STA2201H Methods of Applied Statistics II

Monica Alexander

Week 2: Generalized Linear Models

GLMs

Where we were at:

- General linear models (e.g. multivariate linear regression) not appropriate for some outcome variables
- In particular, when the outcome Y has a restricted range or the variance depends on the mean
- Generalized Linear Models extend the classical set-up to allow for a wider range of distributions
- GLMs have three pieces
    1. **random component**: $Y_i \sim$ some distribution with $E[Y_i|\mathbf{X}_i] = \mu_i$
    2. **systematic component**: $\mathbf{X}_i^T\beta$
    3. The **link function** that links the random and systematic components $g(\mu_i) = \mathbf{X}_i^T\beta$

What can Y be distributed as? In principle, anything. In practice (and original formulation), distributions come from the **exponential family**.

Exponential Family

Exponential Family

The random variable Y belongs to the exponential family of distributions if its support does not depend upon any unknown parameters and its density or probability mass function takes the form

$$p(y|\theta, \phi) = \exp\left(\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right)$$

- $\theta = h(\mu)$ depends on the expected value of y and is the **canonical parameter**
- ϕ is the scale parameter (if known: one-parameter family)
- b and c are arbitrary functions

Mean and variance for exponential families

$$p(y|\theta, \phi) = \exp\left(\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right)$$

It can be shown that

$$E(Y|\theta, \phi) = b'(\theta) = \mu$$

and

$$Var(Y|\theta, \phi) = \phi b''(\theta)$$

Note the variance of Y depends not only on the scale parameter but also on a function of the mean.

Examples:

$$E(Y|\theta, \phi) = b'(\theta)$$

and

$$Var(Y|\theta, \phi) = \phi b''(\theta)$$

- Poisson:
  - $\theta = \log \mu$
  - $b(\theta) = e^{\theta}$
  - $\phi = 1$
  - $E(Y|\theta, \phi) = e^{\theta} = \mu, \; Var(Y|\theta, \phi) = 1 \times e^{\theta} = \mu$
- Normal:
  - $\theta = \mu$
  - $b(\theta) = \frac{1}{2}\theta^2$
  - $\phi = \sigma^2$
  - $E(Y|\theta, \phi) = \theta = \mu, \; Var(Y|\theta, \phi) = \sigma^2 \times 1 = \sigma^2$

The canonical link

The link function $g(\mu)$ could in theory be any function linking the linear predictor to the distribution of the outcome variable, which is also is **monotonic** and **smooth**.

Recall $\theta = h(\mu)$. If we choose $g = h$, then

$$\theta_i = h(\mu_i) = h(g^{-1}(\mathbf{x}_i^T \beta)) = h(h^{-1}(\mathbf{x}_i^T \beta)) = \mathbf{x}_i^T \beta$$

In other words, it ensures that the systematic component of our model is modeling the parameter of interest. The canonical link function transforms the mean μ_i to the natural/canonical parameter θ_i.

Canonical links

- Normal: identity $\theta = h(\mu) = \mu$
- Poisson: $\theta = h(\mu) = \log \mu$
- Bernoulli: $\theta = h(\mu) = \log(\frac{\mu}{1-\mu})$
- Exponential/Gamma: $\theta = h(\mu) = -\mu^{-1}$
- Inverse Gaussian: $\theta = h(\mu) = \mu^{-2}$

Likelihood-based estimation

- Inference is based on MLE, but cannot derive closed form solutions for regression coefficients

The log-likelihood function is:

$$\ell(\theta) = \sum_i \ell(\theta_i) = \sum_i \frac{Y_i\theta_i - b(\theta_i)}{\phi} + c(Y_i, \phi)$$

- Differentiate with respect to β to get the score function $\mathbf{S}(\beta)$ and then set this equal to 0
- Use Method of Scoring to estimate $\hat{\beta}$

