☹ VECTOR DIFFERENTIATION ☹

1) Every point (x,y,z) in R^3 can be represented by its ***position vector*** denoted by

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along co-ordinate axes x,y,z respectively.

2) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then its absolute value or '***modulus***' is $r = |\bar{r}| = \sqrt{(x^2 + y^2 + z^2)}$.

3) A '***unit vector***' is a vector whose absolute value is '1'.

4) The unit vector in the direction of a vector $\bar{a}$ is $\hat{n} = \frac{\bar{a}}{|\bar{a}|}$.

5) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ then $\bar{a}.\bar{b} = |\bar{a}|\,|\bar{b}|\cos(\theta)$, where 'θ' is the angle between the vectors, is called '***dot product***' of the vectors.

6) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ then $\bar{a}.\bar{b} = a_1b_1+a_2b_2+a_3b_3$.

<u>Scalor and Vector point functions</u>

1) If for each point 'R' of a region 'E' in space, there corresponds a definite scalar φ(R), then φ(R) is called a scalar point function in 'E'. The region 'E' so defined is called a scalar field.

***<u>Ex</u>*:** The temperature at any instance, Density of a body, Potential due to gravitational matter is examples of scalar point functions.

2) If for each point 'R' of a region 'E' in space, there corresponds a definite vector F(R), then F(R) is called a vector point function in 'E'. The region 'E' so defined is called a vector field.

***<u>Ex</u>*:** The velocity of a moving fluid at any instant is the example of vector point function.

<u>Note</u>: Differentiation of vector point function 'F(x,y,z)' follows the same rules as those of ordinary calculus.

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \quad \text{and} \quad dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz .$$

Level Surface: If a surface φ(x,y,z) = c be drawn through any point 'P' such that at each point on it, function has the same value as at 'P', then such a surface is called a level surface of the function φ through 'P'.

<u>Ex</u>: Equipotential or isothermal surface.

<u>Vector Differential Operator</u> : It is denoted by the symbol '∇' and defined as

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

<u>Gradient of a scalar point function</u>: Let φ(x,y,z) be a scalar point function. Then the gradient of 'φ' is defined as $grad(\varphi) = \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$.

<u>Physical interpretation</u>: grad(φ) is a vector in the direction of normal to the level surface φ(x,y,z) = c and is in increasing direction. Its absolute value is equal to the greatest rate of increase of φ.

<u>PROBLEMS:</u>

1) Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point (-1,-1,2).

<u>Sol:</u> Let $\varphi(x,y,z) = xy^3z^2$

$\frac{\partial\varphi}{\partial x} = y^3z^2.1 = y^3z^2$; $\qquad \frac{\partial\varphi}{\partial y} = xz^2.3y^2 = 3xy^2z^2$; $\qquad \frac{\partial\varphi}{\partial z} = xy^3.2z = 2xy^3z$.

∴ Vector in the direction of normal to the surface φ(x,y,z) = 4 is grad(φ) = ∇φ

$$= \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}\,(y^3z^2) + \bar{j}\,(3xy^2z^2) + \bar{k}\,(2\,xy^3z)$$

At the point (-1,-1,2), $\nabla\varphi = \bar{i}\,((-1)^3 2^2) + \bar{j}\,(3(-1)(-1)^2 2^2) + \bar{k}\,(2(-1)(-1)^3\,2)$

$$\Rightarrow \nabla\varphi = -4\,\bar{i} - 12\,\bar{j} + 4\,\bar{k}.$$

$$\Rightarrow |\nabla\varphi| = \sqrt{((-4)^2 + (-12)^2 + 4^2)} = \sqrt{176}$$

$\therefore$ unit vector normal to the surface $= \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-4\,\bar{i} - 12\,\bar{j} + 4\,\bar{k}}{\sqrt{176}} = \frac{-4(\bar{i} + 3\,\bar{j} - \bar{k})}{\sqrt{11.16}} = \frac{\bar{i} + 3\,\bar{j} - \bar{k}}{-\sqrt{11}}$.

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2) The temperature of points in space is given by T = $x^2 + y^2 - z$. A mosquito located at (1,1,2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

Sol: The rate of change of temperature is maximum in the direction of normal to the surface.

$T = x^2 + y^2 - z \quad \Rightarrow \quad \frac{\partial T}{\partial x} = 2x; \quad \frac{\partial T}{\partial y} = 2y; \quad \frac{\partial T}{\partial z} = -1.$

$\therefore$ Vector in the direction of normal to the surface T(x,y,z) = 0 is grad(T) = ∇T

$$= \bar{i}\frac{\partial T}{\partial x} + \bar{j}\frac{\partial T}{\partial y} + \bar{k}\frac{\partial T}{\partial z} = \bar{i}\,(2x) + \bar{j}\,(2y) + \bar{k}\,(-1)$$

At the point (1,1,2), $\nabla T = \bar{i}\,(2) + \bar{j}\,(2) - \bar{k}$

$\therefore$ The mosquito should move in the direction of the vector $2\,\bar{i} + 2\,\bar{j} - \bar{k}$

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3) Find the angle between the normals to the surface xy = z^2 at the points (4, 1, 2) and (3,3,-3).

Sol: Let $\varphi(x,y,z) = xy - z^2$.

$\frac{\partial\varphi}{\partial x} = y.1 - 0 = y; \quad \frac{\partial\varphi}{\partial y} = x.1 - 0 = x; \quad \frac{\partial\varphi}{\partial z} = 0 - 2z = -2z.$

$\therefore$ Vector in the direction of normal to the surface $\varphi(x,y,z) = 0$ is grad(φ) = $\nabla\varphi$

$$= \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}\,(y) + \bar{j}\,(x) + \bar{k}\,(-2z)$$

At (4,1,2), normal vector $\bar{a} = \bar{i}\,(1) + \bar{j}\,(4) + \bar{k}\,(-2.2) = \bar{i} + 4\,\bar{j} - 4\,\bar{k}$

$$\Rightarrow |\bar{a}| = \sqrt{(1^2 + (4)^2 + (-4)^2)} = \sqrt{33}.$$

At (3,3, –3), normal vector $\bar{b} = \bar{i}\,(3) + \bar{j}\,(3) + \bar{k}\,(-2.-3) = 3\,\bar{i} + 3\,\bar{j} + 6\,\bar{k}$

$$\Rightarrow |\bar{b}| = \sqrt{(3^2 + (3)^2 + (6)^2)} = \sqrt{54}.$$

Let 'θ' be the angle between the normals $\bar{a}$ and $\bar{b}$. Then

$\bar{a}.\bar{b} = |\bar{a}|\,|\bar{b}|\cos(\theta) \qquad \Rightarrow \cos(\theta) = \frac{\bar{a}.\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{1.3 + 4.3 + 6.(-4)}{\sqrt{33}.\sqrt{54}.} = \frac{-9}{\sqrt{11.3}.\sqrt{27.2}} = \frac{-9}{9\sqrt{11}.\sqrt{2}}$

$$\therefore \cos(\theta) = \frac{-1}{\sqrt{22}} \quad \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$$

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4) Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2 - 3$ at (2, –1,2).

Sol: We know that the angle between the surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\varphi = x^2+y^2+z^2 \quad \Rightarrow \frac{\partial\varphi}{\partial x} = 2x+0+0 = 2x; \quad \frac{\partial\varphi}{\partial y} = 0+2y+0 = 2y; \quad \frac{\partial\varphi}{\partial z} = 0 + 0 + 2z = 2z.$

$\therefore$ Vector in the direction of normal to the surface $\varphi(x,y,z) = 9$ is $\bar{a} = \nabla\varphi$

$$= \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}\,(2x) + \bar{j}\,(2y) + \bar{k}\,(2z)$$

At the point (2, –1, 2), $\bar{a} = \bar{i}\,(2.2) + \bar{j}\,(2(-1)) + \bar{k}\,(2.2)$

$\Rightarrow \bar{a} = 4\,\bar{i} - 2\,\bar{j} + 4\,\bar{k}$.

$\Rightarrow |\,\bar{a}\,| = \sqrt{((4)^2 + (-2)^2 + 4^2)} = \sqrt{36} = 6$

Let $\psi = x^2+y^2 - z$ $\quad \Rightarrow \frac{\partial\psi}{\partial x} = 2x+0-0 = 2x; \quad \frac{\partial\psi}{\partial y} = 0+2y-0 = 2y; \quad \frac{\partial\psi}{\partial z} = 0 + 0 - 1 = -1$

$\therefore$ Vector in the direction of normal to the surface $\psi(x,y,z) = 3$ is $\bar{b} = \nabla\psi$

$$= \bar{i}\frac{\partial\psi}{\partial x} + \bar{j}\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z} = \bar{i}\,(2x) + \bar{j}\,(2y) + \bar{k}\,(-1)$$

At the point $(2, -1, 2)$, $\bar{b} = \bar{i}\,(2.2) + \bar{j}\,(2(-1)) + \bar{k}\,(-1)$

$\Rightarrow \bar{b} = 4\,\bar{i} - 2\,\bar{j} - \bar{k}$.

$\Rightarrow |\,\bar{b}\,| = \sqrt{((4)^2 + (-2)^2 + (-1)^2)} = \sqrt{21}$

Let 'θ' be the angle between the normals $\bar{a}$ and $\bar{b}$. Then

$\bar{a}.\,\bar{b} = |\,\bar{a}|\,|\,\bar{b}|\cos(\theta) \qquad \Rightarrow \cos(\theta) = \frac{\bar{a}.\bar{b}}{|\,\bar{a}|\,|\,\bar{b}|} = \frac{4.4+(-2)(-2)+4.(-1)}{6.\sqrt{21}.} = \frac{16}{6.\sqrt{21}} = \frac{8}{3\sqrt{21}}$

$$\therefore \cos(\theta) = \frac{8}{3\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

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5) Find the constants 'a' and 'b' so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1,-1,2).

<u>Sol</u>: We know that the angle between the surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\varphi = ax^2 - byz - (a+2)x$

$\frac{\partial\varphi}{\partial x} = a.2x\text{-}0\text{-}(a+2).1 = 2ax - (a+2); \quad \frac{\partial\varphi}{\partial y} = 0\text{-}\ bz.1\text{-}\ 0 = \text{-}bz; \quad \frac{\partial\varphi}{\partial z} = 0 - by.1 + 0 = \text{-}by.$

$\therefore$ Vector in the direction of normal to the surface $\varphi(x,y,z) = 0$ is $\bar{a} = \nabla\varphi$

$$= \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}\,(2ax - (a+2)) + \bar{j}\,(\text{-}bz) + \bar{k}\,(\text{-}by)$$

At the point $(1, -1, 2)$, $\bar{a} = \bar{i}\,(2a.1 - (a+2)) + \bar{j}\,((-b.2)) + \bar{k}\,(-b.(-1))$

$\Rightarrow \bar{a} = (a-2)\,\bar{i} - 2b\,\bar{j} + b\,\bar{k}$.

Let $\psi = 4x^2y + z^3$

$\frac{\partial\psi}{\partial x} = 4y.2x+0 = 8xy; \qquad \frac{\partial\psi}{\partial y} = 4x^2+ 0 = 4x^2; \qquad \frac{\partial\psi}{\partial z} = 0 + 3z^2 = 3z^2$

$\therefore$ Vector in the direction of normal to the surface $\psi(x,y,z) = 4$ is $\bar{b} = \nabla\psi$

$$= \bar{i}\frac{\partial\psi}{\partial x} + \bar{j}\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z} = \bar{i}\,(8xy) + \bar{j}\,(4x^2) + \bar{k}\,(3z^2)$$

At the point $(1, -1, 2)$, $\bar{b} = \bar{i}\,(8.1.(-1)) + \bar{j}\,(4(1)^2) + \bar{k}\,(3.2^2)$

$\Rightarrow \bar{b} = -8\,\bar{i} + 4\,\bar{j} + 12\bar{k}$.

Since the two surfaces are orthogonal, the angle between them is 90^0 .i.e $\theta = 90^0$

$\therefore\ \bar{a}.\,\bar{b} = |\,\bar{a}|\,|\,\bar{b}|\cos(\theta) = |\,\bar{a}|\,|\,\bar{b}|\cos(90) = 0$

$\Rightarrow (a-2).(-8) + (-2b).4 + b.12 = 0 \ \Rightarrow -8a + 16 + 4b = 0$

$\Rightarrow -2a + b + 4 = 0$…………(1)

Now the two surfaces meet at (1,-1,2)

$\Rightarrow a.1^2 - b(-1).2 = (a+2).1 \ \Rightarrow a + 2b = a + 2 \ \Rightarrow b = 1$

From (1) $-2a + 1 + 4 = 0 \ \Rightarrow a = 5/2 \qquad \therefore a = 5/2,\ b = 1$

6) If the temperature at any point in space is given by t = xy + yz + zx, find the direction in which temperature changes most rapidly from the point (1,1,1) and determine the maximum rate of change.

Sol: The greatest rate of increase of ‘t’ at any point is in the direction of ∇t, and the maximum rate is equal to $|\nabla t|$. Now $t = xy + yz + zx$

$\frac{\partial t}{\partial x} = y.1 + 0 + z.1 = y + z$; $\frac{\partial t}{\partial y} = x.1 + z.1 + 0 = x + z$; $\frac{\partial t}{\partial z} = 0 + y.1 + x.1 = x + y$

$$\nabla t = \bar{i}\frac{\partial t}{\partial x} + \bar{j}\frac{\partial t}{\partial y} + \bar{k}\frac{\partial t}{\partial z} = \bar{i}\,(y + z) + \bar{j}\,(x + z) + \bar{k}\,(x + y)$$

At (1,1,1), $\nabla t = \bar{i}\,(1 + 1) + \bar{j}\,(1 + 1) + \bar{k}\,(1 + 1) = 2\,\bar{i} + 2\,\bar{j} + 2\,\bar{k}$.

$$|\nabla t| = \sqrt{(2^2 + 2^2 + 2^2)} = \sqrt{12} = 2\sqrt{3}.$$

$\therefore$ The temperature changes most rapidly in the direction of the vector $2\,\bar{i} + 2\,\bar{j} + 2\,\bar{k}$ and the greatest rate of increase = $2\sqrt{3}$.

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7) Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at (1,– 2,1).

Sol: We know that the angle between the surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\varphi = xy^2z - 3x - z^2$

$\frac{\partial \varphi}{\partial x} = y^2z.1 - 3.1 - 0 = y^2z - 3$; $\frac{\partial \varphi}{\partial y} = xz.2y - 0 - 0 = 2xyz$; $\frac{\partial \varphi}{\partial z} = xy^2.1 - 0 - 2z = xy^2 - 2z$;

$\therefore$ Vector in the direction of normal to the surface $\varphi(x,y,z) = 0$ is $\bar{a} = \nabla\varphi$

$$= \bar{i}\frac{\partial \varphi}{\partial x} + \bar{j}\frac{\partial \varphi}{\partial y} + \bar{k}\frac{\partial \varphi}{\partial z} = \bar{i}\,(y^2z - 3) + \bar{j}\,(2xyz) + \bar{k}\,(xy^2 - 2z)$$

At the point (1, –2,1), $\bar{a} = \bar{i}\,((-2)^2.1 - 3) + \bar{j}\,((2.1.(-2).1) + \bar{k}\,(1.(-2)^2 - 2.1)$

$\Rightarrow \bar{a} = \bar{i} - 4\,\bar{j} + 2\,\bar{k} \Rightarrow |\,\bar{a}\,| = \sqrt{(1^2 + (-4)^2 + 2^2)} = \sqrt{21}$.

Let $\psi = 3x^2 - y^2 + 2z$

$\frac{\partial \psi}{\partial x} = 3.2x - 0 + 0 = 6x$; $\frac{\partial \psi}{\partial y} = 0 - 2y + 0 = -2y$; $\frac{\partial \psi}{\partial z} = 0 - 0 + 2.1 = 2$

$\therefore$ Vector in the direction of normal to the surface $\psi(x,y,z) = 1$ is $\bar{b} = \nabla\psi$

$$= \bar{i}\frac{\partial \psi}{\partial x} + \bar{j}\frac{\partial \psi}{\partial y} + \bar{k}\frac{\partial \psi}{\partial z} = \bar{i}\,(6x) + \bar{j}\,(-2y) + \bar{k}\,(2)$$

At the point (1, –2, 1), $\bar{b} = \bar{i}\,(6.1.) + \bar{j}\,(-2.-2) + \bar{k}\,(2)$

$\bar{b} = 6\,\bar{i} + 4\,\bar{j} + 2\,\bar{k}$. $\Rightarrow |\,\bar{b}\,| = \sqrt{(6^2 + 4^2 + 2^2)} = \sqrt{56}$

Let ‘θ’ be the angle between the normals $\bar{a}$ and $\bar{b}$. Then

$$\bar{a}.\bar{b} = |\bar{a}|\,|\bar{b}|\cos(\theta) \Rightarrow \cos(\theta) = \frac{\bar{a}.\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{1.6+(-4).4+2.2}{\sqrt{56}\,\sqrt{21}.} = \frac{-3}{7.\sqrt{6}}$$

$$\therefore \cos(\theta) = \frac{-3}{7.\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\frac{-3}{7.\sqrt{6}}\right)$$

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8) Show that grad(r^n) = $\nabla r^n = n\,r^{n-2}R$ where $R = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\,\bar{r}|$

Sol: $r = |\,\bar{r}| = \sqrt{(x^2 + y^2 + z^2)}$ Squaring on both sides

$r^2 = x^2 + y^2 + z^2$ ……………..(1)

Diff. (1) partially w.r.t ‘x’

$2r.\frac{\partial r}{\partial x} = 2x + 0 + 0 = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\nabla r^n = \bar{i}\frac{\partial r^n}{\partial x} + \bar{j}\frac{\partial r^n}{\partial y} + \bar{k}\frac{\partial r^n}{\partial z} = \bar{i}\,(n.\,r^{n-1}.\frac{\partial r}{\partial x}) + \bar{j}\,(n.\,r^{n-1}.\frac{\partial r}{\partial y}) + \bar{k}\,(n.\,r^{n-1}.\frac{\partial r}{\partial z})$

$= n.\,r^{n-1}\,(\bar{i}\,\frac{x}{r} + \bar{j}\,\frac{y}{r} + \bar{k}\,\frac{z}{r}) = \frac{n.r^{n-1}}{r}\,(x\bar{i} + y\bar{j} + z\bar{k})$

$\Rightarrow \nabla r^n = n\, r^{n-2}\,\bar{r}$

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9) Show that grad(f(r)) = $\nabla f(r) = \frac{f'(r)}{r}\,\bar{r}$

Sol: $\nabla f(r) = \bar{i}\frac{\partial f(r))}{\partial x} + \bar{j}\frac{\partial f(r))}{\partial y} + \bar{k}\frac{\partial f(r))}{\partial z} = \bar{i}\,(f'(r).\frac{\partial r}{\partial x}) + \bar{j}\,(f'(r).\frac{\partial r}{\partial y}) + \bar{k}\,(f'(r).\frac{\partial r}{\partial z})$

$= f'(r)\,(\bar{i}\,\frac{x}{r} + \bar{j}\,\frac{y}{r} + \bar{k}\,\frac{z}{r}) = \frac{f'(r)}{r}\,(x\bar{i} + y\bar{j} + z\bar{k})$

$\Rightarrow \nabla f(r) = \frac{f'(r)}{r}\,\bar{r}$

10) If '$\bar{a}$' is a constant vector, then prove that grad($\bar{r}.\bar{a}$) = $\bar{a}$

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ where a_1, a_2, a_3 are constants

$\therefore \bar{r}.\bar{a} = a_1x + a_2y + a_3z$.

$\Rightarrow \frac{d(\bar{r}.\bar{a})}{dx} = a_1.1 + 0 + 0 = a_1$. Similarly $\frac{d(\bar{r}.\bar{a})}{dy} = a_2$ and $\frac{d(\bar{r}.\bar{a})}{dz} = a_3$

$\therefore$ grad($\bar{r}.\bar{a}$) $= \bar{i}\,\frac{\partial(\bar{r}.\bar{a})}{\partial x} + \bar{j}\,\frac{\partial(\bar{r}.\bar{a})}{\partial y} + \bar{k}\,\frac{\partial(\bar{r}.\bar{a})}{\partial z} = \bar{i}\,(a_1) + \bar{j}\,(a_2) + \bar{k}\,(a_3)$

$\Rightarrow$ grad($\bar{r}.\bar{a}$) = $\bar{a}$

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11) Prove that $\nabla\log(r) = \frac{1}{r^2}\,\bar{r}$

Sol: $\nabla\log(r) = \bar{i}\frac{\partial\log(r))}{\partial x} + \bar{j}\frac{\partial\log(r))}{\partial y} + \bar{k}\frac{\partial\log(r))}{\partial z} = \bar{i}\,(\frac{1}{r}.\frac{\partial r}{\partial x}) + \bar{j}\,(\frac{1}{r}.\frac{\partial r}{\partial y}) + \bar{k}\,(\frac{1}{r}.\frac{\partial r}{\partial z})$

$= \frac{1}{r}(\bar{i}\,\frac{x}{r} + \bar{j}\,\frac{y}{r} + \bar{k}\,\frac{z}{r}) = \frac{1}{r^2}\,(x\bar{i} + y\bar{j} + z\bar{k}) = \frac{1}{r^2}\,\bar{r}$.

☹ Directional derivative ☹

Let φ(x,y,z) be a scalar function defined in a region of space. Let 'φ' be the value of this function at a point 'P' whose position vector is '$\bar{r}$'. Let φ + δφ be the value of the above function at neighboring point 'Q' whose position vector is $\bar{r} + \delta\bar{r}$. Then $\overline{PQ} = \delta\bar{r}$. Let δr be the length of $\delta\bar{r}$. $\frac{\delta\varphi}{\delta r}$ gives the rate at which 'φ' changes. As we move from 'P' to 'Q', the limiting value of $\frac{\delta\varphi}{\delta r}$ as δr→0 is called the directional derivative of 'φ' in the direction of $\overline{PQ}$. It is denoted by $\frac{d\varphi}{dr}$, and it is equal to "$\nabla\varphi\,.\,\hat{n}$", where '$\hat{n}$' is unit vector in the direction of $\overline{PQ}$.

Note: 1) $|\nabla\varphi|$ gives the maximum rate of increase of 'φ'.

2) The directional derivative is maximum in the direction of 'grad(φ)' and the maximum value is equal to | grad(φ)|.

PROBLEMS:

1) Find the rate of change of φ = xyz in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1,1,1).

