QUANTUM ALGORITHM

Deutsch’s algorithm, Deutsch-Jozsa Algorithm

Phase Kickback

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$CNOT(H \otimes H) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\begin{aligned} H \otimes H(|00\rangle) &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= |++\rangle \end{aligned}$$

$$\begin{aligned} CNOT(H \otimes H(|00\rangle)) &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= |++\rangle \end{aligned}$$

$$\begin{aligned} H \otimes H(|01\rangle) &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= |+-\rangle \end{aligned}$$

$$\begin{aligned} CNOT(H \otimes H(|01\rangle)) &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= |--\rangle \end{aligned}$$

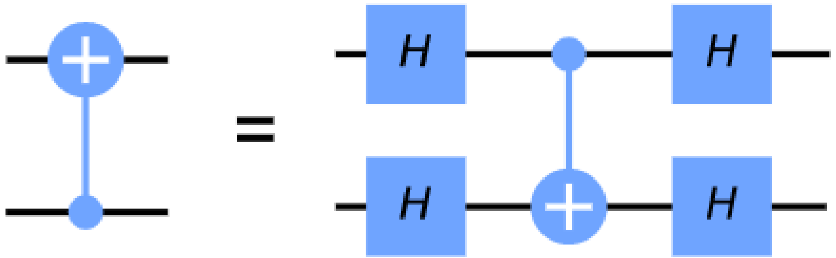
$$H \otimes H(|10\rangle) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|0\rangle - |1\rangle|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= |-+\rangle$$

$$H \otimes H(|11\rangle) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= |--\rangle$$

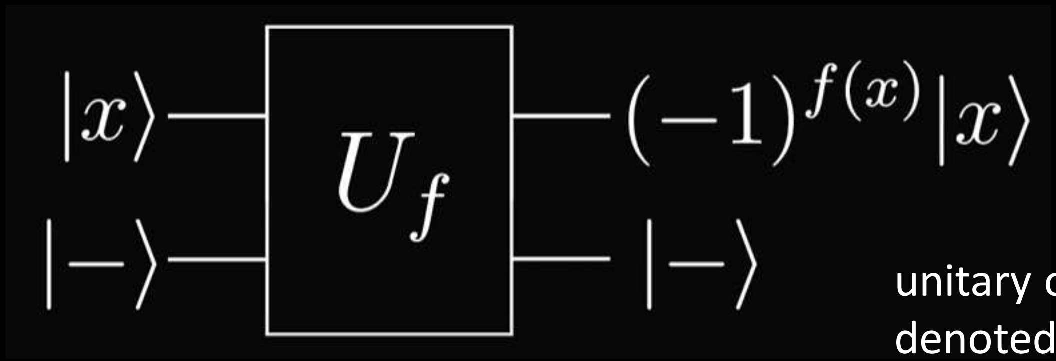
$$CNOT(|++\rangle) = |++\rangle$$
$$CNOT(|+-\rangle) = |--\rangle$$
$$CNOT(|-+\rangle) = |-+\rangle$$
$$CNOT(|--\rangle) = |+-\rangle$$

$$CNOT(H \otimes H(|10\rangle)) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|0\rangle - |1\rangle|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$= |-+\rangle$$

$$CNOT(H \otimes H(|11\rangle)) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= |+-\rangle$$



Phase Kickback



unitary operation denoted by U_f which is linear

$$|x\rangle|-\rangle$$

$$= |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|x\rangle|0\rangle - |x\rangle|1\rangle)$$

$$\rightarrow U_f \frac{1}{\sqrt{2}}(|x\rangle|0\rangle - |x\rangle|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(U_f|x\rangle|0\rangle - U_f|x\rangle|1\rangle)$$

U_f works on both 0 and 1 as function f(x)

$$= \frac{1}{\sqrt{2}}(|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle)$$

$$= \frac{1}{\sqrt{2}}(|x\rangle|f(x)\rangle - |x\rangle|\overline{f(x)}\rangle)$$

$$\frac{1}{\sqrt{2}}(|x\rangle|f(x)\rangle - |x\rangle|\overline{f(x)}\rangle)$$

$$= \begin{cases} \frac{1}{\sqrt{2}}(|x\rangle|0\rangle - |x\rangle|1\rangle) & : f(x) = 0 \\ \frac{1}{\sqrt{2}}(|x\rangle|1\rangle - |x\rangle|0\rangle) & : f(x) = 1 \end{cases}$$

$$= \begin{cases} |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & : f(x) = 0 \\ -\frac{1}{\sqrt{2}}(|x\rangle|0\rangle - |x\rangle|1\rangle) & : f(x) = 1 \end{cases}$$

$$= \begin{cases} |x\rangle|-\rangle & : f(x) = 0 \\ -|x\rangle|-\rangle & : f(x) = 1 \end{cases}$$

$$= (-1)^{f(x)}|x\rangle|-\rangle$$

Deutsch's algorithm

- Deutsch's algorithm combines quantum parallelism with a property of quantum mechanics known as interference.
- Quantum parallelism can be described as the ability to evaluate the function f (x) at many values of x simultaneously.
- Let's consider very simple functions, one that accepts a single bit as input and produces a single bit as output.
- The identity and bit flip functions are called balanced because the outputs are opposite for half the inputs. Constant functions give same output regardless of inputs.

