☹ DOUBLE INTEGRALS ☹

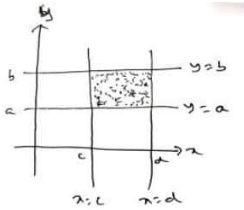
Def: Let f(x, y) be a function defined at each point in a finite region 'R' of xy-plane. Divide 'R' in to 'n' elementary areas $\delta A_1, \delta A_2, \delta A_3, \ldots, \delta A_n$. Let (x_r, y_r) be any point within the r^{th} elementary area δA_r.

Consider the sum $\sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r$. The limit of this sum, if exists, as the number of sub-divisions increases indefinitely and area of each sub division decreases to zero, is defined as the double integral of f(x, y) over the region 'R' and is denoted by

$$\iint_R f(x,y)dA = \iint_R f(x,y)dx\,dy$$

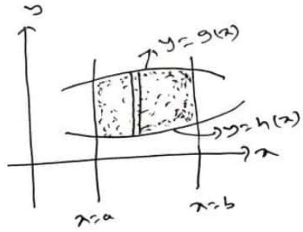
Evaluation Of Double Integrals:

I) $\int_a^b \int_c^d f(x,y)dxdy$ **; where a, b, c, d are constants:**



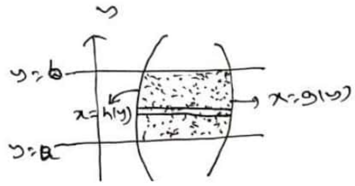
Integrate first w.r.t 'x' keeping 'y' constant between the limits x = c to x = d, and then integrate w.r.t 'y' between the limits y = a to y = b.

II) $\int_a^b \int_{g(x)}^{h(x)} f(x,y)dxdy$ **; where g(x) and h(x) are functions of 'x':**



Integrate first w.r.t 'y' keeping 'x' constant between the limits y = g(x) to y = h(x), and then integrate w.r.t 'x' between the limits x = a to x = b.

III) $\int_a^b \int_{g(y)}^{h(y)} f(x,y)dxdy$ **; where g(y) and h(y) are functions of 'y':**



Integrate first w.r.t 'x' keeping 'y' constant between the limits x = g(y) to x = h(y), and then integrate w.r.t 'y' between the limits y = a to y = b.

PROBLEMS:

1) Evaluate the following integrals

(i) $\int_1^2 \int_1^3 xy^2\, dxdy$

Sol: $\int_1^2 \int_1^3 xy^2\, dxdy = \int_1^2 \left[\int_1^3 xy^2\, dx\right] dy$

$$= \int_1^2 y^2 \left(\frac{x^2}{2}\right)_1^3 dy = \int_1^2 y^2 \left(\frac{9}{2} - \frac{1}{2}\right) dy$$

$$= 4 \int_1^2 y^2\, dy = 4\left(\frac{y^3}{3}\right)_1^2 = 4\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{28}{3}.$$

(ii) $\int_0^a \int_0^b (x^2 + y^2)dydx$

Sol: $\int_0^a \int_0^b (x^2+y^2)dydx = \int_0^a \left[\int_0^b (x^2+y^2)\, dy\right] dx$

$$= \int_0^a \left[x^2 (y)_0^b + \left(\frac{y^3}{3}\right)_0^b\right] dx = \int_0^a \left[x^2(b-0) + \left(\frac{b^3}{3} - 0\right)\right] dx$$

$$= \int_0^a \left[b\, x^2 + \frac{b^3}{3}\right] dx = b\left(\frac{x^3}{3}\right)_0^a + \frac{b^3}{3}(x)_0^a$$

$$= b\left(\frac{a^3}{3} - 0\right) + \frac{b^3}{3}(a-0) = b\frac{a^3}{3} + a\frac{b^3}{3} = \frac{ab}{3}(a^2 + b^2).$$

(iii) $\int_0^5 \int_0^{x^2} x(x^2+y^2)dx\,dy$

Sol: $\int_0^5 \int_0^{x^2} x(x^2+y^2)dxdy = \int_0^5 \left[\int_0^{x^2} (x^3+xy^2)\,dy\right] dx$

$$= \int_0^5 \left[x^3(y)_0^{x^2} + x\left(\frac{y^3}{3}\right)_0^{x^2}\right] dx = \int_0^5 \left[x^3(x^2-0) + x\left(\frac{x^6}{3}-0\right)\right] dx$$

$$= \int_0^5 \left[x^5 + \frac{x^7}{3}\right] dx = \left(\frac{x^6}{6}\right)_0^5 + \left(\frac{x^8}{8.3}\right)_0^5$$

$$= \frac{5^6}{6} - 0 + \frac{5^8}{24} - 0 = 5^6\left(\frac{1}{6} + \frac{25}{24}\right).$$

(iv) $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$

Sol: $\int_0^4 \int_0^{x^2} e^{y/x}\, dydx = \int_0^4 \left[\int_0^{x^2} e^{y/x}\, dy\right] dx$

$$= \int_0^4 \left(\frac{e^{y/x}}{1/x}\right)_0^{x^2} dx = \int_0^4 x\left(e^{x^2/x} - e^0\right) dx$$

$$= \int_0^4 (x\,e^x - x)dx = \left(x\,e^x - e^x - \frac{x^2}{2}\right)_0^4$$

$$= 4\,e^4 - e^4 - \frac{16}{2} - (0 - e^0 - 0) = 3e^4 - 7.$$

(v) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\, dy\right] dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right)_0^{\sqrt{1+x^2}} dx = \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1} 0\right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}(\tan^{-1} 1 - 0)dx = \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} dx$$

$$= \frac{\pi}{4}\left(\log[x + \sqrt{1+x^2}]\right)_0^1 = \frac{\pi}{4}\left(\log[1+\sqrt{1+1}] - \log[0+\sqrt{1+0}]\right)$$

$$= \frac{\pi}{4}\left(\log[1+\sqrt{2}]\right).$$

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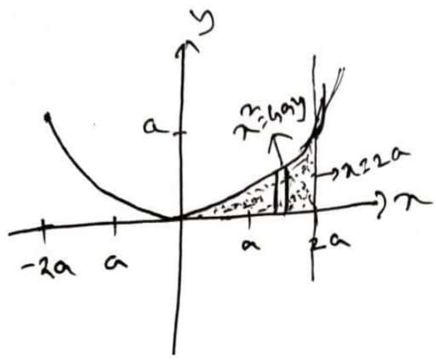
2) Evaluate $\iint_R xy\, dx\, dy$, where R is the domain bounded by x-axis, ordinate x = 2a and the curve $x^2 = 4ay$.

Sol: Consider $x^2 = 4ay$

x	0	2a	-2a
y	0	a	a



∴ 'x' limits are : $x = 0$ to $x = 2a$ (constants)

'y' limits are : $y = 0$ to $y = x^2/4a$. (in terms of 'x')

$$\therefore \iint_R xy\, dx\, dy = \int_0^{2a} \left[\int_0^{x^2/4a} xy\, dy\right] dx$$

$$= \int_0^{2a} x\left(\frac{y^2}{2}\right)_0^{x^2/4a} dx = \int_0^{2a} \frac{x}{2}\left[\left(\frac{x^2}{4a}\right)^2 - 0\right] dx$$

$$= \frac{1}{32\,a^2}\int_0^{2a} x^5 dx = \frac{1}{32\,a^2}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{1}{192\,a^2}(64\,a^6) = \frac{a^4}{3}.$$

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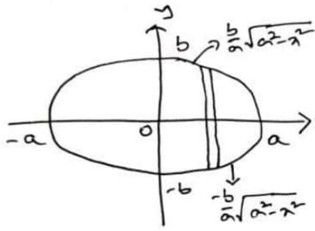
3) Evaluate $\iint_R (x+y)^2\, dx\, dy$, over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol: 'x' limits are : x = -a to x = a

(constants)

'y' limits are : $y = -\frac{b}{a}\sqrt{a^2 - x^2}$ to $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

(in terms of 'x')



$$\therefore \iint_R (x+y)^2\ dx\,dy = \int_{-a}^{a}\left[\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\ dy\right]dx$$

$$= \int_{-a}^{a}\left[\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\ dy\right]dx$$

$$= \int_{-a}^{a}\left[\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\ dy + \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} 2xy\ dy\right]dx$$

$$= 2\int_{-a}^{a}\left[\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\ dy\right]dx \quad \{\because x^2+y^2 \text{ is even and } 2xy \text{ is odd }\}$$

$$= 2\int_{-a}^{a}\left[\left(x^2y+\frac{y^3}{3}\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}}\right]dx$$

$$= 2\int_{-a}^{a}\left(x^2\left[\frac{b}{a}\sqrt{a^2-x^2}-0\right]+\frac{1}{3}\left[\frac{b^3}{a^3}(a^2-x^2)^{\frac{3}{2}}-0\right]\right)dx$$

$$= 2\int_{-a}^{a}\left(\frac{b}{a}\,x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{\frac{3}{2}}\right)dx$$

$$= 4\int_{0}^{a}\left(\frac{b}{a}\,x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{\frac{3}{2}}\right)dx \quad \{\because \text{ integrand is even function }\}$$

Put $x = a\sin\theta \quad \Rightarrow dx = a\cos\theta.\ d\theta$

If $x = 0$, then $\theta = 0$

If $x = a$, then $\theta = \pi/2$

$$= 4\int_{0}^{\pi/2}\left(\frac{b}{a}\,a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}+\frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{\frac{3}{2}}\right)a\cos\theta\ d\theta$$

$$= 4\int_{0}^{\pi/2}\left(b\,a^2\sin^2\theta\ a\cos\theta+\frac{ab^3}{3}\cos^3\theta\right)\ \cos\theta\ d\theta$$

$$= 4\,ba^3\int_{0}^{\pi/2}\sin^2\theta\ \cos^2\theta\ d\theta+\frac{4ab^3}{3}\int_{0}^{\pi/2}\cos^4\theta\ d\theta$$

$$= 4\,ba^3\ \frac{(2-1)(2-1)}{(2+2)(2+2-2)}\,\frac{\pi}{2}+\frac{4ab^3}{3}\ \frac{(4-1)(4-3)}{4(4-2)}\,\frac{\pi}{2}$$

$$= \frac{\pi}{2}\left(\frac{ba^3}{2}+\frac{ab^3}{2}\right) = \frac{\pi}{4}\,ab(\ a^2+b^2\ ).$$

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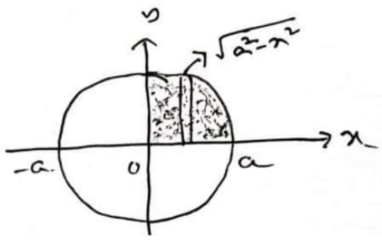
4) Evaluate $\iint_R xy\ dx\,dy$, over the positive quadrant of the circle $x^2 + y^2 = a^2$

Sol: 'x' limits are : x = 0 to x = a

(constants)

'y' limits are : $y = 0$ to $y = \sqrt{a^2 - x^2}$.

