

Nanyang Technological University  
SPMS/DIVISION OF MATHEMATICAL SCIENCES

2015/16 Semester 1

MH1810 Mathematics I

Tutorial 11

**Topics:** Reduction Formulae, Improper Integrals, Areas, Volumes.

1. Let  $I_n = \int \cos^n x \, dx$  for  $n = 0, 1, 2, 3, \dots$ .

(a) Prove the reduction formula

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \text{ for } n \geq 2.$$

(b) Use part(a) to evaluate

(i)  $\int \cos^3 x \, dx$ .

(ii)  $\int_0^{\pi/2} \cos^4 x \, dx$ .

[Solution]

(a) For  $n \geq 2$ , we have

$$\begin{aligned} I_n &= \int \cos^n x \, dx = \int \underbrace{(\cos^{n-1} x)}_u \underbrace{\cos x}_{v'} \, dx \\ &= \underbrace{(\cos^{n-1} x)}_u \underbrace{(\sin x)}_v - \int \underbrace{(n-1)(\cos^{n-2} x)(-\sin x)}_{u'} \underbrace{\sin x}_v \, dx \\ &= (\cos^{n-1} x)(\sin x) + \int (n-1)(\cos^{n-2} x)(\sin^2 x) \, dx \\ &= (\cos^{n-1} x)(\sin x) + \int (n-1)(\cos^{n-2} x)(1 - \cos^2 x) \, dx \\ &= (\cos^{n-1} x)(\sin x) + (n-1) \underbrace{\int (\cos^{n-2} x) \, dx}_{I_{n-2}} - (n-1) \underbrace{\int (\cos^{n-2} x) \cos^2 x \, dx}_{I_n}. \end{aligned}$$

Therefore, we have

$$I_n = (\cos^{n-1} x)(\sin x) + (n-1)I_{n-2} - (n-1)I_n$$

which gives

$$nI_n = (\cos^{n-1} x)(\sin x) + (n-1)I_{n-2},$$

and hence

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

(b) Use part(a) to evaluate

$$\begin{aligned} \text{(i)} \quad \int \cos^3 x \, dx &= I_3 = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} I_1 \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C. \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int \cos^4 x \, dx &= I_4 = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} I_2 \\
 &= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left( \frac{1}{2} \cos x \sin x + \frac{1}{2} I_0 \right) \text{ where } I_0 = \int 1 \, dx = x + C_1.
 \end{aligned}$$

Thus, we have

$$\int_0^{\pi/2} \cos^4 x \, dx = \left[ \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left( \frac{1}{2} \cos x \sin x + \frac{1}{2} x \right) \right]_0^{\pi/2} = \frac{3\pi}{16}$$

2. Evaluate each of the following improper integrals.

- (a)  $\int_0^1 x \ln x \, dx$
- (b)  $\int_0^1 \frac{4r}{\sqrt{1-r^4}} dr$
- (c)  $\int_1^\infty \frac{1}{x^3} dx$
- (d)  $\int_5^\infty \frac{1}{\sqrt{x-1}} dx$
- (e)  $\int_2^\infty \frac{1}{x(\ln x)^2} dx$

[Solution]

- (a)  $\int_0^1 x \ln x \, dx$  is improper since  $\ln 0$  is undefined. We have

$$\int_0^1 x \ln x \, dx = \lim_{a \rightarrow 0^+} \int_a^1 x \ln x \, dx.$$

Using integration by parts, we have

$$\int_a^1 x \ln x \, dx = \left[ \frac{x^2}{2} (\ln x) \right]_a^1 - \int_a^1 \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{-a^2}{2} (\ln a) - \left[ \frac{x^2}{4} \right]_a^1 = \frac{-a^2}{2} (\ln a) - \frac{1}{4} + \frac{a^2}{4}$$

Therefore, we have

$$\int_0^1 x \ln x \, dx = \lim_{a \rightarrow 0^+} \left( \frac{-a^2}{2} (\ln a) - \frac{1}{4} + \frac{a^2}{4} \right).$$

We evaluate the first limit as follows:

$$\underbrace{\lim_{a \rightarrow 0^+} \left( \frac{a^2}{2} (\ln a) \right)}_{\text{indet. '0} \cdot \infty'} = \underbrace{\lim_{a \rightarrow 0^+} \left( \frac{\ln a}{2/a^2} \right)}_{\frac{-\infty}{\infty'}} \stackrel{L'H}{=} \lim_{a \rightarrow 0^+} \left( \frac{\frac{1}{a}}{-4/a^3} \right) = \lim_{a \rightarrow 0^+} \left( \frac{a^2}{-4} \right) = 0.$$

Thus, we have  $\int_0^1 x \ln x \, dx = -\frac{1}{4}.$

The improper integral converges to  $-\frac{1}{4}.$

$$\begin{aligned}
\text{(b)} \quad \int_0^1 \frac{4r}{\sqrt{1-r^4}} dr &= \lim_{b \rightarrow 1^-} \underbrace{\left( \int_0^b \frac{4r}{\sqrt{1-r^4}} dr \right)}_{\text{subst. } u=r^2} = \lim_{b \rightarrow 1^-} \left( 2 \int_0^{b^2} \frac{1}{\sqrt{1-u^2}} du \right) = \lim_{b \rightarrow 1^-} 2 [\sin^{-1}(u)]_0^{b^2} \\
&= \lim_{b \rightarrow 1^-} 2 (\sin^{-1}(b^2)) = 2\left(\frac{\pi}{2}\right) = \pi
\end{aligned}$$