Four possible functions $f(x)$:

$$\underbrace{f(x) = 0 \quad f(x) = 1}_{\text{Constant functions}} \; \underbrace{f(x) = x \quad f(x) = \bar{x}}_{\text{Balanced functions}}$$

Deutsch’s Problem statement

- Deutsch’s algorithm will let us put together a state that has all of the output values of the function associated with each input value in a superposition state. Then we will use quantum interference to find out if the given function is constant or balanced.
- The first step in developing this algorithm is to imagine a unitary operation denoted by U_f that acts on two qubits.

David Deutsch

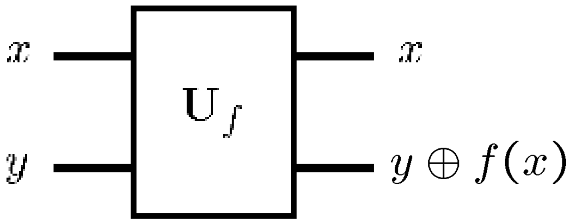
Falcon

Deutsch's Problem (1985)

two qubits

$$\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle \qquad \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

Oracle



$x, y, f(x) \in \{0, 1\}$

Example $f(x) = x$:

$$\mathbf{U}_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Four possible functions $f(x)$:

$$\underbrace{f(x) = 0 \quad f(x) = 1}_{\text{Constant functions}} \quad \underbrace{f(x) = x \quad f(x) = \bar{x}}_{\text{Balanced functions}}$$

Deutsch's Problem

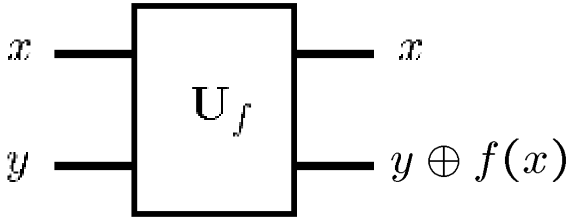
Determine whether f(x) is constant or balanced using as few queries to the oracle as possible.

Classical Deutsch

Four possible functions $f(x)$:

$$\underbrace{f(x)=0 \quad f(x)=1}_{\text{Constant functions}} \quad \underbrace{f(x)=x \quad f(x)=\bar{x}}_{\text{Balanced functions}}$$

Oracle



$$x, y, f(x) \in \{0, 1\}$$

Query input of x_0 and y_0 only gives information about $f(x_0)$.

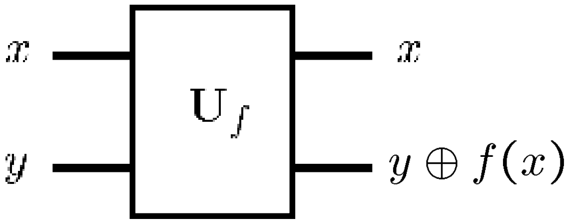
Knowing $f(x_0)$ not enough to distinguish constant from balanced.

Classically we need to query the oracle **two times** to solve Deutsch's Problem. f(0) and f(1) both need to be calculated.

Four possible functions $f(x)$:

$$\underbrace{f(x)=0 \quad f(x)=1}_{\text{Constant functions}} \; \underbrace{f(x)=x \quad f(x)=\bar{x}}_{\text{Balanced functions}}$$

Oracle



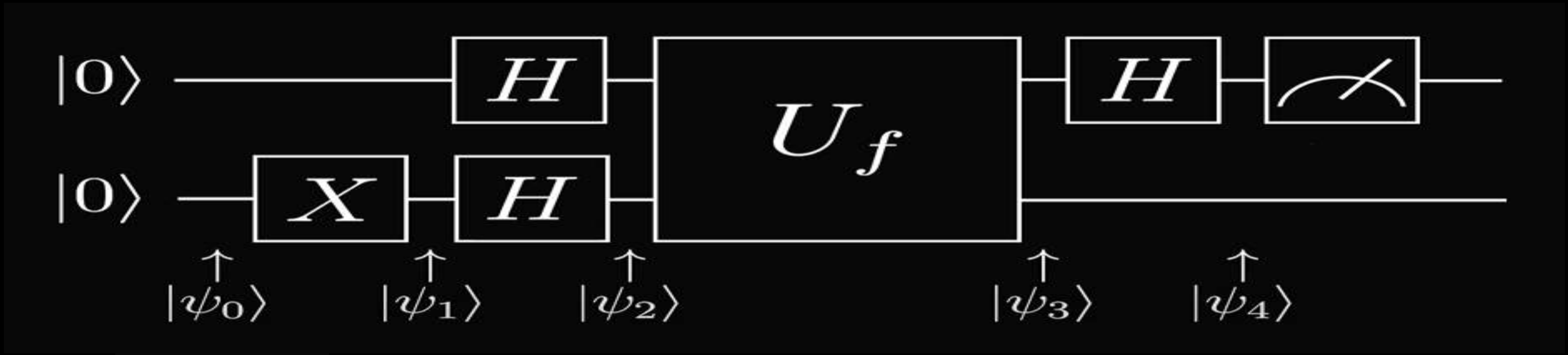
$x, y, f(x) \in \{0, 1\}$

input $y = 0$:

$$|x\rangle|0\rangle \xrightarrow{U_f} |x\rangle|0 \oplus f(x)\rangle$$
$$= |x\rangle|f(x)\rangle$$