Sol: φ(x,y,z) = xyz $\Rightarrow \frac{\partial\varphi}{\partial x} = yz$; $\frac{\partial\varphi}{\partial y} = xz$; $\frac{\partial\varphi}{\partial z} = xy$

$\therefore\ \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}\,(yz) + \bar{j}\,(xz) + \bar{k}\,(xy)$

At the point (1,1,1), $\nabla\varphi = \bar{i} + \bar{j} + \bar{k}$

Let $\psi = x^2y + y^2x + yz^2$

$\Rightarrow \frac{\partial\psi}{\partial x} = 2xy + y^2$; $\frac{\partial\psi}{\partial y} = x^2 + 2xy + z^2$; $\frac{\partial\psi}{\partial z} = 2yz$

Vector normal to the surface $\psi = 3$ is $\bar{a} = \nabla\psi$

$$= \bar{i}\frac{\partial\psi}{\partial x} + \bar{j}\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z} = \bar{i}\,(2xy + y^2) + \bar{j}\,(x^2 + 2xy + z^2) + \bar{k}\,(2yz)$$

At the point (1,1,1), $\bar{a} = \bar{i}\,(2 + 1) + \bar{j}\,(1 + 2 + 1) + \bar{k}\,(2) = 3\,\bar{i} + 4\,\bar{j} + 2\,\bar{k}$

$$\Rightarrow |\,\bar{a}\,| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$$

$\therefore$ Unit vector in the direction of $\bar{a}$ is $\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{29}}(3\bar{i} + 4\bar{j} + 2\bar{k})$

Rate of change of $\varphi = \nabla\varphi\,.\,\hat{n} = \frac{1}{\sqrt{29}}(1.3 + 1.4 + 1.2) = \frac{9}{\sqrt{29}}$

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2) Find the directional derivative of $\varphi(x,y,z) = xy^2 + yz^3$ at (2,-1,1) in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$.

Sol: $\varphi(x,y,z) = xy^2 + yz^3$

$\Rightarrow \frac{\partial\varphi}{\partial x} = y^2.1 + 0 = y^2$; $\frac{\partial\varphi}{\partial y} = x.2y + z^3.1 = 2xy + z^3$; $\frac{\partial\varphi}{\partial z} = 0 + y.3z^2 = 3yz^2$

$\therefore \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}\,(y^2) + \bar{j}\,(2xy + z^3) + \bar{k}\,(3yz^2)$

At the point (2, –1,1), $\nabla\varphi = \bar{i}\,((-1)^2) + \bar{j}\,((2.2.(-1) + 1^3) + \bar{k}\,(3.(-1)\,1^2)$

$$= \bar{i} - 3\,\bar{j} - 3\,\bar{k}$$

Now $\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k} \Rightarrow |\,\bar{a}\,| = \sqrt{(1^2 + (2)^2 + 2^2)} = \sqrt{9} = 3$

$\therefore$ Unit vector in the direction of $\bar{a}$ is $\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$

Directional derivative $= \nabla\varphi\,.\,\hat{n} = \frac{1}{3}(1.1 + 2(-3) + 2(-3)) = -11/3$

----

3) What is the directional derivative of $\varphi = xy^2 + yz^3$ at the point (2,-1,1) in the direction of the normal to the surface $x\log(z) - y^2 = -4$ at (-1,2,1).

Sol: Let $f = x\log(z) - y^2$

$\Rightarrow \frac{\partial f}{\partial x} = \log(z)\,.1 = \log(z)$; $\frac{\partial f}{\partial y} = 0 - 2y = -2y$; $\frac{\partial f}{\partial z} = x.\frac{1}{z} - 0 = \frac{x}{z}$

$\therefore$ Vector in the direction of normal to the surface $f(x,y,z) = -4$ is $\bar{a} = \nabla f$

$$= \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \bar{i}\,(\log(z)) + \bar{j}\,(-2y) + \bar{k}\,(\tfrac{x}{z})$$

At the point (-1, 2, 1), $\bar{a} = \bar{i}\,(\log(1).) + \bar{j}\,(-2.\,2) + \bar{k}\,(\frac{-1}{1})$

$$\Rightarrow \bar{a} = -4\,\bar{j} - \bar{k} \quad \Rightarrow |\,\bar{a}\,| = \sqrt{(0^2 + (-4)^2 + (-1)^2)} = \sqrt{17}$$

$\therefore$ Unit vector in the direction of $\bar{a}$ is $\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{17}}(-4\,\bar{j} - \bar{k})$

Now $\varphi = xy^2 + yz^3$

$\Rightarrow \nabla\varphi = \bar{i} - 3\,\bar{j} - 3\,\bar{k}$ [$\because$ by problem (1)]

Directional derivative $= \nabla\varphi\,.\,\hat{n} = \frac{1}{\sqrt{17}}(0.1 + (-4)(-3) + (-1)(-3)) = 15/\sqrt{17}$

----

4) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P = (1,2,3) in the direction of the line 'PQ' where Q = (5,0,4).

Sol: $f = x^2 - y^2 + 2z^2$

$\Rightarrow \frac{\partial f}{\partial x} = 2x - 0 + 0 = 2x$; $\frac{\partial f}{\partial y} = 0 - 2y + 0 = -2y$; $\frac{\partial f}{\partial z} = 0 - 0 + 2.2z = 4z$ ∇f

$\therefore \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \bar{i}(2x) + \bar{j}(-2y) + \bar{k}(4z)$

At the point (1, 2, 3), $\nabla f = \bar{i}(2.1) + \bar{j}(-2.2) + \bar{k}(4.3) = 2\bar{i} - 4\bar{j} + 12\bar{k}$.

Now $\overline{PQ} = \overline{OQ} - \overline{OP} = (5\bar{i} + 4\bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k}) = 4\bar{i} - 2\bar{j} + \bar{k}$

$\Rightarrow |\overline{PQ}| = \sqrt{(4^2 + (-2)^2 + 1^2)} = \sqrt{21}$

$\therefore$ Unit vector in the direction of $\overline{PQ}$ is $\hat{n} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k})$

Directional derivative $= \nabla\varphi \,.\, \hat{n} = \frac{1}{\sqrt{21}}(2.4 + (-4)(-2) + (12.1)) = 28/\sqrt{21}$.

----

5) In what direction from (3,1,-2) is the directional derivative of $\varphi = x^2y^2z^4$ maximum. Find also the magnitude of this maximum.

Sol: The directional derivative is maximum in the direction of 'grad(φ)' and the maximum value is equal to |grad(φ)|.

Now $\varphi = x^2y^2z^4$

$\Rightarrow \frac{\partial \varphi}{\partial x} = y^2z^4.2x = 2xy^2z^4$; $\frac{\partial \varphi}{\partial y} = x^2z^4.2y = 2y\,x^2z^4$; $\frac{\partial \varphi}{\partial z} = x^2y^2.4z^3 = 4\,x^2y^2z^3$

$\therefore \nabla\varphi = \bar{i}\frac{\partial \varphi}{\partial x} + \bar{j}\frac{\partial \varphi}{\partial y} + \bar{k}\frac{\partial \varphi}{\partial z} = \bar{i}(2xy^2z^4) + \bar{j}(2y\,x^2z^4) + \bar{k}(4\,x^2y^2z^3)$

At the point (3, 1, –2), $\nabla\varphi = \bar{i}(2.3.1(-2)^4) + \bar{j}((2.1.3^2.(-2)^4) + \bar{k}(4.3^2.1.(-2)^3)$

$= 96(\bar{i} + 3\bar{j} - 3\bar{k})$

$\Rightarrow |\nabla\varphi| = 96\sqrt{(1^2 + (3)^2 + (-3)^2)} = 96\sqrt{19}$

$\therefore$ The directional derivative is maximum in the direction of the vector $96(\bar{i} + 3\bar{j} - 3\bar{k})$ and the maximum value $= 96\sqrt{19}$.

----

6) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point (1,1,1)

Sol: Let $\varphi = xy^2 + yz^2 + zx^2$

$\Rightarrow \frac{\partial \varphi}{\partial x} = y^2.1 + 0 + z.2x = y^2 + 2xz$; $\frac{\partial \varphi}{\partial y} = x.2y + z^2.1 + 0 = 2xy + z^2$;

$\frac{\partial \varphi}{\partial z} = 0 + y.2z + x^2.1 = 2yz + x^2$

$\therefore \nabla\varphi = \bar{i}\frac{\partial \varphi}{\partial x} + \bar{j}\frac{\partial \varphi}{\partial y} + \bar{k}\frac{\partial \varphi}{\partial z} = \bar{i}(y^2 + 2xz) + \bar{j}(2xy + z^2) + \bar{k}(2yz + x^2)$

At the point (1,1, 1), $\nabla\varphi = \bar{i}(1 + 2.1.1) + \bar{j}((2.1.1 + 1) + \bar{k}(2.1.1 + 1) = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let '$\bar{r}$' be the position vector of any point on the curve $x = t, y = t^2, z = t^3$

Then $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$.

We know that $\frac{d\bar{r}}{dt}$ is the vector tangent to the curve.

$\therefore \frac{d\bar{r}}{dt} = 1.\bar{i} + 2t\,\bar{j} + 3t^2\bar{k} = \bar{i} + 2t\,\bar{j} + 3t^2\bar{k}$

At the point (1,1, 1), $\frac{d\bar{r}}{dt} = 1.\bar{i} + 2.1\,\bar{j} + 3.1\bar{k} = \bar{i} + 2\,\bar{j} + 3\,\bar{k}$

Now $\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(1^2 + 2^2 + 3^2)} = \sqrt{14}$

Unit vector along the tangent $\hat{n} = \dfrac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{1}{\sqrt{14}}(\bar{i} + 2\bar{j} + 3\bar{k})$

Directional derivative = $\nabla\varphi \cdot \hat{n} = \frac{1}{\sqrt{14}}(1.3 + 2.3 + 3.3) = 18/\sqrt{14}$.

----

7) Find the directional derivative of $\varphi = xyz^2 + xz$ at the point (1,1,1) in the direction of the normal to the surface $3xy^2 + y = z$ at (0,1,1).

Sol: Let $f = 3xy^2 + y - z$

$\Rightarrow \frac{\partial f}{\partial x} = 3y^2.1+0-0 = 3y^2$; $\frac{\partial f}{\partial y} = 3x.2y + 1 - 0 = 6xy + 1$; $\frac{\partial f}{\partial z} = 0 + 0 - 1 = -1$

$\therefore$ Vector in the direction of normal to the surface $f(x,y,z) = 0$ is $\bar{a} = \nabla f$

$$= \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \bar{i}(3y^2) + \bar{j}(6xy+1) + \bar{k}(-1)$$

At the point (0, 1, 1), $\bar{a} = \bar{i}(3.1) + \bar{j}(0+1) - \bar{k} = 3\bar{i} + \bar{j} - \bar{k}$

$\Rightarrow |\bar{a}| = \sqrt{(3^2 + 1^2 + (-1)^2)} = \sqrt{11}$

$\therefore$ Unit vector in the direction of $\bar{a}$ is $\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{11}}(3\bar{i} + \bar{j} - \bar{k})$

Now $\varphi(x,y,z) = xyz^2 + xz$

$\Rightarrow \frac{\partial\varphi}{\partial x} = yz^2.1 + z.1 = yz^2 + z$; $\frac{\partial\varphi}{\partial y} = xz^2.1 + 0 = xz^2$; $\frac{\partial\varphi}{\partial z} = xy.2z + x.1 = 2xyz + x$

$\therefore \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}(yz^2 + z) + \bar{j}(xz^2) + \bar{k}(2xyz + x)$

At the point (1,1,1), $\nabla\varphi = \bar{i}(1+1) + \bar{j}(1.1) + \bar{k}(2.1.1.1 + 1) = 2\bar{i} + \bar{j} + 3\bar{k}$

Directional derivative = $\nabla\varphi \cdot \hat{n} = \frac{1}{\sqrt{11}}(2.3 + 1.1 + 3.(-1)) = 4/\sqrt{11}$.

----

8) Find the directional derivative of $\varphi(x,y,z) = x^2yz + 4xz^2$ at (1, -2, -1) in the direction of vector $2\bar{i} - \bar{j} - 2\bar{k}$.

Sol: $\varphi(x,y,z) = x^2yz + 4xz^2$

$\Rightarrow \frac{\partial\varphi}{\partial x} = yz.2x + 4z^2.1 = 2xyz + 4z^2$; $\frac{\partial\varphi}{\partial y} = x^2z.1 + 0 = x^2z$; $\frac{\partial\varphi}{\partial z} = x^2y.1 + 4x.2z = x^2y + 8xz$

$\therefore \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y + 8xz)$

At the point (1, –2, –1), $\nabla\varphi = \bar{i}(2.1.(-2).(-1)+4(-1)^2) + \bar{j}(1.(-1)) + \bar{k}(1.(-2) + 8.1.(-1))$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Now $\bar{a} = 2\bar{i} - \bar{j} - 2\bar{k}$ $\Rightarrow |\bar{a}| = \sqrt{(2^2 + (-1)^2 + (-2)^2)} = \sqrt{9} = 3$

$\therefore$ Unit vector in the direction of $\bar{a}$ is $\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{3}(2\bar{i} - \bar{j} - 2\bar{k})$

Directional derivative = $\nabla\varphi \cdot \hat{n} = \frac{1}{3}(8.2 + (-1).(-1) + (-10).(-2)) = 37/3$

----

9) Find the constants a, b, c so that the directional derivative of $f = ax^2y + by^2z + cz^2x$ at (1,1,1) is maximum value 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$

Sol: The directional derivative is maximum in the direction of 'grad(f)' and the maximum value is equal to |grad(f)|.

$f = ax^2y + by^2z + cz^2x$ $\Rightarrow \frac{\partial f}{\partial x} = 2axy + cz^2$; $\frac{\partial f}{\partial y} = ax^2 + 2byz$; $\frac{\partial f}{\partial z} = by^2 + 2czx$

$\therefore\ \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \bar{i}\,(2axy + cz^2) + \bar{j}\,(ax^2 + 2byz) + \bar{k}\,(by^2 + 2czx)$

At the point (1,1,1), $\nabla f = \bar{i}\,(2a + c) + \bar{j}\,(a + 2b) + \bar{k}\,(b + 2c)$

Given that directional derivative is maximum in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$

A vector in the direction of this line is $2\bar{i} - 2\bar{j} + \bar{k}$

$\Rightarrow$ Vector parallel to this line = $2p\,\bar{i} - 2p\,\bar{j} + p\,\bar{k}$, where 'p' is constant

$\therefore \bar{i}\,(2a + c) + \bar{j}\,(a + 2b) + \bar{k}\,(b + 2c) = 2p\,\bar{i} - 2p\,\bar{j} + p\,\bar{k}$

$\Rightarrow 2a + c = 2p$ ………..(1), $a + 2b = -2p$ ……………(2), $b + 2c = p$ ……….(3)

(1) – 2.(3) $\Rightarrow c - 4b = 6p$ ……………(4)

(3) – 2.(4) $\Rightarrow 9b = -11p$ $\Rightarrow b = \frac{-11}{9}p$

Substitute in (2), $a = \frac{4}{9}p$. substitute in (1), $c = \frac{10}{9}p$

Now maximum value of directional derivative = 15

$\Rightarrow \sqrt{(2p)^2 + (-2p)^2 + p^2} = 15$ $\Rightarrow \sqrt{9p^2} = 15$

$\Rightarrow \pm 3p = 15$ $\Rightarrow p = \pm 5$

$\therefore a = \pm\frac{20}{9};\quad b = \mp\frac{55}{9};\quad c = \pm\frac{50}{9}$

☹ DIVERGENT OF A VECTOR FUNCTION ☹

Def: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a continuously differentiable vector point function. Then the Divergent of '$\bar{F}$' is defined as $\text{div}\bar{F} = \nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{F}$

$$= \bar{i}\,.\frac{\partial \bar{F}}{\partial x} + \bar{j}\,.\frac{\partial \bar{F}}{\partial y} + \bar{k}\,.\frac{\partial \bar{F}}{\partial z}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \sum \frac{\partial F_1}{\partial x}$$, which is a scalar.

Physical interpretation:

1) If $\bar{F}$ represents velocity of a fluid at a point, then $\text{div}\bar{F}$ gives the rate at which the fluid is originating at that point per unit volume.

2) If $\bar{F}$ represents heat flux at a point, then $\text{div}\bar{F}$ gives the rate at which heat is issuing from that point per unit volume.

Solenoidal vector: A vector point function $\bar{F}$ is said to be Solenoidal vector if $\text{div}\bar{F} = 0$.

PROBLEMS:

1) Find $\text{div}\bar{F}$ where $\bar{F}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$.

Sol: $\varphi = x^3 + y^3 + z^3 - 3xyz$

$\Rightarrow \frac{\partial \varphi}{\partial x} = 3x^2 - 3yz.1 = 3x^2 - 3yz;$ $\frac{\partial \varphi}{\partial y} = 3y^2 - 3xz.1 = 3y^2 - 3xz;$

$\frac{\partial \varphi}{\partial z} = 3z^2 - 3xy.1 = 3z^2 - 3xy$

$\therefore \text{grad}(\varphi) = \nabla\varphi = \bar{i}\frac{\partial \varphi}{\partial x} + \bar{j}\frac{\partial \varphi}{\partial y} + \bar{k}\frac{\partial \varphi}{\partial z} = \bar{i}\,(3x^2 - 3yz) + \bar{j}\,(3y^2 - 3xz) + \bar{k}\,(3z^2 - 3xy)$

$\Rightarrow \bar{F} = (3x^2 - 3yz)\,\bar{i} + (3y^2 - 3xz)\,\bar{j} + (3z^2 - 3xy)\bar{k}$

Now $\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(3x^2 - 3yz) = 3.2x - 0 = 6x;$ $\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(3y^2 - 3xz) = 3.2y - 0 = 6y;$

$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(3z^2 - 3xy) = 3.2z - 0 = 6z;$

$\therefore \text{div}\overline{F} = \nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x + 6y + 6z = 6(x + y + z).$

----

2) If $\overline{F} = (x + 3y)\,\bar{i} + (y - 2z)\,\bar{j} + (x + pz)\overline{k}$ is solenoidal vector, then find 'p'.

<u>Sol:</u> We know that $\overline{F}$ is Solenoidal vector if $\text{div}\overline{F} = 0$.

Now $\overline{F} = (x + 3y)\,\bar{i} + (y - 2z)\,\bar{j} + (x + pz)\overline{k}$

$\Rightarrow \frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(x + 3y) = 1 + 0 = 1;\quad \frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(y - 2z) = 1 - 0 = 1;\quad \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(x + pz) = 0 + p.1 = p;$

$\text{div}\overline{F} = 0 \Rightarrow \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$

$\Rightarrow 1 + 1 + p = 0 \Rightarrow p = -2$

----

3) Show that $(3y^4 z^2)\,\bar{i} + (z^3 x^2)\,\bar{j} - (3x^2y^2)\overline{k}$ is solenoidal vector

<u>Sol</u>: Let $\overline{F} = (3y^4 z^2)\,\bar{i} + (z^3 x^2)\,\bar{j} - (3x^2y^2)\overline{k}$

$\Rightarrow \frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(3y^4 z^2) = 0\,;\qquad \frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(z^3 x^2) = 0;\qquad \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(-3x^2y^2) = 0;$

$\therefore \text{div}\overline{F} = \nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 0 = 0$

$\Rightarrow \overline{F}$ is solenoidal vector.

----

4) Find $\text{div}\overline{F}$ (or) $\nabla.\overline{F}$ where $\overline{F} = r^n\bar{r}$. Find 'n' if it is solenoidal.

<u>Sol</u>: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\,\bar{r}| = \sqrt{(x^2 + y^2 + z^2)}$ Squaring on both sides

$r^2 = x^2 + y^2 + z^2$ ……………..(1)

Diff. (1) partially w.r.t 'x' $\Rightarrow 2r\frac{\partial r}{\partial x} = 2x + 0 + 0 = 2x \;\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\overline{F} = r^n\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k}) = r^n x\,\bar{i} + r^n y\,\bar{j} + r^n z\,\bar{k}$

$\Rightarrow \frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(r^n x) = r^n.1 + nr^{n-1}\frac{\partial r}{\partial x}.x = r^n + nr^{n-1}\frac{x}{r}.x = r^n + nx^2r^{n-2};$

Similarly $\frac{\partial F_2}{\partial y} = r^n + ny^2r^{n-2};\quad \frac{\partial F_3}{\partial z} = r^n + nz^2r^{n-2}.$

$\therefore \text{div}\overline{F} = \nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = r^n + nx^2r^{n-2} + r^n + ny^2r^{n-2} + r^n + nz^2r^{n-2}$

$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$

$= 3r^n + nr^{n-2}r^2 = 3r^n + nr^n$

$\therefore \text{div}\overline{F} = (n + 3)\,r^n.$

If $\overline{F}$ is solenoidal, then $\text{div}\overline{F} = 0 \Rightarrow (n + 3)\,r^n = 0 \quad \Rightarrow n + 3 = 0 \Rightarrow n = -3$

----

5) Find $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ (or) $\text{div}\left(\frac{\bar{r}}{r^3}\right)$

<u>Sol:</u> $\overline{F} = \frac{\bar{r}}{r^3} = r^{-3}(x\bar{i} + y\bar{j} + z\bar{k}) = r^{-3}x\,\bar{i} + r^{-3}y\,\bar{j} + r^{-3}z\,\bar{k}$

$\Rightarrow \frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(r^{-3}x) = r^{-3}.1 + (-3)r^{-3-1}\frac{\partial r}{\partial x}.x = r^{-3} - 3r^{-4}\frac{x}{r}.x = r^{-3} - 3x^2r^{-5}$

Similarly $\frac{\partial F_2}{\partial y} = r^{-3} - 3y^2r^{-5}$ and $\frac{\partial F_3}{\partial z} = r^{-3} - 3z^2r^{-5}$

$\therefore \text{div}\overline{F} = \nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = r^{-3} - 3x^2r^{-5} + r^{-3} - 3y^2r^{-5} + r^{-3} - 3z^2r^{-5}$

$= 3r^{-3} - 3r^{-5}(x^2 + y^2 + z^2)$

$= 3r^{-3} - 3r^{-5}r^2 = 3r^{-3} - 3r^{-3} = 0$

$\therefore \text{div}\left(\frac{\bar{r}}{r^3}\right) = 0$

Laplacian operator: The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator and $\nabla^2 = 0$ is called Laplace equation.

Now div(grad(φ)) = $\nabla.\nabla\varphi = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(\bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z})$

$$\Rightarrow \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

----

6) Show that $\nabla^2(r^n) = n(n+1)\, r^{n-2}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$.

(or) Div(grad(r^n)) = $n(n+1)\, r^{n-2}$

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}| = \sqrt{(x^2+y^2+z^2)}$ Squaring on both sides

$r^2 = x^2 + y^2 + z^2$ ……………..(1)

Diff. (1) partially w.r.t 'x' $\Rightarrow 2r\frac{\partial r}{\partial x} = 2x + 0 + 0 = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\frac{\partial}{\partial x}(r^n) = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nxr^{n-2}$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(r^n) = n\left[1.r^{n-2} + x.(n-2)\, r^{n-3}\frac{\partial r}{\partial x}\right]$$

$$= n\left[r^{n-2} + x.(n-2)\, r^{n-3}\frac{x}{r}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(r^n) = n[r^{n-2} + x^2.(n-2)\, r^{n-4}]$$

Similarly $\frac{\partial^2}{\partial y^2}(r^n) = n[r^{n-2} + y^2.(n-2)\, r^{n-4}]$ and $\frac{\partial^2}{\partial z^2}(r^n) = n[r^{n-2} + z^2.(n-2)\, r^{n-4}]$

$$\therefore \nabla^2(r^n) = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial^2}{\partial y^2}(r^n) + \frac{\partial^2}{\partial z^2}(r^n)$$

$$= n[r^{n-2} + x^2.(n-2)\, r^{n-4}] + n[r^{n-2} + y^2.(n-2)\, r^{n-4}] + n[r^{n-2} + z^2.(n-2)\, r^{n-4}]$$

$$= n[3r^{n-2} + (n-2)\, r^{n-4}(x^2+y^2+z^2)] = n[3r^{n-2} + (n-2)\, r^{n-4}r^2]$$

$$= n\,[3r^{n-2} + (n-2)\, r^{n-2}]$$

$$= n\, r^{n-2}(3 + n - 2)$$

$$\therefore \nabla^2(r^n) = n(n+1)\, r^{n-2}.$$

----

7) Show that $\nabla^2(f(r)) = \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r}f^1(r)$. where $r = |\bar{r}|$.

