STA2201H Methods of Applied Statistics II

Monica Alexander

Week 3: Intro to Bayesian Inference

Announcements

- A1
- Office hours this week

Introduction to Bayesian Inference

Readings

- Gelman, Carlin, Stern, Dunson, Ventari and Rubin (2013). Bayesian Data Analysis (Third Edition) Chapman and Hall/CRC
  - Aki's slides on BDA are useful: https://github.com/avehtari/BDA_course_Aalto
- Gelman and Hill (2006). Data Analysis Using Regression and Multilevel/Hierarchical Models. Cambridge University
- Hoff (2010). A first course in Bayesian statistical methods
- Fienberg (2006) "When Did Bayesian Inference Become 'Bayesian'?" Bayesian Analysis, 1(1). 1-40.
- If interested in something a bit more philosophical:
  - Stark (2015). Constraints versus priors. SIAM/ASA Journal on Uncertainty Quantification, 3(1), 586-598.
  - Nau (2001). De Finetti was right: probability does not exist. https://people.duke.edu/~rnau/definettiwasright.pdf

Back to linear regression

We model the relationship between the (potentially transformed) data and covariates as a linear regression model

$$g(y_i) = {x_i}^T \beta + \epsilon_i$$

- Previously, you have probably written down the likelihood and found the MLE estimate(s) for β. Look something like

  $$\hat{\beta} = (X^T W X)^{-1} X^T W z$$

  where $z = f(y, X, \beta, g)$ and for usual linear regression, the weights W are the identity and $z = y$.
- Once we have $\hat{\beta}$s, can assume asymptotic normality and do some inference

What are we doing here

This type of classical inference (= **frequentist** inference) has an underlying probabilistic framework:

- The data y are random
- The estimator $\hat{\beta}$ is a function of the data
- We can then make probability statements about how often the true value is within some interval around the estimator.
- So we are always making probabilistic statements about the true value of β and how uncertain we are as a function of the data

Let's ask a different question

Which values of β are consistent with the data we have observed?

"*Given* the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of it happening in a single trial lies somewhere between any two degrees of probability that can be named."

Bayesian versus frequentist

Frequentist

- Parameter(s) θ is a fixed but unknown quantity
- Probability: to describe the relative frequency of an outcome in an infinitely repeatable but unpredictable experiment
- Uncertainties typically involve expectations with respect to the distribution of the data, holding the parameter fixed

Bayesian

- Parameter(s) θ is a random variable
- Probability statements reflect a state of knowledge
- Uncertainties typically involve expectations with respect to the distribution of the parameter, holding the data fixed

Laplace

Laplace and inverse probability

- Pierre-Simon Laplace was particularly interested in data and inferential probability
- Both Bayes and Laplace were aware of what's now called **Bayes rule**:

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto p(y|\theta)p(\theta)$$

- For a long time, inferential probability known as 'inverse probability'

Laplace was the original Bayesian demographer

- Studying the sex ratio at birth in Paris (1781)
- Over 1745-1770, observed 251,527 boys and 241,945 girls
- Denote $X =$ the probability that a given birth was male
- Laplace was interested in estimating $P(X \leq 0.5)$
- Using inverse probability he computed this was very very small (around zero)

> As it is exceedingly small, we can assert, with the same certainty as any other moral truth, that the difference observed in Paris between births of boys and those of girls is due to a greater likelihood for births of boys (Laplace 1781).

Bayesian inference

Bayesian inference

The process of learning via Bayes rule.

Example: breast cancer screening (using mammograms) in Germany. Imagine we know

- The probability an asymptomatic woman has breast cancer is 0.8%.
- If she has breast cancer, the probability is 90% that she has a positive mammogram
- If she does not have breast cancer, the probability is 7% that she still has a positive mammogram.

Suppose a woman has a positive mammogram: What is the probability she actually has breast cancer?

Breast cancer

Use Bayes rule for events:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Let

- C be the cancer outcome (=1 if cancer, 0 otherwise)
- M be the mammogram outcome (=1 if mammogram is positive, 0 otherwise)

"Suppose a woman has a positive mammogram: What is the probability she actually has breast cancer?"

A somewhat famous example because physicians had no idea what the answer should be.

