Basic Probability Theory
HW2 Solution

October 1st 2024

1 Question 3.4

a.

$$\text{P(never married)} = \frac{53,216}{214,523} = 0.2481$$

b.

$$\text{P(widow | female)} = \frac{\text{P(female, widow)}}{\text{P(female)}} = \frac{11,141}{110,882} = 0.1005$$

c.

$$\text{P(male | divorced)} = \frac{\text{P(male, divorced)}}{\text{P(divorced)}} = \frac{8,956}{21,760} = 0.4116$$

d.

$$\text{P(widow | male)} = \frac{\text{P(male, widow)}}{\text{P(male)}} = \frac{2,641}{103,641} = 0.0255 < \text{P(widow | female)} = 0.1005$$

$\Rightarrow$ Therefore, females are more likely to be widowed.

2 Question 3.5

Let B: breakfast, W: work on time

$$P(B,W) = 0.2$$
$$P(B) = 0.4$$

$$P(W \mid B) = \frac{P(B, W)}{P(B)} = \frac{0.2}{0.4} = 0.5$$

$\Rightarrow$ Answer : 0.5

3 Question 3.6

Let S: studies, P: pass

$$\begin{aligned} P(S, P) &= 0.8 \\ P(S) &= 0.9 \end{aligned}$$

$$P(P \mid S) = \frac{P(S, P)}{P(S)} = \frac{0.8}{0.9} = 0.889$$

$\Rightarrow$ Answer : 0.899

4 Question 3.8

Let C: buys cat food, S: stops for grocery

$$\begin{aligned} P(C \mid S) &= 0.5 \\ P(S) &= 0.6 \end{aligned}$$

$$P(C, S) = P(C \mid S)P(S) = 0.5 \cdot 0.6 = 0.3$$

$\Rightarrow$ Answer : 0.3
(sol2)

$P(B + C) = \frac{13.2+2.7}{6.5+13.2+2.7} = 0.7098$

X: # of male or co-owned by male $\sim$ Bin(4, 0.7098)

$$\begin{aligned} P(X = 1) &= \binom{4}{1} \cdot (0.7098)^1 \cdot (1 - 0.7098)^3 \\ &= 0.0695 \\ &= blue\texttt{dbinom(1, 4, 0.7098)} \quad \text{(R Code)} \end{aligned}$$

Question 4.45

$\mathrm{P_x(x)} = \binom{50}{\mathrm{x}}\mathrm{p^x(1-p)^{50-x}}$

a. $X \sim \text{Bin}(n = 50,\, p = 0.05)$

$$
\begin{aligned}
P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\
&= \binom{50}{0}(0.05)^0(0.95)^{50} + \binom{50}{1}(0.05)^1(0.95)^{49} + \binom{50}{2}(0.05)^2(0.95)^{48} \\
&= (0.95)^{48}[(0.95)^2 + 50 \times 0.05 \times 0.95 + \frac{49 \times 50}{2} \times (0.05)^2] \\
&= 0.5405 \\
&= \texttt{pbinom(2, 50, 0.05)} \quad \text{(R Code)}
\end{aligned}
$$

b. $X \sim \text{Bin}(n = 50,\, p = 0.1)$

$$
\begin{aligned}
P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\
&= \binom{50}{0}(0.1)^0(0.9)^{50} + \binom{50}{1}(0.1)^1(0.9)^{49} + \binom{50}{2}(0.1)^2(0.9)^{48} \\
&= (0.9)^{48}\left[(0.9)^2 + 50 \times 0.1 \times 0.9 + \frac{49 \times 50}{2} \times (0.1)^2\right] \\
&= 0.1117 \\
&= \texttt{pbinom(2, 50, 0.1)} \quad \text{(R Code)}
\end{aligned}
$$

Question 4.48

A: Female-owned, B: Male-owned, C: Jointly male and female-owned

$$
\begin{aligned}
P(A) &= \frac{6.5}{6.5 + 13.2 + 2.7} = 0.29 \\
P(B) &= \frac{13.2}{6.5 + 13.2 + 2.7} = 0.5892 \\
P(C) &= \frac{2.7}{6.5 + 13.2 + 2.7} = 0.121
\end{aligned}
$$

a. X: Number of female-owned businesses $\sim \text{Bin}(4, 0.29)$

$$
\begin{aligned}
P(X = 4) &= \binom{4}{4} \cdot (0.29)^4 \cdot (1 - 0.29)^0 \\
&= 0.0071 \\
&= \texttt{dbinom(4, 4, 0.29)} \quad \text{(R Code)}
\end{aligned}
$$

b. X: # of male-owned or co-owned by male businesses $\sim$ Bin(4, 0.7098)

