Solutions of the Dirac equation

Matthew Kirk

11th August 2025

The Dirac equation is

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i \sum_{j=1}^{3} \left(\gamma^j \frac{\partial}{\partial x_j}\right) - m\mathbb{1}_{4x4}\right)\psi = 0_{4x1}\,, \tag{0.1}$$

where $\partial/\partial x_1 = \partial/\partial x, \partial/\partial x_2 = \partial/\partial y, \partial/\partial x_3 = \partial/\partial z$, $\mathbb{1}_{4x4}$ is a 4x4 matrix that is all zeroes except for 1's down the diagonal, and 0_{4x1} is a column of 4 zeroes.

In the Dirac representation, the Gamma matrices have the following form:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{0.2}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{0.3}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{0.4}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{0.5}$$

Exercise: Show that

$$\psi_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} \times e^{-i(Et-pz)} \tag{0.6}$$

is a solution of the Dirac equation, as long as $E^2 = p^2 + m^2$.

In fact, there are four independent solutions of the Dirac equation. In addition to the one above, there are also the following solutions:

$$\psi_2 = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E+m} \end{pmatrix} \times e^{-i(Et-pz)} \tag{0.7}$$

$$\psi_3 = \sqrt{E+m} \begin{pmatrix} 0 \\ \frac{-p}{E+m} \\ 0 \\ 1 \end{pmatrix} \times e^{i(Et-pz)} \tag{0.8}$$

$$\psi_4 = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \times e^{i(Et-pz)} \tag{0.9}$$

Bonus exercise: Show that these other three are also solutions of the Dirac equation.