We want to know $P(C = 1|M = 1)$.

Breast cancer

We want to know $P(C = 1|M = 1)$.

- $P(C = 1) = 0.008$.
- $P(M = 1|C = 1) = 0.9$.
- $P(M = 1|C = 0) = 0.07$.
- so $P(M = 1) =?$

Use Bayes rule, get $P(C = 1|M = 1) = 9.4\%$.

What did we do? Updated **prior** probability $P(C = 1)$ based on observing **data** (mammograms) to get the **posterior** probability $P(C = 1|M = 1)$.

Bayesian inference about parameters

Happiness example

Hoff (Chapter 3):

- Each female aged 65+ in 1998 General Social Survey was asked about being happy.
- Data: Out of $n = 129$ women, $y = 118$ women (91%) reported being happy.
- What is $\theta =$ the proportion of 65+ women who are happy?
- Goal: inference about $\theta =$ happiness parameter.

Happiness example

What's our usual approach? (frequentist)

1. Relate data to parameter of interest through a likelihood function, e.g. assume $Y|\theta \sim Bin(n, \theta)$ where y is the number of women who report to be happy out of the sample of n women.
2. Maximum likelihood estimate: Find a point estimate θ that maximizes the likelihood function ($\hat{\theta} = 0.91$)
3. Construct a confidence interval for θ (CI: [0.87, 0.96])
4. Interpretation of frequentist CI: If repeated samples were taken and the 95% confidence interval was computed for each sample, 95% of the intervals would contain the population mean.

Happiness example

The Bayesian approach:

- Also assume a likelihood, as before $Y|\theta \sim Bin(n, \theta)$

But now we proceed differently. In Bayesian inference, unknown parameters (like θ) are considered **random variables**. This means information/knowledge about these random variables can be summarized using probability distributions.

- Have existing knowledge/info about θ, summarized by the prior probability distribution
- Observe some data that gives more info about θ
- Update our previous knowledge to obtain the posterior distribution using Bayes' rule

Happiness example

The Bayesian approach:

1. Also assume a likelihood $p(y|\theta)$, as before $Y|\theta \sim Bin(n, \theta)$
2. Set a prior distribution for θ, $p(\theta)$
3. Use Bayes rule to update the prior into the posterior distribution
$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$
4. Use the posterior to provide summaries of interest, e.g. point estimates and uncertainty intervals, called credible intervals.

Happiness example

1. Likelihood is $Y|\theta \sim Bin(n, \theta)$ so

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

2. Now we need to pick a prior $p(\theta)$

- Suppose any outcome between 0 and 1 for θ is equally likely, what prior can be used to describe these beliefs?
- $\theta \sim U(0, 1)$ so $p(\theta) = 1$

3. Now we calculate the posterior distribution

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta')d\theta'}$$

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta')d\theta'}$$

In the happiness case,

$$p(\theta|y) = \frac{\binom{n}{y}\theta^y(1-\theta)^{n-y}}{\int_0^1 \binom{n}{y}\theta'^y(1-\theta')^{n-y}d\theta'} = \frac{1}{Z}\theta^y(1-\theta)^{n-y}$$

where

$$Z = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

So posterior is

$$\theta|y \sim \mathrm{Beta}(y+1, n-y+1)$$

Up to a constant

To recognize the posterior as a Beta distribution, it would have been sufficient to consider only the terms that include θ

$$p(\theta|y) \propto p(y|\theta)p(\theta)$$

i.e.

$$p(\theta|y) \propto \theta^y(1-\theta)^{n-y}$$

because $p(\theta|y)$ is a pdf so must integrate to one. So the marginal distribution $p(y)$ is just a scaling factor.

Inference about θ based on posterior distribution

Bayesian point estimates are often given by:

- The posterior mean $E(\theta|y)$
- The posterior median θ^* $P(\theta < \theta^*|y) = 0.5$.

Uncertainty is quantified with credible intervals (CIs), e.g. for 95% CIs:

- An interval is called a 95% Bayesian CI if the posterior probability that θ is contained in the interval is 0.95.
- More formally a $1 - \alpha$ credible interval for θ is an interval C_n satisfying $P(\theta \in C_n|Y_1, \ldots, Y_n) = 1 - \alpha$.
  - a probability statement about θ, not C_n.