$P(B+C) = \frac{13.2+2.7}{6.5+13.2+2.7} = 0.7098$

$$\begin{aligned} P(X=1) &= \binom{4}{1} \cdot (0.7098)^1 \cdot (1-0.7098)^3 \\ &= 0.0695 \\ &= \texttt{dbinom(1, 4, 0.7098)} \quad \text{(R Code)} \end{aligned}$$

c. X: Number of jointly owned business $\sim \text{Bin}(4, 0.121)$

$$\begin{aligned} P(X=0) &= \binom{4}{0} \cdot (0.121)^0 \cdot (1-0.121)^4 \\ &= 0.597 \\ &= \texttt{dbinom(0, 4, 0.121)} \quad \text{(R Code)} \end{aligned}$$

Question 4.55

X: # of defectives Bin(4, 0.12), (X = 0,1,2,3,4)

$$\begin{aligned} E(X) &= \text{np} \\ &= 0.48 \\ E(X^2) &= Var(X) + [E(x)]^2 \\ &= 4 \cdot 0.12 \cdot 0.88 + (0.48)^2 \\ &= 0.6528 \\ E(C) &= E(2X^2 + X + 3) \\ &= 2E(X^2) + E(X) + 3 \\ &= 2 \times 0.6528 + 0.48 + 3 \\ &= 4.7856 \end{aligned}$$

$\therefore$ 4.7856

Question 4.58

X: # of defectives $\sim \text{Bin}(n, p)$

$P(X \geq 1) = 0.95$

a.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \binom{n}{0} \cdot (0.1)^0 \cdot (0.9)^n \\ &= 1 - (0.9)^n = 0.95 \end{aligned}$$

$$\begin{aligned} \therefore\ n &= \log_{0.9}(0.05) \\ &= 28.4332 \\ &= \texttt{log(0.05, base = 0.9)} \quad \text{(R Code)} \end{aligned}$$

b.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \binom{n}{0} \cdot (0.05)^0 \cdot (0.95)^n \\ &= 1 - (0.95)^n = 0.95 \end{aligned}$$

$$\begin{aligned} \therefore\ n &= \log_{0.95}(0.05) \\ &= 58.404 \\ &= \texttt{log(0.05, base = 0.95)} \quad \text{(R Code)} \end{aligned}$$

Question 4.65

X : # of failures before 1st success $\sim \text{Geom}(p = 0.1)$

$P_x(x) = (0.9)^x \cdot (0.1)^1$

a.

$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 1 - [P(X = 0) + P(X = 1)] \\ &= 1 - [(0.1) + (0.1) \cdot (0.9)] \\ &= 0.81 \\ &= \texttt{1 - pgeom(1, 0.1)} \quad \text{(R Code)} \end{aligned}$$

b.

$$
\begin{aligned}
P(X \geq 4 \mid X \geq 2) &= \frac{P(X \geq 4, X \geq 2)}{P(X \geq 2)} \\
&= \frac{P(X \geq 4)}{P(X \geq 2)} \\
&= \frac{1 - P(X \leq 3)}{1 - P(X \leq 1)} \\
&= \frac{1 - [P(X = 0) + ... + P(X = 3)]}{1 - [P(X = 0) + P(X = 1)]} \\
&= \frac{1 - (0.1 + 0.1 \cdot 0.9 + 0.1 \cdot 0.9^2 + 0.1 \cdot 0.9^3)}{1 - [0.1 + 0.1 \cdot 0.9]} \\
&= \frac{0.6561}{0.81} \\
&= 0.81 \\
&= \texttt{(1 - pgeom(3, 0.1)) / (1 - pgeom(1, 0.1))} \quad \text{(R Code)}
\end{aligned}
$$

Question 4.67

X : # of failures(defectives) before 1st success(good engine found)
$\sim$ Geom($p = 0.9$)

$P_x(x) = (0.1)^x \cdot (0.9)^1$

$$
\begin{aligned}
P(X = 2) &= (0.1)^2 \cdot (0.9) \\
&= 0.009 \\
\therefore\ 0.009
\end{aligned}
$$

Question 4.68

$$
\begin{aligned}
P(X \geq 4 \mid X \geq 2) &= \frac{P(X \geq 4, X \geq 2)}{P(X \geq 2)} \\
&= \frac{P(X \geq 4)}{P(X \geq 2)} \\
&= \frac{1 - P(X \leq 3)}{1 - P(X \leq 1)} \\
&= \frac{1 - [P(X = 0) + ... + P(X = 3)]}{1 - [P(X = 0) + P(X = 1)]} \\
&= \frac{1 - [(0.9) + (0.1) \cdot (0.9) + (0.1)^2 \cdot (0.9) + (0.1)^3 \cdot (0.9)]}{1 - [(0.9) + (0.1) \cdot (0.9)]} \\
&= 0.01 \\
&= \texttt{(1 - pgeom(3, 0.9)) / (1 - pgeom(1, 0.9))} \quad \text{(R Code)} \\
\therefore 0.01
\end{aligned}
$$