The improper integral converges to  $\pi$ .

$$\text{(c)} \quad \int_1^\infty \frac{1}{x^3} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^3} dx = \lim_{b \rightarrow \infty} \left[ \frac{1}{-2x^2} \right]_1^b = \lim_{b \rightarrow \infty} \left( \frac{1}{-2b^2} + \frac{1}{2} \right) = \frac{1}{2}$$

The improper integral converges to  $\frac{1}{2}$ .

$$\text{(d)} \quad \int_5^\infty \frac{1}{\sqrt{x-1}} dx = \lim_{b \rightarrow \infty} \left( \int_5^b \frac{1}{\sqrt{x-1}} dx \right) = \lim_{b \rightarrow \infty} [2\sqrt{x-1}]_5^b = \lim_{b \rightarrow \infty} (2\sqrt{b-1} - 4) = \infty$$

The improper integral diverges.

$$\text{(e)} \quad \int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \left( \int_2^b \frac{1}{x(\ln x)^2} dx \right) = \lim_{b \rightarrow \infty} \left[ \frac{-1}{\ln x} \right]_2^b = \lim_{b \rightarrow \infty} \left( \frac{-1}{\ln b} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

The improper integral converges to  $\frac{1}{\ln 2}$ .

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3. Sketch each of the region enclosed by the given lines and curves. Find the area of the enclosed region.

(a)  $y = 2x - x^2$  and  $y = -3$

(b)  $y = x^2 - 2x$  and  $y = x$

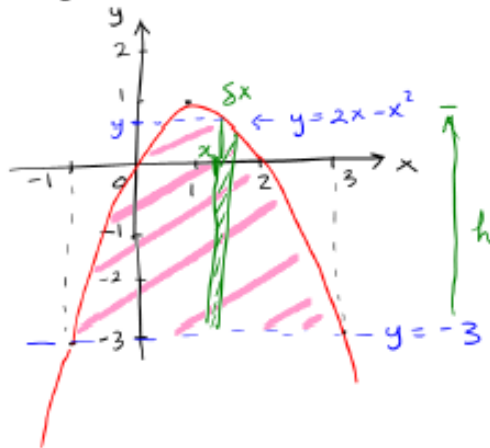
(c)  $x = y^2$  and  $x = y + 2$

(d)  $x = y^3 - y^2$  and  $x = 2y$

(Answers: (a)  $\frac{32}{3}$  (b)  $\frac{9}{2}$  (c)  $\frac{9}{2}$  (d)  $\frac{37}{12}$ )

[Solution] For the above, sketch the graphs to find the enclosed regions.

(a)  $y = 2x - x^2$ ,  $y = -3$

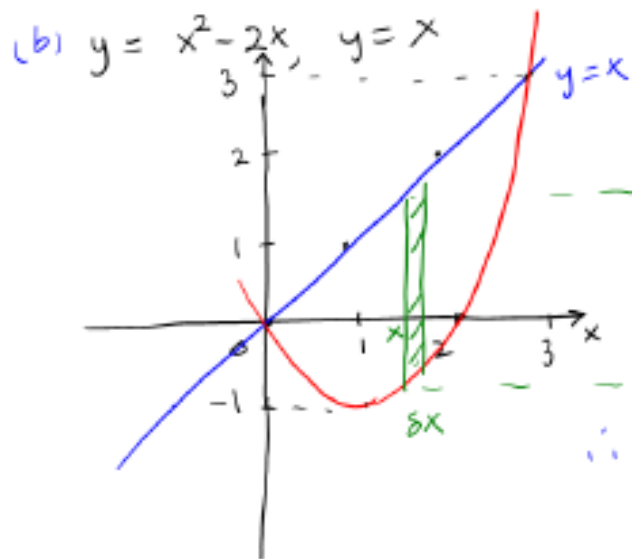


For  $-1 \leq x \leq 3$ ,  
area of a typical  
strip is

$$\underbrace{(2x - x^2 - (-3))}_{h} (\delta x)$$

So, area of region

$$\begin{aligned} \text{is } & \int_{-1}^3 (2x - x^2 + 3) dx \\ &= \left[ x^2 - \frac{x^3}{3} + 3x \right]_{-1}^3 \\ &= \frac{32}{3} \end{aligned}$$



For each  $0 \leq x \leq 3$   
area of typical  
strip is

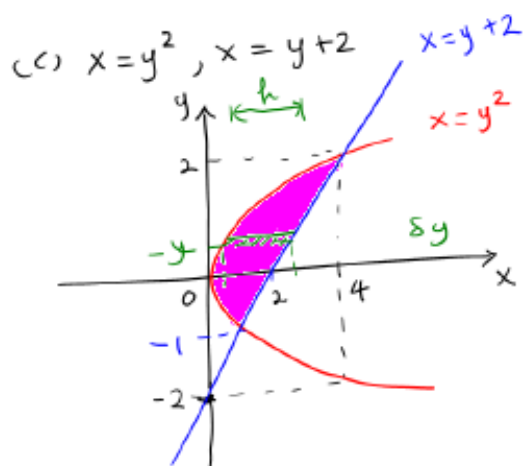
$$(x - (x^2 - 2x)) (\delta x)$$

$\therefore$  Required area

$$\text{is } \int_0^3 x - (x^2 - 2x) dx$$

$$= \int_0^3 x^2 - 3x dx = \left[ \frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3$$