Sol: $\frac{\partial}{\partial x}(f(r)) = f^1(r)\frac{\partial r}{\partial x} = f^1(r)\frac{x}{r} = \frac{x.f^1(r)}{r}$

$$\Rightarrow \nabla^2(f(r)) = \sum\left[\frac{\partial^2}{\partial x^2}(f(r))\right] = \sum\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(f(r))\right)\right]$$

$$= \sum\left[\frac{\partial}{\partial x}\left(\frac{x.f^1(r)}{r}\right)\right] = \sum\left[\frac{r.\frac{\partial}{\partial x}\left(x.f^1(r)\right) - x.f^1(r).\frac{\partial r}{\partial x}}{r^2}\right]$$

$$= \sum\left[\frac{r.\left(x.f^{11}(r)\frac{\partial r}{\partial x} + f^1(r)\right) - x.f^1(r).\frac{x}{r}}{r^2}\right] = \sum\left[\frac{r.\left(x.f^{11}(r)\frac{x}{r} + f^1(r)\right) - f^1(r).\frac{x^2}{r}}{r^2}\right]$$

$$= \sum\left[\frac{\left(x^2.f^{11}(r) + r.f^1(r)\right) - f^1(r).\frac{x^2}{r}}{r^2}\right] = \sum\left[\frac{x^2}{r^2}f^{11}(r) + \frac{1}{r}f^1(r) - \frac{x^2}{r^3}f^1(r)\right]$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \sum\frac{1}{r}f^1(r) - \frac{f^1(r)}{r^3}\sum x^2 = \frac{f^{11}(r)}{r^2}r^2 + 3\frac{1}{r}f^1(r) - \frac{f^1(r)}{r^3}r^2$$

$$= f^{11}(r) + \frac{3}{r}f^1(r) - \frac{f^1(r)}{r} = f^{11}(r) + \frac{2}{r}f^1(r).$$

☹ CURL OF A VECTOR FUNCTION ☹

Def: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a continuously differentiable vector point function. Then the Curl of '$\bar{F}$' is defined as $\text{curl}\bar{F} = \nabla \times \bar{F}$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \bar{F}$$

$$= \left(\bar{i} \times \frac{\partial \bar{F}}{\partial x}\right) + \left(\bar{j} \times \frac{\partial \bar{F}}{\partial y}\right) + \left(\bar{k} \times \frac{\partial \bar{F}}{\partial z}\right)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \bar{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] = \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \bar{j}\left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$= \sum \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] \quad \text{which is a vector.}$$

Physical interpretation:

Angular velocity of a rigid body rotating about a fixed axis at any point is equal to half the curl of velocity vector.

Irrotational Motion, Irrotational Vector:

1) Any motion in which curl of the velocity vector is a null vector i.e $\text{curl}\bar{v} = \bar{0}$ is said to be '*Irrotational motion'*.

2) A vector $\bar{F}$ is said to be Irrotational vector if $\text{curl}\bar{F} = \bar{0}$. In this case there exists a scalar 'φ' such that $\bar{F} = \nabla\varphi$. Then 'φ' is called "***scalar potential***".

PROBLEMS:

1) If $\bar{F} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ then find $\text{curl}\bar{F}$ at the point (1,-1,1)

Sol: $\bar{F} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$

Then $\text{curl}\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$

$$= \bar{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + \bar{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \bar{i}\,(-3z^2.1 - 2x^2y.1) - \bar{j}\,(0 - 0) + \bar{k}\,(2yz.2x - x.2y) = \bar{i}\,(-3z^2 - 2x^2y) + \bar{k}\,(4xyz - 2xy)$$

$\therefore \text{curl}\bar{F}$ at $(1,-1,1) = \bar{i}\,(-3.1^2 - 2.1^2(-1)) + \bar{k}\,(4.1.(-1).1 - 2.1.(-1)) = -\bar{i} - 2\bar{k}$

----

2) Find $\text{curl}\bar{F}$, where $\bar{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.

Sol: $\bar{F} = (3x^2 - 3yz)\,\bar{i} + (3y^2 - 3xz)\,\bar{j} + (3z^2 - 3xy)\bar{k}$ {by above problem}

Then $\text{curl}\bar{F} = \nabla \times \bar{F} = \sum \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right]$

$$= \sum \bar{i}\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right]$$

$$= \sum \bar{i}\,[(0 - 3x) - (0 - 3x)] = \sum \bar{i}\,[(-3x) + 3x)] = \bar{0}$$

$\therefore \text{curl}\bar{F} = \bar{0}$.

----

3) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then find '$\text{curl}\bar{r}$' and '$\text{div}\bar{r}$'.

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\operatorname{div}\bar{r} = \nabla.\bar{r} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$

$\operatorname{curl}\bar{r} = \nabla \times \bar{r} = \sum \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] = \sum \bar{i}\left[\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right] = \bar{0}$.

----

4) Find the constants a, b and c if the vector $\bar{F} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$ is Irrotational.

Sol: $\bar{F}$ is Irrotational vector $\Rightarrow$ $\operatorname{curl}\bar{F} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{0} \quad \Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 3y + az & bx + 2y + 3z & 2x + cy + 3z \end{vmatrix} = \bar{0}$$

$\Rightarrow \bar{i}\left[\frac{\partial}{\partial y}(2x + cy + 3z) - \frac{\partial}{\partial z}(bx + 2y + 3z)\right] - \bar{j}\left[\frac{\partial}{\partial x}(2x + cy + 3z) - \frac{\partial}{\partial z}(2x + 3y + az)\right] +$
$\bar{k}\left[\frac{\partial}{\partial x}(bx + 2y + 3z) - \frac{\partial}{\partial y}(2x + 3y + az)\right] = \bar{0}$

$\Rightarrow \bar{i}\,(c - 3) - \bar{j}\,(2 - a) + \bar{k}\,(b - 3) = \bar{0}$ $\qquad \Rightarrow c = 3;\ a = 2;\ b = 3.$

----

5) Show that the vector $(x^2 - yz)\,\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is Irrotational and find the scalar potential.

Sol: Let $\bar{F} = (x^2 - yz)\,\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$\operatorname{curl}\bar{F} = \nabla \times \bar{F} = \sum \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] = \sum \bar{i}\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right]$

$= \sum \bar{i}\,[(0 - x.1) - (0 - x.1)] = \bar{0}$.

$\Rightarrow \operatorname{curl}\bar{F} = \bar{0}$. $\Rightarrow$ $\bar{F}$ is Irrotational.

Since $\bar{F}$ is Irrotational, there exists a scalar function 'φ' such that $\bar{F} = \nabla\varphi$.

$\Rightarrow \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = (x^2 - yz)\,\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$\Rightarrow \frac{\partial\varphi}{\partial x} = x^2 - yz; \quad \frac{\partial\varphi}{\partial y} = y^2 - zx; \quad \frac{\partial\varphi}{\partial z} = z^2 - xy;$

Now $\frac{\partial\varphi}{\partial x} = x^2 - yz$. Integrating w.r.t 'x', keeping y,z as constants

$\Rightarrow \varphi = \frac{x^3}{3} - yz.x + f(y,z) = \frac{x^3}{3} - xyz + f(y,z)$………(1)

$\Rightarrow \frac{\partial\varphi}{\partial y} = 0 - xz.1 + \frac{\partial f}{\partial y} \quad \Rightarrow \frac{\partial\varphi}{\partial y} = -xz + \frac{\partial f}{\partial y}$.

$\Rightarrow y^2 - zx = -xz + \frac{\partial f}{\partial y} \quad \Rightarrow \frac{\partial f}{\partial y} = y^2$. Integrate w.r.t 'y' keeping z constant.

$\Rightarrow f(y,z) = \frac{y^3}{3} + g(z)$. Substitute in (1)

$\therefore \varphi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + g(z)$…………..(2)

$\Rightarrow \frac{\partial\varphi}{\partial z} = 0 - xy.1 + 0 + \frac{\partial g}{\partial z}$

$\Rightarrow z^2 - xy = -xy + \frac{dg}{dz} \quad \Rightarrow \frac{dg}{dz} = z^2$. Integrate w.r.t.'z'.

$\Rightarrow g(z) = \frac{z^3}{3} + c$. Substitute in (2)

$\therefore \varphi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c.$

Hence scalar potential is $\quad \varphi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c.$

----

6) A fluid motion is given by $\bar{v} = (y\,Sinz - Sinx)\bar{i} + (x\,Sinz + 2yz)\bar{j} + (xy\,Cosz + y^2)\bar{k}$. Is the motion irrotational? If so find the velocity potential.

Sol: $curl\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\,Sinz - Sinx & x\,Sinz + 2yz & xy\,Cosz + y^2 \end{vmatrix}$

$= \bar{i}\left[\frac{\partial}{\partial y}(xy\,Cosz + y^2) - \frac{\partial}{\partial z}(x\,Sinz + 2yz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(xy\,Cosz + y^2) - \frac{\partial}{\partial z}(y\,Sinz - Sinx)\right]$

$+ \bar{k}\left[\frac{\partial}{\partial x}(x\,Sinz + 2yz) - \frac{\partial}{\partial y}(y\,Sinz - Sinx)\right]$

$= \bar{i}\,[(x\,Cosz + 2y) - (x\,Cosz + 2y)] - \bar{j}\,[y\,Cosz - y\,Cosz] + \bar{k}\,[\,Sinz - Sinz] = \bar{0}$

$\therefore$ The motion is irrotational.

Let φ be the velocity potential. Then $\bar{v} = \nabla\varphi$

$\Rightarrow (y\,Sinz - Sinx)\bar{i} + (x\,Sinz + 2yz)\bar{j} + (xy\,Cosz + y^2)\bar{k} = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$

$\Rightarrow \frac{\partial\varphi}{\partial x} = y\,Sinz - Sinx;\qquad \frac{\partial\varphi}{\partial y} = x\,Sinz + 2yz;\qquad \frac{\partial\varphi}{\partial z} = xy\,Cosz + y^2$

Now $\frac{\partial\varphi}{\partial x} = y\,Sinz - Sinx$. Integrate w.r.t 'x' keeping 'y' and 'z' as constants

$\Rightarrow \varphi = y\,Sinz.\,x + Cosx + f(y,z)$ ………(1)

$\Rightarrow \frac{\partial\varphi}{\partial y} = Sinz.\,x + 0 + \frac{\partial f}{\partial y}$

$\Rightarrow x\,Sinz + 2yz = x\,Sinz + \frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial y} = 2yz$. Integrate w.r.t 'y' keeping z constant.

$\Rightarrow f(y,z) = 2z\frac{y^2}{2} + g(z) = zy^2 + g(z)$ Substitute in (1)

$\therefore \varphi = xy\,Sinz + Cosx + zy^2 + g(z)$…………..(2)

$\Rightarrow \frac{\partial\varphi}{\partial z} = xy\,Cosz + 0 + y^2 + \frac{\partial g}{\partial z}$

$\Rightarrow xy\,Cosz + y^2 = xy\,Cosz + y^2 + \frac{dg}{dz} \Rightarrow \frac{dg}{dz} = 0$. Integrate w.r.t.'z'.

$\Rightarrow g(z) = c$. Substitute in (2)

$\therefore$ Velocity potential is $\varphi = xy\,Sinz + Cosx + zy^2 + c$

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7) If f(r) is differentiable, Show that '$\bar{r}$ f(r)' is Irrotational.

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}| = \sqrt{(x^2 + y^2 + z^2)}$ Squaring on both sides

$r^2 = x^2 + y^2 + z^2$ ……………..(1)

Diff. (1) partially w.r.t 'x' $\Rightarrow 2r\frac{\partial r}{\partial x} = 2x + 0 + 0 = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\bar{r}\,f(r) = xf(r)\bar{i} + yf(r)\bar{j} + zf(r)\bar{k}$

$Curl(\bar{r}\,f(r)) = \nabla \times \bar{r}\,f(r) = \sum \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] = \sum \bar{i}\left[\frac{\partial}{\partial y}(z\,f(r)) - \frac{\partial}{\partial z}(y\,f(r))\right]$

$= \sum \bar{i}\left[(z\,.\,f^1(r)\frac{\partial r}{\partial y}) - (y.\,f^1(r)\frac{\partial r}{\partial z})\right] = \sum \bar{i}\,f^1(r)\left[(z\,.\frac{y}{r}) - (y.\frac{z}{r})\right]$

$= \sum \bar{i}\,f^1(r)\left[\frac{yz}{r} - \frac{yz}{r})\right] = \bar{0}$

$\Rightarrow curl(\bar{r}\,f(r)) = \bar{0}$. $\Rightarrow \bar{r}\,f(r)$ is Irrotational.

----

8) Show that '$r^n\,\bar{r}$' is Irrotational.

Sol: Now $r^n\,\bar{r} = xr^n + yr^n\bar{j} + zr^n\bar{k}$

$\text{Curl}(r^n\bar{r}) = \nabla \times r^n\bar{r} = \sum \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] = \sum \bar{i}\left[\frac{\partial}{\partial y}(z\,r^n) - \frac{\partial}{\partial z}(y\,r^n)\right]$

$= \sum \bar{i}\left[(z\,.\,nr^{n-1}\frac{\partial r}{\partial y}) - (y.\,nr^{n-1}\frac{\partial r}{\partial z})\right] = \sum \bar{i}\,nr^{n-1}\left[(z\,.\frac{y}{r}) - (y.\frac{z}{r})\right]$

$= \sum \bar{i}\,nr^{n-1}\left[\frac{yz}{r} - \frac{yz}{r})\right] = \bar{0}$

$\Rightarrow \text{curl}(r^n\bar{r}) = \bar{0}\,. \quad \Rightarrow\ r^n\bar{r}$ is Irrotational.

----

9) Evaluate $\nabla.\left[r\,\nabla(\frac{1}{r^3})\right]$, where $r = |\,\bar{r}| = \sqrt{(x^2+y^2+z^2)}$.

<u>Sol</u>: $\nabla(\frac{1}{r^3}) = \sum \bar{i}\,\frac{\partial}{\partial x}(\frac{1}{r^3}) = \sum \bar{i}\,\frac{\partial}{\partial x}(\,r^{-3}\,) = \sum \bar{i}\,(-3\,r^{-3-1}\frac{\partial r}{\partial x}) = \sum \bar{i}\,(-3\,r^{-4}\frac{x}{r})$

$\therefore\ \nabla(\frac{1}{r^3}) = -3x\,r^{-5}\,\bar{i} - 3y\,r^{-5}\bar{j} - 3z\,r^{-5}\bar{k}$

$\Rightarrow r\,\nabla(\frac{1}{r^3}) = r(-3x\,r^{-5}\,\bar{i} - 3y\,r^{-5}\bar{j} - 3z\,r^{-5}\bar{k}\,) = -3x\,r^{-4}\,\bar{i} - 3y\,r^{-4}\bar{j} - 3z\,r^{-4}\bar{k}\,.$

Now $\nabla.\left[r\,\nabla(\frac{1}{r^3})\right] = \text{div}\left[r\,\nabla(\frac{1}{r^3})\right] = \sum\frac{\partial F_1}{\partial x} = \sum\frac{\partial}{\partial x}(\,-3x\,r^{-4}\,)$

$= \sum -3(\,1.r^{-4} + x.(-4)r^{-4-1}\frac{\partial r}{\partial x}) = \sum -3\,r^{-4} + 12x.\,r^{-5}.\frac{x}{r})$

$= \sum(-3\,r^{-4} + 12x^2.\,r^{-6})$

$= (-3\,r^{-4} + 12x^2.\,r^{-6}) + (-3\,r^{-4} + 12y^2.\,r^{-6}) + (-3\,r^{-4} + 12z^2.\,r^{-6})$

$= -9\,r^{-4} + 12\,r^{-6}\,(x^2+y^2+z^2) = -9\,r^{-4} + 12\,r^{-6}\,r^2 = -9\,r^{-4} + 12\,r^{-4} = 3r^{-4}$

$\therefore \nabla.\left[r\,\nabla(\frac{3}{r^4})\right] = \frac{3}{r^4}$

----

10) Show that $\nabla\left[\nabla.(\frac{\bar{r}}{r})\right] = \frac{-2}{r^3}\,\bar{r}$.

<u>Sol</u>: $\frac{\bar{r}}{r} = \frac{1}{r}(x\bar{i} + y\bar{j} + z\bar{k}\,) = xr^{-1}\bar{i} + yr^{-1}\bar{j} + zr^{-1}\bar{k}$

$\therefore \nabla.(\frac{\bar{r}}{r}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(xr^{-1}) + \frac{\partial}{\partial y}(yr^{-1}) + \frac{\partial}{\partial z}(zr^{-1})$

$= [1.\,r^{-1} + x.(-1)\,r^{-2}\frac{\partial r}{\partial x}] + [1.\,r^{-1} + y.(-1)\,r^{-2}\frac{\partial r}{\partial y}] + [1.\,r^{-1} + z.(-1)\,r^{-2}\frac{\partial r}{\partial z}]$

$= 3r^{-1} - r^{-2}(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}) = 3r^{-1} - r^{-2}(\frac{r^2}{r}) = \frac{2}{r}\,.$

$\therefore \nabla\left[\nabla.(\frac{\bar{r}}{r})\right] = \nabla\left[\frac{2}{r}\right] = \sum \bar{i}\,\frac{\partial}{\partial x}(\frac{2}{r}) = \sum \bar{i}\,(\frac{-2}{r^2}\frac{\partial r}{\partial x}) = \sum \bar{i}\,(\frac{-2}{r^2}\frac{x}{r}) = \frac{-2}{r^3}\sum \bar{i}\,x$

$\Rightarrow\ \nabla\left[\nabla.(\frac{\bar{r}}{r})\right] = \frac{-2}{r^3}\,\bar{r}\,.$

☹ <u>VECTOR OPERATORS</u> ☹

(1) *Scalar differential operator* $\bar{a}.\nabla$:

The operator $\bar{a}.\nabla = (\bar{a}.\,\bar{i}\,)\frac{\partial}{\partial x} + (\bar{a}.\,\bar{j}\,)\frac{\partial}{\partial y} + (\bar{a}.\,\bar{k}\,)\frac{\partial}{\partial z}$ is defined as

$(\bar{a}.\nabla)\varphi = (\bar{a}.\,\bar{i}\,)\frac{\partial\varphi}{\partial x} + (\bar{a}.\,\bar{j}\,)\frac{\partial\varphi}{\partial y} + (\bar{a}.\,\bar{k}\,)\frac{\partial\varphi}{\partial z}$

and $(\bar{a}.\nabla)\,\bar{F} = (\bar{a}.\,\bar{i}\,)\frac{\partial\bar{F}}{\partial x} + (\bar{a}.\,\bar{j}\,)\frac{\partial\bar{F}}{\partial y} + (\bar{a}.\,\bar{k}\,)\frac{\partial\bar{F}}{\partial z}$

(2) *Vector differential operator* $\bar{a}\times\nabla$:

The operator $\bar{a}\times\nabla = (\bar{a}\times\bar{i}\,)\frac{\partial}{\partial x} + (\bar{a}\times\bar{j}\,)\frac{\partial}{\partial y} + (\bar{a}\times\bar{k}\,)\frac{\partial}{\partial z}$ is defined as

$(\bar{a}\times\nabla)\,\varphi = (\bar{a}\times\bar{i}\,)\frac{\partial\varphi}{\partial x} + (\bar{a}\times\bar{j}\,)\frac{\partial\varphi}{\partial y} + (\bar{a}\times\bar{k}\,)\frac{\partial\varphi}{\partial z}$

$(\bar{a}\times\nabla).\bar{F} = (\bar{a}\times\bar{i}).\frac{\partial\bar{F}}{\partial x} + (\bar{a}\times\bar{j}).\frac{\partial\bar{F}}{\partial y} + (\bar{a}\times\bar{k}).\frac{\partial\bar{F}}{\partial z}$

and $(\bar{a}\times\nabla)\times\bar{F} = [(\bar{a}\times\bar{i})\times\frac{\partial\bar{F}}{\partial x}] + [(\bar{a}\times\bar{j})\times\frac{\partial\bar{F}}{\partial y}] + [(\bar{a}\times\bar{k})\times\frac{\partial\bar{F}}{\partial z}]$

VECTOR IDENTITIES

Let '$\bar{a}$' be a differentiable vector function and 'φ' is a differentiable scalar function.

1) div($\varphi\bar{a}$) = (grad φ). $\bar{a}$ + φ div$\bar{a}$.   (or) $\nabla.(\varphi\bar{a}) = \nabla\varphi.\bar{a} + \varphi(\nabla.\bar{a})$.

Proof: div($\varphi\bar{a}$) = $\nabla.(\varphi\bar{a})$

$$= \sum(\bar{i}.\frac{\partial}{\partial x}(\varphi\bar{a})) = \sum\bar{i}.\left(\frac{\partial\varphi}{\partial x}\bar{a} + \varphi\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum\left(\bar{i}.\frac{\partial\varphi}{\partial x}\bar{a}\right) + \sum\left(\bar{i}.\varphi\frac{\partial\bar{a}}{\partial x}\right) = \sum\left(\bar{i}\frac{\partial\varphi}{\partial x}.\bar{a}\right) + \varphi\sum\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum\left(\bar{i}\frac{\partial\varphi}{\partial x}\right).\bar{a} + \varphi\,\text{div}\bar{a} = (\text{grad}\,\varphi).\bar{a} + \varphi\,\text{div}\bar{a}.$$

----

2) curl($\varphi\bar{a}$) = [(grad φ)× $\bar{a}$] + φ curl $\bar{a}$ (or) $\nabla\times(\varphi\bar{a}) = (\nabla\varphi\times\bar{a}) + \varphi(\nabla\times\bar{a})$.

Proof: curl($\varphi\bar{a}$) = $\nabla\times(\varphi\bar{a})$

$$= \sum\left(\bar{i}\times\frac{\partial}{\partial x}(\varphi\bar{a}\right) = \sum\bar{i}\times\left(\frac{\partial\varphi}{\partial x}\bar{a} + \varphi\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum\left(\bar{i}\times\frac{\partial\varphi}{\partial x}\bar{a}\right) + \sum\left(\bar{i}\times\varphi\frac{\partial\bar{a}}{\partial x}\right) = \sum\left(\bar{i}\frac{\partial\varphi}{\partial x}\times\bar{a}\right) + \varphi\sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum\left(\bar{i}\frac{\partial\varphi}{\partial x}\right)\times\bar{a} + \varphi\,\text{curl}\,\bar{a} = [(\text{grad}\,\varphi)\times\bar{a}] + \varphi\,\text{curl}\,\bar{a}.$$

----

3) grad $(\bar{a}.\bar{b}) = (\bar{b}.\nabla)\bar{a} + (\bar{a}.\nabla)\bar{b} + [\bar{b}\times\text{curl}\,\bar{a}] + [\bar{a}\times\text{curl}\,\bar{b}]$.