Bayesian credible intervals

An interval is called a 95% Bayesian CI if the posterior probability that θ is contained in the interval is 0.95.

- More formally a $1 - \alpha$ credible interval for θ is an interval C_n satisfying $P(\theta \in C_n | Y_1, \ldots, Y_n) = 1 - \alpha$.
- a probability statement about θ (given the data), not C_n.
- "the probability that θ is in C_n given the data is 95%"

This interpretation differs from a frequentist CI:

- c.f. confidence interval: a $1 - \alpha$ confidence interval for θ is an interval C_n satisfying $P(\theta \ni C_n) \geq 1 - \alpha$
- a probability statement about C_n, not θ
- "if I repeat the experiment over and over, the interval will contain the parameter 95% of the time."

Bayesian credible intervals

C_n is not uniquely defined. Interval options:

- Quantile-based Bayesian $100(1-\alpha)\%$ CIs are used, which are given by posterior quantiles $(\theta_{\alpha/2}, \theta_{1-\alpha/2})$, with $P(\theta < \theta_{\alpha/2}|y) = P(\theta > \theta_{1-\alpha/2}|y) = \alpha/2$. (focus here)
- Highest posterior density (HPD) intervals (see here for more details).

In this class, we will essentially be only ever calculating quantile-based CIs.

Happinesss findings

Bayesian estimates: Mean

```
## [1] 0.91
```

95% Credible interval:

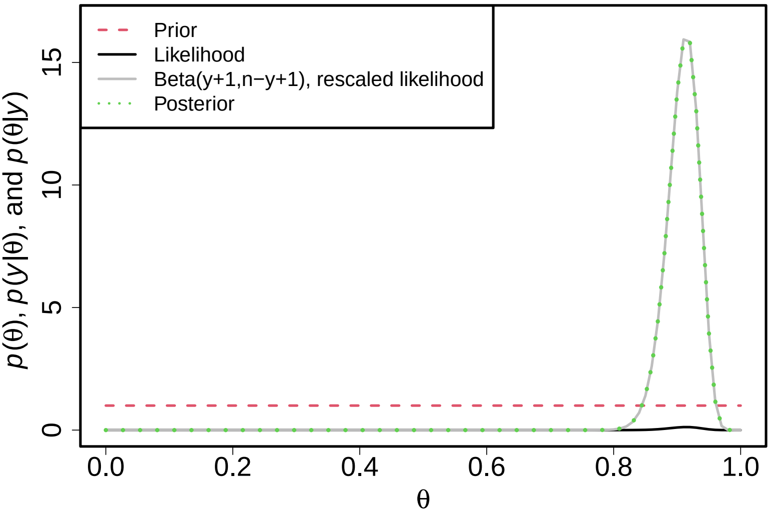
```
## [1] 0.85 0.95
```

Frequentist estimates (mean, 95% CI)

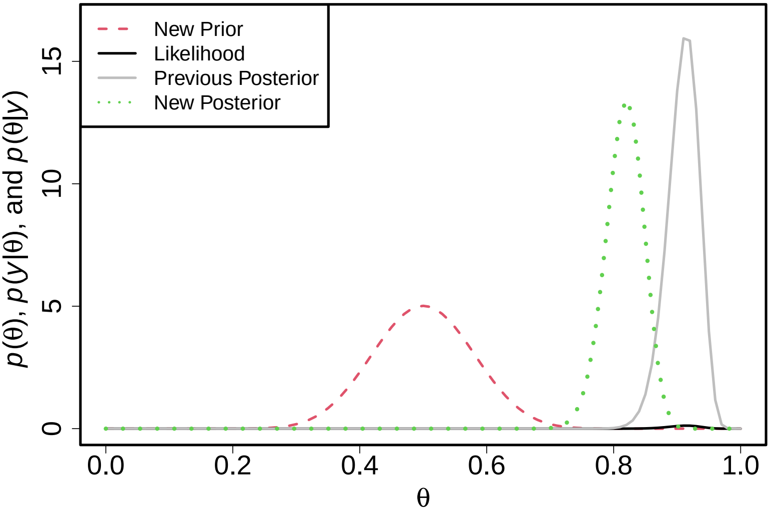
```
## [1] 0.91
```

```
## [1] 0.87
```

```
## [1] 0.96
```


Prior
Likelihood
Beta(y+1,n−y+1), rescaled likelihood
Posterior
p(θ), p(y|θ), and p(θ|y)
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0
θ

What if instead I used Beta$(20, 20)$ as the prior?



