Partitions of n with k parts

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A partition of n with k parts is a vector $\mathbf{x} = (x_1, \ldots, x_k)$ with non-negative integer coordinates that sum to n. Observe that there are only finitely many such objects because each coordinate must be a number in $\{0, 1, \ldots, n\}$ and so there are at most n^k partitions of n with k parts. But how many are there exactly?

To solve this question, we start with an easier one which is very similar. How many partitions of n with $k parts are there with no zero entries? More formally, how many elements are there in the set

$$\mathbf{V} = \mathbf{V}(n,k) = \left\{ \mathbf{x} = (x_1, \ldots, x_k) : x_i \in \{1, 2, \ldots\} \text{ and } \sum_1^k x_i = n \right\}.$$

To figure this out, we use a famous argument called **stars and bars argument**.
Imagine we have n stars arranged in a raw

$$\underbrace{* * * \ldots\ldots * **}_{n*}.$$

Now, suppose we have $k-1$ bars $|$ which can be inserted in the $n-1$ empty spaces between the n stars with at most one bar in one slot. Inserting the bars give configurations of stars and bars which looks as follows (for the sake of the illustration we use $n = 9$ and $k = 5$)

$$*| * * * | * | * * * |*$$

or (an other example)

$$*| * | * | * * | * * * *.$$

How many such distinct configurations are there? That is not too hard to figure out because, in order to create a configuration, we just need to choose the $k-1$ slots in which we insert the $k-1$ bars that define the configuration. As already noted, there are $n-1$ slots between the stars.
It follows that the total number of such configurations in $\binom{n-1}{k-1}$.

Now, observe that any one of these configurations is associated with a unique element in $\mathbf{V}$ and vice-versa.
This is simply because the inserted bars decompose the n stars into k parts. For instance

$$*|***|*|***|* \text{ corresponds to } (1,3,1,3,1)$$

and

$$*|*|*|**|**** \text{ corresponds to } (1,1,1,2,4).$$

This gives a one -to-one and onto map from configurations of stars and bars" and the set $\mathbf{V}$ and it follows that they both have the same number of elements, $\binom{n-1}{k-1}$.

Now, our original question was to count the number of elements of (compare to the definition of $\mathbf{V}$)

$$\mathbf{U} = \mathbf{U}(n,k) = \left\{ \mathbf{x} = (x_1, \ldots, x_k) : x_i \in \{0,1,2,\ldots\} \text{ and } \sum_1^k x_i = n \right\}.$$

To do that it suffices to observe that adding 1 to each coordinate of an element $\mathbf{x} = (x_1, \ldots, x_k)$ of $\mathbf{U}(n,k)$ automatically give us an element of $\mathbf{V}(n+k,k)$, that is a k-vector with integer coordinates all greater or equal to 1 and summing to $n+k$. The map

$$\mathbf{U}(n,k) \to \mathbf{V}(n,k), \quad \mathbf{x} = (x_1, \ldots, x_k) \mapsto \mathbf{y} = (x_1+1, \ldots, x_k+1)$$

is on-to-one and onto. It follows that $|\mathbf{U}(n,k)|$, the number of elements in $\mathbf{U}(n,k)$, is equal to

$$|\mathbf{U}(n,k)| = |\mathbf{V}(n+k,k)| = \binom{n+k-1}{k-1}.$$

The number of partitions of n with k non-zero parts, that is, the number of vectors $\mathbf{x} = (x_1, \ldots, x_k)$, $x_i \in \{1,2,\ldots\}$, $1 \le i \le k$ and $\sum_{i=0}^k x_i = n$, is $\binom{n-1}{k-1}$.

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Exercise: We pick a k-tuple in $\{0,1,\ldots,n\}^k$ uniformly at random . What is the probability that the picked k-tuple is a partition of n with k parts?

Set partitions A partition of a set A is a set of disjoint non-empty subsets of A whose union is A. Let us restrict ourselves to the case of a finite set A, say with n elements. Any given set partition is made of a certain number, say k, of non-empty disjoint subsets $A_i \subseteq A, 1 \leq i \leq k$. So, for instance, there is only one set partition of A in one subset: $A_1 = A$. And there is n set partitions of A into 2 subsets one of which is a singleton, namely, if $A = \{a_1, \ldots, a_n\}$,
each of $A_1 = \{a_i\}, A_2 = A_1^c, 1 \leq i \leq n$, is a set partition of A with two subsets one of which is a singleton. How many set partition of A have exactly $n-1$ subsets?

Call B_n the number of set partitions of a set A with n elements. Why does B_n depend only on n and not on what kind of elements the set A is made of? Compute B_1, B_2, B_3, B_4. Find a counting proof that

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$