Chapter 8: Performance Surfaces and Optimum Points

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E8.1

We are given the following scalar function

$$F(x) = \frac{1}{x^3 - 0.75x - 0.5}$$

The second-order Taylor series approximation for a scalar function, $F(x)$, is given by:

$$F(x) = F(x^*) + \frac{d}{dx}F(x)|_{x=x^*}(x - x^*) + \frac{1}{2}\frac{d^2}{dx^2}F(x)|_{x=x^*}(x - x^*)^2$$

Here we get the following first and second derivatives of the function:

$$F'(x) = -\frac{3x^2 - 0.75}{(x^3 - 0.75x - 0.5)^2}$$

$$F''(x) = -\frac{2(-6x^4 + 2.25x^2 - 1.5x - 0.5625)}{(x^3 - 0.75x - 0.5)^3}$$

Thus, we end up with the following expansion:

$$F(x) = \frac{1}{(x^*)^3 - 0.75(x^*) - 0.5} - \frac{3(x^*)^2 - 0.75}{((x^*)^3 - 0.75(x^*) - 0.5)^2}(x-(x^*)) - \frac{1}{2}\frac{2(-6(x^*)^4 + 2.25(x^*)^2 - 1.5(x^*) - 0.5625)}{((x^*)^3 - 0.75(x^*) - 0.5)^3}(x-(x^*))^2$$

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When $x^* = -0.5$, our Taylor expansion reduces down to:

$$F(x) = -4 + 0(x + 0.5) + 24(x + 0.5)^2 = -4 + 24(x + 0.5)^2$$

2

When $x^* = 1.1$, our Taylor expansion reduces down to:

$$F(x) = 166.67 - 80000(x - 1.1) + 3.83 * 10^7(x - 1.1)^2$$

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We can now plot our scalar function and its second order Taylor expansion at each point to obersve its accuracy.

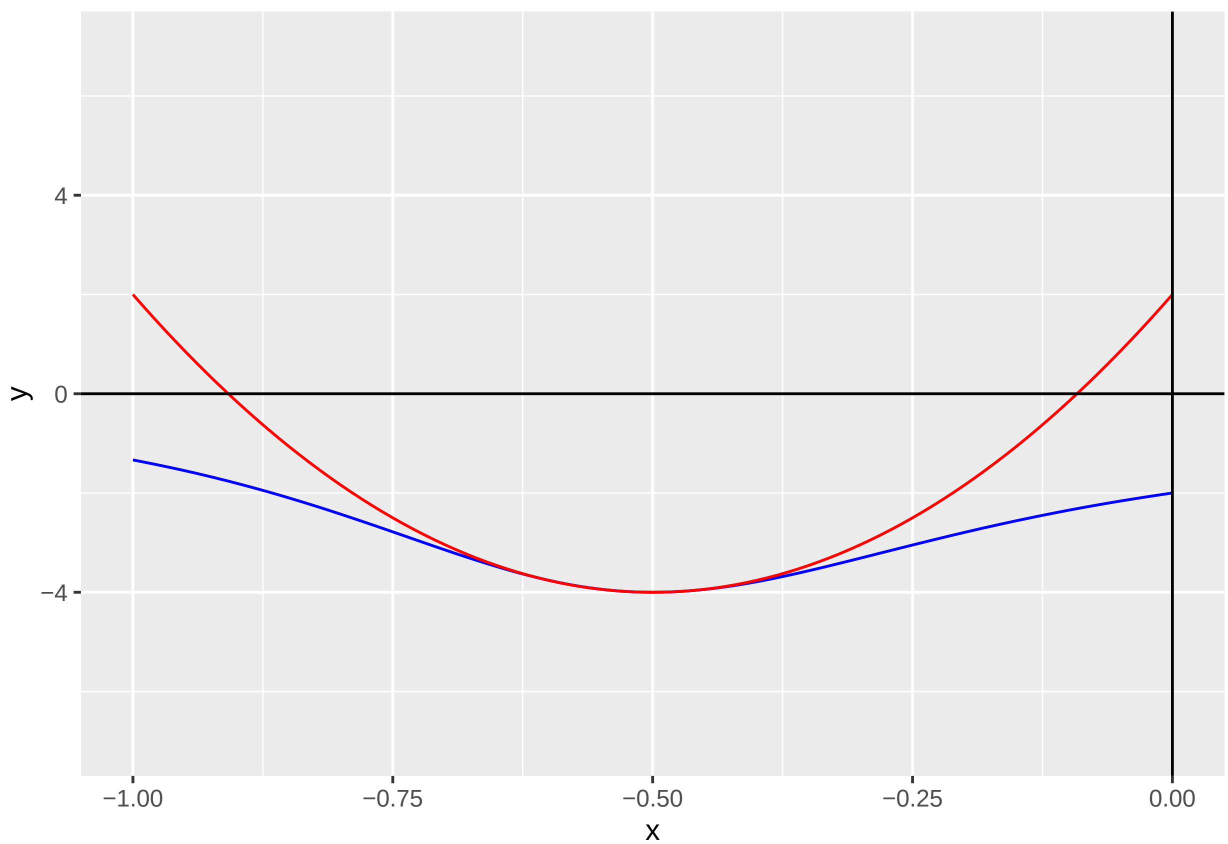
```
library(ggplot2)
```

```
## Warning: package 'ggplot2' was built under R version 3.6.3
```

```
fun = function(x) { # our original function
  1/(x^3-0.75*x-0.5)
}
secondDegree1 = function(x) { # taylor expansion at -0.5
  -4+24*(x+0.5)^2
}
secondDegree2 = function(x) { # taylor expansion at 1.1
  166.67-80000*(x-1.1)+3.83*(10^7)*(x-1.1)^2
}
```

