

# Solving ~~QFTs~~ convection-diffusion equations with finite volume methods

Kurganov and Tadmor (KT)  $O(x^2)$  central scheme - An appetizer

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QCD & beyond with the functional renormalisation group,  
Heidelberg, July 17, 2019



- ▶ FRG flow eqs. especially LPA flow eqs. of low-energy effective models can be written as **Convection–diffusion equations<sup>1</sup>**:

$$\partial_t u(t, x) + \partial_x F[t, u(t, x)] = \partial_x Q[t, x, u(t, x), \partial_x u(t, x)] + \partial_x S(t, x)$$

- Very common class of PDEs in Physics, Engineering and Numerical Mathematics
- Well established numerical methods available (see talks of N. Wink and D. Rischke)  $\Rightarrow$  among them are **Finite volume (FV) methods**

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<sup>1</sup>E. Grossi and N. Wink (2019), arXiv: 1903.09503 [hep-th]

$$\partial_t u(t, x) + \partial_x F[t, u(t, x)] = \partial_x Q[t, x, u(t, x), \partial_x u(t, x)] + \partial_x S(t, x) \quad (1)$$

- ▶  $u(t, x)$  vector of conserved quantities
- ▶  $F[t, u(t, x)]$  nonlinear convection flux
- ▶  $Q[t, x, u(t, x), \partial_x u(t, x)]$  dissipation flux
- ▶  $S(t, x)$  source term

## Examples:

- ▶ Linear Advection and (inviscid/viscous) Burgers' eqs. with non-smooth ICs
- ▶ Euler Equations of Gas Dynamics (Shock tube problem)
- ▶ **FRG LPA flow equations**

Journal of Computational Physics **160**, 241–282 (2000)

doi:10.1006/jcph.2000.6459, available online at <http://www.idealibrary.com> on  IDEAL®

## New High-Resolution Central Schemes for Nonlinear Conservation Laws and Convection–Diffusion Equations

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Received April 8, 1999; revised December 8, 1999

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A. Kurganov and E. Tadmor, J. Comput. Phys. **160** (2000)

**Semi-discrete central scheme**

- ▶ Equidistant spatial finite volume grid with  $n$  cells

$$\{x_j\} = \{x_0, x_0 + \Delta x, \dots, x_1\} \quad (2)$$

$$\{u_j\} = \{u(t, x_0), u(t, x_0 + \Delta x), \dots, u(t, x_1)\} \quad (3)$$

with extrapolated ghost points e.g.  $u_{-1} = 2u_0 - u_1$

- ▶ Conservation form:

$$\begin{aligned} \frac{du_j}{dt} = & -\frac{H_{j+1/2}(t) - H_{j-1/2}(t)}{\Delta x} + \frac{P_{j+1/2}(t) - P_{j-1/2}(t)}{\Delta x} \\ & + \frac{S_{j+1/2}(t) - S_{j-1/2}(t)}{\Delta x} \end{aligned} \quad (4)$$

- $H_{j\pm 1/2}(t)$  Numerical convection flux on left/right cell boundary
- $P_{j\pm 1/2}(t)$  Numerical diffusion flux on left/right cell boundary
- $S_{j\pm 1/2}(t)$  Numerical source term on left/right cell boundary

$$H_{j+1/2}(t) = \frac{F[t, u_{j+1/2}^+] + F[t, u_{j+1/2}^-]}{2} - \frac{a_{j+1/2}}{2} (u_{j+1/2}^+ - u_{j+1/2}^-), \quad (5)$$

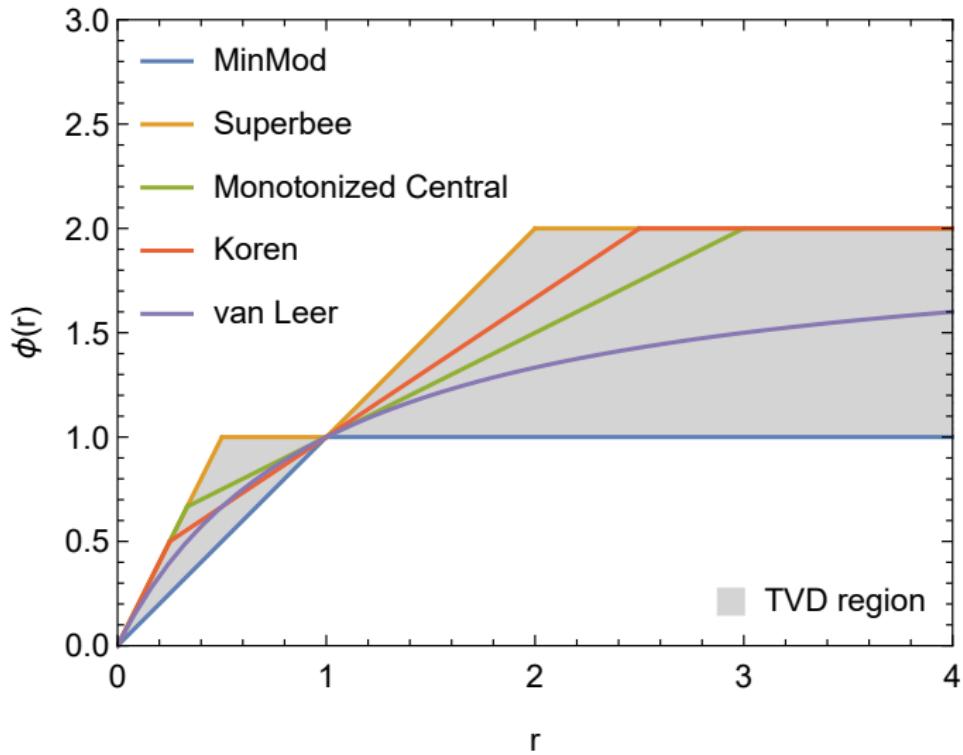
- ▶ Maximal local speed

$$a_{j+1/2} = \max \left( \rho \left( \frac{\partial F}{\partial u}(u_{j+1/2}^+) \right), \rho \left( \frac{\partial F}{\partial u}(u_{j+1/2}^-) \right) \right) \quad (6)$$

- ▶ Approximate derivatives (TVD reconstruction) with limiter  $\phi$

$$u_{j+1/2}^+ = u_{j+1} - \frac{\Delta x}{2} \partial_x u_{j+1} = u_{j+1} - \frac{1}{2} \phi \left( \frac{u_{j+1} - u_j}{u_{j+2} - u_{j+1}} \right) (u_{j+2} - u_{j+1}) \quad (7)$$

$$u_{j+1/2}^- = u_j + \frac{\Delta x}{2} \partial_x u_j = u_j + \frac{1}{2} \phi \left( \frac{u_j - u_{j-1}}{u_{j+1} - u_j} \right) (u_{j+1} - u_j) \quad (8)$$



expressions from: [en.wikipedia.org/wiki/Flux\\_limiter](https://en.wikipedia.org/wiki/Flux_limiter) - 2019.07.10 - 16:30

- Central difference approximation of diffusion flux:

$$\begin{aligned} P_{j+1/2}(t) = & \frac{1}{2} Q[t, x_j, u_j, (u_{j+1} - u_j)/\Delta x] + \\ & + \frac{1}{2} Q[t, x_{j+1}, u_{j+1}, (u_{j+1} - u_j)/\Delta x] \end{aligned} \quad (9)$$

- Diffusion flux for  $\dot{u}_j$  is based on the 3-point stencil  $\{u_{j-1}, u_j, u_{j+1}\}$
- Analogous central difference approximation of source flux:

$$S_{j+1/2}(t) = \frac{1}{2} S(t, x_j) + \frac{1}{2} S(t, x_{j+1}) \quad (10)$$

- ▶ Method of Lines (MoL) finite volume PDE discretization  $\Rightarrow$  sparse (banded) ODE system
- ▶ Total RHS flux for  $\dot{u}_j$  is based on the 5-point stencil  $\{u_{j-2}, u_{j-1}, u_j, u_{j+1}, u_{j+2}\}$
- ▶ Approximate derivatives are reconstructed from the computed cell averages using  $\phi$ -Limiter
- ▶ Only additional information (apart from PDE and grid): spectral radius of jacobian  $\rho = \max |\lambda_i(\partial F / \partial u)|$  to approximate local speeds

# Godunov first order upwind scheme

$$\begin{aligned} \frac{du_j}{dt} = & -\frac{F[t, u_{j+1}] - F[t, u_j]}{\Delta x} \\ & + \frac{Q[t, x_{j+1}, u_{j+1}, (u_{j+1} - u_j)/\Delta x] - Q[t, x_j, u_j, (u_j - u_{j-1})/\Delta x]}{\Delta x} \\ & + \frac{S[t, x_{j+1}] - S[t, x_j]}{\Delta x} \end{aligned} \quad (11)$$

- ▶ Applicable only to convection–diffusion eqs. with  $\partial F / \partial u < 0 \forall t, u$ 
  - Linear Advection eq. with negative velocity
  - FRG LPA flow eqs.
  - ...

# Classical examples

- ▶ Linear Advection eq.:

$$\partial_t u(t, x) - \partial_x u(t, x) = 0, \quad u(0, x) = \begin{cases} 1 & 3 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \quad (\text{LA})$$

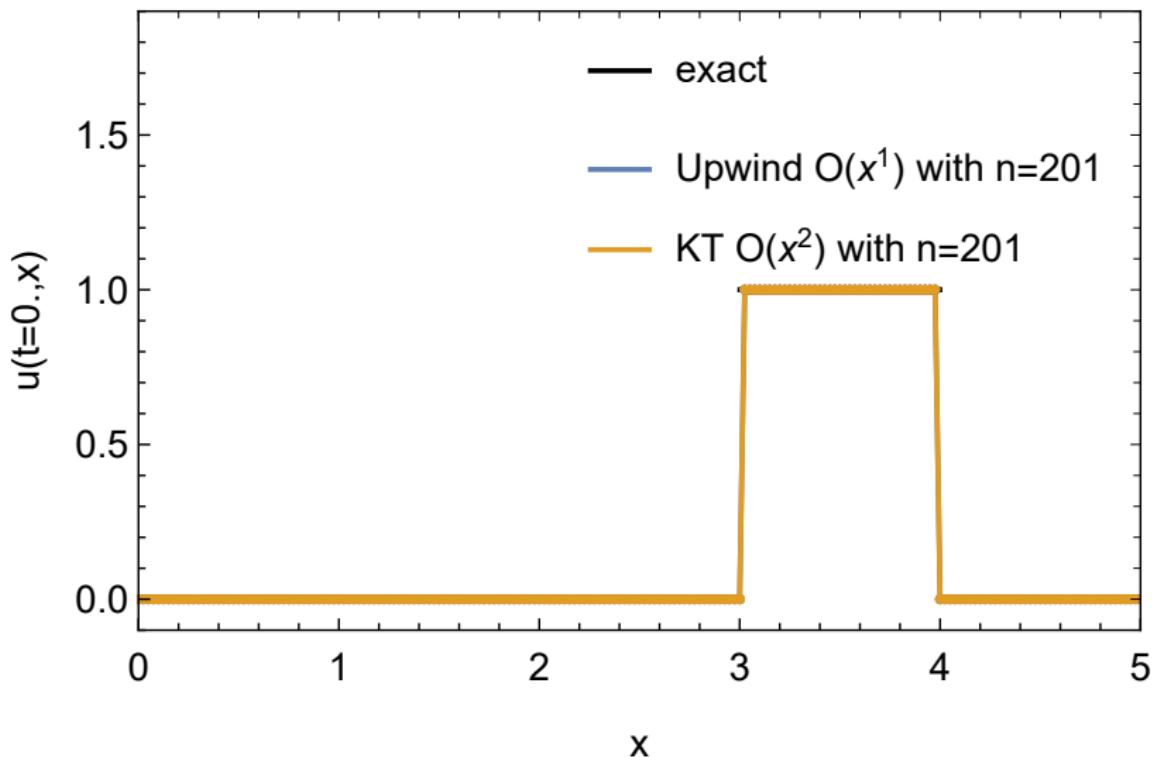
- ▶ Burgers' eq.:

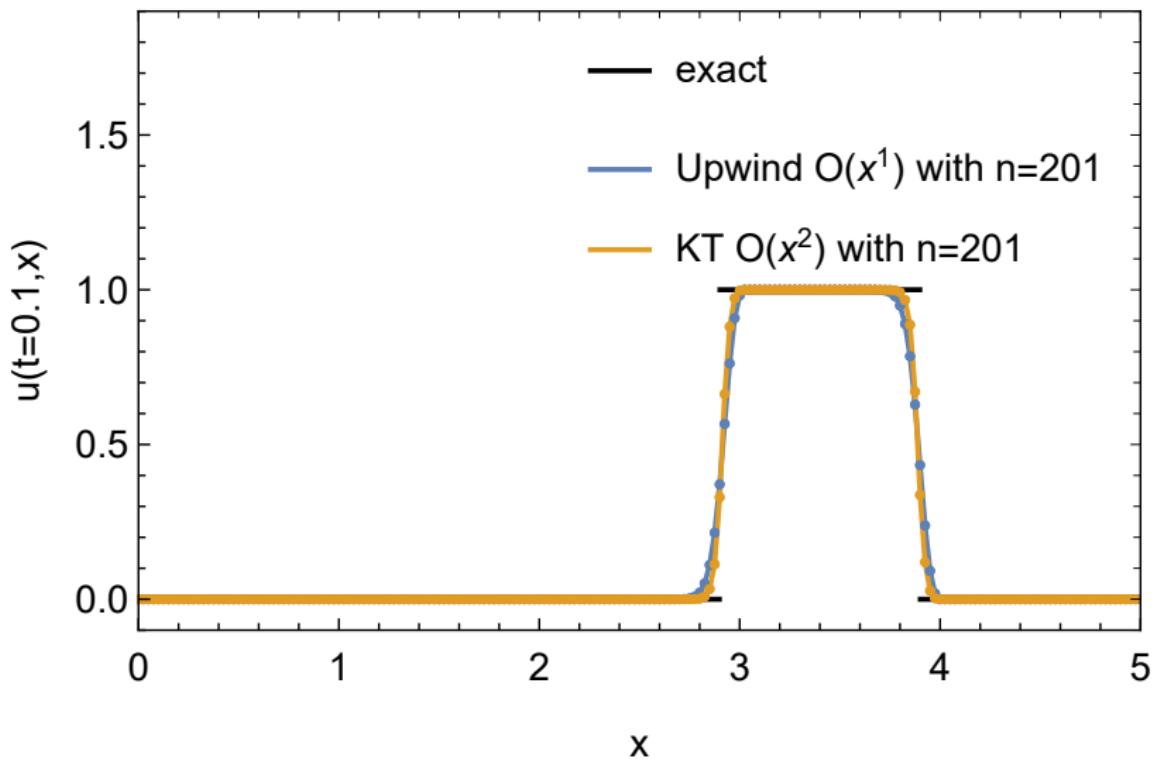
$$\partial_t u(t, x) + \partial_x \frac{u(t, x)^2}{2} = \nu \partial_x^2 u(t, x) \quad (\text{B})$$

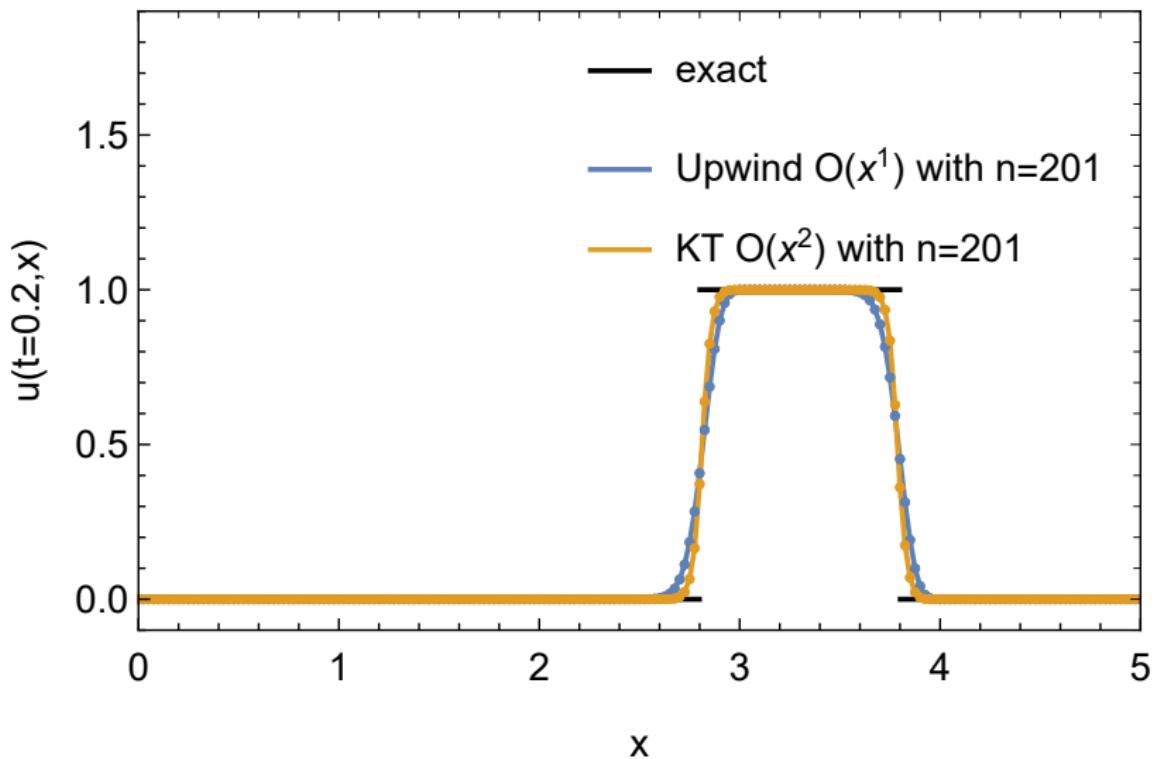
$$u(0, x) = 0.5 + \sin(x) \quad \text{and inviscid: } \nu = 0 \quad (\text{B.1})$$

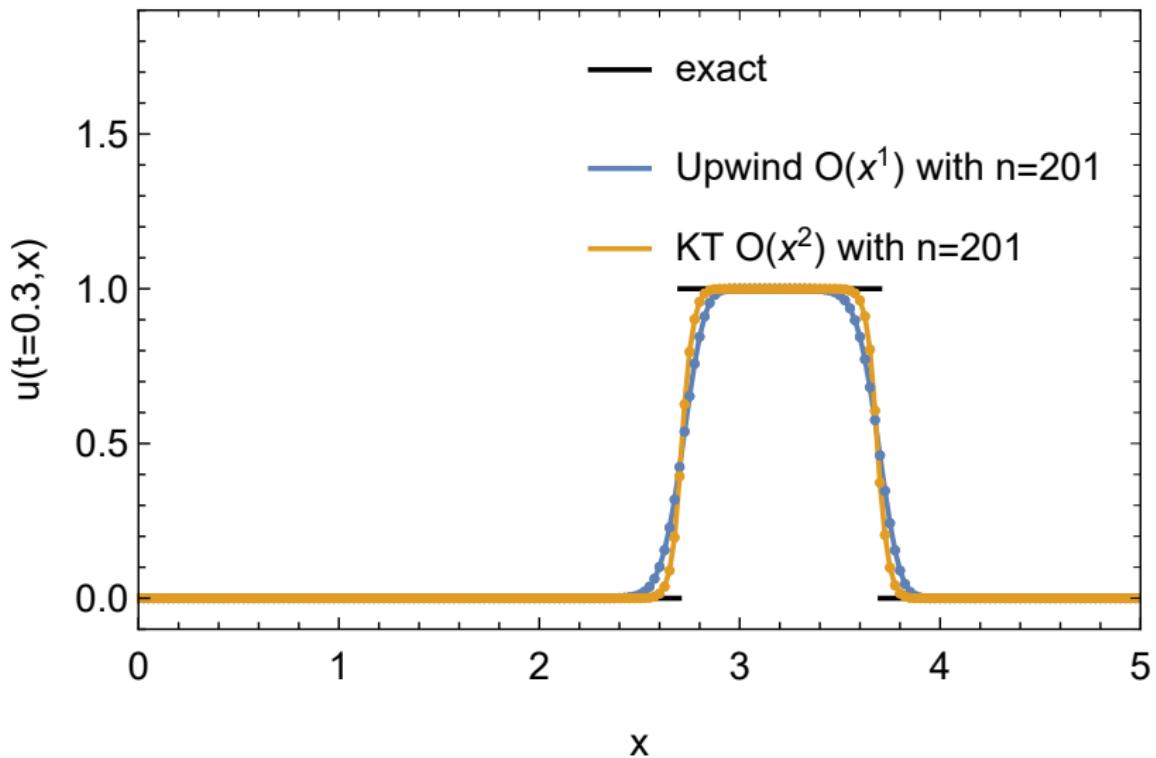
$$u(0, x) = 0.5 + \sin(x) \quad \text{and viscous: } \nu = 0.1 \quad (\text{B.2})$$

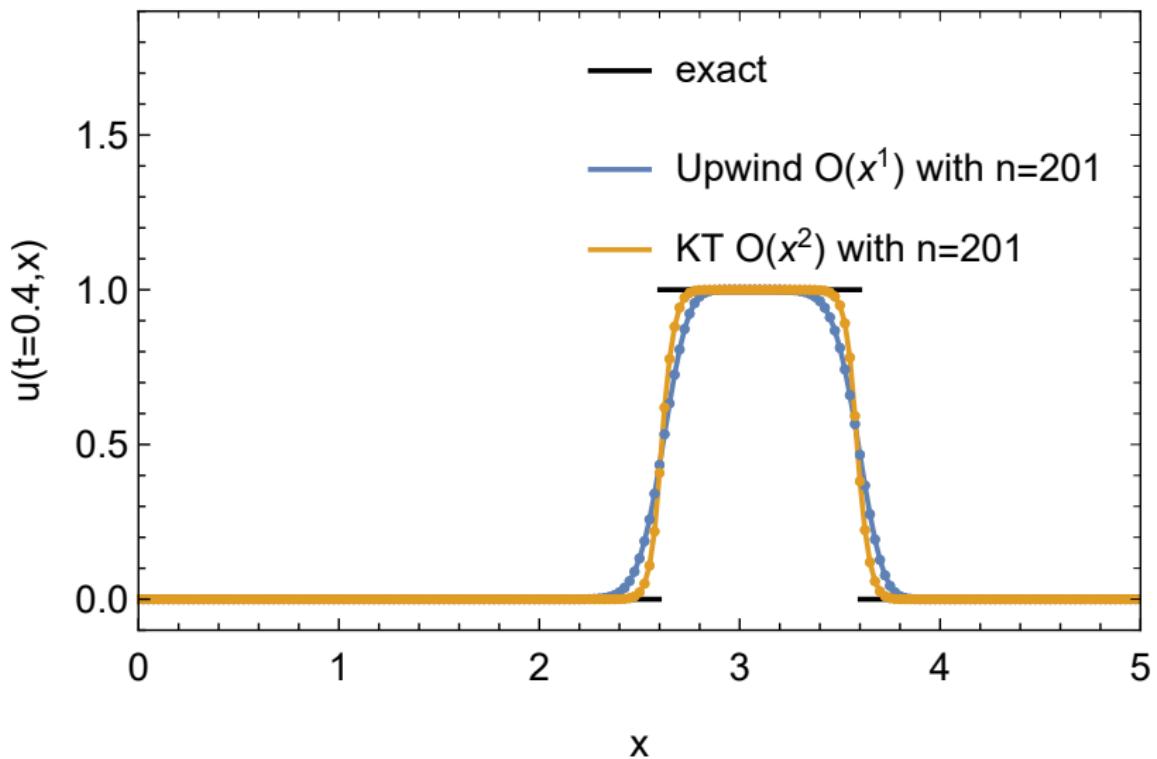
$$u(0, x) = \begin{cases} 1 & x \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and viscous: } \nu = 0.01 \quad (\text{B.3})$$