(in terms of 'x')



$$\therefore \iint_R xy\ dx\,dy = \int_0^a\left[\int_0^{\sqrt{a^2-x^2}} xy\ dy\right]dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx = \int_0^a \frac{x}{2}\left[\left(\sqrt{a^2-x^2}\right)^2-0\right]dx$$

$$= \frac{1}{2}\int_0^a x(a^2-x^2)dx = \frac{1}{2}\int_0^a (xa^2-x^3)dx$$

$$= \frac{1}{2}\left(a^2\frac{x^2}{2}-\frac{x^4}{4}\right)_0^a = \frac{1}{2}\left(a^2\frac{a^2}{2}-\frac{a^4}{4}-(0-0)\right) = \frac{a^4}{8}.$$

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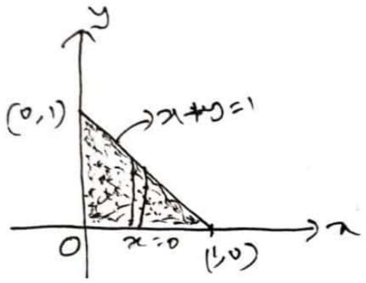
5) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ in the positive quadrant for which $x + y \le 1$.

Sol: Consider $x + y = 1$

x	0	1
y	1	0



$\therefore$ 'x' limits are : $x = 0$ to $x = 1$

(constants)

'y' limits are : $y = 0$ to $y = 1 - x$.

(in terms of 'x')

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_0^{1-x} (x^2+y^2)\,dy\right] dx$$

$$= \int_0^1 \left[\left(x^2y + \frac{y^3}{3}\right)_0^{1-x}\right] dx = \int_0^1 \left[\left(x^2(1-x) + \frac{(1-x)^3}{3} - (0+0)\right)\right] dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3}\right) dx = \left(\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{3(-4)}\right)_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0 - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}.$$

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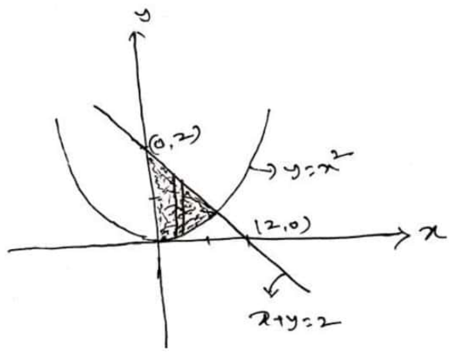
6) Evaluate $\iint_R y\,dx\,dy$, where R is the domain bounded by y-axis, the curve $y = x^2$ and the line $x + y = 2$ in the first quadrant.

Sol: Consider $y = x^2$

x	0	1	-1
y	0	1	1

Consider $x + y = 2$

x	0	2	1
y	2	0	1



$\therefore$ 'x' limits are : $x = 0$ to $x = 1$

(constants)

'y' limits are : $y = x^2$ to $y = 2 - x$.

(in terms of 'x')

$$\therefore \iint_R y\,dx\,dy = \int_0^1 \left[\int_{x^2}^{2-x} y\,dy\right] dx = \int_0^1 \left[\left(\frac{y^2}{2}\right)_{x^2}^{2-x}\right] dx$$

$$= \int_0^1 \left(\frac{(2-x)^2}{2} - \frac{x^4}{2}\right) dx = \left(\frac{(2-x)^3}{2(-3)} - \frac{x^5}{2.5}\right)_0^1$$

$$= -\frac{1}{6} - \frac{1}{10} - \left(-\frac{8}{6} - 0\right) = \frac{16}{15}.$$

Change Of Order Of Integration:

In a double integration with variable limits, the change of order of integration changes the limits of variables.

PROBLEMS:

1) Change the order of the integration and hence evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$.

Sol: 'y' limits are $y = x^2/4a$ to $y = 2\sqrt{ax}$

i.e $x^2 = 4ay$ to $y^2 = 4ax$

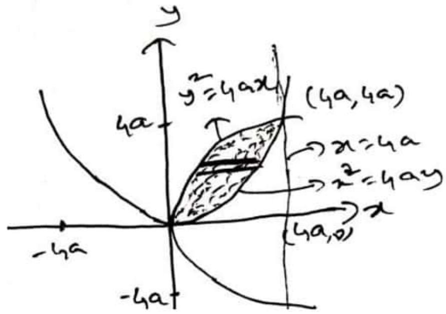
'x' limits are $x = 0$ to $x = 4a$

Consider $x^2 = 4ay$

x	0	4a	-4a
y	0	4a	4a

Consider $y^2 = 4ax$

x	0	4a	4a
y	0	4a	-4a



The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

∴ 'y' limits are : $y = 0$ to $y = 4a$ (constants)

'x' limits are : $x = y^2/4a$ to $x = 2\sqrt{ay}$ (in terms of 'y')

$$\therefore \iint_R dy\,dx = \int_0^{4a}\left[\int_{y^2/4a}^{2\sqrt{ay}} dx\right]dy = \int_0^{4a}\left[(x)_{y^2/4a}^{2\sqrt{ay}}\right]dy$$

$$= \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy = \int_0^{4a}\left[2\sqrt{a}\,y^{1/2} - \frac{y^2}{4a}\right]dy$$

$$= \left(2\sqrt{a}\,\frac{y^{\left(\frac{1}{2}+1\right)}}{\frac{1}{2}+1} - \frac{y^3}{4a.3}\right)_0^{4a} = \left(\frac{4}{3}\sqrt{a}\,y^{\frac{3}{2}} - \frac{y^3}{12a}\right)_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12a} - (0-0) = \frac{4}{3}\,8\,a^{\frac{1}{2}}\,a^{\frac{3}{2}} - \frac{64\,a^3}{12a}$$

$$= \frac{32}{3}a^2 - \frac{64}{12}a^2 = \frac{16}{3}a^2.$$

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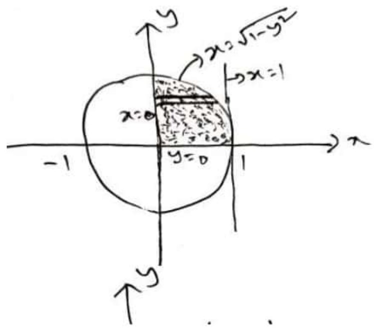
2) Evaluate the integral by changing the order of integration $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$.

Sol: 'y' limits are $y = 0$ to $y = \sqrt{1-x^2}$

i.e $y = 0$ to $y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$

'x' limits are $x = 0$ to $x = 1$



The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

∴ 'y' limits are : $y = 0$ to $y = 1$ (constants)

'x' limits are : $x = 0$ to $x = \sqrt{1-y^2}$ (in terms of 'y')

$$\therefore \iint_R y^2\,dx\,dy = \int_0^1\left[\int_0^{\sqrt{1-y^2}} y^2\,dx\right]dy = \int_0^1 y^2\left[(x)_0^{\sqrt{1-y^2}}\right]dy$$

$$= \int_0^1 y^2\left[\sqrt{1-y^2} - 0\right]dy$$

$$= \int_0^1 y^2\sqrt{1-y^2}\,dy$$

put $y = \sin t \Rightarrow dy = \cos t.\,dt$

If $y = 0 \Rightarrow t = 0$; if $y = 1 \Rightarrow t = \pi/2$

$$= \int_0^{\pi/2} \sin^2 t\,\sqrt{1-\sin^2 t}\,\cos t.\,dt$$

$$= \int_0^{\pi/2} \sin^2 t\,\cos^2 t\,dt = \frac{(2-1)(2-1)}{(2+2)(2+2-2)}\frac{\pi}{2} = \frac{\pi}{16}.$$

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3) Change the order of the integration and hence evaluate $\int_0^3\int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$.