Estimation

Estimator can be written in the form:

$$\widehat{\beta}^{(\mathbf{t+1})} = (\mathbf{x^T W x})^{-1} \mathbf{x^T W z}$$

where $\mathbf{W}$ is diagonal with $w_i = (\frac{\partial \mu_i}{\partial \eta_i})^2 / \phi b''(\theta_i)$, $\eta_i = g(\mu_i)$, and

$$z_i = x_i \beta + (Y_i - \mu_i) \frac{\partial \eta_i}{\partial \mu_i}$$

- $\mathbf{W}$ and $\mathbf{z}$ change depending on $\hat{\beta}$ and vice versa
- Use iteratively weighted least squares (IWLS)
  1. Choose initial value $\hat{\beta}^{(0)}$
  2. Calculate $\mathbf{W}$ and $\mathbf{z}$
  3. Repeat until convergence

Likelihood-based inference

- Inference based on the limiting distribution for MLE

$$\hat{\beta} \sim N(\beta, I(\hat{\beta})^{-1})$$

  where

$$\mathbf{I}_n(\hat{\beta}) = (\mathbf{x}^\mathbf{T}\mathbf{W}\mathbf{x})$$

  Standard errors are the square roots of the inverse of the information matrix.
- Use this for the classic Wald Tests e.g. $\sqrt{W} = \frac{\hat{\beta} - \beta_0}{se(\hat{\beta})}$ follows z distribution.

Likelihood ratio test

Testing nested models, ω_1 and ω_2, $\omega_1 \in \omega_2$ and number of parameters $p_2 > p_1$

$$2[\log \ell(\widehat{\beta}_{\mathbf{1}}|\mathbf{y}) - \log \ell(\widehat{\beta}_{\mathbf{2}}|\mathbf{y})] \sim \chi_{p_1 - p_2}$$

- Comparing fit of two models
- Model with more predictors will almost always fit better, but is the difference significant?

Poisson regression

Review

- mean ?
- variance ?
- link: ?

What's a problem with just looking at counts?

Offsets

$$
\begin{aligned}
Y_i &\sim \text{Poisson}(\lambda_i) \\
\text{or } Y_i &\sim \text{Poisson}(\mu_i O_i) \\
\log \mu_i &= \mathbf{x_i}^T \beta
\end{aligned}
$$

Offset controls for exposure to risk/making inferences to some baseline. e.g.

- population size
- age
- time since exposed

Note! In R, you should include the log of your offset in the `glm` call

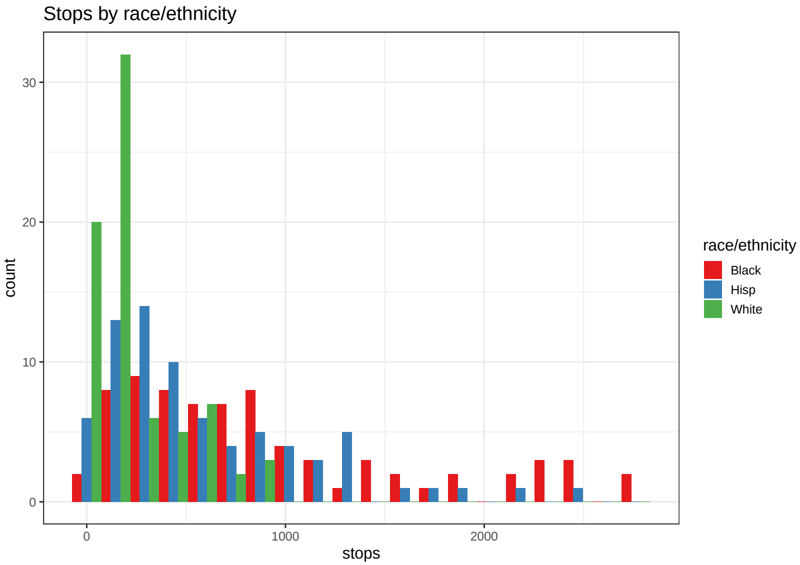
Example: Police stops

Police stop and frisks in NYC (Gelman Hill Chapter 6). Is there a difference in the number of stops by race/ethnicity?

