Solution Sketch for PS5

1. **Solution**

(a) Market clearing prices are (5, 9, 0) and y gets a, x gets b and z gets c. There are other market clearing prices, for example raising each price by 1 works, any market clearing prices are OK.

(b) The prices are again (5, 9, 0) and again y gets a, x gets b and x gets c. The VCG procedure produces these prices and not other market clearing prices.

2. **Solution**

(a) In the second price auction each bidder bids truthfully. So x bids 8, y bids 10 and z bids 6. Bidder y has the highest bid so y wins. The price that y pays is the second highest bid which is 8.

(b) In the VCG procedure y will be assigned item a and the two fictional items, b and c, will be assigned to bidders x and z. (It does not matter which of fictional items is assigned to x and to z.) The price that y pays is the value that x and z could obtain without y but with the item that y receives, which is 8, minus the value that x and z could obtain without y and without the item that y receives, which is 0. So y pays 8 just as in the second price auction. This price is the harm that y creates for the other bidders as it is the reduction in the maximum value that they could obtain that is created by y taking item a. Bidders x and z pay nothing as they do no harm by taking their fictional items.

3. **Solution**

(a) Create a fourth item d with a value of 0 for all individuals. The optimal allocation is to give a to w, b to x, c to y and d to z (or nothing to z). The VCG prices are 4, 4, 4, and 0.

(b) Individuals w, x, and y have different ideal items, individual z has a lower value of 4 for each of these favorite items than the value placed on them by the individuals who want them. So the only harm that w, x or y do by taking their favorite item is to leave nothing for z. Since z values any item at 4 this harm is 4 in every case.

4. **Solution**

(a) The assignment is $x--a, y--b, z--c$ and the prices are $p_{ax} = 5, p_{by} = 2, p_{cz} = 0$.

(b) (b.1) w has the smallest value per click and so will be assigned the fictitious slot d.

(b.2) The harm that x does to the other advertisers by taking slot a has increased by 3v as if x is not assigned a slot then slot c is assigned to advertiser w (without advertiser w no one got slot c when x was removed). So v=1. A proof is: Let the values per click for advertisers y and z, be y and z respectively. Then the old price for slot a to advertiser x was $y + z$. The new price is $y + z + 3v = (y + z) + 3$. So $v = 1$.

5. **Solution**

(a) NO, it is not a NE. Advertiser y can bid below the lowest bid of 6 and get slot c at a price of 0. His payoff from truthful bidding is 36-24=12 and his payoff from a lower bid is 18. Advertiser x also can deviate below 6 and improve their payoff. x currently gets 60-54=6 and would get 20 by getting the last slot for free. Or x can deviate by bidding more than 6 and less than 9 to get the second slot. This generates a profit of 16.

(b) NO, it is not a NE. Advertiser y can bid below the lowest bid of 5 and get slot c at a price of 0. His payoff from a bid of 6 is 36-20=16 and his payoff from a bid below 5 is 18.

(c) Yes, these are bids constructed from market clearing prices with x bidding more than the market clearing price per click of slot a, y bidding the market clearing price per click of slot a, and z bidding the market clearing price per click of slot b. Can also prove that it's a Nash equilibrium by showing that no advertiser can find a profitable deviation.