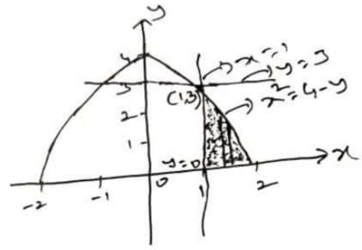
Sol: 'x' limits are $x = 1$ to $x = \sqrt{4-y}$

i.e $x = 1$ to $x^2 = 4 - y$

'y' limits are $y = 0$ to $y = 3$

Consider $x^2 = 4 - y$

x	0	2	-2	1	-1
y	4	0	0	3	3



The change of order of integration changes 'x' limits as constants and 'y' limits in terms of 'x'

∴ 'x' limits are : $x = 1$ to $x = 2$ (constants)

'y' limits are : $y = 0$ to $y = 4 - x^2$ (in terms of 'x')

$$\therefore \iint_R (x+y)\,dx\,dy = \int_1^2 \left[\int_0^{4-x^2} (x+y)\,dy\right] dx = \int_1^2 \left[\left(xy + \frac{y^2}{2}\right)_0^{4-x^2}\right] dx$$

$$= \int_1^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2} - 0\right) dx$$

$$= \int_1^2 \left(4x - x^3 + \frac{16+x^4-8x^2}{2}\right) dx = \int_1^2 \left(4x - x^3 + 8 + \frac{x^4}{2} - 4x^2\right) dx$$

$$= \left(4\,\frac{x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{2.5} - 4\frac{x^3}{3}\right)_1^2$$

$$= 8 - 4 + 16 + \frac{32}{10} - \frac{32}{3} - \left(2 - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3}\right) = \frac{241}{60}.$$

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4) Evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy$, by changing the order of the integration.

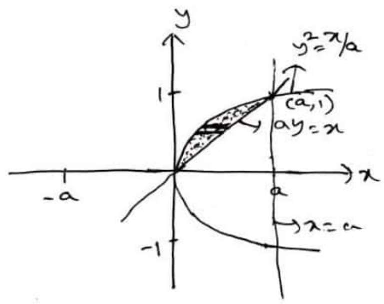
<u>Sol:</u> 'y' limits are $y = x/a$ to $y = \sqrt{x/a}$

i.e $ay = x$ to $y^2 = x/a$

'x' limits are $x = 0$ to $x = a$

Consider $y^2 = x/a$

x	0	a	a
y	0	1	-1



Consider $ay = x$

x	0	a
y	0	1

The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

∴ 'y' limits are : $y = 0$ to $y = 1$ (constants)

'x' limits are : $x = ay^2$ to $x = ay$ (in terms of 'y')

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_{ay^2}^{ay} (x^2+y^2)\,dx\right] dy = \int_0^1 \left[\left(\frac{x^3}{3} + y^2x\right)_{ay^2}^{ay}\right] dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + y^2ay - \left(\frac{a^3y^6}{3} + y^2ay^2\right)\right) dy$$

$$= \int_0^1 \left(\left[\frac{a^3}{3} + a\right] y^3 - \frac{a^3}{3} y^6 - ay^4\right) dy$$

$$= \left(\left[\frac{a^3}{3} + a\right]\frac{y^4}{4} - \frac{a^3}{3}\frac{y^7}{7} - a\frac{y^5}{5}\right)_0^1 = \left[\frac{a^3}{3} + a\right]\frac{1}{4} - \frac{a^3}{3}\frac{1}{7} - a\frac{1}{5} - (0 - 0 - 0)$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a}{5} - \frac{a^3}{21} = a^3\left(\frac{1}{12} - \frac{1}{21}\right) + a\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{a^3}{28} + \frac{a}{20}.$$

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5) Evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dy\,dx$, by changing the order of the integration.

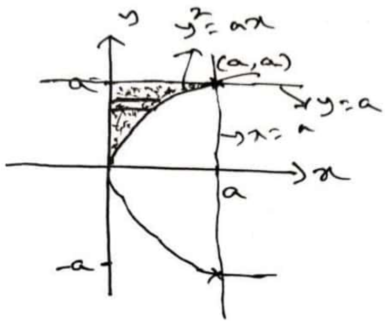
Sol: 'y' limits are $y = \sqrt{ax}$ to $y = a$

i.e $y^2 = ax$ to $y = a$

'x' limits are $x = 0$ to $x = a$

Consider $y^2 = ax$

x	0	a	a
y	0	a	-a



The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

∴ 'y' limits are : $y = 0$ to $y = a$

(constants)

'x' limits are : $x = 0$ to $x = y^2/a$

(in terms of 'y')

$$\therefore \iint_R \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dx\,dy = \int_0^a \left[\int_0^{y^2/a} y^2 \frac{1}{\sqrt{(y^2)^2-(ax)^2}}\,dx\right] dy$$

$$= \int_0^a \left[\int_0^{y^2/a} y^2 \,\frac{1}{a}\frac{a}{\sqrt{(y^2)^2-(ax)^2}}\,dx\right] dy$$

$$= \int_0^a \left[y^2 \frac{1}{a}\left(\sin^{-1}\frac{ax}{y^2}\right)_0^{y^2/a}\right] dy = \int_0^a \left[y^2 \frac{1}{a}(\sin^{-1} 1 - \sin^{-1} 0)\right] dy$$

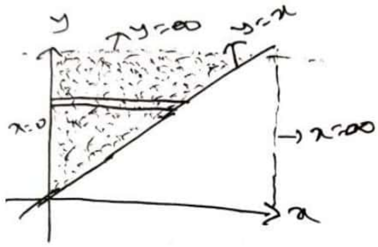
$$= \int_0^a \left[y^2 \frac{1}{a}(\pi/2)\right] dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a = \frac{\pi}{6a}(a^3 - 0) = \frac{\pi}{6}a^2 .$$

----

6) Change the order of integration and hence evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$.

Sol: 'y' limits are $y = x$ to $y = \infty$

'x' limits are $x = 0$ to $x = \infty$



The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

∴ 'y' limits are : $y = 0$ to $y = \infty$

(constants)

'x' limits are : $x = 0$ to $x = y$

(in terms of 'y')

$$\therefore \iint_R \frac{e^{-y}}{y}\,dy\,dx = \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y}\,dx\right] dy = \int_0^\infty \left[\frac{e^{-y}}{y}(x)_0^y\right] dy$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y}(y - 0)\right] dy = \int_0^\infty e^{-y} dy$$

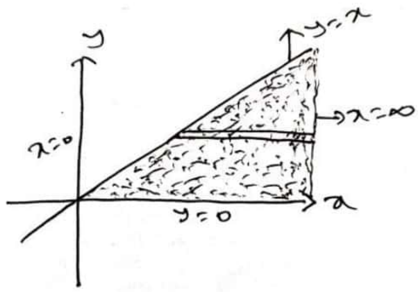
$$= \left(\frac{e^{-y}}{-1}\right)_0^\infty = \frac{e^{-\infty}}{-1} - \frac{e^0}{-1} = 1.$$

----

7) Change the order of integration and hence evaluate $\int_0^\infty \int_0^x x\,e^{-x^2/y}\,dy\,dx$.

Sol: 'y' limits are $y = 0$ to $y = x$

'x' limits are $x = 0$ to $x = \infty$



The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

∴ 'y' limits are : $y = 0$ to $y = \infty$

(constants)

'x' limits are : $x = y$ to $x = \infty$

(in terms of 'y')

$$\therefore \iint_R x\, e^{-x^2/y}\, dy\, dx = \int_0^\infty \left[\int_y^\infty x\, e^{-x^2/y}\, dx\right] dy$$ put $x^2/y = t \Rightarrow x^2 = yt$

$\Rightarrow 2x\, dx = y\, dt$

$$= \int_0^\infty \left[\int_y^\infty e^{-t}\, \frac{y}{2}\, dt\right] dy$$ If $x = y$, then $t = y$

$$= \int_0^\infty \frac{y}{2}\left[\left(\frac{e^{-t}}{-1}\right)_y^\infty\right] dy$$ If $x = \infty$, then $t = \infty$

$$= \int_0^\infty \frac{y}{2}\left[\left(\frac{e^{-\infty}}{-1} - \frac{e^{-y}}{-1}\right)_y^\infty\right] dy = \frac{1}{2}\int_0^\infty y e^{-y} dy$$

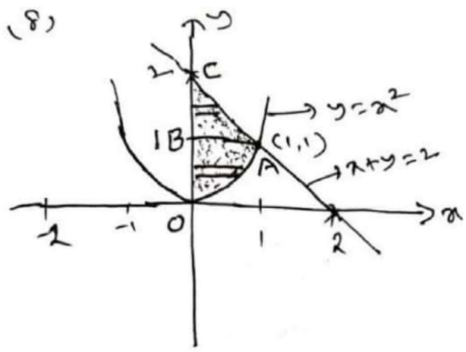
$$= \frac{1}{2}\left(y\frac{e^{-y}}{-1} - \frac{e^{-y}}{(-1)(-1)}\right)_0^\infty = \frac{1}{2}\,(0 - 0 - [0 - 1]) = \frac{1}{2}.$$

----

8) Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$, and hence evaluate the same.