The data look like:

precinct	stops	arrests	race_eth
1	202	980	Black
1	102	295	Hisp
1	81	381	White
2	132	753	Black
2	144	557	Hisp
2	71	431	White
3	752	2188	Black
3	441	627	Hisp
3	410	1238	White
4	385	471	Black

Distribution



GLM

Use arrests as exposure

```r
mod1 <- glm(stops~race_eth,family=poisson,offset=log(arrests),data=d)
summary(mod1)
```

```
##
## Call:
## glm(formula = stops ~ race_eth, family = poisson, data = d, offset = log(arrests))
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -47.327   -7.740   -0.182   10.241   39.140
##
## Coefficients:
##                Estimate Std. Error z value Pr(>|z|)
## (Intercept)   -0.588086   0.003784 -155.40   <2e-16 ***
## race_ethHisp   0.070208   0.006061   11.58   <2e-16 ***
## race_ethWhite -0.161581   0.008558  -18.88   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 46120  on 224  degrees of freedom
## Residual deviance: 45437  on 222  degrees of freedom
## AIC: 47150
##
## Number of Fisher Scoring iterations: 5
```

GLM

Add in factors for precinct

```
mod2 <- glm(stops~race_eth + factor(precinct), family=poisson,offset=log(arrests),data=d)
summary(mod2)[["coefficients"]][1:10,]
```

```
##                      Estimate  Std. Error    z value      Pr(>|z|)
## (Intercept)       -1.37886803 0.051019006 -27.026556 7.205634e-161
## race_ethHisp       0.01018798 0.006802045   1.497782  1.341899e-01
## race_ethWhite     -0.41900122 0.009434996 -44.409261  0.000000e+00
## factor(precinct)2 -0.14904964 0.074030344  -2.013359  4.407691e-02
## factor(precinct)3  0.55995498 0.056758425   9.865583  5.869222e-23
## factor(precinct)4  1.21063605 0.057548994  21.036615  3.032678e-98
## factor(precinct)5  0.28286532 0.056794015   4.980548  6.340447e-07
## factor(precinct)6  1.14420375 0.058047383  19.711547  1.716374e-86
## factor(precinct)7  0.21817307 0.064335032   3.391202  6.958688e-04
## factor(precinct)8 -0.39056473 0.056867814  -6.867940  6.513564e-12
```

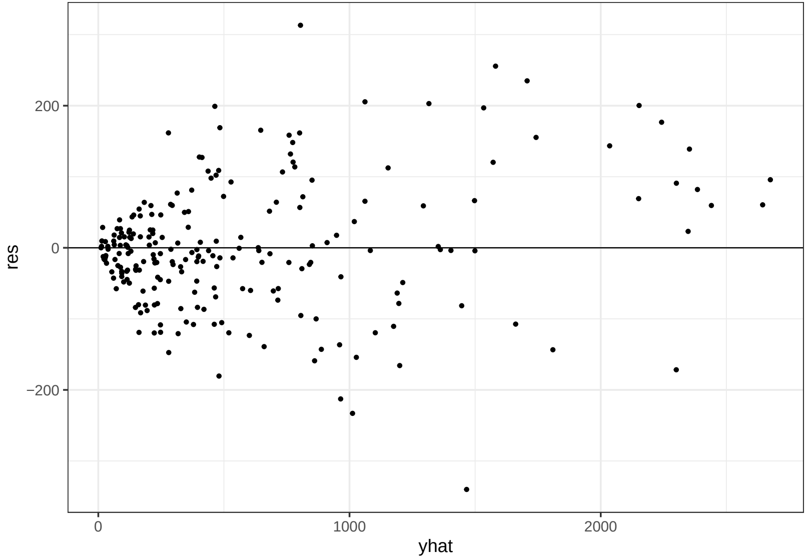
Coefficient interpretation

- e.g. after controlling for precinct, compared to blacks, whites have $1 - exp(-0.42) = 34\%$ less chance of being stopped.
- be wary of exposure variable: stops are compared to the number of arrests in the previous year
- so that the coefficient 'whites' will be less than 1 if the people in that group are stopped disproportionately less than their rates of arrest, as compared to blacks.

Is this a reasonable model?

Look at predicted values versus residuals $(y_i - \hat{y}_i)$. What do we expect?

