Game Theory

INFO 4220 – Networks II: Market Design

In this lecture:

We will continue talking about Game Theory:

- Introduce the concept of equilibrium in dominant strategies
- Introduce Nash Equilibrium
- Explain what is a Pareto superior outcome
- Introduce games of incomplete information
- Explain how we find a Nash Equilibrium in games of incomplete information

We will discuss why we talk about game theory in a market design course

Definition of a Game

A Normal (or Strategic) Form Game consists of:

- A set of n players denoted by: N$= \{i_1, i_2, \dots, i_n\}$
- Each player $i, i \in N$, has a pure strategy set $\mathrm{S_i}$
- For each player i there is payoff function π^i that assigns a real number representing the utility player i will obtain for all possible combinations of strategies chosen be all players of the game $S = S_1 \times S_2 \times \cdots \times S_n$
- We will denote by $s_i \in S_i$ a strategy for player i, and by $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ the strategies of the other players
- In this notation, $(s_i, s_{i-1}) \in S$ denotes a strategy profile (or outcome) of the game (i.e. the specific strategies chosen by all the players)

Example of a Game

Alice and Bob are playing the following card game. Each of them has a red card and a black card. Playing the red card give 400 points to the other player (i.e. if Alice plays the red card, Bob gets 400 points). Playing the black card give 300 points to yourself (i.e. if Alice plays the black card, she gets 300 points). Both players reveal the card they play at the same time.

Example of a Game

- Consider the game $G = \{N, (S_i)_{i \in N}, (\pi_i)_{i \in N}\}$
  - N={Alice,Bob}
  - $S_{Alice} = S_{Bob} = \{Red, Black\}$
  - There were 4 possible outcomes *(strategy profiles)*: (Red, Red), (Red, Black), (Black, Red), (Black, Black)
  - The fuction π_i needs to specify how much each player would earn in each of the possible outcomes
- We can represent all this in a table

Representation in Matrix Form

		Bob	
Alice		Red	Black
	Red	(400, 400)	(0, 700)
	Black	(700, 0)	(300, 300)

- Normal form games are usually represented in a matrix. The matrix above shows the game in the example
- This game above is typically referred as the prisoners' dilemma
- This simple game applies to many settings. For example, firms engaging in price wars, or the game we played earlier today

Equilibrium

- Outlining all the possible outcomes and payoffs tells us little about the game
- What we want is to be able to predict how a game is going to end
- We also need to define characteristics that equilibrium outcomes should have:
    - Ideally, we want a single outcome, i.e. a *unique equilibrium*
    - Equilibrium may not be unique, or may not exist at all

Notation

- We said that the outcome of a game $s = \left(s^1, s^2, \dots, s^N\right)$ is a list of what the N players of the game are playing
- If we pick player i, their strategy would be s^i
- If we remove s^i from the outcome, we denote the strategies of all other players as:
  - $s^{-i} = \left(s^1, s^2, \dots, s^{i-1}, s^{i+1}, \dots, s^N\right)$
- Using this notation, we can express the outcome s as $s = (s^i, s^{-i})$

Dominant Strategies

A strategy $\tilde{s}^i \in S^i$ is a weakly dominant strategy for player i if no matter what strategies all the other players are using, playing $\tilde{s}^i$ always maximizes player $i's$ profits.

Formally:

$\pi^i(\tilde{s}^i, s^{-i}) \geq \pi^i(s^i, s^{-i})$, for every $s^i \in S^i$

$\pi^i(\tilde{s}^i, s^{-i}) > \pi^i(s^i, s^{-i})$, for at least one stratetgy $s^i \in S^i$

Strictly dominant strategies does not admit any equality. All payoffs must be strictly greater.

Dominant Strategies

Let's go back to the red/black game and propose that black is a dominant strategy

Let's see what happens with Alice's payoffs for the different strategies Bob may use:

- If Bob plays $a^{Bob} = black$
  - $\pi^{Alice}(black, black) = 300 > 0 = \pi^{Alice}(red, black)$
- If Bob plays $a^{Bob} = red$
  - $\pi^{Alice}(black, red) = 700 > 300 = \pi^{Alice}(red, red)$
- Thus, black is a dominant strategy for Alice

Note this game is symmetric, thus black is also a dominant strategy for Bob

Example 1

Alice and Bob are deciding what to do during the weekend. Alice would prefer to go skiing and Bob would prefer to stay home and watch a movie. They both want to do something together and would not enjoy their weekend if they go separate ways. Let's say that the person that gets to do their favorite activity gets a utility of 200 if the other person joins. However, the person that joins their less preferred activity only get a utility of 100. If they go separate ways, they get a utility of zero.