Second-order term aat $x = -0.5$

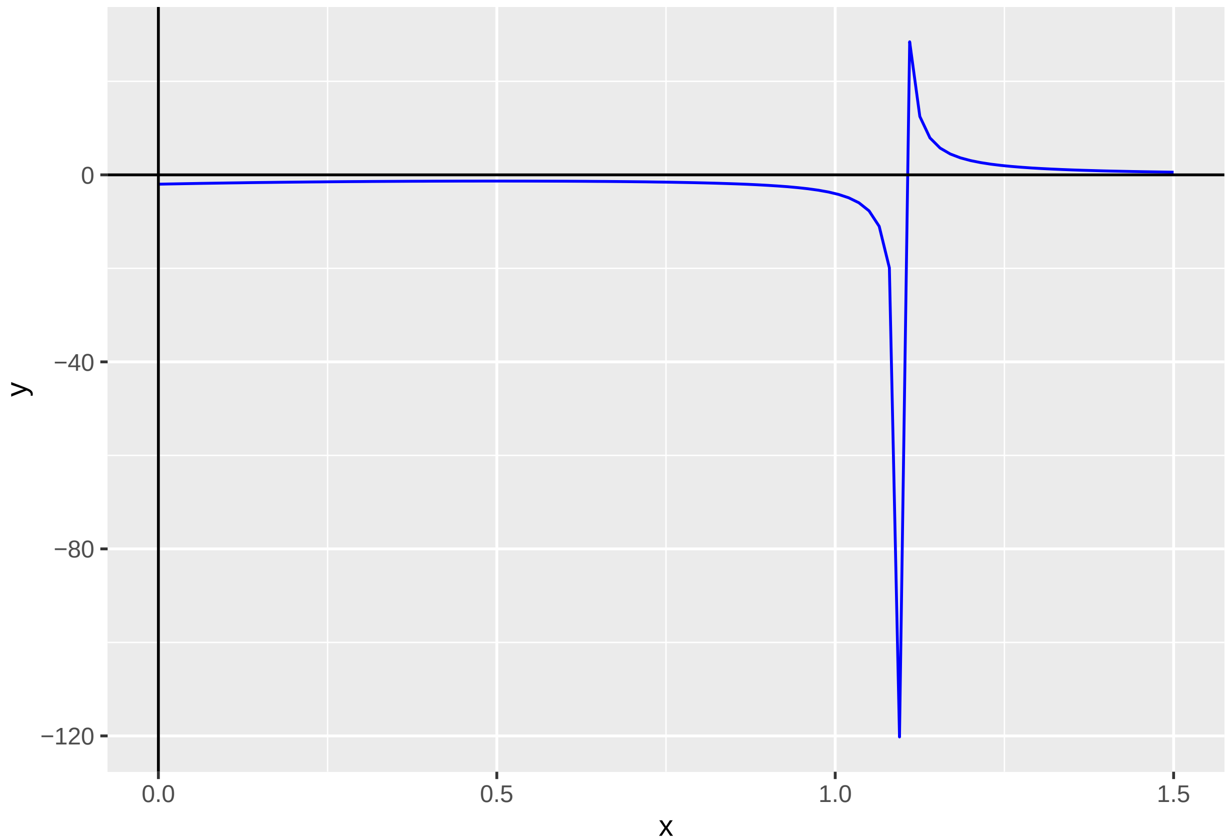
```
ggplot(data.frame(x=c(-1, 0)), aes(x=x))+stat_function(fun=fun, color = "blue")+
  stat_function(fun=secondDegree1, color = "red")+
  scale_y_continuous(limits = c(-7,7))+geom_hline(yintercept=0)+geom_vline(xintercept = 0)
```