Question 4.133

Probability-Generating Function : $\mathbf{G_X(t) = E(t^x) = \sum_{x=0}^{\infty} t^x \cdot P(X = x)}$

$X \sim Poi(\lambda), \quad P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \; x = 0, 1, 2, ...$

$$\begin{aligned} G_X(t) &= \sum_{x=0}^{\infty} t^x \cdot \frac{\lambda^x e^{-\lambda}}{x!} \\ &= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(t\lambda)^x}{x!} \\ &= e^{-\lambda} \cdot e^{t\lambda} \quad (\because) \text{ Exponential Series} \\ &= e^{\lambda(t-1)}, \; {}^{\forall}t \in \mathbb{R} \end{aligned}$$

Question 4.134

$X \sim Binom(n, p), \quad P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$\begin{aligned} G_X(t) &= \sum_{x=0}^{\infty} t^x \cdot \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{\infty} \binom{n}{x} (tp)^x (1-p)^{n-x} \\ &= (tp + 1 - p)^n \quad (\because) \text{Binomial Theorem} \\ &= (p(t-1) + 1)^n, \; {}^{\forall}t \in \mathbb{R} \end{aligned}$$

Question 4.135

Probability-Generating Function and it's 1st, 2nd Factorial Moments :

- Probability-Generating Function : $G_X(t) = \sum_{x=0}^{\infty} t^x P(X = x)$
- 1st Factorial Moment : $G'_X(t) = \sum_{x=0}^{\infty} x \cdot t^{x-1} P(X = x)$
  - $E(X) = G'_X(1) = \sum_{x=0}^{\infty} x P(X = x)$
- 2nd Factorial Moment : $G''_X(t) = \sum_{x=0}^{\infty} x(x-1) \cdot t^{x-2} P(X = x)$
  - $G''_X(1) = \sum_{x=0}^{\infty} x(x-1) P(X = x) = E(X^2 - X) = E(X^2) - E(X)$
  - $Var(X) = E(X^2) - (E(X))^2 = G''_X(1) + G'_X(1) - (G'_X(1))^2$

Applying Probability-Generating Function on Poisson Distribution :

- $G_X(t) = \sum_{x=0}^{\infty} t^x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = e^{\lambda(t-1)} \quad \cdots$ (Question 4.133)
- $G'_X(t) = \frac{d}{dt} G_x(t) = \lambda e^{\lambda(t-1)}$
  - $G'_x(1) = \lambda$
- $G''_X(t) = \frac{d^2}{dt^2} G_x(t) = \lambda^2 e^{\lambda(t-1)}$
  - $G''_X(1) = \lambda^2$

$E(X) = G'_X(1) = \lambda$

$Var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda$

Question 4.136

$X \sim Binom(n,p), \quad P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

From Question 4.134, the Probability-Generating Function of Binomial Distribution is $G_X(t) = (p(t-1)+1)^n$

- $G'_X(t) = \frac{d}{dt} G_X(t) = np(tp+1-p)^{n-1}$
  - $G'_X(1) = np$
- $G''_X(t) = \frac{d^2}{d^2t} G_X(t) = n(n-1)p^2(tp+1-p)^{n-2}$
  - $G''_X(1) = n(n-1)p^2$

$E(X) = G'_X(1) = np$

$Var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$

Question 4.137

$M_X(t) = E(e^{tX})$

$M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E(e^{atX+tb}) = E(e^{atX} \cdot e^{tb}) = e^{tb}E(e^{atX}) = e^{tb}M_X(at)$

Question 4.138

1. Mean

$M_Y(t) = E(e^{tY}) = \sum_{y=0}^{\infty} e^{ty}P(Y=y)$

$\rightarrow M_Y(0) = \sum_{y=0}^{\infty} P(Y=y) = 1$

$M'_Y(t) = \sum_{y=0}^{\infty} ye^{ty}P(Y=y)$

$\rightarrow M'_Y(0) = \sum_{y=0}^{\infty} yP(Y=y) = E(Y) \quad \cdots (1)$

$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}P(X=x)$

$\rightarrow M_X(0) = \sum_{x=0}^{\infty} P(X=x) = 1$

$M'_X(t) = \sum_{x=0}^{\infty} xe^{tx}P(X=x)$

$\rightarrow M'_X(0) = \sum_{x=0}^{\infty} xP(X=x) = E(X) \quad \cdots (2)$

$$\begin{aligned} M_Y(t) &= e^{tb}M_X(at) \quad \cdots \text{(Question 4.137)} \\ M'_Y(t) &= b \cdot e^{tb}M_X(at) + e^{tb}M'_X(at) \cdot a \\ \rightarrow M'_Y(0) &= b \cdot M_X(0) + M'_X(0) \cdot a \\ &= \mathbf{a\,E(X) + b = E(Y)} \end{aligned}$$