$$= \frac{9}{2} //$$

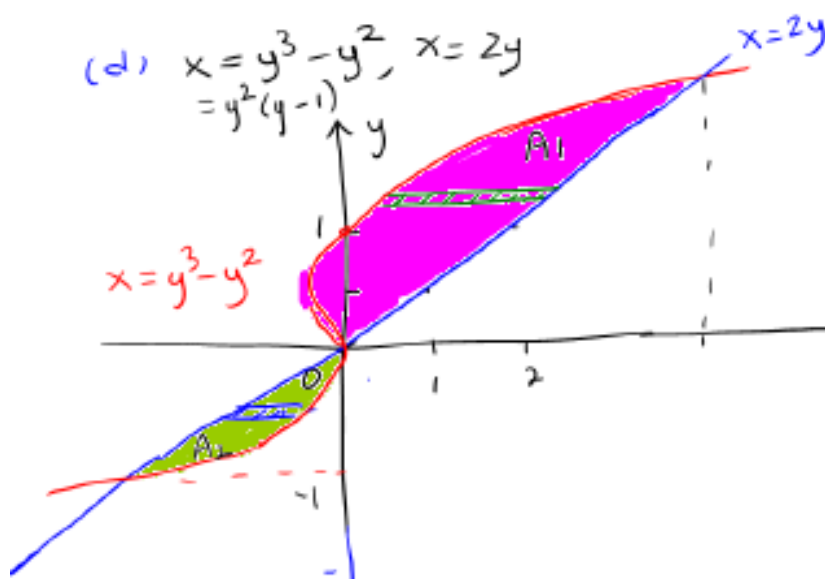


We use horizontal strips :-  
 For each  $-1 \leq y \leq 2$ ,  
 area of a typical strip is  
 $\underbrace{(y+2) - y^2}_{h} (\delta y)$

So, required area of region

$$\text{is } \int_{-1}^2 (y+2) - y^2 dy = \left[ \frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \left( \frac{4}{2} + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right)$$

$$= \cancel{(2+2-\frac{8}{3})} - \cancel{(\frac{1}{2}-2+\frac{1}{3})} = \frac{9}{2} = 9/2$$



Solve for point of intersection :-  
 $y^3 - y^2 = 2y$   
 $\Leftrightarrow y^3 - y^2 - 2y = 0$   
 $\Leftrightarrow y(y^2 - y - 2) = 0$   
 $\Leftrightarrow y(y+1)(y-2) = 0$   
 $\Leftrightarrow y = 0 \text{ or } y = -1 \text{ or } y = 2$

(d):

For region  $A_1$ , note that  $0 \leq y \leq 2$ , the area of a typical horizontal strip is  $(2y - (y^3 - y^2))(\delta y)$ .

Thus, the area of the region  $A_1$  is

$$\int_0^2 (2y - (y^3 - y^2)) dy = \left[ y^2 - \frac{y^4}{4} + \frac{y^3}{3} \right]_0^2 = \frac{8}{3}$$

For region  $A_2$ , note that  $-1 \leq y \leq 0$ , the area of a typical horizontal strip is  $((y^3 - y^2) - 2y)(\delta y)$ .

Thus, the area of the region  $A_1$  is

$$\int_{-1}^0 ((y^3 - y^2) - 2y) \, dy = \left[ \frac{y^4}{4} - \frac{y^3}{3} - y^2 \right]_{-1}^0 = -(1/4) - 1/3 + 1 = \frac{5}{12}$$

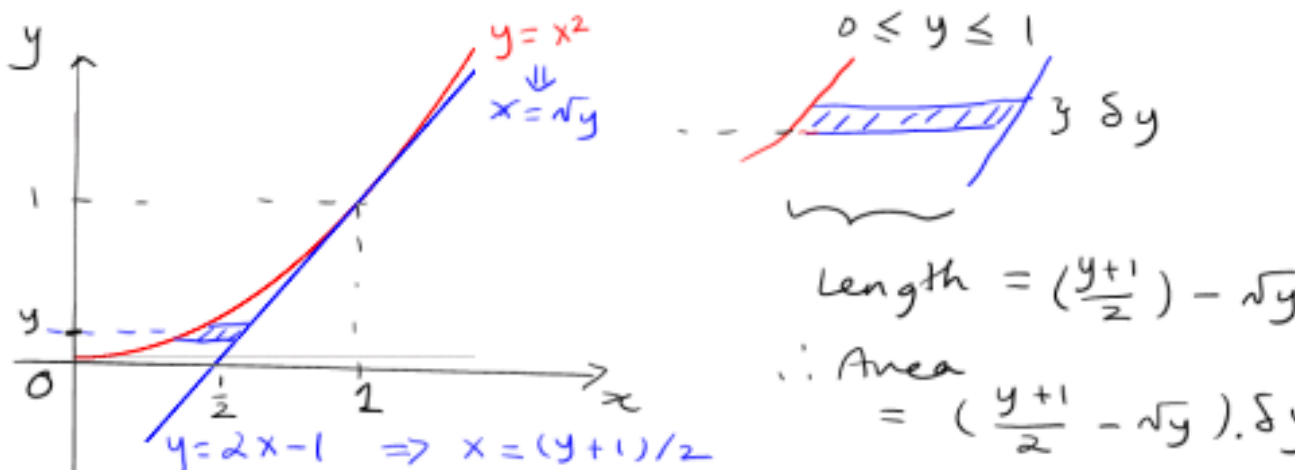
Total area is  $\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$ .

