Econ 4390 Notes - Economics of Education

Michael Lee

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Contents

1 Syllabus / How to Class Works

- Use this email: guillaumevdb@gmail.com. Put Econ 4390 in the SUBJECT
- There are no OH. Email Guillaume to figure out when to talk to him about class questions
- No HW or participation grade. Grade is the midterm, group project, and final
- See syllabus
- **Group Project**
  - Groups are 2-4 people
  - Topic can be on whatever you want BUT we will go over various topics you could present on
  - All group members will ahve the same grade in the end
  - These will be toward the end of the semester. 30 min / presentation
  - Groups will randomly be drawn day of to present. Thus, be ready to present on Day 1
  - Need to give Guillaume a slide print out BEFORE the presentation begin
- Exams should take 20 min
- He will frequently leave open problems on the board. These are practice midterm / final questions
- The final will contain 1 midterm question
- Math will involve power functions and basic derivates. Interpreting the math will give you problems on the exams — not the maths
- Exams are conceptual + math. No MC or TF

2 Intro

- This course will discuss education and what impacts education has on:
  1. Macroeconomy
  2. Human experience
  3. Other topics
- This course will cover K-12 and college / university education
- **Def.** Human Capital, ($h \in \mathbb{R}$): Input into the production function of any FIRM based on hardskills and softskills a person can RENT to a FIRM
- **Def.** Time worked: $n \in \mathbb{R}$
- **Def.** Effective Labor Supply: $h \cdot n$
- **Def.** Wage per Effective Unit of Labor per Time Period: (w)
- **Def.** 'Wage Rate': $w \cdot h$
  - wh together are observable
  - However, w and h independently are not observable
- **Def.** Paycheck: $w \cdot h \cdot n$

3 Model 1 - Why are more people going to college?

- Suppose "life" last for 2 periods
- Period 1:
  - Time in college
  - Tuition, $e > 0$ – "opportunity cost of college"
- Period 2:
  - Time after the college
- People who attend college don't work in Period 1 and work in Period 2
- People who do NOT college college work in Period 1 and Period 2

- People differ in their ability, $a > 0$. a is given at the start of Period 1
- Assume there are perfect credit markets (you can borrow and lend AS MUCH AS YOU WANT)
- Income for non-college people, $I_{HS} = w \cdot a$
- Income for college grads, $I_C = w \cdot h \cdot a$
- Wealth vs. Income
  - Income, I, is how much you get paid (typically)

$$I = \text{Price of skills} \times \text{Quantity of skills}$$

  - Wealth is the total remaining amount of Income accrued
  - The value of wealth decreases over time due to inflation
  - **Def.** Interest rate (r) - The amount you would pay today to get a dollar tomorrow
  - Hence, the present value of a dollar is

$$\frac{1}{1+r}$$

  - Suppose I invest $\frac{1}{1+r}$

$$\frac{1}{1+r} \cdot (1+r) = 1$$

- Suppose Income in Period i is y_i
- Thus, the present value of Income is

$$y_1 + \frac{y_2}{1+r}$$

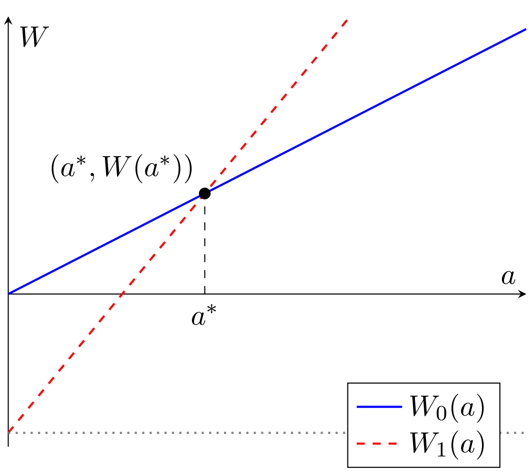
- Hence, for HS grads, the present value of Income is:

$$W_0(a) = aw + \frac{aw}{1+r}$$

- And, for college grads, the present value of Income is:

$$W_1(a) = -e + \frac{ahw}{1+r}$$

- Graphing this:



- Here, a^* is a s.t. $W_0 = W_1$
  - If your $a = a^*$, you are indifferent financially going to college
  - If your $a > a^*$, you are better off financially going to college
  - If your $a < a^*$, you are worse off financially going to college
- Are we certain $\exists(a^* > 0)$?

- Yes $\iff W_1(a)$ is steeper than $W_0(a)$

$$\frac{dW_0(a)}{da} = w + \frac{w}{1+r} \quad \text{and} \quad \frac{dW_1(a)}{da} = \frac{wh}{1+r}$$

$$\frac{dW_1(a)}{da} = \frac{wh}{1+r} > \frac{dW_0(a)}{da} = w + \frac{w}{1+r} \Rightarrow h-1 > 1+r$$

- Hence, our condition is

$$\frac{wh}{1+r} > w + \frac{w}{1+r} \iff h-1 > 1+r$$

- What is a^*?
  - a^* is defined as some a s.t. $W_0(a^*) = W_1(a^*)$

$$W_0(a^*) = a^*w + \frac{a^*w}{1+r} = W_1(a^*) = -e + \frac{a^*wh}{1+r}$$

$$a^*w + \frac{a^*w}{1+r} - \frac{a^*wh}{1+r} = a^*w\left(1 + \frac{1}{1+r} - \frac{h}{1+r}\right) = -e$$

$$a^*w\left(\frac{1+r+1-h}{1+r}\right) = a^*w\left(\frac{(h-1)-(r+1)}{1+r}\right) = -e$$

$$a^* = \frac{e}{w} \cdot \frac{1+r}{(h-1)-(r+1)}$$

  - Using this equation, we notice the following statics:
    - e and r are **directly proportional** to a^*. Equivalently, $\frac{da^*}{dw}, \frac{da^*}{dh} < 0$
    - w and h are **inversely proportional** to a^*. Equivalently, $\frac{da^*}{de}, \frac{da^*}{dr} > 0$
  - So what are the key drivers right now?
- So what is the rate of return on this investment?
  - **<u>Def.</u>** Rate of return:

$$\frac{\text{“what it pays”} - \text{“what it costs”}}{\text{“what it costs”}}$$