$0 \oplus 0=0$, $0 \oplus 1=1$

Now let's see how **Deutsch's Algorithm works.**



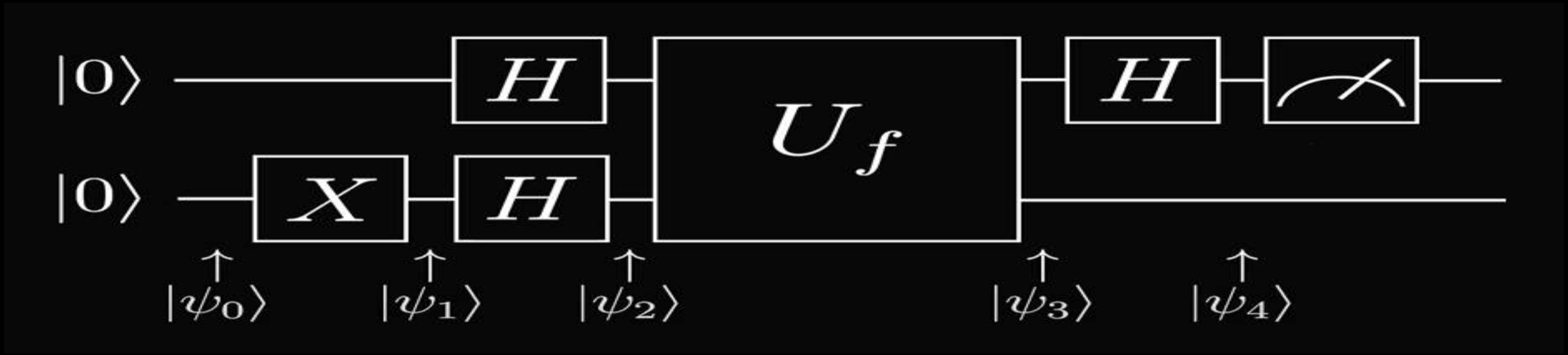
$|\psi_0\rangle = |00\rangle$

$|\psi_1\rangle = |01\rangle$

$$|\psi_2\rangle = |+-\rangle$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|-\rangle$$
$$= \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle)$$

Deutsch's circuit

Now check for next state



$$|\psi_3\rangle = U_f \frac{1}{\sqrt{2}}(|0\rangle|-\rangle + |1\rangle|-\rangle)$$

$$= \frac{1}{\sqrt{2}}(U_f|0\rangle|-\rangle + U_f|1\rangle|-\rangle)$$

$$= \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle|-\rangle + (-1)^{f(1)}|1\rangle|-\rangle)$$

From phase kickback

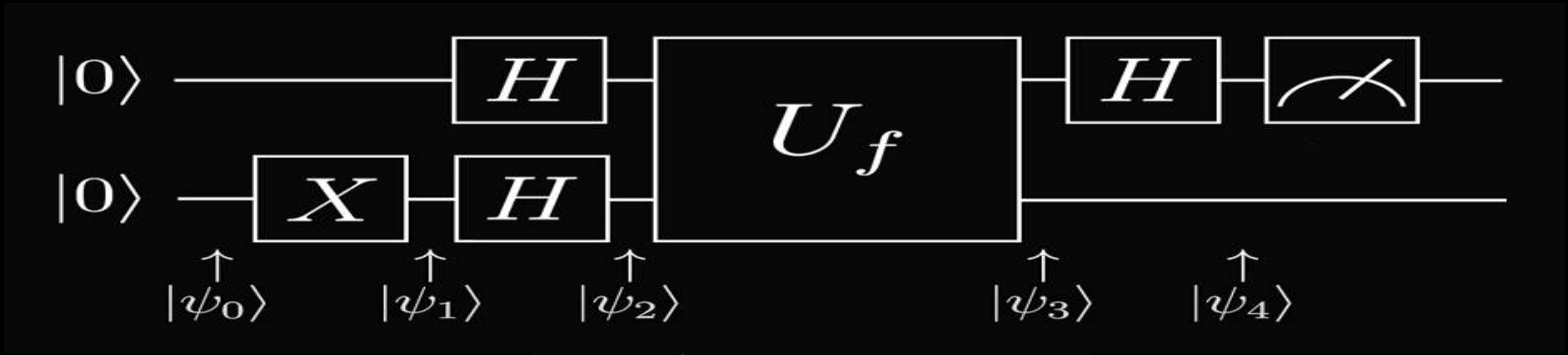
$$(-1)^{f(x)}|x\rangle|-\rangle$$

$$= \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)|-\rangle$$

ignore - state

$$= \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)$$

Now check for two conditions



$$|\psi_3\rangle = \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)$$

$$f(0) = f(1) \text{ or } f(0) \neq f(1)$$

$$f(0) = f(1)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \text{ if } f(0) = 0 \text{ and } f(1) = 0$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(-|0\rangle - |1\rangle), \text{ if } f(0) = 1 \text{ and } f(1) = 1$$