Equilibrium in Dominant Strategies

An outcome $\left(\tilde{s}^1, \tilde{s}^2, \ldots, \tilde{s}^N\right)$ (where $\tilde{s}^i \in S^i$ for every $i = 1,2, \ldots, N$) is an equilibrium in dominant strategies if $\tilde{s}^i$ is a dominant strategy for each player i

	Bob		
Alice		Skiing	Movie
	Skiing	(200, 100)	(0, 0)
	Movie	(0, 0)	(100, 200)

Although an equilibrium in dominant strategies is a reasonable prediction of how people may interact, in many games this equilibrium does not exist

Example 3

In the following game, assume you are player 2.
What strategy will you play:

		Player 2	
Player 1		Left	Right
	Up	(8, 10)	(-100, 9)
	Down	(7, 6)	(6, 5)

Dominated Strategies

An strategy $\tilde{s}^i \in S^i$ is a weakly dominated strategy for player i if:

$\pi^i\left(\tilde{s}^i, s^{-i}\right) \leq \pi^i\left(s^i, s^{-i}\right)$, for every $s^i \in S^i$

$\pi^i\left(\tilde{s}^i, s^{-i}\right) < \pi^i\left(s^i, s^{-i}\right)$, for at least one strategy $s^i \in S^i$

A strictly dominated strategy does not admit any equality.

When you have strictly dominated strategies, you can eliminate them.

If they are weakly dominated, you can eliminate them, although that may lead you to eliminate some potential equilibria.

Iterative Elimination of Dominated Strategies

	Bob			
Alice		Left	Mid	Right
	Up	(10,4)	(5,3)	(4,3)
	Mid	(6,6)	(3,2)	(3,2)
	Down	(3,3)	(4,5)	(8,2)

- In this game there are no dominant strategies:
  - If Bob plays Left or Mid, Alice should play Up
  - If Bob plays Right, Alice should play Down
  - If Alice plays Up or Mid, Bob should play Left
  - If Alice Plays Down, Bob should play mid
- Bob will never play Right. We can just delete it
- Alice will now never play Mid or Down. We delete them
- Bob will not play Mid. We are left with (Up, Left)
- **Note this is not an equilibrium in dominant strategies,** but it is a reasonable prediction of how the game my play out.

Nash Equilibrium

- The concept of equilibrium in dominant strategies fails to predict an equilibrium in many games
  - Ans you won't always find an equilibrium by deleting dominated strategies
- John Nash developed the following concept of equilibrium (which is the most commonly used):
  - An outcome $\left(\hat{s}^1, \hat{s}^2, \ldots, \hat{s}^N\right)$ (where $\hat{s}^i \in S^i$ for every $i = 1,2, \ldots, N$) is a Nash Equilibrium if no player would find it beneficial to deviate provided that all other players do not deviate from their strategies played at the Nash Equilibrium. Formally $(\hat{s}^i, \hat{s}^{-i})$ is a N.E. if:

$$\boldsymbol{\pi^i\left(\hat{s}^i, \hat{s}^{-i}\right) \geq \pi^i\left(s^i, \hat{s}^{-i}\right)}, \textbf{ for every } \boldsymbol{s^i \in S^i}$$

Nash Equilibrium

- Most common way to find if an outcome is a Nash Equilibrium is to check if any player has incentives to deviate

	Bob		
Alice		Skiing	Movie
	Skiing	(200, 100)	(0, 0)
	Movie	(0, 0)	(100, 200)

- Nash Equilibrium need not be unique
- Nash Equilibrium does not always exist

Non-Existence of Nash Eq.