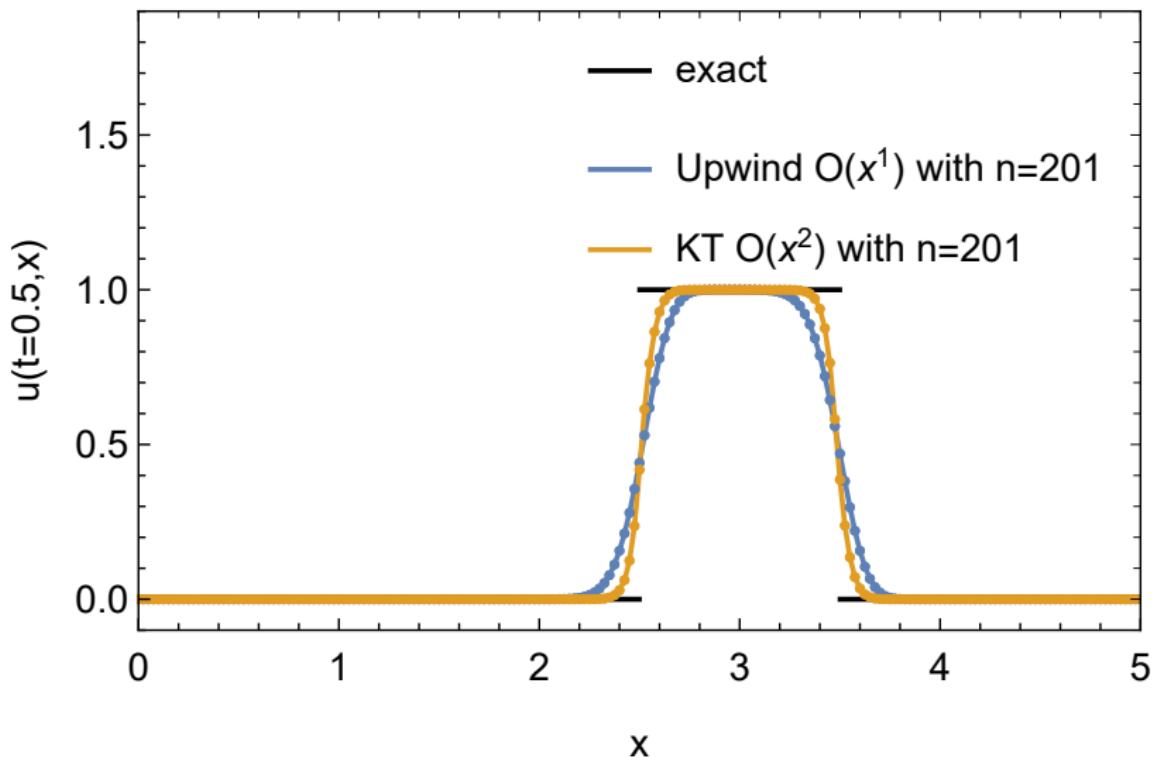


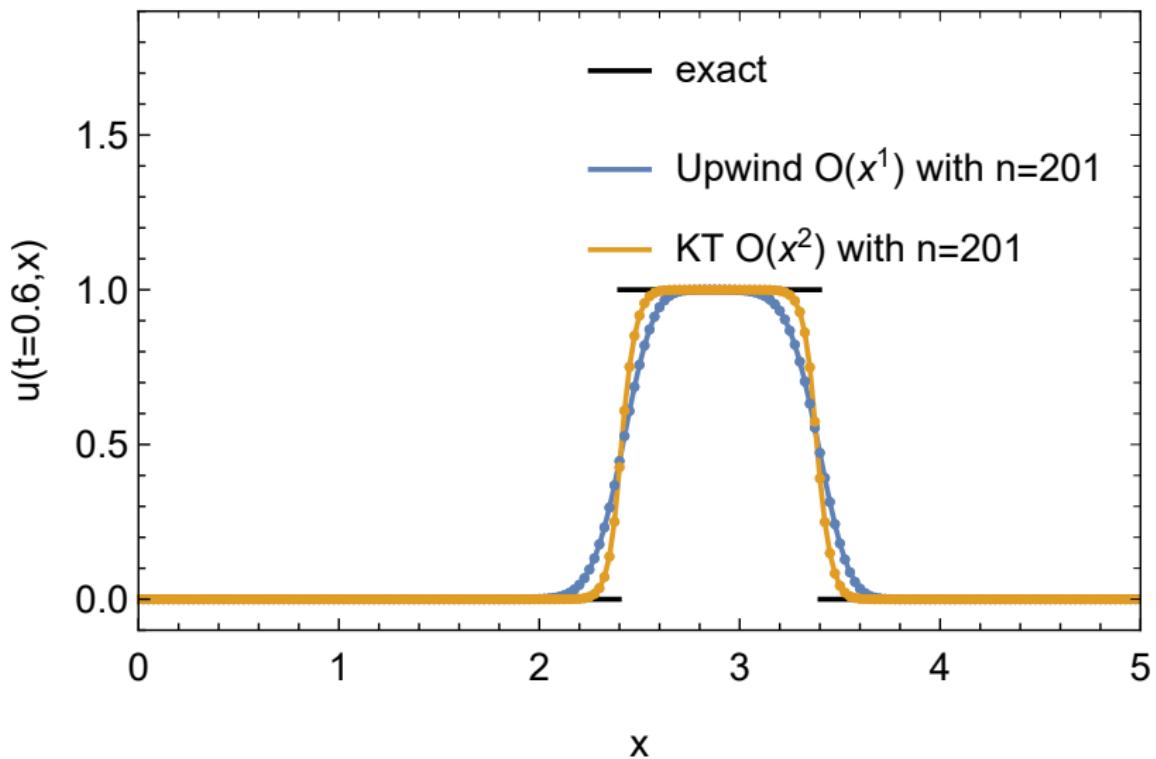


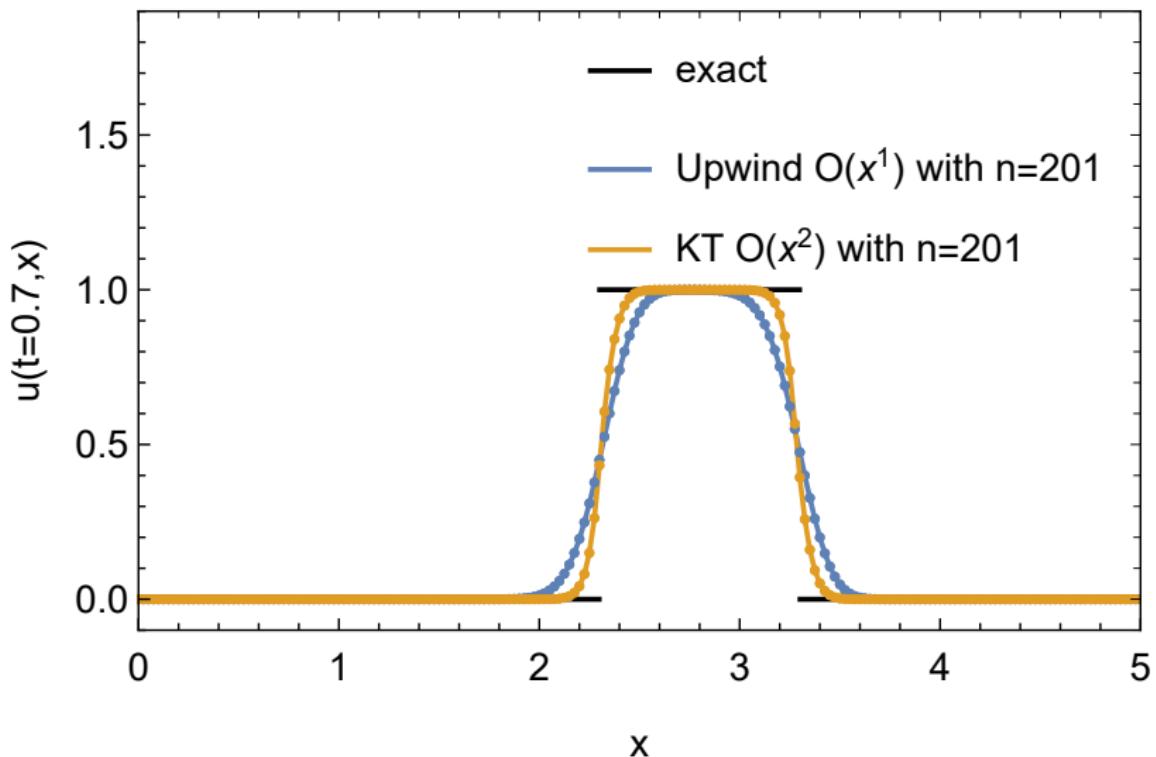


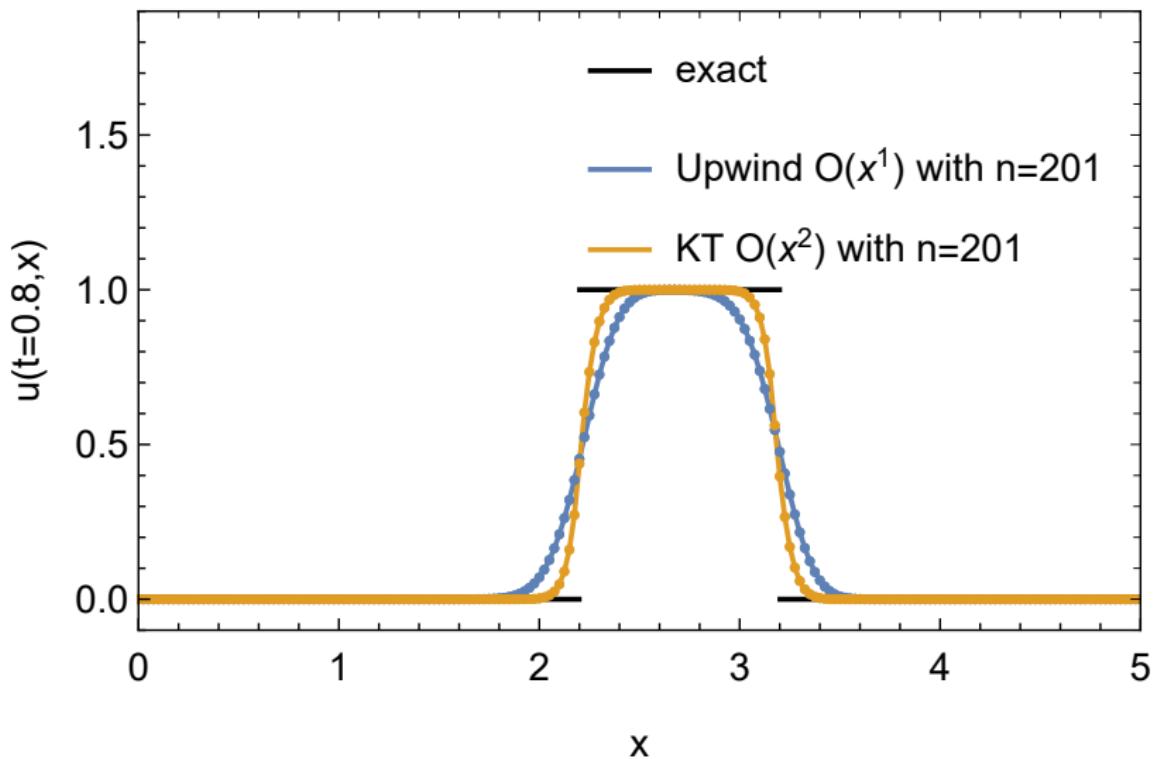


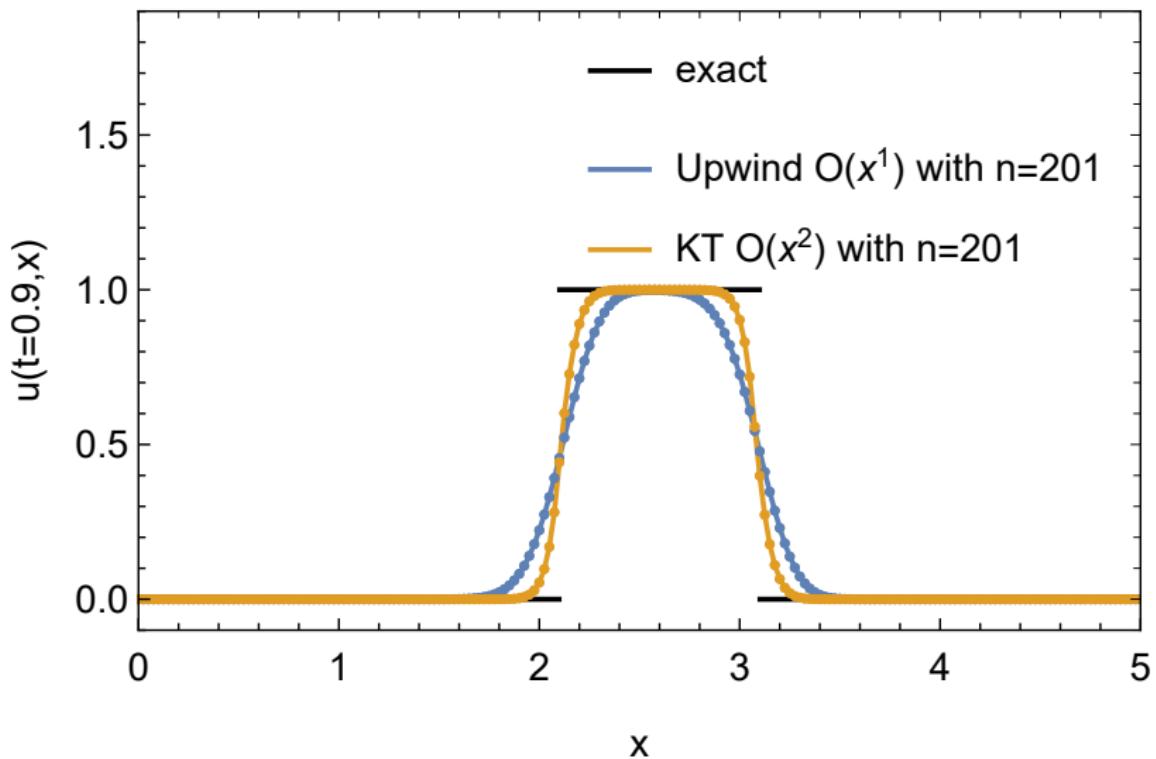


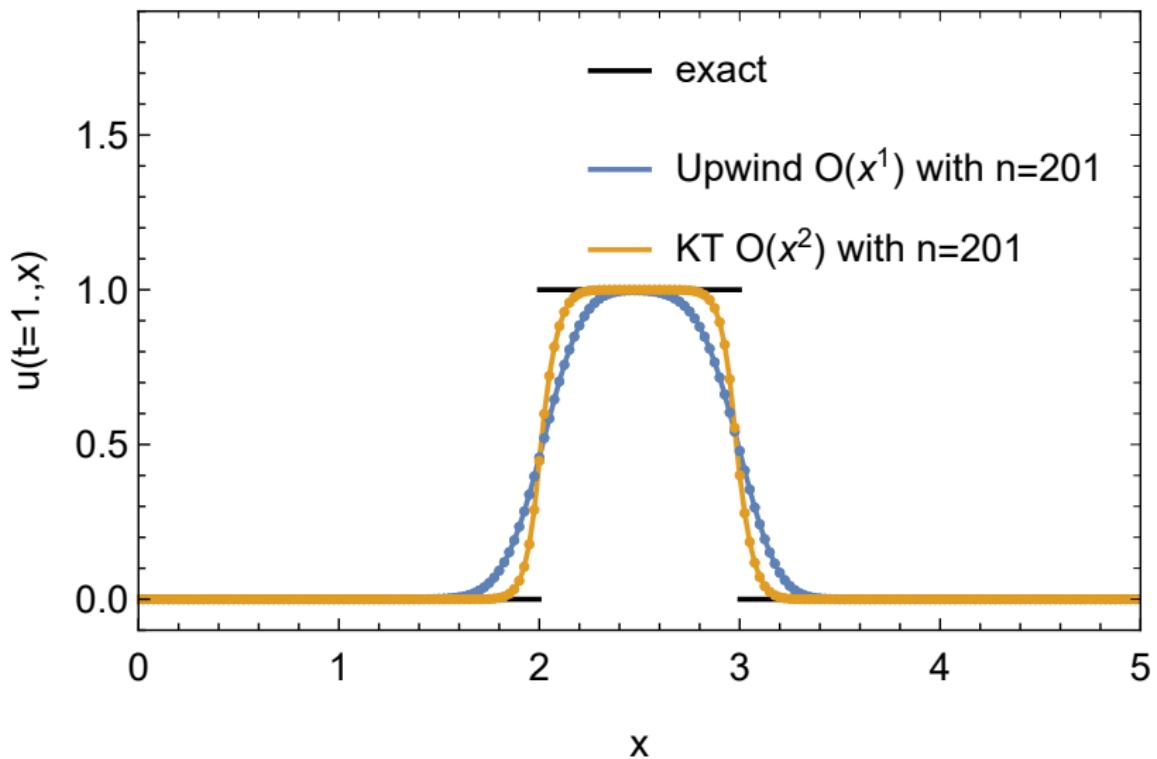


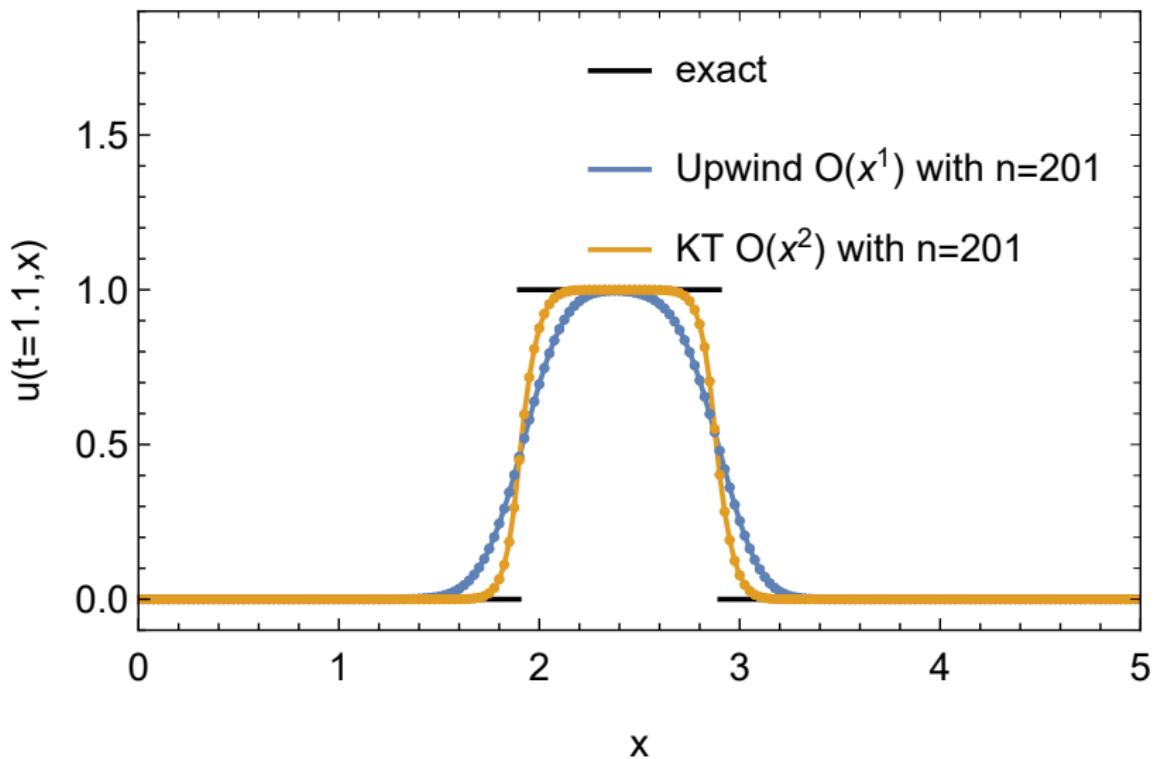


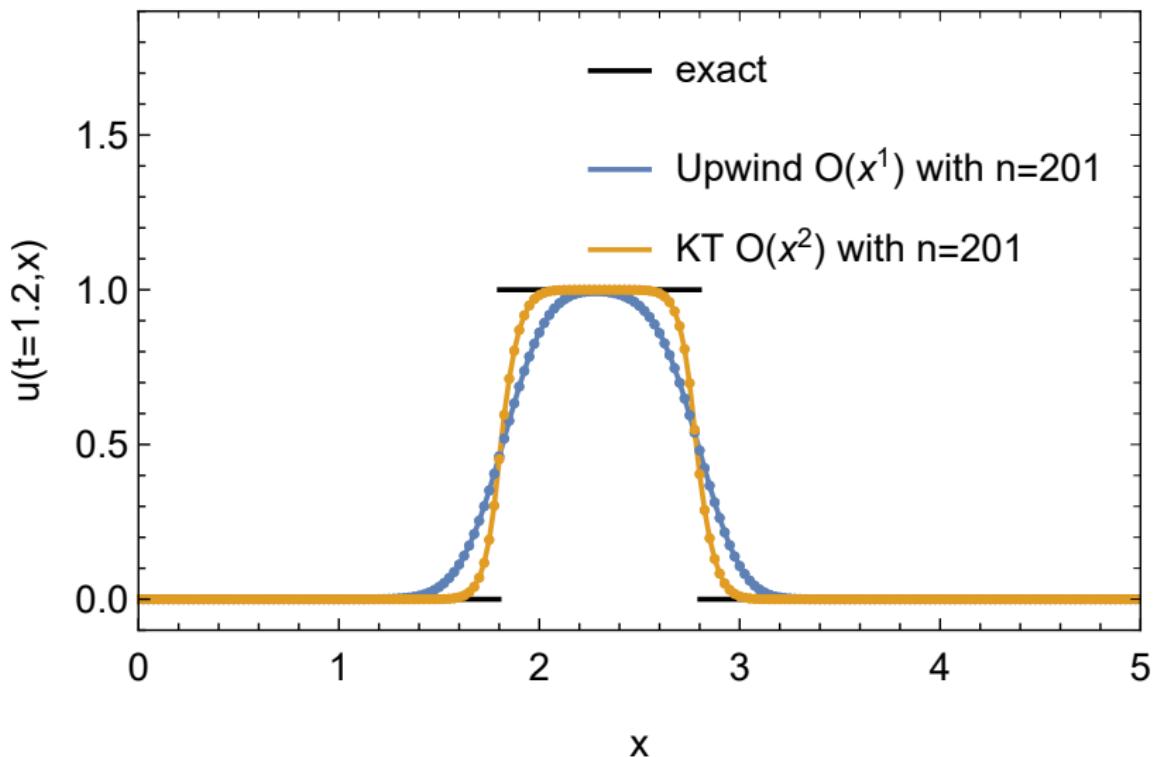


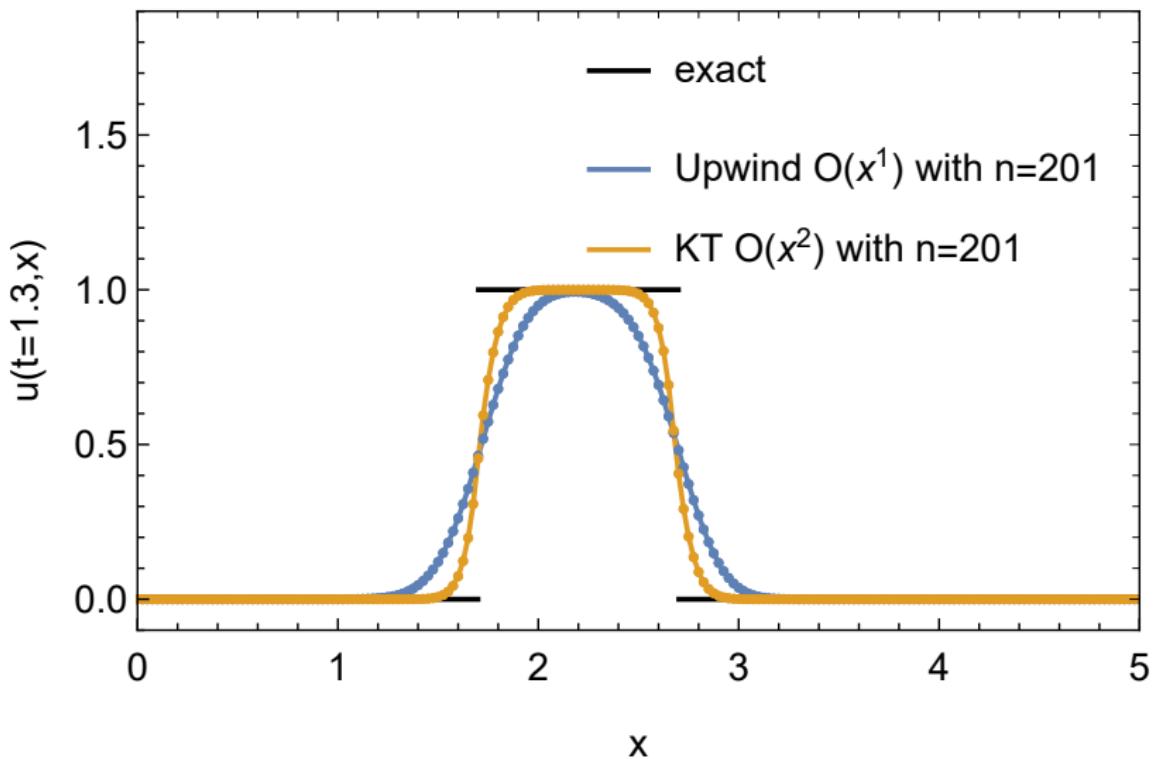


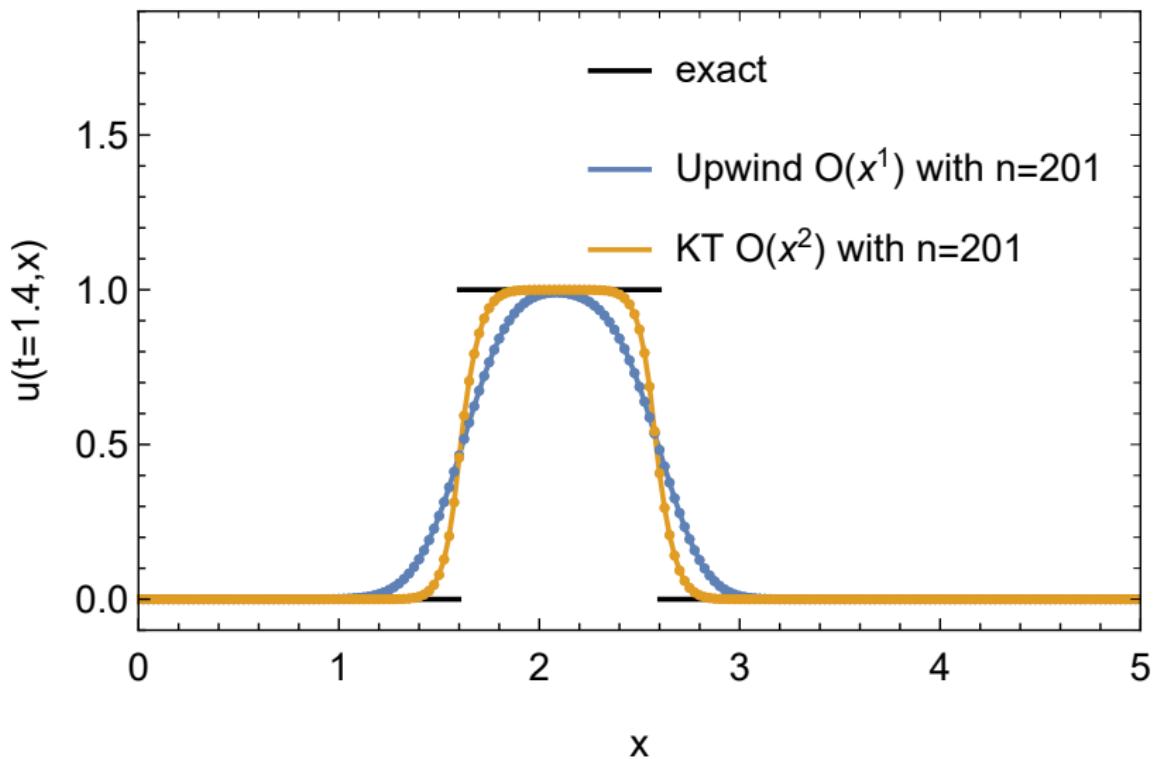


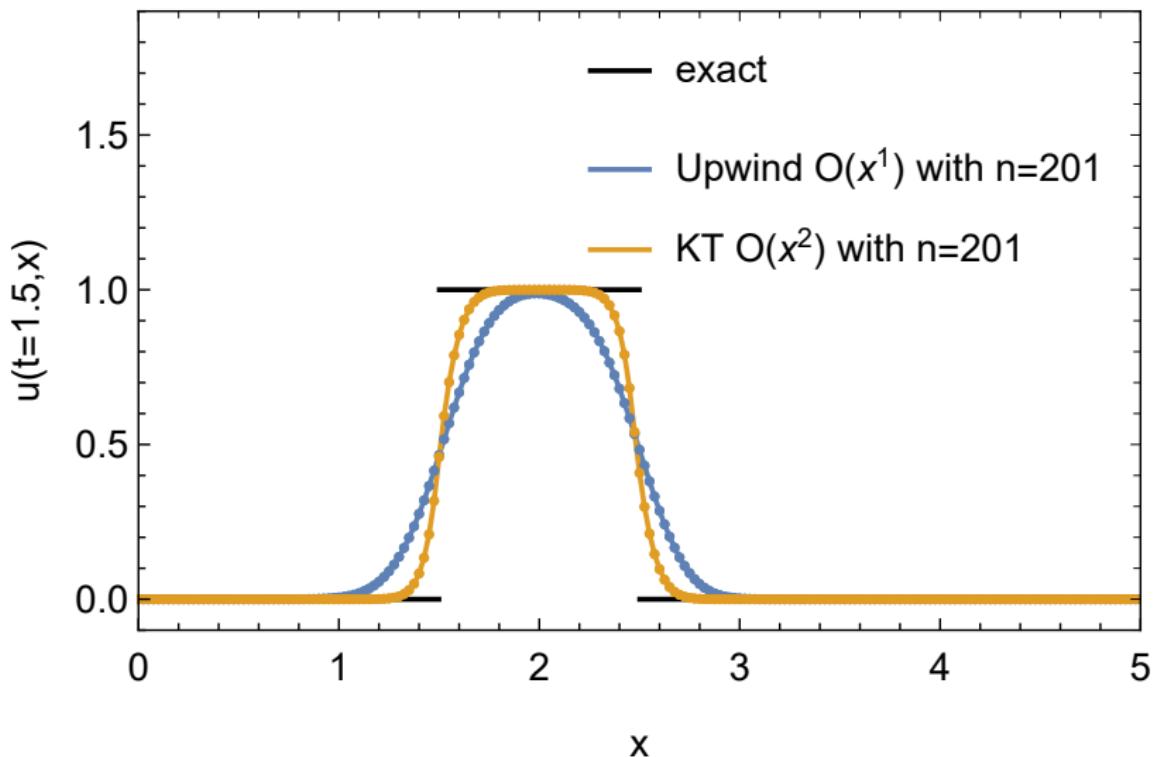


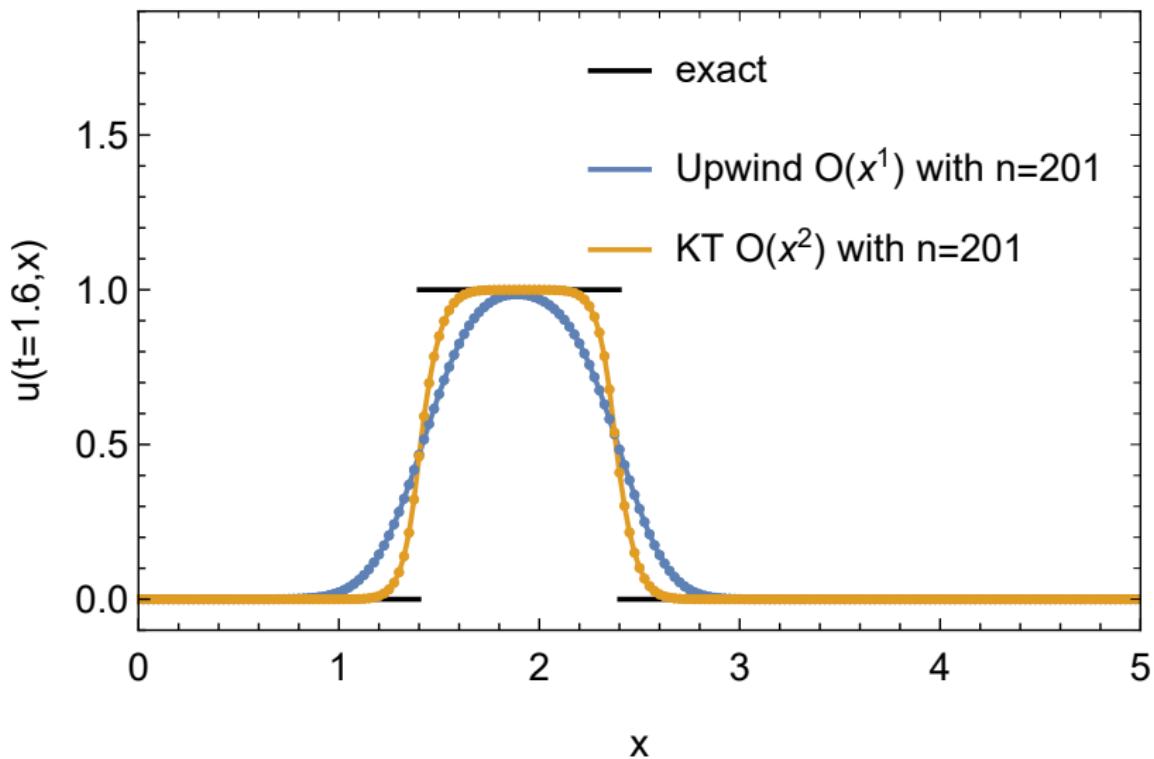


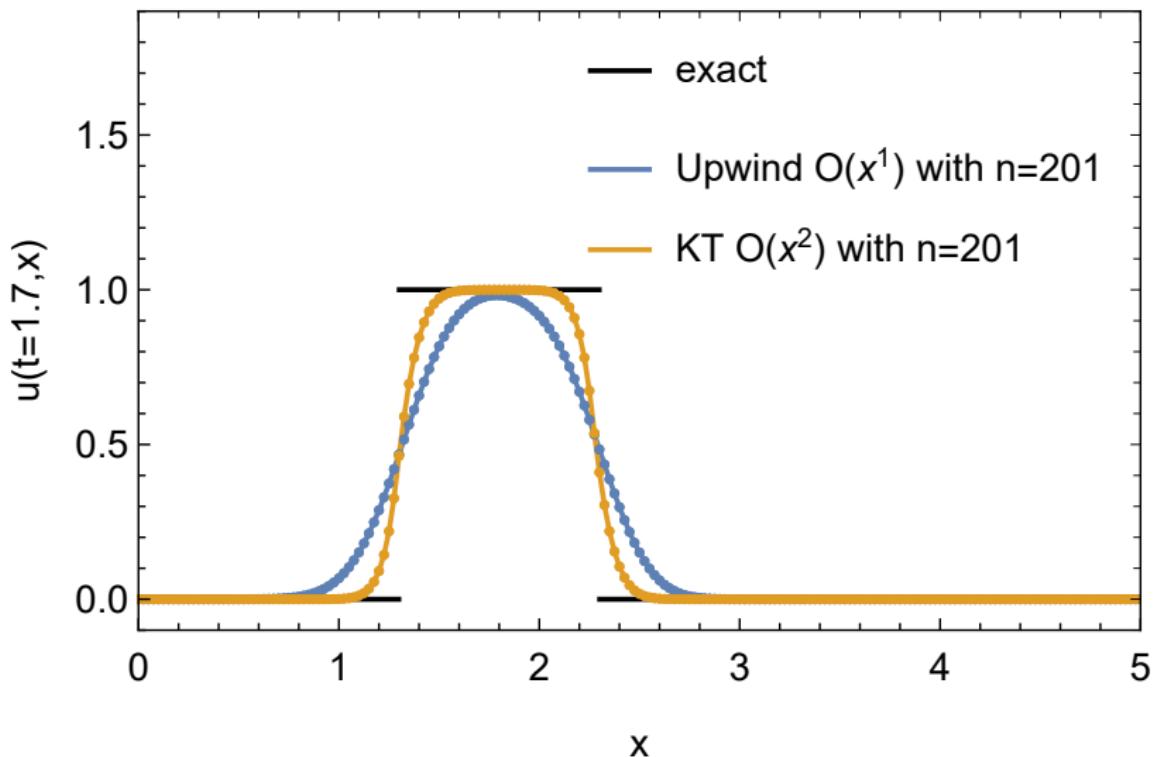


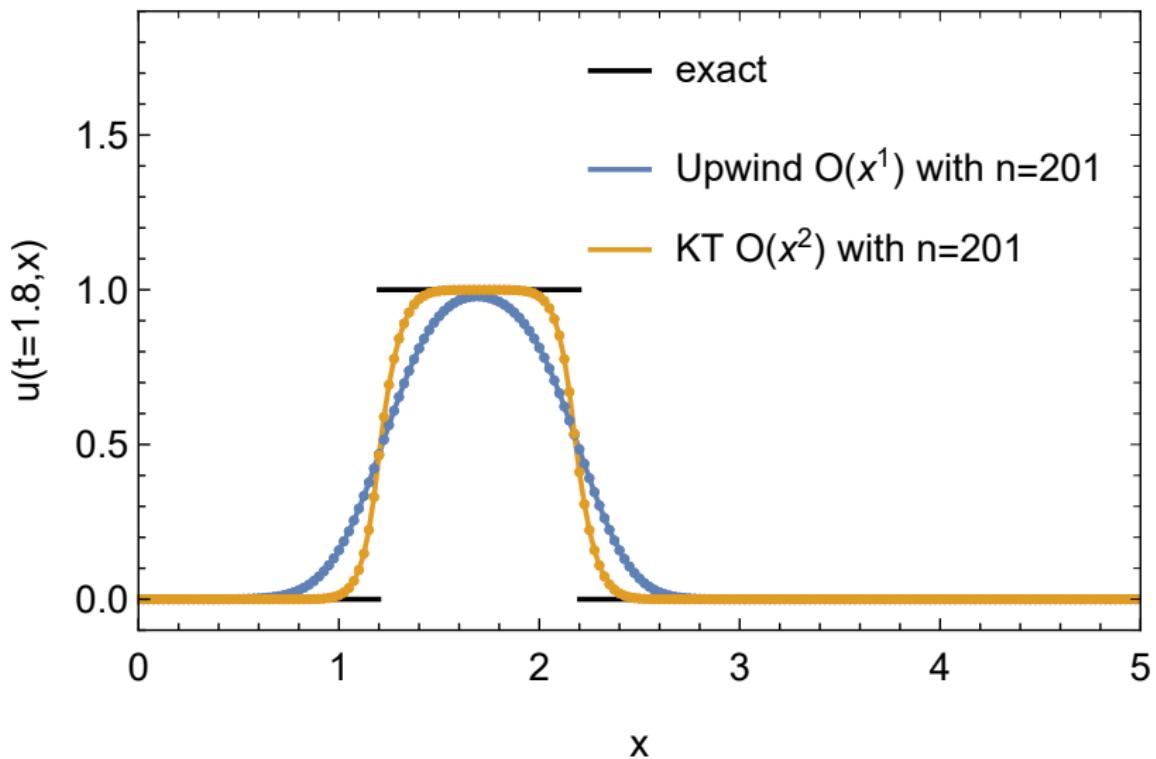


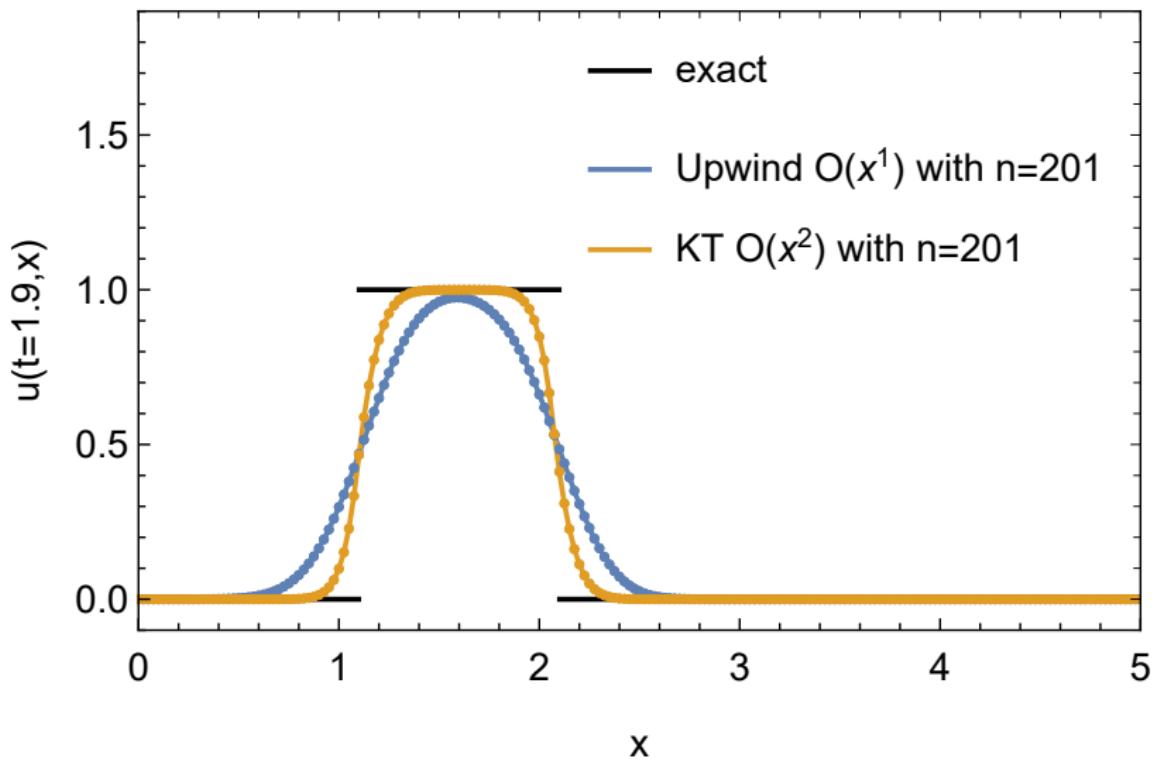


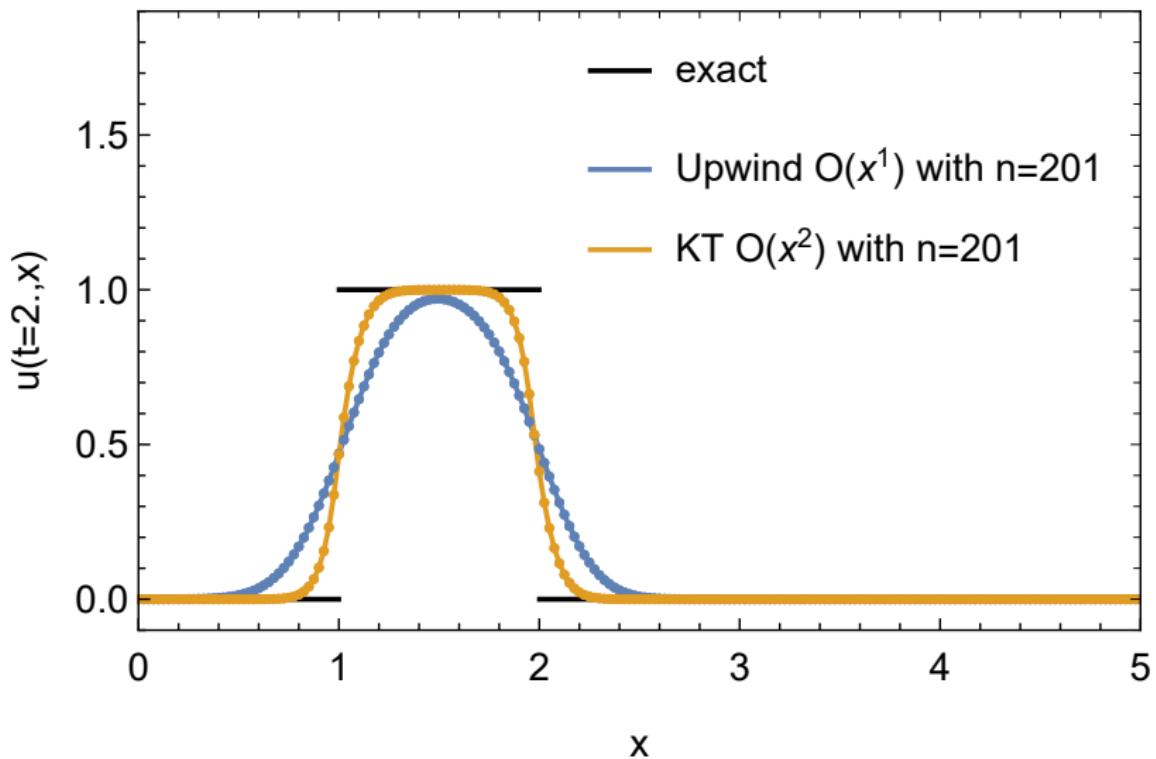


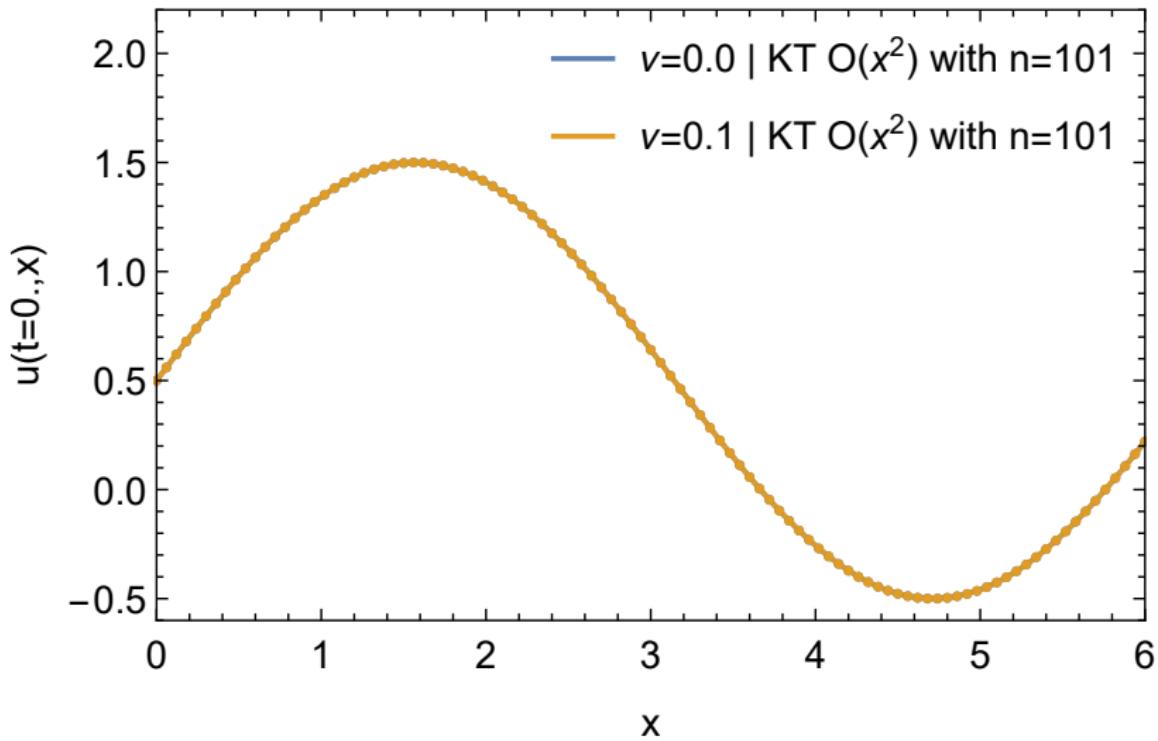


