

MH1810 Math 1 Part 3 Differentiation

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Tangent

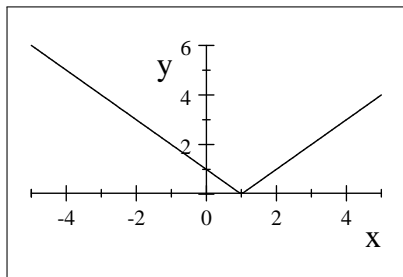
Consider a curve and a fixed point P on a curve. What is a tangent to a curve at a point?

Tangent

- (a) A straight line passing any two points of a curve is called a **chord** or **segment**.
- (b) The **tangent** to a curve at a point P is a straight line which 'touches' the curve at P . The tangent at P is a line which cuts the curve in one and only one point, namely the point P , in a sufficiently small neighbourhood around P , although it may cut the curve at more than one point.

Example

Consider the following curve:



There is no (unique) tangent at P , when there is a 'kink'.

Describe Tangent Mathematically

Question How to describe mathematically the tangent to a curve at a point P ?

Describe Tangent Mathematically

To describe the tangent at P , we have to find **the equation of the tangent**.
We have to determine the **gradient** (or the **slope**) of the tangent.
Since

$$\text{tangent at } P = \lim_{Q \rightarrow P} \text{Chord } PQ,$$

we shall determine the slope of the chord PQ :

Describe Tangent Mathematically

$$\text{Slope of tangent at } P = \lim_{Q \rightarrow P} \frac{y_Q - y_P}{x_Q - x_P}.$$

Denoting by Δx the change in x and Δy the corresponding change in y , we have

$$\Delta x = x_Q - x_P \quad \text{and} \quad \Delta y = y_Q - y_P.$$

Thus we have

$$\text{Slope of tangent at } P = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

Slope of a Tangent

If the curve is the graph of a function $f(x)$, we have

$$\begin{aligned}\text{Slope of tangent at } P &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{x \rightarrow x_P} \frac{f(x) - f(x_P)}{x - x_P}.\end{aligned}$$

if limit exists.

Instantaneous Rate of Change

Increments

- Suppose the state of a physical or chemical system is described by a set of variables. E.g.
 - (x, t) = position of a sprinter along a 100m-race at time t track.
 - (m, T) = molarity and temperature of an aqueous chemical solution.
 - (p, V, T) = pressure, volume and temperature of a gas

Increments

- When the system undergoes a change (e.g. when a sprinter runs or the temperature of the gas changes), the values of those state variables (e.g. position x or temperature T) change. We denote a change, or increment, in the variable x as

$$\Delta x = \text{new value of } x - \text{old value of } x.$$

- Note that Δx can be positive, negative or zero, depending on if x increases, decreases or stays the same.

- Suppose $f(x)$ is a function of x . As x changes from x_1 to x_2 , the corresponding $f(x)$ changes from $f(x_1)$ to $f(x_2)$. We have

$$\Delta x = x_2 - x_1, \text{ and } \Delta f = f(x_2) - f(x_1).$$

- Most often, we are interested in the ratio $\frac{\Delta f}{\Delta x}$, which is known as the **average change in f** as x changes from x_1 to x_2 .

The limit $\lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$ is the **instantaneous rate of change in f with respect to x** .

Definition

- (1) The **derivative** of a function f at a number c , denoted by $f'(c)$, is defined as follows:

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c},$$

if this limit exists.

We say f is **differentiable** at c if $f'(c)$ exists.

Definition

- (1) A function f is said to be **differentiable** on an open interval (a, b) if it is differentiable at every number c in (a, b) , i.e., $f'(c)$ exists.
- (2) The **derivative function** (or simply **derivative**) f' is defined by

$$f'(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}.$$

The domain of f' is the set of real numbers x such that $f'(x)$ exists.

- ① Using the symbol $\Delta x = x - c$ for the change in x , we note that $\Delta x \rightarrow 0$ whenever $x \rightarrow c$. Thus we may express $f'(c)$ as

$$f'(c) = \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

- ② Sometimes, we use $h = x - c$ instead of Δx to denote the change. Thus, we can also write

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h}.$$

Remark

- (3) In the expression below,

$$f'(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}.$$

The variable y on the right is a dummy variable which can be replaced by other symbol except x . For example, we have

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}.$$

- (4) Leibniz notation for derivative: $\left. \frac{df}{dx} \right|_{x=a}$, $\frac{df}{dx}(a)$.
- (5) To differentiate f (with respect to x) means to determine the derivative f' (i.e., $f'(x)$).