  - College pays: $awh - aw$ (because aw is guaranteed to all HS grads)
  - College costs: $e + aw$
  - Let the rate of return be $\rho(a)$

$$\rho(a) = \frac{aw(h-1)-(e+aw)}{e+aw}$$

  - Thus,

$$\rho(a) = \frac{aw(h-1)-(e+aw)}{e+aw} = \frac{aw(h-1)}{e+aw} - \frac{e+aw}{e+aw} = \frac{aw(h-1)}{e+aw} - 1$$

$$\Rightarrow 1 + \rho(a) = \frac{h-1}{1+\frac{e}{aw}} = \frac{aw(h-1)}{aw+e}$$

  - Recall:

$$a^* = \frac{e}{w} \cdot \frac{1+r}{(h-1)-(r+1)}$$

  - Hence,

$$1+\rho(a^*) = \frac{h-1}{1+\frac{e}{a^*w}} = \frac{h-1}{1+\frac{e}{\frac{e}{w}\cdot\frac{1+r}{(h-1)-(r+1)}w}} = \frac{h-1}{1+\frac{e}{e\cdot\frac{1+r}{(h-1)-(r+1)}}} = \frac{h-1}{1+\frac{(h-1)-(r+1)}{1+r}}$$

$$= \frac{(h-1)(1+r)}{(1+r)+(h-1)-(r+1)} = 1+r \Rightarrow \rho(a^*) = r$$

- Conceptually, this means that **rate of return** on education for people whose ability, a, makes them indifferent to going to college or not going to college have an ability equal to the interest rate, r.

- $\forall a > a^*$, **rate of return** on education be better than the interest rate (so it's a good investment)
- $\forall a < a^*$, **rate of return** on education be worse than the interest rate (so it's a bad investment)
- Now, we will interpret $h - 1 > 1 + r$
  - Recall
$$1 + \rho(a^*) = \frac{h-1}{1 + \frac{e}{a^* w}}$$
  - Thus,
$$\lim_{a \to \infty} (1 + \rho(a)) = h - 1 > 1 - r$$
  - The interpretation is that for individuals with exceptional ability, their rate of return is ALWAYS better than the interest rate (so it's a worthwhile investment)
- **Practice Drills**
  1. In Excel, create a graph where you fix e, h, r, have a w_{min}, w_{max} and graph a function showing $f : w \to a^*$. Play around with the graph and think about conceptually WHY changing the input variables impacts a^*
- How do we incorporate family wealth into the model?
  - For this model, we will assume your family transfer some amount of wealth, $A \in \mathbb{R}$ after high school
$$W_0(a) = wa + \frac{aw}{1+r} + A \quad \text{and} \quad W_1(a) = -e + \frac{awh}{1+r} + A$$
  - If we make A conditional a, this implies that ability is a trait that is carried from generation to generation. Therefore, people with higher wealth and likely to have high ability and vice versa
  - This could be because of ...
    1. Richer parents can give their kids more resources, leading to a higher a
    2. Parents with high ability create high ability kids (because genetics and/or culture)
$$W_0(a) = wa + \frac{aw}{1+r} + A(a) \quad \text{and} \quad W_1(a) = -e + \frac{awh}{1+r} + A(a)$$
  - A has no impact on a^*, $\rho(a)$, or $\frac{dW_0(a)}{da} > \frac{dW_1(a)}{da}$
- Credit Constraints
  - **Def.** Early Credit Constraint - Capability to investment in a child's education before college. Impacts a student's ability (a) to attend college
  - **Def.** Late Credit Constraint - Capability to investment in a child's education in college
  - Early credit constraint is much more important in determining college attainment than late credit constraints (according to the literature). This is why people pay so much for good preschool and elementary school
  - **This relationship could be a good presentation topic**

4 Deterministic Wage Growth

- **Def.** Deterministic - Something is impacted in a determinable, non-random way
- **Def.** Stochastic - Something is impacted in an indeterminable, randomy way
- Thus, a **Deterministic Wage Growth model** shows how wages grow in a determinable, non-random way
- Suppose wages, w, increase a a constant rate, g:

$$w_{t+1} = (1+g)w_t \Rightarrow w_t = (1+g)w_{t-1} = (1+g)^t w_0 \Rightarrow \ln(w_t) = \ln\left((1+g)^t w_0\right) = t \cdot \ln(1+g) + \ln(w_0)$$

- Thus,

$$\frac{d\ln(w_t)}{dt} = \ln(1+g)$$

- And so, the log-wage is a straight line (over time). We will assume that w increases in a step-wise manner
- Factoring this into our previous wealth model,

$$W_0(a) = wa + \frac{aw(1+g)}{1+r} \quad \text{and} \quad W_1(a) = -e + \frac{awh(1+g)}{1+r}$$

- Now, let's find a^* (**Good practice drill to do at home!!**)

$$W_0(a^*) = wa + \frac{a^*w(1+g)}{1+r} = W_1(a^*) = -e + \frac{a^*wh(1+g)}{1+r} \Rightarrow \boxed{a^* = \frac{e}{w} \cdot \frac{1+r}{(h-1)(1+g)-(1+r)}}$$

- Hence, g and a^* are **inversely proportional**
- Now we will look at the rate of return and see if $\rho(a^*) = r$
  - College pays: $awh(1+g) - aw(1+g) = aw(1+g)(h-1)$
  - College costs: $e + aw$
  - Hence,

$$1 + \rho(a) = \frac{(1+g)aw(h-1) - (e+aw)}{e+aw} = \frac{(1+g)(h-1)}{1+\frac{e}{aw}}$$