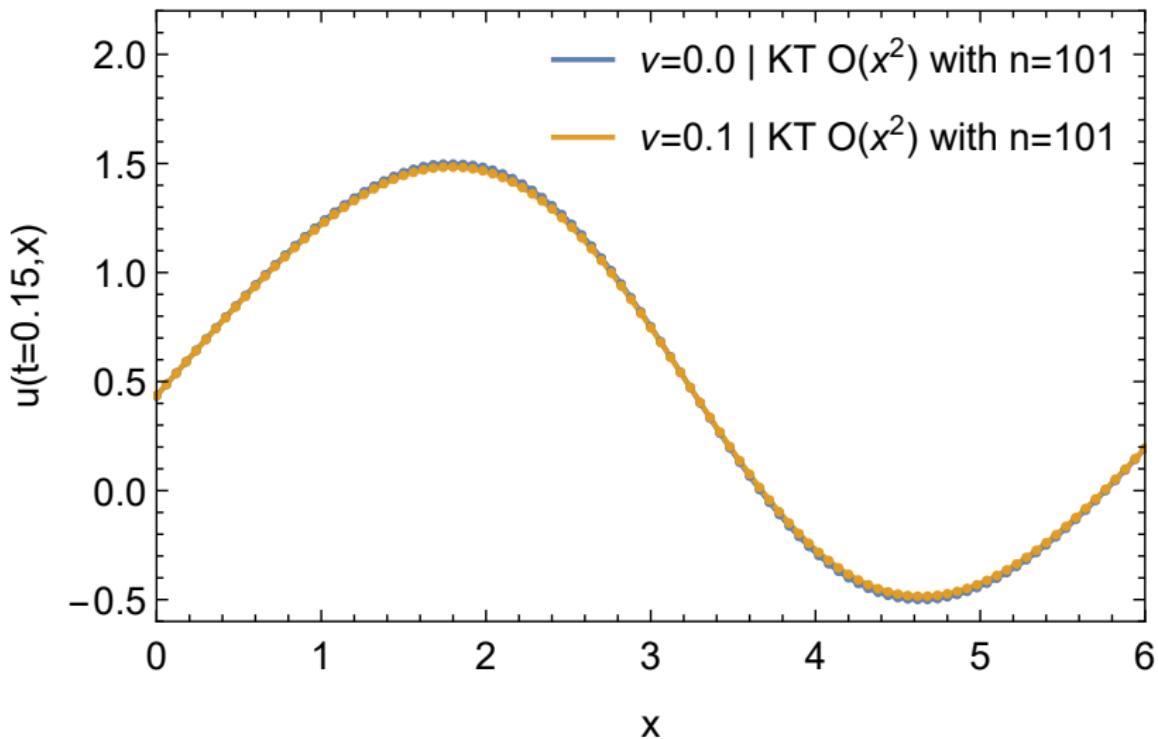


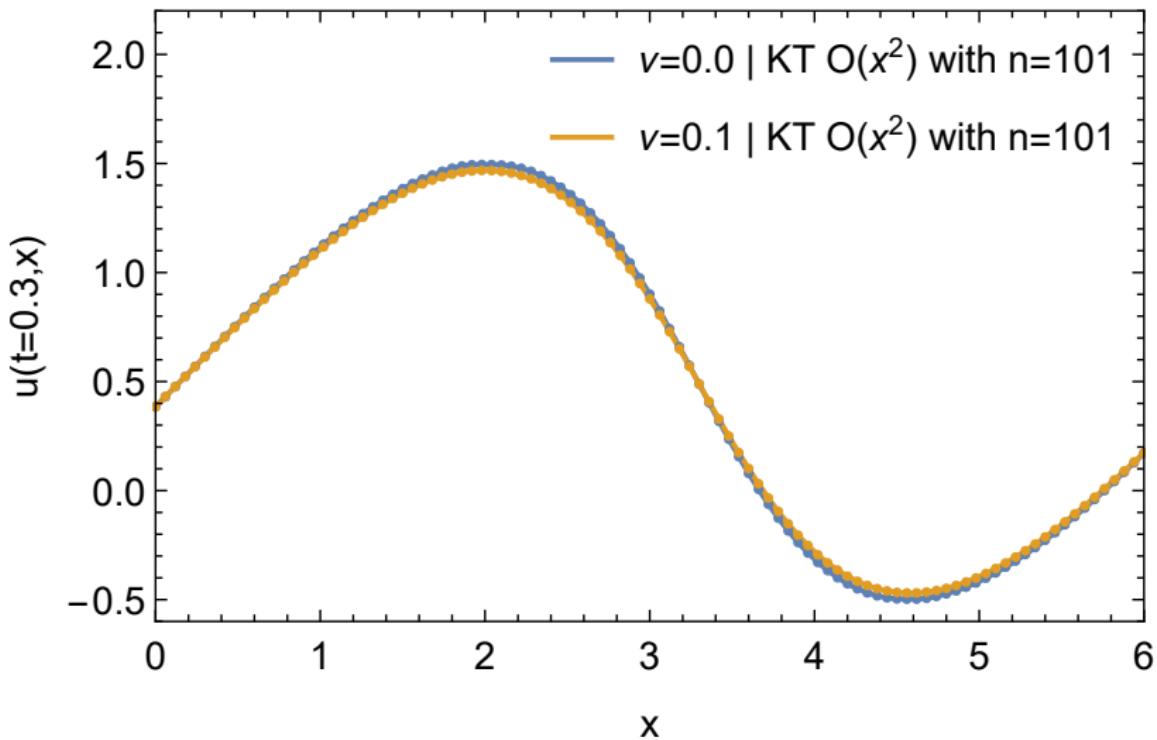


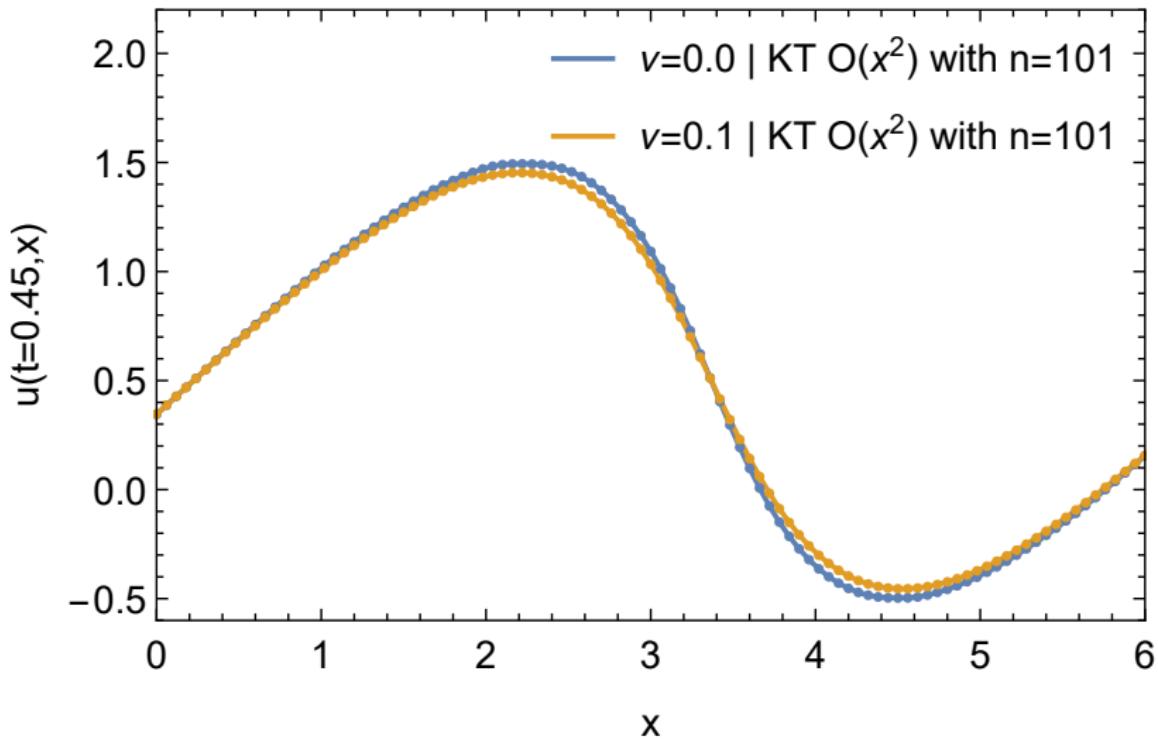


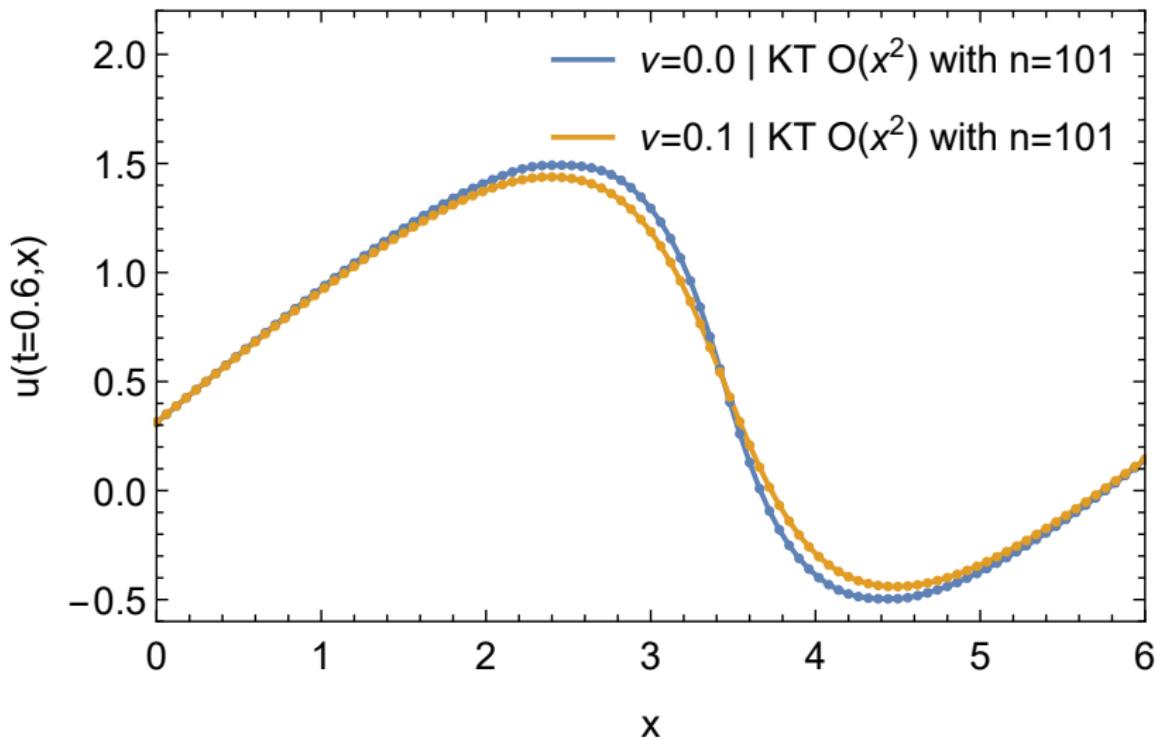


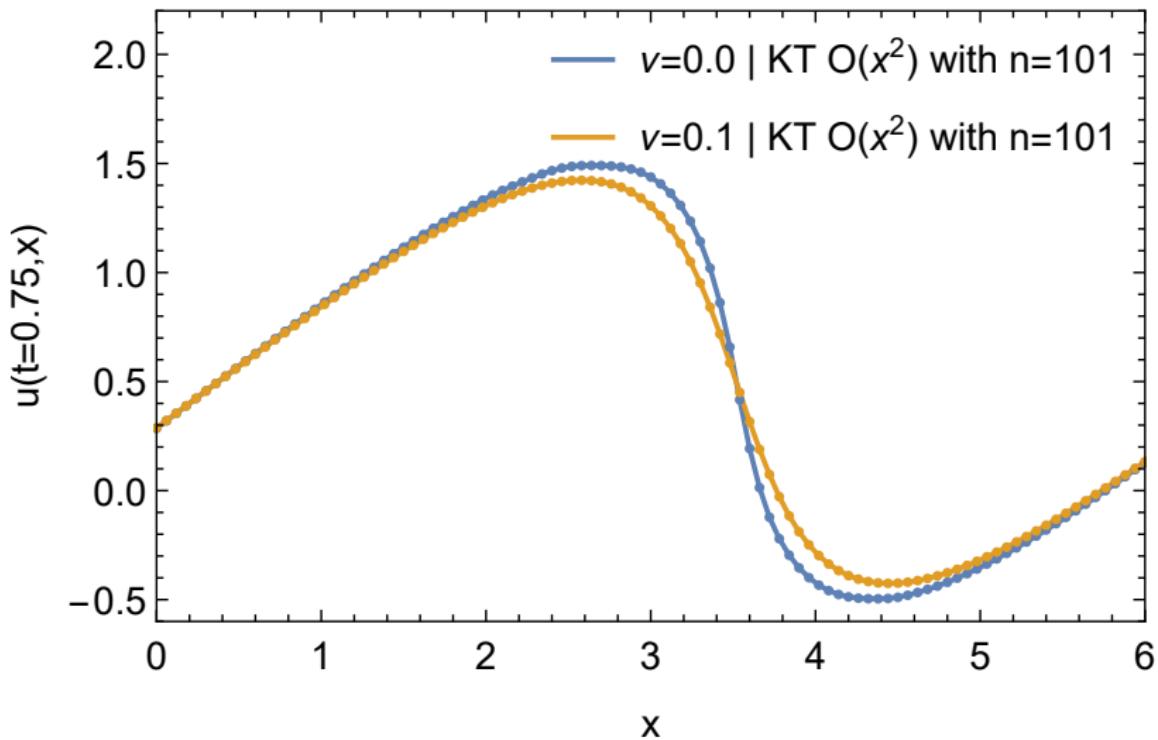


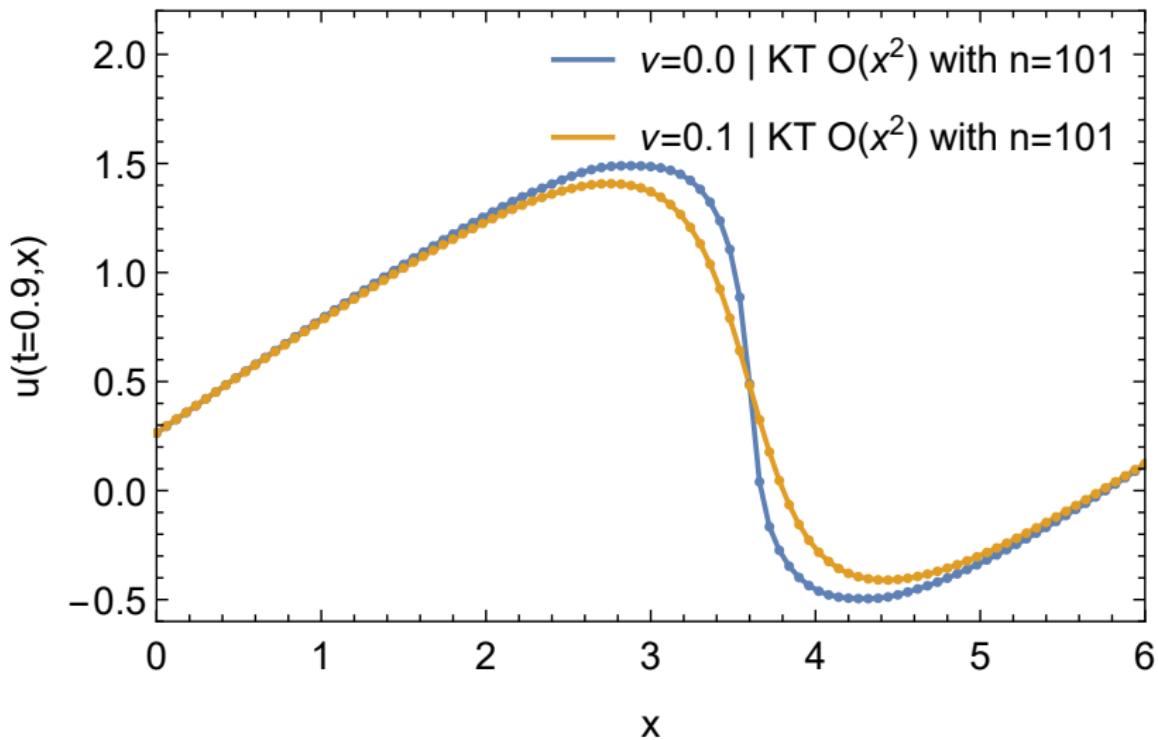


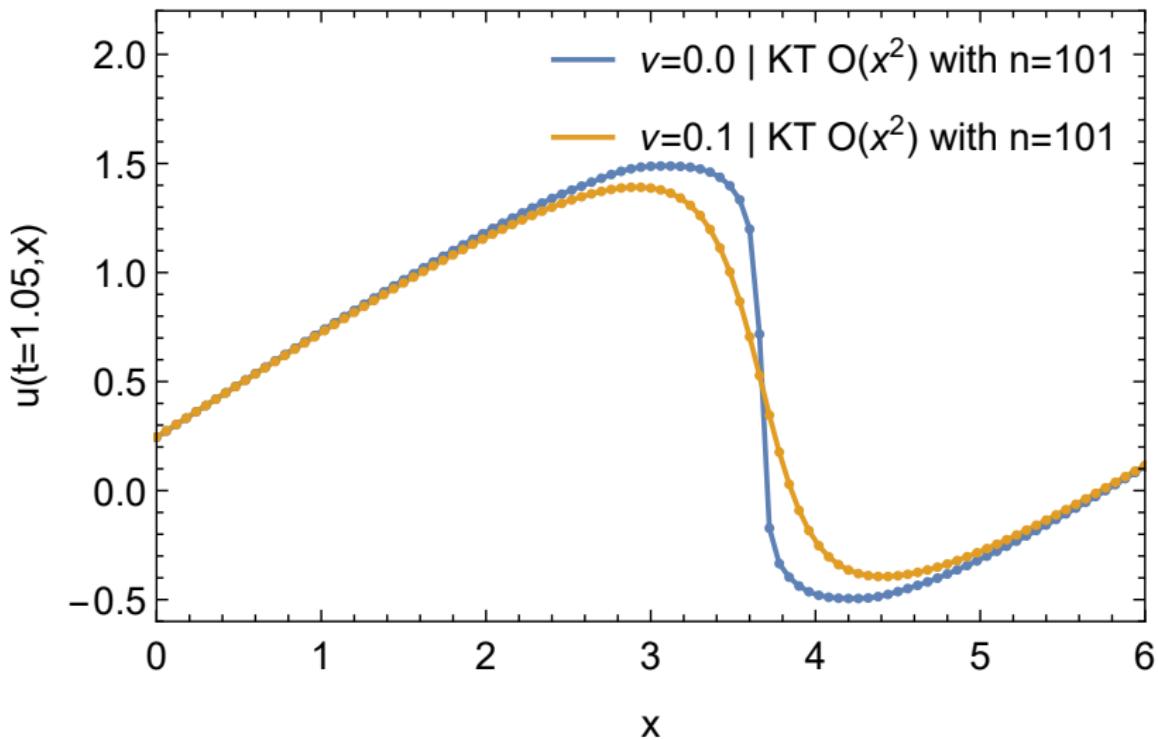


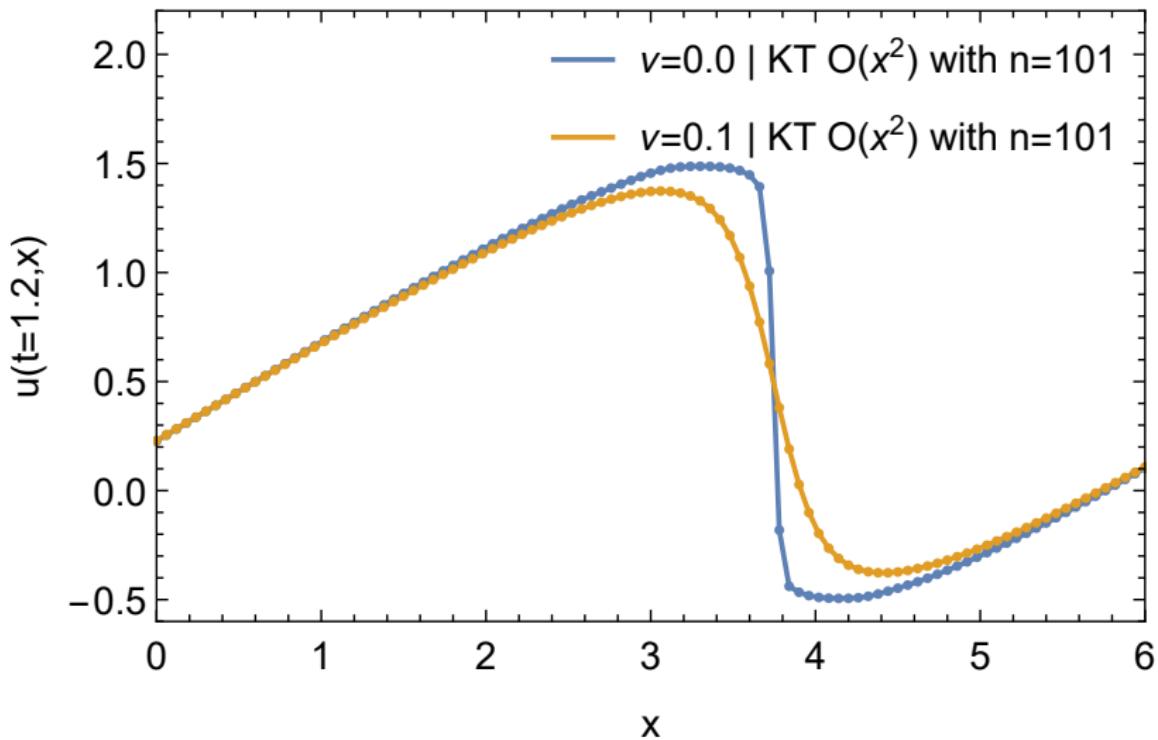


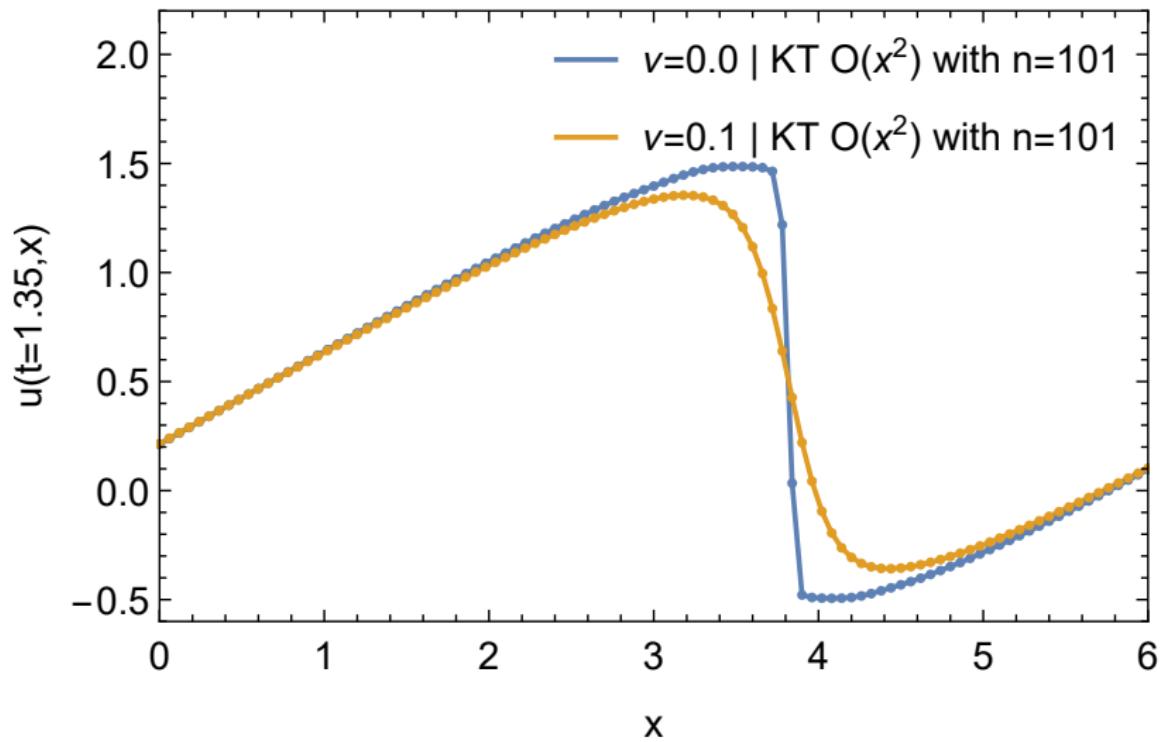


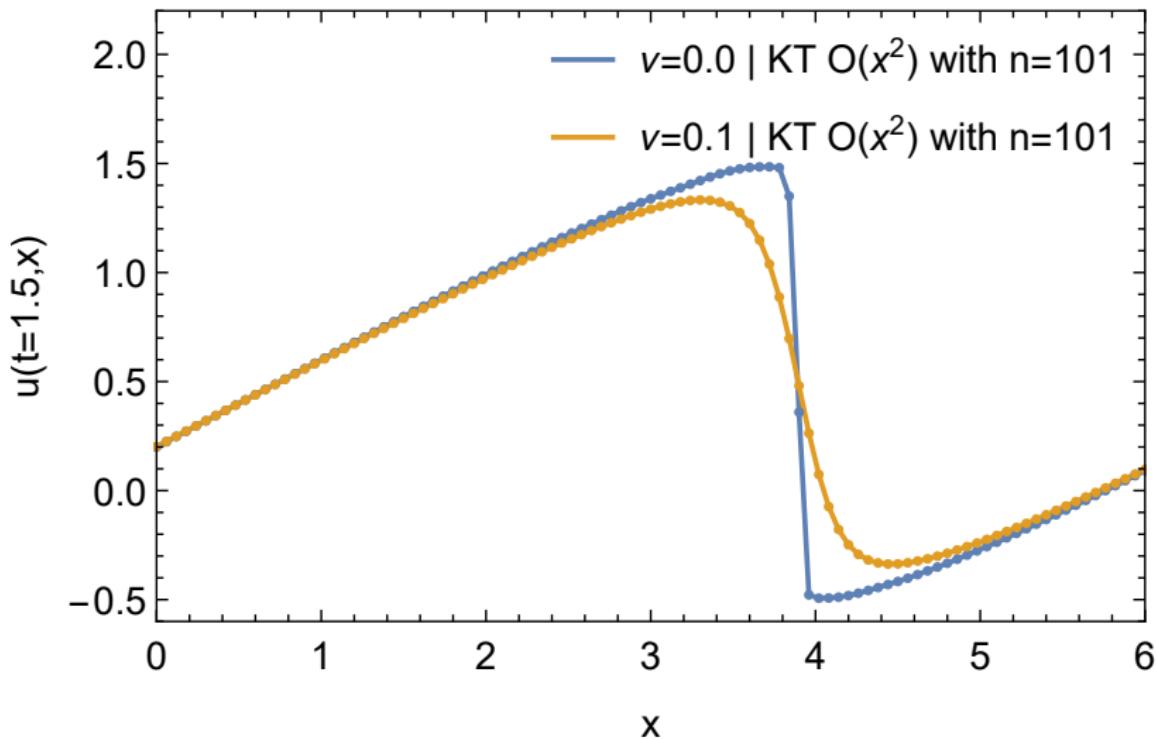


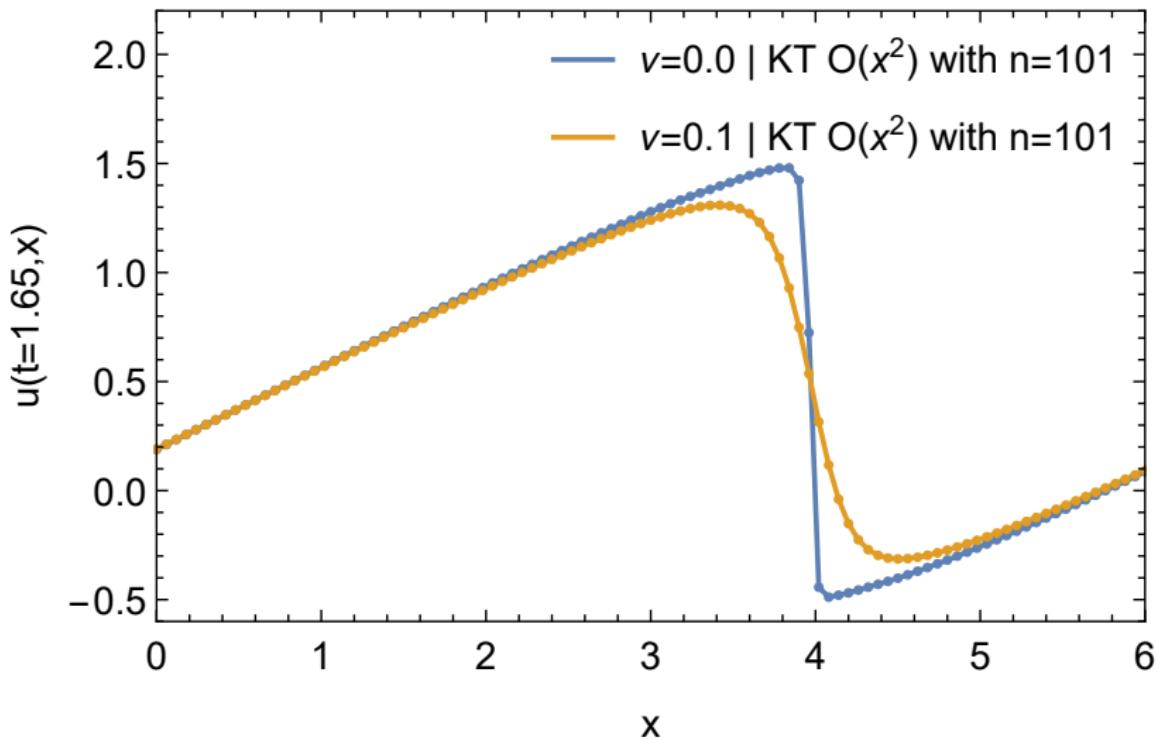


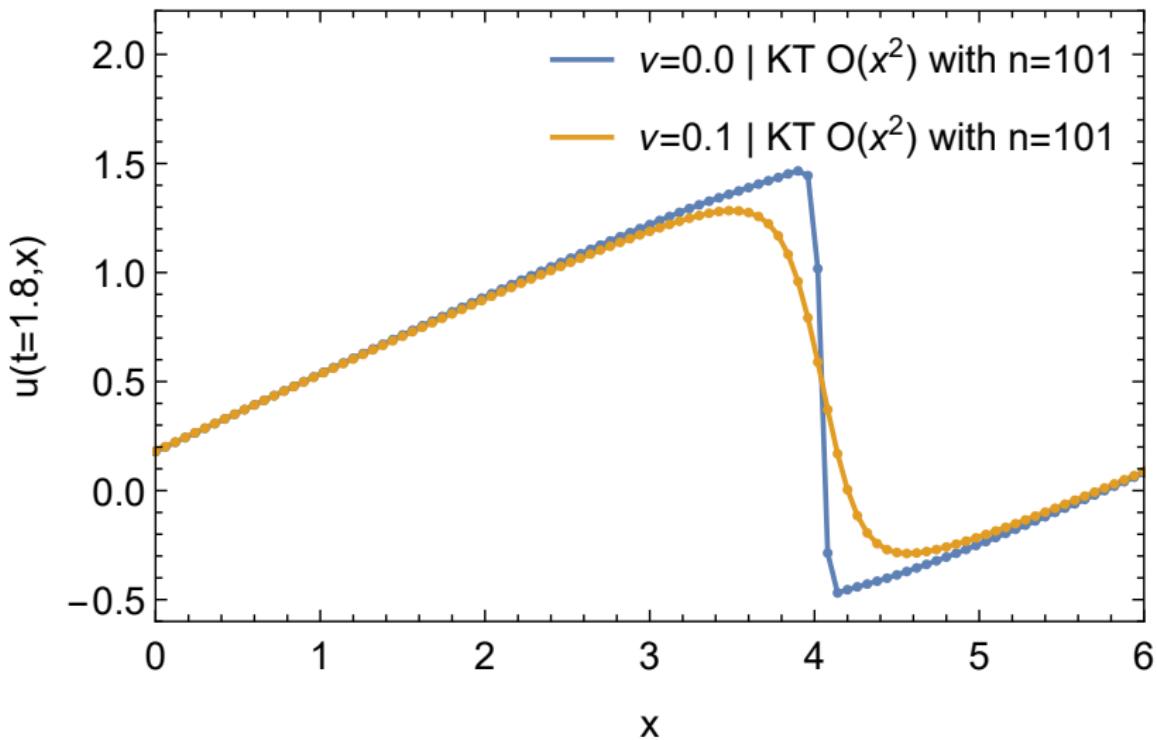


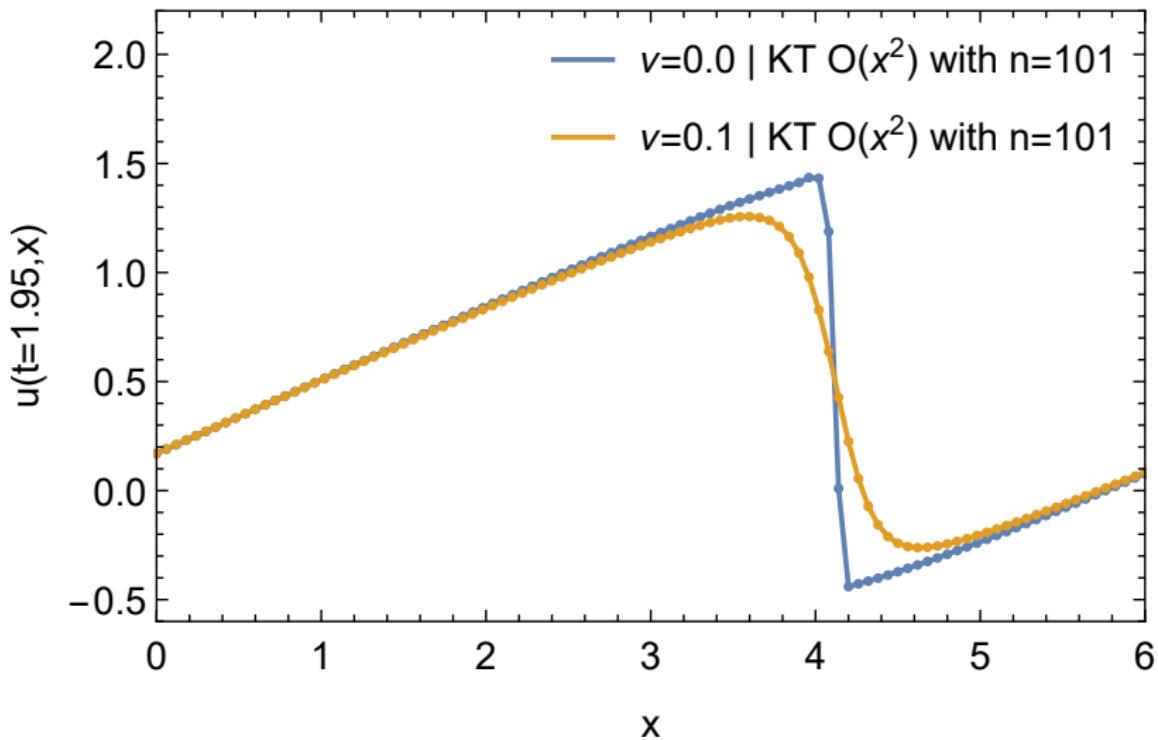


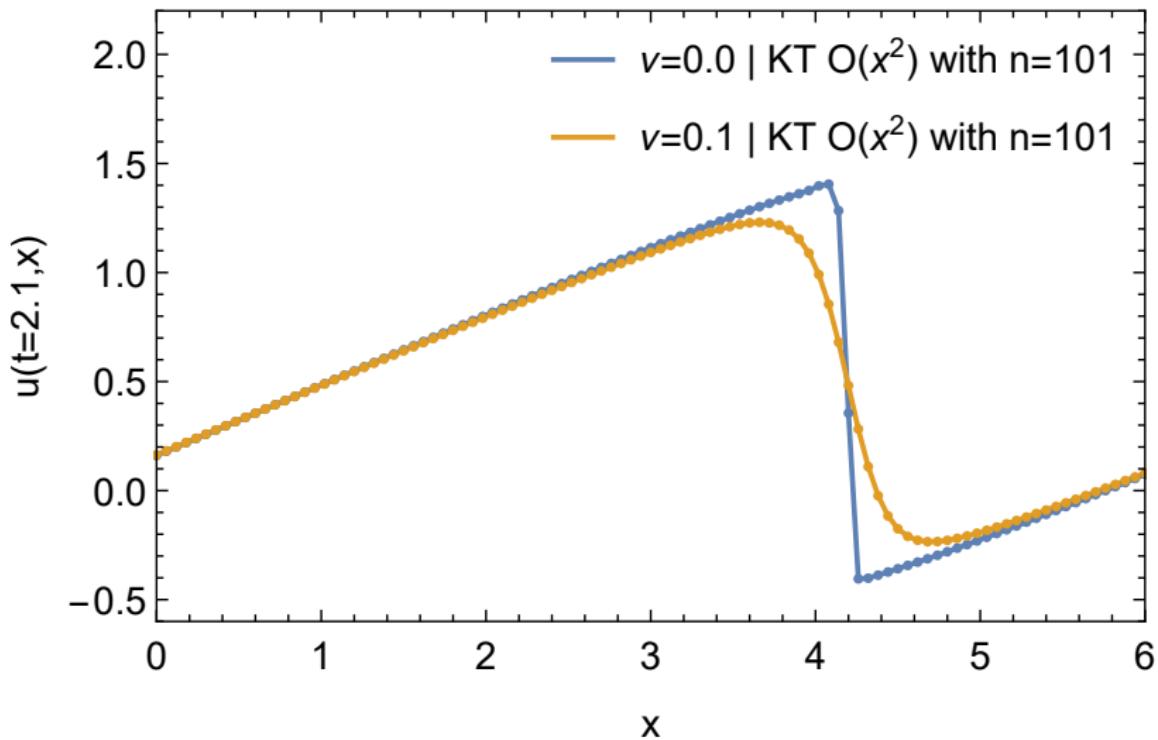


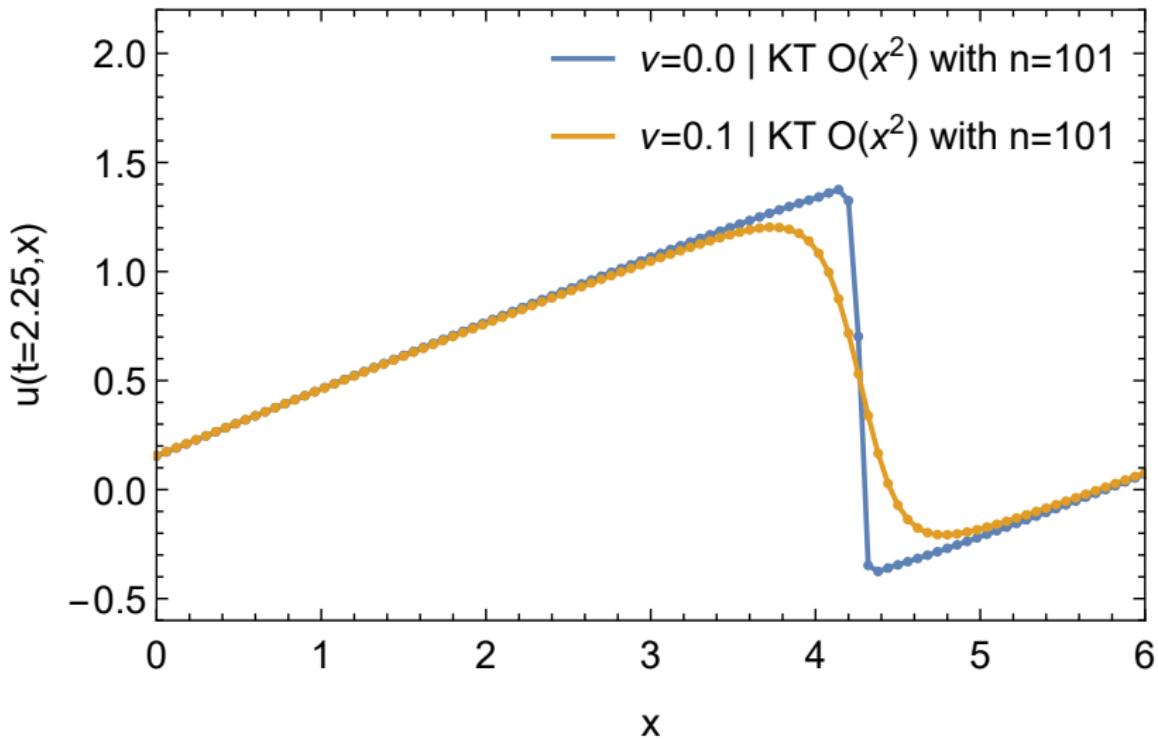


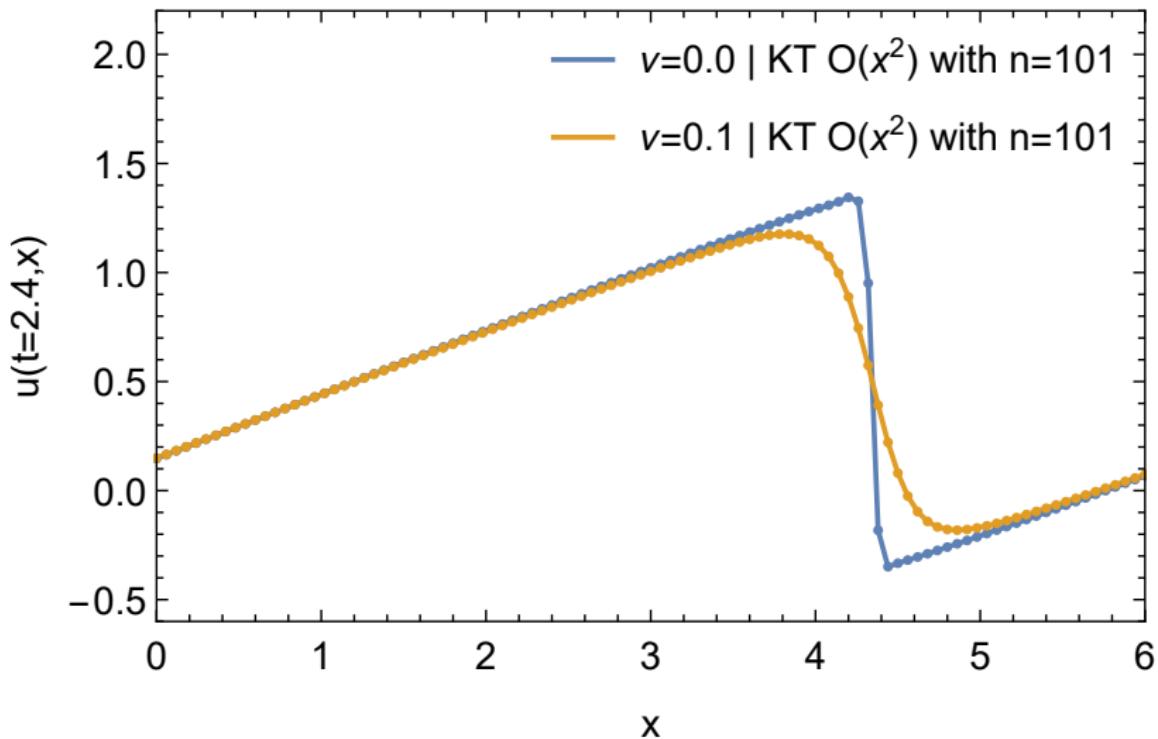


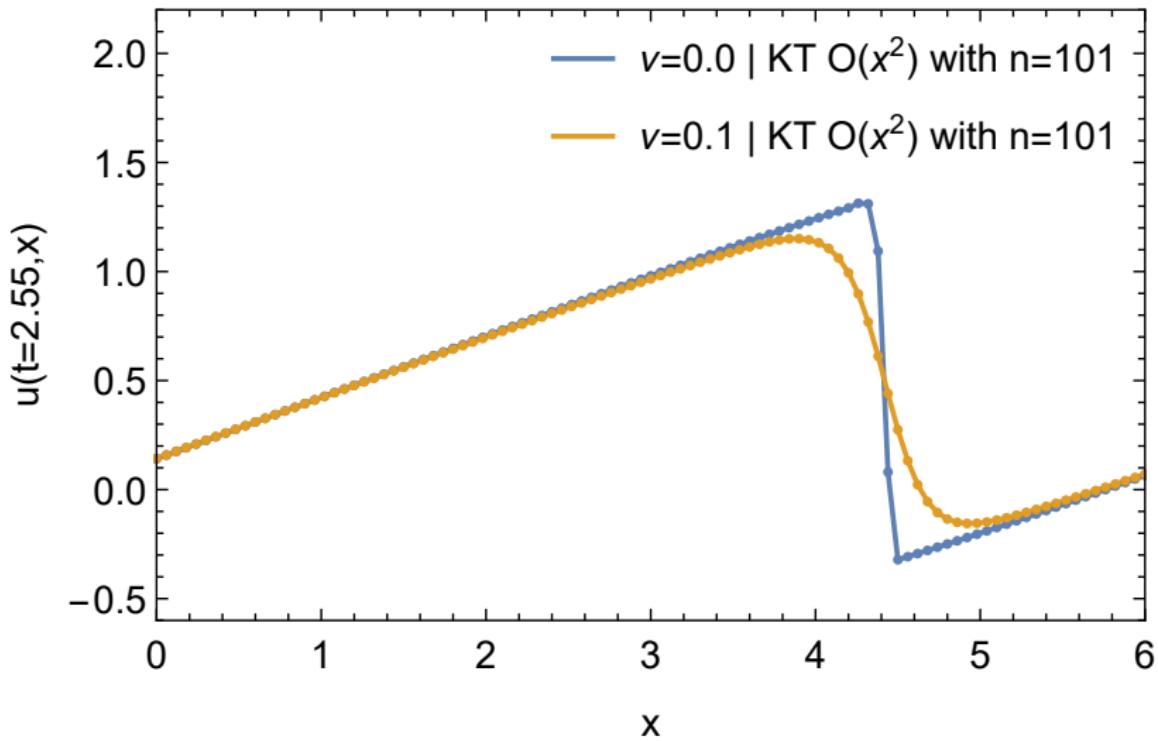