Examples

Example

- (a) Is $f(x) = x^2$ differentiable at $x = 1$?
(b) Find $f'(x)$ and the domain of f' .

Solution (a)

To check whether f is differentiable at $x = 1$, we have to *check whether the limit* $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$ *exists*. We proceed to compute this limit:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1} \frac{x^2 - (1)^2}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)} = \lim_{x \rightarrow 1} (x + 1) = 2.\end{aligned}$$

Since the limit exists, f is differentiable at $x = 1$ and hence $f'(1) = 2$.

Solution (b)

We determine $f'(x)$ for $f(x) = x^2$ as follows:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2x + h) = 2x. \end{aligned}$$

Therefore, we have $f'(x) = 2x$ for $x \in \mathbb{R}$.

Examples

Example

- (a) Is the modulus function $f(x) = |x|$ differentiable at 0? Is there a tangent to the graph of $y = |x|$ at $x = 0$?
- (b) Find $f'(x)$ and the domain of f' .

Solution (a)

Note that $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x}$ (*).

To determine the limit (*) we note that $f(x)$ takes different expression according to $x > 0$ or $x < 0$ because

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For x close to 0, x can be negative or positive.

Solution (a)

Solution (a (con'td))

Thus, we have to consider both $\lim_{x \rightarrow 0^+} \frac{f(x)}{x}$ and $\lim_{x \rightarrow 0^-} \frac{f(x)}{x}$, and check whether these limits are equal.

- $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$
- $\lim_{x \rightarrow 0^-} \frac{f(x)}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$

Since $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} \neq \lim_{x \rightarrow 0^-} \frac{f(x)}{x}$, we conclude that $\lim_{x \rightarrow 0} \frac{f(x)}{x}$ does not exist. Therefore, the function $f(x) = |x|$ is **not** differentiable at 0.

Solution (b)

Solution (b)

For $x > 0$, note that

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = \lim_{y \rightarrow x} \frac{y - x}{y - x} = 1.$$

We have used $f(y) = y$ because we are interested in y very close to x , we may thus consider $0 < x/2 < y < 3x/2$ so that $f(y) = y$. Similarly, for $x < 0$, note that

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = \lim_{y \rightarrow x} \frac{-y - (-x)}{y - x} = -1.$$

We have used $f(y) = -y$ because we are interested in y very close to x , we may thus consider $3x/2 < y < x/2 < 0$ so that $f(y) = -y$.

Solution (b)

Solution (b (cont'd))

As discussed in part (a), f is not differentiable as $f'(0)$ does not exist. In conclusion, we have

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f' is $\mathbb{R} \setminus \{0\}$

- ① If the derivative $f'(c)$ exists, then the graph of f has a tangent at $x = c$. And, the finite number $f'(c)$ is the slope of this tangent. The equation of the tangent is given by

$$\frac{y - f(c)}{x - c} = f'(c),$$

- ② It is however possible for the graph of f to have a tangent even when the derivative does not exist.

Example

Example

- (a) Is $f(x) = x^{1/3}$ differentiable at $x = 0$? Is there a tangent to the graph at $x = 0$?
- (b) Find the derivative of $f(x) = x^{1/3}$.

Solution (a)

Solution (a)

At $x = 0$, we look at the following limit of the fraction:

$$\frac{f(0+h) - f(0)}{h} = \frac{h^{1/3} - 0}{h} = \frac{1}{h^{2/3}} = \frac{1}{(h^{1/3})^2}.$$

For h close to 0, the denominator $(h^{1/3})^2$ is close to 0 and positive (because of the square). Hence $\frac{1}{(h^{1/3})^2}$ is large and positive and we have

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \infty.$$

Since this is not a finite number, $f'(0)$ does not exist.

However, the graph of f has a (vertical) tangent at $x = 0$.

