

Solutions of the Dirac equation

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The Dirac equation is

$$\left(i\gamma^0 \frac{\partial}{\partial t} - \sum_{i=1}^3 \gamma^i \frac{\partial}{\partial x_i} - m\mathbb{1}_{4 \times 4} \right) \psi = 0, \quad (0.1)$$

where $\partial/\partial x_1 = \partial/\partial x$, $\partial/\partial x_2 = \partial/\partial y$, $\partial/\partial x_3 = \partial/\partial z$, and $\mathbb{1}_{4 \times 4}$ is a 4x4 matrix that is all zeroes except for 1's down the diagonal.

In the Dirac representation, the Gamma matrices have the following form:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (0.2)$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (0.3)$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad (0.4)$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (0.5)$$

Exercise: Show that

$$\psi_1 = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} \times e^{-i(Et-pz)} \quad (0.6)$$

is a solution of the Dirac equation, as long as $E^2 = p^2 + m^2$.

In fact, there are four independent solutions of the Dirac equation. In addition to the one above, there are also the following solutions:

$$\psi_2 = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E+m} \end{pmatrix} \times e^{-i(Et-pz)} \quad (0.7)$$

$$\psi_3 = \sqrt{E+m} \begin{pmatrix} 0 \\ \frac{-p}{E+m} \\ 0 \\ 1 \end{pmatrix} \times e^{i(Et-pz)} \quad (0.8)$$

$$\psi_4 = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \times e^{i(Et-pz)} \quad (0.9)$$

Bonus exercise: Show that these other three are also solutions of the Dirac equation.