Conjugate priors

Note that $\theta \sim U(0, 1)$ is the same as $\theta \sim Beta(1, 1)$.

For the binomial likelihood, a Beta prior results in a Beta posterior distribution: we say that the beta prior is **conjugate** for the binomial likelihood.

More generally, for a certain likelihood, a prior distribution which results in a posterior distribution of the same form is called a **conjugate** prior distribution.

Priors

Probability does not exist

> *Probabilistic reasoning –always to be understood as subjective– merely stems from our being uncertain about something. It makes no difference whether the uncertainty relates to an unforeseeable future, or to an unnoticed past, or to a past doubtfully reported or forgotten... The only relevant thing is uncertainty - the extent of our own knowledge and ignorance.*

de Finetti, 1974

Different types of priors

BDA Chapter 2

- Conjugate prior
- Noninformative prior
- Proper and improper prior
- Weakly informative prior
- Informative prior

Conjugate priors

- Prior and posterior have the same form
- only for exponential family distributions (plus for some irregular cases)
- Used to be important for computational reasons

e.g beta for binomial. What's the interpretation of a Beta(1,1) prior?

Noninformative prior, proper and improper prior

- Vague, flat, diffuse of noninformative
- "let the data speak for themselves"

But flat is not non-informative!

Proper prior: $\int p(\theta) = 1$

Improper prior: doesn't have finite integral (but the posterior can still sometimes be proper)

- e.g. The uniform distribution on an infinite interval (i.e., a half-line or the entire real line).

Weakly informative priors

- Quite often there's at least some knowledge about the scale
- The idea is that the prior rules out unreasonable parameter values but is not so strong as to rule out values that might make sense
- Weakly informative priors produce computationally better behaving posteriors
- Generic weakly informative prior: $N(0, 1)$
- Good example in the Gabry et al paper on air pollution
- More on this in a couple of lectures

Informative priors

Prior distributions giving numerical information that is crucial to estimation of the model. Information might come from a literature review or explicitly from an earlier data analysis.

- Example from Gelman (linked): Mass of liver as a fraction of lean body mass is known to vary very little.
- E.g. Gompertz models for mortality: can only have a restricted range on α and β that lead to plausible values of life expectancy

Bias-variance tradeoff

- Effect of incorrect priors: Introduce bias, but often still produce smaller estimation error because the variance is reduced
- Misleading certainty in results?

How to choose?

Some good practical advice: https://github.com/stan-dev/stan/wiki/Prior-Choice-Recommendations

- if you do have prior info, include it!
- make sure it has appropriate range (e.g. prior on variance needs to have positive support)
- prior predictive checks with simulated data
- check sensitivity of model findings to model choice

More on this later.

More than one parameter

More than one parameter

What if the data model (likelihood function) includes more than 1 unknown parameter, e.g. do inference for μ if

$$y_i \sim N(\mu, \sigma^2)$$

What do we want? $p(\mu, \sigma|\mathbf{y})$. If only mean is of interest, then want $p(\mu|\mathbf{y})$.

Bayes rule for multiple continuous RVs

Let RVs $\theta = (\theta_1, \theta_2, \dots \theta_p)$ and $\mathbf{Y} = (Y_1, Y_2, \dots Y_n)$

Bayes rule:

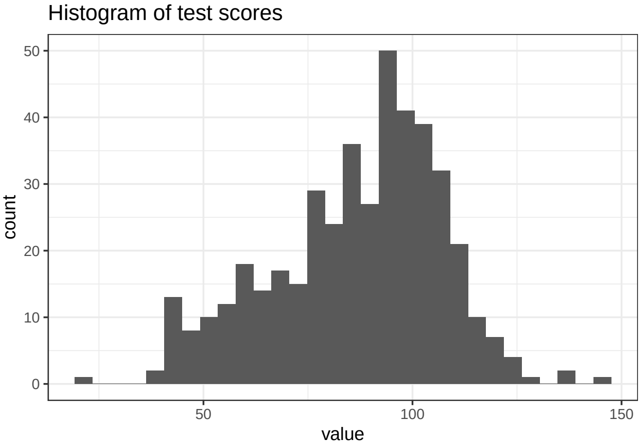
$$p(\theta|\mathbf{y}) = \frac{p(\mathbf{y}|\theta)p(\theta)}{p(\mathbf{y})}$$