Predicted values versus residuals



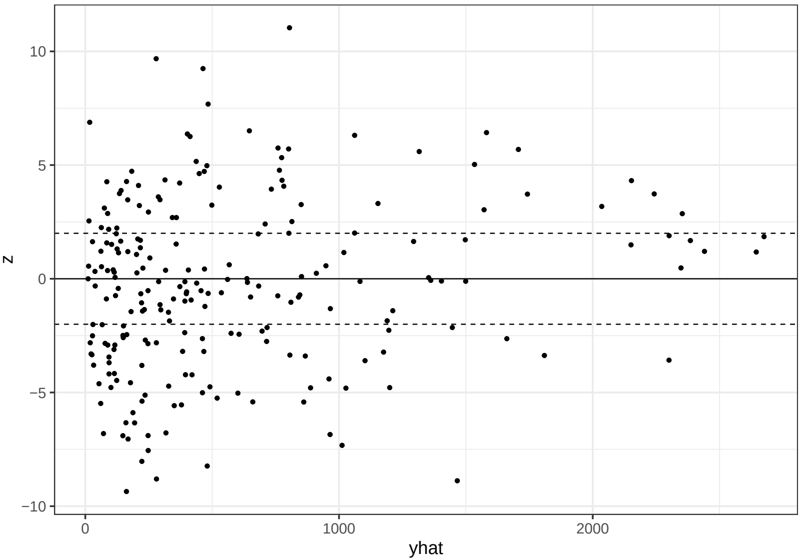
Is this a reasonable model?

Consider standardized residuals

$$z_i = \frac{y_i - \hat{y}_i}{sd(\hat{y}_i)}$$

If Poisson is a good model then these should have mean 0 and sd 1.

Predicted values versus standardized residuals



Overdispersion

- Extra variation in the data beyond what is allowed for in statistical model
- Poisson does not have independent variance parameter

Test for overdispersion: compare sum of squares of standardized residuals to χ^2_{n-k} distribution.

Estimated overdispersion factor is

$$\frac{1}{n-k}\sum_i z_i^2$$

Overdispersion

overdispersion factor is

```
sum(res_df$z^2)/(n-k)
```

```
## [1] 21.88505
```

Probability that we observe a factor at least as big as this is

```
1- pchisq(sum(res_df$z^2), n-k)
```

```
## [1] 0
```

But what's a problem here?

Fit overdispersed Poisson

- General form includes extra dispersion parameter θ
- Quasi-poisson: assume variance is proportion to the mean, rather than equal to the mean $Var[Y] = \mu\theta$

```
mod3 <- glm(stops~race_eth + factor(precinct), family=quasipoisson,offset=log(arrests),data=d)
summary(mod3)[["coefficients"]][1:10,]
```

```
##                       Estimate Std. Error    t value     Pr(>|t|)
## (Intercept)        -1.37886803 0.23867441 -5.7771925 4.326149e-08
## race_ethHisp        0.01018798 0.03182097  0.3201657 7.492943e-01
## race_ethWhite      -0.41900122 0.04413830 -9.4929170 5.489337e-17
## factor(precinct)2 -0.14904964 0.34632483 -0.4303753 6.675488e-01
## factor(precinct)3  0.55995498 0.26552425  2.1088656 3.664011e-02
## factor(precinct)4  1.21063605 0.26922265  4.4967837 1.384310e-05
## factor(precinct)5  0.28286532 0.26569075  1.0646412 2.887722e-01
## factor(precinct)6  1.14420375 0.27155419  4.2135374 4.352372e-05
## factor(precinct)7  0.21817307 0.30096874  0.7249028 4.696562e-01
## factor(precinct)8 -0.39056473 0.26603599 -1.4680898 1.442019e-01
```

Notice

```
summary(mod3)[["dispersion"]]
```

```
## [1] 21.88506
```

. . . and the SEs are inflated $\sim \sqrt{21.9}$.

Overdisperson

Downside to quasi-Poisson it's not true MLE so you don't get likelihood etc to compare models.