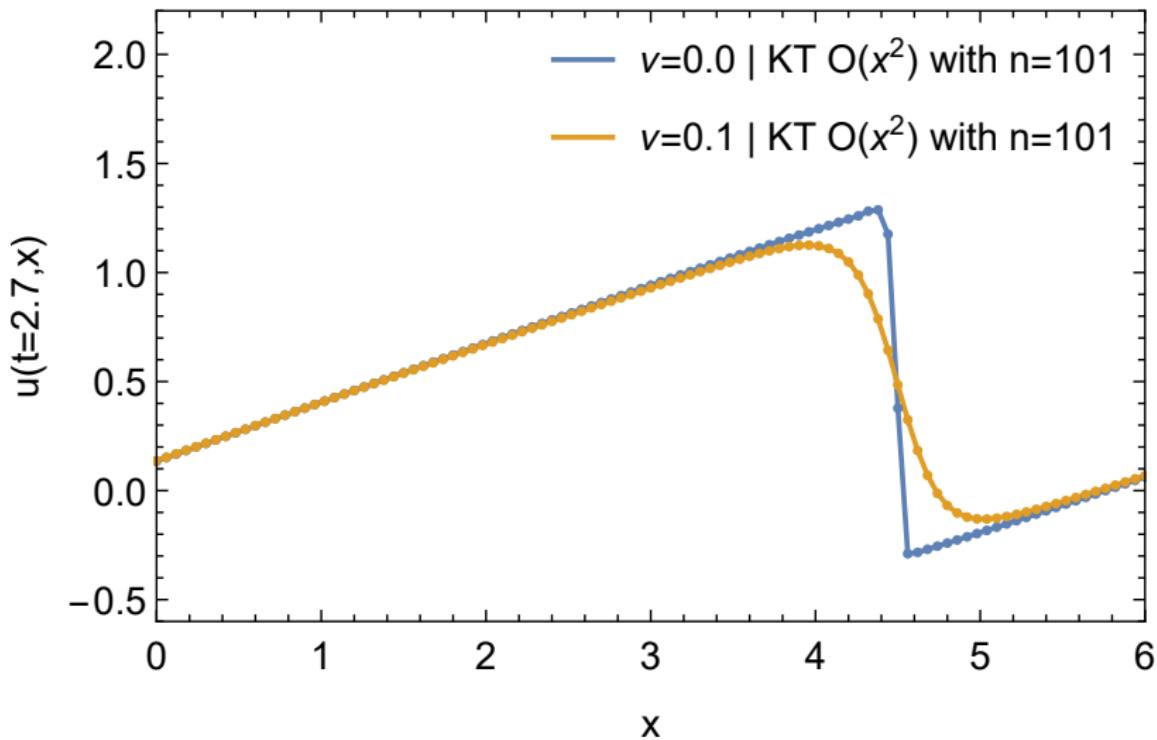


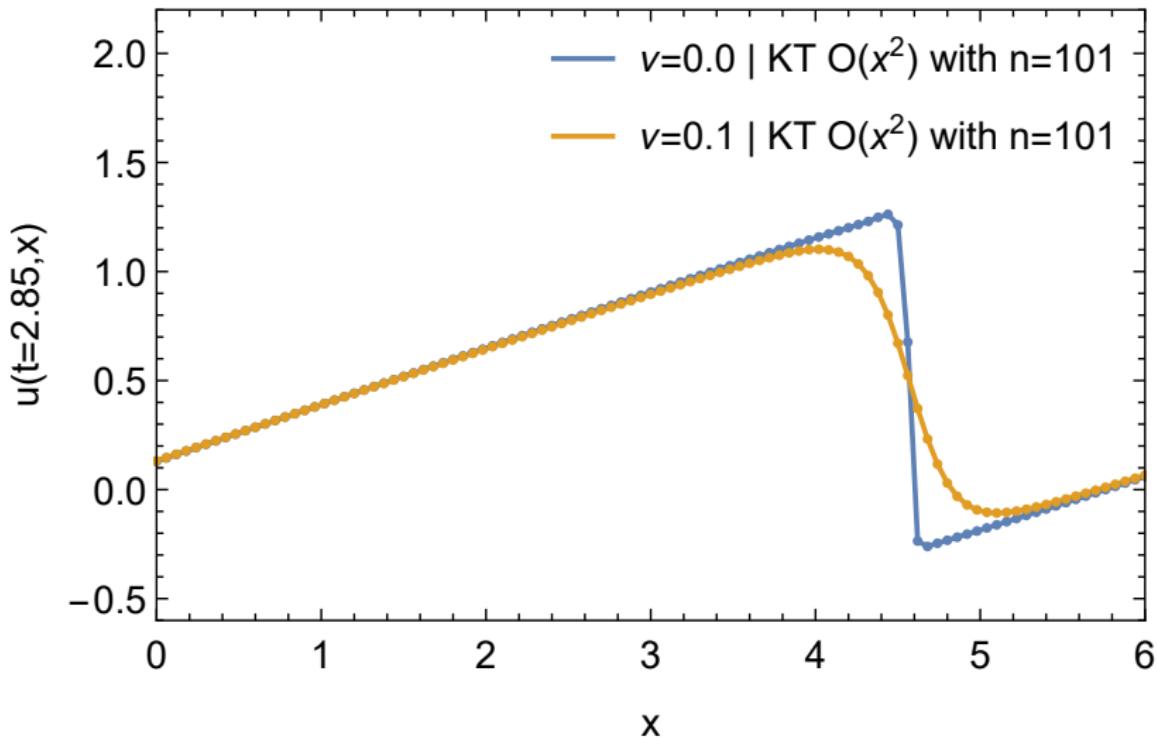


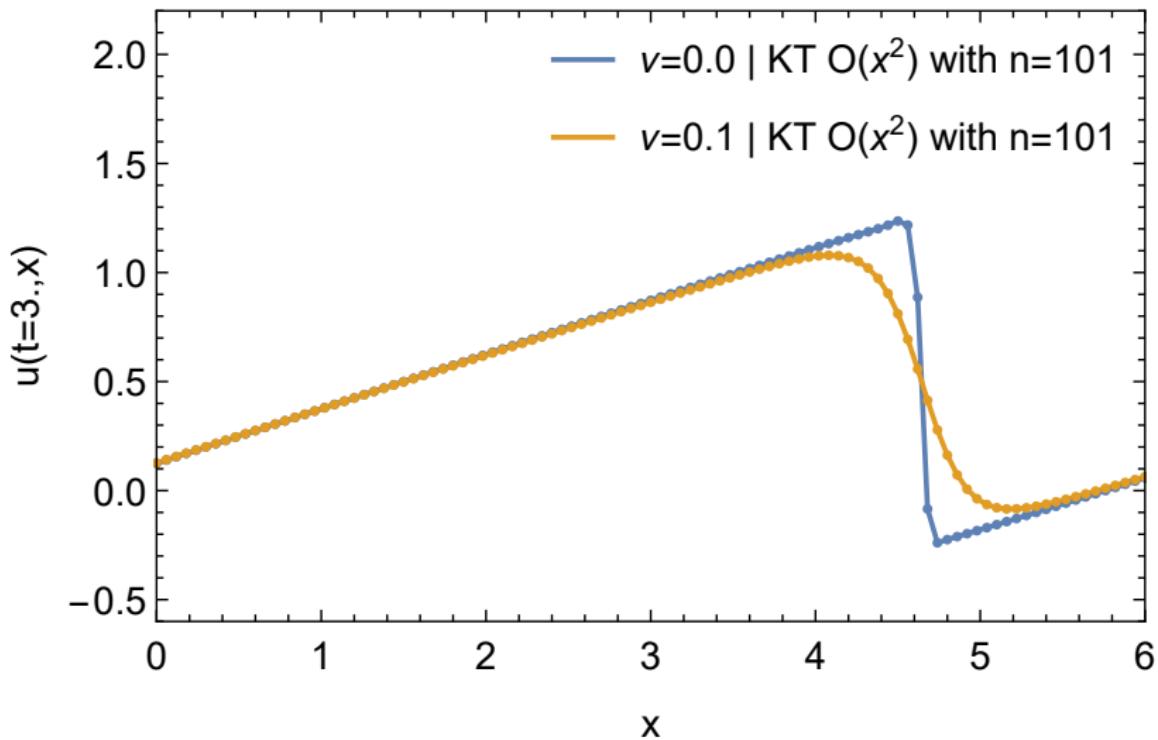


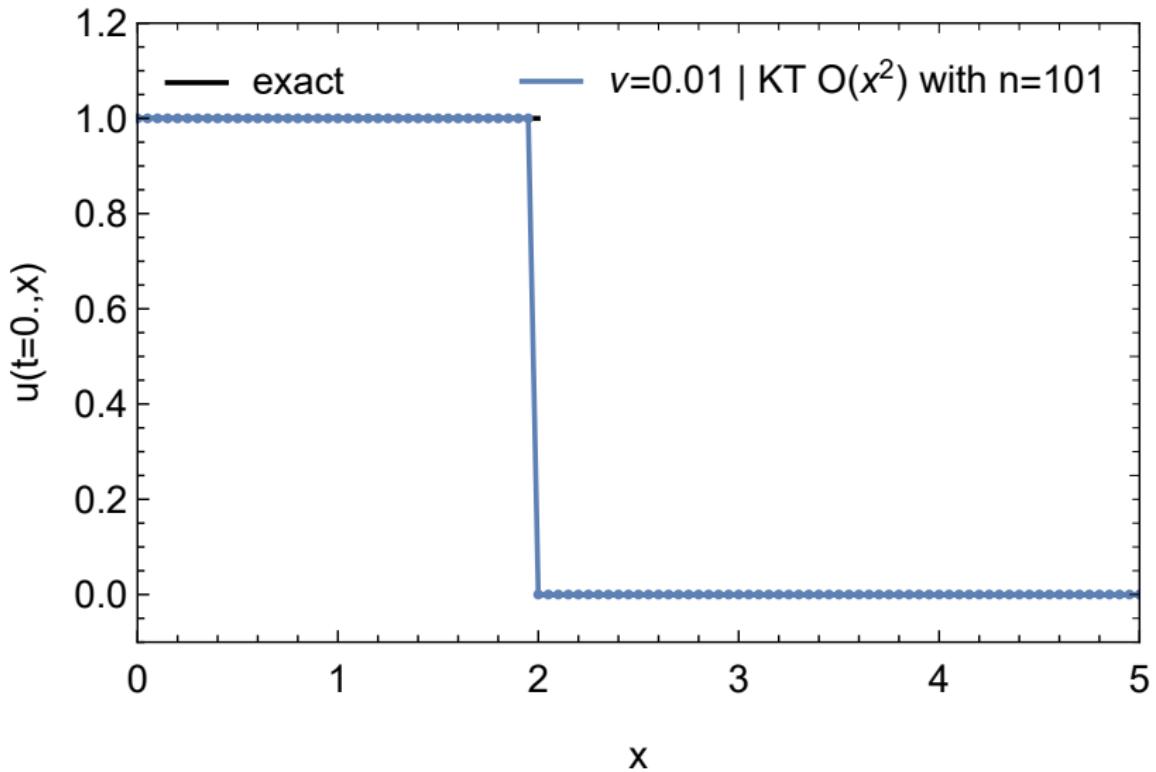


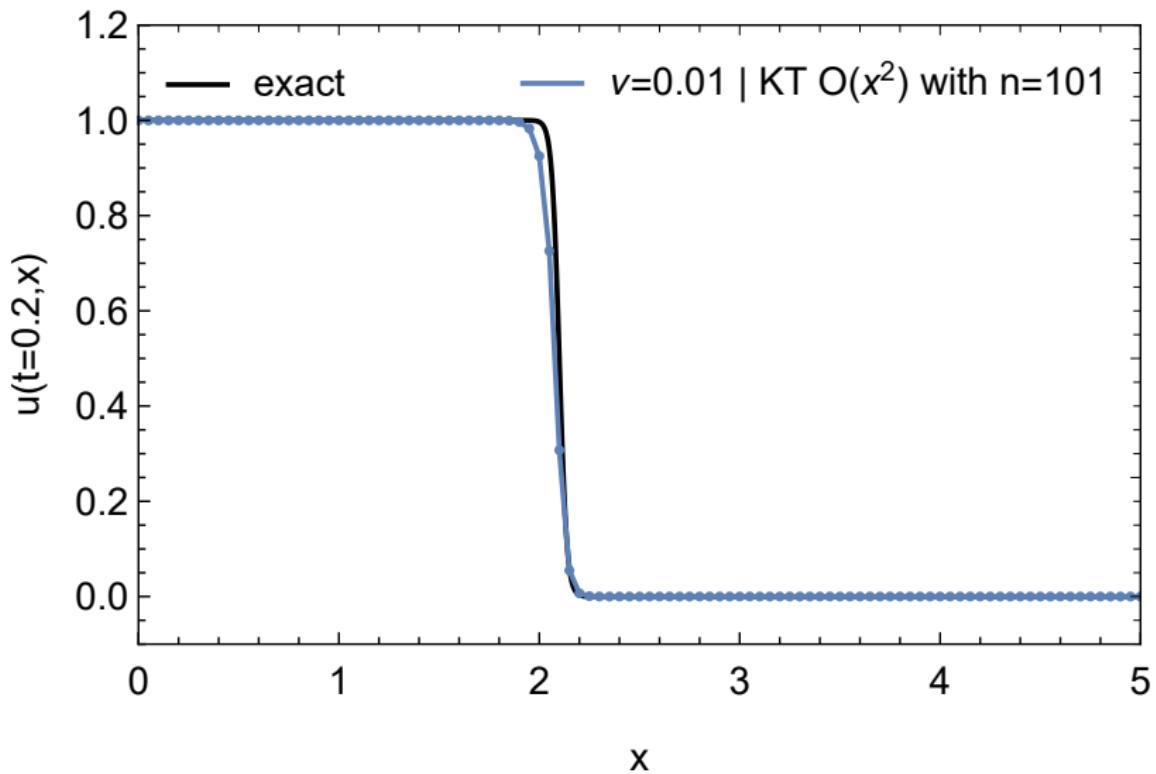


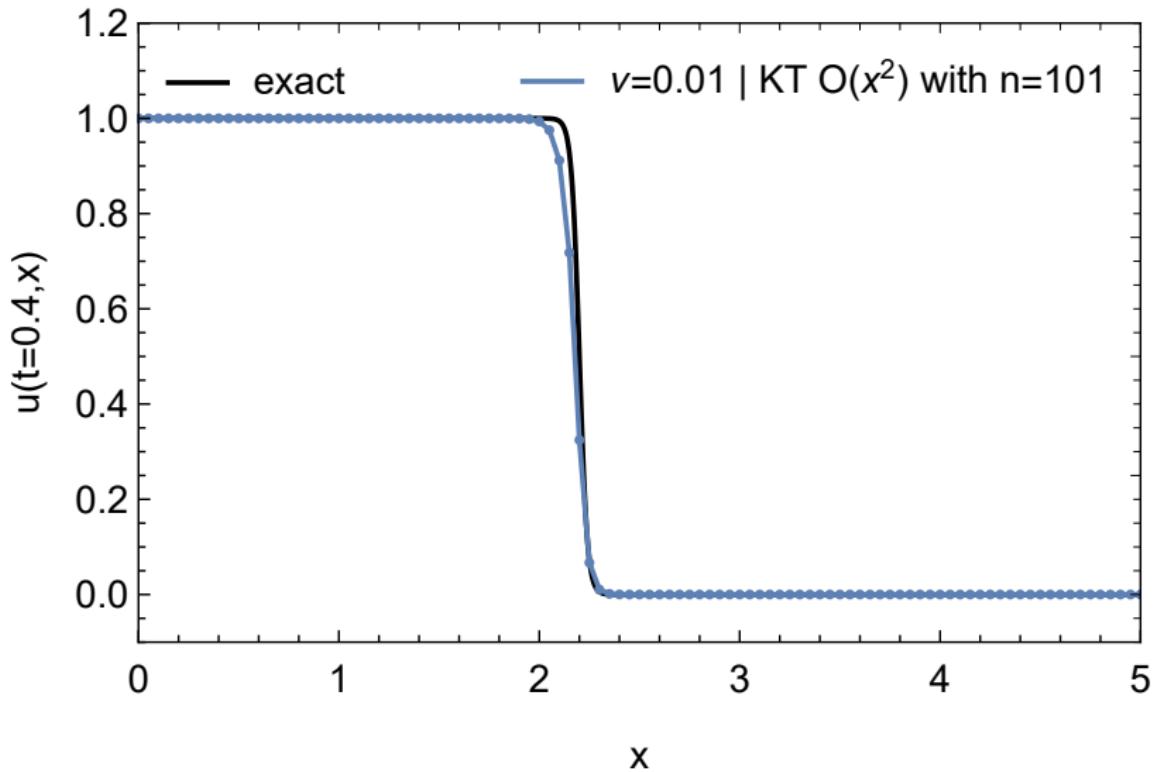


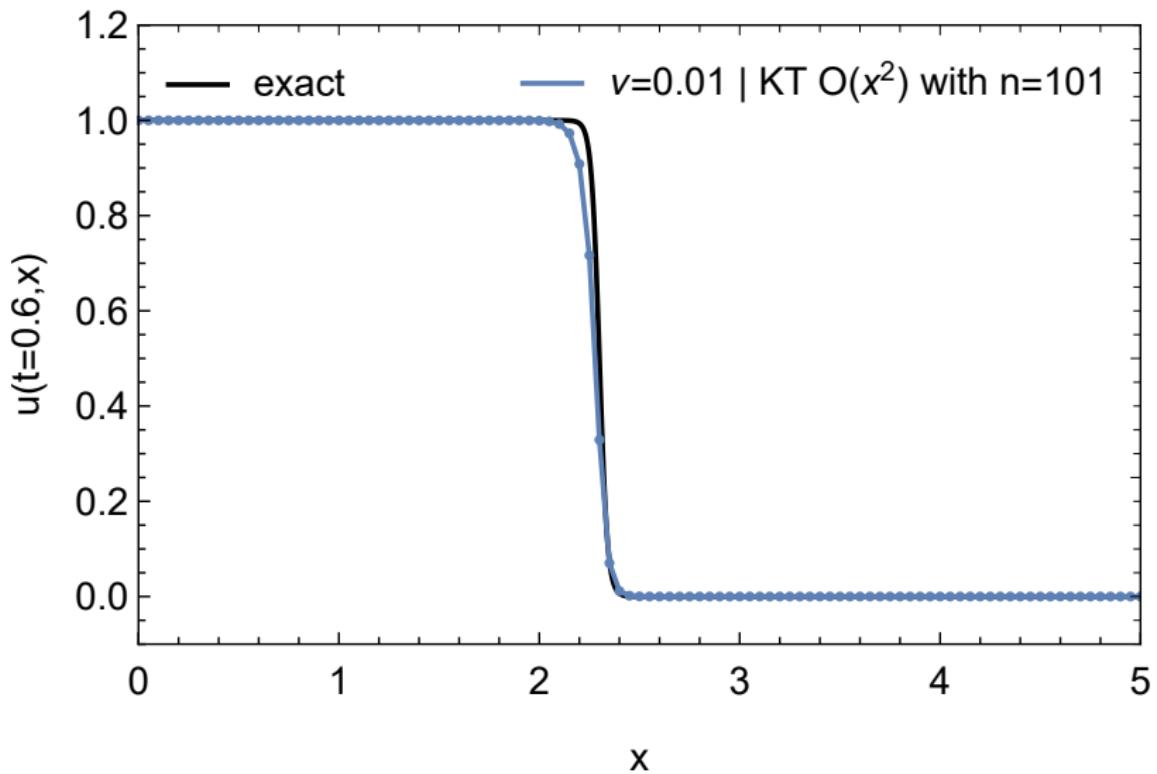


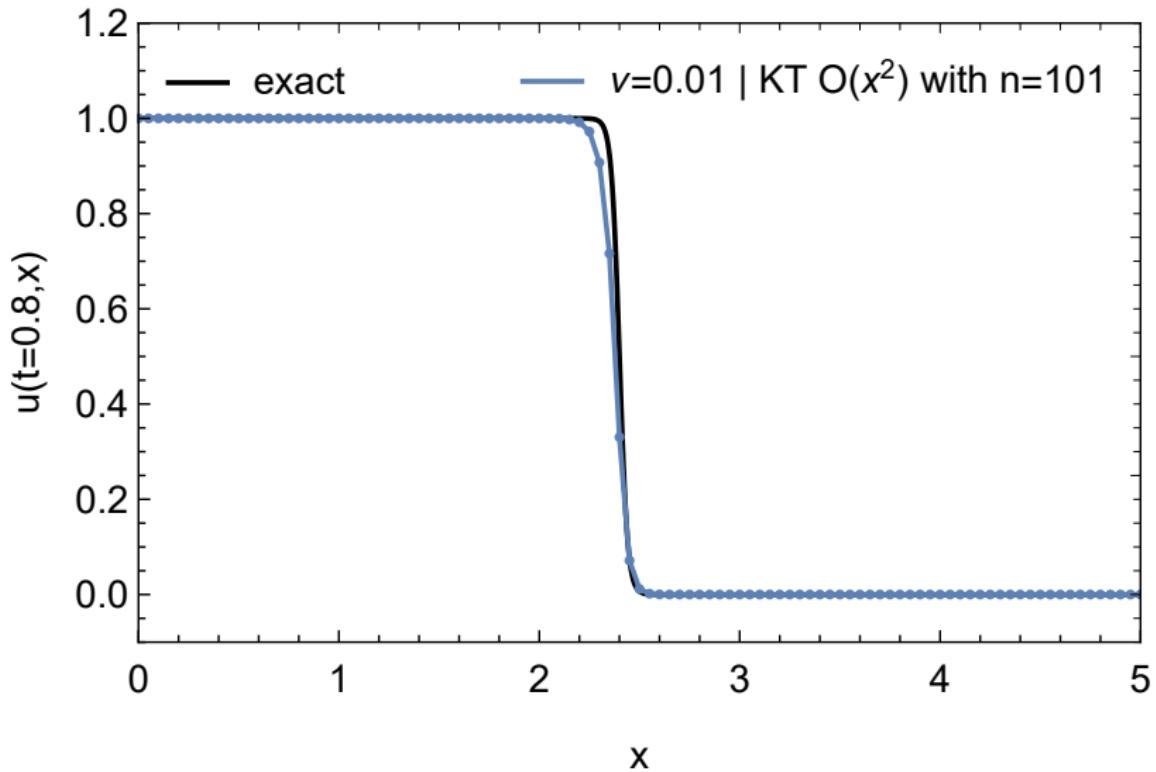


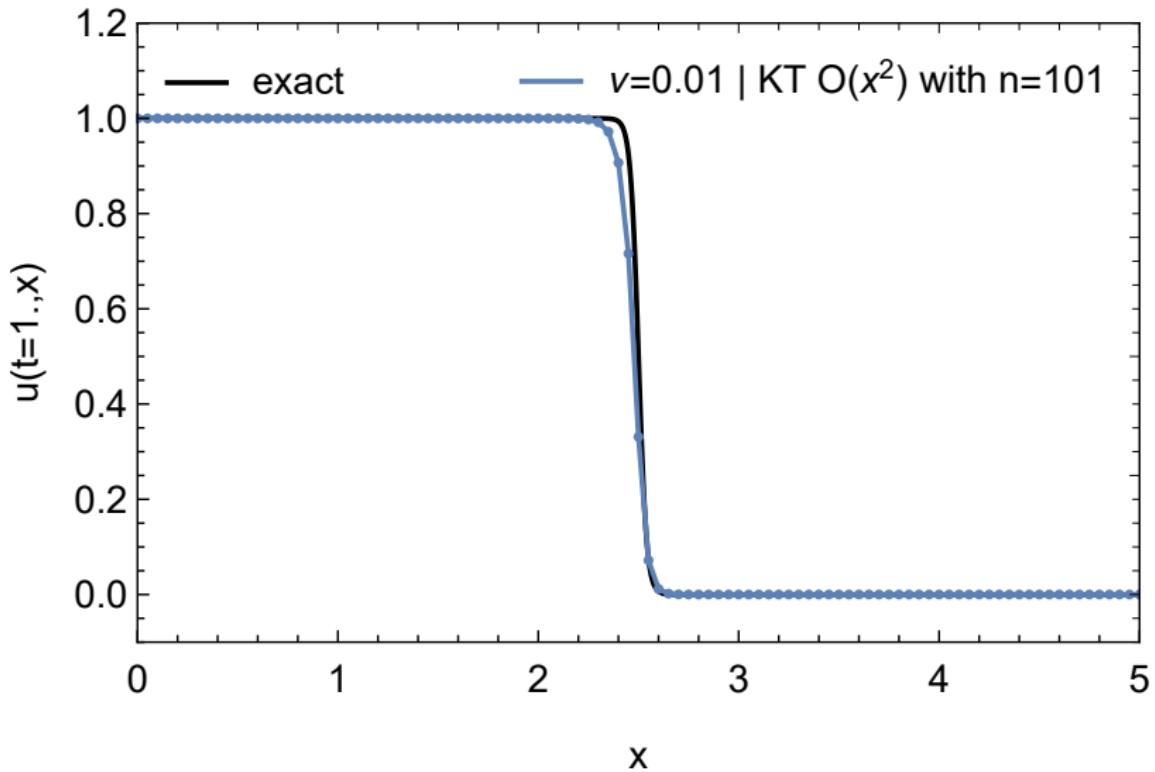


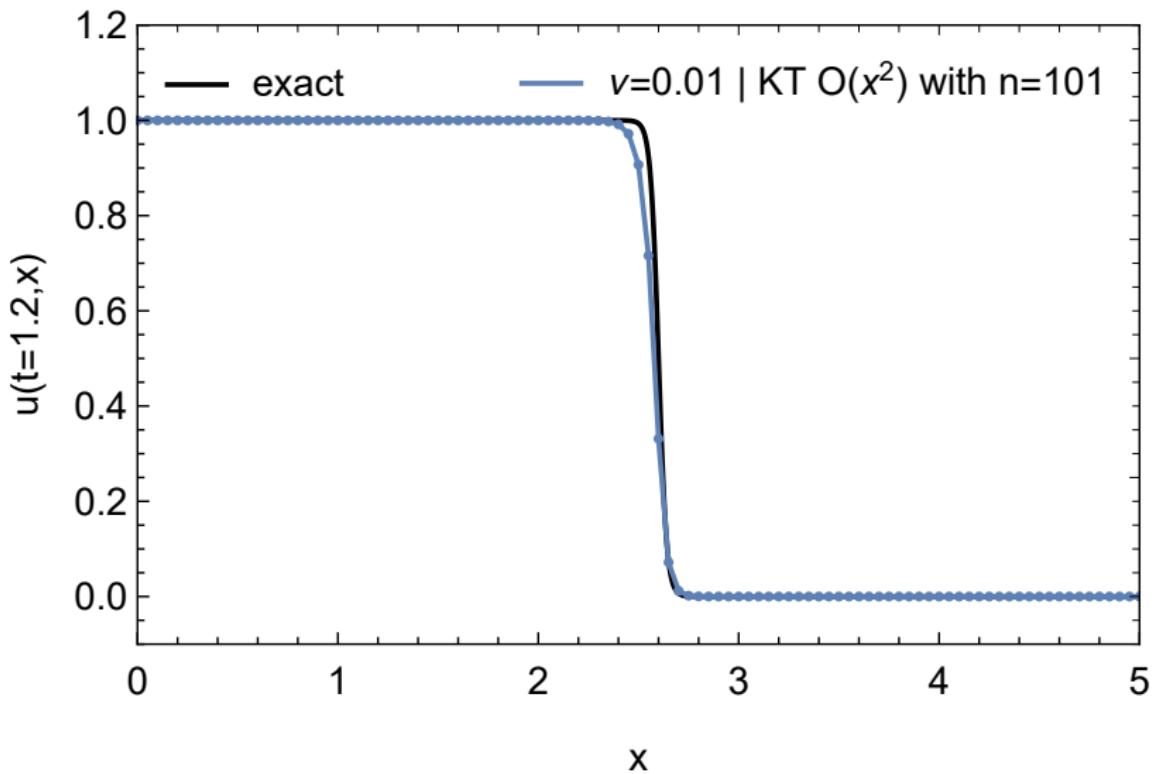


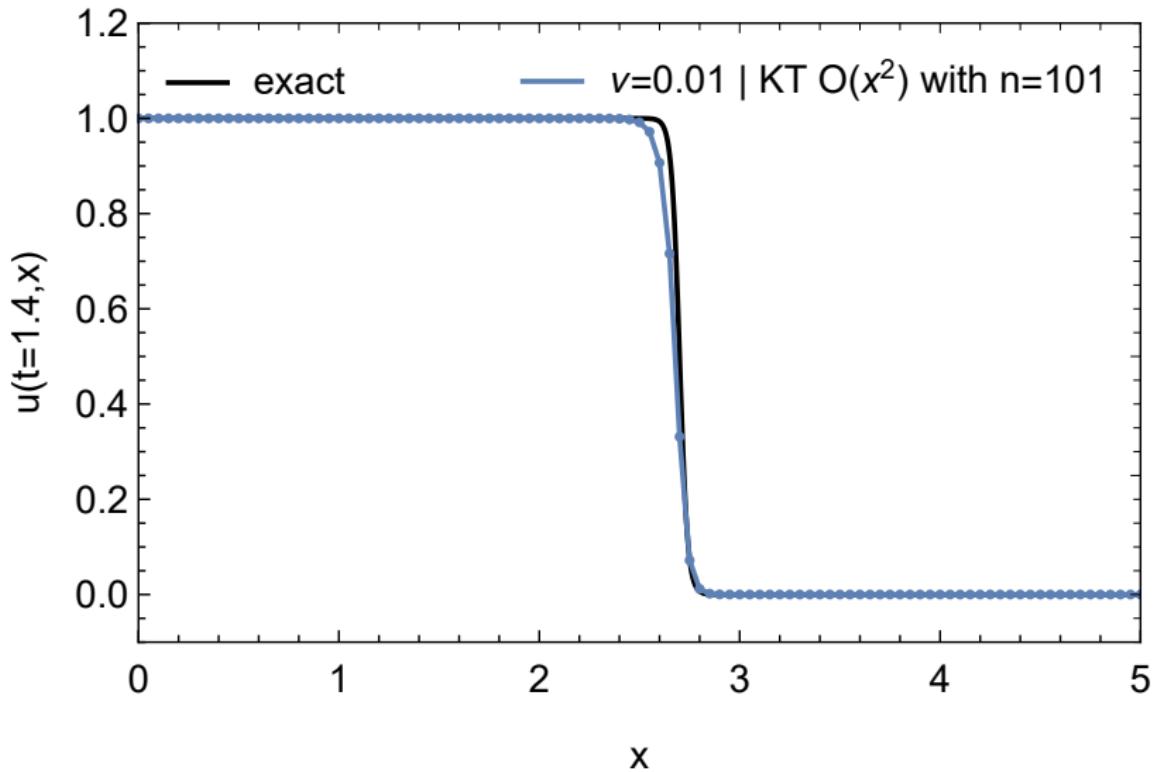


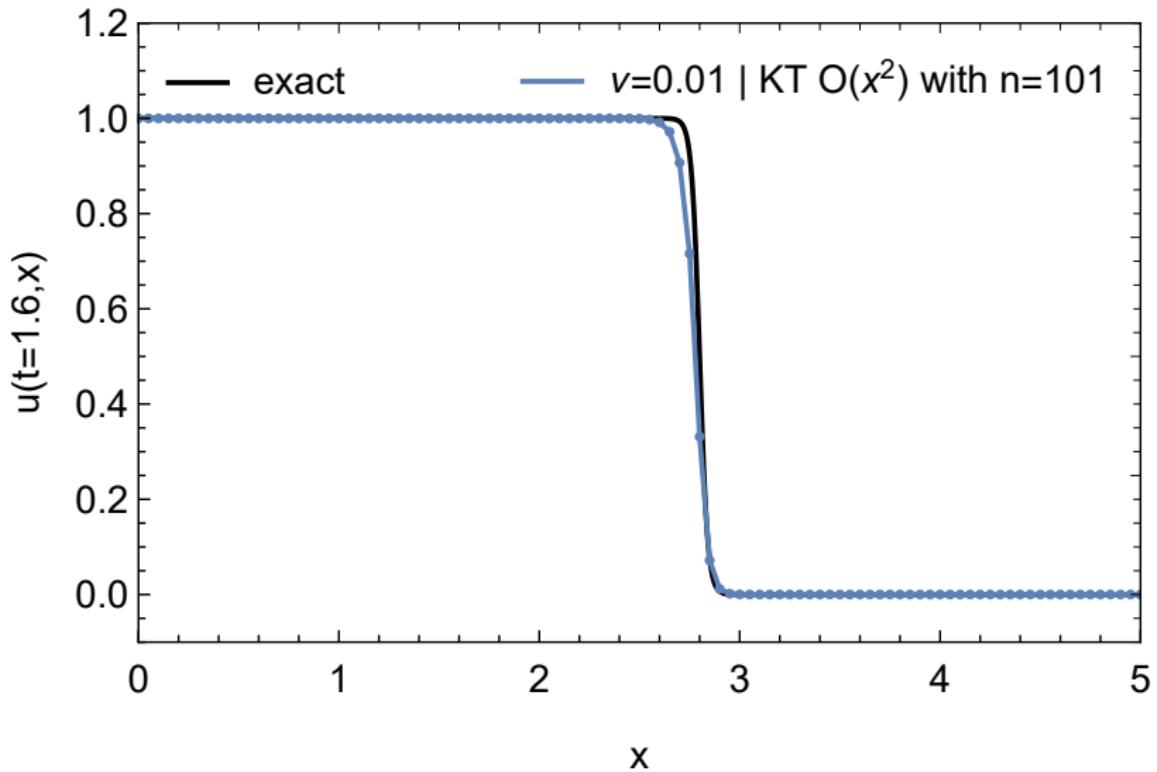


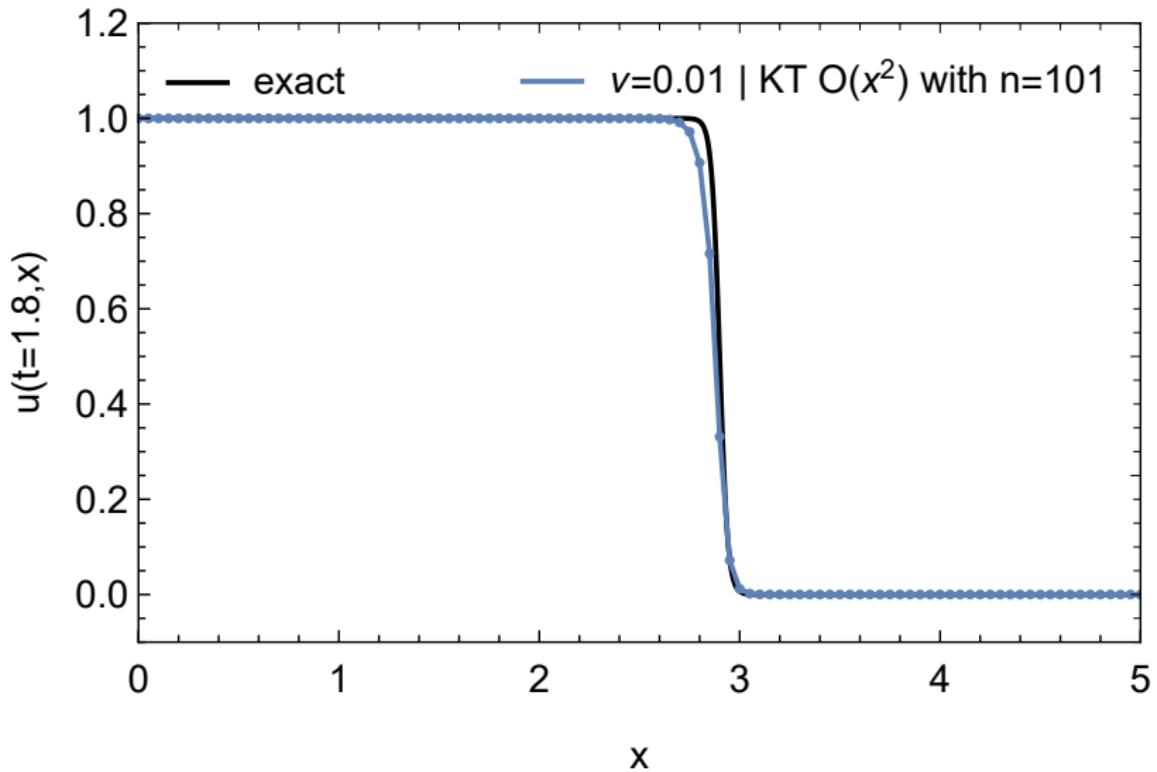


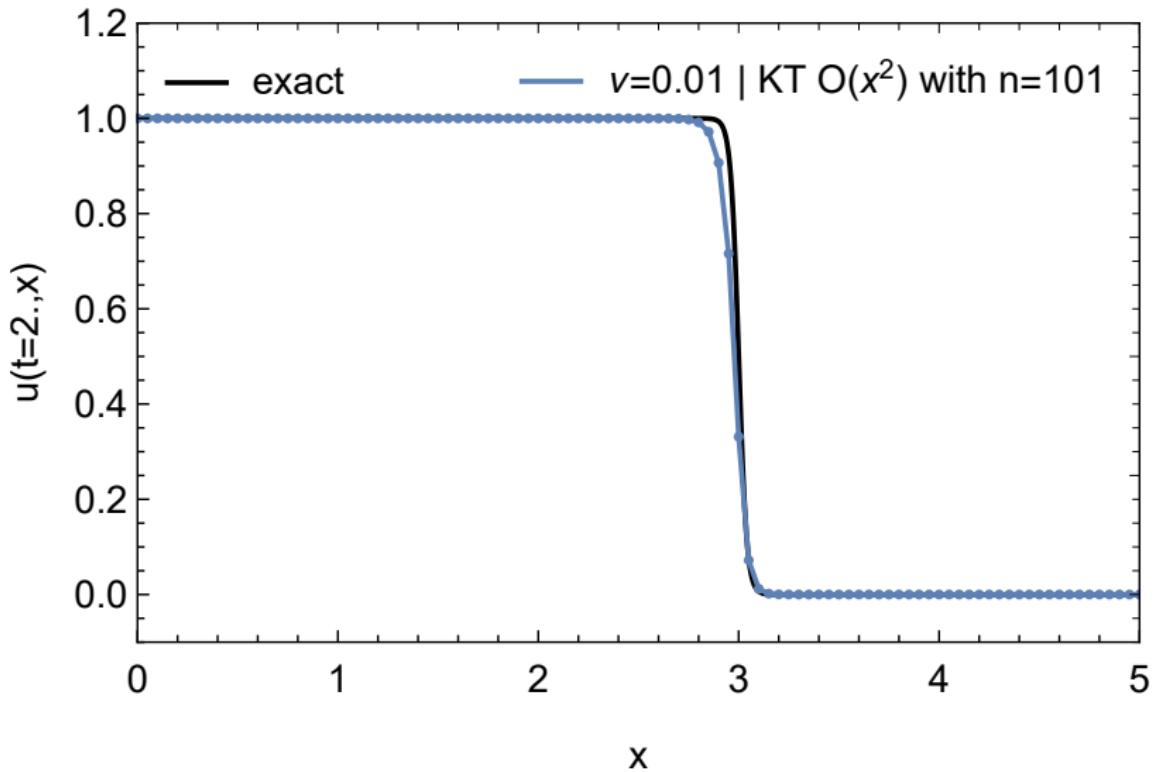


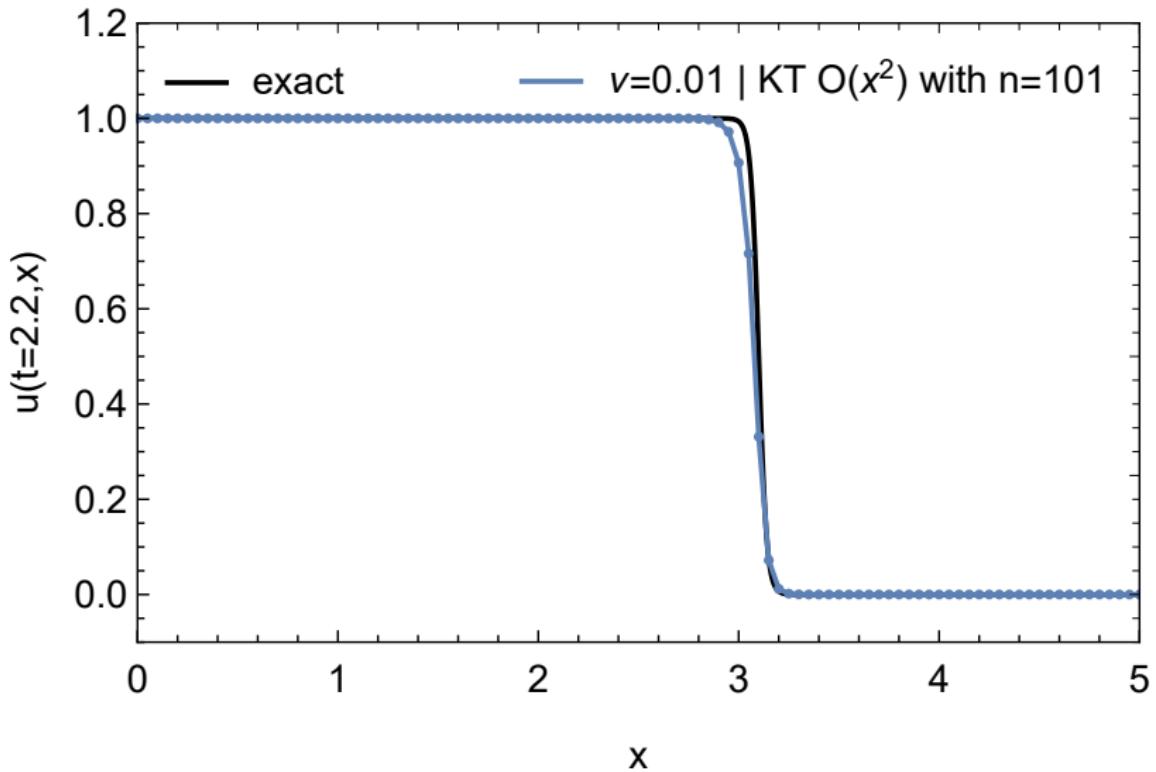


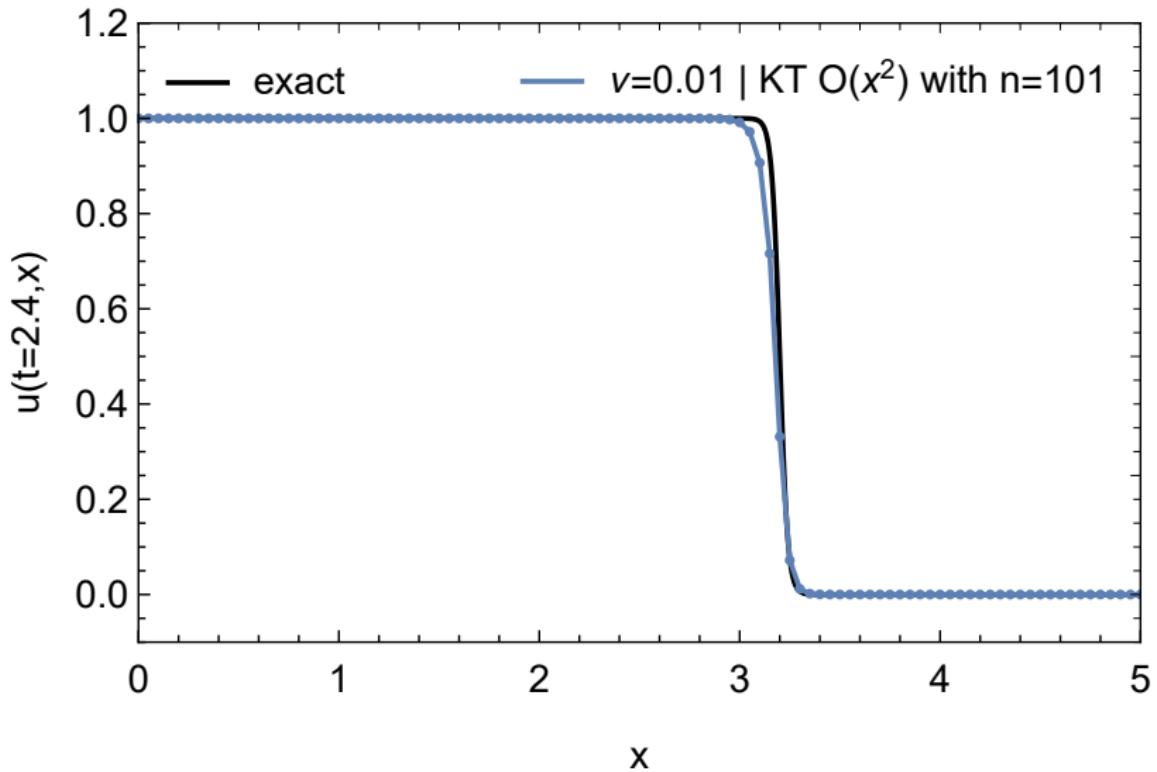


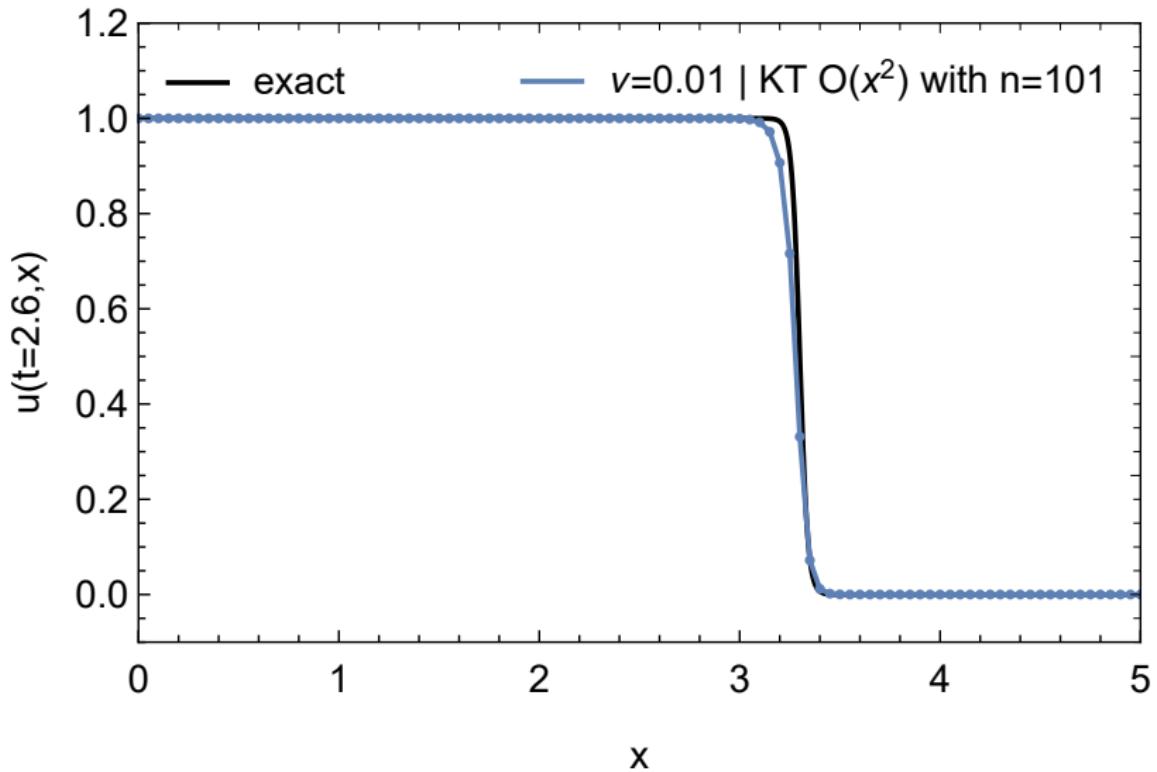