__Sol:__ 'y' limits are $y = x^2$ to $y = 2 - x$ i.e $x + y = 2$

'x' limits are $x = 0$ to $x = 1$



Consider $y = x^2$

x	0	1	-1
y	0	1	1

Consider $x + y = 2$

x	0	2	1
y	2	0	1

The change of order of integration changes 'y' limits as constants and 'x' limits in terms of 'y'

In x-axis direction, one edge of the strip remains on $x = 0$ but the other edge of the strip does not remain on a single curve. Hence split the region as 'OAB' and 'BAC'

(i) over the sub region 'OAB':

'y' limits are : $y = 0$ to $y = 1$

(constants)

'x' limits are : $x = 0$ to $x = \sqrt{y}$

(in terms of 'y')

$$\therefore \iint_{OAB} xy\, dx\, dy = \int_0^1 \left[\int_0^{\sqrt{y}} x\, y\, dx\right] dy = \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy$$

$$= \int_0^1 y\left(\frac{y}{2} - 0\right) dy = \frac{1}{2}\int_0^1 y^2\, dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 = \frac{1}{6}(1 - 0) = \frac{1}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

(ii) over the sub region 'BAC':

'y' limits are : $y = 1$ to $y = 2$

(constants)

'x' limits are : $x = 0$ to $x = 2 - y$

(in terms of 'y')

$$\therefore \iint_{BAC} xy\, dx\, dy = \int_1^2 \left[\int_0^{2-y} x\, y\, dx\right] dy$$

$$= \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy = \frac{1}{2}\int_1^2 y((2 - y)^2 - 0)dy$$

$$= \frac{1}{2}\int_1^2 y(y^2 - 4y + 4)dy = \frac{1}{2}\int_1^2 (y^3 - 4y^2 + 4y)dy$$

$$= \frac{1}{2}\left(\frac{y^4}{4} - 4\frac{y^3}{3} + 4\frac{y^2}{2}\right)_1^2 = \frac{1}{2}\left(\frac{16}{4} - \frac{32}{3} + 8 - [\frac{1}{4} - \frac{4}{3} + 2\right)$$

$$= \frac{1}{2}\left(\frac{15}{4} - \frac{28}{3} + 6\right) = \frac{1}{2}\left(\frac{45-112+72}{12}\right) = \frac{5}{24} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$\therefore \iint_R x\,y\,dx\,dy = \iint_{OAB} xy\,dx\,dy + \iint_{BAC} xy\,dx\,dy$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Double Integrals In Polar Co-Ordinates:

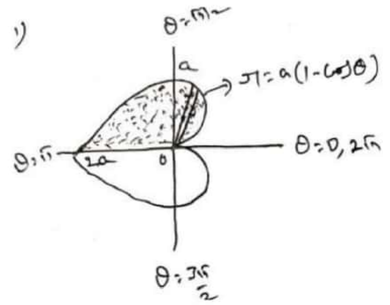
In polar co-ordinates, $x = r\cos\theta$ and $y = r\sin\theta$

PROBLEMS:

1) Evaluate $\iint r\sin\theta\,dr\,d\theta$, over the cardioids $r = a(1-\cos\theta)$ above the initial line.

Sol: Consider $r = a(1-\cos\theta)$

θ	0	$\pi/3$	$\pi/2$	π
r	0	$a/2$	a	2a



The cardioids is symmetrical about the initial line.

'θ' limits are $\theta = 0$ to $\theta = \pi$

'r' limits are $r = 0$ to $r = a(1-\cos\theta)$

$$\therefore \iint r\sin\theta\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a(1-\cos\theta)} r\sin\theta\,dr\right]d\theta$$

$$= \int_0^{\pi}\left[\sin\theta\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)}\right]d\theta = \frac{1}{2}\int_0^{\pi}\left[\sin\theta\left\{\left(a(1-\cos\theta)\right)^2 - 0\right\}\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\sin\theta\,(1-\cos\theta)^2 d\theta \qquad \text{put } 1-\cos\theta = t \Rightarrow \sin\theta\,d\theta = dt$$

$$= \frac{a^2}{2}\int_0^{\pi} t^2 dt$$

$$= \frac{a^2}{2}\left(\frac{t^3}{3}\right)_0^{\pi} = \frac{a^2}{6}\left((1-\cos\theta)^3\right)_0^{\pi}$$

$$= \frac{a^2}{6}\left((1-\cos\pi)^3 - (1-\cos 0)^3\right) = \frac{a^2}{6}(8-0) = \frac{4a^2}{3}.$$

----

2) Evaluate $\iint r^3\,dr\,d\theta$, over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

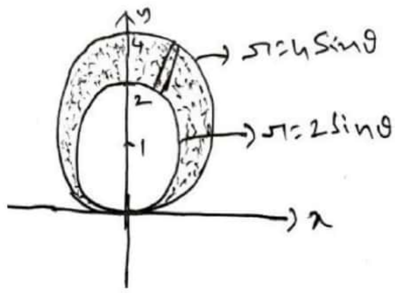
Sol: $r = 2\sin\theta = 2\frac{y}{r}$

$r^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0$, which represents circle with center (0, 1) and radius 1

lly $r = 4\sin\theta$ represents circle with center (0, 2) and radius 2.

$\therefore$ 'θ' limits are $\theta = 0$ to $\theta = \pi$

'r' limits are $r = 2\sin\theta$ to $r = 4\sin\theta$



$$\therefore \iint r^3\,dr\,d\theta = \int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\right]d\theta$$

$$= \int_0^{\pi}\left[\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta}\right]d\theta$$

$$= \frac{1}{4}\int_0^{\pi}(256\sin^4\theta - 16\sin^4\theta)d\theta$$

$$= \frac{240}{4}\int_0^{\pi}(\sin^4\theta)d\theta = 60.\ 2\int_0^{\pi/2}(\sin^4\theta)d\theta$$

$$= 120\,\frac{(4-1)(4-3)}{4(4-2)}\,\frac{\pi}{2} = \frac{45\pi}{2}.$$

Change Of Variables In Double Integrals:

Then $\iint F(x,y)\,dx\,dy = \iint G(u,v)\,J\,du\,dv$

Where $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ which is called the 'Jakobian' of the coordinate transformation.

<u>Change Of Variables From Cartesian To Polar Coordinates:</u>

In this case $x = r\cos\theta,\ y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$\therefore \iint F(x,y)\,dx\,dy = \iint F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

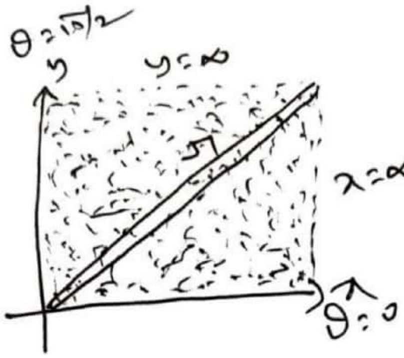
PROBLEMS:

1) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$, by changing into polar coordinates. Hence show that $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

<u>Sol:</u> 'x' limits are $x = 0$ to $x = \infty$

'y' limits are $y = 0$ to $y = \infty$

The region of integration is the first quadrant.



In polar coordinates

Put $x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$

$\therefore$ 'θ' limits are $\theta = 0$ to $\theta = \pi/2$

'r' limits are $r = 0$ to $r = \infty$

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{\infty} e^{-r^2}\,r\,dr\right]d\theta = \frac{-1}{2}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-r^2}\,(-2r\,dr)\right]d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-r^2}\,d(-r^2)\right]d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2}\left[\left(e^{-r^2}\right)_0^{\infty}\right]d\theta = \frac{-1}{2}\int_0^{\pi/2}[(e^{-\infty} - e^0)]d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$\therefore \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$.

Now $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$

$\Rightarrow \int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \frac{\pi}{4} \quad \Rightarrow \int_0^{\infty} e^{-x^2}dx \,.\, \int_0^{\infty} e^{-y^2}dy = \frac{\pi}{4}$

$\Rightarrow \left(\int_0^{\infty} e^{-x^2}dx\right)^2 = \frac{\pi}{4}$

$\Rightarrow \left(\int_0^{\infty} e^{-x^2}dx\right) = \frac{\sqrt{\pi}}{2}$.

----

2) Show that $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$, by changing into polar coordinates.

<u>Sol:</u> 'x' limits are $x = y^2/4a$ to $x = y$

i.e $y^2 = 4ax$ to $x = y$

'y' limits are $y = 0$ to $y = 4a$

Consider $y^2 = 4ax$

x	0	4a	4a
y	0	4a	-4a

Consider $x = y$

x	0	4a	1
y	0	4a	1

In polar coordinates

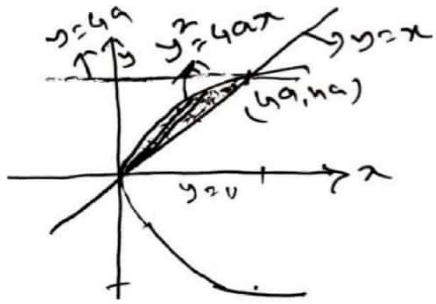
Put $x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$

Now $y^2 = 4ax \Rightarrow r^2\sin^2\theta = 4a\, r\cos\theta$

$\Rightarrow r = 4a\cos\theta/\sin^2\theta$

$\therefore$ 'θ' limits are $\theta = \pi/4$ to $\theta = \pi/2$

'r' limits are $r = 0$ to $r = 4a\cos\theta/\sin^2\theta$



$$\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}\, r\,dr d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\, r\,dr\right]d\theta$$

$$= \int_{\pi/4}^{\pi/2}(\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \int_{\pi/4}^{\pi/2}(\cos^2\theta - \sin^2\theta)\left(\frac{16a^2\cos^2\theta}{2\sin^4\theta} - 0\right)d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left(\frac{\cos^4\theta}{\sin^4\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right)d\theta$$

$$= 8a^2\left(\int_{\pi/4}^{\pi/2}\cot^4\theta\,d\theta - \int_{\pi/4}^{\pi/2}\cot^2\theta\,d\theta\right)$$

put $\cot\theta = t \Rightarrow -\csc^2\theta\,d\theta = dt$

$\Rightarrow d\theta = \frac{-dt}{\csc^2\theta} = \frac{-dt}{1+\cot^2\theta} = \frac{-dt}{1+t^2}$

if $\theta = \pi/4 \Rightarrow t = 1$

if $\theta = \pi/2 \Rightarrow t = 0$

$$= 8a^2\left(\int_1^0 t^4\,\frac{-dt}{1+t^2} - \int_1^0 t^2\,\frac{-dt}{1+t^2}\right)$$

$$= 8a^2\left(\int_0^1 \frac{(t^2)^2+1-1}{1+t^2}\,dt - \int_0^1 \frac{t^2+1-1}{1+t^2}\,dt\right)$$

$$= 8a^2\left(\int_0^1\left\{\frac{(t^2)^2-1}{1+t^2} + \frac{1}{1+t^2}\right\}dt - \int_0^1\left\{\frac{t^2+1}{1+t^2} - \frac{1}{1+t^2}\right\}dt\right)$$

$$= 8a^2\left(\int_0^1 \frac{(t^2-1)(t^2+1)}{1+t^2}dt + \int_0^1 \frac{1}{1+t^2}dt - \int_0^1 dt + \int_0^1 \frac{1}{1+t^2}dt\right)$$

$$= 8a^2\left(\int_0^1 (t^2-1)dt + 2\int_0^1 \frac{1}{1+t^2}dt - \int_0^1 dt\right)$$

$$= 8a^2\left(\left(\frac{t^3}{3} - t\right)_0^1 + 2(\tan^{-1}t)_0^1 - (t)_0^1\right) = 8a^2\left(\left(\frac{1}{3} - 1\right) + 2\frac{\pi}{4} - 1\right)$$

$$= 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right).$$

----

3) By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ ($b > a$).