$$= -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

$$|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

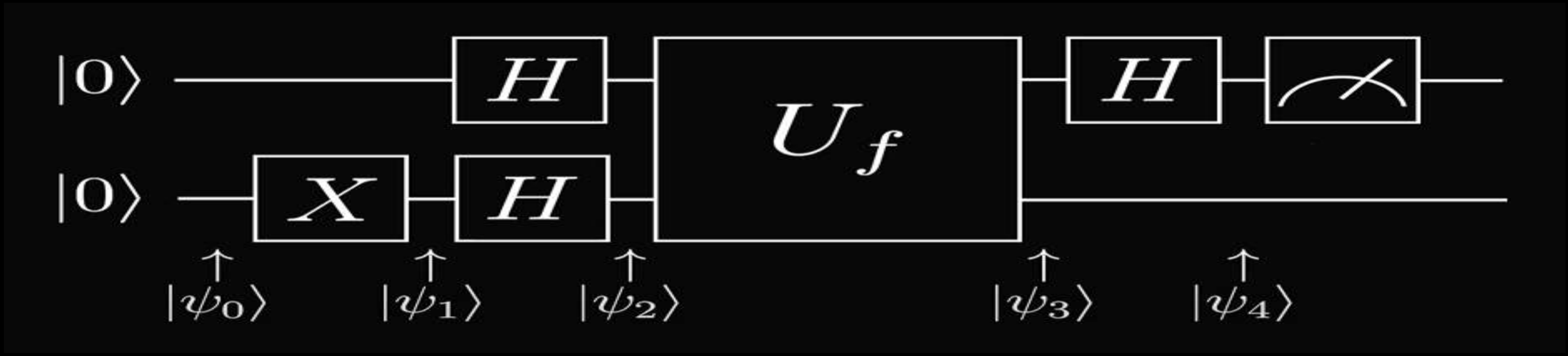
$$f(0) \neq f(1)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \text{ if } f(0) = 0 \text{ and } f(1) = 1$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle), \text{ if } f(0) = 1 \text{ and } f(1) = 0$$

$$= -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



$$\text{if } f(0) = f(1): \quad |\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |\psi_3\rangle = \pm|+\rangle$$

$$\text{if } f(0) \neq f(1): \quad |\psi_3\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad |\psi_3\rangle = \pm|-\rangle$$

After Applying H gate at state 4:

$$\text{if } f(0) = f(1): \quad |\psi_4\rangle = \pm|0\rangle$$

$$\text{if } f(0) \neq f(1): \quad |\psi_4\rangle = \pm|1\rangle$$

if we measure 0 then the function is constant, since $f(0) = f(1)$

if we measure 1 then the function is balanced, since $f(0) \neq f(1)$

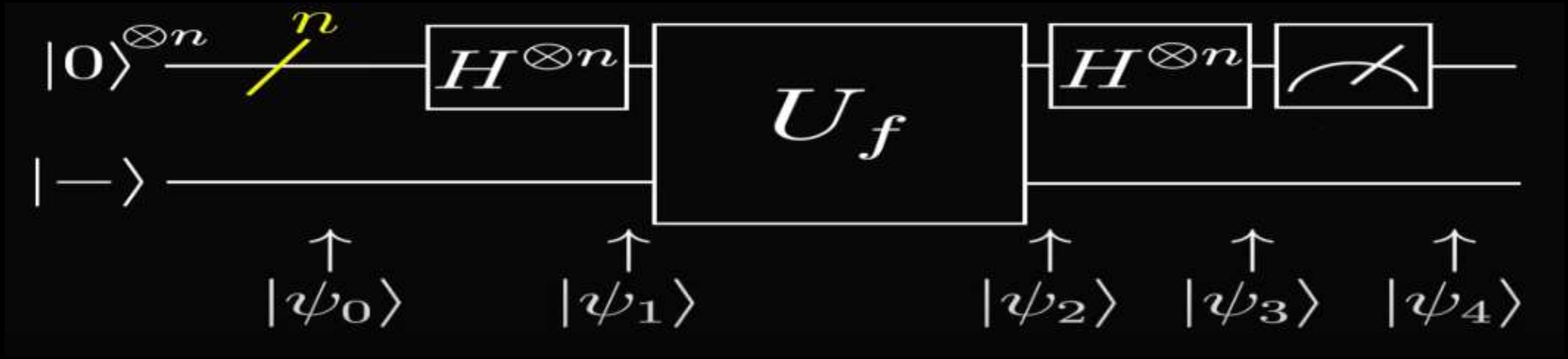
In words Deutsch's algorithm is implemented by the following steps:

1. Apply Hadamard gates to the input state $|0\rangle|1\rangle$ to produce a product state of two superpositions.
2. Apply U_f to that product state.
3. Apply a Hadamard gate to the first qubit leaving the second qubit alone.