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4. (a) Find the area of the region bounded by the parabola  $y = x^2$ , the tangent line to this parabola at  $(1, 1)$ , and the  $x$ -axis.

[Solution] For  $y = x^2$ , we have  $\frac{dy}{dx} = 2x$ . Thus the equation of tangent line to this parabola at  $(1, 1)$  is  $y - 1 = 2(x - 1)$ , i.e.,  $y = 2x - 1$ .

From the sketch of the graph of  $y = x^2$  and  $y = 2x - 1$ ,



Note:  $y = 2x - 1 \Leftrightarrow x = \frac{y+1}{2}$ .

At each  $y$  where  $0 \leq y \leq 1$ , the area of each horizontal strip is  $(\frac{y+1}{2} - \sqrt{y}) \cdot \delta y$ .

Therefore, the area required is

$$\int_0^1 \left[ \frac{1}{2}(y+1) - \sqrt{y} \right] dy = \frac{1}{4}y^2 + \frac{1}{2}y - \frac{2}{3}(\sqrt{y})^3 \Big|_0^1 = \frac{1}{12}.$$

#### Alternative solution:

The area required is the difference between the area of the region under the curve  $y = x^2$  for  $0 \leq x \leq 1$  and the area of the region (which is a triangle) under the line  $y = 2x - 1$ , for  $1/2 \leq x \leq 1$ .

Thus we have area required is

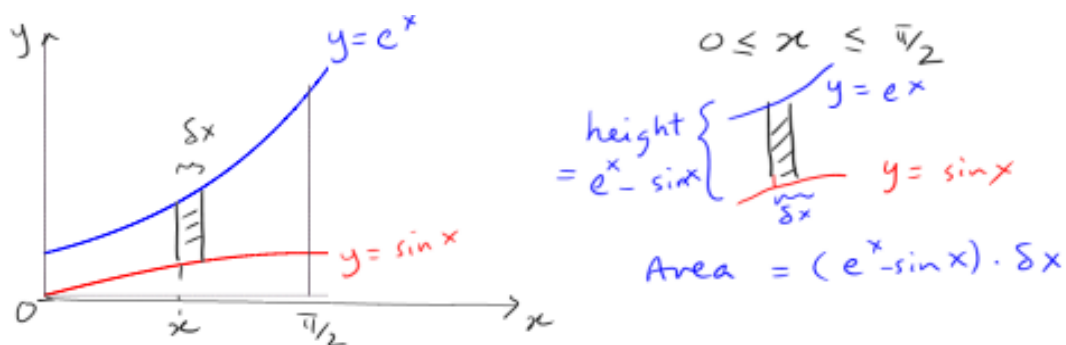
$$\int_0^1 x^2 dx - \frac{1}{2}(1)\left(\frac{1}{2}\right) = \left[ \frac{x^3}{3} \right]_0^1 - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

- (b) Sketch the region bounded by the given curves and find the area of the region:

$$y = \sin x, y = e^x, x = 0, x = \pi/2$$

[Solution] You should sketch the graphs of  $y = \sin x, y = e^x$  on  $[0, \pi/2]$ :





Note that for each  $0 \leq x \leq \frac{\pi}{2}$ , the area of the vertical strip is  $(e^x - \sin x)\delta x$ .

The area of the region is thus given by

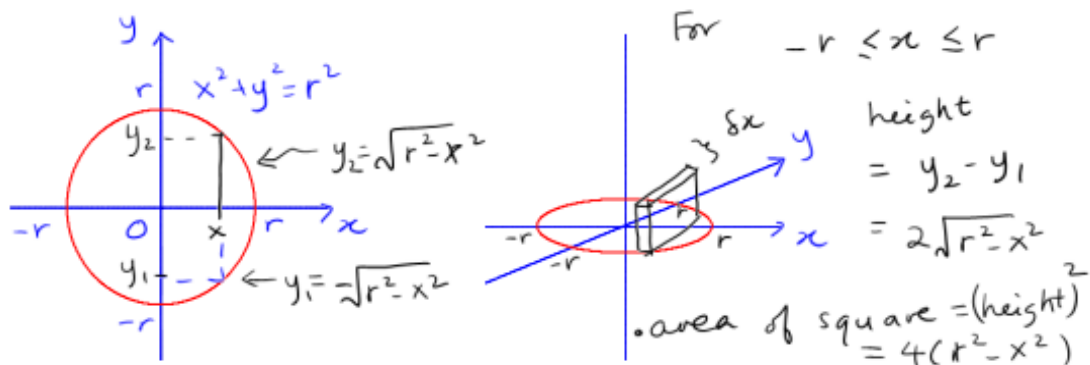
$$\int_0^{\frac{\pi}{2}} e^x - \sin x \, dx = (e^x + \cos x) \Big|_0^{\frac{\pi}{2}} = e^{\frac{\pi}{2}} - 2.$$


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5. The base of a solid  $S$  is circular disk with radius  $r$ . Parallel cross-sections perpendicular to the base are squares. Show that the volume of the solid  $S$  is  $\frac{16}{3}r^3$ .

(Note that the equation of a circle with radius  $r$  and centered at  $(0,0)$  is  $x^2 + y^2 = r^2$ .)

[Solution] Consider the circular disc with equation  $x^2 + y^2 = r^2$ , centered at  $(0,0)$  with radius  $r$ .