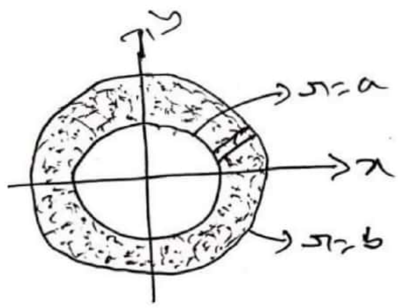
Sol: In polar coordinates, $x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$

$x^2 + y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$

$x^2 + y^2 = b^2 \Rightarrow r^2 = b^2 \Rightarrow r = b$

$\therefore$ 'θ' limits are $\theta = 0$ to $\theta = 2\pi$

'r' limits are $r = a$ to $r = b$



$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_a^b \frac{r^2\cos^2\theta.\, r^2\sin^2\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_a^b (\cos^2\theta.\sin^2\theta)\, r^3\,dr\right]d\theta$$

$$= \int_0^{2\pi}(\cos^2\theta.\sin^2\theta)\left(\frac{r^4}{4}\right)_a^b d\theta$$

$$= \int_0^{2\pi}(\cos^2\theta.\sin^2\theta)\left(\frac{b^4}{4} - \frac{a^4}{4}\right)d\theta$$

$$= \frac{b^4-a^4}{4}\int_0^{2\pi} \cos^2\theta.\sin^2\theta\, d\theta \;=\frac{b^4-a^4}{4.4}\int_0^{2\pi} (2\cos\theta.\sin\theta)^2\, d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi} \sin^2 2\theta\, d\theta = \frac{b^4-a^4}{16}\int_0^{2\pi} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{b^4-a^4}{32}\left(\theta - \frac{\sin 4\theta}{4}\right)_0^{2\pi} = \frac{b^4-a^4}{32}\left(2\pi - \frac{\sin 8\pi}{4} - (0-0)\right) = \frac{\pi(b^4-a^4)}{16}.$$

Area Enclosed By Plane Curves

1) The area enclosed by the curves in Cartesian coordinates is $\iint dx\,dy$

2) The area enclosed by the curves in polar coordinates is $\iint r\,dr\,d\theta$

PROBLEMS:

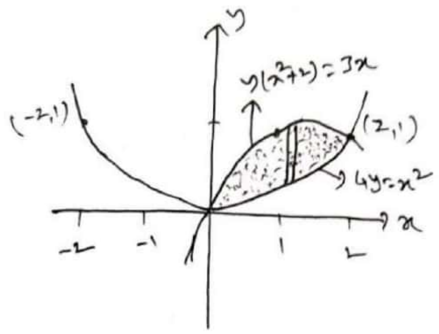
1) Find by double integration the area enclosed by the curves $y = \frac{3x}{x^2+2}$ and $4y = x^2$.

Sol: Consider $y = \frac{3x}{x^2+2} \Rightarrow y(x^2+2) = 3x$

x	0	1	2
y	0	1	1

Consider $4y = x^2$

x	0	2	-2	1
y	0	1	1	1/4



∴ 'x' limits are : $x = 0$ to $x = 2$ (constants)

'y' limits are : $y = x^2/4$ to $y = \frac{3x}{x^2+2}$ (in terms of 'x')

$$\therefore \text{Area} = \iint dx\,dy = \int_0^2 \left[\int_{x^2/4}^{\frac{3x}{x^2+2}} dy\right] dx$$

$$= \int_0^2 (y)_{x^2/4}^{\frac{3x}{x^2+2}} dx = \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4}\right) dx$$

$$= \int_0^2 \left(\frac{3}{2}\frac{2x}{x^2+2} - \frac{x^2}{4}\right) dx = \left(\frac{3}{2}\log(x^2+2) - \frac{x^3}{4.3}\right)_0^2$$

$$= \left(\frac{3}{2}\log(6) - \frac{8}{4.3} - \{\frac{3}{2}\log(2) - 0\}\right) = \frac{3}{2}\log 3 - \frac{2}{3}.$$

----

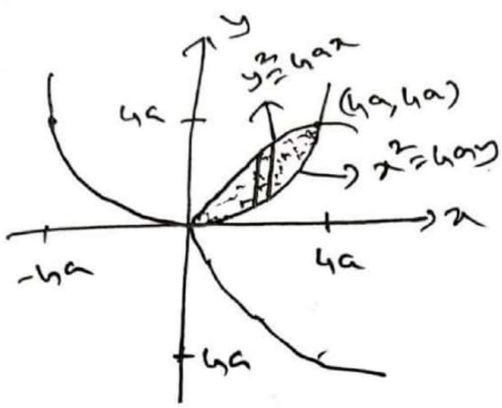
2) Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Sol: Consider $y^2 = 4ax$

x	0	4a	4a
y	0	4a	-4a

Consider $x^2 = 4ay$

x	0	4a	-4a
y	0	4a	4a



∴ 'x' limits are : $x = 0$ to $x = 4a$ (constants)

'y' limits are : $y = x^2/4a$ to $y = \sqrt{4ax}$ (in terms of 'x')

$$\therefore \text{Area} = \iint dx\,dy = \int_0^{4a} \left[\int_{x^2/4a}^{\sqrt{4ax}} dy\right] dx$$

$$= \int_0^{4a} (y)_{x^2/4a}^{\sqrt{4ax}} dx = \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx$$

$$= \left(\sqrt{4a}\ \frac{x^{3/2}}{3/2} - \frac{x^3}{4a.3}\right)_0^{4a} = \left((4a)^{1/2}\ \frac{(4a)^{3/2}}{3/2} - \frac{64a^3}{12a} - (0-0)\right)$$

$$= \frac{2}{3}\, 16a^2 - \frac{16}{3}\, a^2 = \frac{16}{3}\, a^2 .$$

----

3) Find the area of the quadrant of the ellipse $4x^2 + 9y^2 = 36$.

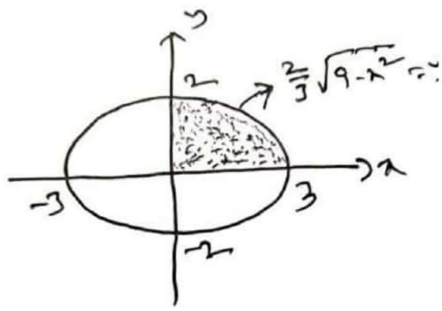
Sol: $4x^2 + 9y^2 = 36 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$

∴ 'x' limits are : $x = 0$ to $x = 3$

(Constants)

'y' limits are : $y = 0$ to $y = \frac{2}{3}\sqrt{9 - x^2}$

(in terms of 'x')



$$\therefore \text{Area} = \iint dx\, dy = \int_0^3 \left[\int_0^{\frac{2}{3}\sqrt{9-x^2}} dy\right] dx$$

$$= \int_0^3 (y)_0^{\frac{2}{3}\sqrt{9-x^2}} dx$$

$$= \int_0^3 \left(\frac{2}{3}\sqrt{9-x^2} - 0\right) dx$$

$$= \frac{2}{3}\left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^3 = \frac{2}{3}\left(0 + \frac{9}{2}\sin^{-1}(1) - (0+0)\right)$$

$$= \frac{2}{3}\,\frac{9}{2}\,\frac{\pi}{2} = \frac{3\pi}{2}.$$

Note: Area of the above ellipse = 4(area of the quadrant of the ellipse)

$$= 4\,\frac{3\pi}{2} = 6\pi .$$

----

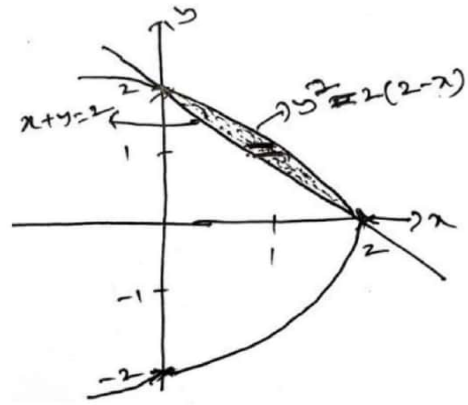
4) Find by double integration the area enclosed by the curves $y = 2 - x$ and $y^2 = 2(2 - x)$.