Deutsch's Algorithm: $f : \{0, 1\} \to \{0, 1\}$

Deutsch-Jozsa Algorithm: $f : \{0, 1\}^n \to \{0, 1\}$

Classical computers need to query the function at worst $2^{n-1} + 1$ times



n means n number of qubits

$$
\begin{aligned}
|\psi_0\rangle &= |00...0\rangle|-\rangle \\
&= |0\rangle^{\otimes n}|-\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\psi_1\rangle &= H^{\otimes n}|0\rangle^{\otimes n}|-\rangle \\
&= H|0\rangle H|0\rangle ... H|0\rangle|-\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes ... \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|-\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle|-\rangle
\end{aligned}
$$

Refer next slide for solution

Now check for next state

$$H^{\otimes 2}|0\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}\sum_{x\in\{0,1\}^2} |x\rangle$$

Combination of string of length 2

$$H^{\otimes 3}|0\rangle^{\otimes 3} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2^3}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

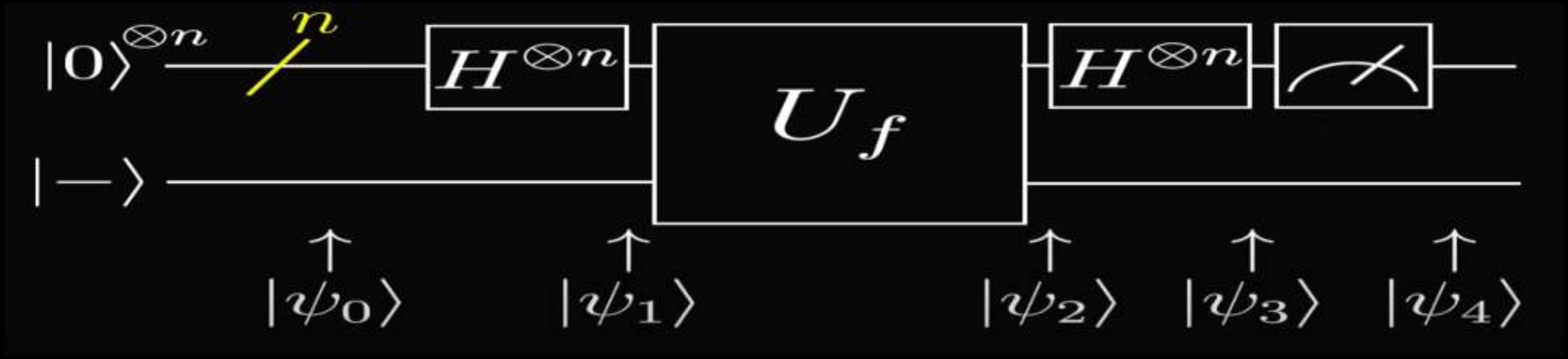
$$= \frac{1}{\sqrt{2^3}}\sum_{x\in\{0,1\}^3} |x\rangle$$

Combination of string of length 3

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n} |x\rangle$$

Combination of string of length n

Remember: This is only for 0 input in H gate, not for general input



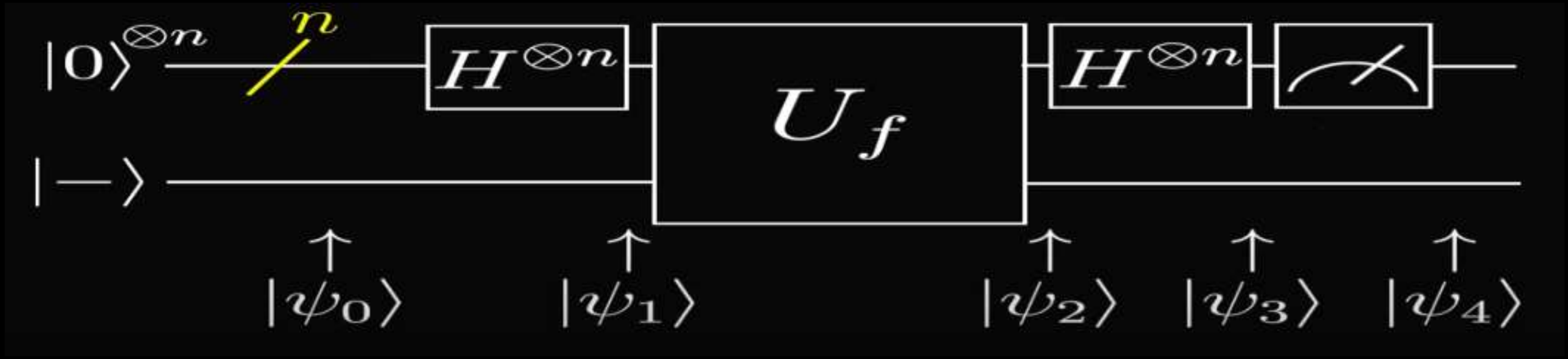
$$|\psi_2\rangle = U_f \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle|-\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} U_f|x\rangle|-\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle|-\rangle$$ Phase Kickback

$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle$$ Ignore - state

Now check for next state



$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} H^{\otimes n}|x\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle$$

For explanation read next 2 slides

$$= \frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)+x\cdot z}|z\rangle$$

Amplitude of the $|0\rangle^{\otimes n}$ state: (All 0 state)

$$\frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)+x\cdot 000\ldots 0} = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)+0} = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}$$