For each  $x$  where  $-r \leq x \leq r$ , note that the area of each square cross-section is

$$A(x) = (2y)^2 = (2\sqrt{r^2 - x^2})^2 = 4(r^2 - x^2).$$

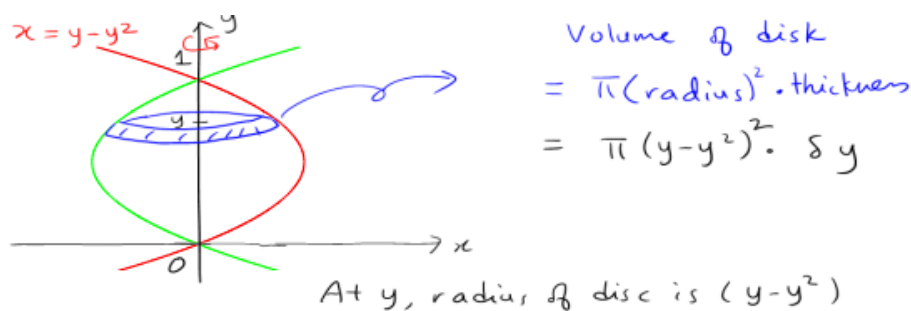
Thus, for  $-r \leq x \leq r$ , the volume of a square slice is  $4(r^2 - x^2)\delta x$ .

So, the volume of solid is

$$\int_{-r}^r 4(r^2 - x^2) dx = \int_{-r}^r 4(r^2 - x^2) dx = 4 \left[ r^2x - \frac{1}{3}x^3 \right]_{-r}^r = \frac{16}{3}r^3$$

6. (a) Find the volume of the solid obtained by revolving the region bounded by the curves  $x = y - y^2$  and  $x = 0$  about the  $y$ -axis. (Answer:  $\frac{\pi}{30}$ )

[Solution] Sketch a diagram and note that:



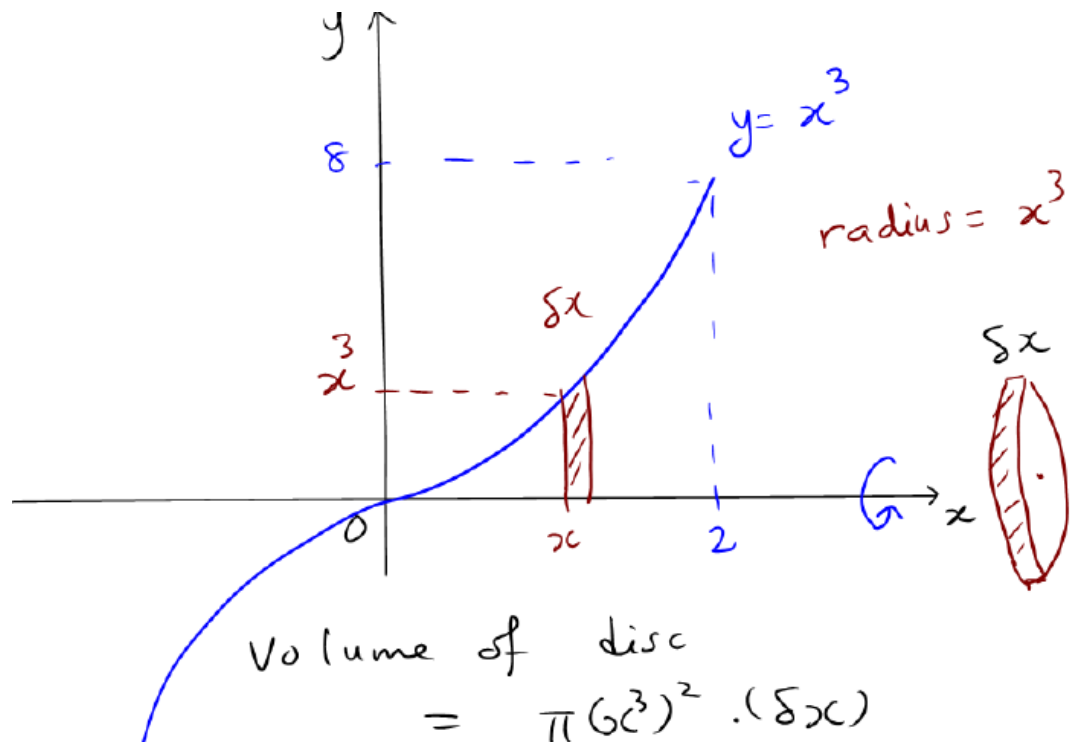
For each  $y$  with  $0 \leq y \leq 1$ : a cross section is a disk with radius  $y - y^2$  and thickness  $\delta y$  so that its volume is  $\pi(y - y^2)^2(\delta y)$ .

Thus, the volume of the solid is  $\int_0^1 \pi(y - y^2)^2 dy = \pi \int_0^1 y^2 - 2y^3 + y^4 dy$

$$= \pi \left( y^3/3 - 2y^4/4 + y^5/5 \right) \Big|_0^1 = \frac{\pi}{30}.$$

- (b) Find the volume of the solid generated by revolving the regions bounded by the curve  $y = x^3$  and lines  $y = 0$  and  $x = 2$  about the  $x$ -axis.

[Solution]



Using the disc method, we have the volume of the solid is given by

$$\int_0^2 \pi (x^3)^2 dx = \left[ \pi \frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}.$$

**Method using Cylindrical shell:** We shall use a horizontal slice with  $0 \leq y \leq 8$ . This horizontal slice generates a cylindrical shell when it is rotated about the  $x$ -axis.