2. Variance

$M''_Y(t) = \sum_{y=0}^{\infty} y^2e^{ty}P(Y=y)$

$\rightarrow M''_Y(0) = \sum_{y=0}^{\infty} y^2P(Y=y) = E(Y^2)$

$M''_X(t) = \sum_{x=0}^{\infty} x^2e^{tx}P(X=x)$

$\rightarrow M''_X(0) = \sum_{x=0}^{\infty} x^2P(X=x) = E(X^2)$

$$\begin{aligned} M''_Y(t) &= b(be^{tb}M_X(at) + e^{tb}M'_X(at) \cdot a) + a(be^{tb}M'_X(at) + e^{tb}M''_x(at) \cdot a) \\ &= b^2e^{tb}M_X(at) + 2abe^{tb}M'_X(at) + a^2e^{tb}M''_X(at) \\ \rightarrow M''_Y(0) &= b^2M_X(0) + 2abM'_X(0) + a^2M''_X(0) \\ &= b^2 + 2abE(X) + a^2E(X^2) \quad \cdots (3) \end{aligned}$$

$$
\begin{aligned}
Var(Y) &= E(Y^2) - (E(Y))^2 = M''_Y(0) - (M'_Y(0))^2 \\
&= a^2E(X^2) + 2abE(X) + b^2 - (aE(X) + b)^2 \qquad \cdots (\because)\ E(Y) = M'_Y(0) \\
&= a^2E(X^2) + 2abE(X) + b^2 - a^2(E(X))^2 - b^2 - 2abE(X) \\
&= a^2 \cdot (E(X^2) - (E(X))^2) \\
&= \mathbf{a^2} \cdot \mathbf{Var}(\mathbf{X})
\end{aligned}
$$

Question 5.85

X : Amount spent for maintenance and repairs $\sim N(600, 40^2)$

$$\begin{aligned} P(X \geq 700) &= P(Z \geq \frac{700-600}{40}) \\ &= P(Z \geq 2.5) \\ &= 1 - P(Z \leq 2.5) \\ &= \texttt{1 - pnorm(2.5)} \quad \text{(R Code)} \\ &= \mathbf{0.0062} \end{aligned}$$

Question 5.91

X : Resistance $\sim N(10000, 4000^2)$

$$\begin{aligned} P(8000 \leqslant X \leqslant 15000) &= P(\frac{8000-10000}{4000} \leqslant Z \leqslant \frac{15000-10000}{4000}) \\ &= P(-0.5 \leqslant Z \leqslant 1.25) \\ &= P(Z \leqslant 1.25) - P(Z \leqslant -0.5) \\ &= \texttt{pnorm(1.25) - pnorm(-0.5)} \quad \text{(R Code)} \\ &= \mathbf{0.5858} \end{aligned}$$

Question 5.94

X : Length of Trout $\sim N(22, 4^2)$

a.

$$\begin{aligned} P(14 \leq X \leq 24) &= p(\frac{14-22}{4} \leq Z \leq \frac{24-22}{4}) \\ &= P(-2 \leq Z \leq 0.5) \\ &= \texttt{pnorm(0.5) - pnorm(-2)} \quad \text{(R code)} \\ &= \mathbf{0.6687} \end{aligned}$$

b.

$$P(X \geq a) = 0.05 \Longrightarrow P(Z \geq \frac{a-22}{4}) = 0.05$$

$$\texttt{qnorm(0.95) = 1.645} \quad \text{(R code)}$$

$$\frac{a-22}{4} = 1.645 \Longrightarrow (\therefore) \quad \mathbf{a = 28.58}$$

c. y : # of failures (catching trouts outside legal limits) before the 1st success (catching trouts within legal limits) $\sim Geom(p = 0.6687)$

$$\begin{aligned} P(Y = 3) &= (1 - 0.6687)^3 \cdot 0.6687 \\ &= \mathbf{0.0243} \end{aligned}$$

Question 5.97

X : Filling Cereal Boxes $\sim N(\mu, 1^2)$

a.

