# Abstract

The Monte Carlo method is a statistical technique which can simulate a mathematical or physical experiment on a computer. A Monte Carlo code is developed to simulate the case study of nuclear level gauge which consist of a Cs-137 source, a tank with water/air and detector. Tabulated photoelectric cross section for water are used in the energy range 0.001-1 Mev. Compton cross section calculated form Klein-Nishina (KN) formula and for scattered photon angle distribution and Rejection sampling used for the angle distribution. With the variation of tank width, particles are tracked and counted which reached the detector. The detector response depicts a decreasing trend with increase in the tank width.

# Introduction

The Monte Carlo Method is concerned with the application of random sampling to problems of applied mathematics. While subtle or difficult questions may arise in applications, most problem can be treated without using much statistical theory. Nevertheless, statistical theory can be very helpful.[1]

The Monte Carlo method is a numerical solution to a problem that models objects interacting with other objects or their environment based upon simple object-object or object environment relationships. It represents an attempt to model nature through direct simulation of the essential dynamics of the system in question. In this sense the Monte Carlo method is essentially simple in its approach—a solution to a macroscopic system through simulation of its microscopic interactions.[2]

A solution is determined by random sampling of the relationships, or the microscopic interactions, until the result converges. Thus, the mechanics of executing a solution involves repetitive action or calculation. To the extent that many microscopic interactions can be modelled mathematically, the repetitive solution can be executed on a computer. However, the Monte Carlo method predates the computer (more on this later) and is not essential to carry out a solution although in most cases computers make the determination of a solution much faster. There are many examples of the use of the Monte Carlo method that can be drawn from social science, traffic flow, population growth, finance, genetics, quantum chemistry, radiation sciences, radiotherapy, and radiation dosimetry but our discussion will concentrate on the simulation of neutrons, photons and electrons being transported in condensed materials, gases and vacuum. We will make brief excursions into other kinds of Monte Carlo methods when they serve to elucidate some point or when there may be a deeper connection to particle-matter interactions or radiation transport in general.[2]

In general, a particle is emanated from a source (fixed or fission) with a random spatial position, random direction, and random energy. Each particle has a chance of traveling freely in a medium before undergoing an interaction with nuclei. Different types of interactions may occur, depending on the particle type and energy, and the composition of the medium. These interactions, which can be described by pdfs that are established using nuclear data and physics models, may lead to production of one or more particles, termination of the particle, and/or a change in particle energy or direction. A particle also may be terminated if it escapes from the medium. In a Monte Carlo simulation, the “history” of each particle from birth to death is followed, expected tallies of interest are estimated by simulation of numerous histories, and the associated variance and/or relative uncertainties for these tallies are evaluated.[3]

## Monte Carlo Methods

Monte Carlo methods can be used to solve[4]

1. The problems that are stochastic (probabilistic) by nature:
   1. particle transport,
   2. telephone and other communication systems,
   3. population studies based on the statistics of survival and reproduction.
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Monte Carlo methods are divided into:[4]

1. ANALOG, where the natural laws are PRESERVED
   1. the game played is the analog of the physical problem of interest
2. (i.e., the history of each particle is simulated exactly)
3. NON-ANALOG, where in order to reduce required computational time the strict analog simulation of particle histories is abounded (i.e., we CHEAT!)
4. Variance-reduction techniques:
   1. Absorption suppression
   2. History termination and Russian Roulette
   3. Splitting and Russian Roulette
   4. Forced Collisions
   5. Source Biasing

Major Components of a Monte Carlo Algorithm [4]

* Probability distribution functions (pdf’s) - the physical (or mathematical) system must be described by a set of pdf’s.
* Random number generator - a source of random numbers uniformly distributed on the unit interval must be available.
* Sampling rule - a prescription for sampling from the specified pdf, assuming the availability of random numbers on the unit interval.
* Scoring (or tallying) - the outcomes must be accumulated into overall tallies or scores for the quantities of interest.
* Error estimation - an estimate of the statistical error (variance) as a function of the number of trials and other quantities must be determined.
* Variance reduction techniques - methods for reducing the variance in the estimated solution to reduce the computational time for Monte Carlo simulation. • Parallelization and vectorization - efficient use of advanced computer architectures

## Probability Distribution Function and Sampling

Random Variable, x, - a variable that takes on particular values with a frequency that is determined by some underlying probability distribution.

Continuous Probability Distribution

Discrete Probability Distribution

If f(x) and F(x) represent PDF and CDF od a random variable x, and if is a random number distributed uniformly on [0,1] with PDF g()=1, and if x is such that

than for each there is a corresponding x, and the variable x is distribute according to the probability density function f(x).

## Direct Sampling

If the function F(x) is invertible then we can write

Then generate and determine x from as from above relation

Shape, rectangle

Description automatically generated

Figure 1: Direct Sampling

This is straight forward technique and often referred to as ‘high level’ approach. But often we came across the situation in which functions are complicated and cannot be inverted (e.g., Klien-Nishina)

## Rejection Sampling

When the abovementioned scenario occur in which inverse of CDF is costly are impossible then we employ the technique of random sampling.

Select a bounding function, g(x), such that

Sample x from g(x)

Then check

Diagram

Description automatically generated

Figure 2: Rejection Sampling

This is low level approach and sometimes hard to understand but involve simple computer operations.