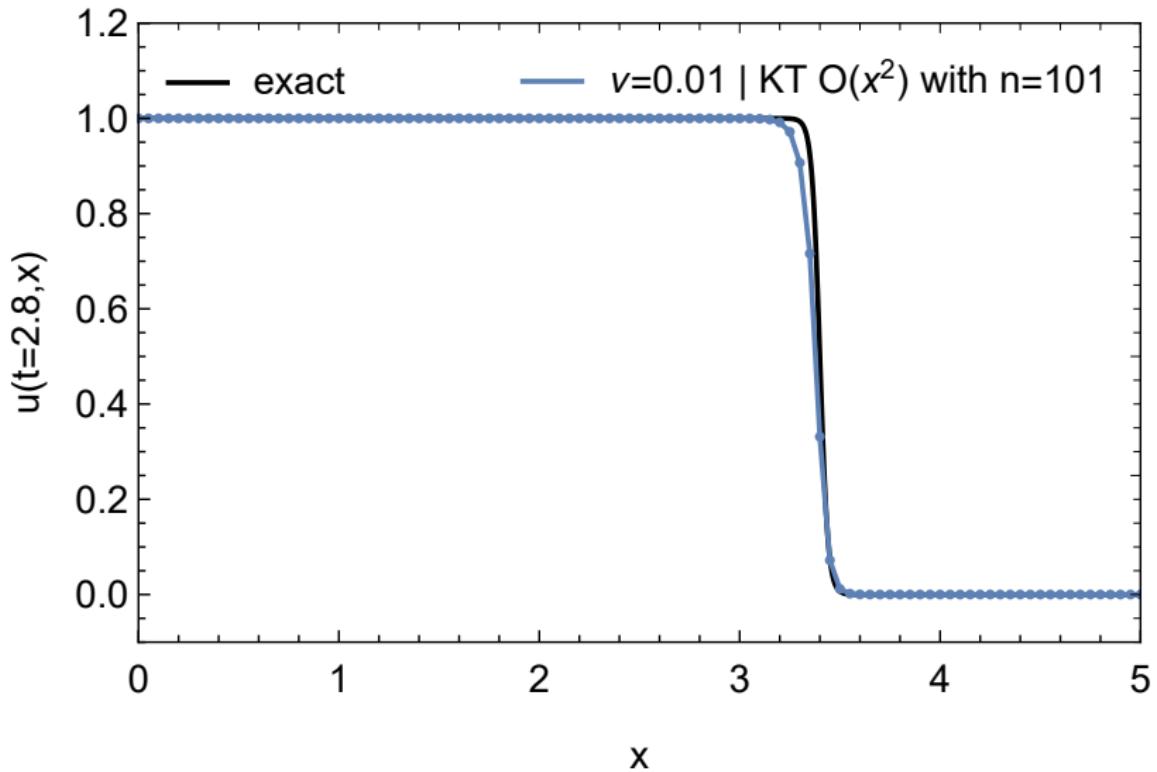


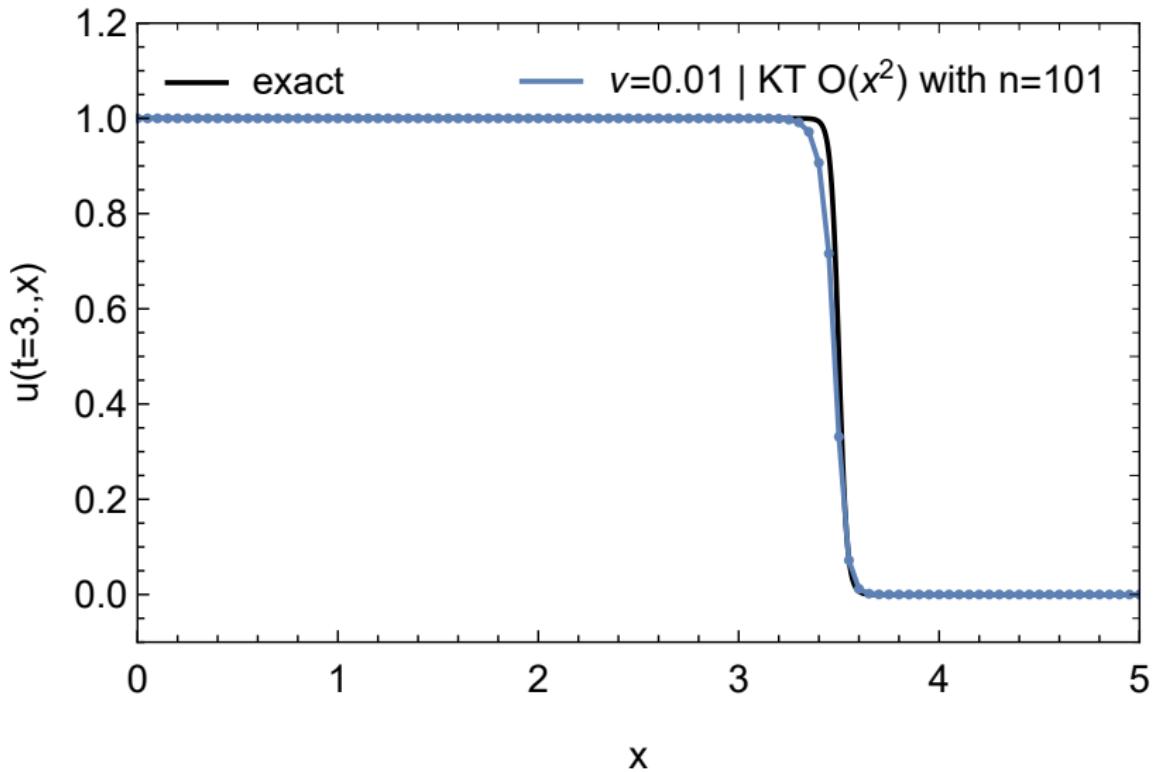


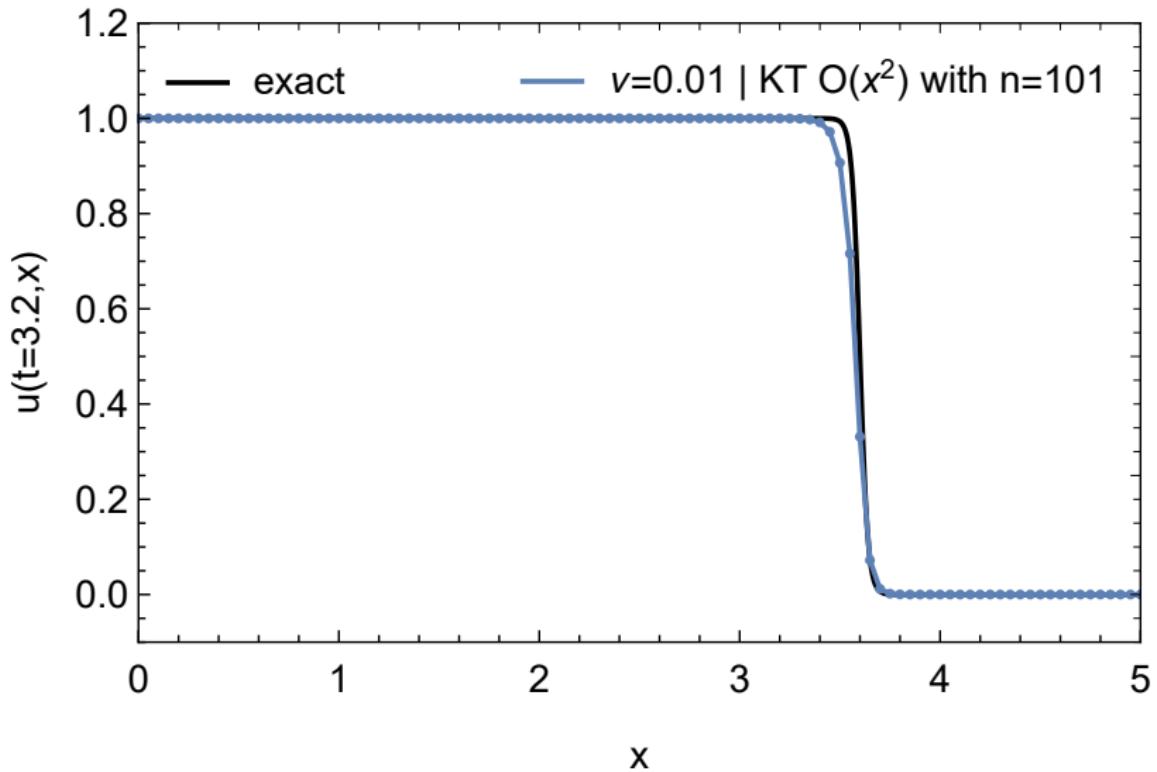


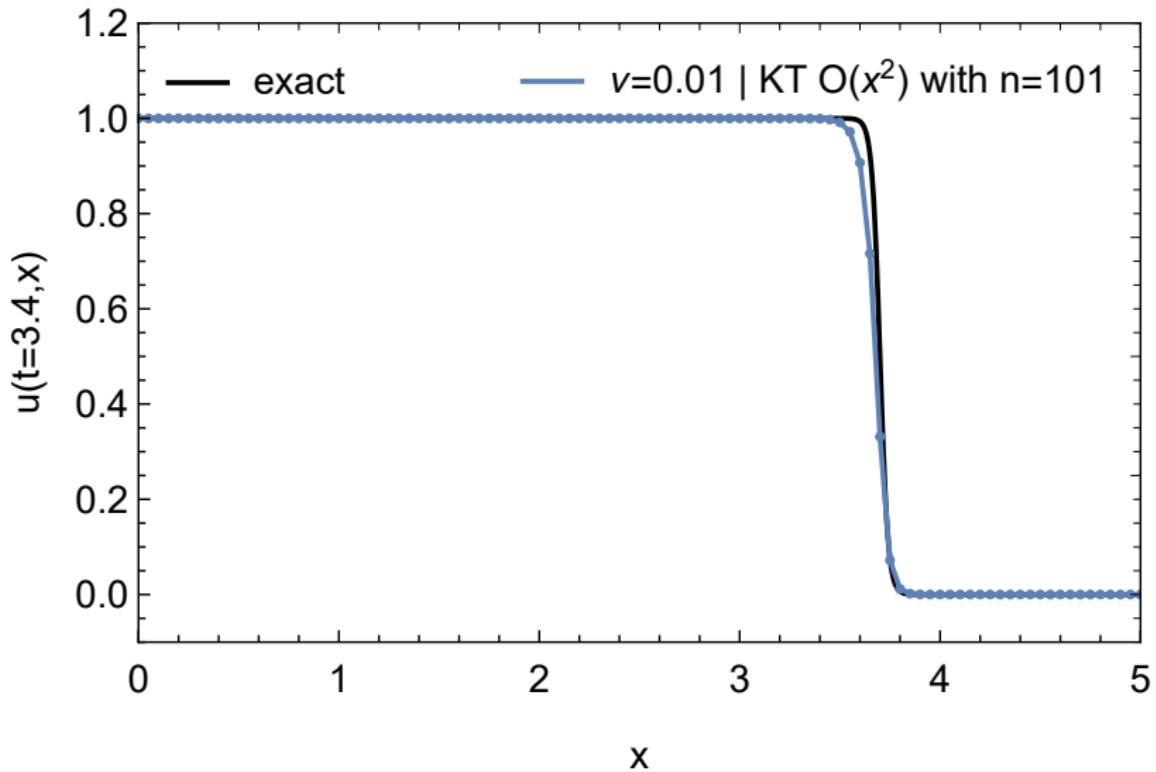


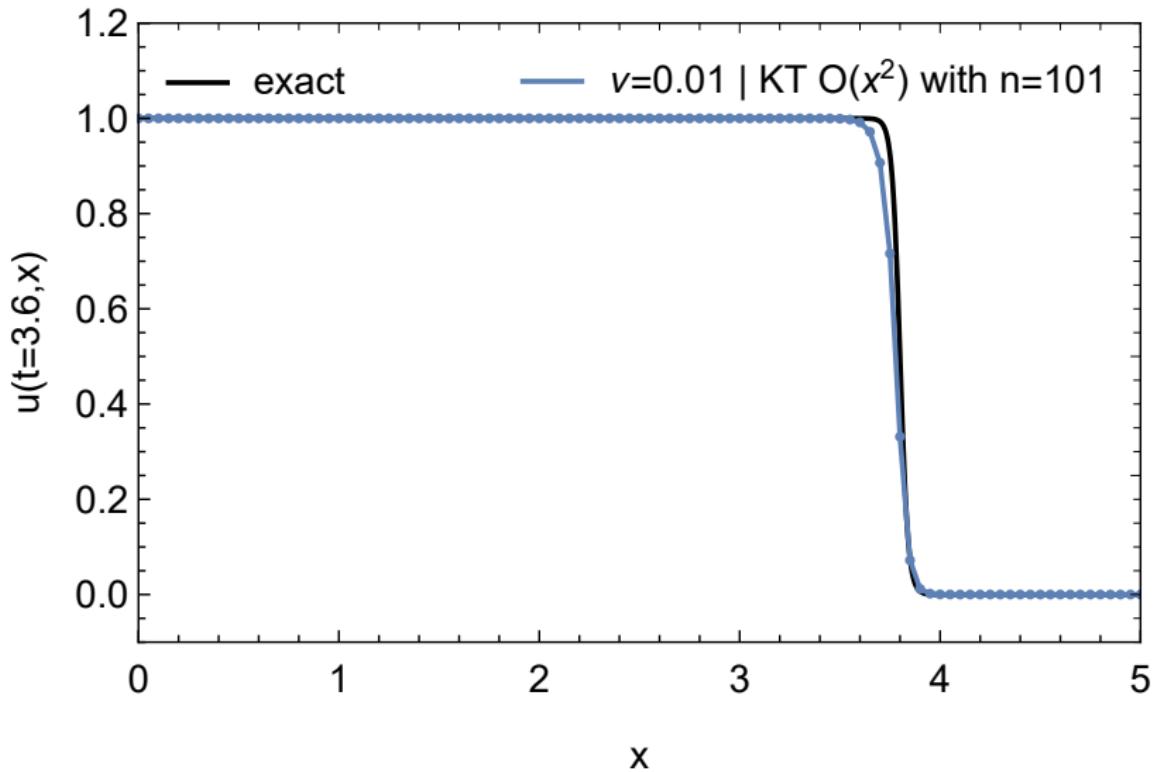


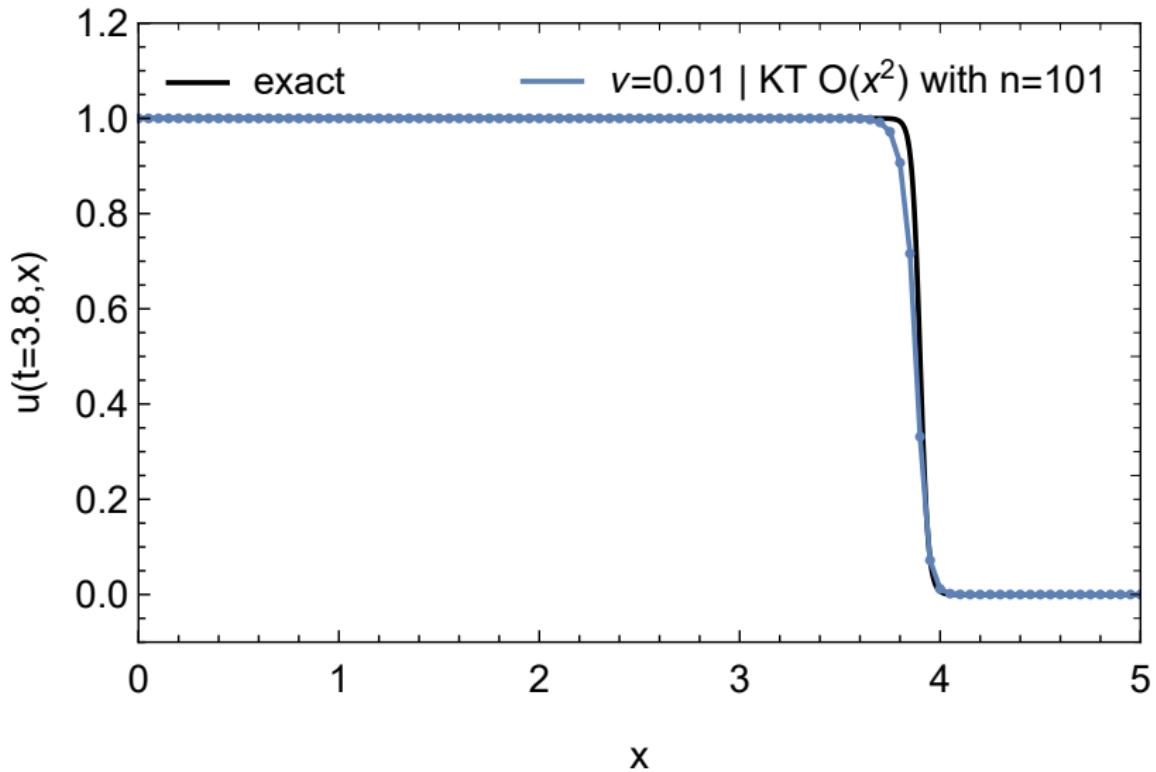


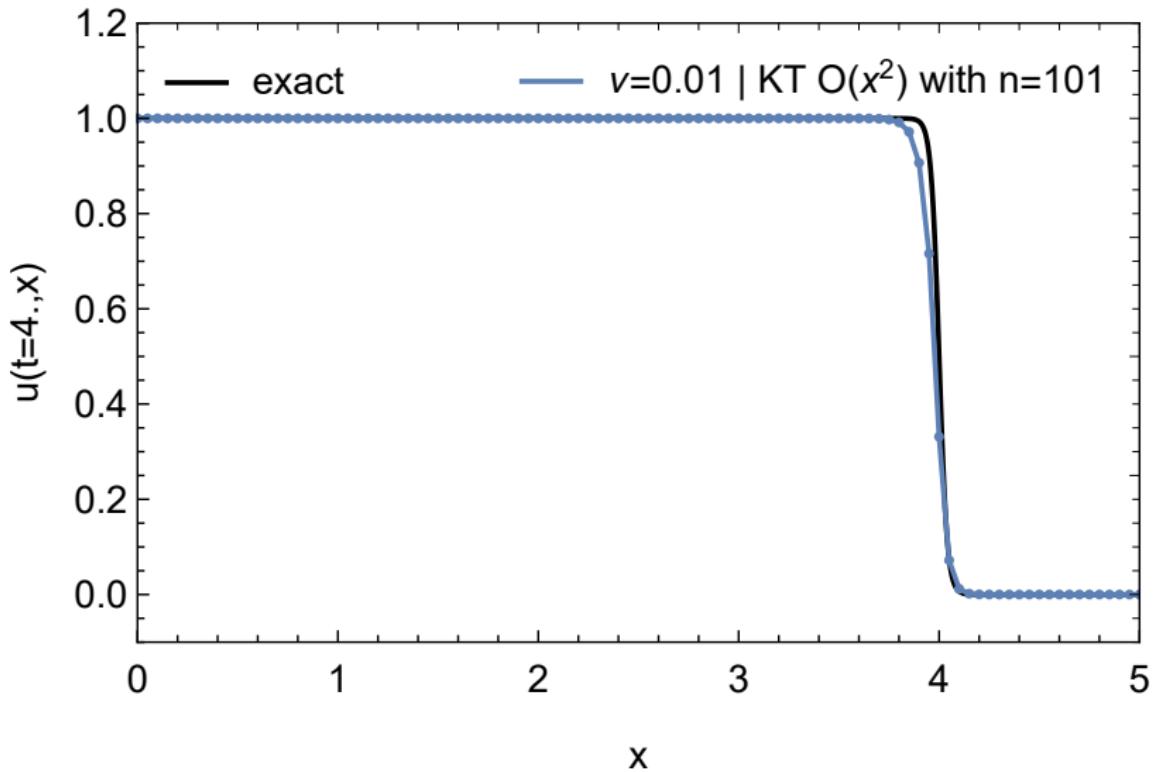












One-dimensional Euler System:

$$\partial_t \begin{pmatrix} \rho \\ \mu \\ \epsilon \end{pmatrix} + \partial_x \begin{pmatrix} \mu \\ \rho v^2 + p \\ (\epsilon + p)v \end{pmatrix} \equiv \partial_t u + \partial_x F[u] = 0, \quad (\text{E})$$

with  $u = (\rho, \mu, \epsilon)^T$ .

- ▶ Conserved quantities/densities: mass  $\rho$ , momentum  $\mu = \rho v$  and energy  $\epsilon$