Sol: Consider $y^2 = 2(2 - x)$

x	0	0	2	1
y	2	-2	0	$\sqrt{2}$

Consider $y = 2 - x$

x	0	2	1
y	2	0	1



∴ 'x' limits are : $x = 0$ to $x = 2$

(Constants)

'y' limits are : $y = 2 - x$ to $y = \sqrt{2(2-x)}$

(in terms of 'x')

$$\therefore \text{Area} = \iint dx\, dy = \int_0^2 \left[\int_{2-x}^{\sqrt{2(2-x)}} dy\right] dx$$

$$= \int_0^2 (y)_{2-x}^{\sqrt{4-2x}} dx = \int_0^2 \left(\sqrt{4-2x} - (2-x)\right) dx$$

$$= \left(\frac{(4-2x)^{3/2}}{\left(\frac{3}{2}\right)(-2)} - \frac{(2-x)^2}{2(-1)}\right)_0^2 = \left(0 - 0 - \left\{\frac{(4)^{\frac{3}{2}}}{-3} - \frac{2^2}{-2}\right\}\right)$$

$$= \frac{8}{3} - 2 = \frac{2}{3}.$$

----

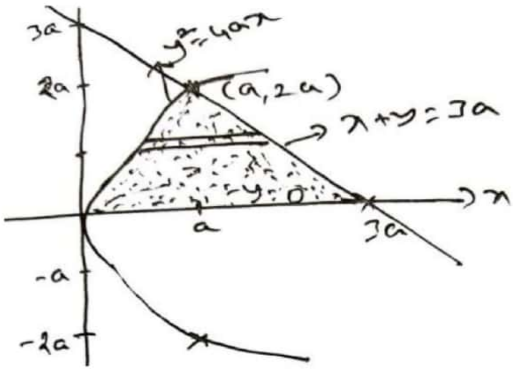
5) Using double integral determine the area of the region bounded by the curves $y^2=4ax$, $x + y = 3a$ and $y = 0$.

Sol: Consider $y^2 = 4ax$

x	0	a	a	1
y	0	2a	-2a	$a\sqrt{2}$

Consider $x + y = 3a$

x	0	3a	a
y	3a	0	2a



∴ 'y' limits are : $y = 0$ to $y = 2a$ (Constants)

'x' limits are : $x = y^2/4a$ to $x = 3a - y$ (in terms of 'y')

$$\therefore \text{Area} = \iint dx\,dy = \int_0^{2a}\left[\int_{y^2/4a}^{3a-y} dx\right]dy$$

$$= \int_0^{2a} (x)_{y^2/4a}^{3a-y} dy = \int_0^{2a}\left(3a - y - \frac{y^2}{4a}\right)dy$$

$$= \left(3ay - \frac{y^2}{2} - \frac{y^3}{3(4a)}\right)_0^{2a} = \left(3a.2a - \frac{4a^2}{2} - \frac{8a^3}{12a} - \{0-0\}\right)$$

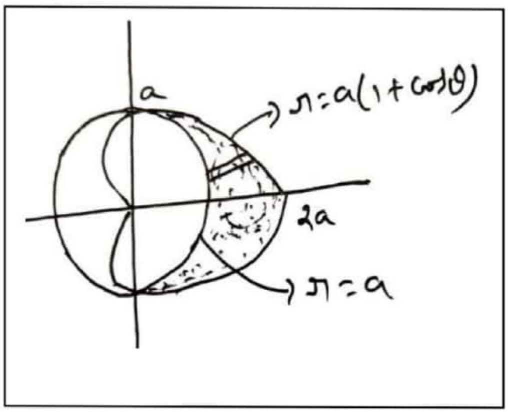
$$= 4a^2 - \frac{2a^2}{3} = \frac{10a^2}{3}.$$

----

6) Find by double integration the area outside the circle 'r = a' and inside the cardioid r = a (1+ cosθ).

Sol: Consider $r = a(1 + \cos\theta)$

θ	0	π/3	π/2	π
r	2a	3a/2	a	0



Consider $r = a \Rightarrow r^2 = a^2$

$\Rightarrow x^2 + y^2 = a^2$, circle with center (0, 0) and radius 'a'

The region is symmetrical about x-axis.

Total area = 2(double of the upper region area)

Now 'θ' limits are $\theta = 0$ to $\theta = \pi$

'r' limits are $r = a$ to $r = a(1 + \cos\theta)$

$$\therefore \text{Area} = \iint r\,dr\,d\theta = 2\int_0^{\pi}\left[\int_a^{a(1+\cos\theta)} r\,dr\right]d\theta$$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta = \int_0^{\pi}\left(a^2(1+\cos\theta)^2 - a^2\right)d\theta$$

$$= a^2\int_0^{\pi}(1 + \cos^2\theta + 2\cos\theta - 1)d\theta$$

$$= a^2\int_0^{\pi}\left(\frac{1+\cos2\theta}{2} + 2\cos\theta\right)d\theta$$

$$= a^2\left(\frac{\theta}{2} + \frac{\sin2\theta}{2.2} + 2\sin\theta\right)_0^{\pi} = a^2\left(\frac{\pi}{2} + 0 + 0 - (0+0)\right) = a^2\frac{\pi}{2}.$$

----

7) Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$.

Sol: $r = a\sin\theta = a\frac{y}{r}$

$r^2 = ay \Rightarrow x^2 + y^2 - ay = 0,$

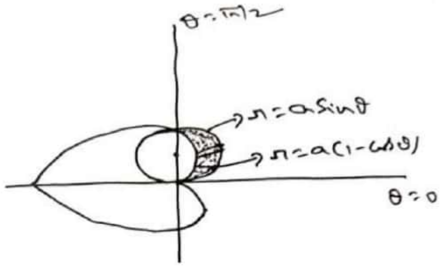
which represents circle with center (0, a/2) and radius a/2

Consider r = a(1 – cosθ)

θ	0	π/3	π/2	π
r	0	a/2	a	2a

Now ‘θ’ limits are θ = 0 to θ = π/2

‘r’ limits are r = a sinθ to r = a(1 – cosθ)



$\therefore$ Area = $\iint$ r dr dθ = $\int_0^{\pi/2}\left[\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\right]d\theta$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(a^2\sin^2\theta - a^2(1-\cos\theta)^2\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(-\cos 2\theta - 1 + 2\cos\theta\right)d\theta$$

$$= \frac{a^2}{2}\left(-\theta - \frac{\sin 2\theta}{2} + 2\sin\theta\right)_0^{\pi/2}$$

$$= \frac{a^2}{2}\left(-\frac{\pi}{2} - \frac{\sin\pi}{2} + 2\sin\frac{\pi}{2} - (-0-0+0)\right) = \frac{a^2}{2}\left(2 - \frac{\pi}{2}\right).$$

----

8) Find by double integration, the area of the lemniscate $r^2 = a^2 \cos 2\theta$.

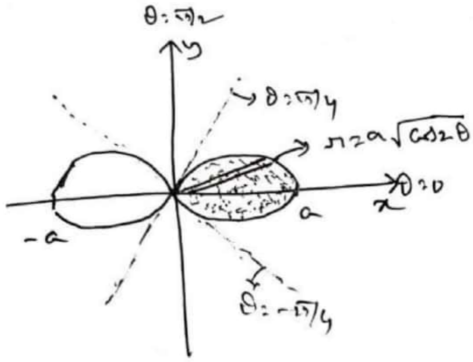
<u>Sol:</u> Consider $r^2 = a^2 \cos 2\theta$

θ	0	0	π/4	-π/4	π/2
r	a	-a	0	0	Not exists

The region is symmetrical about x-axis and y-axis

Now ‘θ’ limits are θ = -π/4 to θ = π/4

‘r’ limits are r = 0 to r = $a\sqrt{\cos 2\theta}$



$\therefore$ Area = $\iint$ r dr dθ = $2\int_{-\pi/4}^{\pi/4}\left[\int_0^{a\sqrt{\cos 2\theta}} r\,dr\right]d\theta$

$$= 2\int_{-\pi/4}^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left(a^2\cos 2\theta - 0\right)d\theta = a^2\left(\frac{\sin 2\theta}{2}\right)_{-\pi/4}^{\pi/4}$$

$$= \frac{a^2}{2}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = a^2.$$

----

9) Calculate the area included between the curve r = a(secθ + cosθ) and its asymptote.

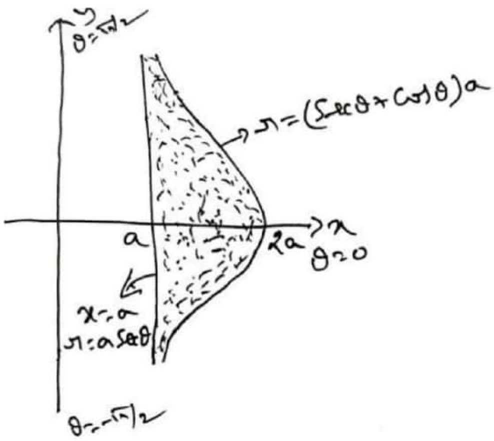
<u>Sol:</u> The curve is symmetrical about x-axia

and has an asymptote r = a secθ

now r = a secθ = $\frac{a}{\cos\theta} = \frac{a}{x/r} = \frac{ra}{x}$

$\Rightarrow$ x = a

Consider r = a(secθ + cosθ)



θ	0	π/2
r	2a	∞

Total area = 2(area of the upper half of the region)

‘θ’ limits are θ = 0 to θ = π/2

‘r’ limits are r = a secθ to r = a(secθ + cosθ)

$\therefore$ Area = 2 $\iint$ r dr dθ = $2\int_0^{\pi/2}\left[\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\right]d\theta$

$$= 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} (a^2(\sec\theta + \cos\theta)^2 - a^2\sec^2\theta)d\theta$$

$$= a^2\int_0^{\pi/2} (\sec^2\theta + \cos^2\theta + 2 - \sec^2\theta)d\theta$$

$$= a^2\int_0^{\pi/2}\left(\frac{1+\cos2\theta}{2} + 2\right)d\theta$$

$$= a^2\left(\frac{\theta}{2} + \frac{\sin2\theta}{2.2} + 2\theta\right)_0^{\pi/2} = a^2\left(\frac{\pi}{4} + 0 + 2\frac{\pi}{2} - (0+0)\right)$$

$$= a^2\left(\frac{\pi}{4} + \pi\right) = \frac{5\pi}{4}a^2.$$

☹ TRIPLE INTEGRALS ☹

Def: Let f(x, y, z) be a function defined at each point of the three dimensional finite region 'V'. Divide 'V' in to 'n' elementary volumes $\delta v_1, \delta v_2, \delta v_3, \ldots, \delta v_n$. Let (x_r, y_r, z_r) be any point within the r^{th} elementary volume δv_r.

Consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta v_r$. The limit of this sum, if exists, as the number of sub-divisions increases indefinitely and volume of each sub division decreases to zero, is defined as the triple integral of f(x, y, z) over the region 'V' and is denoted by

$$\iiint_V f(x,y,z)\,dv = \iiint_V f(x,y,z)\,dx\,dy\,dz .$$

PROBLEMS:

1) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dx\,dy\,dz$.

Sol: $= \int_{-c}^{c}\int_{-b}^{b}\left[\int_{-a}^{a}(x^2+y^2+z^2)dx\right]dy\,dz$

$= \int_{-c}^{c}\int_{-b}^{b}\left[2\int_{0}^{a}(x^2+y^2+z^2)dx\right]dy\,dz$ {∵ integrand is even function in x}

$= 2\int_{-c}^{c}\int_{-b}^{b}\left(\frac{x^3}{3} + (y^2+z^2)x\right)_0^a dy\,dz = 2\int_{-c}^{c}\int_{-b}^{b}\left(\frac{a^3}{3} + (y^2+z^2)a - (0+0)\right)dy\,dz$

$= 2\int_{-c}^{c}\left[\int_{-b}^{b}\left(\frac{a^3}{3} + ay^2 + az^2\right)dy\right]dz$

$= 2\int_{-c}^{c}\left[2\int_{0}^{b}\left(\frac{a^3}{3} + ay^2 + az^2\right)dy\right]dz$ {∵ integrand is even function in y}

$= 4\int_{-c}^{c}\left(\frac{a^3}{3}y + a\frac{y^3}{3} + az^2y\right)_0^b dz = 4\int_{-c}^{c}\left(\frac{a^3}{3}b + a\frac{b^3}{3} + az^2b - (0+0)\right)dz$

$= 4\int_{-c}^{c}\left(\frac{a^3}{3}b + a\frac{b^3}{3} + az^2b\right)dz$

$= 4.\,2\int_{0}^{c}\left(\frac{a^3}{3}b + a\frac{b^3}{3} + az^2b\right)dz$ {∵ integrand is even function in z}

$= 8\left(\frac{a^3b}{3}z + \frac{ab^3}{3}z + ab\frac{z^3}{3}\right)_0^c = 8\left(\frac{a^3b}{3}c + \frac{ab^3}{3}c + ab\frac{c^3}{3} - (0+0)\right)$

$= \frac{8abc}{3}(a^2+b^2+c^2).$

----

2) Evaluate $\int_0^1\int_1^2\int_2^3 xyz\,dz\,dy\,dx$.

Sol: $= \int_0^1\int_1^2\left[\int_2^3 xyz\,dz\right]dy\,dx = \int_0^1\int_1^2\left(xy\frac{z^2}{2}\right)_2^3 dy\,dx$

$= \int_0^1\int_1^2\left(xy\left(\frac{9}{2} - \frac{4}{2}\right)\right)dy\,dx$

$= \frac{5}{2}\int_0^1\left[\int_1^2 xy\,dy\right]dx = \frac{5}{2}\int_0^1\left(x\frac{y^2}{2}\right)_1^2 dx$

$$= \frac{5}{2}\int_0^1 \left(x\left(\frac{4}{2}-\frac{1}{2}\right)\right) dx = \frac{15}{4}\int_0^1 x\, dx = \frac{15}{4}\left(\frac{x^2}{2}\right)_0^1 = \frac{15}{4}\left(\frac{1}{2}-0\right) = \frac{15}{8}.$$

----

3) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$.

Sol: $= \int_0^a \int_0^x \left[\int_0^{x+y} e^{x+y+z}\, dz\right] dy\, dx = \int_0^a \int_0^x \left(e^{x+y+z}\right)_0^{x+y} dy\, dx$

$$= \int_0^a \int_0^x \left(e^{x+y+x+y} - e^{x+y+0}\right) dy\, dx$$

$$= \int_0^a \left[\int_0^x \left(e^{2x+2y} - e^{x+y}\right) dy\right] dx$$

$$= \int_0^a \left(\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1}\right)_0^x dx = \int_0^a \left(\frac{e^{2x+2x}}{2} - e^{x+x} - \left[\frac{e^{2x+0}}{2} - e^{x+0}\right]\right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{3\, e^{2x}}{2} + e^x\right) dx$$

$$= \left(\frac{e^{4x}}{2.4} - \frac{3e^{2x}}{2.2} + e^x\right)_0^a = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \left(\frac{1}{8} - \frac{3}{4} + 1\right) = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}.$$

----

4) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$.

Sol: $= \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \log z.1\, dz\right] dx\, dy$

$$= \int_1^e \int_1^{\log y} \left(\log z.z - \int \frac{1}{z}.z\, dz\right)_1^{e^x} dx\, dy = \int_1^e \int_1^{\log y} (z.\log z - z)_1^{e^x}\, dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left(e^x.\log e^x - e^x - [1.\log 1 - 1]\right) dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left(e^x.x - e^x + 1\right) dx\, dy$$

$$= \int_1^e \left[\int_1^{\log y} \left(e^x.x - e^x + 1\right) dx\right] dy = \int_1^e \left(x.e^x - 1.e^x - e^x + x\right)_1^{\log y} dy$$

$$= \int_1^e \left(\log y.e^{\log y} - 2.e^{\log y} + \log y - [1.e - 2e + 1]\right) dy$$

$$= \int_1^e (y.\log y - 2y + \log y + e - 1)\, dy$$

$$= \int_1^e \left((y+1)\log y - 2y + e - 1\right) dy$$

$$= \left(\log y.\left(\frac{y^2}{2} + y\right) - \int \frac{1}{y}.\left(\frac{y^2}{2} + y\right) dy - 2\frac{y^2}{2} + (e-1)y\right)_1^e$$

$$= \left(\log y.\left(\frac{y^2}{2} + y\right) - \int \left(\frac{y}{2} + 1\right) dy - y^2 + (e-1)y\right)_1^e$$

$$= \left(\log y.\left(\frac{y^2}{2} + y\right) - \frac{y^2}{2.2} - y - y^2 + (e-1)y\right)_1^e$$

$$= \log e.\left(\frac{e^2}{2} + e\right) - \frac{e^2}{4} - e - e^2 + (e-1)e - \left[0 - \frac{1}{4} - 1 - 1 + (e-1)1\right]$$

$$= \frac{e^2}{2} + e - \frac{e^2}{4} - e - e^2 + e^2 - e + \frac{13}{4} - e = \frac{e^2}{4} - 2e + \frac{13}{4}.$$

----

5) Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z)\, dx\, dy\, dz$.

Sol: $= \int_{-1}^1 \int_0^z \left[\int_{x-z}^{x+z} (x+y+z) dy\right] dx\, dz$

$$= \int_{-1}^1 \int_0^z \left((x+z)y + \frac{y^2}{2}\right)_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^1 \int_0^z \left((x+z)(x+z) + \frac{(x+z)^2}{2} - \left\{(x+z)(x-z) + \frac{(x-z)^2}{2}\right\}\right) dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left(x^2+z^2+2xz-x^2+z^2+\frac{(x+z)^2-(x-z)^2}{2}\right)dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left(2z^2+2xz+\frac{4xz}{2}\right)dx\,dz = \int_{-1}^{1}\left[\int_{0}^{z}(2z^2+4xz)\,dx\right]dz$$

$$= \int_{-1}^{1}\left(2z^2x+4z\frac{x^2}{2}\right)_0^z dz = \int_{-1}^{1}(2z^2z+2z\,z^2-0)\,dz$$

$$= \int_{-1}^{1}4z^3\,dz = \left(4\frac{z^4}{4}\right)_{-1}^{1} = 1-1=0.$$

6) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\ dx\,dy\,dz$.

Sol: $= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\right]dy\,dx$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left(xy\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\,dx = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}}(xy(1-x^2-y^2)-0)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\left[\int_0^{\sqrt{1-x^2}}(xy-x^3y-xy^3)\,dy\right]dx$$

$$= \frac{1}{2}\int_0^1\left(x\frac{y^2}{2}-x^3\frac{y^2}{2}-x\frac{y^4}{4}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{4}\int_0^1\left(x(1-x^2)-x^3(1-x^2)-x\frac{(1-x^2)^2}{2}-0\right)dx$$

$$= \frac{1}{4}\int_0^1\left(x-x^3-x^3+x^5-x\frac{1+x^4-2x^2}{2}\right)dx = \frac{1}{8}\int_0^1(2x-4x^3+2x^5-x-x^5+2x^3)\,dx$$

$$= \frac{1}{8}\int_0^1(x-2x^3+x^5)\,dx = \frac{1}{8}\left(\frac{x^2}{2}-2\frac{x^4}{4}+\frac{x^6}{6}\right)_0^1 = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}-0\right) = \frac{1}{48}.$$

----

7) Evaluate the triple integral $\iiint xy^2z\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Sol: 'x' limits are : $x = 0$ to $x = a$
(Constants)
'y' limits are : $y = 0$ to $y = \sqrt{a^2-x^2}$
(in terms of 'x')
'z' limits are : $z = 0$ to $z = \sqrt{a^2-x^2-y^2}$
(in terms of 'x, y')

$$\therefore \iiint xy^2z\,dx\,dy\,dz = \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\int_0^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz\right]dy\,dx$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left(xy^2\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}}(xy^2(a^2-x^2-y^2)-0)\,dy\,dx$$

$$= \frac{1}{2}\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}((a^2-x^2)xy^2-xy^4)\,dy\right]dx$$

$$= \frac{1}{2}\int_0^a\left((a^2-x^2)x\frac{y^3}{3}-x\frac{y^5}{5}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a\left((a^2-x^2)x\frac{(a^2-x^2)^{3/2}}{3}-x\frac{(a^2-x^2)^{5/2}}{5}-0\right)dx$$

$$= \frac{1}{2}\int_0^a\left(x\frac{(a^2-x^2)^{5/2}}{3}-x\frac{(a^2-x^2)^{5/2}}{5}\right)dx$$

put $a^2 - x^2 = t$

$$= \frac{1}{2}\int_0^a\left(\frac{1}{3}-\frac{1}{5}\right)x(a^2-x^2)^{5/2}\,dx$$