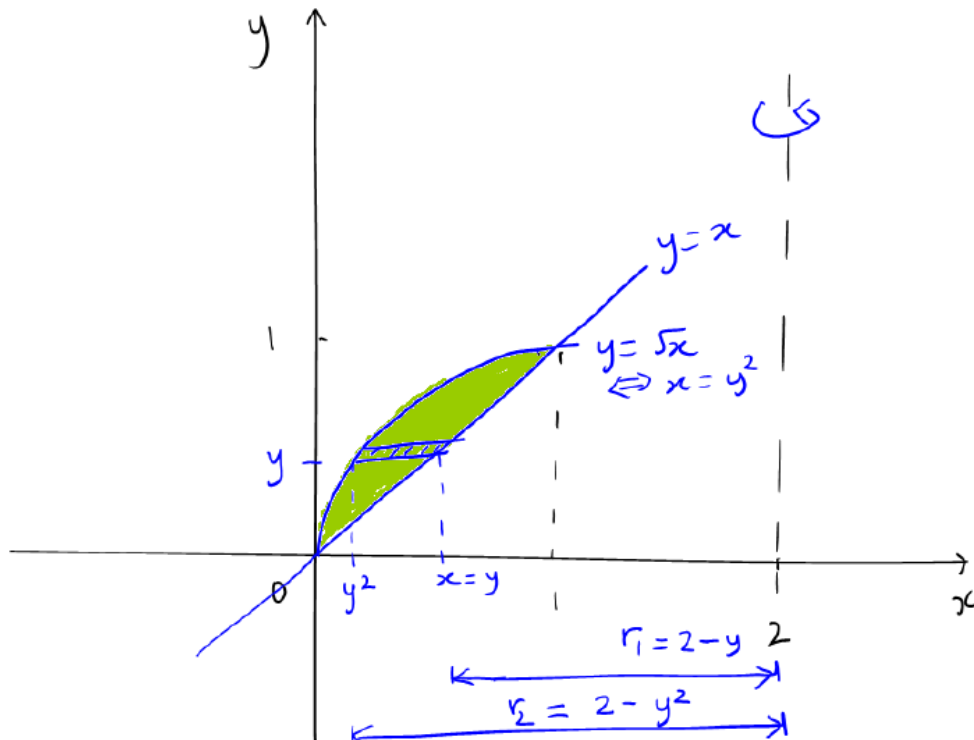
Volume of a typical cylindrical shell is  $2\pi(y)(2 - y^{1/3})(\delta y)$ .

Thus the volume of the solid is

$$\int_0^8 2\pi(y)(2 - y^{1/3})dy = 2\pi \int_0^8 (2y - y^{5/3}) dy = 2\pi \left[ y^2 - \frac{3y^{7/3}}{7} \right]_0^8 = \frac{128\pi}{7}.$$

- (c) Find the volume of the solid obtained by revolving the region bounded by the curves  $y = x$  and  $y = \sqrt{x}$  about the line  $x = 2$ . (Answer:  $\frac{8}{15}\pi$ )

[Solution] Sketch the graph and note that



At each  $y$  between 0 and 1:

each horizontal slice is ring (center from the line  $x = 2$ ): with thickness  $\delta y$ , with outer radius  $2 - y^2$  and inner radius is  $2 - y$ .

Thus the volume of typical slice is

$$\pi ((2 - y^2)^2 - (2 - y)^2) \delta y.$$

Therefore volume of the solid is

$$\begin{aligned} \int_0^1 \pi [(2 - y^2)^2 - (2 - y)^2] dy &= \pi \int_0^1 [(4 - 4y^2 + y^4) - (4 - 4y + y^2)] dy \\ &= \pi \int_0^1 (-5y^2 + y^4 + 4y) dy = \pi \left[ \frac{-5y^3}{3} + \frac{y^5}{5} + 2y^2 \right]_0^1 = \left( \frac{-5}{3} + \frac{1}{5} + 2 \right) \pi = \frac{8}{15} \pi. \end{aligned}$$

#### Method by Cylindrical shell:

For each  $x$  where  $0 \leq x \leq 1$ , each vertical strip rotated about the line  $x = 2$  will generate a cylindrical shell with radius  $(2 - x)$  and height  $(\sqrt{x} - x)$ . Thus, the volume of a typical cylindrical shell is  $2\pi(2 - x)(\sqrt{x} - x)(\delta x)$ .

Therefore, the volume of the solid is

$$\int_0^1 2\pi(2 - x)(\sqrt{x} - x) dx = 2\pi \int_0^1 (2 - x)(\sqrt{x} - x) dx$$

$$\begin{aligned}
&= 2\pi \int_0^1 \left( 2\sqrt{x} - 2x - x^{3/2} + x^2 \right) dx = 2\pi \left[ \frac{4x^{3/2}}{3} - x^2 - \frac{2x^{5/2}}{5} + \frac{x^3}{3} \right]_0^1 \\
&= 2\pi \left( \frac{4}{3} - 1 - \frac{2}{5} + \frac{1}{3} \right) = \frac{8}{15}\pi.
\end{aligned}$$


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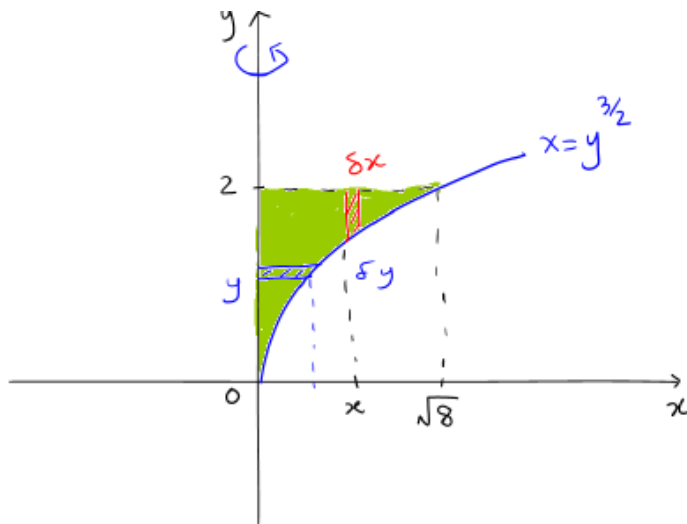
7. Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the  $y$ -axis.