Now check 2 conditions after 2 slides

$$H|x_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_i}|1\rangle)$$

We already know

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^0|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^1|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Our main aim is to find the output for n qubit generalized input to H gate

$$H^{\otimes n}|x\rangle = H|x_0\rangle H|x_1\rangle ... H|x_{n-1}\rangle$$

Testing $H^{\otimes 3}|x\rangle$

$$H^{\otimes 3}|x\rangle = H|x_0\rangle H|x_1\rangle H|x_2\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_0}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_1}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_2}|1\rangle)$$

$$= \frac{1}{\sqrt{2^3}}(|000\rangle + (-1)^{x_2}|001\rangle + (-1)^{x_1}|010\rangle + (-1)^{x_1}(-1)^{x_2}|011\rangle +$$

$$(-1)^{x_0}|100\rangle + (-1)^{x_0}(-1)^{x_2}|101\rangle + (-1)^{x_0}(-1)^{x_1}|110\rangle + (-1)^{x_0}(-1)^{x_1}(-1)^{x_2}|111\rangle)$$

Order: x_0 bit x_1 bit x_2 bit

Let's solve the equation

$$H^{\otimes 3}|x\rangle = \frac{1}{\sqrt{2^3}}(|000\rangle+(-1)^{x_2}|001\rangle+ (-1)^{x_1}|010\rangle+(-1)^{x_1+x_2}|011\rangle+(-1)^{x_0}|100\rangle+$$

$$(-1)^{x_0+x_2}|101\rangle + (-1)^{x_0+x_1}|110\rangle + (-1)^{x_0+x_1+x_2}|111\rangle)$$

Check that the position of 1 in the state and value of x is same. Rewrite x with the superposition of the position of 1

$$= \frac{1}{\sqrt{2^3}}(|000\rangle + (-1)^{x\cdot 001}|001\rangle + (-1)^{x\cdot 010}|010\rangle + (-1)^{x\cdot 011}|011\rangle + (-1)^{x\cdot 100}|100\rangle$$

$$+ (-1)^{x\cdot 101}|101\rangle + (-1)^{x\cdot 110}|110\rangle + (-1)^{x\cdot 111}|111\rangle)$$

Order: x_0 bit x_1 bit x_2 bit

$$= \frac{1}{\sqrt{2^3}} \sum_{z\in\{0,1\}^3} (-1)^{x\cdot z}|z\rangle$$

$$x_1.x_2.x_3...x_n * z_1.z_2.z_3....z_n = x_1z_1+x_2z_2+...+x_nz_n$$

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle$$

Now let’s go back 2 slides to find the output in state 3.

Amplitude of the $|0\rangle^{\otimes n}$ state: $\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$

if $f(x)$ is constant:

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} = \begin{cases} \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^0 & : f(x) = 0 \\ \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^1 & : f(x) = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{2^n} \sum_{x \in \{0,1\}^n} 1 & : f(x) = 0 \\ \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^1 & : f(x) = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{2^n} 2^n & : f(x) = 0 \\ \frac{1}{2^n} (-2^n) & : f(x) = 1 \end{cases}$$

$$= \begin{cases} 1 & : f(x) = 0 \\ -1 & : f(x) = 1 \end{cases}$$

In General combing these two conditions

if $f(x)$ is constant:

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} = \pm 1$$

Now check if f(x) is balanced

Amplitude of the $|0\rangle^{\otimes n}$ state: $\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$

if $f(x)$ is balanced:

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$

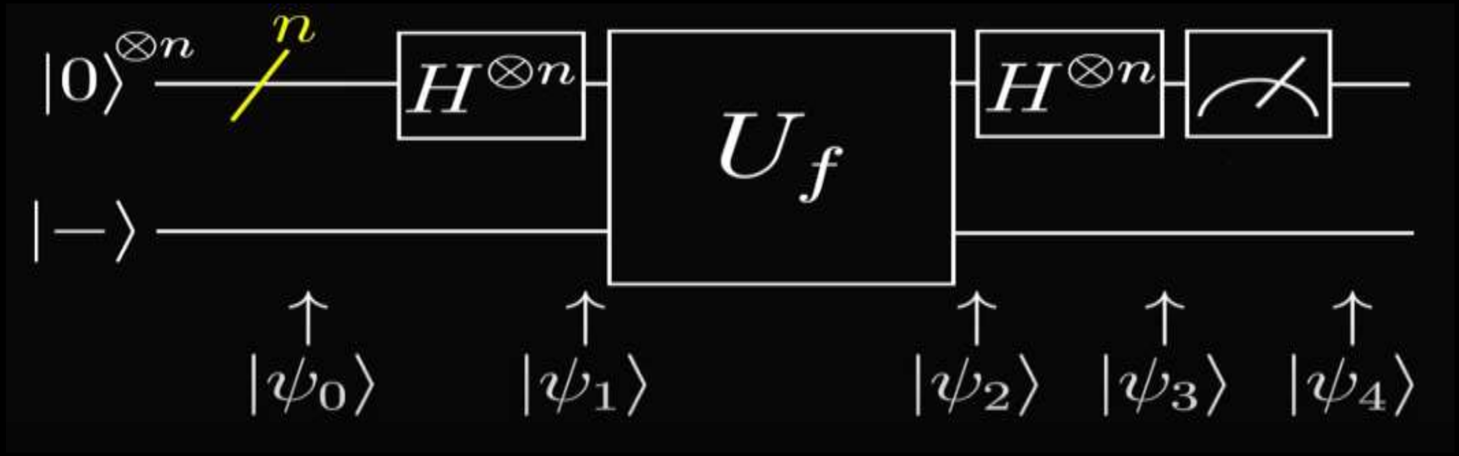
$$= \frac{1}{2^n}((-1)^0 + (-1)^1 + (-1)^1 + (-1)^0 + \ldots + (-1)^0)$$

