Chapter 13: Generalization

Brandon Morgan

1/26/2021

13.3

We are given the first order polynomial: $g_1(0) = x_0 + x_1 p$ to the following data set:

$$p_1 = 1,\ t_1 = 4$$

$$p_2 = 2,\ t_2 = 6$$

1

We want to find the least squares solution to the weights that minimize the sum squared error performance index:

$$F(x) = \Sigma(t - g_1(p))^2$$

Where in our example:

$$t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix},\ G = \begin{bmatrix} 1 & p_1 \\ 1 & p_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},\ x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}$$

Our performance index can be rewritten as:

$$F(x) = [t - GX]^T[t - GX] = t^T t - 2X^T G^T t + X^T G^T G X$$

We can find the minimum where the gradient of the function is zero:

$$\nabla F(X) = -2G^T t + 2G^T G X = 0$$

Solving for X we find: $X = [G^T G]^{-1} G^T t$:

$$X = [G^T G]^{-1} G^T t = (\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Thus, our first order polynomial which minimizes the least sum squared error is: $g = 2 + 2p$.

2

When we have the squared weight penalty,

$$F(x) = \Sigma(t - g_1(p))^2 + \Sigma\ x^2$$

Our performance index can be written as:

$$F(x) = [t - GX]^T[t - GX] + X^TX = t^Tt - 2X^TG^Tt + X^TG^TGX + X^TX$$

We can find the minimum where the gradient of the function is zero:

$$\nabla F(X) = -2G^Tt + 2G^TGX + 2X = 0$$

Now we can solve for X:

$$\begin{gathered}
-2G^Tt + 2G^TGX + 2X = 0 \\
-G^Tt + G^TGX + X = 0 \\
-G^Tt + (G^TG + I)X = 0 \\
(G^TG + I)X = G^Tt \\
X = (G^TG + I)^{-1}G^Tt
\end{gathered}$$

$$X = (G^TG + I)^{-1}G^Tt = (\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1.\bar{3} \\ 2 \end{bmatrix}$$

Thus, our first order polynomial which minimizes the least sum squared error with weight penalty is: $g = 1.\bar{3} + 2p$.