$-2x\,dx = dt \Rightarrow x\,dx = dt/\text{-}2$

$$= \frac{1}{2}\frac{2}{15}\int_{a^2}^{0} t^{\frac{5}{2}}\left(-\frac{dt}{2}\right) = \frac{1}{30}\int_0^{a^2} t^{\frac{5}{2}}\,dt$$

if $x = 0$, $t = a^2$; if $x = a$, $t = 0$

$$= \frac{1}{30}\left(\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right)_0^{a^2} = \frac{1}{30}\frac{2}{7}\left(x^{\frac{7}{2}}\right)_0^{a^2}$$

$$= \frac{1}{105}(a^7 - 0) = \frac{a^7}{105}.$$

VOLUME AS TRIPLE INTEGRAL:

Volume of a given solid is given by $\iiint dx\,dy\,dz$ with appropriate limits of integration.

----

1) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$

Sol: 'x' limits are : $x = -2$ to $x = 2$

(Constants)

'y' limits are : $y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$

(in terms of 'x')

'z' limits are : $z = 0$ to $z = 4 - y$

$$\therefore \text{Volume} = \iiint dx\,dy\,dz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[\int_0^{4-y} dz\right]dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(z)_0^{4-y}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y-0)\,dy\,dx = \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\right]dx$$

$$= \int_{-2}^{2}\left(4y - \frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$= \int_{-2}^{2}\left(4\sqrt{4-x^2} - \frac{4-x^2}{2} - \{-4\sqrt{4-x^2} - \frac{4-x^2}{2}\}\right)dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx = 8.2\int_0^2\sqrt{4-x^2}\,dx \qquad \{\because \text{integrand is even function}\}$$

$$= 16\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2$$

$$= 16\,(0 + 2\sin^{-1}(1) - [0+0]) = 16(2\,\tfrac{\pi}{2}) = 16\pi.$$

----

2) Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol: The ellipsoid is cut into 8 equal parts by the three coordinate planes

$\therefore$ Volume = 8 (volume of the first octant)

Hence 'x' limits are : $x = 0$ to $x = a$

(Constants)

'y' limits are : $y = 0$ to $y = b\sqrt{1 - \frac{x^2}{a^2}} = p$ where $1 - \frac{x^2}{a^2} = \frac{p^2}{b^2}$

(in terms of 'x')

'z' limits are : $z = 0$ to $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = \frac{c}{b}\sqrt{p^2 - y^2}$

(in terms of 'x, y')

$$\therefore \text{Volume} = 8\iiint dx\,dy\,dz = 8\int_0^a\int_0^p\left[\int_0^{\frac{c}{b}\sqrt{p^2-y^2}} dz\right]dy\,dx$$

$$= 8\int_0^a\int_0^p (z)_0^{\frac{c}{b}\sqrt{p^2-y^2}}\,dy\,dx$$

$$= 8\int_0^a\int_0^p\left(\frac{c}{b}\sqrt{p^2-y^2} - 0\right)dy\,dx = \frac{8c}{b}\int_0^a\left[\int_0^p\sqrt{p^2-y^2}\,dy\right]dx$$

$$= \frac{8c}{b} \int_0^a \left(\frac{y}{2}\sqrt{p^2 - y^2} + \frac{p^2}{2} \sin^{-1}(\frac{y}{p}) \right)_0^p dx$$

$$= \frac{8c}{b} \int_0^a \left(0 + \frac{p^2}{2} \sin^{-1}(1) - [0+0] \right) dx$$

$$= \frac{8c}{b} \int_0^a \left(\frac{p^2}{2}\frac{\pi}{2} \right) dx = \frac{2\pi c}{b} \int_0^a b^2 (1 - \frac{x^2}{a^2}) dx$$

$$= 2\pi bc \left(x - \frac{x^3}{3a^2} \right)_0^a$$

$$= 2\pi bc \left(a - \frac{a^3}{3a^2} - [0-0] \right) = 2\pi bc \left(a - \frac{a}{3} \right) = \frac{4\pi}{3} abc.$$

Note: If a = b = c, then volume of the sphere $x^2 + y^2 + z^2 = a^2$ is $\frac{4\pi}{3} a^3$.

----

3) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol: The required volume is in 8 octants.

Hence 'x' limits are : x = -a to x = a

(Constants)

'y' limits are : $y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$

(in terms of 'x')

'z' limits are : $z = -\sqrt{a^2 - x^2}$ to $z = \sqrt{a^2 - x^2}$

$$\therefore \text{Volume} = \iiint dx\,dy\,dz = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz \right] dy\,dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (z)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2 - x^2} + \sqrt{a^2 - x^2} \right) dy\,dx$$

$$= 2 \int_{-a}^{a} \left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy \right] dx = 2 \int_{-a}^{a} \sqrt{a^2 - x^2} (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= 2 \int_{-a}^{a} \sqrt{a^2 - x^2} \left(\sqrt{a^2 - x^2} + \sqrt{a^2 - x^2} \right) dx$$

$$= 4 \int_{-a}^{a} (a^2 - x^2)\, dx = 4.2 \int_0^a (a^2 - x^2)\, dx$$

$$= 8 \left(a^2 x - \frac{x^3}{3} \right)_0^a = 8 \left(a^3 - \frac{a^3}{3} - 0 \right) = \frac{16\, a^3}{3}.$$

----

4) Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Sol: The volume of the tetrahedron is in the first octant only.

Hence 'x' limits are : x = 0 to x = a

(Constants)

'y' limits are : y = 0 to $y = b(1 - \frac{x}{a})$

(in terms of 'x')

'z' limits are : z = 0 to $z = c\,(1 - \frac{x}{a} - \frac{y}{b})$

(in terms of 'x, y')

$$\therefore \text{Volume} = \iiint dx\,dy\,dz = \int_0^a \int_0^{b(1-\frac{x}{a})} \left[\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz \right] dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} (z)_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \left(c\left(1 - \frac{x}{a} - \frac{y}{b}\right) - 0 \right) dy\,dx$$

$$= c\int_0^a \left[\int_0^{b(1-\frac{x}{a})}(1-\frac{x}{a}-\frac{y}{b})\,dy\right]dx$$

$$= c\int_0^a \left(y-\frac{x}{a}y-\frac{y^2}{2b}\right)_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left(b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}-0\right)dx$$

$$= bc\int_0^a \left(1-\frac{x}{a}-\frac{x}{a}+\frac{x^2}{a^2}-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right)dx = bc\int_0^a \left(1-2\frac{x}{a}+\frac{x^2}{a^2}-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right)dx$$

$$= bc\left(x-2\frac{x^2}{2a}+\frac{x^3}{3a^2}-\frac{1}{2}\frac{\left(1-\frac{x}{a}\right)^3}{3(-\frac{1}{a})}\right)_0^a$$

$$= bc\left(a-a+\frac{a}{3}-0-\{0-0+\frac{a}{6}1\}\right) = bc\left(\frac{a}{3}-\frac{a}{6}\right) = \frac{abc}{6}.$$

----

1) Evaluate $\int_0^4 \int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx\,dy$.

<u>**Sol:**</u> $\int_0^4 \int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx\,dy = \int_0^4 \left[\int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx\right]dy$

$$= \int_0^4 \left[y\int_{y^2/4}^{y} \frac{1}{y^2+x^2}\,dx\right]dy$$

$$= \int_0^4 y\frac{1}{y}\left(\tan^{-1}\frac{x}{y}\right)_{y^2/4}^{y} dy = \int_0^4 \left(\tan^{-1}(\frac{y}{y})-\tan^{-1}(\frac{y^2/4}{y})\right)dy$$

$$= \int_0^4 \left(\frac{\pi}{4}-\tan^{-1}(\frac{y}{4})\right)dy = \frac{\pi}{4}\int_0^4 dy - \int_0^4 1.\tan^{-1}(\frac{y}{4})\,dy$$

$$= \frac{\pi}{4}(y)_0^4 - \left(\tan^{-1}\frac{y}{4}\,.\,y - \int \frac{1}{1+(\frac{y}{4})^2}\frac{1}{4}.\,y\,dy\right)_0^4$$

$$= \frac{\pi}{4}(4-0) - \left(y.\tan^{-1}\frac{y}{4} - \int \frac{1}{(\frac{16+y^2}{16})}\frac{y}{4}\,dy\right)_0^4$$

$$= \pi - \left(y.\tan^{-1}\frac{y}{4} - \int \frac{4y}{16+y^2}\,dy\right)_0^4$$

$$= \pi - \left(y.\tan^{-1}\frac{y}{4} - 2\int \frac{2y}{16+y^2}\,dy\right)_0^4 = \pi - \left(y.\tan^{-1}\frac{y}{4} - 2\log(16+y^2)\right)_0^4$$

$$= \pi - \left(4.\tan^{-1}\frac{4}{4} - 2\log(16+16) - [0-2\log(16+0)]\right)$$

$$= \pi - (\pi - 2\log(32) + 2\log(16))$$

$$= 2\log(32) - 2\log(16) = 2\log(\frac{32}{16}) = 2\log 2.$$