$$= \frac{1}{2^n}(0)$$

$$= 0$$

Half of the f(x) should be 0 and another half should be 1

Next slide check the effect on measurement



if we measure 000...0 then $f(x)$ is constant

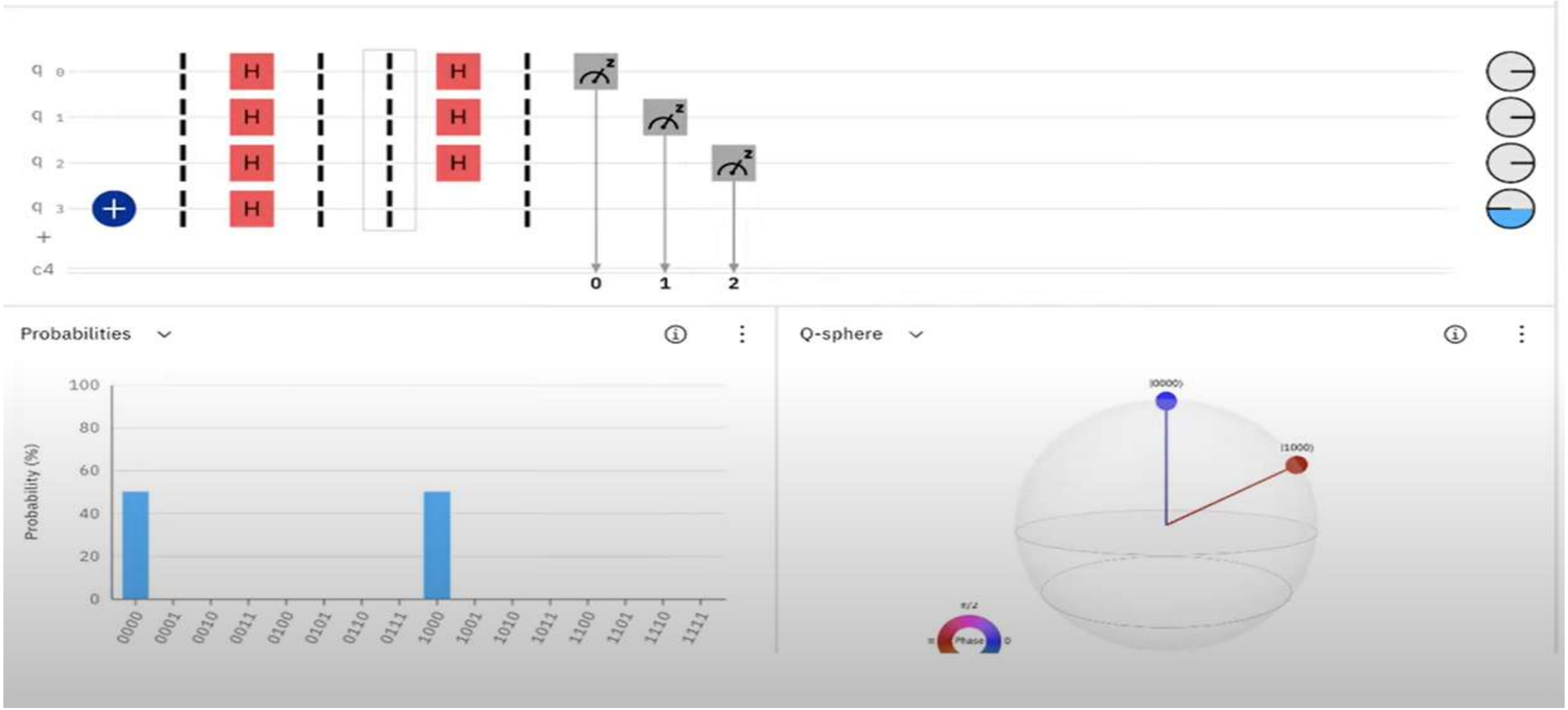
Since the amplitude of the all 0's state if $f(x)$ is constant is ± 1