$$P(X > 14) = 0.01$$

$$P(X > 14) = P(Z > \frac{14-\mu}{1}) = P(Z > 14-\mu) = 0.01$$
$$\implies 14 - \mu = 2.326 = \texttt{qnorm(0.99)} \quad \text{(R code)}$$

$(\therefore)$ $\mu = \mathbf{11.674}$ ounces

b.

$$\begin{aligned} P(X < 12.8) &= P(Z < \frac{12.8 - 11.674}{1}) \\ &= P(Z < 1.126) \\ &= \mathbf{0.87} \end{aligned}$$

Question 5.105

a.

$$\begin{aligned} \int_0^1 f(x)\,dx = \int_0^1 kx^4(1-x)^2\,dx &= k\int_0^1 x^6 - 2x^5 + x^4\,dx \\ &= k[\frac{1}{7}x^7 - \frac{1}{3}x^6 + \frac{1}{5}x^5]_0^1 \\ &= k(\frac{1}{7} - \frac{1}{3} + \frac{1}{5}) = 1 \end{aligned}$$

$(\therefore)$ $\mathbf{k = 105}$

b.

$$\begin{aligned} \mathbf{E(X)} &= \int_0^1 x \cdot f(x)\,dx = 105\int_0^1 x^5(1-x)^2\,dx = 105\int_0^1 x^7 - 2x^6 + x^5\,dx \\ &= 105[\frac{1}{8}x^8 - \frac{2}{7}x^7 + \frac{1}{6}x^6]_0^1 = \frac{5}{8} = \mathbf{0.625} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 \cdot f(x)\,dx = 105\int_0^1 x^6(1-x)^2\,dx = 105\int_0^1 x^8 - 2x^7 + x^6\,dx \\ &= 105[\frac{1}{9}x^9 - \frac{1}{4}x^8 + \frac{1}{7}x^7]_0^1 = \frac{5}{12} = 0.4167 \end{aligned}$$

$$\mathbf{Var(X)} = E(X^2) - (E(X))^2 = \frac{5}{12} - (\frac{5}{8})^2 = \mathbf{0.026}$$

Question 5.106

a. X is a Beta Distribution with $\alpha = 2, \beta = 2$

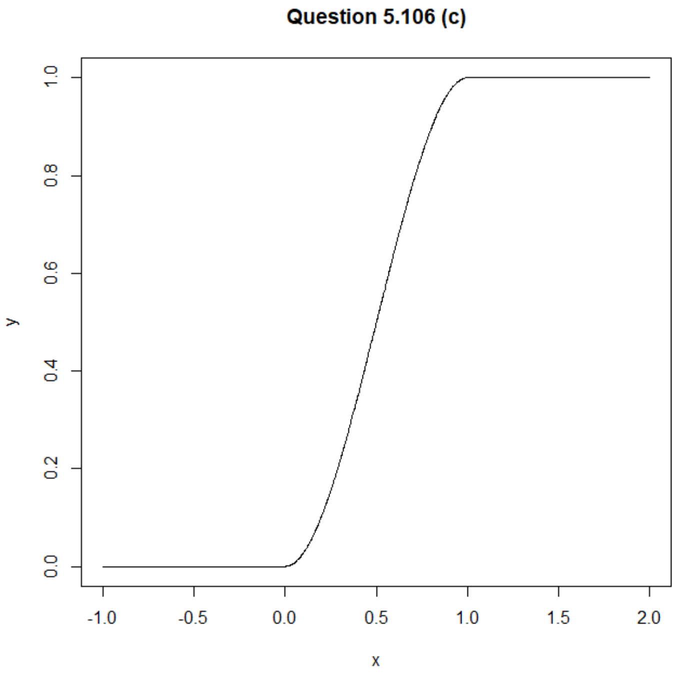
$$\begin{aligned} f_X(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} x^1 \cdot (1-x)^1 \\ &= 6x(1-x) \end{aligned}$$

b. from (a), we can conclude that value of $\mathbf{c = 6}$.

Proof.

$$\int_0^1 f(x)\,dx = c\int_0^1 x(1-x)\,dx = c\cdot[\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1$$
$$= c\cdot(\frac{1}{2}-\frac{1}{3}) = c\cdot\frac{1}{6} = \mathbf{1}$$
$$(\therefore)\ \mathbf{c = 6}$$

c. Distribution Function of X



d. \$75 = 0.75 (with measurements in \$100)

$$P(X \geq 0.75) = \int_{0.75}^1 6x(1-x)\,dx$$
$$= 6\cdot[\frac{1}{2}x^2 - \frac{1}{3}x^3]_{0.75}^1$$
$$= \mathbf{0.1563}$$

Question 5.110

a.