Proof: $\bar{a}\times\text{curl}\,\bar{b} = \bar{a}\times(\nabla\times\bar{b}) = \bar{a}\times\sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)$

$$= \sum\bar{a}\times\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right) = \sum\left\{\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right)\bar{i} - (\bar{a}.\bar{i})\frac{\partial\bar{b}}{\partial x}\right\} = \sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right) - \sum(\bar{a}.\bar{i})\frac{\partial\bar{b}}{\partial x}$$

$(\because \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c})$

$\therefore\ \bar{a}\times\text{curl}\,\bar{b} = \sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}.\nabla)\bar{b}$ ………………(1)

Similarly $\bar{b}\times\text{curl}\,\bar{a} = \sum\bar{i}\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}.\nabla)\bar{a}$ ……….(2)

$\therefore\ (\bar{a}\times\text{curl}\,\bar{b}) + (\bar{b}\times\text{curl}\,\bar{a}) = \sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}.\nabla)\bar{b} + \sum\bar{i}\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}.\nabla)\bar{a}$

$\Rightarrow (\bar{a}\times\text{curl}\,\bar{b}) + (\bar{b}\times\text{curl}\,\bar{a}) + (\bar{a}.\nabla)\bar{b} + (\bar{b}.\nabla)\bar{a} = \sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right) + \sum\bar{i}\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right)$

$$= \sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x} + \bar{b}.\frac{\partial\bar{a}}{\partial x}\right) = \sum\bar{i}\left(\frac{\partial}{\partial x}(\bar{a}.\bar{b})\right)$$

$\Rightarrow (\bar{a}\times\text{curl}\,\bar{b}) + (\bar{b}\times\text{curl}\,\bar{a}) + (\bar{a}.\nabla)\bar{b} + (\bar{b}.\nabla)\bar{a} = \text{grad}\,(\bar{a}.\bar{b})$.

----

4) div$(\bar{a}\times\bar{b}) = \bar{b}$ .curl $\bar{a} - \bar{a}$ .curl $\bar{b}$ (or) $\nabla.(\bar{a}\times\bar{b}) = \bar{b}.(\nabla\times\bar{a}) - \bar{a}.(\nabla\times\bar{b})$

Proof: div$(\bar{a}\times\bar{b}) = \nabla.(\bar{a}\times\bar{b})$

$$= \sum(\bar{i}.\frac{\partial}{\partial x}(\bar{a}\times\bar{b})) = \sum\bar{i}.\left([\frac{\partial\bar{a}}{\partial x}\times\bar{b}] + [\bar{a}\times\frac{\partial\bar{b}}{\partial x}]\right)$$

$$= \sum\bar{i}.[\frac{\partial\bar{a}}{\partial x}\times\bar{b}] + \sum\bar{i}.[\bar{a}\times\frac{\partial\bar{b}}{\partial x}] = \sum\bar{i}.\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right] - \sum\bar{i}.[\frac{\partial\bar{b}}{\partial x}\times\bar{a}]$$

$$= \sum\left[\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right].\bar{b} - \sum\left[\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right].\bar{a} \qquad (\because \bar{a}.(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b}).\bar{c})$$

$= (\text{curl}\,\bar{a}).\bar{b} - (\text{curl}\,\bar{b}).\bar{a}$

$\Rightarrow \text{div}(\bar{a}\times\bar{b}) = \bar{b}.\text{curl}\,\bar{a} - \bar{a}.\text{curl}\,\bar{b}$

----

5) $\text{curl}(\bar{a}\times\bar{b}) = \bar{a}\ \text{div}\bar{b} - \bar{b}\ \text{div}\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

(or) $\nabla\times(\bar{a}\times\bar{b}) = \bar{a}(\nabla.\bar{b}) - \bar{b}(\nabla.\bar{a}) + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

Proof: $\text{curl}(\bar{a}\times\bar{b}) = \nabla\times(\bar{a}\times\bar{b}) = \sum\left(\bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})\right)$

$$= \sum\bar{i}\times\left[\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)\right] = \sum\bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum\bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left\{(\bar{i}.\bar{b})\frac{\partial\bar{a}}{\partial x} - \left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\} + \sum\left\{\left(\bar{i}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (\bar{i}.\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$= \sum(\bar{b}.\bar{i})\frac{\partial\bar{a}}{\partial x} - \sum\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{i}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - \sum(\bar{a}.\bar{i})\frac{\partial\bar{b}}{\partial x}$$

$$= (\bar{b}.\nabla)\bar{a} - (\text{div}\bar{a})\bar{b} + (\text{div}\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$\Rightarrow \text{curl}(\bar{a}\times\bar{b}) = \bar{a}\ \text{div}\bar{b} - \bar{b}\ \text{div}\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

----

6) $\text{curl}(\text{grad}\,\varphi) = \bar{0}$ (grad φ is always Irrotational vector.)

(or) $\nabla\times\nabla\varphi = \bar{0}$

Proof: $\text{grad}\,\varphi = \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$

$$\therefore \text{curl}(\text{grad}\,\varphi) = \text{curl}(\nabla\varphi) = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z}\end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}\frac{\partial\varphi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\varphi}{\partial y}\right] - \bar{j}\left[\frac{\partial}{\partial x}\frac{\partial\varphi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\varphi}{\partial x}\right] + \bar{k}\left[\frac{\partial}{\partial x}\frac{\partial\varphi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial x}\right]$$

$$= \bar{i}\left[\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right] - \bar{j}\left[\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right] + \bar{k}\left[\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right]$$

$$= \bar{i}\left[\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial y\,\partial z}\right] - \bar{j}\left[\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial x\,\partial z}\right] + \bar{k}\left[\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial x\,\partial y}\right] = \bar{0}\,.$$

----

7) $\text{div}(\text{curl}\,\bar{F}) = 0$. (or) $\nabla.(\nabla\times\bar{F}) = 0$

Proof: $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix} = \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \bar{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\therefore \text{div}(\text{curl}\,\bar{F}) = \nabla.(\text{curl}\,\bar{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_1}{\partial y\,\partial z} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} - \frac{\partial^2 F_3}{\partial x\,\partial y} + \frac{\partial^2 F_1}{\partial y\,\partial z} + \frac{\partial^2 F_2}{\partial x\,\partial z} - \frac{\partial^2 F_1}{\partial y\,\partial z} = 0.$$

----

8) $\text{curl}(\text{curl}\,\bar{F}) = \text{grad}(\text{div}\,\bar{F}) - \nabla^2\bar{F}$ (or) $\nabla\times(\nabla\times\bar{F}) = \nabla(\nabla.\bar{F}) - \nabla^2\bar{F}$

Proof: $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\text{curl}\overline{F} = \nabla\times\overline{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{i}\left[\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right] - \bar{j}\left[\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right]$$

$$\nabla\times\overline{F} = \bar{i}\left[\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right] + \bar{j}\left[\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right].$$

$$\therefore \text{curl}(\text{curl }\overline{F}) = \nabla\times(\nabla\times\overline{F}) = \sum\bar{i}\left[\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right]$$

$$= \sum\bar{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right] = \sum\bar{i}\left[\frac{\partial^2 F_2}{\partial y\,\partial x}-\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}+\frac{\partial^2 F_3}{\partial z\,\partial x}\right]$$

$$= \sum\bar{i}\left[\frac{\partial^2 F_2}{\partial x\,\partial y}-\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}-\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_3}{\partial x\,\partial z}\right]$$

$$= \sum\bar{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$= \sum\bar{i}\left[\frac{\partial}{\partial x}(\text{div }\overline{F})-(\nabla^2 F_1)\right] = \sum\bar{i}\frac{\partial}{\partial x}(\text{div }\overline{F}) - \nabla^2\sum\bar{i}F_1$$

$$\therefore \text{curl}(\text{curl }\overline{F}) = \text{grad}(\text{div }\overline{F}) - \nabla^2\overline{F}\ .$$

PROBLEMS:

1) Find $(\bar{a}\times\nabla)\varphi$, where $\bar{a} = yz^2\,\bar{i} - 3xz^2\,\bar{j} + 2xyz\,\bar{k}$ and $\varphi = xyz$.

Sol: $\bar{a}\times\nabla = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$

$$= \bar{i}\left[\frac{\partial}{\partial z}(-3xz^2)-\frac{\partial}{\partial y}(2xyz)\right] - \bar{j}\left[\frac{\partial}{\partial z}(yz^2)-\frac{\partial}{\partial x}(2xyz)\right] + \bar{k}\left[\frac{\partial}{\partial y}(yz^2)-\frac{\partial}{\partial x}(-3xz^2)\right]$$

$$= \bar{i}\,[(-3x.2z)-(2xz.1)] - \bar{j}\,[(y.2z)-(2yz.1)] + \bar{k}\,[(1.z^2)-(-3.1.z^2)]$$

$$= \bar{i}\,[-8xz] - \bar{j}\,[0] + \bar{k}\,[4z^2] = -8xz\,\bar{i} + 4z^2\,\bar{k}\ .$$

$$\therefore (\bar{a}\times\nabla)\varphi = (-8xz\,\bar{i} + 4z^2\,\bar{k})xyz = -8x^2yz^2\,\bar{i} + 4xyz^3\,\bar{k}\ .$$

----

2) Find $(\bar{a}\,.\,\nabla)\varphi$ at (1,-1,1) if $\bar{a} = 3xyz^2\,\bar{i} + 2xy^3\,\bar{j} - x^2yz\,\bar{k}$ and $\varphi = 3x^2 - yz$.

Sol: $(\bar{a}.\nabla)\varphi = (\bar{a}.\,\bar{i}\,)\frac{\partial\varphi}{\partial x} + (\bar{a}.\,\bar{j}\,)\frac{\partial\varphi}{\partial y} + (\bar{a}.\,\bar{k}\,)\frac{\partial\varphi}{\partial z}$

Now $\frac{\partial\varphi}{\partial x} = 3.2x - 0 = 6x$; $\frac{\partial\varphi}{\partial y} = 0 - z.1 = -z$; $\frac{\partial\varphi}{\partial z} = 0 - y.1 = -y$;

$\bar{a}.\,\bar{i} = 3xyz^2$; $\bar{a}.\,\bar{j} = 2xy^3$; $\bar{a}.\,\bar{k} = -x^2yz$;

$\therefore (\bar{a}.\nabla)\varphi = (3xyz^2)6x + (2xy^3)(-z) + (-x^2yz)(-y)$

$= 18x^2yz^2 - 2xy^3z + x^2y^2z$.

At (1,-1,1) , $(\bar{a}.\nabla)\varphi = 18.1.(-1).1 - 2.1.(-1)^3.1 + 1.(-1)^2.1 = -15$.

----

3) If $u\overline{F} = \nabla v$, where u,v are scalar fields and $\overline{F}$ is a vector field, then show that $\overline{F}.\text{curl}\overline{F} = 0$.

Sol: $u\overline{F} = \nabla v \Rightarrow \overline{F} = \frac{1}{u}\nabla v$

$\therefore \text{curl}\overline{F} = \nabla\times\overline{F} = \nabla\times(\frac{1}{u}\nabla v) = [(\nabla\frac{1}{u})\times\nabla v] + \frac{1}{u}[\nabla\times(\nabla v)]$

$= (\nabla\frac{1}{u})\times\nabla v + \overline{0}$ ($\because \text{curl}(\text{grad }\varphi) = \overline{0}$)

$\therefore \text{curl}\overline{F} = (\nabla\frac{1}{u})\times\nabla v$.

Now $\quad \overline{F}.\text{curl}\overline{F} = \overline{F}\,.\,(\nabla\frac{1}{u} \times \nabla v)$

$$= \frac{1}{u}\nabla v\,.\,(\nabla\frac{1}{u} \times \nabla v) = \frac{1}{u}[\nabla v\,.\,(\nabla\frac{1}{u} \times \nabla v)]$$

$$= \frac{1}{u}(0) \qquad (\because \text{ two vectors are same in the triple product})$$

$$\Rightarrow \overline{F}.\text{curl}\overline{F} = 0.$$

----

4) If 'r' and '$\overline{r}$' have their usual meanings and '$\overline{a}$' is a constant vector, then show that $\nabla \times \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \frac{2-n}{r^n}\overline{a} + \frac{n(\overline{a}.\overline{r})}{r^{n+2}}\overline{r}$.

<u>Sol</u>: $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ and $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$

$$\nabla \times \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \sum\left[\overline{i} \times \frac{\partial}{\partial x}\left(\frac{\overline{a}\times\overline{r}}{r^n}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now $\frac{\partial}{\partial x}\left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \overline{a} \times \frac{\partial}{\partial x}\left(\frac{\overline{r}}{r^n}\right) = \overline{a} \times \left[\frac{1}{r^n}\frac{\partial\overline{r}}{\partial x} - \frac{n}{r^{n+1}}\frac{\partial r}{\partial x}\overline{r}\right]$

$$= \overline{a} \times \left[\frac{1}{r^n}\overline{i} - \frac{n}{r^{n+1}}\frac{x}{r}\overline{r}\right] = \overline{a} \times \left[\frac{1}{r^n}\overline{i} - \frac{nx}{r^{n+2}}\overline{r}\right]$$

$$= \frac{\overline{a}\times\overline{i}}{r^n} - \frac{nx}{r^{n+2}}(\overline{a}\times\overline{r})$$

$$\Rightarrow \overline{i} \times \frac{\partial}{\partial x}\left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \overline{i} \times \left(\frac{\overline{a}\times\overline{i}}{r^n} - \frac{nx}{r^{n+2}}(\overline{a}\times\overline{r})\right)$$

$$= \frac{\overline{i}\times(\overline{a}\times\overline{i})}{r^n} - \frac{nx}{r^{n+2}}[\overline{i}\times(\overline{a}\times\overline{r})] = \frac{(\overline{i}.\overline{i})\overline{a}-(\overline{i}.\overline{a})\overline{i}}{r^n} - \frac{nx}{r^{n+2}}[(\overline{i}.\overline{r})\overline{a} - (\overline{i}.\overline{a})\overline{r}]$$

$$= \frac{\overline{a}-a_1\overline{i}}{r^n} - \frac{nx}{r^{n+2}}[x\,\overline{a} - a_1\overline{r}]$$

From (1) $\quad \nabla \times \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \sum\left\{\frac{\overline{a}-a_1\overline{i}}{r^n} - \frac{nx}{r^{n+2}}[x\,\overline{a} - a_1\overline{r}]\right\}$

$$= \sum\left\{\frac{\overline{a}-a_1\overline{i}}{r^n} - \frac{n\overline{a}}{r^{n+2}}\,x^2 + \frac{n\overline{r}}{r^{n+2}}a_1x\right\}$$

$$= \sum\left\{\frac{\overline{a}-a_1\overline{i}}{r^n}\right\} - \frac{n\overline{a}}{r^{n+2}}\sum\{x^2\} + \frac{n\overline{r}}{r^{n+2}}\sum\{a_1x\}$$

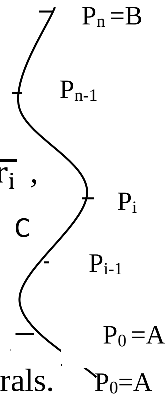
$$= \frac{3\,\overline{a}-\overline{a}}{r^n} - \frac{n\overline{a}}{r^{n+2}}r^2 + \frac{n\overline{r}}{r^{n+2}}(\overline{a}.\overline{r}) = \frac{2\,\overline{a}}{r^n} - \frac{n\overline{a}}{r^n} + \frac{n\overline{r}}{r^{n+2}}(\overline{a}.\overline{r})$$

$$\Rightarrow \nabla \times \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \frac{2-n}{r^n}\overline{a} + \frac{n(\overline{a}.\overline{r})}{r^{n+2}}\overline{r}\,.$$

----

5) If $\overline{a}$ is a constant vector, show that Curl$\left(\frac{\overline{a}\times\overline{r}}{r^3}\right) = \frac{-\overline{a}}{r^3} + \frac{3\overline{r}}{r^5}(\overline{a}.\overline{r})$

☹ LINE INTEGRAL ☹



Def: Let $\overline{F}$ be a vector function defined at each point of the curve 'C' in a space. Divide 'C' in to n parts at points A = P_0 , P_1 ,…., P_{i-1} , P_i ,…., P_n =B. Let the position vectors of these points be $\overline{r_0}$, $\overline{r_1}$,…., $\overline{r_i}$, $\overline{r_{i-1}}$,…., $\overline{r_n}$. Let U_i be the position vector of any point on the arc $P_{i-1}P_i$. Consider the sum S= $\sum_{i=0}^{n} \overline{F}(U_i).\delta\,\overline{r_i}$, where $\delta\overline{r_i} = \overline{r_i} - \overline{r_{i-1}}$. Now "$\lim\limits_{n\to\infty}$ S" in such a way that $|\delta\overline{r_i}| \to 0$, provided it exists, is called the tangential line integral of $\overline{F}$ along 'C'.

It is denoted by $\int_C \overline{F}\,.\,d\overline{r}$.

Note: The other two line integrals are $\int_C \overline{F}\times d\overline{r}$ and $\int_c \varphi d\overline{r}$ which are both integrals.

Physical application:

1) If $\overline{F}$ represents the velocity of a fluid particle then the line integral $\int_C \overline{F}.d\overline{r}$ is called the circulation of $\overline{F}$ around the curve.

2) If $\overline{F}$ represents the force acting on a particle moving along an arc AB, then the line integral $\int_C \overline{F}.d\overline{r}$ gives the work done by $\overline{F}$ during the displacement from A to B.

3) If force $\overline{F}$ is conservative (i.e curl$\overline{F}$ = 0), then the work done is independent of the path and vice versa.

PROBLEMS:

1) If $\overline{F}$= 3xy $\overline{i}$ – $y^2\overline{j}$, evaluate $\int_C \overline{F}.d\overline{r}$ where 'c' is the curve in the xy-plane y = $2x^2$ from (0,0) to (1,2).

Sol: In xy-plane z = 0. ∴ $\overline{r}= x\overline{i} + y\overline{j} \Rightarrow d\overline{r} = dx\,\overline{i} + dy\,\overline{j}$

Now $\overline{F}= 3xy\,\overline{i} - y^2\,\overline{j} \Rightarrow \overline{F}.d\overline{r} = 3xy\,dx - y^2\,dy$

From curve eq. $y = 2x^2 \Rightarrow dy = 2.2x\,dx = 4x\,dx$

'x' limits are x=0 to x=1

$$\therefore \int_C \overline{F}.d\overline{r} = \int_c (3xy\,dx - y^2\,dy) = \int_0^1 (3x\,2x^2 dx - (2x^2)^2 4x\,dx = \int_0^1 (\,6x^3 - 16x^5)\,dx$$

$$= \left(6\frac{x^4}{4} - 16\frac{x^6}{6}\right)_0^1 = \frac{3}{2}.1 - \frac{8}{3}.1 - (\,0 - 0) = \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}$$

----

2) Find the work done in moving a particle in the force field $\overline{F}$= $3x^2\,\overline{i}$ + (2xz – y)$\,\overline{j}$ + z $\overline{k}$, along

(a) the straight line from (0,0,0) to (2,1,3).

(b) the curve defined by x^2 = 4y, $3x^3$ = 8z from x = 0 to x = 2.

Sol: $\overline{r}= x\overline{i} + y\overline{j} + z\overline{k} \Rightarrow d\overline{r}= dx\,\overline{i} + dy\,\overline{j} + dz\,\overline{k}$.

Now $\overline{F}= 3x^2\,\overline{i} + (2xz - y)\,\overline{j} + z\,\overline{k} \Rightarrow \overline{F}.d\overline{r} = 3x^2\,dx + (2xz - y)\,dy + z\,dz$.

(a) Eq. of the straight line from (0,0,0) to (2,1,3) is $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$

$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ = t (say)

⇒ x = 2t, y = t, z = 3t are its parametric eq.s.

⇒ dx = 2 dt, dy = dt, dz = 3 dt.

If x = 0 then 2t = 0 ⇒ t = 0

If x = 2 then 2t = 2 ⇒ t = 1

∴ work done = $\int_C \overline{F}.d\overline{r} = \int_c (3x^2\, dx + (2xz - y)\, dy + z\, dz)$

$$= \int_0^1 [3(2t)^2 2dt + (2.2t.3t - t)dt + 3t.3dt] = \int_0^1 [24t^2 + (12t^2 - t) + 9t]dt$$

$$= \int_0^1 [36t^2 + 8t]dt] = \left(36\frac{t^3}{3} + 8\frac{t^2}{2}\right)_0^1 = 12 + 4 - (0 - 0) = 16.$$

∴ work done = 16

(b) curve defined by $x^2 = 4y$, $3x^3 = 8z$ from x = 0 to x = 2

$$y = \frac{x^2}{4} \text{ and } z = \frac{3x^3}{8}$$

$$\Rightarrow dy = \frac{2x}{4}dx = \frac{x}{2}dx \text{ and } dz = \frac{3.3x^2}{8}dx = \frac{9x^2}{8}dx$$

'x' limits are x=0 to x=2.

∴ work done = $\int_C \overline{F}.d\overline{r} = \int_c (3x^2\, dx + (2xz - y)\, dy + z\, dz)$

$$= \int_0^2 [3x^2 dx + \left(2x.\frac{3x^3}{8} - \frac{x^2}{4}\right)\frac{x}{2}\, dx + \frac{3x^3}{8}.\frac{9x^2}{8}dx] = \int_0^2 [3x^2 - \frac{x^3}{8} + \frac{51}{64}x^5]dx$$

$$= \left(3\frac{x^3}{3} - \frac{x^4}{8.4} + \frac{51}{64}\frac{x^6}{6}\right)_0^2 = 8 - \frac{1}{2} + \frac{17}{2} = 16.$$

∴ work done = 16

----

3) A vector field is given by $\overline{F}$= siny $\overline{i}$ + x(1 + cosy) $\overline{j}$. Evaluate $\int_C \overline{F}.d\overline{r}$, where 'C' is a circular path given by $x^2 + y^2 = a^2$, z = 0.

__Sol:__ Since z = 0, dz = 0. ∴ $\overline{r} = x\overline{i} + y\overline{j} \Rightarrow d\overline{r} = dx\,\overline{i} + dy\,\overline{j}$

Now $\overline{F}$= siny $\overline{i}$ + x(1 + cosy) $\overline{j}$ ⇒ $\overline{F}.d\overline{r}$ = siny dx + x(1 + cosy) dy.