- $p(\mathbf{y}) = \int_{\theta'} p(\mathbf{y}|\theta')p(\theta')d\theta'$
- The marginal posterior for just one parameter is given by $p(\theta_1|\mathbf{y}) = \int_{\theta'_2} \cdots \int_{\theta'_p} p(\theta_1, \theta'_2, \dots \theta'_p|\mathbf{y})d\theta'_2 \dots d\theta'_p$.

Example: kid's test scores

Gelman-Hill Chapter 3 Outcome of interest: cognitive tests scores for 3-4 year old kids. Denote the unknown mean test score by μ, and observed test score by y_i for kid i, with $i = 1, \ldots, n$.

Goal: estimate μ.



Example: kid's test scores

Let's assume Normal likelihood

$$y_i \sim N(\mu, \sigma^2)$$

- If we put a joint prior $p(\mu, \sigma)$ on the parameters, Bayes' rule tells us how to get the joint posterior distribution:

$$p(\mu, \sigma | \mathbf{y}) = \frac{p(\mathbf{y}|\mu, \sigma)p(\mu, \sigma)}{p(\mathbf{y})}$$

- And if inference about μ is our goal, we can get the marginal posterior distribution

$$p(\mu | \mathbf{y}) = \int_\sigma p(\mu, \sigma | \mathbf{y}) d\sigma$$

Example: kid's test scores

- What priors to set for μ and σ^2? Let's assume that μ and σ^2 are independent a priori, $p(\mu, \sigma) = p(\mu)p(\sigma)$, and use

$$\mu \sim N(\mu_0, \sigma^2_{\mu_0})$$

and

$$1/\sigma^2 \sim Gamma(\nu_0/2, \nu_0/2 \cdot \sigma_0^2)$$

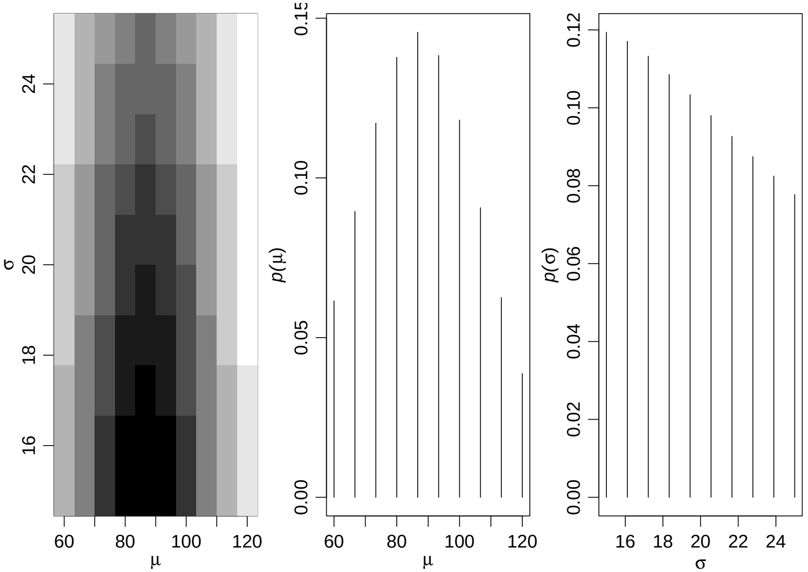
For illustrative purposes, will set **hyperparameters** to be $\mu_0 = 86.8$, $\sigma_{\mu_0} = \sigma_0 = 20.4$ and $\nu_0 = 1$.