  - Now we plug in a^*

$$1+\rho(a^*) = \frac{(1+g)(h-1)}{1+\frac{e}{a^*w}} = \frac{(1+g)(h-1)}{1+\frac{e}{\frac{e}{w}\cdot\frac{1+r}{(h-1)(1+g)-(1+r)}\cdot w}} = \frac{(1+g)(h-1)}{1+\frac{(h-1)(1+g)-(1+r)}{1+r}} = \frac{(1+g)(h-1)}{1+\frac{(h-1)(1+g)}{1+r}-1}$$

$$1+\rho(a^*) = \frac{(1+g)(h-1)}{\frac{(h-1)(1+g)}{1+r}} = 1+r$$

  - Thus,

$$\rho(a^*) = r$$

  - The lesson here is that we made things closer to reality, but it did not lead to much additional insight
  - More complicated model $\not\Rightarrow$ More insightful model

5 Stochastic Wage Growth

Defining Random Variables

- Suppose X is a random variable that can take value x_i with probability p_i $(i \in [n] \subseteq \mathbb{N})$
- All x_i are possible realizations of x, so $\sum_i p_i = 1$
- Recall $E(X) = \sum_i x_i \cdot p_i$
- E is linear. In other words:

$$E(\alpha X + \beta) = \sum_i p_i(\alpha x_i + \beta) = \alpha \cdot E(X) + \beta \quad (\alpha, \beta \in \mathbb{R})$$

Going into the model . . .

- Suppose a wage rate in the second period, w' is not known in Period 1. Thus, in Period 1:

$$W_0(a) = E\left(aw + \frac{aw'}{1+r}\right) \quad \text{and} \quad W_1(a) = E\left(-e + \frac{ahw'}{1+r}\right)$$

- which is equivalent to . . .

$$W_0(a) = aw + \frac{aE(w')}{1+r} \quad \text{and} \quad W_1(a) = -e + \frac{ahE(w')}{1+r}$$

- Hence, if $W_0(a^*) = W_1(a^*)$

$$W_0(a^*) = a^*w + \frac{a^*E(w')}{1+r} = W_1(a^*) = -e + \frac{a^*hE(w')}{1+r}$$

$$-e = a^*w\left(1 + \frac{E(w')/w - hE(w')/w}{1+r}\right) = a^*w\left(\frac{(1+r) + E(w')/w - hE(w')/w}{1+r}\right)$$

$$\boxed{a^* = \frac{-e}{w} \cdot \frac{1+r}{(h-1)E(w')/w - (1+r)}}$$

- Recall the solution for the **Deterministic Wage Growth** model

$$a^* = \frac{e}{w} \cdot \frac{1+r}{(h-1)(1+g) - (1+r)}$$

- Thus, if $\frac{E(w')}{w} = 1 + g$, we can expect a^* to be the same in deterministic and stochastic models
- If $E(w') = w$, we expect no growth and so a^* is the same as in the basic model

Utility Functions

- Recall utility function $U : \mathbb{R}^n \to \mathbb{R}, c \to r$
- Typicaly, we think of utility as continuous. Imagine it's discrete:
- Suppose that consumption, c_i occurs with probability p_i $(i \in [n])$
- Thus, the expected utility of consumption is:

$$E(U(c)) = \sum_i p_i \cdot U(c_i)$$

- However, the utility of expected consumption is:

$$U(E(c)) = U\left(\sum_i p_i \cdot c_i\right)$$

- When $U(c)$ is monotonic and $\forall c$, $U''(c) < 0$,

$$U(E(c)) > E(U(c))$$

- This proves the utility from the expected consumption is better than the expected value of the utility given discrete consumption options

6 Work in College

- Suppose people work a fraction of their time, $1-\theta$ in college ($\theta \in [0,1]$)
- Hence, our value functions become:

$$W_0(a) = aw + \frac{aw}{1+r} \quad \text{and} \quad W_1(a) = -e + aw(1-\theta) + \frac{ahw}{1+r}$$

- Which is not super true as the more time you spend on working in college, the less time you spend studying. Thus, h is conditional on θ
- Hence, your goal should be:

$$W_1(a) = \max_\theta \left\{ -e + aw(1-\theta) + \frac{ah(\theta)w}{1+r} \right\}$$

- **Practice Drills**:
  - **Prove:** $\rho(a) = \frac{aw(h-1)-(e+aw\theta)}{e+aw\theta}$
  - **Prove:** $a^* = \frac{e}{w} \cdot \frac{1+r}{(h-1)-(1+r)\theta}$
  - **Prove:** For θ that maximizes $W_1(a) = \max_\theta \left\{ -e + aw(1-\theta) + \frac{ah(\theta)w}{1+r} \right\}$ is independent of a

$$W_1 \quad \Rightarrow \quad \frac{\partial W_1}{\partial \theta} = -aw + \frac{aw}{1+r}h'(\theta) = aw\left(\frac{h'(\theta)}{1+r} - 1\right) = 0$$

$$\frac{h'(\theta)}{1+r} - 1 = 0 \quad \Rightarrow \quad h'(\theta) = 1+r \quad \Rightarrow \quad \theta = h^{-1}(1+r)$$