(a)  $x = y^{3/2}$ ,  $x = 0$ ,  $y = 2$

(b)  $x = \sqrt{2 \sin 2y}$ ,  $0 \leq y \leq \pi/2$ ,  $x = 0$

[Solution]

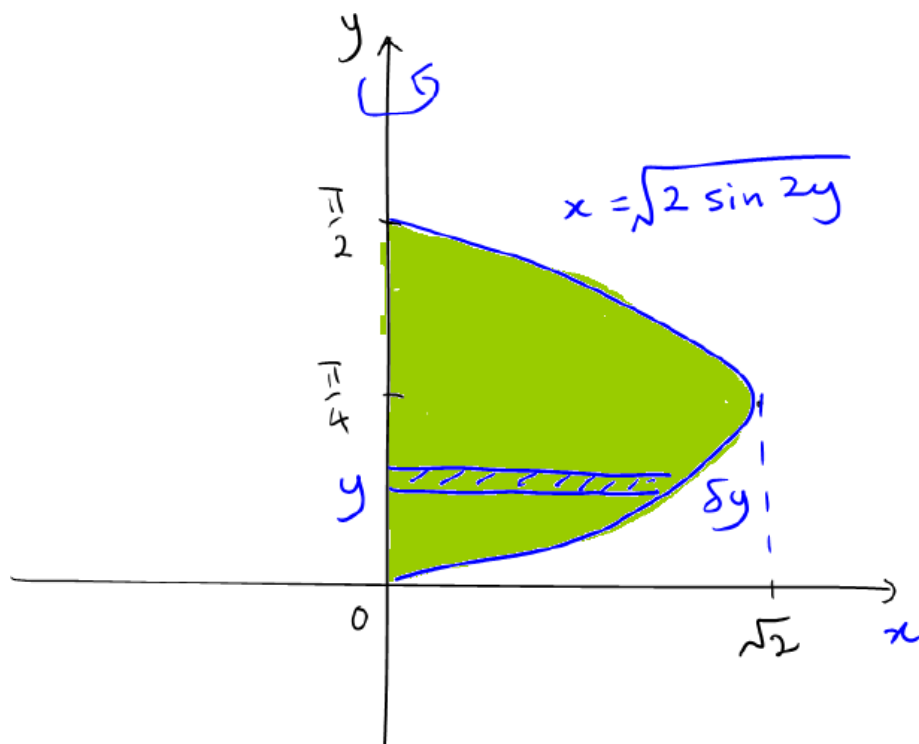
(a) Volume of the solid is  $\int_0^2 (\pi y^3) dy = \pi \left[ \frac{y^4}{4} \right]_0^2 = 4\pi$



① Disc method  
 Volume of solid  $= \int_0^2 \pi (y^{3/2})^2 dy$   
 $= \pi \int_0^2 y^3 dy = \pi \left( \frac{y^4}{4} \right) \Big|_0^2$   
 $= 4\pi$

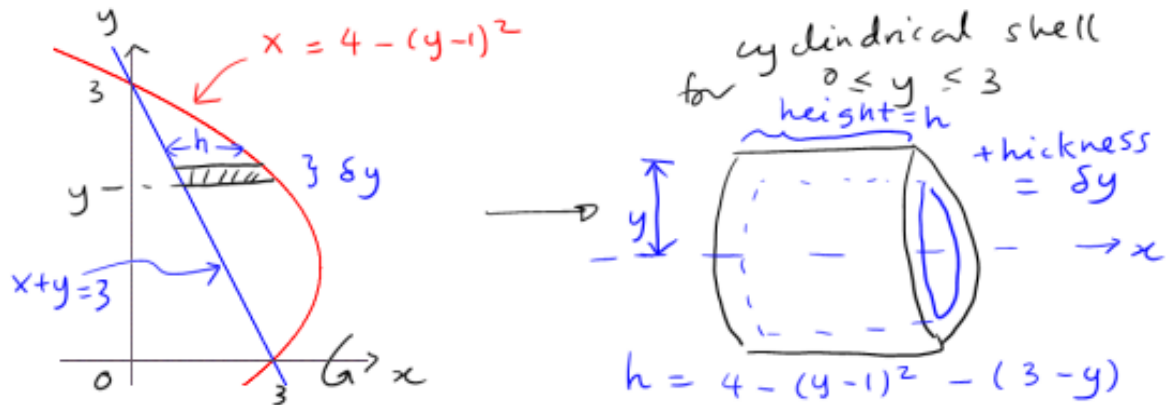
② Cylindrical shell  
 Volume of solid  $= \int_0^{\sqrt{8}} 2\pi(x) (2 - x^{2/3}) dx$   
 $= 2\pi \int_0^{\sqrt{8}} 2x - x^{5/3} dx$   
 $= 2\pi \left[ x^2 - \frac{x^{8/3}}{8/3} \right]_0^{\sqrt{8}}$   
 $= 2\pi \left( 8 - \frac{16(3)}{8} \right) = 4\pi$

(b) Volume of the solid is  $\int_0^{\pi/2} \pi(\sqrt{2 \sin(2y)})^2 dy = \pi \int_0^{\pi/2} 2 \sin(2y) dy = \pi [-\cos(2y)]_0^{\pi/2} = 2\pi$ .



8. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves  $x + y = 3$  and  $x = 4 - (y - 1)^2$  about the  $x$ -axis.

[Solution] For  $0 \leq y \leq 3$ ,



The volume  $\delta V$  of a typical cylindrical shell is

$$\delta V = 2\pi y \underbrace{[4 - (y - 1)^2 - (3 - y)]}_{\text{Height of cylindrical shell}} \delta y$$

Thus, the volume  $V$  of the solid is

$$\begin{aligned} \int_0^3 2\pi y [4 - (y - 1)^2 - (3 - y)] dy &= 2\pi \int_0^3 y(-y^2 + 3y) dy \\ &= 2\pi \int_0^3 -y^3 + 3y^2 dy = 2\pi \left[ \frac{-y^4}{4} + y^3 \right]_0^3 = \frac{27\pi}{2} \end{aligned}$$

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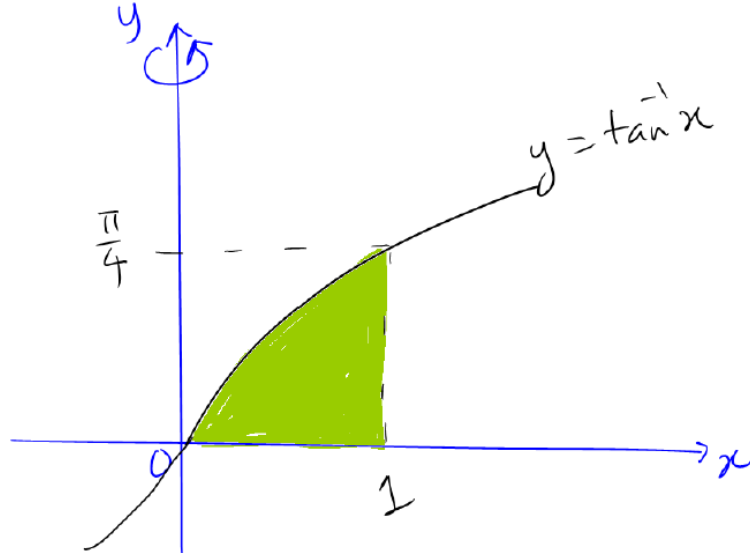
9. Consider the region bounded by the graphs of  $y = \tan^{-1} x$ ,  $y = 0$  and  $x = 1$ .

(a) Find the area of the region.

(b) Find the volume of the solid formed by revolving this region about the  $y$ -axis.

[Solution]

(a) Note that for  $0 \leq x \leq 1$  and  $0 \leq y \leq \pi/4$ , we have  $y = \tan^{-1} x \iff x = \tan y$ .



$$\begin{aligned} \text{Area of the region is } \int_0^1 \tan^{-1} x \, dx &= [x \tan^{-1} x]_0^1 - \int_0^1 x \left( \frac{1}{1+x^2} \right) dx \\ &= \frac{\pi}{4} - \left( \frac{1}{2} \right) [\ln(1+x^2)]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2. \end{aligned}$$

**Alternative Method:** Area of the enclosed region is

$$\int_0^{\pi/4} 1 - \tan y \, dy = [y - \ln |\sec x|]_0^{\pi/4} = \frac{\pi}{4} - (\ln \sqrt{2} - \ln 1) = \frac{\pi}{4} - \ln \sqrt{2} = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

(b) Volume of each cylindrical shell is  $(2\pi x)(\tan^{-1} x)\delta x$  where  $0 \leq x \leq 1$ .

Thus the volume of the solid is  $\int_0^1 (2\pi x)(\tan^{-1} x) \, dx = 2\pi \int_0^1 x \tan^{-1} x \, dx$ .

Now, we apply integration by parts to  $\int_0^1 x \tan^{-1} x \, dx$  :

$u = \tan^{-1} x$ ,  $v' = x$  so that  $u' = \frac{1}{1+x^2}$  and  $v = \frac{x^2}{2}$

$$\begin{aligned} \int_0^1 \underbrace{x}_{v'} \underbrace{\tan^{-1} x}_u \, dx &= \left[ \frac{x^2}{2} \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x^2}{2} \left( \frac{1}{1+x^2} \right) dx \\ &= \frac{\pi}{8} - \frac{1}{2} \int_0^1 \left( 1 - \frac{1}{1+x^2} \right) dx = \frac{\pi}{8} - \frac{1}{2} [x - \tan^{-1}(x)]_0^1 = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}. \end{aligned}$$

Thus, the volume is  $2\pi \left( \frac{\pi}{4} - \frac{1}{2} \right) = \pi \left( \frac{\pi}{2} - 1 \right)$ .

**Alternative Method:** The volume of the solid is

$$\begin{aligned} \int_0^{\pi/4} (\pi(1)^2 - \pi(\tan y)^2) \, dy &= \pi \int_0^{\pi/4} (1 - \tan^2 y) \, dy \\ &= \pi \int_0^{\pi/4} (1 - (\sec^2 y - 1)) \, dy = \pi [2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) \end{aligned}$$