Solution (b)

Solution (b)

For $x \neq 0$, we evaluate the following limit

$$\begin{aligned}\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} &= \lim_{y \rightarrow x} \frac{y^{1/3} - x^{1/3}}{y - x} \\&= \lim_{y \rightarrow x} \frac{y^{1/3} - x^{1/3}}{y - x} \cdot \frac{(y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2}{(y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2} \\&= \lim_{y \rightarrow x} \frac{y - x}{(y - x)} \left((y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2 \right)^{-1} \\&= \lim_{y \rightarrow x} \frac{1}{(y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2} = \frac{1}{3x^{2/3}}.\end{aligned}$$

Thus, $f'(x) = \frac{1}{3x^{2/3}}$, i.e., $\frac{d}{dx} \left(x^{1/3} \right) = \frac{1}{3x^{2/3}}$, for $x \in \mathbb{R} \setminus \{0\}$.

Example

Example

Use the definition of derivative to prove $\frac{dC}{dx} = 0$ for any constant C .

Theorem

Let $f(x) = x^n$, where n is a positive integer. Then,

$$f'(x) = nx^{n-1}, \text{ i.e., } \frac{dx^n}{dx} = nx^{n-1}.$$

Derivatives of powers of x

For $f(x) = x^n$, we have

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

$$\begin{aligned} & \overbrace{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n \right)}^{\text{Binomial Expansion}} - x^n \\ = & \lim_{h \rightarrow 0} \frac{}{h} \\ = & \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n}{h} \end{aligned}$$

Derivatives of powers of x

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1} \right) h}{h} \\ &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1} \right) \\ &= nx^{n-1}. \end{aligned}$$

Thus, f is differentiable at every real number x and $f'(x) = nx^{n-1}$.

Theorem

Let α be a real constant. Consider the function αf , defined by $(\alpha f)(x) = \alpha \cdot f(x)$. If f is differentiable at $x = c$, i.e., $f'(c)$ exists, then the function αf is differentiable at $x = c$ and its derivative at c is

$$(\alpha f)'(c) = \alpha \cdot f'(c).$$

Derivatives of a multiple

Proof.

We shall prove that the limit $\lim_{x \rightarrow c} \frac{\alpha f(x) - \alpha f(c)}{x - c}$ exists as follows:

$$\begin{aligned}\lim_{x \rightarrow c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{\alpha \cdot f(x) - \alpha \cdot f(c)}{x - c} \\&= \lim_{x \rightarrow c} \frac{\alpha \cdot (f(x) - f(c))}{x - c} = \lim_{x \rightarrow c} \alpha \left(\frac{f(x) - f(c)}{x - c} \right) \\&= \alpha \lim_{x \rightarrow c} \frac{(f(x) - f(c))}{x - c} = \alpha f'(c).\end{aligned}$$

Therefore, (αf) is differentiable at $x = c$ and

$$(\alpha f)'(x) = \alpha \cdot f'(x).$$



Example

We have proved that $\frac{d}{dx}(x^{1/3}) = \frac{1}{3x^{2/3}}, x \neq 0$. By the above proposition, we have

$$\frac{d}{dx}(179x^{1/3}) = 179 \frac{d}{dx}(x^{1/3}) = \frac{179}{3x^{2/3}}.$$

Higher Derivatives

Once we obtain the derivative f' of f , we may proceed to discuss the derivative of f' and obtain the second derivative of f' , and so on. These are known as **higher Derivatives** of f . Higher derivatives are used in Taylor Series.

Example

Example

Consider $f(x) = x^5$. Then $f'(x) = \frac{dy}{dx} = 5x^4$.

The derivative of f' is the **second derivative** f'' (also denoted by $f^{(2)}(x)$, $\frac{d^2y}{dx^2}$). We have

$$f^{(2)}(x) = (f')'(x) = \frac{d}{dx}(5x^4) = 5\frac{d}{dx}(x^4) = 5(4x^3) = 20x^3.$$

Similarly, we may also have other higher derivatives:

$$f'(x), f^{(2)}(x), f^{(3)}(x), f^{(4)}(x), \dots$$

By letting $y = f(x)$, the higher derivatives are denoted by

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots$$

Remark

- (a) In the study of kinematics, suppose $x(t)$ is the distance travelled by an object at time t . Then the speed of the object is $x'(t)$ and the acceleration at time t is $x''(t)$.
- (b) The shape of the graph of a function $f(x)$ (which is twice differentiable) can be determined by the first and second derivatives. This will be discussed in the next chapter.
- (c) Higher derivatives are very useful in approximation. The Taylor series of a function involves higher derivatives.