Chapter 8: Performance Surfaces and Optimum Points

Brandon Morgan

1/15/2021

E8.4

We are given the following scalar function

$$F(x) = x^4 - \frac{1}{2}x^2 + 1$$

1

Stationary points are defined by any set of points that make the gradient, $\nabla F(x)$, equal to zero. In scalar form, stationary points are the same as extrema, points that make the first derivative equal to zero.

Here we have the first derivative of our function:

$$F'(x) = 4x^3 - x$$

We can now solve, algebraically, for when $F'(x) = 0$.

$$F'(x) = 4x^3 - x = x(4x^2 - 1) = 0$$

From this, we can see that $x = [-\sqrt{\frac{1}{4}}, 0, \sqrt{\frac{1}{4}}]$.

2

Testing the stationary points to check whether they are minima or maxima's is determined by the second derivative test. The second derivative test is as follows:

$$\text{If } F''(x) < 0 \text{ then x is local maximum}$$
$$\text{If } F''(x) > 0 \text{ then x is local minimum}$$
$$\text{If } F''(x) = 0 \text{ then test is inconclusive}$$

Our second derivative is calculated to be:

$$F''(x) = 12x^2 - 1$$

Now we test our stationary points:

$$F''(-\frac{1}{4}) = -0.25 < 0 \ \therefore \text{ This point is local minimum}$$

$$F''(0) = -1 < 0 \ \therefore \text{ This point is local minimum}$$

$$F''(\frac{1}{4}) = -0.25 < 0 \ \therefore \text{ This point is local minimum}$$

To determine the global minimum, we plug these stationary points back into the function and the value which yields the smallest value is the global minimum.

$$F(-0.25) = F(0.25) = 0.9375$$
$$F(0) = 1$$

As we can see, we have a tie for the global minimum at the stationary points $x = [-0.25, 0.25]$

Even though the question does not ask for it, the inflection points of a function are defined where $F''(x) = 0$, thus we would get $x = [-\sqrt{\frac{1}{12}}, \sqrt{\frac{1}{12}}]$. Inflection point denotes the point where the function changes from concave up to concave down, or visa versa.

3

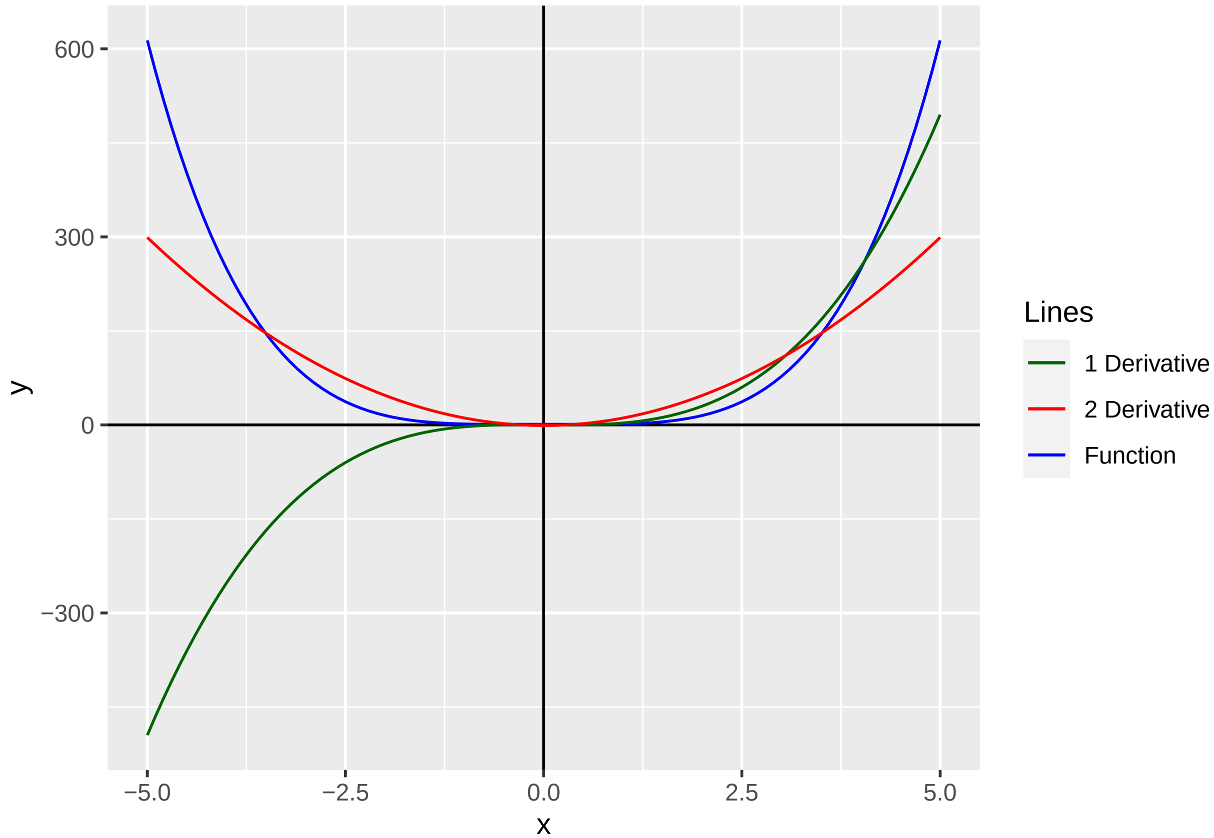
```
library(ggplot2)
```

```
## Warning: package 'ggplot2' was built under R version 3.6.3
```

```
fun = function(x) {
  x^4-0.5*x^2+1
}
deriv1 = function(x) {
  4*x^3-x
}
deriv2 = function(x) {
  12*x^2-1
}
```