= siny dx + x cosy dy + x dy = d(x siny) + x dy

The parametric eq.s of the circular path are

x = a cost, y = a sint, z = 0,where $0 \le t \le 2\pi$.

$$\therefore \int_C \overline{F}.d\overline{r} = \int_c (\, d(x \text{ siny}) + x\, dy) = \int_0^{2\pi} [d(x \text{ siny}) + a\cos t\,.\,a\cos t\, dt]$$

$$= (x \text{ siny})_0^{2\pi} + \int_0^{2\pi} [a^2\cos^2 t]dt$$

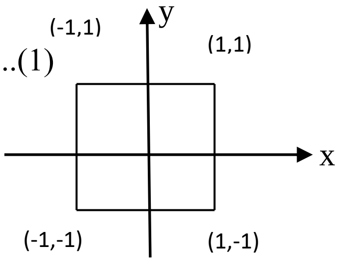
$$= \left(a\, \text{cost}.\sin(a\, \text{sint})\right)_0^{2\pi} + a^2 \int_0^{2\pi} [\frac{1+\cos 2t}{2}]dt$$

$$= a\cos 2\pi\,.\sin(a\sin 2\pi) - a\cos 0\,.\sin(a\sin 0) + \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi}$$

$$= 0 - 0 + \frac{a^2}{2}(2\pi + \frac{\sin 4\pi}{2} - 0 - 0) = \pi a^2.$$

----

4) Evaluate the line integral $\int_c [\,(x^2 + xy)dx + (\,x^2 + y^2)dy\,]$ where 'c' is the square formed by the lines x = ±1 and y = ±1.



__Sol:__ $\int_C \overline{F}.d\overline{r} = \int_{AB} \overline{F}.d\overline{r} + \int_{BC} \overline{F}.d\overline{r} + \int_{CD} \overline{F}.d\overline{r} + \int_{DA} \overline{F}.d\overline{r}$ …..(1)

(i) along 'AB':

Eq. of AB is y = -1 ⇒ dy = 0

∴ x limits are '-1' to '1'.

$$\int_{AB} \overline{F}.d\overline{r} = \int_{-1}^{1} [\,(x^2 + x(-1))dx + [\,(x^2 + y^2)0]$$

$$= \int_{-1}^{1} (x^2 - x)dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_{-1}^{1} = \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{2}{3} \text{ …………(2)}$$

(ii) along 'BC':

Eq. of BC is $x = 1 \Rightarrow dx = 0$

$\therefore$ y limits are '-1' to '1'.

$\Rightarrow \int_{BC} \overline{F}.d\overline{r} = \int_{-1}^{1}[\,(1^2 + 1.y)0 + [\,(1^2 + y^2)dy]$

$= \int_{-1}^{1}(1 + y^2)dy = \left(y + \frac{y^3}{3}\right)_{-1}^{1} = \left(1 + \frac{1}{3}\right) - (-1 + \frac{(-1)^3}{3})$

$= \frac{4}{3} + 1 + \frac{1}{3} = \frac{8}{3}$ ....(3)

(iii) along 'CD':

Eq. of CD is $y = 1 \quad \Rightarrow dy = 0$

$\therefore$ x limits are '1' to '-1'.

$\Rightarrow \int_{CD} \overline{F}.d\overline{r} = \int_{1}^{-1}[\,(x^2 + x(1))dx + [\,(x^2 + y^2)0]$

$= \int_{1}^{-1}(x^2 + x)dx = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{1}^{-1} = \left(\frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right)$

$= \frac{-1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} = \frac{-2}{3}$ ....(4)

(iv) along 'DA':

Eq. of DA is $x = -1 \quad \Rightarrow dx = 0$

$\therefore$ y limits are '1' to '-1'.

$\Rightarrow \int_{DA} \overline{F}.d\overline{r} = \int_{1}^{-1}[\,((-1)^2 + (-1).y)0 + [\,((-1)^2 + y^2)dy]$

$= \int_{1}^{-1}(1 + y^2)dy = \left(y + \frac{y^3}{3}\right)_{1}^{-1} = \left(-1 + \frac{(-1)^3}{3}\right) - (1 + \frac{1}{3})$

$= -1 - \frac{1}{3} - 1 - \frac{1}{3} = \frac{-8}{3}$ ....(5)

Substitute (2),(3),(4) and (5) in (1)

$\therefore \int_C \overline{F}.d\overline{r} = \frac{2}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3} = 0$

----

5) If $\overline{F} = (5xy - 6x^2)\,\overline{i} + (2y - 4x)\,\overline{j}$, evaluate $\int_C \overline{F}.d\overline{r}$ along the curve 'C' in the xy-plane, $y = x^3$ from the point (1,1) to (2,8).

<u>Sol</u>: In xy-plane z = 0. $\therefore\ \overline{r} = x\overline{i} + y\overline{j} \Rightarrow d\overline{r} = dx\,\overline{i} + dy\,\overline{j}$

Now $\overline{F} = (5xy - 6x^2)\,\overline{i} + (2y - 4x)\,\overline{j} \Rightarrow \overline{F}.d\overline{r} = (5xy - 6x^2)\,dx + (2y - 4x)\,dy$

From curve eq. $y = x^3 \Rightarrow dy = 3x^2\,dx$

'x' limits are x=1 to x=2

$\therefore \int_C \overline{F}.d\overline{r} = \int_c[(5xy - 6x^2)dx + (2y - 4x)dy]$

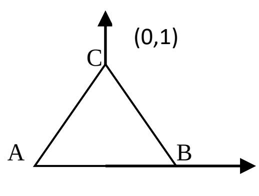
$= \int_1^2[(5x.x^3 - 6x^2)dx + (\,2x^3 - 4x)3x^2dx]$

$= \int_1^2[(5x^4 - 6x^2) + (\,6x^5 - 12x^3)]dx$

$= \left(5\frac{x^5}{5} - 6\frac{x^3}{3} + 6\frac{x^6}{6} - 12\frac{x^4}{4}\right)_1^2 = (x^5 - 2x^3 + x^6 - 3x^4)_1^2$

$= 32–16+64–48 – (1–2+1–3) = 35.$

6) Compute the line integral $\int_c(y^2dx - x^2dy)$ about the triangle whose vertices are (1,0), (0,1) and (-1,0).



<u>Sol</u>: $\int_C \overline{F}.d\overline{r} = \int_{AB} \overline{F}.d\overline{r} + \int_{BC} \overline{F}.d\overline{r} + \int_{CA} \overline{F}.d\overline{r}$ ....(1)

(i) along 'AB':

Eq. of AB is $y = 0 \quad \Rightarrow dy = 0$

$\therefore$ x limits are '-1' to '1'. (-1,0) (1,0)

$\Rightarrow \int_{AB} \bar{F}.\, d\bar{r} = \int_{-1}^{1}[\, 0dx - x^2 0] = 0$…………..(2)

(ii) along 'BC':

Eq. of BC is $x + y = 1 \Rightarrow x = 1 - y \quad \Rightarrow dx = 0 - dy = -dy$

$\therefore$ y limits are '0' to '1'.

$\Rightarrow \int_{BC} \bar{F}.\, d\bar{r} = \int_{0}^{1} (y^2(-dy) - (1-y)^2 dy)$

$= \int_0^1 [-y^2 - (1-y)^2] dy = \left(-\frac{y^3}{3} - \frac{(1-y)^3}{-3}\right)_0^1 = \left(-\frac{1}{3} + 0\right) - (-0 + \frac{(1)^3}{3})$

$= -\frac{2}{3}$ …………(3)

(iii) along 'CA':

Eq. of CA is $y - x = 1 \Rightarrow y = 1 + x \quad \Rightarrow dy = 0 + dx = dx$

$\therefore$ x limits are '0' to '-1'.

$\Rightarrow \int_{CA} \bar{F}.\, d\bar{r} = \int_{0}^{-1} [(1+x)^2 dx - x^2 dx)$

$= \left(\frac{(1+x)^3}{3} - \frac{x^3}{3}\right)_0^{-1} = \left(0 - \frac{-1}{3}\right) - (\frac{1}{3} - 0) = 0$ ………..(4)

Substitute (2),(3) and(4) in (1)

$\therefore \int_{C} \bar{F}.\, d\bar{r} = 0 - \frac{2}{3} - 0 = -\frac{2}{3}$.

----

7) Compute the work done by the force $\bar{F} = (2y + 3)\, \bar{i} + xz\, \bar{j} + (yz - x)\, \bar{k}$ when it moves a particle from the point (0,0,0) to the point (2,1,1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\, \bar{i} + dy\, \bar{j} + dz\, \bar{k}$.

Now $\bar{F} = (2y + 3)\, \bar{i} + xz\, \bar{j} + (yz - x)\, \bar{k} \quad \Rightarrow \bar{F}.\, d\bar{r} = (2y + 3)\, dx + xz\, dy + (yz - x)\, dz$.

Eq.s of the curve are $x = 2t^2$, $y = t$, $z = t^3$

$\Rightarrow dx = 4t\, dt; \quad dy = dt; \quad dz = 3t^2\, dt$

If $x = 0$, then $2t^2 = 0 \Rightarrow t = 0$

If $x = 2$, then $2t^2 = 2 \Rightarrow t = 1$

$\therefore$ work done $= \int_{C} \bar{F}.\, d\bar{r} = \int_{c}[(2y + 3)\, dx + xz\, dy + (yz - x)\, dz\,]$

$= \int_0^1 [(2t+3)4tdt + (2t^2.\, t^3)dt + (t.\, t^3 - 2t^2)3t^2 dt]$

$= \int_0^1 [8t^2 + 12t + 2t^5 + 3t^6 - 6t^4] dt$

$= \left(8\frac{t^3}{3} + 12\frac{t^2}{2} + 2\frac{t^6}{6} + 3\frac{t^7}{7} - 6\frac{t^5}{5}\right)_0^1$

$= \left(\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right) - 0 = \frac{288}{35}$.

----

8) Find the circulation of $\bar{F} = (2x - y + 2z)\, \bar{i} + (x + y - z)\, \bar{j} + (3x - 2y - 5z)\, \bar{k}$ along the circle $x^2 + y^2 = 4$ in the xy-plane.

Sol: In xy-plane $z = 0$. $\Rightarrow dz = 0 \quad \therefore \bar{r} = x\bar{i} + y\bar{j} \Rightarrow d\bar{r} = dx\, \bar{i} + dy\, \bar{j}$

Now $\bar{F} = (2x - y + 2z)\, \bar{i} + (x + y - z)\, \bar{j} + (3x - 2y - 5z)\, \bar{k}$

$= (2x - y)\, \bar{i} + (x + y)\, \bar{j} + (3x - 2y)\, \bar{k}$

$\Rightarrow \bar{F}.\, d\bar{r} = (2x - y)\, dx + (x + y)\, dy$.

Circulation $= \int_{C} \bar{F}.\, d\bar{r} = \int_{c}[(2x - y)\, dx + (x + y)\, dy\,]$.

Now parametric eq.s of the circle $x^2 + y^2 = 4$ are

$$x = 2\cos\theta;\quad y = 2\sin\theta;\ \text{where } 0 \le \theta \le 2\pi.$$

$$dx = -2\sin\theta.\, d\theta \quad \text{and} \quad dy = 2\cos\theta.\, d\theta$$

$$\therefore \text{Circulation} = \int_0^{2\pi}[(2.2\cos\theta - 2\sin\theta)(-2\sin\theta.\,d\theta) + (2\cos\theta + 2\sin\theta)2\cos\theta.\,d\theta\,]$$

$$= \int_0^{2\pi}[-8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta]d\theta$$

$$= \int_0^{2\pi}[4 - 4\sin\theta\cos\theta]d\theta = \int_0^{2\pi}[4 - 2\sin 2\theta\,]d\theta$$

$$= \left(4\theta - 2.\left(\frac{-\cos 2\theta}{2}\right)\right)_0^{2\pi} = (4\theta + \cos 2\theta)_0^{2\pi}$$

$$= 8\pi + \cos(4\pi) - (0 + \cos(0)) = 8\pi + (-1)^4 - 1 = 8\pi.$$

----

9) Show that the force field is $\overline{F} = 2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k}$ is conservative. Find the work done by moving a particle from (1,-1,2) to (3,2,-1) in this force field.

Sol: $\overline{F} = 2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k}$

$$\text{curl}\overline{F} = \nabla\times\overline{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3x^2yz^2) - \frac{\partial}{\partial z}(x^2z^3)\right] - \bar{j}\left[\frac{\partial}{\partial x}(3x^2yz^2) - \frac{\partial}{\partial z}(2xyz^3)\right] + \bar{k}\left[\frac{\partial}{\partial x}(x^2z^3) - \frac{\partial}{\partial y}(2xyz^3)\right]$$

$$= \bar{i}\,(3x^2z^2.1 - x^2.3z^2) - \bar{j}\,(3yz^2.2x - 2xy.3z^2) + \bar{k}\,(z^3.2x - 2xz^3.1)$$

$$= \bar{i}\,(3x^2z^2 - 3x^2z^2) - \bar{j}\,(6xyz^2 - 6xyz^2) + \bar{k}\,(2xz^3 - 2xz^3) = 0$$

$\therefore \text{curl}\overline{F} = 0 \quad\Rightarrow\quad \overline{F}$ is conservative.

Hence the work done is independent of the path.

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\overline{F} = 2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k}$

$$\Rightarrow \overline{F}.\,d\bar{r} = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$$

$$= yz^3\,(2x\,dx) + x^2z^3\,(dy) + x^2y\,(3z^2\,dz)$$

$$= yz^3\ d(x^2) + x^2z^3\,(dy) + x^2y\ d(z^3) = d(x^2yz^3)$$

$$\therefore \text{work done} = \int_C \overline{F}.\,d\bar{r} = \int_{(1,-1,2)}^{(3,2,-1)}[d(x^2yz^3)]$$

$$= (x^2yz^3)_{(1,-1,2)}^{(3,2,-1)} = 3^2.2.(-1)^3 - 1^2.(-1).2^3 = -18 + 8 = -8.$$

----

10) Calculate the work done by the force $\overline{F} = (3x^2 + 6y)\,\bar{i} - 14yz\ \bar{j} + 20xz^2\,\bar{k}$ along the lines from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1).

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\overline{F} = (3x^2 + 6y)\,\bar{i} - 14yz\ \bar{j} + 20xz^2\,\bar{k} \quad \Rightarrow \overline{F}.\,d\bar{r} = (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$.

(i) along the line AB from (0,0,0) to (1,0,0):

Eq. of the line is $y=0$ and $z=0$ $\Rightarrow dy=0;\ dz=0$

'x' limits are $x=0$ to $x=1$.

$$\therefore \int_{AB}\overline{F}.\,d\bar{r} = \int_{AB}[(3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz\,]$$

$$= \int_0^1[(3x^2 + 0)dx + 0 + 0] = \int_0^1[3x^2dx] = \left(3\frac{x^3}{3}\right)_0^1 = 1 - 0 = 1$$

$$\therefore \int_{AB}\overline{F}.\,d\bar{r} = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

(ii) along the line BC from (1,0,0) to (1,1,0):

Eq. of the line is x=1 and z=0 ⇒ dx=0; dz=0

'y' limits are y=0 to y=1.

∴ $\int_{BC}\overline{F}.d\overline{r} = \int_{BC}[(3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz]$

$= \int_0^1[0 - 14y(0)dy + 0] = \int_0^1 0\,dy = 0$

∴ $\int_{BC}\overline{F}.d\overline{r} = 0$ ………………..(2)

(iii) along the line CD from (1,1,0) to (1,1,1):

Eq. of the line is x=1 and y=1 ⇒ dx=0; dy=0

'z' limits are y=0 to y=1.

∴ $\int_{CD}\overline{F}.d\overline{r} = \int_{CD}[(3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz]$

$= \int_0^1[0 - 0 + 20.1.z^2dz] = 20\int_0^1 z^2dz$

$=20\left(\frac{z^3}{3}\right)_0^1 = 20\left(\frac{1}{3} - 0\right) = \frac{20}{3}$

∴ $\int_{CD}\overline{F}.d\overline{r} = \frac{20}{3}$ ………………..(3)

∴ work done = $\int_C\overline{F}.d\overline{r} = \int_{AB}\overline{F}.d\overline{r} + \int_{BC}\overline{F}.d\overline{r} + \int_{CD}\overline{F}.d\overline{r}$

$= 1 + 0 + \frac{20}{3} = \frac{23}{3}$.

----

11) If $\overline{F} = (4xy - 3x^2z^2)\,\overline{i} + 2x^2\,\overline{j} - 2x^3z\,\overline{k}$, then show that the work done by the force field $\overline{F}$ is independent of the curve joining two points.

Sol: We know that the work done by the force field $\overline{F}$ is independent of the path if

curl$\overline{F}$= = 0

Now curl$\overline{F} = \nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$

$=\overline{i}\left[\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)\right] - \overline{j}\left[\frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy - 3x^2z^2)\right] +$

$\overline{k}\left[\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z^2)\right]$

$= \overline{i}\,(0 - 0) - \overline{j}\,[-2z.3x^2 - (0 - 3x^2.2z)] + \overline{k}\,[2.2x - (4x.1 - 0)]$

$= -\overline{j}\,(6x^2z - 6x^2z) + \overline{k}\,(4x - 4x) = 0$

∴ curl$\overline{F} = 0$ ⇒ $\overline{F}$ is conservative.

Hence the work done is independent of the path.

☹ <u>SURFACE INTEGRAL</u> ☹

<u>Def</u>: Let $\overline{F}$ be a continuous vector function and 'S' be the surface. Divide 'S' into a finite number of sub-surfaces. Let δS be the position vector of a sub-surface whose magnitude being the area and its direction that of the outward normal to the sub-surface. Consider the sum $\sum\overline{F}.\delta S$ over all the sub-surfaces. The limit of this sum as the number of sub-surfaces tends to infinity and the area of each sub-surface tends to zero, is called the 'normal surface integral of $\overline{F}$ over 'S'. It is denoted by $\int_S \overline{F}.dS$ or $\int_S \overline{F}.Nds$ where 'N' is a unit outward normal vector to 'S'.

Note: Other type of integrals is $\int_S \overline{F} \times dS$

Flux across a surface: If $\overline{F}$ represents the velocity of a fluid particle, then the surface integral $\int_S \overline{F}.dS$ gives the total outward flux of $\overline{F}$ across a closed surface 'S'.

When the flux of $\overline{F}$ across every closed surface 'S' in a region 'E' is zero, then $\overline{F}$ is said to be a Solenoidal vector in the region 'E'.

Cartesian form: If $\overline{F} = F_1\,\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}$; then $\int_S \overline{F}.Nds = \iint_{R_1} F_1 dydz + F_2 dzdx + F_3 dxdy$

Note: 1) Let R_1 be the projection of 'S' on xy-plane. Then $\int_S \overline{F}.Nds = \iint_{R_1} (\overline{F}.N) \frac{dx\,dy}{|N.\overline{k}|}$.

2) Let R_2 be the projection of 'S' on yz-plane. Then $\int_S \overline{F}.Nds = \iint_{R_2} (\overline{F}.N) \frac{dy\,dz}{|N.\overline{i}|}$.

3) Let R_3 be the projection of 'S' on xz-plane. Then $\int_S \overline{F}.Nds = \iint_{R_3} (\overline{F}.N) \frac{dx\,dz}{|N.\overline{j}|}$.

Problems:

1) Evaluate $\int_S \overline{F}.Nds$ where $\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$ and 'S' is the closed surface of the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes x = 0, x = 2, y = 0, z = 0.

Sol: The closed surface 'S' is comprised of

S_1 : The rectangular face 'OAEB' in xy-plane

S_2 : The rectangular face 'OADC' in xz-plane

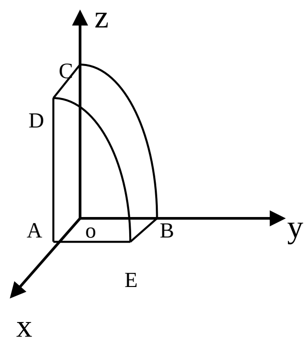
S_3 : The circular quadrant 'OBC' in yz-plane

S_4 : The circular quadrant 'AED' in the first octant

S_5 : The curved surface 'BCDE' of the cylinder in the first octant.

$\therefore \int_S \overline{F}.Nds = \int_{S_1} \overline{F}.Nds + \int_{S_2} \overline{F}.Nds + \int_{S_3} \overline{F}.Nds + \int_{S_4} \overline{F}.Nds + \int_{S_5} \overline{F}.Nds$ ……………….(1)

$\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$



(i) over S_1 : eq. of S_1 is $z = 0$ and $N = -\overline{k}$

$\therefore\ \overline{F}.N = -1(4xz^2) = -4x.0 = 0$

$\Rightarrow \int_{S_1} \overline{F}.Nds = 0$ …………….(2)

(ii) over S_2 : eq. of S_2 is $y = 0$ and $N = -\overline{j}$

$\therefore\ \overline{F}.N = -1(-y^2) = 0$

$\Rightarrow \int_{S_2} \overline{F}.Nds = 0$ …………….(3)

(iii) over S_3 : eq. of S_3 is $x = 0$ and $N = -\overline{i}$

$\therefore\ \overline{F}.N = -1(2x^2y) = 0$

$\Rightarrow \int_{S_3} \overline{F}.Nds = 0$ …………….(4)

(iv) over S_4 : eq. of S_4 is $x = 2$ and $N = \overline{i}$

$\therefore\ \overline{F}.N = 1(2x^2y) = 2.2^2y = 8y$

Let R_1 be the projection on yz-plane.

Then $\int_{S_4} \overline{F}.Nds = \iint_{R_1} (\overline{F}.N) \frac{dy\,dz}{|N.\overline{i}|} = \iint_{R_1} (8y) \frac{dy\,dz}{|\overline{i}.\overline{i}|} = 8 \iint_{R_1} y\,dy\,dz$

Now 'z' limits are z=0 to z=3

'y' limits are y=0 to y=$\sqrt{(9-z^2)}$

$$\therefore \int_{S_4} \overline{F}.Nds = 8\int_0^3 [\int_0^{\sqrt{(9-z^2)}} y\,dy]\,dz$$

$$= 8\int_0^3 \left(\frac{y^2}{2}\right)_0^{\sqrt{(9-z^2)}} dz = 4\int_0^3 [(\sqrt{(9-z^2)}\,)^2 - 0]dz$$

$$= 4\int_0^3 (9-z^2)\,dz = 4\left(9y - \frac{z^3}{3}\right)_0^3 = 4(27-9) = 72$$

$$\Rightarrow \int_{S_4} \overline{F}.Nds = 72 \ldots\ldots\ldots\ldots\ldots(5)$$

(v) over S_5 : eq. of S_5 is $y^2 + z^2 = 9$. It is not parallel to any co-ordinate plane.

Now grad$(y^2 + z^2) = \bar{i}\frac{\partial}{\partial x}(y^2+z^2) + \bar{j}\frac{\partial}{\partial y}(y^2+z^2) + \bar{k}\frac{\partial}{\partial z}(y^2+z^2)$

$$= \bar{i}\,.0 + \bar{j}\,(2y) + \bar{k}\,(2z) = 2y\,\bar{j} + 2z\,\bar{k}$$

$$\therefore\ N = \frac{2y\,\bar{j} + 2z\,\bar{k}}{\sqrt{((2y)^2+(2z)^2)}} = \frac{2(y\,\bar{j} + z\,\bar{k})}{2\sqrt{(y^2+z^2)}} = \frac{y\,\bar{j} + z\,\bar{k}}{\sqrt{9}} \qquad \therefore\ N = \frac{y\,\bar{j} + z\,\bar{k}}{3}$$

$$\therefore\ \overline{F}.N = -y^2\left(\frac{y}{3}\right) + 4xz^2\left(\frac{z}{3}\right) = -\frac{y^3}{3} + \frac{4xz^3}{3}$$

Let R_2 be the projection on xz-plane.