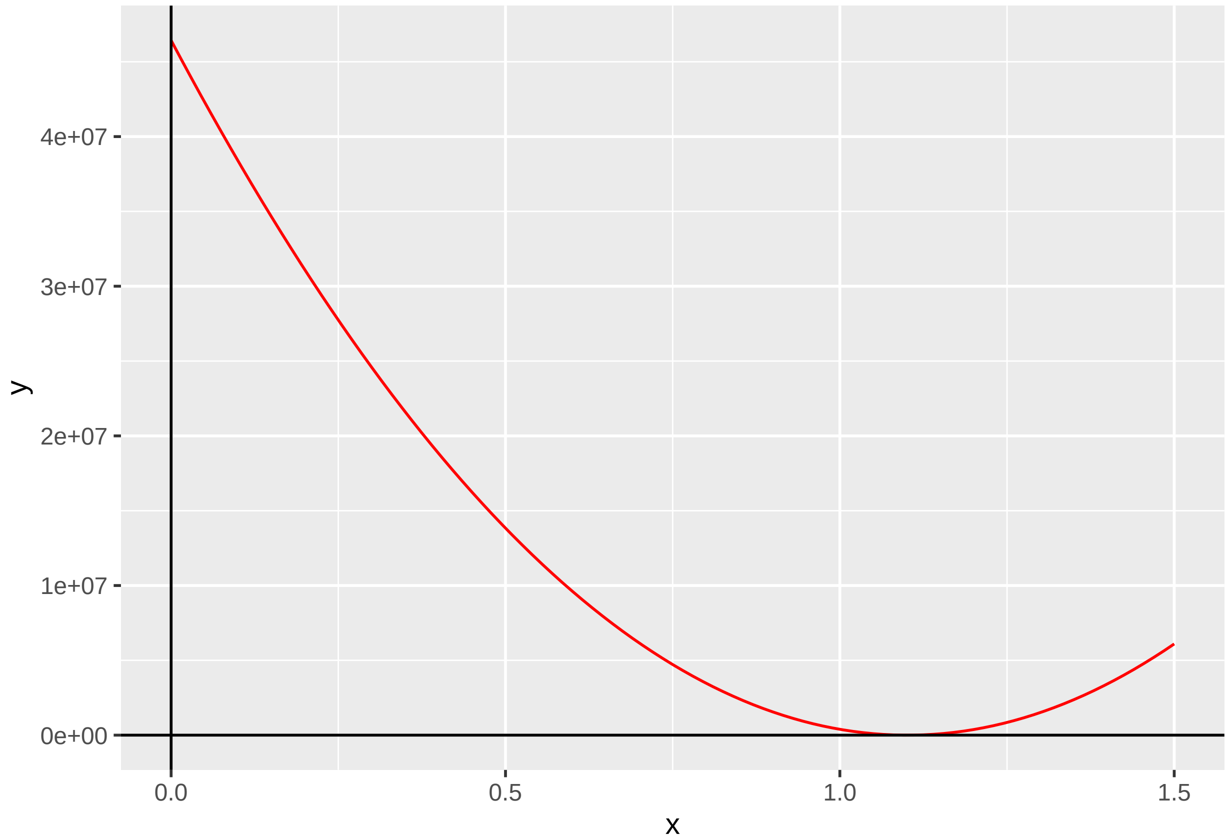
As we can see from the plot above, our second order Taylor approximation does a very good job of approximating our function at the point $x = -0.5$ within a small interval.

Second-order term at $x = 1.1$

```
ggplot(data.frame(x=c(0.5, 1.5)), aes(x=x))+stat_function(fun=fun, color = "blue")+geom_hline(yintercep
```




```
ggplot(data.frame(x=c(0.5, 1.5)), aes(x=x))+stat_function(fun=secondDegree2, color = "red")+geom_hline(y
```



Unfortunately, I was unable to plot both functions on top of each other; however, we can see that our function approaches infinity as $x \to 1.1$ from both sides. Because this point is undefined, our Taylor approximation does a poor job at approximating this point, as one can tell from its graph.