

Problem 2

Consider the following matrix:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

- (a) Find the eigenvalues and corresponding eigenvectors of matrix A

$$|A - \lambda I| = \left| \begin{bmatrix} 5 - \lambda & 4 \\ 4 & 5 - \lambda \end{bmatrix} \right| = (5 - \lambda)^2 - 4^2 = 25 - 10\lambda + \lambda^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 1)(\lambda - 9) = 0$$

$$\Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = 9$$

Let the eigenvector of λ_i be v_i .

First, we will find v_1 :

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} v_1 = \vec{0}$$

Via Gaussian elimination,

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow v_1 = \begin{bmatrix} a \\ -a \end{bmatrix} \quad \text{where } a \in \mathbb{R}$$

Then, we will find v_2 :

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} v_2 = \vec{0}$$

Via Gaussian elimination,

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow v_2 = \begin{bmatrix} b \\ b \end{bmatrix} \quad \text{where } b \in \mathbb{R}$$

Thus,

$$\boxed{\lambda_1 = 1, \quad \lambda_2 = 9, \quad v_1 = \begin{bmatrix} a \\ -a \end{bmatrix}, \quad v_2 = \begin{bmatrix} b \\ b \end{bmatrix} \quad \text{where } (a, b \in \mathbb{R})}$$

- (b) Find the eigendecomposition of matrix $A = Q\Lambda Q^T$ using the results from a)

Given λ_1, λ_2 , let Λ be

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \quad \text{and so} \quad Q = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Verifying our solution,

$$A = Q\Lambda Q^T = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 9 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

- (c) What is the definiteness of the matrix?

A is positive-definite as $\lambda_1, \lambda_2 > 0$.

Problem 4

Proof. Use the definition of a convex function to show that affine functions $f(x) = a^T x + b$ are convex, where $a, x \in \mathcal{R}^n$ and $b \in \mathcal{R}$.

Let $x_1, x_2 \in \mathcal{R}^n$. Recall that a function $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is convex if

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2) \quad \text{for all } x_1, x_2 \text{ and } t \in [0, 1].$$

Looking at f ,

$$\begin{aligned} f(tx_1 + (1-t)x_2) &= a^T(tx_1 + x_2 - tx_2) + b \geq t f(x_1) + (1-t) f(x_2) = t(a^T x_1 + b) \\ a^T(tx_1 + x_2 - tx_2) &\geq t(a^T x_1 + b) + (1-t)(a^T x_2 + b) \\ ta^T x_1 + a^T x_2 - ta^T x_2 &\geq ta^T x_1 + tb + a^T x_2 + b - ta^T x_2 - tb \\ 0 &\geq 0 \end{aligned}$$

The inequality holds, and so affine functions $f(x) = a^T x + b$ are convex. \square

Problem 5

Consider the following objective function:

$$f(x, y) = x^2 + 2y^2 - 2xy$$

- (a) Compute the Hessian matrix H of function f , and find the definiteness of matrix H . First, we will compute the Hessian matrix of f , H :

$$f \Rightarrow \nabla f = \begin{bmatrix} 2x - 2y \\ 4y - 2x \end{bmatrix} \Rightarrow H = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

Now we will determine the definiteness of H by finding its eigenvalues:

$$\begin{aligned} |(H - \lambda I)| &= \left| \begin{bmatrix} 2 - \lambda & -2 \\ -2 & 4 - \lambda \end{bmatrix} \right| = (2 - \lambda)(4 - \lambda) - (-2)^2 = \lambda^2 - 6\lambda + 4 = 0 \\ \lambda &= \frac{6 \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} = 3 \pm \sqrt{5} > 0 \end{aligned}$$

Thus, H is positive-definite

- (b) Use calculus method to find the minimum value of $f(x, y)$

$$\nabla f = \vec{0} \Rightarrow (x, y) = (0, 0)$$

$f_{xx} = 2$ and $|H| = 8 - (-2)^2 = 4 \Rightarrow (0, 0)$ is the global minimum of f