Then $\int_{S_5} \overline{F}.Nds = \iint_{R_2} (\overline{F}.N)\frac{dx\,dz}{|N.\bar{j}|} = \iint_{R_2}\left(-\frac{y^3}{3} + \frac{4xz^3}{3}\right)\frac{dx\,dz}{\left|\frac{y}{3}\right|}$

Now 'x' limits are x=0 to x=2

'z' limits are z=0 to z=3

$$\therefore \int_{S_5} \overline{F}.Nds = \int_0^2 [\int_0^3 (-y^2 + \frac{4xz^3}{y})\,dz]\,dx$$

$$= \int_0^2 [\int_0^3 (z^2 - 9 + \frac{4xz^3}{\sqrt{(9-z^2)}})\,dz]\,dx \qquad \text{(eq. of } s_5)$$

Put z=3 cosθ ⇒ dz= - 3sinθ dθ

If z=0, then θ= $\pi/2$

If z=3, then θ= 0

$$\therefore \int_{S_5} \overline{F}.Nds = \int_0^2 [\int_{\pi/2}^0 (9\cos^2\theta - 9 + \frac{4x.27\cos^3\theta}{3\sin\theta})\,(-3\sin\theta\,d\theta)]\,dx$$

$$= \int_0^2 [\int_{\pi/2}^0 (-9\sin^2\theta + \frac{4x.27\cos^3\theta}{3\sin\theta})\,(-3\sin\theta\,d\theta)]\,dx$$

$$= \int_0^2 [\int_0^{\pi/2} (\frac{-27\sin^3\theta + 4x.27\cos^3\theta}{3\sin\theta})\,3\sin\theta\,d\theta]\,dx$$

$$= \int_0^2 [\int_0^{\pi/2} (-27\sin^3\theta + 108x.\cos^3\theta)\,d\theta]\,dx$$

$$= \int_0^2 (-27\frac{(3-1)}{3(3-1)} + 108x\frac{(3-1)}{3(3-1)})\,dx$$

$$= \frac{2}{3}.27\int_0^2 (-1+4x)\,dx = 18\left(-x + 4\frac{x^2}{2}\right)_0^2 = 18\,(-2 + 8 - (0-0))$$

$$\Rightarrow \int_{S_5} \overline{F}.Nds = 108 \ldots\ldots\ldots\ldots\ldots(6)$$

Substitute (2),(3),(4),(5) and (6) in (1)

$$\therefore \int_S \overline{F}.Nds = 0 + 0 + 0 + 72 + 108 = 180.$$

----

2) Evaluate $\int_S \overline{F}.Nds$ where $\overline{F} = 6z\,\bar{i} - 4\,\bar{j} + y\,\bar{k}$ and 'S' is the portion of the plane 2x + 3y + 6z = 12 in the first octant.

<u>**Sol:**</u> Given portion of the plane is not parallel to any co-ordinate axes.

Let φ = 2x + 3y + 6z ⇒ $\frac{\partial\varphi}{\partial x} = 2;\ \frac{\partial\varphi}{\partial y} = 3;\ \frac{\partial\varphi}{\partial z} = 6.$

$\therefore$ Vector in the direction of normal to the surface $\varphi(x,y,z) = 0$ is grad$(\varphi) = \nabla\varphi$

$$= \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z} = 2\,\bar{i} + 3\,\bar{j} + 6\,\bar{k}$$

$$\Rightarrow |\nabla\varphi| = \sqrt{(2^2 + 3^2 + 6^2)} = \sqrt{49} = 7$$

$\therefore$ Unit out ward normal vector to the plane is $N = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\,\bar{i} + 3\,\bar{j} + 6\,\bar{k}}{7}$

$\bar{F} = 6z\,\bar{i} - 4\,\bar{j} + y\,\bar{k} \Rightarrow \bar{F}.N = 6z(\frac{2}{7}) - 4(\frac{3}{7}) + y(\frac{6}{7}) = \frac{12z-12+6y}{7}$.

Let R be the projection on xy-plane.

Then $\int_S \bar{F}.Nds = \iint_R (\bar{F}.N)\frac{dx\,dy}{|N.\bar{k}|} = \iint_R (\frac{12z-12+6y}{7})\frac{dx\,dy}{|\frac{6}{7}|} = \iint_R (2z - 2 + y)dx\,dy$.

In xy-plane z=0. Hence eq. of R in xy-plane is $2x + 3y = 12$. If $y = 0$, then $x = 6$

Now 'x' limits are x=0 to x=6

'y' limits are y=0 to $y = \frac{12-2x}{3}$

$$\therefore \int_S \bar{F}.Nds = \int_0^6[\int_0^{\frac{12-2x}{3}} (2z - 2 + y)\,dy]\,dx = \int_0^6[\int_0^{\frac{12-2x}{3}} (2\frac{12-2x-3y}{6} - 2 + y)\,dy]\,dx$$

$$= \int_0^6[\int_0^{\frac{12-2x}{3}} (4 - \frac{2}{3}x - y - 2 + y)\,dy]\,dx$$

$$= \int_0^6[\int_0^{\frac{12-2x}{3}} (2 - \frac{2}{3}x)\,dy]\,dx = \int_0^6 \left(2y - \frac{2}{3}xy\right)_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \left(2\frac{12-2x}{3} - \frac{2}{3}x\frac{12-2x}{3} - (0-0)\right)dx$$

$$= \int_0^6 \left(8 - \frac{4x}{3} - \frac{24}{9}x + \frac{4x^2}{9}\right)dx = \int_0^6 \left(8 - 4x + \frac{4x^2}{9}\right)dx$$

$$= \left(8x - 4\frac{x^2}{2} + \frac{4}{9}\frac{x^3}{3}\right)_0^6 = 48 - 72 + 32 = 8.$$

☹ VOLUME INTEGRAL ☹

Def: Let $\bar{F}$ be a continuous vector function defined over a volume 'V' bounded by a surface $\bar{r} = \bar{f}(u,v)$. Divide 'V' into 'm' sub-regions of volumes $\delta v_1, \delta v_2, \ldots, \delta v_i, \ldots, \delta v_m$. Let $P_i(r_i)$ be a point in δv_i. Then form the sum $I_m = \sum_{i=1}^{i=m} \bar{F}(r_i)\,\delta v_i$. The limit of I_m if it exists, as $m\to\infty$ in such a way that δv_i shrinks to a point, is called the volume integral of $\bar{F}(\bar{r})$ in the region 'V'. It is denoted by $\int_V \bar{F}\,dv = \iiint_V \bar{F}\,dxdydz$.

Cartesian form: If $\bar{F} = F_1\,\bar{i} + F_2\,\bar{j} + F_3\,\bar{k}$, then $\int_V \bar{F}\,dv = \bar{i}\iiint F_1 dxdydz + \bar{j}\iiint F_2 dxdydz + \bar{k}\iiint F_3 dxdydz$.

☹ GREEN'S THEOREM IN A PLANE ☹

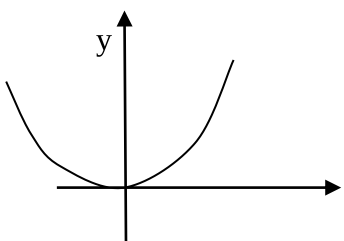
If 'R' is closed region in xy-plane bounded by a simple closed curve 'C' and if 'M' and 'N' are continuous functions of x and y having continuous derivatives in 'R', then

$\oint_C (M\,dx + N\,dy) = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx\,dy$; where 'C' is traversed in the positive(anti clock-wise) direction.

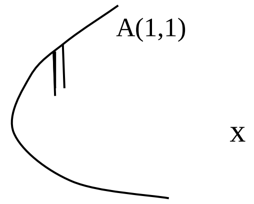
PROBLEMS:

1) Verify Green's theorem in plane for $\oint_C ((3x^2 - 8y^2)dx + (4y - 6xy)dy)$ where 'C' is the region bounded by $y = \sqrt{x}$ and $y = x^2$.

Sol: $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$



$\Rightarrow \frac{\partial M}{\partial y} = 0 - 8.2y = -16y;$ $\quad \frac{\partial N}{\partial x} = 0 - 6y.1 = -6y.$



Consider $y = \sqrt{x} \Rightarrow y^2 = x$

x	0	1	1
y	0	1	-1

Consider $y = x^2$

x	0	1	-1
y	0	1	1

$\therefore$ 'x' limits are $x = 0$ to $x = 1$

'y' limits are $y = x^2$ to $y = \sqrt{x}$.

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (-6y - (-16y))\,dy\right]dx$$

$$= \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (10y)\,dy\right]dx = \int_0^1 \left(10\,\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 \left((\sqrt{x})^2 - (x^2)^2\right)dx = 5\int_0^1 (x - x^4)dx$$

$$= 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5} - 0 - 0\right) = 5\left(\frac{5-2}{10}\right)$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \frac{3}{2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find $\oint_C (\mathbf{M\,dx + N\,dy})$:

(i) along the curve $y = x^2$ from 'O' to 'A'

$dy = 2x\,dx$

'x' limits are $x = 0$ to $x = 1$

$$\therefore \int_{OA} (M\,dx + N\,dy) = \int_0^1 \left((3x^2 - 8(x^2)^2)dx + (4x^2 - 6x\,x^2)2x\,dx\right)$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx = \left(3\frac{x^3}{3} + 8\frac{x^4}{4} - 20\,\frac{x^5}{5}\right)_0^1$$

$$= 1 + 2 - 4 - 0 - 0 = -1 \;\ldots\ldots\ldots\ldots\ldots(2)$$

(ii) along the curve $y = \sqrt{x}$ from 'A' to 'O'

i.e $y^2 = x \quad \Rightarrow dx = 2y\,dy$

'y' limits are $y = 1$ to $y = 0$

$$\therefore \int_{AO} (M\,dx + N\,dy) = \int_1^0 \left((3(y^2)^2 - 8y^2)2y\,dy + (4y - 6y^2y)\,dy\right)$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left(4\frac{y^2}{2} - 22\frac{y^4}{4} + 6\,\frac{y^6}{6}\right)_1^0$$

$$= 0 - 0 + 0 - \left(2 - \frac{11}{2} + 1\right) = \frac{5}{2} \;\ldots\ldots\ldots\ldots\ldots(3)$$

$$\therefore \oint_C (M\,dx + N\,dy) = \int_{OA} (M\,dx + N\,dy) + \int_{AO} (M\,dx + N\,dy)$$

$$= -1 + \frac{5}{2} = \frac{3}{2} \;\ldots\ldots\ldots\ldots\ldots(4)$$

$\therefore$ from (1) and (4) $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

Hence green's theorem is verified.

----

2) Verify Green's theorem in plane for $\oint_C((xy + y^2)dx + x^2dy)$ where 'C' is the region bounded by y = x and $y = x^2$.

Sol: $M = xy + y^2$ and $N = x^2$

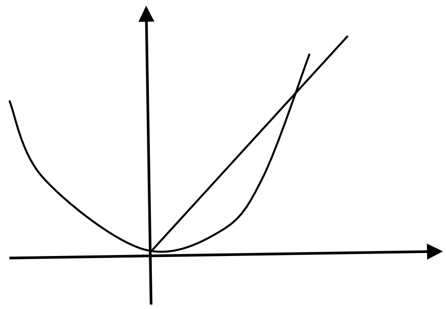
$\Rightarrow \frac{\partial M}{\partial y} = x.1 + 2y = x + 2y$; $\frac{\partial N}{\partial x} = 2x$.

Consider y = x

x	0	1	2
y	0	1	2

Consider $y = x^2$

x	0	1	-1
y	0	1	1



$\therefore$ 'x' limits are x = 0 to x = 1

'y' limits are $y = x^2$ to y = x.

$$\therefore \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_0^1\left[\int_{x^2}^{x}(2x - (x + 2y))\,dy\right]dx$$

$$= \int_0^1\left[\int_{x^2}^{x}(x - 2y)\,dy\right]dx = \int_0^1\left(xy - 2\,\frac{y^2}{2}\right)_{x^2}^{x}dx$$

$$= \int_0^1(x.x - x^2 - (x.x^2 - (x^2)^2)dx = \int_0^1(x^4 - x^3)dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \left(\frac{1}{5} - \frac{1}{4} - 0 + 0\right) = \left(\frac{4-5}{20}\right) = -\frac{1}{20} \text{ ....(1)}$$

To find $\oint_C(M\,dx + N\,dy)$:

(i) along the curve $y = x^2$ from 'O' to 'A'

dy = 2x dx

'x' limits are x = 0 to x = 1

$$\therefore \int_{OA}(M\,dx + N\,dy) = \int_0^1\left((x.x^2 + (x^2)^2)dx + (x^2)2x\,dx\right)$$

$$= \int_0^1(3x^3 + x^4)\,dx$$

$$= \left(3\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \text{ ....(2)}$$

(ii) along the curve y = x from 'A' to 'O'

$\Rightarrow$ dy = dx

'x' limits are y = 1 to y = 0

$$\therefore \int_{AO}(M\,dx + N\,dy) = \int_1^0\left((x.x + x^2)\,dx + (x^2)\,dx\right)$$

$$= \int_1^0(3x^2)\,dx$$

$$= \left(3\frac{x^3}{3}\right)_1^0 = 0 - 1 = -1 \text{ ....(3)}$$

$$\therefore \oint_C(M\,dx + N\,dy) = \int_{OA}(M\,dx + N\,dy) + \int_{AO}(M\,dx + N\,dy)$$

$$= -1 + \frac{19}{20} = -\frac{1}{20} \text{ ....(4)}$$

$\therefore$ from (1) and (4) $\oint_C(M\,dx + N\,dy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

Hence green's theorem is verified.

----

3) Verify Green's theorem for $\oint_C((3x^2 - 8y^2)dx + (4y - 6xy)dy)$ where 'C' is the region bounded by x = 0, y = 0 and x + y = 1.

Sol: $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

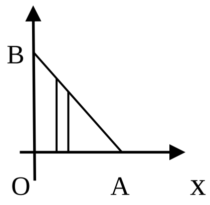
$\Rightarrow \frac{\partial M}{\partial y} = 0 - 8.2y = -16y$; $\quad \frac{\partial N}{\partial x} = 0 - 6y.1 = -6y$

Consider $x + y = 1$

x	0	1
y	1	0

$\therefore$ 'x' limits are $x = 0$ to $x = 1$

'y' limits are $y = 0$ to $y = 1 - x$.



$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \left[\int_0^{1-x} (-6y - (-16y))\, dy\right] dx$

$= \int_0^1 \left[\int_0^{1-x} (10y)\, dy\right] dx = \int_0^1 \left(10\,\frac{y^2}{2}\right)_0^{1-x} dx$

$= 5\int_0^1 ((1-x)^2 - 0)dx = 5\int_0^1 (1-x)^2 dx$

$= 5\left(\frac{(1-x)^3}{-3}\right)_0^1 = 5\left(0 - \frac{1}{-3}\right) = \frac{5}{3}$

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \frac{5}{3}$ ……………………..(1)

To find $\oint_C (M\,dx + N\,dy)$:

(i) along the curve 'OA', $y = 0$

$dy = 0$

'x' limits are $x = 0$ to $x = 1$

$\therefore \int_{OA} (M\,dx + N\,dy) = \int_0^1 \left((3x^2 - 0)dx + 0\right) = \int_0^1 \left((3x^2)dx\right)$

$= \left(3\,\frac{x^3}{3}\right)_0^1 = 1 - 0 = 1$ …………….(2)

(ii) along the curve 'AB' , eq. is $x + y = 1 \Rightarrow x = 1 - y \quad \Rightarrow dx = -dy$

'y' limits are $y = 0$ to $y = 1$

$\therefore \int_{AB} (M\,dx + N\,dy) = \int_0^1 \left((3(1-y)^2 - 8y^2)(-dy) + (4y - 6(1-y)y)\, dy\right)$

$= \int_0^1 \left((-3(1-y)^2 + 8y^2) + (4y - 6y + 6y^2)\right) dy$

$= \int_0^1 (-3(1-y)^2 + 14y^2 - 2y)\, dy$

$= \left(-3\,\frac{(1-y)^3}{-3} + 14\,\frac{y^3}{3} - 2\,\frac{y^2}{2}\right)_0^1$

$= 0 + \frac{14}{3} - 1 - (1 + 0) = \frac{8}{3}$ ……………………..(3)

(iii) along the curve 'BO', $x = 0$

$dx = 0$

'y' limits are $y = 1$ to $y = 0$.

$\therefore \int_{BO} (M\,dx + N\,dy) = \int_1^0 (0 + (4y - 0)\, dy = \int_1^0 ((4y)dy)$

$= \left(4\,\frac{y^2}{2}\right)_1^0 = 0 - 2 = -2$ …………….(4)

$\therefore \oint_C (M\,dx + N\,dy) = \int_{OA} (M\,dx + N\,dy) + \int_{AB} (M\,dx + N\,dy) + \int_{BO} (M\,dx + N\,dy)$

$= 1 + \frac{8}{3} - 2 = \frac{5}{3}$ ………..(5)

$\therefore$ from (1) and (5) $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Hence green's theorem is verified.

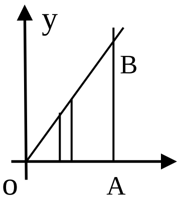
----

4) Evaluate by Green's theorem $\oint_C((y - \sin x)dx + \cos x\, dy)$ where 'C' is the triangle enclosed by the lines y = 0, x = π/2 and πy = 2x.

Sol: $M = y - \sin x$ and $N = \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = 1 - 0 = 1;$ $\frac{\partial N}{\partial x} = -\sin x$.

Consider πy = 2x. It is a straight line passing through orogin



∴ 'x' limits are x = 0 to x = π/2

'y' limits are y = 0 to y = 2x/π

∴ By Green's theorem $\oint_C(M\,dx + N\,dy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

$$= \int_0^{\pi/2}\left[\int_0^{2x/\pi} (-\sin x - 1)\,dy\right]dx$$

$$= \int_0^{\pi/2}(-\sin x - 1)(y)_0^{2x/\pi}\,dx$$

$$= \int_0^{\pi/2}(-\sin x - 1)\left(\frac{2x}{\pi} - 0\right)dx \quad = \frac{-2}{\pi}\int_0^{\pi/2}(x\sin x + x)dx$$

$$= \frac{-2}{\pi}\left\{(x.(-\cos x) - 1.(-\sin x))_0^{\pi/2} + \left(\frac{x^2}{2}\right)_0^{\pi/2}\right\}$$

$$= \frac{-2}{\pi}\left\{0 + \sin\left(\frac{\pi}{2}\right) - 0 - 0 + \frac{1}{2}\left(\frac{\pi^2}{4} - 0\right)\right\}$$

$$= \frac{-2}{\pi}\left\{1 + \frac{\pi^2}{8}\right\} = -\left(\frac{2}{\pi} + \frac{\pi}{4}\right).$$

----

5) Apply Green's theorem to evaluate $\oint_C((2x^2 - y^2)dx + (x^2 + y^2)dy)$ where 'C' is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$.

Sol: $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$\Rightarrow \frac{\partial M}{\partial y} = 0 - 2y = -2y;$ $\frac{\partial N}{\partial x} = 2x - 0 = 2x.$

∴ By Green's theorem $\oint_C(M\,dx + N\,dy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

$$= \iint_R(2x + 2y)dx\,dy$$

Change to polar co-ordinates x=r cosθ, y=r sinθ

⇒ dx dy = r dr dθ

∴ 'r' limits are r = 0 to r = a

'θ' limits are θ = 0 to θ = π

$$\therefore \oint_C(M\,dx + N\,dy) = 2\int_0^{\pi}\left[\int_0^{a} (r\cos\theta + r\sin\theta)\,r\,dr\right]d\theta$$

$$= 2\int_0^{\pi}\left[\int_0^{a} (\cos\theta + \sin\theta)\,r^2\,dr\right]d\theta$$

$$= 2\int_0^{\pi}(\cos\theta + \sin\theta)\left(\frac{r^3}{3}\right)_0^a d\theta$$

$$= 2\int_0^{\pi}(\cos\theta + \sin\theta)\left(\frac{a^3}{3} - 0\right)d\theta$$

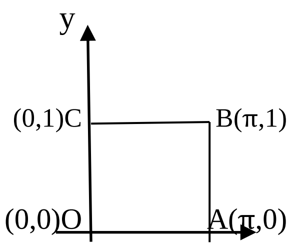
$$= \frac{2a^3}{3}\int_0^{\pi}(\cos\theta + \sin\theta)d\theta = \frac{2a^3}{3}(\sin\theta - \cos\theta)_0^{\pi}$$

$$= \frac{2a^3}{3}(\sin\pi - \cos\pi - (o - \cos 0)) = \frac{2a^3}{3}(1 + 1)$$

$$\therefore \oint_C(M\,dx + N\,dy) = \frac{4a^3}{3}.$$

----

6) Verify Green's theorem for $\oint_C ((x^2 - \cosh y)dx + (y + \sin x)dy)$ where 'C' is the rectangle with vertices (0,0), (π,0), (π,1) and (0,1).



Sol: $M = x^2 - \cosh y$ and $N = y + \sin x$

$\Rightarrow \frac{\partial M}{\partial y} = 0 - \sinh y = -\sinh y$; $\frac{\partial N}{\partial x} = 0 + \cos x = \cos x$.

∴ 'x' limits are $x = 0$ to $x = \pi$

'y' limits are $y = 0$ to $y = 1$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_0^{\pi}\left[\int_0^1 (\cos x + \sinh y)\,dy\right]dx$$

$$= \int_0^{\pi}(\cos x . y + \cosh y)_0^1 dx = \int_0^{\pi}(\cos x.1 + \cosh 1 - (0 + \cosh 0))dx$$

$$= \int_0^{\pi}(\cos x + \cosh 1 - 1)dx$$

$$= (\sin x + (\cosh 1 - 1).x)_0^{\pi} = \sin\pi + (\cosh 1 - 1)\pi - (0+0)$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = (\cosh 1 - 1)\pi \text{ ……………………(1)}$$

To find $\oint_C (M\,dx + N\,dy)$:

(i) along the line 'OA', eq. is $y = 0$

$dy = 0$

∴ 'x' limits are $x = 0$ to $x = \pi$

$$\therefore \int_{OA}(M\,dx + N\,dy) = \int_0^{\pi}\left((x^2 - \cosh 0)dx + 0\right) = \int_0^{\pi}\left((x^2 - 1)dx\right)$$

$$= \left(\frac{x^3}{3} - x\right)_0^{\pi} = \frac{\pi^3}{3} - \pi - 0 = \frac{\pi^3}{3} - \pi \text{ …………….(2)}$$

(ii) along the line 'AB', eq. is $x = \pi$

$dx = 0$

∴ 'y' limits are $y = 0$ to $y = 1$

$$\therefore \int_{AB}(M\,dx + N\,dy) = \int_0^1 (0 + (y + \sin\pi)dy) = \int_0^1 (y\,dy)$$

$$= \left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2} \text{ ………..(3)}$$

(iii) along the line 'BC', eq. is $y = 1$

$dy = 0$

∴ 'x' limits are $x = \pi$ to $x = 0$

$$\therefore \int_{BC}(M\,dx + N\,dy) = \int_{\pi}^0 (x^2 - \cosh 1)dx + 0) = \int_{\pi}^0 (x^2 - \cosh 1)dx)$$

$$= \left(\frac{x^3}{3} - \cosh 1 . x\right)_{\pi}^0 = (0 - 0 - (\frac{\pi^3}{3} - \pi\cosh 1)$$

$$= -\frac{\pi^3}{3} + \pi\cosh 1 \text{ ………..(3)}$$

(iv) along the line 'CO', eq. is $x = 0$

$dx = 0$

∴ 'y' limits are $y = 1$ to $y = 0$

$$\therefore \int_{CO}(M\,dx + N\,dy) = \int_1^0 (0 + (y + \sin 0)dy) = \int_1^0 (y\,dy)$$

$$= \left(\frac{y^2}{2}\right)_1^0 = 0 - \frac{1}{2} = -\frac{1}{2} \text{ ………..(4)}$$

$$\therefore \oint_C (M\,dx + N\,dy) = \int_{OA}(M\,dx + N\,dy) + \int_{AB}(M\,dx + N\,dy) + \int_{BC}(M\,dx + N\,dy) + \int_{CO}(M\,dx + N\,dy)$$

$$= \frac{\pi^3}{3} - \pi + \frac{1}{2} - \frac{\pi^3}{3} + \pi\cosh 1 - \frac{1}{2} = (\cosh 1 - 1)\pi \text{ …………….(5)}$$

$\therefore$ from (1) and (5) $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Hence green's theorem is verified.