- Let's say now that Alice and Bob had an argument over what to do. Alice still wants to something with Bob. But Bob now wants to something alone

		Bob	
		Skiing	Movie
Alice	Skiing	(200, 0)	(0, 200)
	Movie	(0, 100)	(100, 0)

- In this example there isn't an equilibrium (in pure strategies)
- At every possible outcome, one of the players would like to deviate

Best response functions

We can define a best response function as:

- A function that defines the best strategy a player can take for every possible strategy of the other players
- Formally, a function $R^i(s^{-i})$, that for given strategies of players $1,2,\ldots,i-1,i+1,\ldots,N$, assigns a strategy $s^i = R^i\left(s^{-i}\right)$ that maximizes player i's payoff $\pi^i\left(s^i, s^{-i}\right)$
- If $\hat{s}$ is a Nash equilibrium outcome, then $\hat{s}^i = R^i(\hat{s}^{-i})$ for every player i

Best Response Function

- Going back to the skiing/movies game:

		Bob	
Alice		Skiing	Movies
	Skiing	(200, 100)	(0, 0)
	Movies	(0, 0)	(100, 200)

$$R^{Alice}(a^{Bob}) = \begin{cases} skiing\ if\ a^{Bob} = skiing \\ movies\ if\ a^{Bob} = movies \end{cases}$$

and

$$R^{Bob}(a^{Alice}) = \begin{cases} skiing\ if\ a^{Alice} = skiing \\ movies\ if\ a^{Alice} = movies \end{cases}$$

Pareto Comparisons

An outcome is pareto superior to another alternative outcome if it makes at least one player better, and no one else worst. Formally:

- An outcome $\hat{s}$ Pareto dominates (or is Pareto superior) the outcome $\tilde{s}$ if:
  - For every player $i, \pi^i(\hat{s}) \geq \pi^i(\tilde{s})$, and
  - There exists at least one player j for whom $\pi^j(\hat{s}) > \pi^j(\tilde{s})$
- An outcome s^* is called Pareto efficient (or Pareto optimal) if there does not exist any outcome which Pareto dominates s^*

Pareto Efficiency and Equilibrium

- An outcome that is pareto efficiency is not necessary and equilibrium

		Bob	
Alice		Red	Black
	Red	(400, 400)	(0, 700)
	Black	(700, 0)	(300, 300)

- For example, in the Red/Black game, Black/Black is pareto dominated by Red/Red, but the only equilibrium in the game is Black/Black

Let’s Play a Game

This game will give you a bonus for PS1 equal to your score/200

Game: Somehow, after grading PS1 your score and the score of your friend got lost in CANVAS (along with your submissions). Despite our best efforts, we (i.e. the teaching staff) cannot recover them. We know both of you had the exact same score and that it was in between 60 and 100, but we cannot recall the exact number. After a long meeting, we decided that we are going to ask each of you separately how much was your score. If you both say the same number, then we will give you both that score. If you say different numbers, we will know that the one of you saying the highest number is lying. Thus, we will give you both the lower score. But, to reward the honesty of the student reporting the lower score, we will give them a +4 bonus, and to the student reporting the higher score we will give a -4 penalty.

Enter your score in the quiz in CANVAS. To determine your payoff, you will be paired with a random student in the course.

Let’s Play Again

This game will give you a bonus for PS1 equal to your score/200

Game: Somehow, after grading PS1 your score and the score of your friend got lost in CANVAS (along with your submissions). Despite our best efforts, we (i.e. the teaching staff) cannot recover them. We know both of you had the exact same score and that it was in between 60 and 100, but we cannot recall the exact number. After a long meeting, we decided that we are going to ask each of you separately how much was your score. If you both say the same number, then we will give you both that score. If you say different numbers, we will know that the one of you saying the highest number is lying. Thus, we will give you both the lower score. But, to reward the honesty of the student reporting the lower score, we will give them a +30 bonus, and to the student reporting the higher score we will give a -30 penalty.

Enter your score in the quiz in CANVAS. To determine your payoff, you will be paired with a random student in the course.

Game Results

Round 1	Round 2
Mean = 89.48	Mean = 68
Median = 96	Median = 60
Min = 60	Min = 60
Max = 100	Max = 100

Games of Incomplete Information

- So far, we have assumed that all players in the game know the payoff function of all players
- Typically, this is not the case as players may have private information
- The way this is modelled in game theory is by assuming players have a distribution of payoff functions:
  - We say that "Nature" will choose the payoff function
  - For example, two different payoff functions with probability p and (1-p)

Bayesian Games

A Bayesian Games consists of:

- A set of N players: $I = \{1,2,\dots,N\}$
- Each player $i, i \in I$, has an action set A^i
- For each player i, a set T_i of possible signals
- A probability distribution f over the set of signal profiles $T = T_1 \times T_2 \times \cdots \times T_n$
- For each player i, a payoff function u_i corresponding to the payoff player i will receive for each combination of actions chosen by all the players and each possible profile of signals received by the players.
- **A strategy for player $\boldsymbol{i}$ is a function $\boldsymbol{s_i}$ that gives for each signal $\boldsymbol{t_i}$ an action in $\boldsymbol{A_i}$**