$$E(X) = \frac{\alpha}{\alpha+\beta} = 0.75 \implies \alpha = 3\beta$$
$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3\beta^2}{(4\beta)^2(4\beta+1)} = 0.0375$$

$(\therefore) \alpha = \mathbf{3}, \beta = \mathbf{1}$

b. $X \sim \text{Beta}(\alpha = 3, \beta = 1)$

$$\begin{aligned}
f_X(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1} \\
&= \frac{\Gamma(4)}{\Gamma(3)\Gamma(1)} x^2 \\
&= 3x^2
\end{aligned}$$

$$\begin{aligned}
P(X \geq 0.5) &= \int_{0.5}^{1} 3x^2 \; dx = [x^3]_{0.5}^{1} \\
&= 1 - (0.5)^3 \\
&= \mathbf{0.875}
\end{aligned}$$

$$(\therefore)\ \mathbf{0.875}$$

c.

$$\begin{aligned}
P(0.25 \leq X \leq 0.75) &= \int_{0.25}^{0.75} 3x^2 \; dx \\
&= [x^3]_{0.25}^{0.75} \\
&= \mathbf{0.4063}
\end{aligned}$$

$$(\therefore)\ \mathbf{0.4063}$$

Find the Variance of Beta(α, β)

$$\begin{aligned}
E(X) &= \int_0^1 x \cdot f(x) \; dx \\
&= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha}(1-x)^{\beta-1} \; dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)} \cdot \underbrace{\int \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1} \; dx}_{1} \\
&= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)} \cdot \alpha}{\cancel{\Gamma(\alpha+\beta)}(\alpha+\beta)} \quad (\because)\ \Gamma(x+1) = x \cdot \Gamma(x) \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^2 f(x)\ dx \\
&= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha+1}(1-x)^{\beta-1}\ dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+2+\beta)} \cdot \cancelto{1}{\int \frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{(\alpha+2)-1}(1-x)^{\beta-1}\ dx} \\
&= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{(\alpha+\beta+1)(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}} \quad (\because)\ \Gamma(x+2) = (x+1)\ x\ \Gamma(x) \\
&= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - (E(X))^2 \\
&= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - (\frac{\alpha}{\alpha+\beta})^2 \\
&= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha^3 + \alpha^2\beta + \alpha^2 + \alpha\beta - \alpha^3 - \alpha^2\beta - \alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha\beta}{(\alpha+\beta)^{\mathbf{2}}(\alpha+\beta+\mathbf{1})}
\end{aligned}
$$

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HW6 Solution

January 21, 2025

Question 5.119

X: Fatigue Life $\sim$ Weibull($\gamma = 2, \theta = 4$) $\quad f_X(x) = \frac{\gamma}{\theta} x^{\gamma-1} e^{-\frac{x^\gamma}{\theta}}$

$F_X(x) = P(X \leq x) = \int_0^x \frac{1}{2} t e^{-\frac{t^2}{4}}\ dt = [-e^{-\frac{t^2}{4}}]_0^x = 1 - e^{-\frac{x^2}{4}}, \quad x \geq 0$

a.

$$P(X \leq 2) = F_X(2) = 1 - e^{-1}$$
$$= \mathbf{0.632}$$

b.

$$\begin{aligned}
E(X) &= \int_0^\infty x f(x)\ dx = \int_0^\infty (\frac{\gamma}{\theta}) \cdot x^\gamma e^{-\frac{x^\gamma}{\theta}}\ dx \\
&= \int_0^\infty (\frac{\gamma}{\theta}) \cdot (t\theta) \cdot e^{-t} \cdot (\frac{\theta}{\gamma})(\theta t)^{\frac{1}{\gamma}-1}\ dt \quad (\because)\ t = \frac{x^\gamma}{\theta},\ dx = \frac{\theta}{\gamma}(\theta t)^{\frac{1}{\gamma}-1} dt \\
&= \int_0^\infty (\theta t)^{\frac{1}{\gamma}} e^{-t}\ dt = \theta^{\frac{1}{\gamma}} \int_0^\infty t^{(\frac{1}{\gamma}+1)-1} e^{-t}\ dt \\
&= \theta^{\frac{1}{\gamma}} \cdot \Gamma(\frac{1}{\gamma} + 1)
\end{aligned}$$

$(\therefore)$ $\mathbf{E(X)} = 4^{\frac{1}{2}} \Gamma(\frac{1}{2} + 1) = 2 \times \Gamma(3/2) = 2 \times \frac{1}{2}\sqrt{\pi} = \sqrt{\pi} = \mathbf{1.772}$