Let's start with a discrete approximation

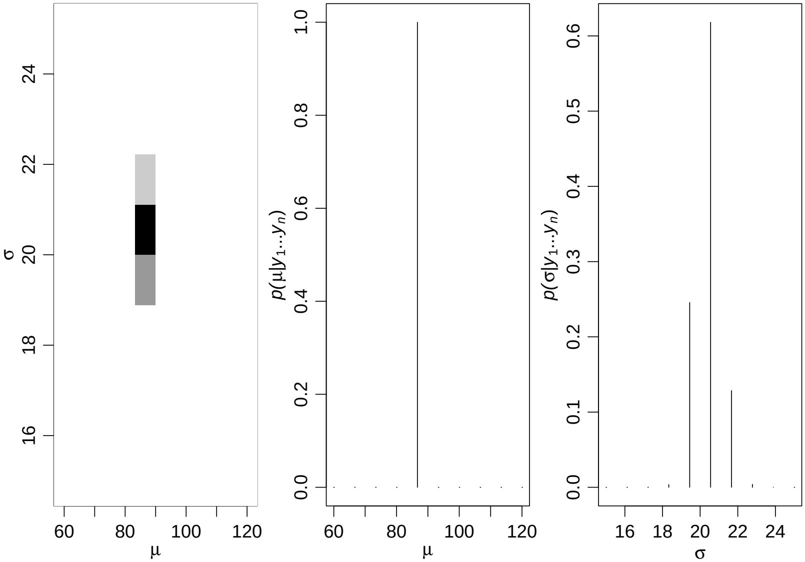
For illustrative purposes, start with a discrete approximation to these priors.

- E.g. use discrete grid of values for μ, and set $p(\mu) = f(\mu)/\sum f(\mu)$ where $f(.)$ is given by the Normal pdf for μ.
- Can use these to calculate discrete likelihood and thus a discrete approximation to the posterior

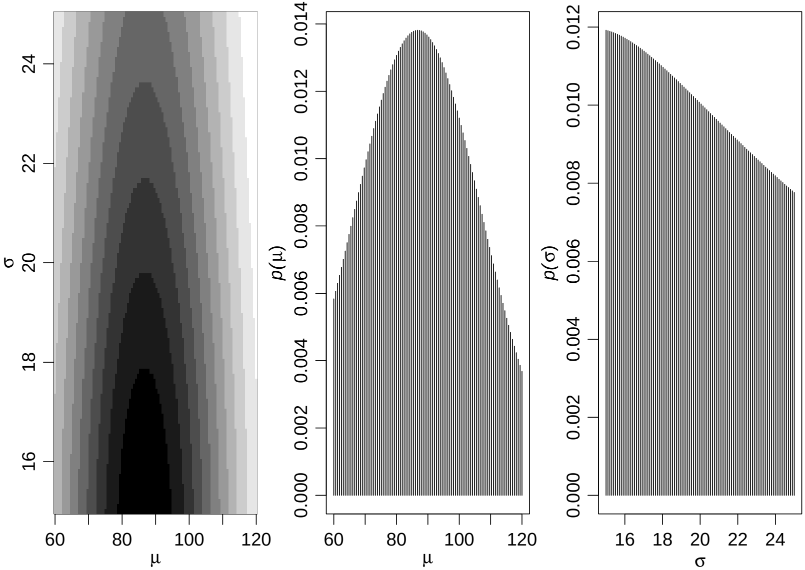
Joint and marginal prior distributions



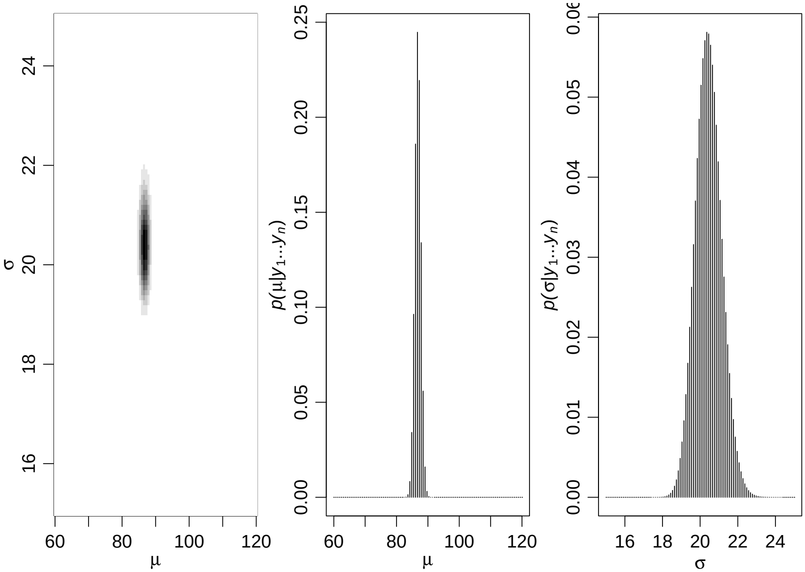
Joint and marginal posterior distributions