$\square$

  - **Prove:** For a monotonic function $U(c)$ where $\forall c_2 > c_1$, $U'(c_1) > U'(c_2)$ and $P(c_i) = p_i$, prove $U(E(c)) > E(U(c))$
- We will assume that $h(\theta)$ in an increasing, monotonic function and first-order derivitable. We call the change in θ and $h(\theta)$ the **marginal productivity**
- Let's find θ that maximizes W_1

$$W_1 \quad \Rightarrow \quad \frac{\partial W_1}{\partial \theta} = 0 = -aw + \frac{awh'(\theta)}{1+r}$$

$$\Rightarrow \quad aw = \frac{awh'(\theta)}{1+r}$$

  - We can think of aw as the opportunity cost of working one more hour and $\frac{awh'(\theta)}{1+r}$ as the opportunity benefit of working one more hour.
  - If $aw \neq \frac{awh'(\theta)}{1+r}$ (i.e. $aw > \frac{awh'(\theta)}{1+r}$ or $aw < \frac{awh'(\theta)}{1+r}$), we are not at the optimum. Thus we can spend less or more time working to reach an optimum, respectively
  - Going back to the maths,

$$h'^{-1}(\theta) = 1+r \quad \Rightarrow \quad \theta = h'^{-1}(1+r)$$

  - In plain English, if a has no impact on θ, this is because
- Consider the following function:

$$W_1(a) = \max_\theta \left\{ -e + aw(1-\theta) + \frac{awh(a\theta)w}{1+r} \right\}$$

- Thus,

$$\frac{\partial W_1}{\partial \theta} = 0 = -aw + \frac{a^2wh'(a\theta)}{1+r} \quad \Rightarrow \quad aw = \frac{a^2wh'(a\theta)}{1+r} \quad \Rightarrow \quad 1 = \frac{ah'(a\theta)}{1+r} \quad \Rightarrow \quad 1+r = ah'(a\theta)$$

$$\Rightarrow \quad \frac{1+r}{a} = h'(a\theta) \quad \Rightarrow \quad h'^{-1}\left(\tfrac{1+r}{a}\right) = a\theta \quad \Rightarrow \quad \theta = \frac{h'^{-1}\left(\frac{1+r}{a}\right)}{a}$$

7 The College Premium

- We notice that wages for workers aged 30-60 is pretty constant over time.
- However, college educated workers have gone from making 30% more than non-college educated workers to 80%.
- Literature suggests this is because of the IT revolution (starting in the 1980s).
- Recall W_0, W_1, and a^*
- The college premium, CP, is defined as:

$$CP = \frac{\text{Earnings of the average college-educated worker}}{\text{Earnings of the average non-college-educated worker}}$$

- Thus, in our model,

$$CP = \frac{E(ahw|a > a^*)}{E(aw|a < a^*)} = \frac{wh \cdot E(a|a > a^*)}{w \cdot E(a|a < a^*)} = h \times \frac{E(a|a > a^*)}{E(a|a < a^*)}$$

- Note that h is what makes CP specific to an individual
- **<u>Def.</u>** Selection Effect: $\frac{E(a|a>a^*)}{E(a|a<a^*)}$
- Recall W_0 and W_1. We can find the **Lifetime College Premium**

$$CP = \frac{W_1}{W_0} = \frac{\frac{E(awh|a>a^*)}{1+r}}{E(aw + \frac{aw}{1+r}|a < a^*)} = \frac{\frac{wh}{1+r} \cdot E(a|a > a^*)}{w\left(1 + \frac{1}{1+r}\right) \cdot E(a|a < a^*)} = \frac{h}{2+r} \cdot \frac{E(a|a > a^*)}{E(a|a < a^*)}$$

- Let the density of ability be $f(a)$. Also, let $a \in [a_m, a_M]$ Note that:

$$p(a) = \int_{a_m}^{a} f(a)\, da$$

- Suppose $f(a) \sim U(a_m, a_M)$. Thus,

$$E(a) = \frac{a_m + a_M}{2}; \quad E(a|a > a^*) = \frac{a^* + a_M}{2}; \quad E(a|a < a^*) = \frac{a^* + a_m}{2}$$

- As a side note,

$$1 = \int_{a_m}^{a_M} f(a)\, da \quad \Rightarrow \quad \forall a,\ f(a) = \frac{1}{a_M - a_m}$$

- This impacts CP

$$CP = \frac{E(a|a > a^*)}{E(a|a < a^*)} = \frac{\frac{a^*+a_M}{2}}{\frac{a^*+a_m}{2}} = \frac{a^* + a_M}{a^* + a_m}$$

$$\frac{\partial}{\partial a^*}\left(\frac{a^* + a_M}{a^* + a_m}\right) = \frac{a_m - a_M}{(a^* + a_m)2} < 0$$

8 Skill-based Technology

- Starting in the 1980s, college educated people began earning way more and by 2000 earned more than non-college educated individuals. We suspect this is due to the IT revolution. We will create a series of models explainin this effect.
- Let our production, $Y = F(K, L)$, be conditional on labor, L, and capital, K.
- Note that $F(0, L) = F(K, 0) = 0$ and $\forall L, K \neq 0$, $F(K, L) \geq 0$. Also, marginal products are decreasing.
- <u>**Def.**</u> Marginal returns: How much does output change when one input changes a little
- <u>**Def.**</u> Retursn to scale: How much does output change when all inputs change in the same proportionally (i.e. $F(\lambda K, \lambda L)$ where $\lambda \in \mathbb{R}$)
- <u>**Def.**</u> Constant returns to scale (CRS): If all inputs are multiplied by $\lambda \in \mathbb{R}$, the output is multiplied by λ (i.e. $\lambda F(K, L) = F(\lambda K, \lambda L)$)
  - NOTE: CRS implies that 1 firm of size k produces as much as k firms of size 1.
- **Practice Drills:** Prove/disprove the following functions show CRS
  - Linear: $F(K, L) = aK + bL$
  - Cobb-Douglas: $F(K, L) = K^\alpha \cdot L^{1-\alpha}$
  - Constant Elasticity Substitution (CES): $F(K, L) = (K^\alpha + L^\alpha)^{1/\alpha}$
- Define the following marginal product of labor and capital (respectively):

$$MP_L = \frac{\partial Y}{\partial L}; \quad MP_K = \frac{\partial Y}{\partial K}$$