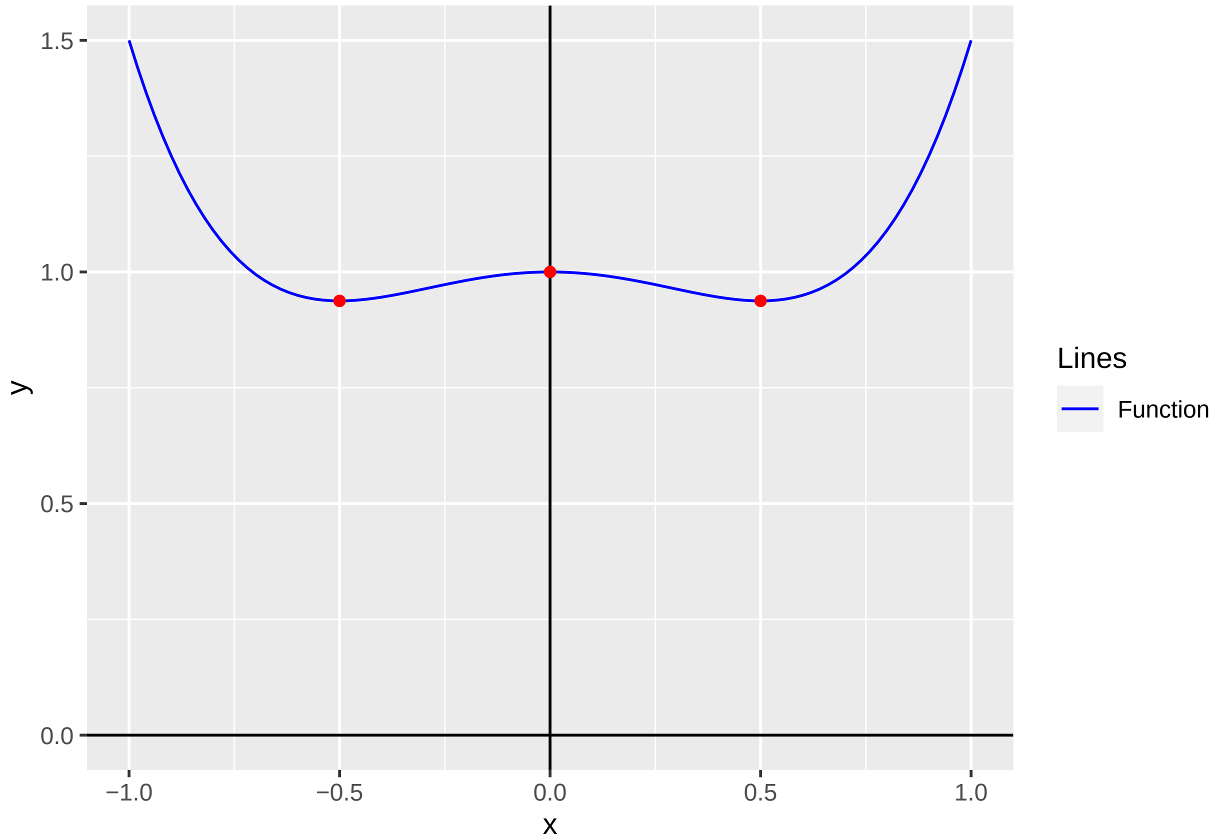
Here we have a plot of our function along with its first and second derivatives

```
ggplot(data.frame(x=c(-5,5)),aes(x=x))+geom_hline(yintercept=0)+geom_vline(xintercept = 0)+
  stat_function(fun=fun, aes(colour="Function"))+
  stat_function(fun=deriv1, aes(colour="1 Derivative"))+
  stat_function(fun=deriv2, aes(colour="2 Derivative"))+
  scale_colour_manual("Lines", values = c("dark green", "red","blue" ))
```



Here we have a zoomed in plot of our function about the domain $x = [-1, 1]$ along with the stationary points.

```
ggplot(data.frame(x=c(-1,1)),aes(x=x))+geom_hline(yintercept=0)+geom_vline(xintercept = 0)+
  stat_function(fun=fun, aes(colour="Function"))+
  scale_colour_manual("Lines", values = c("blue" ))+
  geom_point(aes(x=-0.5, y=fun(-0.5)), colour="red")+
  geom_point(aes(x=.5, y=fun(0.5)), colour="red")+
  geom_point(aes(x=0, y=fun(0)), colour="red")
```



1.5
1.0
0.5
0.0
y
−1.0
−0.5
0.0
0.5
1.0
x
Lines
Function

“‘