☹ STOKE'S THEOREM ☹

If 'S' be an open surface bounded by a closed curve 'C' and $\bar{F} = F_1\,\bar{i} + F_2\,\bar{j} + F_3\,\bar{k}$ be any continuously differentiable vector function, then $\int_C \bar{F}\,.\,d\bar{r} = \int_S \text{Curl}\bar{F}\,.\,N\,ds$ where 'N' is a unit outward normal vector at any point of 'S'.

Note: If $\bar{F} = \varphi\,\bar{i} + \psi\,\bar{j}$ be a vector function in xy-plane, then

$$\int_C \bar{F}\,.\,d\bar{r} = \int_C (\varphi\,\bar{i} + \psi\,\bar{j}).(dx\,\bar{i} + dy\,\bar{j}) = \int_C (\varphi\,dx + \psi\,dy)$$

Unit outward normal vector at any point of xy-plane is $N = \bar{k}$

$$\text{Curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi & \psi & 0 \end{vmatrix} = \bar{i}\,(0\text{-}0) - \bar{j}\,(0\text{-}0) + \bar{k}\left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right) = \bar{k}\left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right)$$

$$\text{Curl}\bar{F}\,.\,N = \bar{k}\left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right).\bar{k} = \frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}$$

$$\therefore \int_S \text{Curl}\bar{F}\,.\,N\,ds = \iint_S \left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right) dx\,dy.$$

$\therefore$ Stoke's theorem is $\int_C (\varphi\,dx + \psi\,dy) = \iint_S \left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right) dx\,dy$ which is Green's theorem.

PROBLEMS:

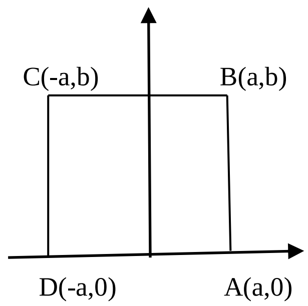
1) Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\,\bar{j}$ taken around the rectangle bounded by the lines $x = \pm\,a$, $y = 0$, $y = b$.

Sol: Stoke's theorem is $\int_C \bar{F}\,.\,d\bar{r} = \int_S \text{Curl}\bar{F}\,.\,N\,ds$. In xy-plane z=0

$$\text{Curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \bar{i}\,(0\text{-}0) - \bar{j}\,(0\text{-}0) + \bar{k}\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 + y^2)\right)$$

$$= \bar{k}\,(-2y.1 - 0 - 2y) = -4y\,\bar{k}.$$



Unit outward normal vector at any point of xy-plane is $N = \bar{k}$

$\text{Curl}\bar{F}\,.\,N = -\,4y\,\bar{k}\,.\,\bar{k} = -\,4y$.

'x' limits are $x = -\,a$ to $x = a$

'y' limits are $y = 0$ to $y = b$

$$\int_S \text{Curl}\bar{F}\,.\,N\,ds = \iint_R (-\,4y)\frac{dx\,dy}{|N.\bar{k}|} = \int_0^b \left[\int_{-a}^{a} (-\,4y)\,dx\right] dy$$

$$= \int_0^b [-\,4y\,(x)_{-a}^{a}]\,dy$$

$$= \int_0^b [-\,4y\,(a + a)]\,dy = -8a\int_0^b [y]\,dy$$

$$= -8a\left(\frac{y^2}{2}\right)_0^b = -4a\,(b^2 - 0)$$

$\therefore \int_S \text{Curl}\overline{F} \cdot N\, ds = -4ab^2$………………..(1)

To evaluate $\int_C \overline{F} \cdot d\overline{r}$:

$\overline{F} \cdot d\overline{r} = [(x^2+y^2)\overline{i} - 2xy\,\overline{j}].[dx\,\overline{i} + dy\,\overline{j}\,] = (x^2+y^2)\,dx - 2xy\,dy.$

(i) along 'AB': Eq. is $x = a \Rightarrow dx=0$

'y' limits are $y=0$ to $y=b$

$$\therefore \int_{AB} \overline{F} \cdot d\overline{r} = \int_0^b [(x^2+y^2)\,dx - 2xy\,dy] = \int_0^b [0 - 2ay\,dy]$$

$$= -2a\left(\frac{y^2}{2}\right)_0^b = -a\,(b^2 - 0) = -ab^2 \text{…………(2)}$$

(ii) along 'BC': Eq. is $y = b \Rightarrow dy=0$

'x' limits are $x=a$ to $x=-a$

$$\therefore \int_{BC} \overline{F} \cdot d\overline{r} = \int_a^{-a} [(x^2+b^2)\,dx - 0] = \left(\frac{x^3}{3} + b^2x\right)_a^{-a}$$

$$=\left(\frac{(-a)^3}{3} + b^2(-a) - [\frac{a^3}{3} + b^2a]\right) = -\frac{2a^3}{3} - 2ab^2 \text{…………(3)}$$

(iii) along 'CD': Eq. is $x = -a \Rightarrow dx=0$

'y' limits are $y=b$ to $y=0$

$$\therefore \int_{CD} \overline{F} \cdot d\overline{r} = \int_b^0 [0 - 2(-a)y] = \int_b^0 [2ay]\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_b^0 = a(0 - b^2) = -ab^2 \text{…………(4)}$$

(iv) along 'DA': Eq. is $y = 0 \Rightarrow dy=0$

'x' limits are $x=-a$ to $x = a$

$$\therefore \int_{DA} \overline{F} \cdot d\overline{r} = \int_{-a}^{a} [(x^2+0)\,dx - 0] = \left(\frac{x^3}{3}\right)_{-a}^{a}$$

$$=\left(\frac{a^3}{3} - \frac{(-a)^3}{3}\right) = \frac{2a^3}{3} \text{………………(5)}$$

$$\int_C \overline{F} \cdot d\overline{r} = \int_{AB} \overline{F} \cdot d\overline{r} + \int_{BC} \overline{F} \cdot d\overline{r} + \int_{CD} \overline{F} \cdot d\overline{r} + \int_{DA} \overline{F} \cdot d\overline{r}$$

$$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \text{…………………..(6)}$$

From (1) and (6), $\int_C \overline{F} \cdot d\overline{r} = \int_S \text{Curl}\overline{F} \cdot N\, ds$. Hence Stoke's theorem is verified.

----

2) Verify Stoke's theorem for the vector field $\overline{F} = (2x - y)\overline{i} - yz^2\,\overline{j} - y^2z\,\overline{k}$ over the upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the xy-plane.

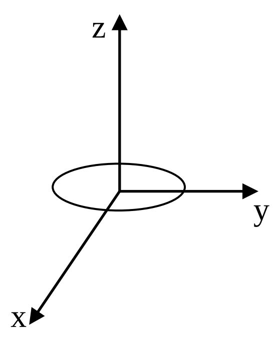
Sol: The boundary 'C' of 'S' is the circle $x^2+y^2 = 1$; $z=0$ in xy-plane.

$$\text{Curl}\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \overline{i}\,[\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)] - \overline{j}\,[\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x - y)]$$

$$+ \overline{k}\,[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x - y)]$$

$$= \overline{i}\,[(-2yz) - (-2yz)] - \overline{j}\,[0 - 0] + \overline{k}\,[0 - (-1)] = \overline{k}\,.$$



Unit outward normal vector at any point of xy-plane is $N = \overline{k}$

$\therefore \int_S \text{Curl}\overline{F} \cdot N\, ds = \iint_R (\overline{k} \cdot \overline{k})ds = \iint_R ds$ = area of the circle = $\pi.1^2$

$\Rightarrow \int_S \text{Curl}\overline{F} \cdot N\, ds = \pi$………………..(1)

To evaluate $\int_C \overline{F}\,.\,d\overline{r}$:

Parametric eq. of 'C' is x = cost, y = sint; $0 \le t \le 2\pi$.

dx = - sint dt, dy = cost dt

$\overline{F}\,.\,d\overline{r} = [(2x - y)\overline{i} - yz^2\,\overline{j} - y^2z\,\overline{k}\,].[\,dx\,\overline{i} + dy\,\overline{j} + dz\,\overline{k}\,]$

$= (2x - y)\,dx - yz^2\,dy - y^2z\,dz = (2x - y)\,dx - 0 - 0$

$\overline{F}\,.\,d\overline{r} = (2x - y)\,dx$

$\therefore \int_C \overline{F}\,.\,d\overline{r} = \int_C [(2x - y)\,dx = \int_0^{2\pi}[(2\cos t - \sin t)\,(-\sin t)dt$

$= \int_0^{2\pi}[(-2\sin t\cos t + \sin^2 t)dt$

$= \int_0^{2\pi}[\left(-\sin 2t + \frac{1-\cos 2t}{2}\right)dt$

$= \left(\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{2.2}\right)_0^{2\pi} = \left(\frac{\cos 4\pi}{2} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} - (\frac{\cos 0}{2} + 0 - 0)\right)$

$= \frac{1}{2} + \pi - 0 - \frac{1}{2} = \pi$

$\therefore \int_C \overline{F}\,.\,d\overline{r} = \pi$ ……………….(2)

From (1) and (2), $\int_C \overline{F}\,.\,d\overline{r} = \int_S \text{Curl}\overline{F}\,.\,N\,ds$. Hence Stoke's theorem is verified.

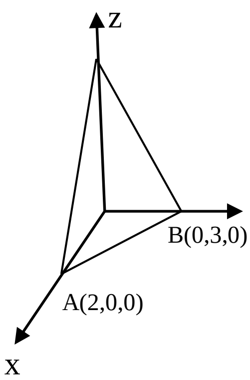
----

3) Using Stoke's theorem evaluate $\int_C[(x + y)dx + (2x - z)dy + (y + z)dz]$ where 'C' is the boundary of the triangle with vertices (2,0,0), (0,3,0), (0,0,6).

<u>Sol:</u> Here $\overline{F}\,.\,d\overline{r} = (x + y)dx + (2x - z)dy + (y + z)dz$

$\Rightarrow \overline{F} = (x + y)\overline{i} + (2x - z)\,\overline{j} + (y + z)\,\overline{k}$

Stoke's theorem is $\int_C \overline{F}\,.\,d\overline{r} = \int_S \text{Curl}\overline{F}\,.\,N\,ds$



Now $\text{Curl}\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & 2x - z & y + z \end{vmatrix}$

$= \overline{i}\,[\frac{\partial}{\partial y}(y + z) - \frac{\partial}{\partial z}(2x - z)] - \overline{j}\,[\frac{\partial}{\partial x}(y + z) - \frac{\partial}{\partial z}(x + y)]$

$+ \overline{k}\,[\,\frac{\partial}{\partial x}(2x - z) - \frac{\partial}{\partial y}(x + y)]$

$= \overline{i}\,[(1 + 0) - (0 - 1)] - \overline{j}\,[0 - 0] + \overline{k}\,[\,2 - 0 - (0 + 1)]$

$\text{Curl}\overline{F} = 2\,\overline{i} + \overline{k}$.

Eq. of plane through A,B,C is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \Rightarrow 3x + 2y + z = 6$

Unit outward normal vector 'N' to this plane = $\frac{\text{grad}(3x + 2y + z\,)}{|\text{grad}(3x + 2y + z\,)|}$

$= \frac{3\,\overline{i}+2\,\overline{j}+\,\overline{k}}{\sqrt{(9+4+1)}} = \frac{3\,\overline{i}+2\,\overline{j}+\,\overline{k}}{\sqrt{14}}$

$\text{Curl}\overline{F}\,.\,N = 2.\,\frac{3}{\sqrt{14}} + 1.\,\frac{1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$.

Let 'R' be the projection of the plane in xy-plane, i.e z=0

'x' limits are x=0 to x=2

'y' limits are y=0 to y= (6 – 3x)/2

$\int_C \overline{F}\,.\,d\overline{r} = \int_S \text{Curl}\overline{F}\,.\,N\,ds = \iint_R (\frac{7}{\sqrt{14}})\frac{dx\,dy}{|N.\overline{k}|} = \iint_R (\frac{7}{\sqrt{14}})\frac{dx\,dy}{\left|\frac{1}{\sqrt{14}}\right|}$

$= 7\int_0^2[\int_0^{(6\,-\,3x)/2} dy]\,dx$

$$=7\int_0^2 [\,(y)_0^{(6-3x)/2}]\,dx$$

$$=7\int_0^2 [\frac{6-3x}{2}]\,dx$$

$$=\frac{7}{2}\left(6x-3\frac{x^2}{2}\right)_0^2=\frac{7}{2}(12-6-(0))=21.$$

----

4) Apply Stoke's theorem to evaluate $\int_C [y\,dx + z\,dy + x\,dz]$ where 'C' is the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $x + z = a$.

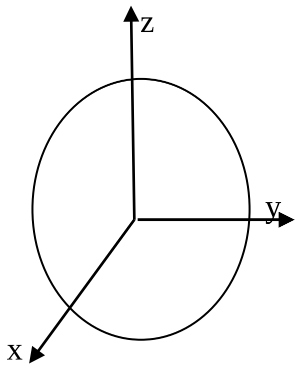
Sol: Here $\bar{F}.d\bar{r} = y\,dx + z\,dy + x\,dz \quad \Rightarrow \bar{F} = y\,\bar{i} + z\,\bar{j} + x\,\bar{k}$

Stoke's theorem is $\int_C \bar{F}.d\bar{r} = \int_S \text{Curl}\bar{F}.N\,ds$

Now $\text{Curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \bar{i}\,[\frac{\partial x}{\partial y}-\frac{\partial z}{\partial z}] - \bar{j}\,[\frac{\partial x}{\partial x}-\frac{\partial y}{\partial z}] + \bar{k}\,[\frac{\partial z}{\partial x}-\frac{\partial y}{\partial y}]$

$= \bar{i}\,(0\text{-}1) - \bar{j}\,(1\text{-}0) + \bar{k}\,(0\text{-}1)$

$\text{Curl}\bar{F} = -\bar{i}-\bar{j}-\bar{k}$.



Clearly 'C' is the circle lying in the plane $x + z = a$ and having $A(a,0,0)$, $B(0,0,a)$ as the extremities of the diameter.

Unit outward normal vector 'N' to the surface $= \frac{\text{grad}(x+z)}{|\text{grad}(x+z)|}$

$$N = \frac{\bar{i}+0\,\bar{j}+\bar{k}}{\sqrt{(1+0+1)}} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$$

$\text{Curl}\bar{F}.N = (-1).\frac{1}{\sqrt{2}} + (-1).\frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}}$.

Let 'R' be the projection of the surface on xz-plane, which is the circle on AB as diameter.

$\int_C \bar{F}.d\bar{r} = \int_S \text{Curl}\bar{F}.N\,ds = \iint_R (\frac{-2}{\sqrt{2}})ds = \frac{-2}{\sqrt{2}}\iint_R ds$

$= \frac{-2}{\sqrt{2}}$.area of the ciecle (diameter = AB = $a\sqrt{2}$)

$= \frac{-2}{\sqrt{2}}.\pi\,(\frac{a\sqrt{2}}{2})^2 = \frac{-\pi a^2}{\sqrt{2}}$.

☹ GAUSS DIVERGENCE THEOREM ☹

If $\bar{F}$ is a continuously differentiable vector function in the region 'E' bounded by the closed surface 'S', then $\int_S \bar{F}.N\,ds = \int_E \text{div}\bar{F}\,dv$, where 'N' is the unit external normal vector.

Note: In Cartesian form if $\bar{F} = F_1\,\bar{i} + F_2\,\bar{j} + F_3\,\bar{k}$, then

$$\int_S [\,F_1 dy\,dz + F_2 dz\,dx + F_3 dx\,dy = \iiint_E \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

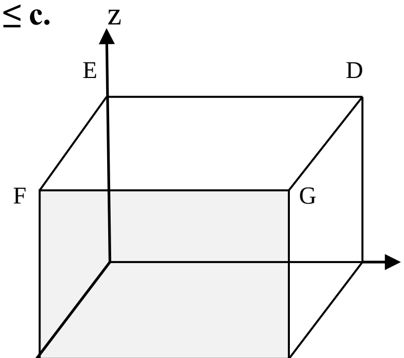
PROBLEMS:

1) Verify Gauss divergence theorem for $\bar{F} = (x^2-yz)\bar{i} + (y^2-xz)\bar{j} + (z^2-xy)\bar{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Sol: $\text{div}\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$

$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-xz) + \frac{\partial}{\partial z}(z^2-xy)$

$= 2x - 0 + 2y - 0 + 2z - 0 = 2(x + y + z)$



'x' limits are x=0 to x=a

'y' limits are y=0 to y=b O C

'z' limits are z=0 to z=c

$$\therefore \int_E \operatorname{div}\overline{F}\ dv = \int_0^c \int_0^b \int_0^a 2(x+y+z)\,dx\,dy\,dz$$

A B

$$= 2\int_0^c \int_0^b \left[\int_0^a (x+y+z)dx\right] dy\,dz$$

x

$$= 2\int_0^c \int_0^b \left(\frac{x^2}{2} + (y+z)x\right)_0^a dy\,dz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + (y+z)a - 0\right) dy\,dz$$

$$= 2\int_0^c \left[\int_0^b \left(\frac{a^2}{2} + ya + za\right) dy\right] dz = 2\int_0^c \left(\frac{a^2}{2}y + a\frac{y^2}{2} + zay\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2}{2}b + a\frac{b^2}{2} + zab - 0\right) dz = 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + zab\right) dz$$

$$= 2\left(\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right)_0^c = 2\left(\frac{a^2b}{2}c + \frac{ab^2}{2}c + ab\frac{c^2}{2} - 0\right)$$

$$\therefore \int_E \operatorname{div}\overline{F}\ dv = abc(a+b+c) \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To evaluate $\int_S \overline{F}\cdot N\,ds$:

The surface has 6 faces.

(i) over the face S_1 (OABC): eq. is $z = 0$.

Unit out ward normal vector $N = -\overline{k}$ and projection is on xy-plane

'x' limits are $x = 0$ to $x = a$

'y' limits are $y = 0$ to $y = b$

$\overline{F}\cdot N = -1(z^2 - xy) = xy - z^2 = xy - 0 = xy.$

$$\therefore \int_{S_1} \overline{F}\cdot N\,ds = \int_0^b \int_0^a xy\frac{dx\,dy}{|N.\overline{k}|} = \int_0^b \int_0^a xy\frac{dx\,dy}{|-\overline{k}.\overline{k}|}$$

$$= \int_0^b \left[\int_0^a xy\,dx\right] dy = \int_0^b \left(y\,\frac{x^2}{2}\right)_0^a dy$$

$$= \int_0^b \frac{y}{2}(a^2 - 0)\,dy = \frac{a^2}{2}\int_0^b y\,dy$$

$$= \frac{a^2}{2}\left(\frac{y^2}{2}\right)_0^b = \frac{a^2}{4}(b^2 - 0) = \frac{a^2b^2}{4}$$

$$\therefore \int_{S_1} \overline{F}\cdot N\,ds = \frac{a^2b^2}{4} \ldots\ldots\ldots\ldots\ldots(2)$$

(ii) over the face S_2 (DEFG): eq. is $z = c$.

Unit out ward normal vector $N = \overline{k}$ and projection is on xy-plane

'x' limits are $x = 0$ to $x = a$

'y' limits are $y = 0$ to $y = b$

$\overline{F}\cdot N = 1(z^2 - xy) = c^2 - xy.$

$$\therefore \int_{S_2} \overline{F}\cdot N\,ds = \int_0^b \int_0^a (c^2 - xy)\frac{dx\,dy}{|N.\overline{k}|} = \int_0^b \int_0^a (c^2 - xy)\frac{dx\,dy}{|\overline{k}.\overline{k}|}$$

$$= \int_0^b \left[\int_0^a (c^2 - xy)\,dx\right] dy = \int_0^b \left(c^2x - y\,\frac{x^2}{2}\right)_0^a dy$$

$$= \int_0^b \left(c^2a - y\frac{a^2}{2} - (0)\right) dy = \int_0^b \left(c^2a - \frac{a^2}{2}y\right) dy$$

$$= \left(c^2a.y - \frac{a^2}{2}\frac{y^2}{2}\right)_0^b = c^2a.b - \frac{a^2}{2}\frac{b^2}{2} - (0)$$

$$\therefore \int_{S_2} \overline{F}\cdot N\,ds = abc^2 - \frac{a^2b^2}{4} \ldots\ldots\ldots\ldots\ldots(3)$$

(iii) over the face S_3 (OCDE): eq. is $x = 0$.

Unit out ward normal vector N= - $\bar{i}$ and projection is on yz-plane

'y' limits are $y = 0$ to $y = b$

'z' limits are $z = 0$ to $z = c$

$$\bar{F} . N = -1(x^2 - yz) = 0 + yz = yz$$

$$\therefore \int_{S_3} \bar{F} . N\, ds = \int_0^b \int_0^c yz \frac{dy\, dz}{|N.\bar{i}|} = \int_0^b \int_0^c yz \frac{dy\, dz}{|-\bar{i}.\bar{i}|}$$

$$= \int_0^b [\int_0^c yz\, dz]\, dy = \int_0^b \left(y\, \frac{z^2}{2}\right)_0^c dy$$

$$= \int_0^b \left(y\frac{c^2}{2} - 0\right) dy = \frac{c^2}{2} \int_0^b y\, dy$$

$$= \frac{c^2}{2} \left(\frac{y^2}{2}\right)_0^b = \frac{c^2}{2} \left(\frac{b^2}{2} - 0 \right)$$

$$\therefore \int_{S_3} \bar{F} . N\, ds = \frac{b^2c^2}{4} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

(iv) over the face S_4 (ABGF): eq. is $x = a$.

