similar calculation or $P(A_1|B) = 1 - P(A_2|B)$.

$P(A_2|B) > P(A_1|B) \implies$ You should always switch!

Independent Events

Sometimes new information doesn't change the probability that an event occurs.

Example: What's the probability that it will rain today, given that the first car I see on the street today is blue?

Intuitively, two events A and B are *independent* if knowing that A occurs doesn't change the probability that B occurs.

Independent Events

Definition: Events A and B are *independent* if

$$P(A \cap B) = P(A)P(B);$$

otherwise, A and B are *dependent*.

Note: Suppose A and B are independent. If $P(B) > 0$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

and if $P(A) > 0$,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

Independence vs. Mutual Exclusivity

Recall that A and B are *mutually exclusive* if $A \cap B = \emptyset$.

Questions:

1. If A and B are mutually exclusive, must they be independent?

   A. Yes
   B. No

2. If A and B are independent, must they be mutually exclusive?

   A. Yes
   B. No

Independent Events: Example

A card is chosen at random from a 52-card deck. Let A be the event that the chosen card is a spade, and let B be the event that the chosen card is an ace.

Question: Are A and B independent?

Independence: More than 2 Events

The events $A_1, \ldots, A_n$ are **independent** if for every subset J of $\{1, \ldots, n\}$,

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j)$$

Example: For A_1, A_2, A_3 to be independent, we need

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1 \cap A_2) &= P(A_1)P(A_2) \\ P(A_1 \cap A_3) &= P(A_1)P(A_3) \\ P(A_2 \cap A_3) &= P(A_2)P(A_3) \end{aligned}$$

Independent Events: Example

Shipments of tablet computers in the 4^{th} quarter of 2010 suggest the following probabilities for a consumer's tablet preferences:

iPad	Android	Other
0.75	0.22	0.03

Suppose the tablet preferences of 3 individuals are independent.

Question: What's the probability that all 3 individuals prefer Androids?

A. 0.78^3

B. 0.22^3

C. $0.03 * 0.22$

D. None of these

Summary

- *Conditional probability* lets you update probabilities using new information.
- The *law of total probability* decomposes the probability of an event into probabilities of smaller events.
- *Bayes' rule* uses conditional probability and the law of total probability to "turn conditional probabilities around".
- Events are *independent* if the probability of their intersection is the product of their individual probabilities.