- Firms care about maximization profit, π.
- Let r be the price at which capital is hired
- Let the **real wage**, w, be the price at which labor is hired
- We will assume that all goods produced are sold

$$\pi = F(K, L) - (rK + wL)$$

- This model assumes the firm has no power over choosing r or w
- The firm's goal, profit maximization, is:

$$\max_{K,L} \{\pi(K, L)\}$$

- If $MP_L > w$, hiring raises profit but lowers MP_L
- If $MP_L < w$, firing raises profit but raises MP_L
- At optimum, $MP_L = w$.
- Firms hire both high school and college educated workers
- Firms hire both types and technology is:

$$Y = F(K, L) \quad \text{where} \quad L = H(L_0, L_1)$$

- Some notes:
  - L aggregates the total labor supplied by both types of workers
  - H is the aggregator for the two types of workers

$$L_0 = \int_{a_m}^{a^*} a \cdot f(a)\, da \quad \text{and} \quad L_1 = \int_{a^*}^{a_M} a \cdot f(a)\, da$$

- The firm's maximizaton problem is:

$$\max_{K, L_0, L_1} \{F(K, H(L_0, L_1)) - rK - wL_0 - w_1 L_1\}$$

- and it requires that:

$$w_0 = F_L(K, H(L_0, L_1)) \cdot H_{L_0}(L_0, L_1)$$
$$w_1 = F_L(K, H(L_0, L_1)) \cdot H_{L_1}(L_0, L_1)$$

- Note that:

$$\frac{w_1}{w_0} = \frac{H_{L_1}(L_0, L_1)F_L(K, H(L_0, L_1))}{H_{L_0}(L_0, L_1)F_L(K, H(L_0, L_1))} = \frac{H_{L_1}(L_0, L_1)}{H_{L_0}(L_0, L_1)}$$

- This equation implies the marginal return, MP, to each type of workers
- Since $1 = \frac{w_1}{w_0}$,

$$1 = \frac{w_1}{w_0} = \frac{H_{L_0}(L_0, L_1)}{H_{L_1}(L_0, L_1)} \quad \Rightarrow \quad H_{L_1}(L_0, L_1) = H_{L_0}(L_0, L_1)$$

- **Practice Drill:** Maximize the firm's output given these functions:

$$F(K, L) = K^{1/3} \cdot L^{2/3} \quad \text{and} \quad L = H(L_0, L_1) = 5L_0 + L_1$$

$$\Rightarrow \quad F_L(K, L) \cdot H_{L_0}(L_0, L_1) = F_L(K, L) \cdot 5 = \frac{10}{3}\left(\frac{K}{L}\right)^{1/3}$$

- For $K, L = 10$, $Y = F(10, 10) = 10$ and $F_L(10, 10)H_{L_0}(10, 10) = 10/3 = MP_{L_0}$.
- Suppose $w_0, w_1, r = 1$. Thus,

$$\pi = 10 - 10(1) - 1(1) - 1(5) = -6$$

- **Practice Drill:** Claim, if I have $F(K, L)$ that is CRS, F_L is a function of K/L and F_K is a function of K/L

- Suppose $w_0 \neq w_1$. Thus,

$$W_0(a) = aw_0 + \frac{aw_0}{1+r} \quad \text{and} \quad W_1(a) = -e + \frac{ahw_1}{1+r}$$

- Recall the marginal individual is defined as a person with ability a^*.

$$W_0(a^*) = a^*w_0 + \frac{a^*w_0}{1+r} = W_1(a^*) = -e + \frac{a^*hw_1}{1+r}$$

$$a^*w_0\left(1 + \frac{1}{1+r} - \frac{hw_1}{w_0(1+r)}\right) = -e$$

$$a^* = \frac{e}{w_0} \cdot \frac{1+r}{(hw_1/w_0 - 1) - (1+r)}$$

- Suppose $H(L_0, L_1) = L_0 + z \cdot L_1$ for some $z \in \mathbb{R}$. Thus,

$$\frac{w_1}{w_0} = z \quad \text{and} \quad a^* = \frac{e}{w_0} \cdot \frac{1+r}{(hz-1) + (1+r)} \quad \text{and} \quad CP = h \cdot z \cdot \frac{E(a|a > a^*)}{E(a|a < a^*)}$$

9 Multi-period Model

- Suppose that life contained T-periods, where s-periods are spent in college. Our model would be:

$$W_0 = \sum_{i=1}^{T} aw \cdot \left(\frac{1+g}{1+r}\right)^{i-1} = aw \cdot \sum_{i=1}^{T} \left(\frac{1+g}{1+r}\right)^{i-1}$$

$$W_s = -e + \sum_{i=s+1}^{T} awh \cdot \left(\frac{1+g}{1+r}\right)^{i-1} = -e + \cdot awh \cdot \sum_{i=s+1}^{T} \left(\frac{1+g}{1+r}\right)^{i-1}$$

- Note that we typically view T as the "life expectancy" of a person. Here, life expectancy is the number of years a person has left to live.
- Where W_s is the model for the college-educated people spending s-periods in college.
- Note that W_s is similar to W_0, but they start earning in period $s+1$. This is because they spent s-periods in college.
- Define the following:

$$D_0 = \sum_{i=1}^{T} \left(\frac{1+g}{1+r}\right)^{i-1} \quad \text{and} \quad D_s = \sum_{i=s+1}^{T} \left(\frac{1+g}{1+r}\right)^{i-1}$$