## The Problem

The main physical processes associated with the transport of rays through matter are coherent scattering, incoherent scattering, the photoelectric effect, and pair creation. the energy range of interest the probability of coherent scattering or Rayleigh scattering, that is, the elastic scattering of a photon with the entire atom, is very low. In addition, the energy threshold for a gamma ray to be converted into an electron–positron pair is twice the rest mass of the electron (1.02 MeV). In the energy range around 1 MeV incoherent scattering or inelastic scattering, with an atom is dominant. In this process the photon transfers energy to an atomic electron which is ejected. At smaller energies the photon can also undergo a photoelectric absorption by an atom, which emits an electron. The subsequent atomic de-excitation takes place by emission of x-ray photons and/or Auger electrons. In both processes, incoherent scattering and the photoelectric effect, the energy is transferred to atomic electrons which are stopped in a very short distance in comparison to the mean free path of the gamma rays.[5]

Incoherent scattering can be approximately described by the Compton effect, that is, the elastic scattering of a photon with a free electron. From conservation of energy and momentum, the following relations can be easily deduced between the energy of the incident photon , the energy of the scattered photon, E, and the deviation angle of the photon

Chart, line chart

Description automatically generated

Figure 3: Compton scattering

Where the parameter is the incident energy in electron mass units, . The energy T transferred to the electron ranges from 0 () to a maximum value of

for backscattered rays at degrees.

The probability distribution of is given by well known Klien-Nishina cross section

which integrated over the solid angle gives a total cross section

where is the classical electron radius. If we neglect the binding energy of the atomic electrons, the cross section for incoherent scattering for an atom is given by multiplying the Klein–Nishina cross section by the atomic number Z. The effect of the binding energy can be considered by reducing the Z value by an amount that increases with atomic number and decreases with both increasing and u. At about 1 MeV this correction is negligible for angles higher than a few degrees, even for heavy elements.[5]

A rigorous treatment of the photoelectric effect leads to a cross section that cannot be expressed by simple analytical formulas. Therefore, photoelectric cross sections are usually tabulated for and Z.

## The Simulation

When a photon is in a medium, it traverses a certain distance in a straight line before it interacts with an atom. As is well known, the magnitude of s follows the statistical distribution

With the mean free path

where n is the number of atoms per unit volume, and is the total cross section per atom, which in our simple model is given by

The cross section for a molecule (or compound) can be obtained by adding up the corresponding cross sections of the atoms.

The interaction of the primary g ray with the medium gives rise either to photoelectric absorption or to Compton scattering with probabilities proportional to the corresponding cross sections. In the first case all the energy is transferred to the medium at the position of the interaction, while in the second case the photon is scattered in a direction given by the polar and azimuthal angles and , respectively. The value of follows a uniform probability distribution between 0 and 2, and follows the probability distribution

As mentioned earlier, the CDF can’t be inverted to find the . So acceptance rejection can be used in this case as described in previous section.

## The Algorithm

The simulation of the random processes is carried out by a Monte Carlo method. The parameters that have to be randomly generated are and the selection of the processes ,photoelectric or Compton, when an interaction occurs. A simple method for the generation of a random variable y with a probability density p(x) defined in the interval (a,b) relies on the fact that r defined as[5]

itself a random variable with a uniformly distributed probability in the interval (0,1). Therefore, y can be obtained from

where r is a random number in the interval (0,1) that can be provided by a computer. This method is feasible as long as Q(y) is a function that can be inverted. In our case we apply this method to generate the path traversed between two interactions, it is easily inferred that the random variable s is related to r by

This method also can be easily applied for the generation of

Unfortunately, this direct method cannot be used for the generation of . Although Eq. (8) can be integrated, cannot be obtained analytically from r. In this case we can apply the acceptance–rejection method. This procedure is less efficient than the direct method in which all random numbers generated by the computer are used, but the acceptance–rejection method can be applied to any bounded function. This method can be applied for the generation of for Compton scattering. The procedure as follows

The Klein-Nishina differential cross section can be written as (ignoring all constants)[6]

where

Random sampling of the Klein-Nishina cross section formula to determine x, by direct (inversion) method is not possible because of its complicated dependence of x. An acceptance-rejection method can be used, but the usual rectangular rejection function is an inefficient boundary for sampling x, especially for high E values[6].

We notice that in limit so we try an envelop function type of

where and can be determined imposing the conditions

From these conditions one can find

Sampling from can be achieved by the relation

where r is the random variable, distributed uniformly in the range (0,1). The energy of the scattered photon can be calculated from

in the units of electron rest mass energy.

Finally, the selection of the process, Compton scattering or photoelectric absorption, can be implemented by splitting the (0,1) interval in two subintervals with a length proportional to the corresponding cross sections. A g ray moves inside the medium until either it is absorbed by the photoelectric effect or leaves the medium, that is, until the position of the next randomly generated interaction occurs outside the medium.[7]

# Results and Discussion

Modeling of the case problem of nuclear level gauge is depicted in Fig. (4). The Cs-137 source resides at location (0,100). Water level and tank width both are variable. Figure (5) shows the tracks of some photon as emitted from the source. In this figure tank width is 20 cm and water level is 120 cm. As depicted, most of the particle reach the detector region where they are added to the detected count. This is not the case when tank width is increased as the particle will experience more and more interactions and thus chance to cover the tank width distance and to reach the detector will be reduced. The detector response with the increase is tank width is depicted in the Fig. (6). As tank width increased the response of the detector is decreased and there isn’t any optimum length only the minimum length at which reasonable particles are being registered on the detector incorporating the efficiency of the detector.

Chart

Description automatically generated

Figure 4: Nuclear level gauge problem

Chart, line chart

Description automatically generated

Figure 5: Particle tracks

Chart, histogram

Description automatically generated

Figure 6: Detector response vs tank width

# Reference

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