Unit out ward normal vector N= $\bar{i}$ and projection is on yz-plane

'y' limits are $y = 0$ to $y = b$

'z' limits are $z = 0$ to $z = c$

$$\bar{F} . N = 1(x^2 - yz) = a^2 - yz$$

$$\therefore \int_{S_4} \bar{F} . N\, ds = \int_0^b \int_0^c (a^2 - yz) \frac{dy\, dz}{|N.\bar{i}|} = \int_0^b \int_0^c (a^2 - yz) \frac{dy\, dz}{|\bar{i}.\bar{i}|}$$

$$= \int_0^b [\int_0^c (a^2 - yz)\, dz]\, dy = \int_0^b \left(a^2 z - y\, \frac{z^2}{2}\right)_0^c dy$$

$$= \int_0^b \left(a^2 c - y\frac{c^2}{2} - 0\right) dy = \int_0^b (a^2 c - \frac{c^2}{2} y)\, dy$$

$$= \left(a^2 c\, y - \frac{c^2}{2}\frac{y^2}{2}\right)_0^b = a^2 cb - \frac{c^2}{2}\frac{b^2}{2} - 0$$

$$\therefore \int_{S_4} \bar{F} . N\, ds = a^2 bc - \frac{b^2c^2}{4} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

(v) over the face S_5 (AOEF): eq. is $y = 0$.

Unit out ward normal vector N= - $\bar{j}$ and projection is on xz-plane

'x' limits are $x = 0$ to $x = a$

'z' limits are $z = 0$ to $z = c$

$$\bar{F} . N = -1(y^2 - zx) = 0 + zx = xz$$

$$\therefore \int_{S_5} \bar{F} . N\, ds = \int_0^c \int_0^a xz \frac{dx\, dz}{|N.\bar{j}|} = \int_0^c \int_0^a xz \frac{dx\, dz}{|-\bar{j}.\bar{j}|}$$

$$= \int_0^c [\int_0^a xz\, dx]\, dz = \int_0^c \left(z\, \frac{x^2}{2}\right)_0^a dz$$

$$= \int_0^c \left(z\frac{a^2}{2} - 0\right) dz = \int_0^b \frac{a^2}{2}\, z\, dz$$

$$= \frac{a^2}{2} \left(\frac{z^2}{2}\right)_0^c = \frac{a^2}{2} \left(\frac{c^2}{2} - 0\right)$$

$$\therefore \int_{S_5} \bar{F} . N\, ds = \frac{a^2c^2}{4} \quad \ldots\ldots\ldots\ldots\ldots(6)$$

(vi) over the face S_6 (BCDG): eq. is $y = b$.

Unit out ward normal vector N= $\bar{j}$ and projection is on xz-plane

'x' limits are $x = 0$ to $x = a$

'z' limits are $z = 0$ to $z = c$

$\overline{F}.N = 1(y^2 - zx) = b^2 - zx$

$\therefore \int_{S_6} \overline{F}.N\,ds = \int_0^a \int_0^c (b^2 - zx)\,\frac{dx\,dz}{|N.\overline{j}|} = \int_0^a \int_0^c (b^2 - zx)\,\frac{dx\,dz}{|\overline{j}.\overline{j}|}$

$= \int_0^a [\int_0^c (b^2 - zx)\,dz]\,dx = \int_0^a \left(b^2 z - x\,\frac{z^2}{2}\right)_0^c dx$

$= \int_0^a \left(b^2 c - x\frac{c^2}{2} - 0\right) dx = \int_0^a (b^2 c - \frac{c^2}{2}x)\,dx$

$= \left(b^2 c\,x - \frac{c^2}{2}\frac{x^2}{2}\right)_0^a = b^2 ca - \frac{c^2}{2}\frac{a^2}{2} - 0$

$\therefore \int_{S_6} \overline{F}.N\,ds = ab^2 c - \frac{a^2c^2}{4}$ ....(7)

$\therefore \int_S \overline{F}.N\,ds$ = (2)+(3)+(4)+(5)+(6)+(7)

$= \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$

$= abc^2 + a^2bc + ab^2c = abc(a+b+c)$ ....(8)

Form (1) and (8) $\int_S \overline{F}.N\,ds = \int_E div\overline{F}\;dv$. Hence Gauss divergent theorem is verified.

2) Verify Gauss divergent theorem for $\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$ taken over the region of the first octant of the cylinder $y^2 + z^2 = 9$ and $x = 0, x = 2$.

<u>Sol:</u> By the previous problem1(pg.no.8), we have $\int_S \overline{F}.Nds = 180$...........(1)

$div\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$

$= 2y.2x - 2y + 4x.2z = 4xy - 2y + 8xz.$

'x' limits are $x = 0$ to $x = 2$

'y' limits are $y = 0$ to $y = 3$

'z' limits are $z = 0$ to $z = \sqrt{(9 - y^2)}$

$\therefore \int_E div\,\overline{F}\,dv = \iiint_E (4xy - 2y + 8xz)\,dx\,dy\,dz$

$= \int_0^2 \int_0^3 [\int_0^{\sqrt{(9-y^2)}} (4xy - 2y + 8xz)dz]\,dy\,dx$

$= \int_0^2 \int_0^3 \left(4xy.z - 2y.z + 8x.\frac{z^2}{2}\right)_0^{\sqrt{(9-y^2)}} dy\,dx$

$= \int_0^2 \int_0^3 \left(4xy.\sqrt{(9-y^2)} - 2y.\sqrt{(9-y^2)} + 4x.(9-y^2)\right) dy\,dx$

$= \int_0^2 [\int_0^3 \left((2x-1)2y.\sqrt{(9-y^2)} + 4x.(9-y^2)\right) dy\,]dx$

Put $9 - y^2 = t \Rightarrow -2y\,dy = dt \Rightarrow 2y\,dy = -dt$

$= \int_0^2 [(2x-1)\int_0^3 (-\sqrt{t})dt + 4x\int_0^3 (9-y^2)dy\,]dx$

$= \int_0^2 [(2x-1)\left(-\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^3 + 4x\left(9y - \frac{y^3}{3}\right)_0^3]dx$

$= \int_0^2 [(2x-1)\left(-\frac{2}{3}(9-y^2)^{\frac{3}{2}}\right)_0^3 + 4x\left(27 - \frac{27}{3} - 0\right)]dx$

$= \int_0^2 [(2x-1)\left(-\frac{2}{3}(0-27)\right) + 4x.18\,]dx$

$= \int_0^2 [(2x-1).18 + 4x.18\,]dx = 18\int_0^2 (6x-1)\,dx$

$= 18\left(6\frac{x^2}{2} - x\right)_0^2 = 18\,(\,12 - 2 - 0\,) = 180$

$\therefore \int_E div\,\overline{F}\,dv = 180$ ....(2)

From (1) and (2) $\int_S \overline{F}.\mathrm{N}ds = \int_E \operatorname{div}\overline{F}\, dv.$

$\Rightarrow$ Gauss divergence theorem is verified.

----

3) Use divergence theorem to evaluate $\int_S \overline{F}.dS$ where $\overline{F} = 4x\,\overline{i} - 2y^2\,\overline{j} + z^2\,\overline{k}$, and 'S' is the surface bounded by the region $x^2+y^2= 4$, z = 0 and z = 3.

<u>**Sol:**</u> Gauss divergence theorem is $\int_S \overline{F}.\mathrm{N}ds = \int_E \operatorname{div}\overline{F}\, dv$.

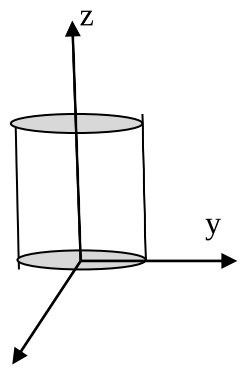
Now $\operatorname{div}\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$

$= 4 - 2.2y + 2z = 4 - 4y + 2z.$

'z' limits are $z = 0$ to $z = 3$

'x' limits are $x = -2$ to $x = 2$

'y' limits are $y = -\sqrt{(4-x^2)}$ to $y = \sqrt{(4-x^2)}$



$\therefore \int_E \operatorname{div}\overline{F}\, dv = \iiint_E (4 - 4y + 2z)\, dx\, dy\, dz$

$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} [\int_0^3 (4 - 4y + 2z)dz]\, dy\, dx$

$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left((4 - 4y)z + 2\frac{z^2}{2}\right)_0^3 dy\, dx$

$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left((4 - 4y)3 + 9\right) dy\, dx$

$= \int_{-2}^{2} [\int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (21 - 12y)dy\,]\, dx = \int_{-2}^{2} \left(21y - 12\frac{y^2}{2}\right)_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$

$= \int_{-2}^{2} \left(21\sqrt{(4-x^2)} - 3(4-x^2) - \{-21\sqrt{(4-x^2)} - 3(4-x^2)\}\right) dx$

$= \int_{-2}^{2} \left(42\sqrt{(4-x^2)}\right) dx = 42.2 \int_0^2 \left(\sqrt{(4-x^2)}\right) dx$

$= 84 \left(\frac{x}{2}\sqrt{(4-x^2)} + \frac{4}{2}\sin^{-1}(\frac{x}{2})\right)_0^2 = 84\,(\,0 + 2\sin^{-1}(1) - 0 + \sin^{-1}(0)\,)$

$= 84\,(\,2\,\frac{\pi}{2} - 0\,) = 84\,\pi.$

$\Rightarrow \int_S \overline{F}.\mathrm{N}ds = 84\,\pi.$

----

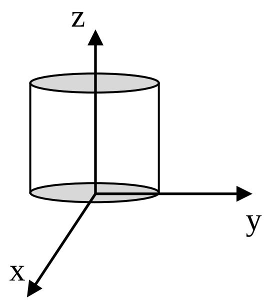
4) Verify Gauss divergent theorem for $\overline{F} = y\,\overline{i} + x\,\overline{j} + z^2\,\overline{k}$ over the cylindrical region bounded by $x^2+y^2= 9$, z = 0 and z = 2.

<u>**Sol:**</u> $\operatorname{div}\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial}{\partial z}(z^2) = 0+0+2z = 2z$

'z' limits are $z = 0$ to $z = 2$

'x' limits are $x = -3$ to $x = 3$

'y' limits are $y = -\sqrt{(9-x^2)}$ to $y = \sqrt{(9-x^2)}$



$\therefore \int_E \operatorname{div}\overline{F}\, dv = \iiint_E 2z\, dx\, dy\, dz$

$= \int_{-3}^{3} \int_{-\sqrt{(9-x^2)}}^{\sqrt{(9-x^2)}} [\int_0^2 2z\, dz]\, dy\, dx$

$= \int_{-3}^{3} \int_{-\sqrt{(9-x^2)}}^{\sqrt{(9-x^2)}} \left(2\frac{z^2}{2}\right)_0^2 dy\, dx$

$= \int_{-3}^{3} \int_{-\sqrt{(9-x^2)}}^{\sqrt{(9-x^2)}} (4)\, dy\, dx$

$=4\int_{-3}^{3}[\int_{-\sqrt{(9-x^2)}}^{\sqrt{(9-x^2)}} dy]\,dx \;=4\int_{-3}^{3}\,(\,y\,)_{-\sqrt{(9-x^2)}}^{\sqrt{(9-x^2)}}\;dx$

$= 4\int_{-3}^{3}\left(\sqrt{(9-x^2)} - \{-\sqrt{(9-x^2)}\}\right)\,dx$

$= 4\int_{-3}^{3}\left(2\sqrt{(9-x^2)}\right)dx \;=8.2\int_{0}^{3}\left(\sqrt{(9-x^2)}\right)dx$

$= \;16\left(\frac{x}{2}\sqrt{(9-x^2)} + \frac{9}{2}\sin^{-1}(\frac{x}{3})\right)_0^3 = 16\,(\,0 + \frac{9}{2}\sin^{-1}(\,1) - 0 + \frac{9}{2}\sin^{-1}(\,0)\,)$

$= 16\,(\frac{9}{2}\frac{\pi}{2} - 0\,) \;= 36\,\pi.$

$\Rightarrow \int_E \text{div}\,\overline{F}\,dv \;= 36\,\pi$..........(1)

To evaluate $\int_S \overline{F}\,.\,N\,ds$:

'S' has three parts which are the bottom and top faces S_1 and S_2 respectively and the cueved portion S_3 .

(i) over the face S_1: eq. is $z = 0$.

Unit out ward normal vector N= - $\overline{k}$ and projection is on xy-plane

$\overline{F}\,.\,N = -\,1.z^2 = -\,z^2 = 0.$

$\therefore \int_{S_1}\overline{F}\,.\,N\,ds = 0$ …………..(2)

(ii) over the face S_2: eq. is $z = 2$.

Unit out ward normal vector N= $\overline{k}$ and projection is on xy-plane

$\overline{F}\,.\,N = 1.z^2 = z^2 = 4.$

$\therefore \int_{S_2}\overline{F}\,.\,N\,ds = \int\int_{S_2} 4\;ds$ = 4(are of the circle $x^2+y^2= 9$)

$= 4(\pi.3^2\,) = 36\pi$ ……..(3)

(iii) over the face S_3 : eq. is $x^2+y^2= 9$

$\nabla(\,x^2+y^2\,) = \overline{i}\frac{\partial}{\partial x}(x^2+y^2) + \overline{j}\frac{\partial}{\partial y}(x^2+y^2) + \overline{k}\frac{\partial}{\partial z}(x^2+y^2)$

$= \overline{i}\;2x + \overline{j}\;2y + \overline{k}\;0 = 2x\;\overline{i} + 2y\;\overline{j}\;.$

Unit out ward normal vector $N = \frac{2x\,\overline{i} + 2y\,\overline{j}}{\sqrt{((2x)^2+(2x)^2)}} = \frac{2(x\,\overline{i} + y\,\overline{j}\,)}{2\sqrt{(x^2+y^2)}} = \frac{(x\,\overline{i} + y\,\overline{j}\,)}{\sqrt{9}} = \frac{x\,\overline{i} + y\,\overline{j}}{3}$.

$\overline{F}\,.\,N = y.\,\frac{x}{3} + x.\,\frac{y}{3} = \frac{2xy}{3}.$

Consider projection on xz-plane.

'x' limits are $x = -\,3$ to $x = 3$

'z' limits are $z = 0$ to $z = 2$

$\therefore \int_{S_3}\overline{F}\,.\,N\,ds = \int_0^2\int_{-3}^{3}\frac{2xy}{3}\,\frac{dx\,dz}{|N.\overline{j}|} = \int_0^2\int_{-3}^{3}\frac{2xy}{3}\,\frac{dx\,dz}{\left|\frac{y}{3}\right|}$

$= \int_0^2[\int_{-3}^{3} 2x\;dx]\,dz = \int_0^2\left(2\,\frac{x^2}{2}\right)_{-3}^{3}dz = \;\int_0^2(3^2-(-3)^2)\,dz$

$\therefore \int_{S_3}\overline{F}\,.\,N\,ds = 0$ ……………(4)

$\therefore \int_S \overline{F}\,.\,N\,ds = (2)+(3)+(4)$

$= 0 + 36\pi + 0 = 36\pi$ …………(5)

From (1) and (5) $\int_S \overline{F}.\,Nds = \int_E \text{div}\,\overline{F}\,dv.$

$\Rightarrow$ Gauss divergence theorem is verified.

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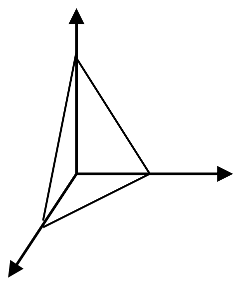
5) verify Gauss divergent theorem for $\overline{F} = x^2\,\overline{i} + y^2\,\overline{j} + z^2\,\overline{k}$ over the surface of the solid cut off by the plane x + y + z = a in the first octant.



Sol: $\operatorname{div}\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$

$= 2x + 2y + 2z = 2(x + y + z)$

'x' limits are $x = 0$ to $x = a$

'y' limits are $y = 0$ to $y = a - x$

'z' limits are $z = 0$ to $z = a - x - y$

$\therefore \int_E \operatorname{div}\overline{F}\, dv = \iiint_E 2(x + y + z)\, dx\, dy\, dz$

$= \int_0^a \int_0^{a-x} [\int_0^{a-x-y} 2(x + y + z)\, dz]\, dy\, dx$

$= 2\int_0^a \int_0^{a-x} \left((x+y)z + \frac{z^2}{2}\right)_0^{a-x-y} dy\, dx$

$= 2\int_0^a \int_0^{a-x} \left((x+y)(a-x-y) + \frac{(a-x-y)^2}{2}\right) dy\, dx$

$= 2\int_0^a \int_0^{a-x} (a-x-y)\left((x+y) + \frac{(a-x-y)}{2}\right) dydx = 2\int_0^a \int_0^{a-x} (a-(x+y))\left(\frac{2(x+y)+(a-(x+y))}{2}\right) dy\, dx$

$= \int_0^a \int_0^{a-x} (a-(x+y))(a+x+y)\, dy\, dx$

$= \int_0^a [\int_0^{a-x} (a^2-(x+y)^2)\, dy]\, dx = \int_0^a \left(a^2 y - \frac{(x+y)^3}{3}\right)_0^{a-x} dx$

$= \int_0^a \left(a^2(a-x) - \frac{(x+a-x)^3}{3} - (0 - \frac{(x+0)^3}{3}\right) dx$

$= \int_0^a \left(a^3 - a^2 x - \frac{a^3}{3} + \frac{x^3}{3}\right) dx = \int_0^a \left(\frac{2a^3}{3} - a^2 x + \frac{x^3}{3}\right) dx = \left(\frac{2a^3}{3}x - a^2\frac{x^2}{2} + \frac{x^4}{3.4}\right)_0^a$

$= \left(\frac{2a^3}{3}a - a^2\frac{a^2}{2} + \frac{a^4}{3.4} - 0\right) = \frac{2a^4}{3} - \frac{a^4}{2} + \frac{a^4}{12} = \frac{8a^4 - 6a^4 + a^4}{12}$

$\therefore \int_E \operatorname{div}\overline{F}\, dv = \frac{a^4}{4}$ ……………………(1)

To evaluate $\int_S \overline{F}\,.\,N\, ds$:

Eq. of the surface is x + y + z = a

Let $\varphi = x + y + z$

$\nabla\varphi = \overline{i}\frac{\partial}{\partial x}(x + y + z) + \overline{j}\frac{\partial}{\partial y}(x + y + z) + \overline{k}\frac{\partial}{\partial z}(x + y + z)$

$= \overline{i}\,1 + \overline{j}\,1 + \overline{k}\,1 = \overline{i} + \overline{j} + \overline{k}$.

Unit out ward normal vector $N = \frac{\overline{i}+\overline{j}+\overline{k}}{\sqrt{(1+1+1)}} = \frac{\overline{i}+\overline{j}+\overline{k}}{\sqrt{3)}}$.

$\overline{F}\,.\,N = \frac{(x^2+y^2+z^2)}{\sqrt{3)}}$.

Consider projection on xy-plane(i.e z = 0)

'x' limits are $x = 0$ to $x = a$

'y' limits are $y = 0$ to $y = a - x$

$\therefore \int_S \overline{F}\,.\,N\, ds = \int_0^a \int_0^{a-x} \frac{(x^2+y^2+z^2)}{\sqrt{3)}} \frac{dx\, dy}{|N.\overline{k}|} = \int_0^a \int_0^{a-x} \frac{(x^2+y^2+z^2)}{\sqrt{3)}} \frac{dx\, dy}{\left|\frac{1}{\sqrt{3)}}\right|}$

$= \int_0^a \int_0^{a-x} (x^2 + y^2 + (a-x-y)^2)\, dx\, dy$ $(\because x+y+z=a)$

$= \int_0^a [\int_0^{a-x} (x^2 + y^2 + (a-x-y)^2)\, dy]\, dx$

$= \int_0^a \left(x^2 y + \frac{y^3}{3} + \frac{(a-x-y)^3}{-3}\right)_0^{a-x} dx$

$$= \int_0^a \left(x^2(a-x) + \frac{(a-x)^3}{3} - \frac{(a-x-(a-x))^3}{3} - \{0 + 0 - \frac{(a-x-0)^3}{3}\}\right) dx$$

$$= \int_0^a \left(ax^2 - x^3 + \frac{(a-x)^3}{3} - 0 + \frac{(a-x)^3}{3}\right) dx$$

$$= \int_0^a \left(ax^2 - x^3 + \frac{2(a-x)^3}{3}\right) dx = \left(a\frac{x^3}{3} - \frac{x^4}{4} + \frac{2}{3}\frac{(a-x)^4}{-4}\right)_0^a$$

$$= \left(a\frac{a^3}{3} - \frac{a^4}{4} - \frac{1}{6}.0 - \{0 - 0 - \frac{1}{6}a^4\}\right) = \frac{a^4}{3} - \frac{a^4}{4} + \frac{a^4}{6}$$

$\therefore \int_S \overline{F} . N\, ds = \frac{a^4}{4}$ …………….(2)

From (1) and (2) $\int_S \overline{F}. Nds = \int_E \text{div}\, \overline{F}\, dv$.

$\Rightarrow$ Gauss divergence theorem is verified.

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6) Use divergence theorem to evaluate $\int_S \overline{F}. dS$ where $\overline{F} = x\,\bar{i} + y\,\bar{j} + z^2\,\bar{k}$, and 'S' is the surface bounded by the cone $x^2+y^2= z^2$, in the plane $z = 4$, in the first octant.

<u>Sol:</u> Gauss divergence theorem is $\int_S \overline{F}. Nds = \int_E \text{div}\, \overline{F}\, dv$.

Now $\text{div}\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial}{\partial z}(z^2)$

$= 1 + 1 + 2z = 2 + 2z.$

On the cone $x^2+y^2= z^2$ and $z = 4 \Rightarrow x^2+y^2= 16$

'z' limits are $z = 0$ to $z = 4$

'x' limits are $x = 0$ to $x = 4$

'y' limits are $y = 0$ to $y = \sqrt{(16 - x^2)}$

$\therefore \int_E \text{div}\, \overline{F}\, dv = \iiint_E (2 + 2z)\, dx\, dy\, dz$

$$= \int_0^4 \int_0^{\sqrt{(16-x^2)}} [\int_0^4 (2 + 2z) dz]\, dy\, dx$$

$$= \int_0^4 \int_0^{\sqrt{(16-x^2)}} \left(2z + 2\frac{z^2}{2}\right)_0^4 dy\, dx$$

$$= \int_0^4 \int_0^{\sqrt{(16-x^2)}} (8 + 16)\, dy\, dx = 24 \int_0^4 [\int_0^{\sqrt{(16-x^2)}} dy]\, dx$$

$$= 24 \int_0^4 (\,y\,)_0^{\sqrt{(16-x^2)}} dx = 24 \int_0^4 \sqrt{(16 - x^2)}\, dx$$

$$= 24 \left(\frac{x}{2}\sqrt{(16 - x^2)} + \frac{16}{2}\sin^{-1}(\frac{x}{4})\right)_0^4 = 24 \left(0 + 8\sin^{-1}(\frac{4}{4}) - 0\right)_0^4$$

$$= 24(\,8.\frac{\pi}{2}\,) = 96\pi.$$