Alternative:

- Could also add a multiplicative random effect θ to represent unobserved heterogeneity.
- Conditional distribution is Poisson $E[Y|\theta] \sim Pois(\mu\theta)$
- Assuming θ is Gamma distributed leads to unconditional distribution being a Negative Binomial distribution
- Can choose parameters so $E(Y) = \mu$ and $Var(Y) = \mu(1 + \sigma^2\mu)$

Overdispersion

Fit Negative Binomial

```
library(MASS)
mod4 <- glm.nb(stops~race_eth + factor(precinct), data = d)
summary(mod3)[["coefficients"]][1:10,]
```

```
##                        Estimate Std. Error    t value     Pr(>|t|)
## (Intercept)         -1.37886803 0.23867441 -5.7771925 4.326149e-08
## race_ethHisp         0.01018798 0.03182097  0.3201657 7.492943e-01
## race_ethWhite       -0.41900122 0.04413830 -9.4929170 5.489337e-17
## factor(precinct)2   -0.14904964 0.34632483 -0.4303753 6.675488e-01
## factor(precinct)3    0.55995498 0.26552425  2.1088656 3.664011e-02
## factor(precinct)4    1.21063605 0.26922265  4.4967837 1.384310e-05
## factor(precinct)5    0.28286532 0.26569075  1.0646412 2.887722e-01
## factor(precinct)6    1.14420375 0.27155419  4.2135374 4.352372e-05
## factor(precinct)7    0.21817307 0.30096874  0.7249028 4.696562e-01
## factor(precinct)8   -0.39056473 0.26603599 -1.4680898 1.442019e-01
```

Binary data

Binary Responses

We have n random variables $Z_1, \ldots, Z_n$ that are binary

$$Z_i = \begin{cases} 1 \text{ if outcome is a success} \\ 0 \text{ if outcome is a failure} \end{cases}$$

with

$$Pr(Z_1 = 1) = \pi_i$$

so

$$Pr(Z_1 = 0) = 1 - \pi_i$$

Logistic regression

We are interested in describing the probability of success π_i with a linear model

$$g(\pi_i) = \mathbf{x}^{\mathsf{T}}\beta$$

The **canonical link** is the logistic function, so

$$\text{logit } \pi_i = \log \frac{\pi_i}{1-\pi_i} = \mathbf{x}^{\mathsf{T}}\beta$$

Latent variable formulation

$$
\begin{aligned}
y_i &= \begin{cases} 1 \text{ if } z_i > 0 \\ 0 \text{ if } z_i < 0 \end{cases} \\
z_i &= X_i\beta + \epsilon_i \\
\epsilon_i &\sim f(.)
\end{aligned}
$$

Latent variable formulation

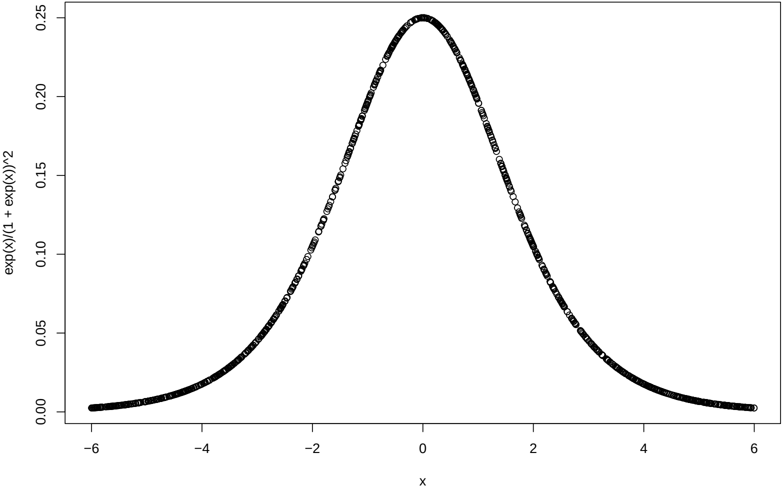
$$
\begin{aligned}
y_i &= \begin{cases} 1 \text{ if } z_i > 0 \\ 0 \text{ if } z_i < 0 \end{cases} \\
z_i &= X_i\beta + \epsilon_i \\
\epsilon_i &\sim f(.)
\end{aligned}
$$

For logistic regression, the errors ϵ have a *logistic* probability distribution

$$p(x) = \frac{e^x}{(1 + e^x)^2}$$

Latent variable formulation

The logistic pdf looks like



Latent variable formulation

Write $\eta_i = X_i\beta$.