- ▶ Equation of state:  $p = (\gamma - 1)(\epsilon - \rho v^2/2)$  with  $\gamma = 1.4$
- ▶ Shock tube ICs:

$$u(0, x) = \begin{cases} (\rho_L, \mu_L, \epsilon_L)^T & x \leq x_0 \\ (\rho_R, \mu_R, \epsilon_R)^T & \text{otherwise} \end{cases} \quad (\text{EST})$$

# Shock tube initial conditions

All initial conditions are given in SI units and have  $x_0 = 0.5$  m:

- ▶ Sod's classical initial condition<sup>2</sup>:

$$u_L = (1.0, 0.0, 2.5)^T \quad u_R = (0.125, 0.0, 0.25)^T \quad (\text{EST.Sod})$$

- ▶ Very strong shock<sup>3</sup>:

$$u_L = (0.445, 0.311, 8.928)^T \quad u_R = (0.5, 0.0, 1.4275)^T \quad (\text{EST.Lax})$$

- ▶ Strong double rarefaction wave and vacuum limit<sup>4</sup>:

$$u_L = (1.0, -2.0, 3.0)^T \quad u_R = (1.0, 2.0, 3.0)^T \quad (\text{EST.Toro2})$$

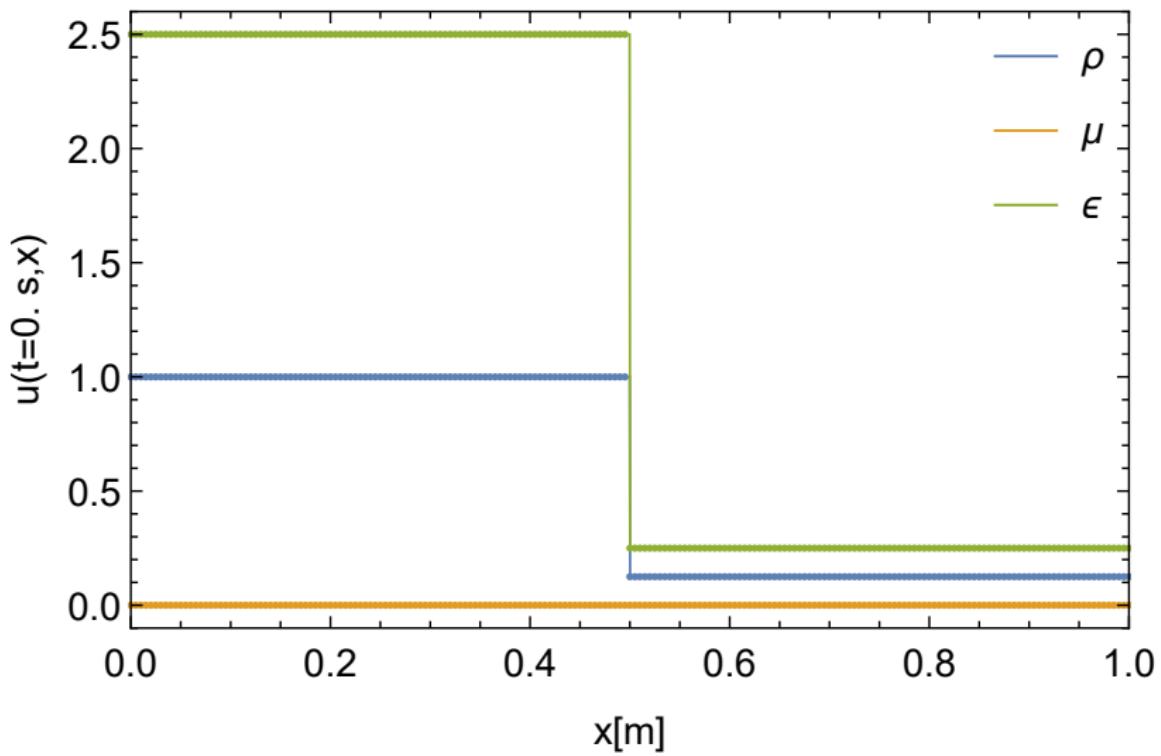
Exact solutions of the respective Riemann problems were computed with<sup>5</sup>

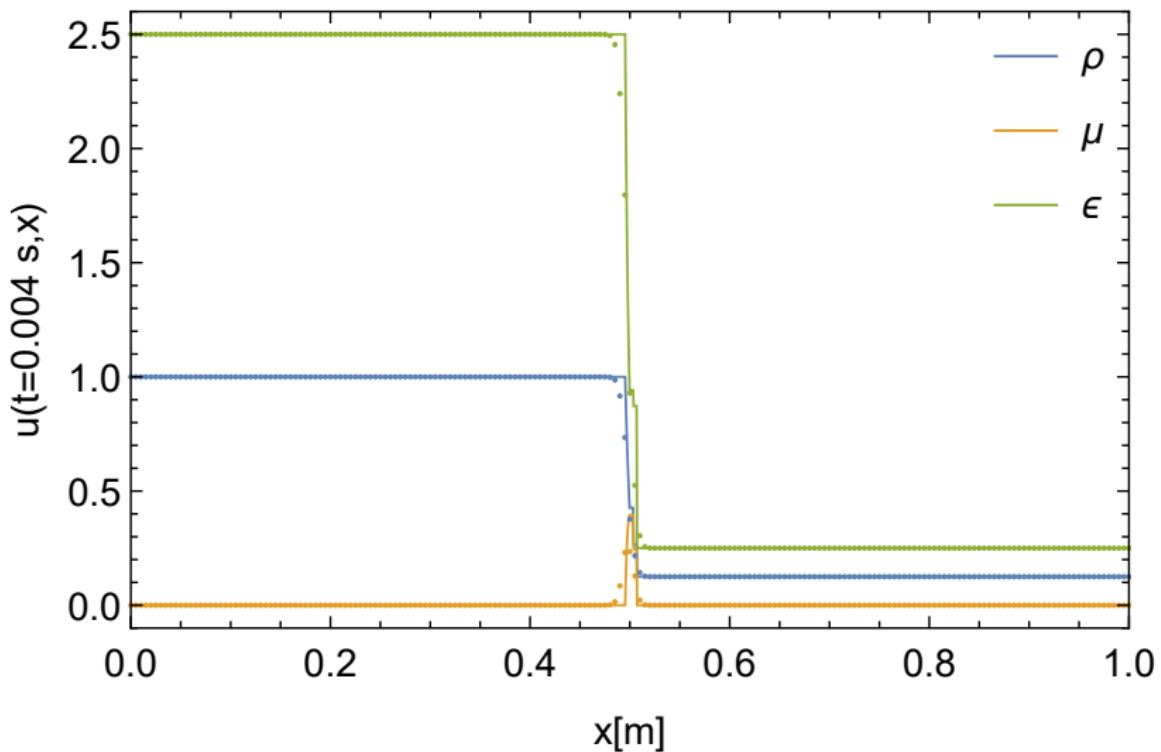
<sup>2</sup>G. A. Sod, J. Comput. Phys. 27.1 (1978)

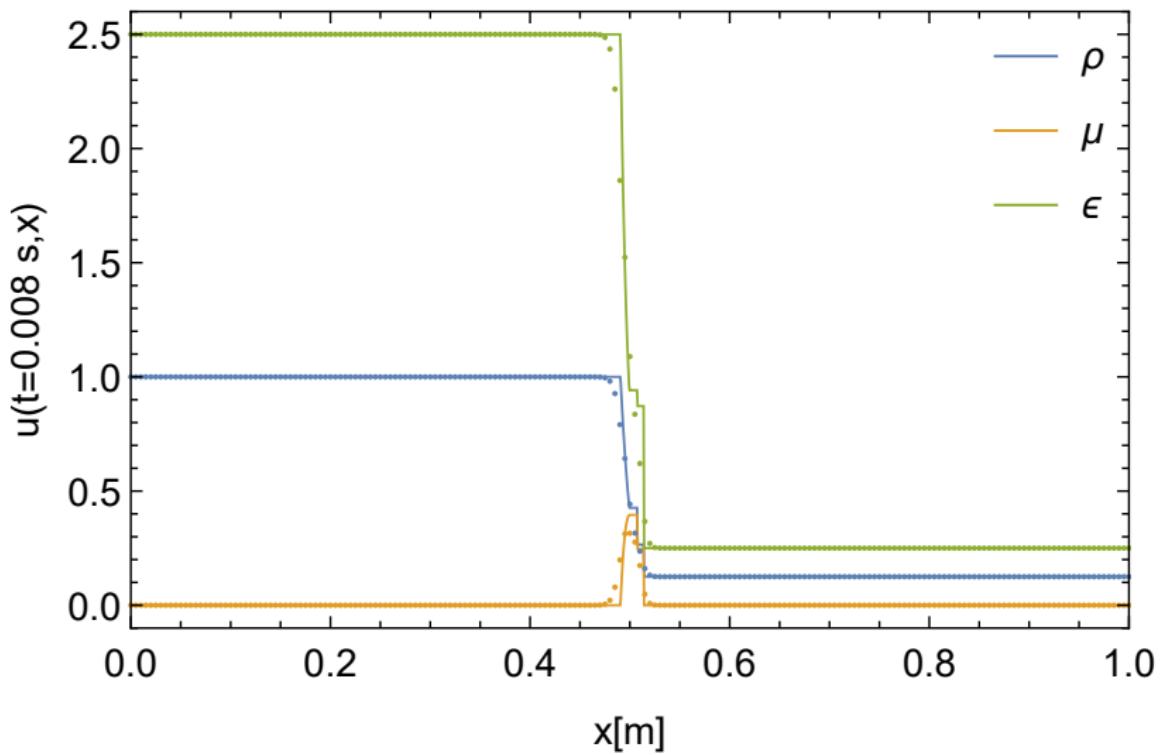
<sup>3</sup>P. D. Lax, Comm. Pure Appl. Math. 7.1 (1954)

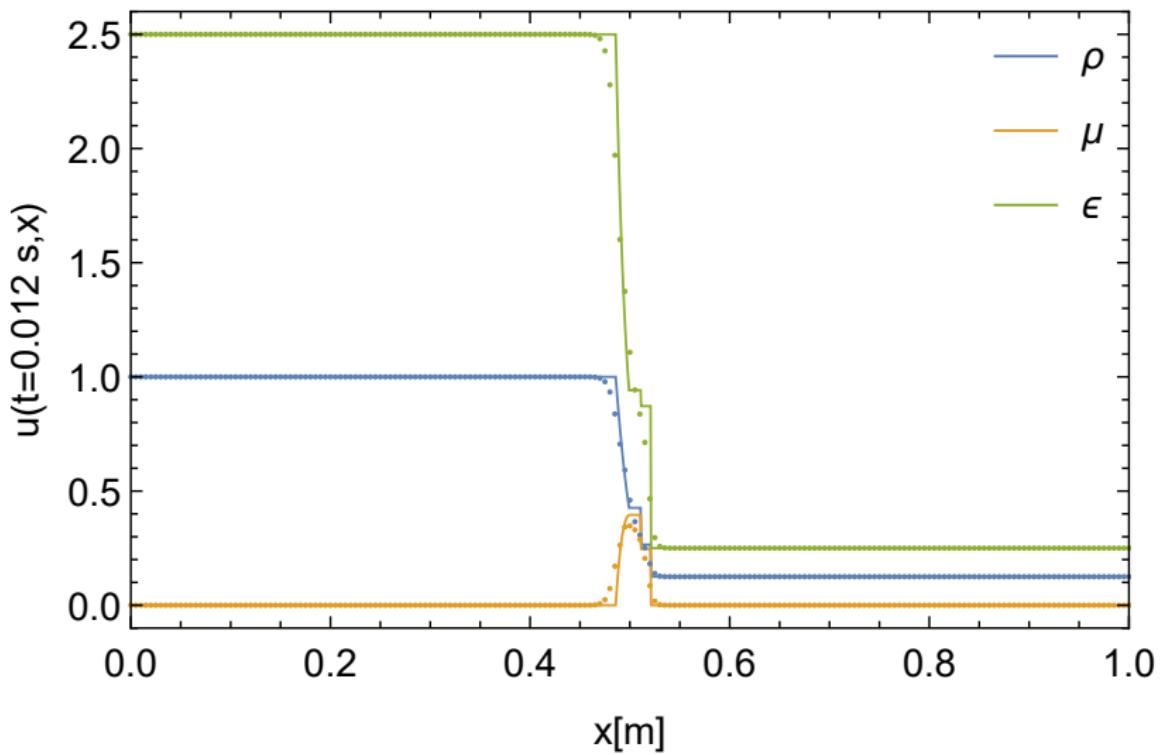
<sup>4</sup>E. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics*, Berlin: Springer, 1999

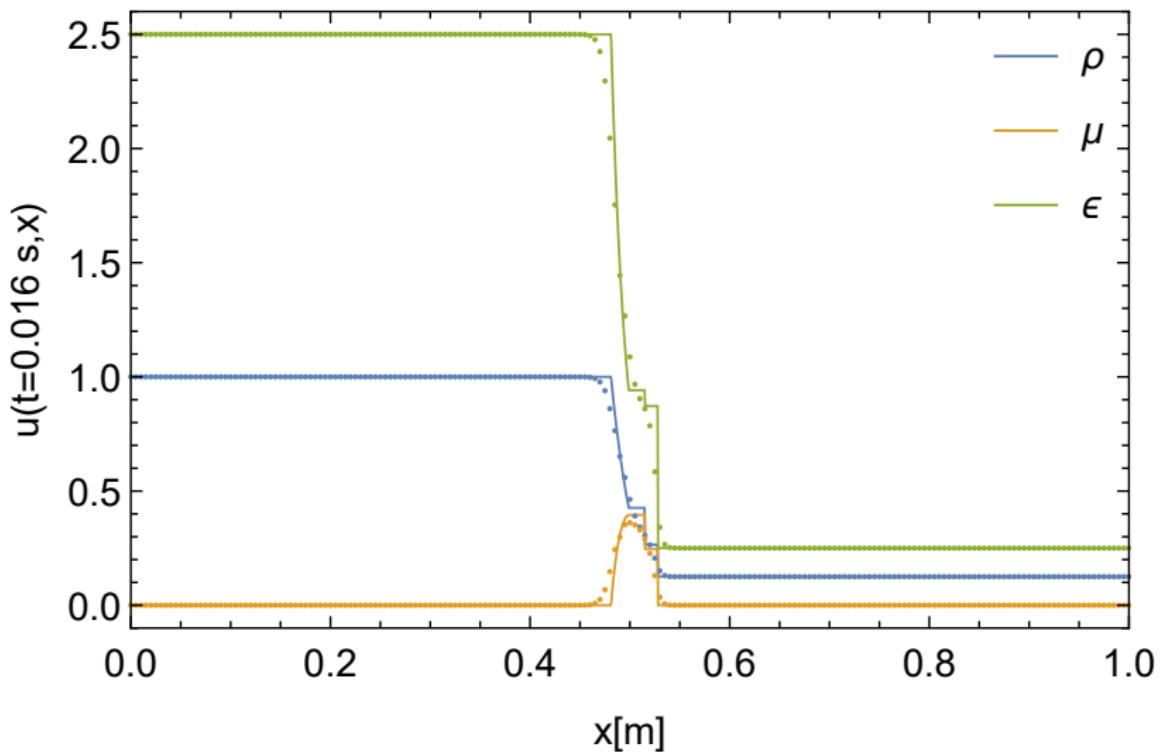
<sup>5</sup>D. I. Ketcheson and et al., Riemann Problems and Jupyter Solutions, [github.com/clawpack/riemann\\_book](https://github.com/clawpack/riemann_book) - 2019.07.09 - 20:30