- Thus,

$$W_0 = aw \cdot D_0 \quad \text{and} \quad W_s = -e + awh \cdot D_s$$

- Solving for a^* yields:

$$a^* = \frac{e}{w} \cdot \frac{1}{hD_s - D_0}$$

- **Practice Drill:** Prove that $a^* = \frac{e}{w} \cdot \frac{1}{hD_s - D_0}$. Expect for this to be on an exam.
- Recall that:

$$\sum_{i=1}^{T} x^i = \frac{1 - x^{T+1}}{1-x}$$

- Thus, for D_0 and D_s, we have:

$$D_0 = \frac{1 - \left(\frac{1+g}{1+r}\right)^T}{1 - \frac{1+g}{1+r}} \quad \text{and} \quad D_s = \frac{\left(\frac{1+g}{1+r}\right)^s - \left(\frac{1+g}{1+r}\right)^T}{1 - \frac{1+g}{1+r}}$$

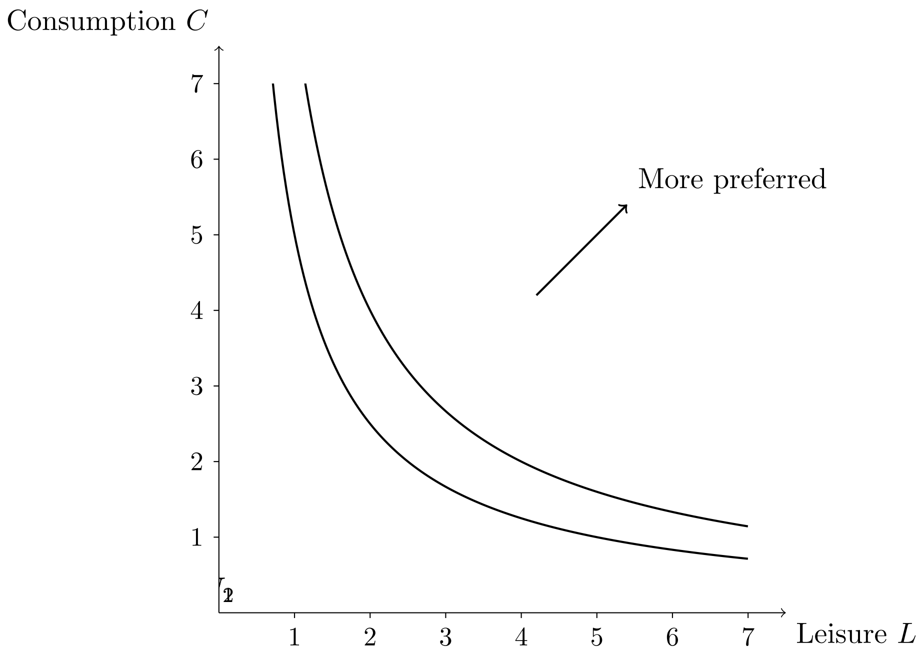
- Hence, looking at $hD_s - D_0$, we have:

$$hD_s - D_0 = h \cdot \frac{\left(\frac{1+g}{1+r}\right)^s - \left(\frac{1+g}{1+r}\right)^T}{1 - \frac{1+g}{1+r}} - \frac{1 - \left(\frac{1+g}{1+r}\right)^T}{1 - \frac{1+g}{1+r}} = \frac{h\left(\frac{1+g}{1+r}\right)^s - 1 - (h-1)\left(\frac{1+g}{1+r}\right)^T}{1 - \frac{1+g}{1+r}}$$

- Note that $g < r \Rightarrow hD_s - D_0$ is increasing with T, so a^* is increasing.
- If $g < r$, then educational attainment is increasing with life expectancy.

10 Labor Supply

- We notice that people are working less per week over time (1820: 72 hr/wk; 2000: 40 hr/wk).
- People also work fewer weeks per year and fewer days per week.x
- We also notice that people spend more time in school over time (1870: 7 yr; 1980: 14 yr).
- We will assume people care about consumption and leisure time. We will assume that more is better for both.
- There are infinitely many indifference curves displaying difference consumption and leisure curve allocations.



- The rate at which someone with substitute consumption and leisure is the marginal rate of substitution (MRS).
- MRS is the slope of the indifference curve.
- Let c be consumption at price p, l be leisure hours, and ω be the wage rate.
- Assume there are $24 \cdot 7 = 168$ hours in a week
- The labor supply is $168 - l$
- This yields the budget curve:

$$p \cdot c = \omega(168 - l) \qquad \Longleftrightarrow \qquad c = \frac{\omega}{p}(168 - l)$$

- If $w = \frac{\omega}{p}$ is the real wage rate and we normalize l to represent the fraction of the week spent working,

$$c = w(1 - l) \qquad \Longleftrightarrow \qquad c + wl = w$$

- Trivially, this budget is a straight line.
- The 'optimum' is where there are no more preferable points
- Let the indifference curve be $U(c, l) = k \in \mathbb{R}$

$$MRS = -\frac{U_l}{U_c}$$

- Recall the Lagrangian:
  - We want to solve $\max_{c,l} U(c, l)$ s.t. $c + wl = w$. We can achieve this by solving the Lagrangian:

$$\mathcal{L} = U(c, l) + \lambda(w - c - wl)$$

- Hence, we obtain the following first-order conditions:

$$\begin{cases} c & : \quad U_c(c,l) - \lambda = 0 \\ l & : \quad U_l(c,l) + \lambda w = 0 \\ \lambda & : \quad w - c - wl = 0 \end{cases}$$

- **Practice Drill:** Find the optimum for $U(c,l) = \frac{1}{2}\ln(c) + \frac{1}{2}\ln(l)$
- Answer: $l = 1/2$ and $c = w/2$
- Interpretation: The set of all optimal choices is conditional only on w. l is always $1/2$
- **Practice Drill:**

$$\max_{c,l}\{\alpha \ln(c) + (1-\alpha)\ln(l)\} \quad \text{s.t.} \quad c + wl = w$$