Example 2

Alice is an investor deciding whether to fund or not Bob's project. Alice doesn't know if Bob will work hard or not if he gets the investment. A random draw will determine the signal that Bob gets. With probability p Bob will get the signal "hard worker". The payoffs each receive depends on this signal

Bob is hard worker			
		Bob	
		Work	Relax
Alice	Invest	9,10	3,4
	Pass	5,5	5,1

Bob is NO hard worker			
		Bob	
		Work	Relax
Alice	Invest	6,3	1,6
	Pass	5,0	5,3

Bayesian Games

A strategy for player i is a function s_i that gives for each signal t_i an action in A_i

In a Bayesian Game:

1. Players receive their signal
2. Plays action prescribed by their strategy

If $t = (t_1, t_2, \dots, t_n)$ is the signal profile, then players play the action profile $s_{1(t_1)}, s_{2(t_2)}, \dots, s_n(t_n)$. Players utility depends on both the action profile and the signal profile:

- If signal t is realized, player i get $u_i(s_t, t)$

Bayesian-Nash Equilibrium

A strategy profile s is a Bayesian-Nash Equilibrium is for each player i and each signal $t_i \in T_i$ and for each action $a_i \in A_i$:

$$E[u_i(s_i(t_i), s_{-i}(t_{-i}), (t_i, t_{-i})|t_i] \geq E[u_i(a_i, s_{-i}(t_{-i}), (t_i, t_{-i})|t_i]$$

This is the action profile: What each player j will do when they receive their signal t_j

The utility of player i also depends on the signal profile

And is conditional on t_i because player i knows the signal they got

This is the only thing that changes w.r.t. the expression on the left. We are comparing the equilibrium ($s_i(t_i)$) against all other possible actions of player i (a_i)

Find the Bayesian-Nash Equilibrium

Alice is an investor deciding whether to fund or not Bob's project. Alice doesn't know if Bob will work hard or not if he gets the investment. A random draw will determine the signal that Bob gets. With probability p Bob will get the signal "hard worker". The payoffs each receive depends on this signal

Bob is hard worker			
		Bob	
		Work	Relax
Alice	Invest	9,10	3,4
	Pass	5,5	5,1

Bob is NO hard worker			
		Bob	
		Work	Relax
Alice	Invest	6,3	1,6
	Pass	5,0	5,3

Find the Bayesian-Nash Equilibrium

Bob’s strategy profile s_B tells us the actions he will take depending on the signal (he knows his signal, and there are 2 possibilities):

- If he gets the signal “hard worker”: Working is a dominant strategy
- If he gets the signal “not hard worker”: Relaxing is a dominant strategy

Thus, s_B simply is: Work if hard worker, relax if not hard worker

Find the Bayesian-Nash Equilibrium

Alice's strategy is harder to determine, because she only gets a signal with a probability of Bob being a hard worker:

- If she invests:
  - With probability p Bob is a hard worker and Alice gets 9
  - With probability $1 - p$ Bob is not a hard worker and Alice gets a payoff of 1
  - Thus: $\mu_{Alice}(invest, s_B) = 9 * p + 1 * (1 - p)$
- If she passes:
  - With probability p (Bob is hard worker) she gets a payoff of 5, and with probability $(1 - p)$ she gets a payoff of 5
  - Thus: $\mu_{Alice}(pass, \mathrm{s_B}) = 5 * \mathrm{p} + 5 * (1 - \mathrm{p})$

Find the Bayesian-Nash Equilibrium

Alice's strategy will depend on the value of p. She will invest as long as:

$$\mu_{Alice}(invest, s_B) \geq \mu_{Alice}(pass, s_B)$$

Replacing from last slide:

$$9 * p + 1 * (1 - p) \geq 5 * p + 5 * (1 - p) \Rightarrow p \geq \frac{1}{2}$$

Thus: Alice's strategy is: Invest if $p \geq \frac{1}{2}$, pass if $p < \frac{1}{2}$

Note that in fact Alice is indifferent if $p = \frac{1}{2}$, so it is also a valid strategy: invest if $p > \frac{1}{2}$, pass if $p \leq \frac{1}{2}$