Note that

$$\begin{aligned}\pi_i &= Pr(z_i > 0) \\ &= Pr(\epsilon_i > -\eta_i) \\ &= 1 - F(-\eta_i) \\ &= F(\eta_i)\end{aligned}$$

For the logistic, $F(\eta_i) = \frac{e^x}{(1+e^x)}$ so $\eta_i = F^{-1}(\pi_i) = \frac{\pi_i}{1-\pi_i}$ as before.

Latent variable formulation

What if we chose the distribution of the errors to be something else, for example, standard Normal?

$$\epsilon \sim N(0, 1)$$

This implies

$$\pi_i = \Phi(\eta_i)$$

or

$$\Phi^{-1}(\pi_i) = \mathbf{X_i}\beta$$

where Φ is the standard normal cdf. This form is called **probit**. What's the interpretation of the β's?

Example: Abortion outcomes in Uganda

- Data from 2018 PMA survey (via IPUMS)
- Outcome of interest: 'ever had abortion (yes/no)'
- Other variables: age, region, urban/rural, education, marital status

What the data look like:

resp	urban	region	marstat	educattgen	abortion
1	rural	north	divorced or separated	primary/middle school	0
2	rural	eastern	currently married	primary/middle school	1
3	urban	kampala	divorced or separated	secondary/post-primary	0
4	rural	western	never married	secondary/post-primary	0
5	rural	karamoja	currently living with partner	never attended	0
6	rural	central 1	currently living with partner	secondary/post-primary	1

Abortion outcomes

Consider the model

$$Y_i \sim \text{Bernoulli}(\pi_i)$$

$$\text{logit}\pi_i = \beta_0 + \beta_1 \text{age}_i + \beta_2 \text{marital}_i + \beta_3 \text{school}_i$$

where school_i is any level of schooling.

What could we use this model for? (i.e. what questions could we ask?)

Aside: what can this model for?

$$\text{logit}\pi_i = \beta_0 + \beta_1 \text{age}_i + \beta_2 \text{marital}_i + \beta_3 \text{school}_i$$

Estimation in R

```
mod <- glm(abortion ~ age + marstat + school, data = d, family = "binomial")
summary(mod)
```

```
##
## Call:
## glm(formula = abortion ~ age + marstat + school, family = "binomial",
##     data = d)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -0.7456  -0.4965  -0.4364  -0.3433   2.5789
##
## Coefficients:
##                              Estimate Std. Error z value Pr(>|z|)
## (Intercept)                  -3.50979    0.26492 -13.248  < 2e-16 ***
## age                           0.02562    0.00644   3.978 6.95e-05 ***
## marstatcurrently married     -0.06072    0.12987  -0.468  0.64012
## marstatdivorced or separated  0.47349    0.15115   3.133  0.00173 **
## marstatnever married         -0.49844    0.22409  -2.224  0.02613 *
## marstatwidow or widower      -0.30133    0.36989  -0.815  0.41528
## school                        0.84777    0.20125   4.212 2.53e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 2564.4  on 3931  degrees of freedom
## Residual deviance: 2501.4  on 3925  degrees of freedom
## AIC: 2515.4
##
## Number of Fisher Scoring iterations: 5
```

Interpretation

```r
coef(mod)
```

```
##                 (Intercept)                         age
##                 -3.50979534                  0.02561998
##     marstatcurrently married marstatdivorced or separated
##                 -0.06072052                  0.47348893
##        marstatnever married      marstatwidow or widower
##                 -0.49844113                 -0.30132839
##                      school
##                  0.84776732
```

```r
exp(coef(mod))
```

```
##                 (Intercept)                         age
##                  0.02990303                  1.02595099
##     marstatcurrently married marstatdivorced or separated
##                  0.94108622                  1.60558621
##        marstatnever married      marstatwidow or widower
##                  0.60747690                  0.73983478
##                      school
##                  2.33442900
```

Questions

- What is the probability of ever had an abortion for a women aged 25, currently living with partner, who has never attended school?

```
estimated_log_odds <- coef(mod)[1] + coef(mod)[2]*25
exp(estimated_log_odds)/(1+exp(estimated_log_odds))
```

```
## (Intercept)
##  0.05369242
```

Questions

- Assume we don't observe women in a particular region. Could we use this model to predict the likelihood of abortion in this region?
- What about if we added region as a covariate?
- Can we use this model to estimate the impact of education on abortion?