- Note: Optimal l is independent of w. Conceptually, this makes sense because when wage increases, people could work more hours to make more money OR work fewer hours and spend more time on leisure.
- **Practice Drill:**

$$\max_{c,l}\{\alpha \ln(c-\bar{c}) + (1-\alpha)\ln(l)\} \quad \text{s.t.} \quad c + wl \geq w$$

- where $\bar{c} > 0$ is 'subsistence.' Think of this as the minimum amount of consumption needed for someone to survive.
- **Answer Conditions:**

$$\begin{cases} \frac{\alpha}{c\cdot\bar{c}} \cdot w = U_c \cdot w = \frac{1-\alpha}{l^*} = U_l \\ c^* + wl^* = w \end{cases}$$

Proof.

$$U(c,l) \quad \Rightarrow \quad \nabla U(c,l) = \begin{bmatrix} \frac{\alpha}{c-\bar{c}} \\ \frac{1-\alpha}{l} \end{bmatrix}$$

Suppose $U_l = U_c \cdot w$. Thus,

$$\frac{1-\alpha}{l} = \frac{\alpha}{c-\bar{c}} \cdot w \quad \Rightarrow \quad wl = \frac{1-\alpha}{\alpha} \cdot (c-\bar{c}) \quad \Rightarrow \quad c - \bar{c} = \frac{1-\alpha}{\alpha} \cdot wl$$

Recall that $c + wl = w$. Thus,

$$c + wl = w \quad \Rightarrow \quad c - \bar{c} + wl = w - \bar{c}$$

Plugging in,

$$\frac{\alpha}{1-\alpha} \cdot l^* + l^* = 1 - \frac{\bar{c}}{w} \quad \Rightarrow \quad 1 - l^* = 1 - (1-\alpha)\left(1 - \frac{\bar{c}}{w}\right) \quad \Rightarrow \quad l^* = (1-\alpha)\left(1 - \frac{\bar{c}}{w}\right)$$

Finally,

$$\lim_{w\to\infty}(l^*) = \lim_{w\to\infty}\left(1 - (1-\alpha)\left(1 - \frac{\bar{c}}{w}\right)\right) = 1 - (1-\alpha) = \alpha$$

$\square$

11 Schooling & Labor Supply

- Suppose life has periods, p, s.t. $p \in [0, T]$
- Time spent in school, s yield humans capital $h(s)$
- Note that: $h'(s) > 0$ and $h''(s) < 0$
- Individuals solve:
$$\max_s w \cdot h(s)(T - s)$$
- Additionally, note that $T - s$ represents time not spent in school (i.e. working).
- Immediately, we obtain the first-order condition (s is indepedent of w):
$$w \cdot h'(s)(T - s) = w \cdot h(s) \iff h'(s)(T - s) = h(s)$$
- Alternatively,
$$w \cdot h'(s)(T - s) = w \cdot h(s) \iff \frac{h'(s)}{h(s)} = \frac{1}{T - s}$$
- One's lifetime labor supply is $(T - s)(1 - l)$
- Problem:
$$\max_{c,l,s} \{U(c) + \alpha V(l) + \beta J(s)\} \quad \text{s.t.} \quad c = wh(s)(1 - l)(T - s)$$
- Hence, the Lagrangian, $\mathcal{L}$, is:
$$\mathcal{L} = U(c) + \alpha V(l) + \beta J(s) + \lambda(wh(s)(1 - l)(T - s) - c)$$
$$\mathcal{L}(c, l, s) \quad \Rightarrow \quad \nabla\mathcal{L}(c, l, s) = \begin{bmatrix} \mathcal{L}_c \\ \mathcal{L}_l \\ \mathcal{L}_s \end{bmatrix} = \begin{bmatrix} U_c - \lambda \\ \alpha V_l - \lambda wh(s)(T - s) \\ \beta J_s - \lambda w(1 - l)[h'(s)(T - s) - h(s)] \end{bmatrix} = \vec{0}$$
- Hence, the first-order conditions are:
$$\begin{cases} c & : \quad U_c - \lambda = 0 \\ l & : \quad \alpha V_l - \lambda wh(s)(T - s) = 0 \\ s & : \quad \beta J_s + \lambda w(1 - l)[h'(s)(T - s) - h(s)] = 0 \end{cases}$$
- **The interpretation of this first-order condition WILL be on the final.**
  - Given $0 = U_c - \lambda \Rightarrow U_c = \lambda$, we can rewrite the conditions for l and s as:
$$aV_l - U_c wh(s)(T - s) = 0 \quad \text{and} \quad \beta J_s + U_c w(1 - l)[h'(s)(T - s) - h(s)] = 0$$
  - or equivalently,
$$aV_l = U_c \cdot wh(s)(T - s) \quad \text{and} \quad \beta J_s = -U_c \cdot w(1 - l)[h'(s)(T - s) - h(s)]$$
  - Looking at the first equation, we see that:
$$aV_l(1 - l) = U_c \cdot wh(s)(T - s)(1 - l) = U_c \cdot c$$
  - If $U(c) = \ln(c)$, then
$$aV_l(1 - l) = U_c \cdot c = \frac{1}{c} \cdot c = 1$$
  - This implies that if utility is logarithmic, then it is not conditional on w (as $c = w - wl$)
  - Looking at the second equation, we see that:
$$\beta J_s = -U_c \cdot w(1 - l)[h'(s)(T - s) - h(s)] \cdot \left(h(s)(T - s) \cdot \frac{1}{h(s)(T - s)} \right)$$
$$\beta J_s = U_c \cdot c \cdot \left[\frac{1}{T - s} - \frac{h'(s)}{h(s)} \right]$$

- Recall that for a person maximizing income, $\frac{h'(s)}{h(s)} = \frac{1}{T-s}$. Thus,

$$\beta J_s = U_c \cdot c \cdot \left[\frac{1}{T-s} - \frac{1}{T-s}\right] = 0$$

- This implies either $\beta = 0$ or $J_s = 0$
  * If $\beta = 0$, then there is no pecuniary cost to going to school.
  * If $J_s = 0$, then your $J(s)$ is at a maximum.
- Additionally, consider the case where $U(c) = \ln(c)$. Then,

$$\beta J_s = \frac{1}{T-s} - \frac{h'(s)}{h(s)}$$

- Hence, it is not conditional on w.

