Calculating polarized scattering

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In the Dirac representation, the Gamma matrices have the following form:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{0.1}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{0.2}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{0.3}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{0.4}$$

and the Dirac spinors look like:

$$u_+(E, \overrightarrow{p}) = \sqrt{E+m} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \\ \frac{|\overrightarrow{p}|}{E+m}\cos(\theta/2) \\ \frac{|\overrightarrow{p}|}{E+m}\sin(\theta/2) \end{pmatrix} \tag{0.5}$$

$$u_-(E, \overrightarrow{p}) = \sqrt{E+m} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \\ \frac{|\overrightarrow{p}|}{E+m}\sin(\theta/2) \\ -\frac{|\overrightarrow{p}|}{E+m}\cos(\theta/2) \end{pmatrix} \tag{0.6}$$

$$v_+(E, \overrightarrow{p}) = \sqrt{E+m} \begin{pmatrix} \frac{|\overrightarrow{p}|}{E+m}\sin(\theta/2) \\ -\frac{|\overrightarrow{p}|}{E+m}\cos(\theta/2) \\ -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \tag{0.7}$$

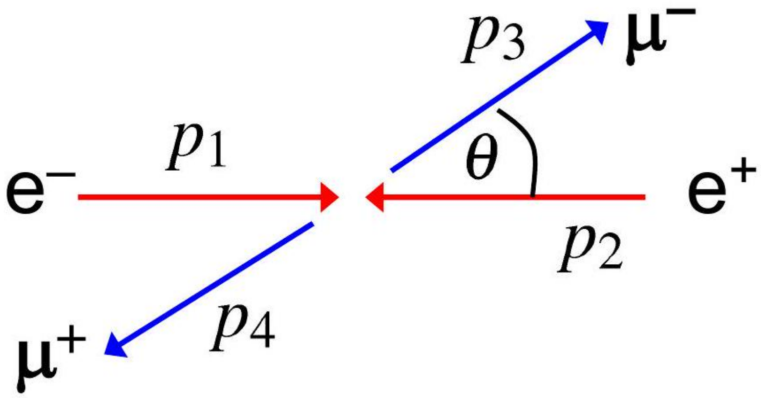

Figure 1: A much simpler version of the diagram you need for Exercise 1 – talk to us for all the details

$$v_-(E, \overrightarrow{p}) = \sqrt{E+m}\begin{pmatrix} \frac{|\overrightarrow{p}|}{E+m}\cos(\theta/2) \\ \frac{|\overrightarrow{p}|}{E+m}\sin(\theta/2) \\ \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \tag{0.8}$$

where $E^2 = |\overrightarrow{p}|^2 + m^2$, and $\overrightarrow{p} = (p_x, p_y, p_z) = (|\overrightarrow{p}|\sin\theta, 0, |\overrightarrow{p}|\cos\theta)$

Bonus exercise 1: Show that $(u_+u_+^\dagger + u_-u_-^\dagger)\gamma^0 = \left(E\gamma^0 - \sum_{i=1}^3 p_i\gamma^i + m\mathbb{1}_{4x4}\right)$

Bonus exercise 2: Show that $(v_+v_+^\dagger + v_-v_-^\dagger)\gamma^0 = \left(E\gamma^0 - \sum_{i=1}^3 p_i\gamma^i - m\mathbb{1}_{4x4}\right)$

Exercise 1: Write down the amplitude $\mathcal{M}$ for $e^-e^+ \to \mu^-\mu^+$, assuming the initial electron and final muon both have "+" spin, and the initial positron and final state muon have "−" spin, just in terms of spinors and the gamma matrices. For this you can assume that both the electron and the muon have zero mass, so that $E = |\overrightarrow{p}|$.

Exercise 2: Using the representation of the spinors and gamma matrices given above, work out an explicit result for the amplitude $\mathcal{M}$.

Exercise 3: Using your previous answer, work out the differential cross-section

$$\frac{d\sigma}{d(\cos\theta)} = \frac{1}{32\pi s}|\mathcal{M}|^2 \tag{0.9}$$

where $s = 4E^2$.

Bonus exercise 3: Do exercises 1–3 again, but now for $e^-e^+ \to e^-e^+$. Similarly assume the initial and final electron both have "+" spin, and the initial and final positron have "−" spin. There is a second diagram now!