What are some potential issues with this analysis?

$$\text{logit}\pi_i = \beta_0 + \beta_1 \text{age}_i + \beta_2 \text{marital}_i + \beta_3 \text{school}_i$$

Model issues

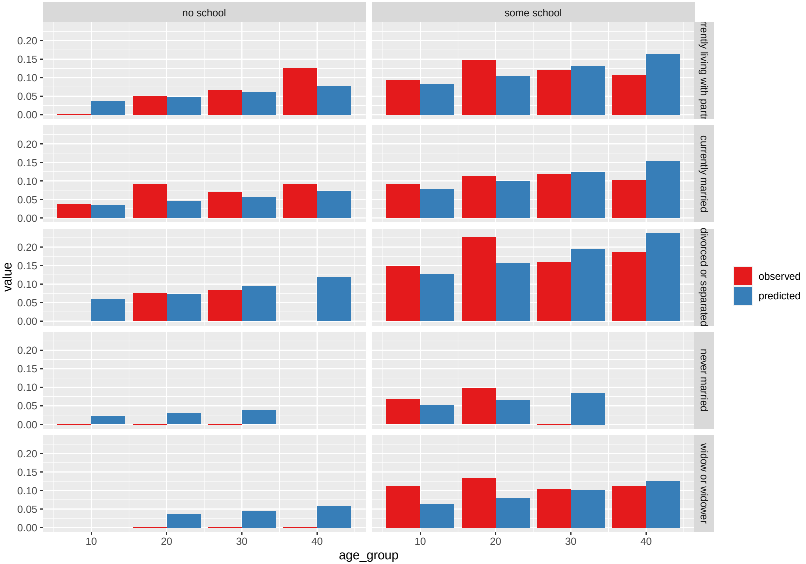
- Omitted variable bias
- Model mis-specification
- Model underfit or overfit
- Multicollinearity

Tools (for now)

- EDA!
- Likelihood ratio tests
- Wald tests
- Assessing predictions/residuals graphically (harder with binary variables)

A good way of assessing model fit

Look at predicted v actual proportions by groups



Issues with causal questions

Consider the situations where we are interested in the impact of education on abortion outcomes.

```r
summary(mod)
```

```
##
## Call:
## glm(formula = abortion ~ age + marstat + school, family = "binomial",
##     data = d)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -0.7456  -0.4965  -0.4364  -0.3433   2.5789
##
## Coefficients:
##                             Estimate Std. Error z value Pr(>|z|)
## (Intercept)                 -3.50979    0.26492 -13.248  < 2e-16 ***
## age                          0.02562    0.00644   3.978 6.95e-05 ***
## marstatcurrently married    -0.06072    0.12987  -0.468  0.64012
## marstatdivorced or separated  0.47349    0.15115   3.133  0.00173 **
## marstatnever married        -0.49844    0.22409  -2.224  0.02613 *
## marstatwidow or widower     -0.30133    0.36989  -0.815  0.41528
## school                       0.84777    0.20125   4.212 2.53e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 2564.4  on 3931  degrees of freedom
## Residual deviance: 2501.4  on 3925  degrees of freedom
## AIC: 2515.4
##
## Number of Fisher Scoring iterations: 5
```

Issues with causal questions

Consider the situations where we are interested in the impact of education on abortion outcomes. This is a causal question. What are some issues that may arise?

Issues with causal questions

- Confounders
  - urbanity
- Colliders (e.g. non-reponse bias)
  - Schooling and abortion both influence survey response
  - Conditioning on survey response creates a noncausal association between schooling and abortion

Data issues

- Non-representative samples
- Non-response (complete survey or specific questions)
- Measurement error
  - in this case, self-reports are a bad survey instrument

Lab

- Using data from Open Data Portal in Toronto
  - `opendatatoronto` package
- EDA
- Questions at end need to be handed in via GitHub