Game Theory Recap Summary

- **What is game:** A game consists of set of players, the rules that dictate the game, a set of actions available to each players, and a payoff function for each player
- **Games with perfect information**: Players are rational and know the structure, rules, and payoff of the game
- **Normal Form games**: All players choose their actions simultaneously
- **Dominant action**: An action that is always the best for player i, regardless of what other players do

Game Theory Recap Summary

- **Nash Equilibrium:** An outcome in which no player has incentives to deviate, provided that all other players do not deviate as well
- **Bayesian Games:** Games of incomplete information. Players do not know the payoffs of other players (only a probability distribution)
- **Bayesian-Nash Equilibrium:** A strategy profile (depends on both actions and signals profiles) in which no player has an incentive to deviate, provided that the others player do not deviate

Why Game Theory?

- We have been looking at game theory as a way to model strategic interactions
- There is a game with certain known rules that dictate how players interact. We would like game theory to predict:
  - Dominant strategies: Actions that are always the best option, despite what others do
  - Nash (or Bayesian Nash) equilibrium: The action that is the best strategy considering the actions of the other players (or a correct probability assessment of what others will do)
- In mechanism design we flip the problem

Mechanism Design

- Goal is to design games that will lead to outcomes with characteristic we prefer
- We have to take into account that people will be strategic in their actions, and choose to withhold or misrepresent information
- Objectives depend on who is designing the game. Examples could include:
  - Maximize sellers’ revenue in an auction
  - Produce the more equitable allocation in a matching problem
- Auctions and matching are special cases of mechanism design

Example

I want to convince my kids to eat vegetable. My options are carrots or peas. Problem is, some of my kids like carrots, other peas. How do we choose:

- I ask them to rank their preferences and choose the vegetable preferred by most
- I ask them to write down a random number. If the sum of all submitted numbers is even, I do carrots, if it odd, I do peas

Some definitions

- Principal: Design the game. In the example: Parent
- Agents: Those playing the game. In the example: Kids
- Agents have some private information. In the example: Each kid knows what they like (carrots or peas)
- Agents give the principal messages, and the principal decides the outcome based on a function of the received messages
  - In the example: Game 1: Possible messages was carrots or peas. Game 2: Message was the number they said

Mechanism

A mechanism consists of:

- A set of possible messages for each player:
$$M_1, M_2, \dots, M_n$$
- An allocation rule:
$$x(m_1, m_2, \dots, m_n)$$
- A payment rule:
$$p_1(m_1, m_2, \dots, m_n), \dots, p_n(m_1, m_2, \dots, m_n)$$

- Some mechanisms will not have payments

Examples of Mechanisms

- Auction Problems:
  - Private information: A person's valuation for the item
  - Messages: Bids
  - Auction mechanism: Determines who gets the item, and how much they have to pay
- Matching Problems:
  - Private information: Person's preferences over matches/items
  - Messages: Stated preference (that may or may not be truthful)
  - Matching mechanism: Algorithm creating the matches

Mechanism and Game Theory

- A mechanism defines a game:
  - The possible actions are the messages: $M_1, \dots, M_n$
  - The payoffs are a function of the messages: $u_i(m_1, \dots, m_n) = v_i\big(x(m_1, \dots, m_n)\big) - p_i(m_1, \dots, m_n)$
- Direct Mechanism: Agents are asked to disclose their private information (also called type)
  - In the example, I asked the kids for their favorite vegetable
- Indirect Mechanism: Agents are not asked for their private information. It could be anything:
  - In the example, if I asked for a random number

Strategy-proof mechanisms

- A strategy for player i specifies the message to send as a function of player $i's$ private information
- A mechanism is strategy-proof if each player i has a dominant strategy (i.e. a strategy that is optimal regardless of what other players do)
- It is not always possible to create strategy-proof mechanisms. Thus, sometimes we have to settle for mechanism with a Nash Equilibrium that creates good enough outcomes

Applying Game Theory to Market Design

- In existing markets:
  - Useful to model how participants will behave based on the “rules of the game”. With this information we can determine if the market is working well or not
- When designing new markets:
  - We need to understand the problem to be solved and the desired outcomes. Then, look at players, their objectives and incentives. With this information we create rules that would help the market arrive to a desired outcome
- Important: Economic theory is just a framework. In market design data and experimentation is typically used to make sure the mechanism works and that the model didn’t miss a critical feature.