Question 5.121

X: time necessary to achieve copper powder $\sim$ Weibull($\gamma = 1.1,\ \theta = 2$)

$$\begin{aligned}
P(X \leq 2) = F_x(2) &= \int_0^2 \frac{1.1}{2} t^{0.1} e^{-\frac{t^{1.1}}{2}}\ dt = [-e^{-\frac{t^{1.1}}{2}}]_0^2 \\
&= 1 - e^{-2^{0.1}} \\
&= \mathbf{0.6575}
\end{aligned}$$

Question 5.137

$X \sim \text{Unif}(a, b)$, $\quad f_x(x) = \frac{1}{b-a} \quad , a \le X \le b$

$$
\begin{aligned}
M_X(t) = E(e^{tx}) &= \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\ dx \\
&= \int_a^b \frac{1}{b-a} e^{tx}\ dx = \frac{1}{b-a}[\frac{1}{t}e^{tx}]_a^b \\
&= \frac{\mathbf{e^{tb} - e^{ta}}}{\mathbf{t(b-a)}}, \quad \mathbf{t \neq 0}
\end{aligned}
$$

Question 5.139

$$
\begin{aligned}
\mathbf{M_x(t)} = E(e^{tx}) &= \int_0^{\sqrt{c}} \frac{2x}{c} \cdot e^{tx}\ dx = \frac{2}{c}\int_0^{\sqrt{c}} xe^{tx}\ dx \\
&= \frac{2}{c} \cdot ([\frac{x}{t}e^{tx}]_0^{\sqrt{c}} - \int_0^{\sqrt{c}} \frac{1}{t}e^{tx}\ dx) \\
&= \frac{2}{c} \cdot (\frac{\sqrt{c}}{t} \cdot e^{t\sqrt{c}} - \frac{1}{t}\int_0^{\sqrt{c}} e^{tx}\ dx) \\
&= \frac{2}{ct^2} \cdot (t\sqrt{c}\ e^{t\sqrt{c}} - e^{t\sqrt{c}} + 1) \\
&= \frac{\mathbf{2}}{\mathbf{ct^2}}(\mathbf{e^{t\sqrt{c}}(t\sqrt{c} - 1) + 1})
\end{aligned}
$$

Question 5.140

$X \sim \text{Gamma}(\alpha, \beta)$, $\quad f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-\frac{x}{\beta}}, \quad x > 0$

$$
\begin{aligned}
\mathbf{M_X(t)} = E(e^{tx}) &= \int_0^{\infty} e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-\frac{x}{\beta}}\ dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^{\infty} x^{\alpha-1}e^{-x(\frac{1}{\beta}-t)}\ dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha)(\frac{\beta}{1-\beta t})^\alpha \cancelto{1}{\int_0^{\infty} \frac{1}{\Gamma(\alpha)(\frac{\beta}{1-t\beta})^\alpha} x^{\alpha-1}e^{-\frac{x}{\frac{\beta}{1-t\beta}}}}\ dx \\
&= (\mathbf{1} - \beta\mathbf{t})^{-\alpha}
\end{aligned}
$$

Question 5.145

Moment-Generating Function: $M_X(t) = E(e^{tx})$

$$
\begin{aligned}
\mathbf{M_Y(t)} &= E(e^{ty}) \\
&= E(e^{t(ax+b)} = E(e^{tax} \cdot e^{tb}) \quad (\because Y = aX + b) \\
&= \mathbf{e^{tb}} \cdot \mathbf{M_X(at)}
\end{aligned}
$$

Question 5.147

X: daily demand by customers $\sim$ Unif(0, 1) $\quad f_X(x) = 1,\ 0 \le X \le 1$
a: amount ordered by grocer

$$
\begin{aligned}
&\text{(i) if} X \ge a \text{ (all groceries have been sold out)} \implies 10a - 6a = 4a \\
&\text{(ii) if} X < a \text{ (couldn't sell all the food)} \implies 10X - 6a
\end{aligned}
$$

$$\text{Daily Profit: } g(x) = \begin{cases} 4a & \text{if } x \geq a, \\ 10X - 6a & \text{if } x < a. \end{cases})$$

$$\begin{aligned} E(g(x)) &= \int_a^1 4a \cdot f_X(x)\ dx + \int_0^a (10x - 6a) \cdot f_X(x)\ dx \\ &= [4ax]_a^1 + [5x^2 - 6ax]_0^a \\ &= (4a - 4a^2) + (5a^2 - 6a^2) = -5a^2 + 4a \end{aligned}$$