- Let s_{max} be s that maximizes U. s_{max} is the s s.t. $\frac{1}{T-s} = \frac{h'(s)}{h(s)}$
- s_{max} is NOT conditional on $U_c \cdot c$
- Questions:
  *

12 Fertility

- Parents need to choose how many children to have. Children cost time, τ.
- Thus, parents want to maximize the following:

$$\max_{c,n}\{U(c) + \alpha V(n)\} \quad \text{s.t.} \quad c = w(1 - \tau n)$$

- Note that this equation is very similar to the labor supply problem.
- Doing the first order conditions, we get:

$$\mathcal{L} = U(c) + \alpha V(n) - \lambda(c - w(1 - \tau n))$$

$$\nabla\mathcal{L} = \begin{bmatrix}\mathcal{L}_c \\ \mathcal{L}_n \\ \mathcal{L}_\lambda\end{bmatrix} = \begin{bmatrix}U_c - \lambda \\ \alpha V_n - \lambda w \tau \\ \lambda c - w(1 - taun)\end{bmatrix} = \vec{0}$$

- Thus,

$$\mathcal{L}_n = 0 \quad \Rightarrow \quad \alpha V_n = \lambda w \tau \quad \Rightarrow \quad \alpha V_n(1 - \tau n) = \lambda w \tau (1 - \tau n) \quad \Rightarrow \quad \frac{\alpha}{\tau} V_n (1 - \tau n) = 1$$

- Children also cost goods, q. Thus,

$$\max_{c,n}\{U(c) + \alpha V(n)\} \quad \text{s.t.} \quad w = c + qn$$

- Finding the first order conditions, we get:

$$\mathcal{L} = U(c) + \alpha V(n) - \lambda(c + qn - w)$$

$$\nabla\mathcal{L} = \begin{bmatrix}\mathcal{L}_c \\ \mathcal{L}_n \\ \mathcal{L}_\lambda\end{bmatrix} = \begin{bmatrix}U_c(c) - \lambda \\ \alpha V_n(n) - \lambda q \\ \lambda c + qn - w\end{bmatrix} = \vec{0}$$

- Hence,

$$\alpha V_n = \lambda q = U_c(w - qn)q$$

- **Practice Drill:** Be able to explain Slides 5 and 6.
- What if parents want to choose how many children to have and how much human capital to give them?

$$\max_{c,n,h} = U(c) + \alpha V(n) + \beta J(h) \quad \text{s.t.} \quad c + qnh = w(1 - \tau n)$$

- **Practice Drill:** Find the first order conditions with $U(c) = \ln(c)$, $V(n) = \ln(n)$ and $J(h) = \ln(h)$. Find optimal c, n, h.

13 Midterm Review

Econ4390_Sample_Questions.pdf

- Question 5
  - (a) Show this exhibits constant returns to scale
  - What does 'constant returns to scale' mean? $\leftarrow$ If we scale all inputs by a factor of λ, the output is scaled by a factor of λ (i.e. $\lambda F(K, L) = F(\lambda K, \lambda L)$)
  - (b) What is the interpretation of z? $\leftarrow$ z is skill-bias (technology)
  - (c) Prove that when $z = \frac{w_0}{w_1}$, the firm maximizes profit

$$\pi = K^\alpha (L_0 + z \cdot L_1)^{1-\alpha} - rK - w_0 L_0 - w_1 L_1$$

Hence,

$$\nabla \pi = \begin{bmatrix} \pi_K \\ \pi_{L_0} \\ \pi_{L_1} \end{bmatrix} = \begin{bmatrix} \alpha K^{\alpha-1}(L_0 + z \cdot L_1)^{1-\alpha} - r \\ K^\alpha (1-\alpha)(L_0 + z \cdot L_1)^{-\alpha} - w_0 \\ K^\alpha (1-\alpha)(L_0 + z \cdot L_1)^{-\alpha} \cdot z - w_1 \end{bmatrix} = \vec{0}$$

Thus,

$$K^\alpha (1-\alpha)(L_0 + z \cdot L_1)^{-\alpha} = w_0 \quad \text{and} \quad K^\alpha (1-\alpha)(L_0 + z \cdot L_1)^{-\alpha} \cdot z = w_1$$

And so,

$$\frac{w_1}{w_0} = \frac{K^\alpha (1-\alpha)(L_0 + z \cdot L_1)^{-\alpha} \cdot z}{K^\alpha (1-\alpha)(L_0 + z \cdot L_1)^{-\alpha}} = z$$

- Consider a world where skill grows at rates g_0 and g_1 respectively. Thus, in a two-period model:

$$W_0(a) = a w_0 + \frac{a w_0 (1 + g_0)}{1 + r} \quad \text{and} \quad W_1(a) = -e + \frac{a h w_1 (1 + g_1)}{1 + r}$$

Suppose $W_0(a) = W_1(a)$. Prove that

$$a^* = \frac{e}{w_0} \cdot \frac{1 + r}{\left(h \frac{w_1}{w_0} \cdot \frac{(1+g_1)}{(1+g_0)} - 1\right)(1 + g_0) - (1 + r)}$$

14 Research:

Tips

- Use https://www.ipums.org/ to gather data
- Login: lee.m.j@wustl.edu
- Password: Ip#9557782

Topics

1. Impact of early credit constraint vs. late credit constraint on educational attainment

2. Impact of Income level on health

3. Impact of income on hours worked on health