Finer grid: priors



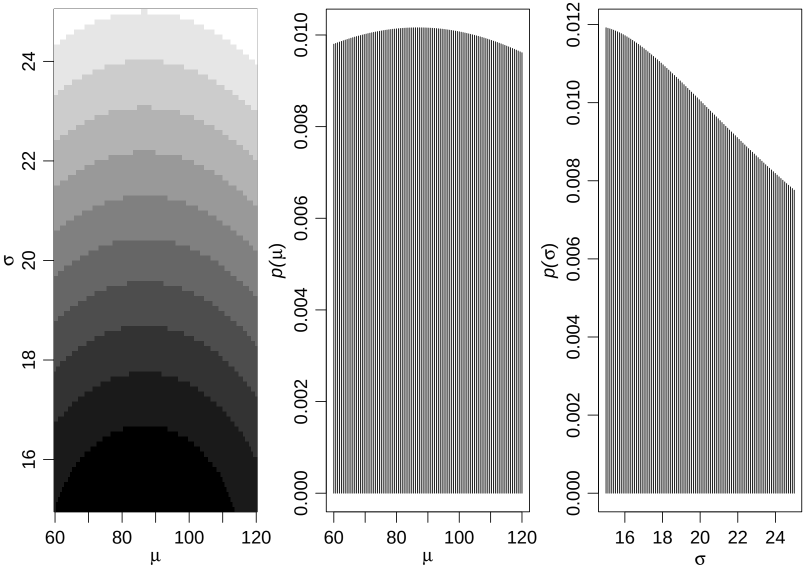
Finer grid: posteriors



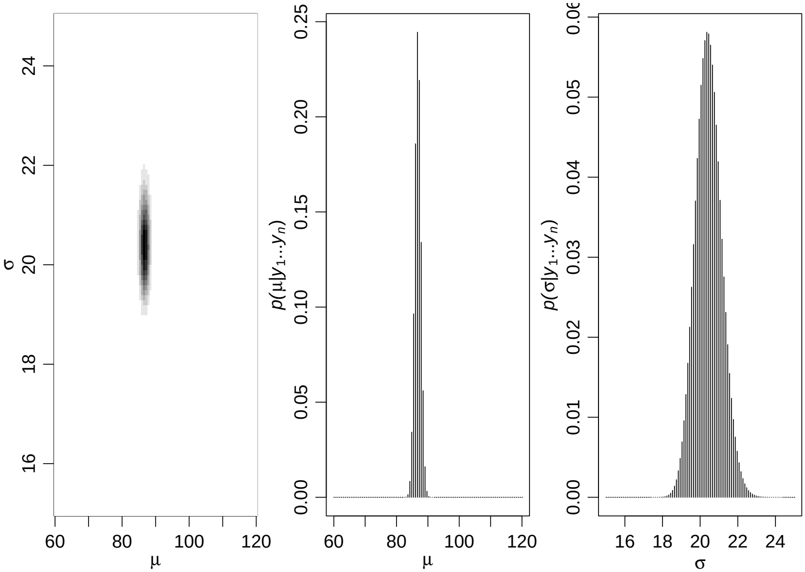
How did I get those previous graphs?

- Defined a grid of μ and σ values
  - e.g. first example for μ was `seq(60,120,length=10)`
- Calculated density at each grid point
  - e.g. using `dnorm`
- Standardized e.g. $p(\mu) = f(\mu)/\sum f(\mu)$
- Calculated prior grid $p(\mu) \cdot p(\sigma)$
- Calculated posterior grid $p(\mu, \sigma|y) = \frac{p(y|\mu,\sigma) \cdot p(\mu) \cdot p(\sigma)}{p(y)}$
- Calculated marginals of posterior grid by summing over relevant parameter e.g. $p(\mu|y) = \sum_\sigma p(\mu, \sigma|y)$

What if we change the prior variance on the mean, $\sigma^2_{\mu_0}$ to be larger?