if we don't measure 000...0 then $f(x)$ is balanced

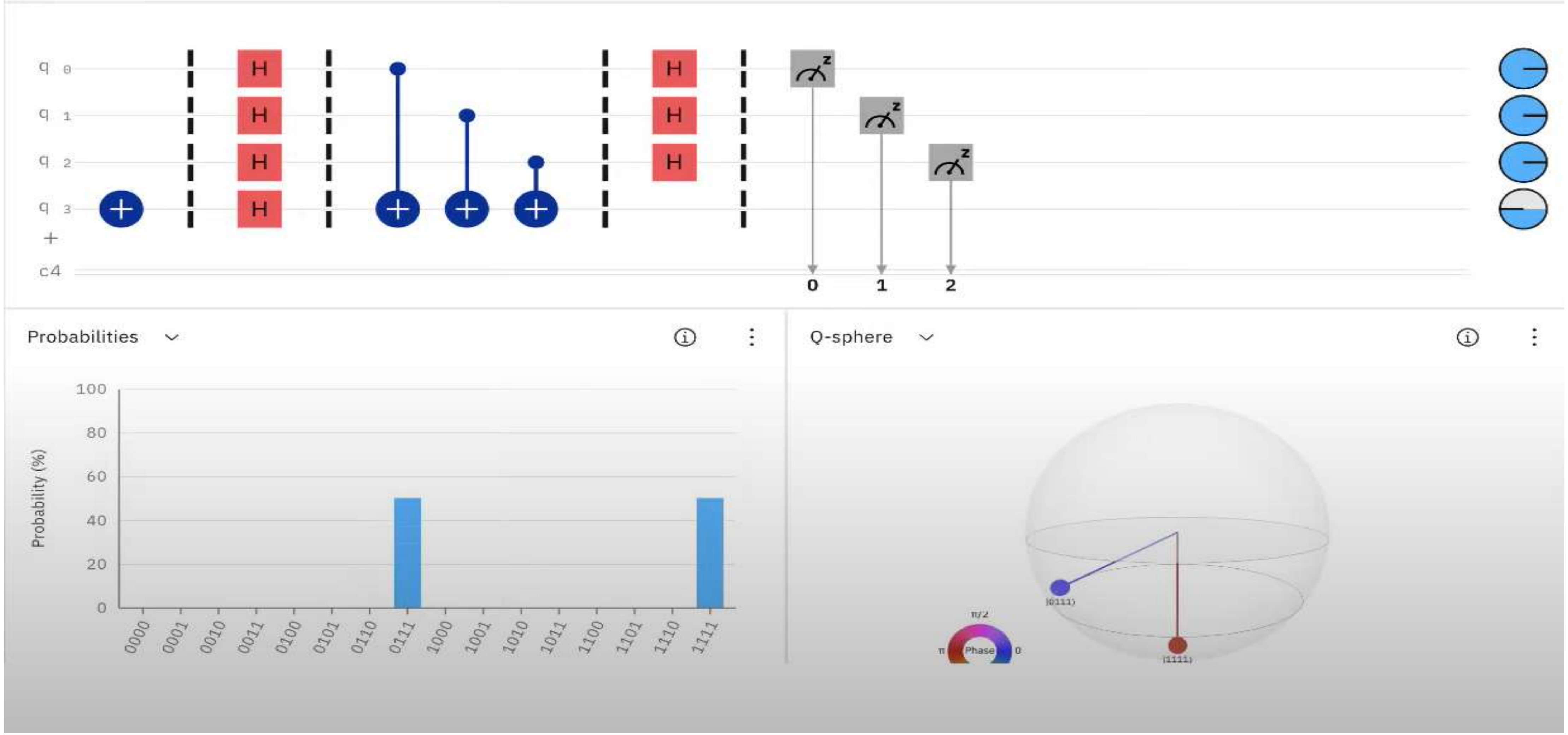
Since the amplitude of the all 0's state if $f(x)$ is balanced is 0

In the case where f is constant the amplitude for all 0 state is +1 or −1, depending on the constant value f(x) takes. Because $|\psi_3\rangle$ is of unit length it follows that all the other amplitudes must be zero, and an observation will yield 0s for all qubits in the query register. If f is balanced then the positive and negative contributions to the amplitude for $|0\rangle^{\otimes n}$ cancel, leaving an amplitude of zero, and a measurement must yield a result other than 0 on at least one qubit in the query register.

For Constant Function: Qiskit Code (Random Example of f(x))



For Balanced Function: Qiskit Code (Random Example of f(x))



Algorithm: Deutsch–Jozsa

Inputs: (1) A black box U_f which performs the transformation $|x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$, for $x \in \{0, \dots, 2^n - 1\}$ and $f(x) \in \{0, 1\}$. It is promised that $f(x)$ is either *constant* for all values of x, or else $f(x)$ is *balanced*, that is, equal to 1 for exactly half of all the possible x, and 0 for the other half.

Outputs: 0 if and only if f is constant.

Runtime: One evaluation of U_f. Always succeeds.

Procedure:

1.	$\lvert 0\rangle^{\otimes n}\lvert 1\rangle$	initialize state
2.	$\rightarrow \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \lvert x\rangle \left[\frac{\lvert 0\rangle - \lvert 1\rangle}{\sqrt{2}}\right]$	create superposition using Hadamard gates
3.	$\rightarrow \sum_x (-1)^{f(x)} \lvert x\rangle \left[\frac{\lvert 0\rangle - \lvert 1\rangle}{\sqrt{2}}\right]$	calculate function f using U_f
4.	$\rightarrow \sum_z \sum_x \frac{(-1)^{x \cdot z + f(x)} \lvert z\rangle}{\sqrt{2^n}} \left[\frac{\lvert 0\rangle - \lvert 1\rangle}{\sqrt{2}}\right]$	perform Hadamard transform
5.	$\rightarrow z$	measure to obtain final output z

Reference

- Phase Kickback
  - https://qiskit.org/textbook/ch-gates/phase-kickback.html
  - https://towardsdatascience.com/quantum-phase-kickback-bb83d976a448
- YouTube Link:

  https://www.youtube.com/watch?v=jfJckA7Amik
- **Pg: 203, Quantum Computing Explained, David McMahon**
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- **https://qiskit.org/textbook/ch-algorithms/deutsch-jozsa.html**