$$\frac{d}{da}(-5a^2 + 4a) = -10a + 4 = 0$$
$$(\therefore)\ \mathbf{a = 0.4}$$

Expected daily profit is maximized when 0.4 lbs is ordered

Question 5.148

a. (i) Discrete

$$\text{when x} = 0,\ F(x) \text{ jumps from } 0 \longrightarrow 0.1 \text{ by } 0.1$$
$$\text{when x} = 0.5,\ F(x) \text{ jumps from } 0.35 \longrightarrow 0.5 \text{ by } 0.15$$

$$\text{discrete : } c_1 p_1(x) = \begin{cases} 0.1 & \text{if } x = 0 \\ 0.15 & \text{if } x = 0.5 \end{cases})$$

$$c_1 \cancelto{1}{F_1(0.5)} = 0.25 \implies \mathbf{c_1 = 0.25}$$

$$\text{discrete pmf : } p_1(x) = \begin{cases} 0.4 & \text{if } x = 0 \\ 0.6 & \text{if } x = 0.5 \end{cases})$$

$$\therefore\ \textbf{discrete CMF : } F_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.4 & \text{if } 0 \leq x < 0.5 \\ 1 & \text{if } x \geq 0.5 \end{cases})$$

(ii) Continuous

$c_1 + c_2 = 1 \longrightarrow c_2 = 0.75 = \frac{3}{4}$

$$\text{continuous : } c_2 F_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 0.5 \\ x - 0.25 & \text{if } 0.5 \leq x < 1 \\ 0.75 & \text{if } x \geq 1 \end{cases})$$

$$\therefore\ \textbf{discrete CDF : } F_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{4}{3}x^2 & \text{if } 0 \leq x < 0.5 \\ \frac{4}{3}(x - 0.25) & \text{if } 0.5 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases})$$

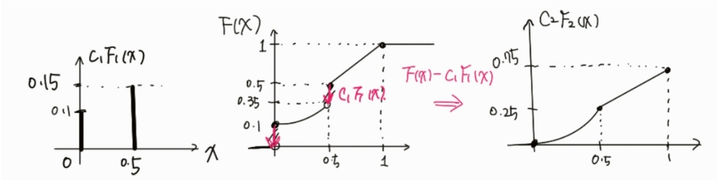

Figure 1: (a) Image Explanation

$$f_2(x) = \begin{cases} \frac{8}{3}x & \text{if } 0 \leq x < 0.5 \\ \frac{4}{3} & \text{if } 0.5 \leq x < 1 \end{cases})$$

b. $\mathbf{0.25 \cdot F_1(x) + 0.75 \cdot F_2(x)}$
c. Sketch of F(x)

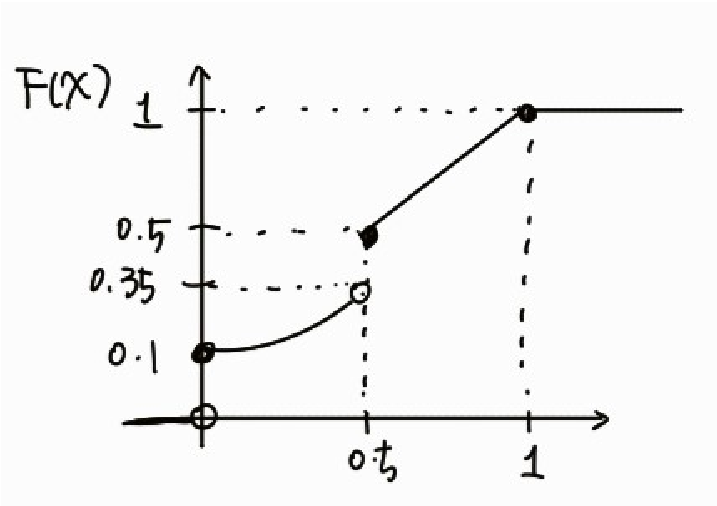

Figure 2: (c) Sketch of F(x)

d.

$$\begin{aligned}
\mathbf{E(X)} &= 0.25E(X_1) + 0.75E(X_2) \\
&= 0.25\sum x \cdot p_1(x) + 0.75\int x \cdot f_2(x) \\
&= 0.25 \cdot (0 \times 0.4 + 0.5 \times 0.6) + 0.75 \cdot (\int_0^{0.5} \frac{8}{3}x^2 \; dx + \int_{0.5}^1 \frac{4}{3}x \; dx) \\
&= 0.075 + 0.75 \cdot ([\frac{8}{9}x^3]_0^{0.5} + [\frac{2}{3}x^2]_{0.5}^1) \\
&= 0.075 + 0.75 \cdot (\frac{1}{9} + \frac{2}{3} - \frac{1}{6}) \\
&= \mathbf{0.5333}
\end{aligned}$$