Posteriors don't change noticeably, why?



Now with continuous priors

$$p(\mu, \sigma|\mathbf{y}) = \frac{p(\mathbf{y}|\mu, \sigma)p(\mu, \sigma)}{p(\mathbf{y})} = \frac{p(\mathbf{y}|\mu, \sigma)p(\mu, \sigma)}{\int_{\mu}\int_{\sigma} p(\mathbf{y}|\mu, \sigma)p(\mu, \sigma)d\sigma d\mu}$$

The bad news:

- Common choices of priors (e.g. what we have chosen) do not result in a closed-form expression for these posterior distributions

The good news:

- Not a problem if we can obtain a sample from the posterior distribution, which is very common in Bayesian inference.

Simulation based inference and Monte Carlo

The general idea in simulation-based inference: we can make inference about a random variable μ, using a sample $\mu^{(1)}, \ldots, \mu^{(S)}$ from its probability distribution. This is called a **Monte Carlo (MC)** approximation.

```
my_sample <- rnorm(5000, mean = 0, sd = 1)
mean(my_sample)
```

```
## [1] -0.01346811
```

```
sd(my_sample)
```

```
## [1] 0.9839806
```

Monte Carlo

- Why can we use a sample mean as an approximation to the mean of a random variable?
- Just about any aspect of the distribution of μ can be approximated arbitrarily exactly with a large enough Monte Carlo sample, e.g.
  - the α-percentile of $\mu^{(1)}, \ldots, \mu^{(S)} \rightarrow$ the α-percentile of the distribution, e.g. the median
  - We can approximate $Pr(\mu \geq x)$ for any constant x by the proportion of samples for which $\mu \geq x$, because

$$1/S \sum_{s=1}^{S} I(\mu^{(s)} \geq x) \rightarrow Pr(\mu \geq x)$$

Monte Carlo

- With a simulation, it also becomes very easy to analyze the distributions of any function of 1 or more random variables, e.g.
  - use $1/\mu^{(s)}$ to study $1/\mu$
- Samples from marginal distributions may be obtained from samples from joint distributions, e.g.
  - the distribution of μ_1, where $(\mu_1, \mu_2) \sim N_2(\mathbf{0}, \Sigma)$ can be studied using samples of $\mu_1^{(s)}$ where $(\mu_1^{(s)}, \mu_2^{(s)}) \sim N_2(\mathbf{0}, \Sigma)$

Back to example

- Problem: For common choices of the priors on μ and σ, there is no closed-form expression for $p(\mu|y)$.
- Solution: let's obtain a posterior sample $\mu^{(1)}, \ldots, \mu^{(S)}$
- How to do this? Next week gives an overview.

Summary

Bayes rule for more than one parameter

$$p(\theta|\mathbf{y}) = \frac{p(\mathbf{y}|\theta)p(\theta)}{p(\mathbf{y})}$$

- $p(\mathbf{y}) = \int_{\theta'} p(\mathbf{y}|\theta')p(\theta')d\theta'$
- The marginal posterior for just one parameter is given by $p(\theta_1|\mathbf{y}) = \int_{\theta'_2} \cdots \int_{\theta'_p} p(\theta_1, \theta'_2, \ldots \theta'_p|\mathbf{y})d\theta'_2 \ldots d\theta'_p$.
- Problem: often don't have a closed form solution for posterior

Summary

- Solution: We can make inference about any random variable θ, using a sample from its probability distribution. This is called a Monte Carlo (MC) approximation.
- If we are able to obtain a sample from posterior $p(\theta|\mathbf{y})$ then
  - for each parameter we have a sample of its marginal e.g. $\theta_1^{(1)}, \ldots, \theta_1^{(S)} \sim p(\theta_1|y)$.
  - we can report any summary we'd like, e.g. posterior mean (sample mean), posterior median or other percentiles (sample percentiles).