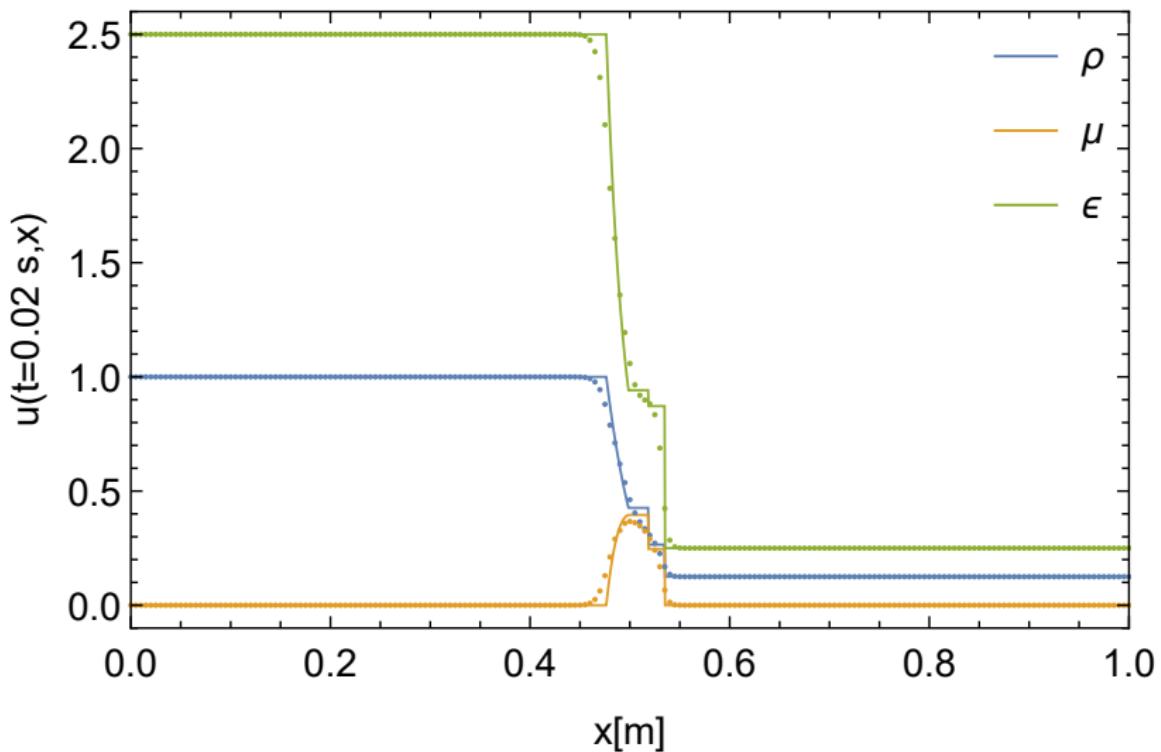


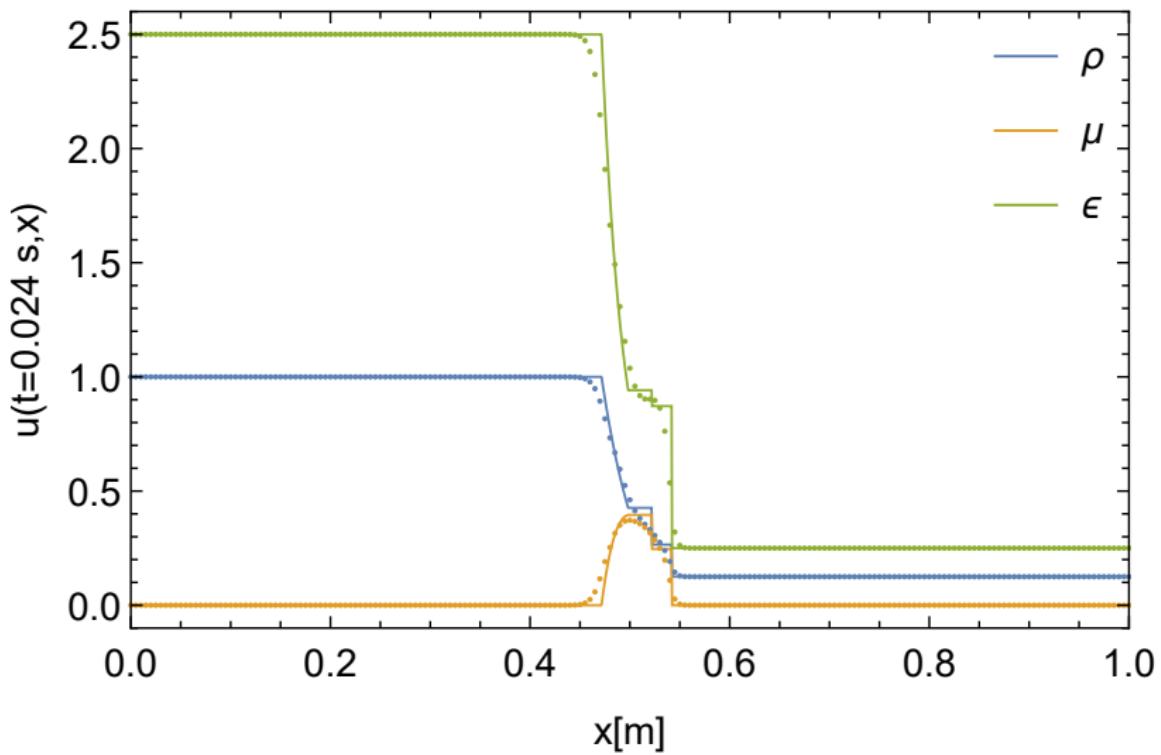


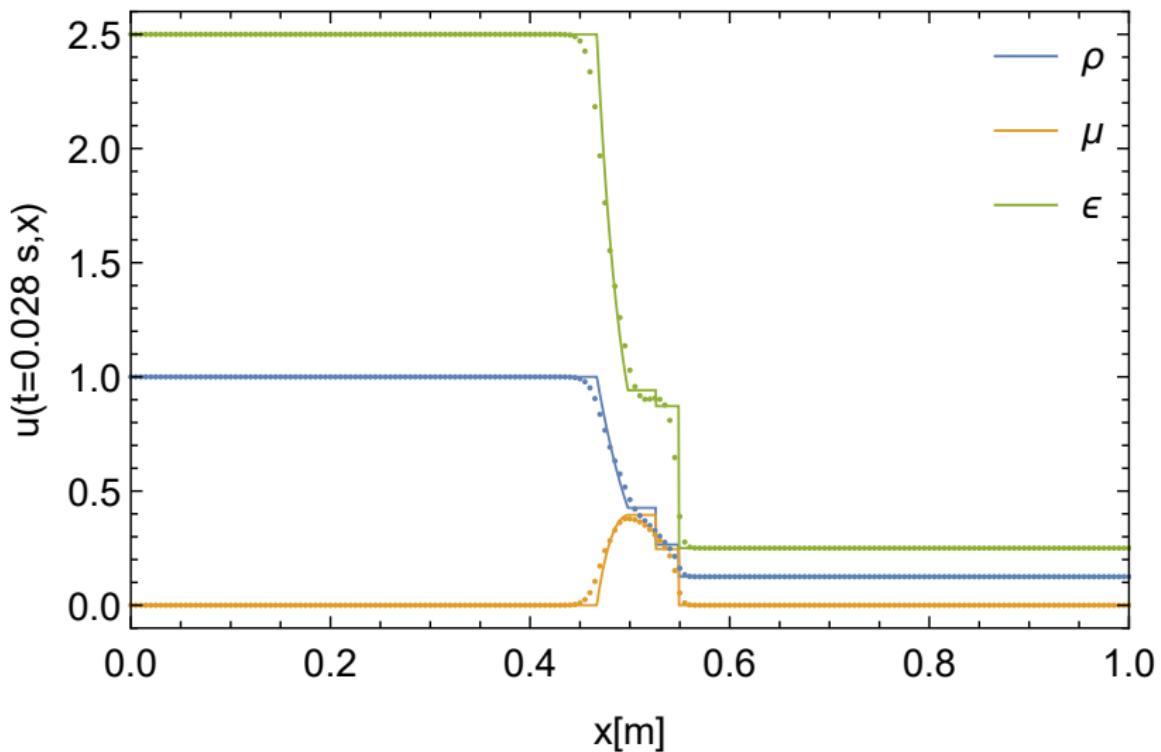


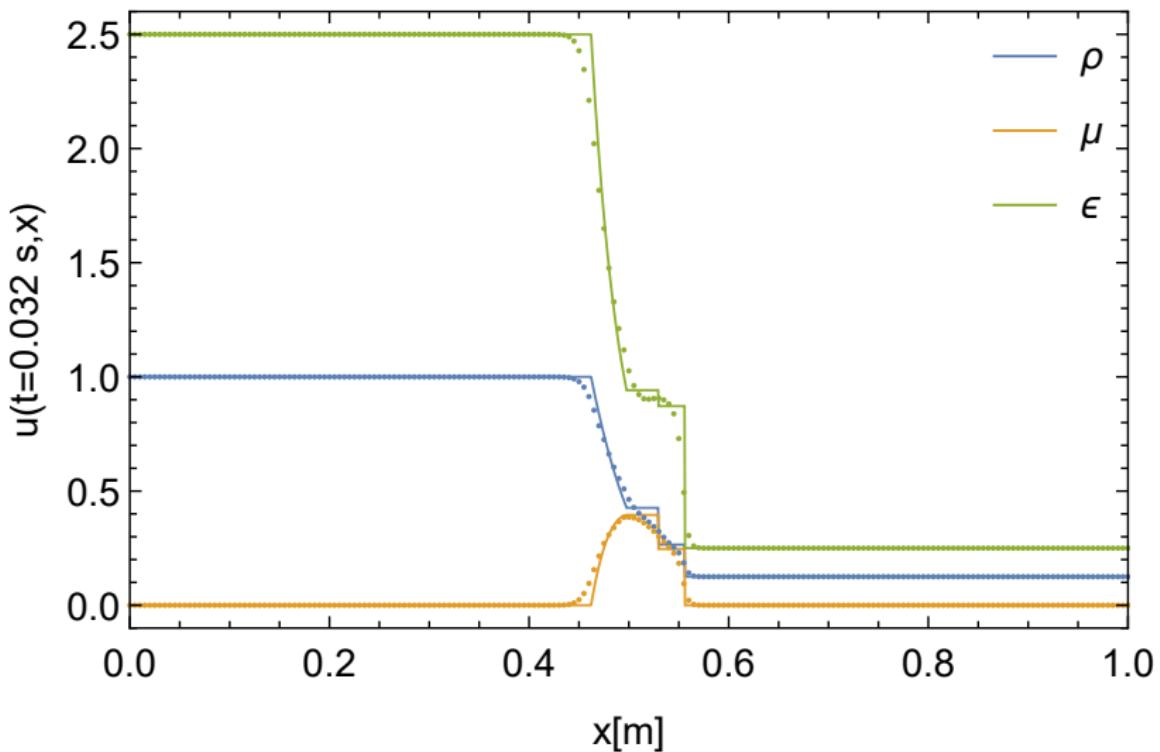


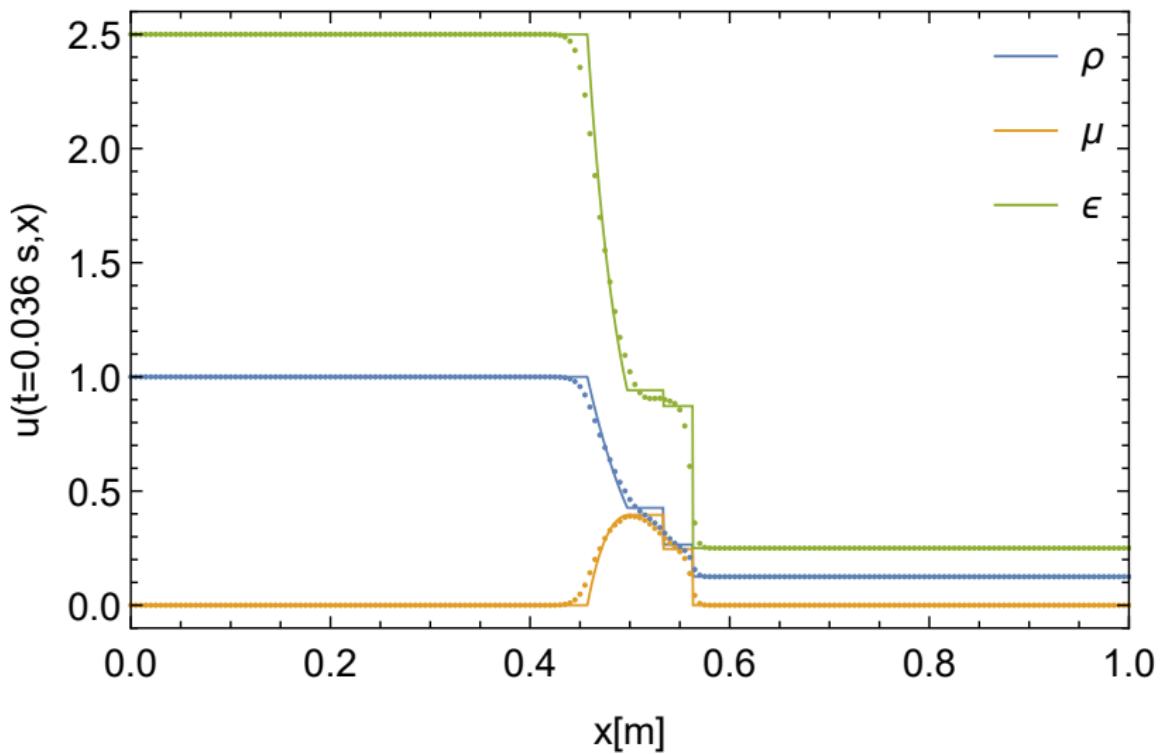


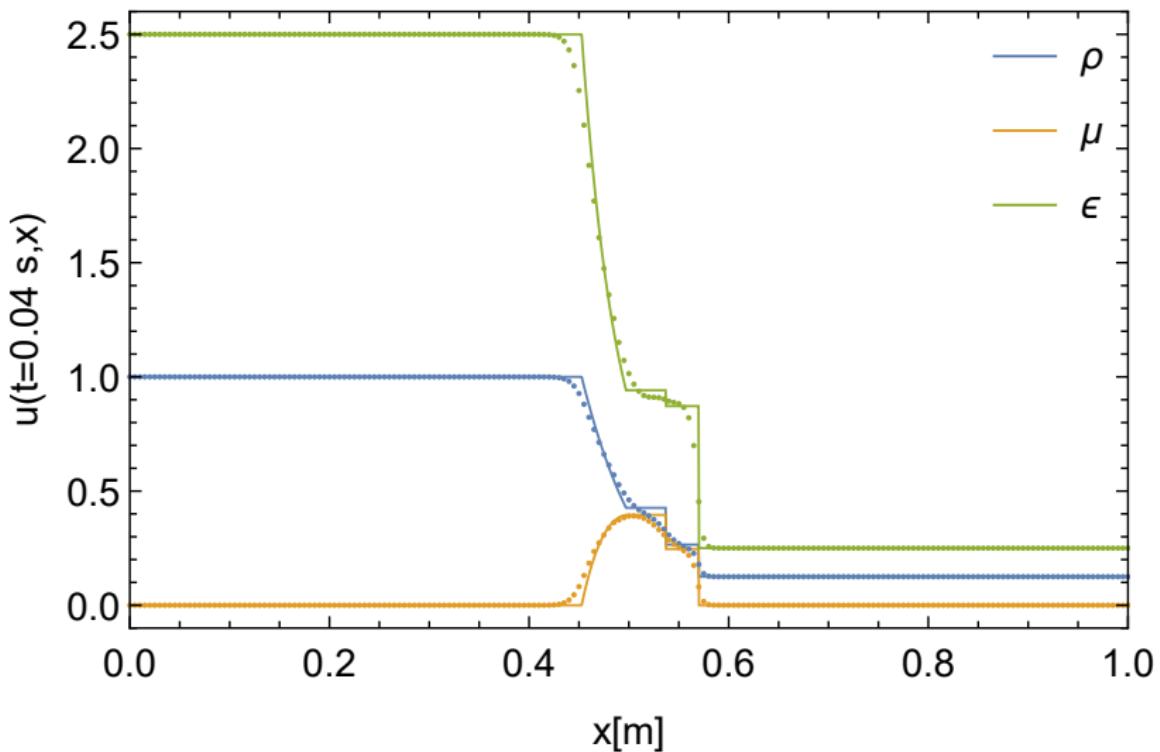


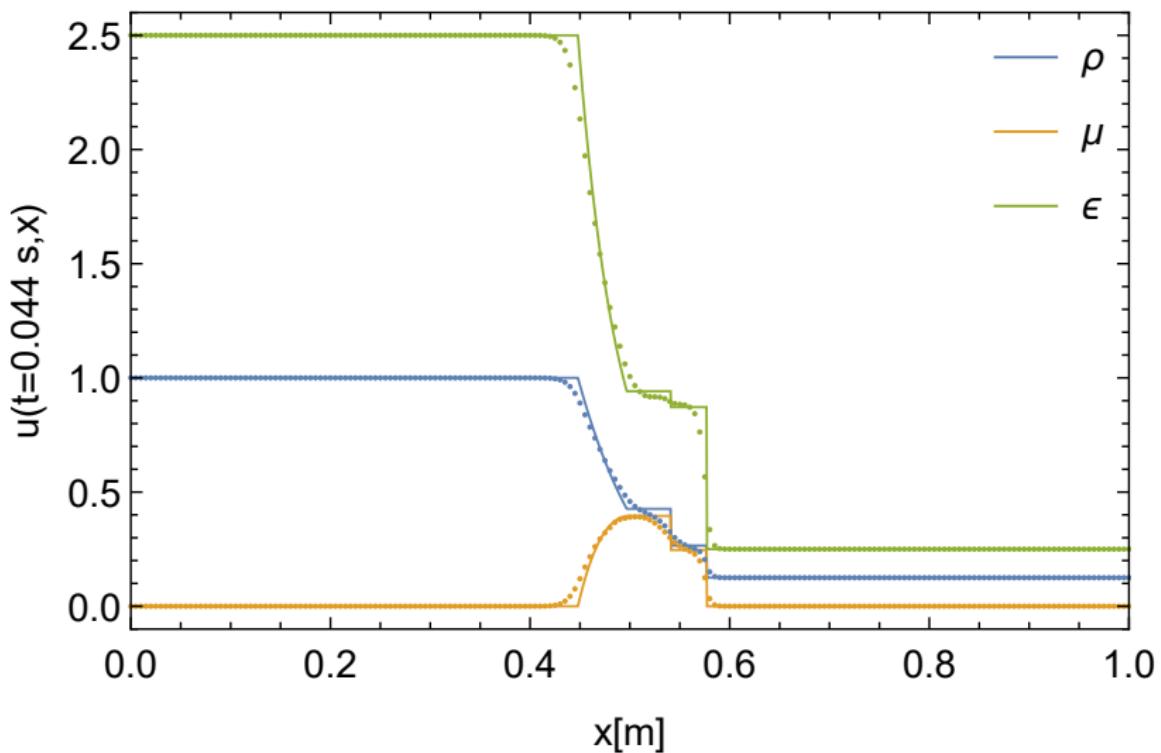


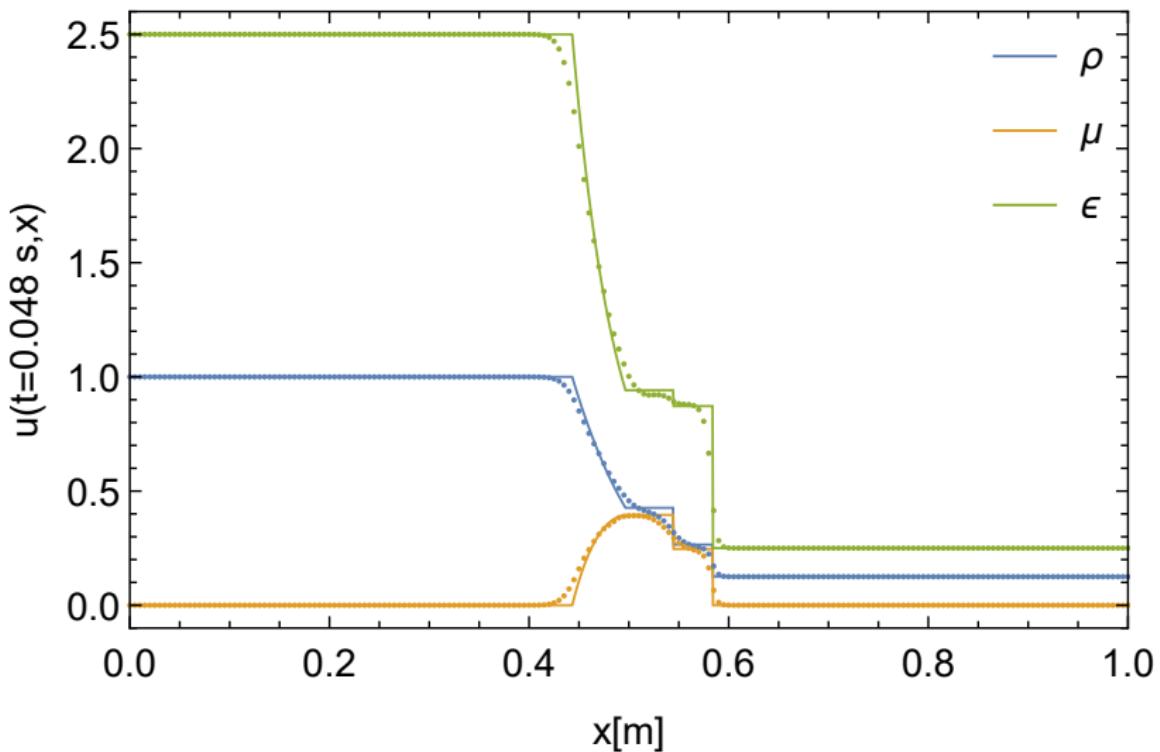


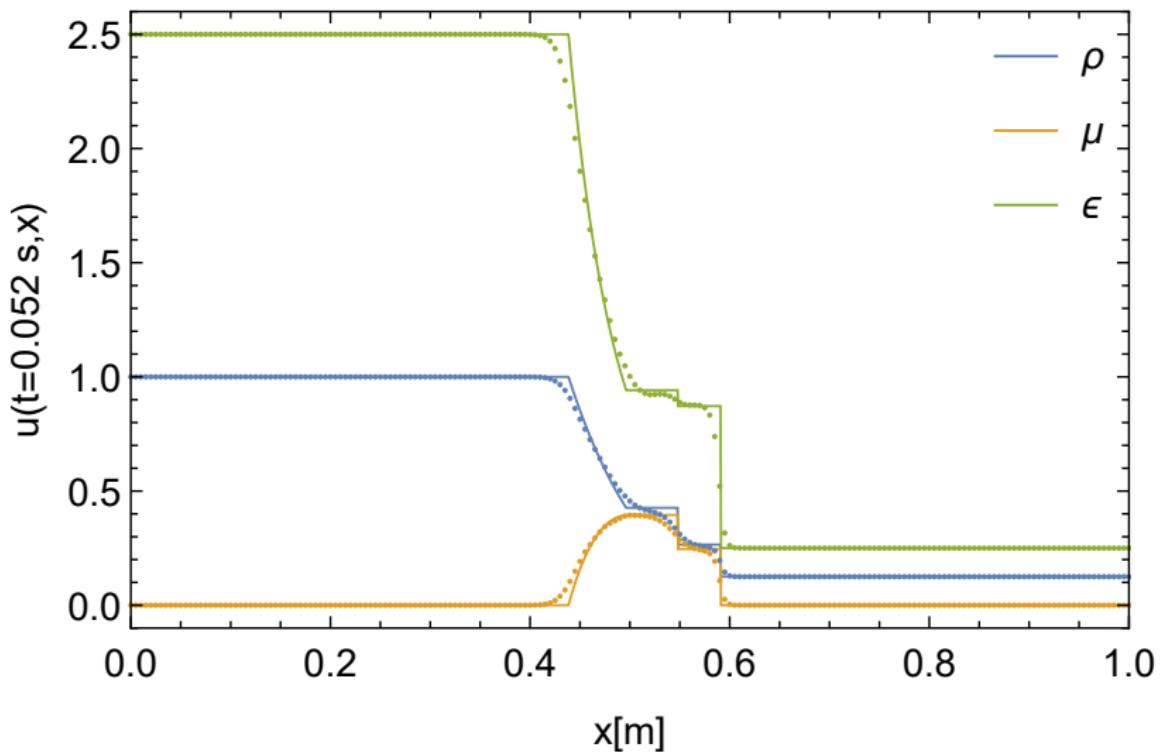


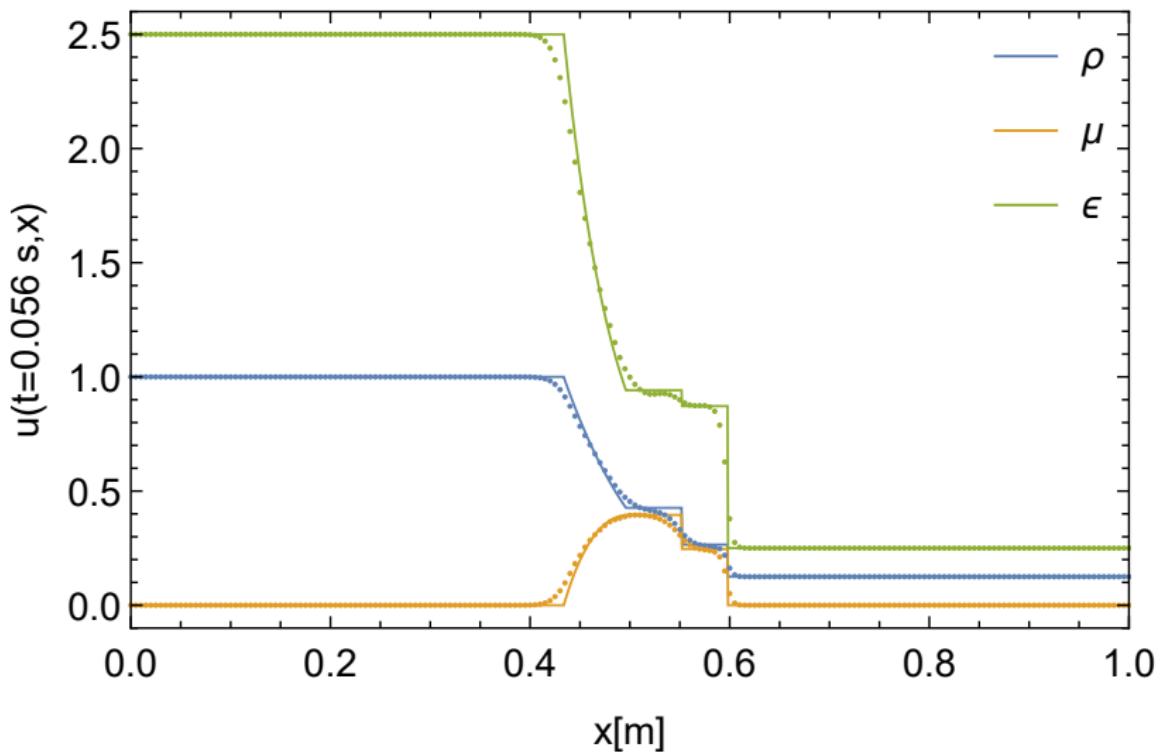


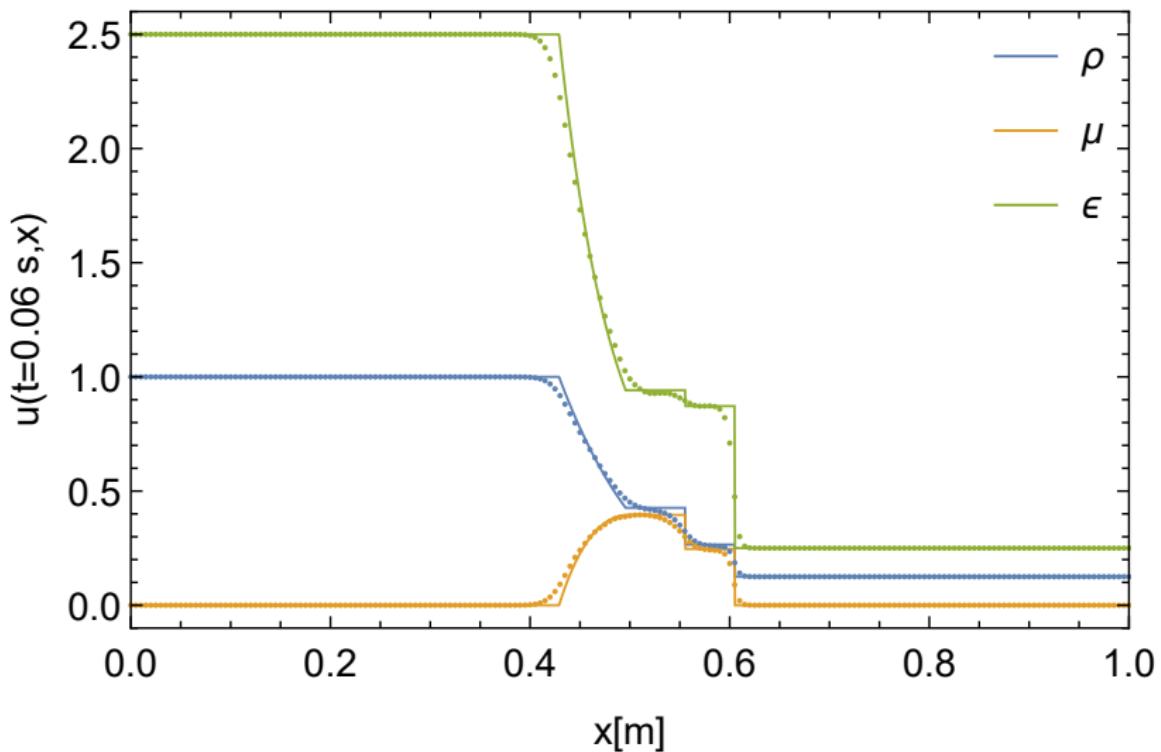


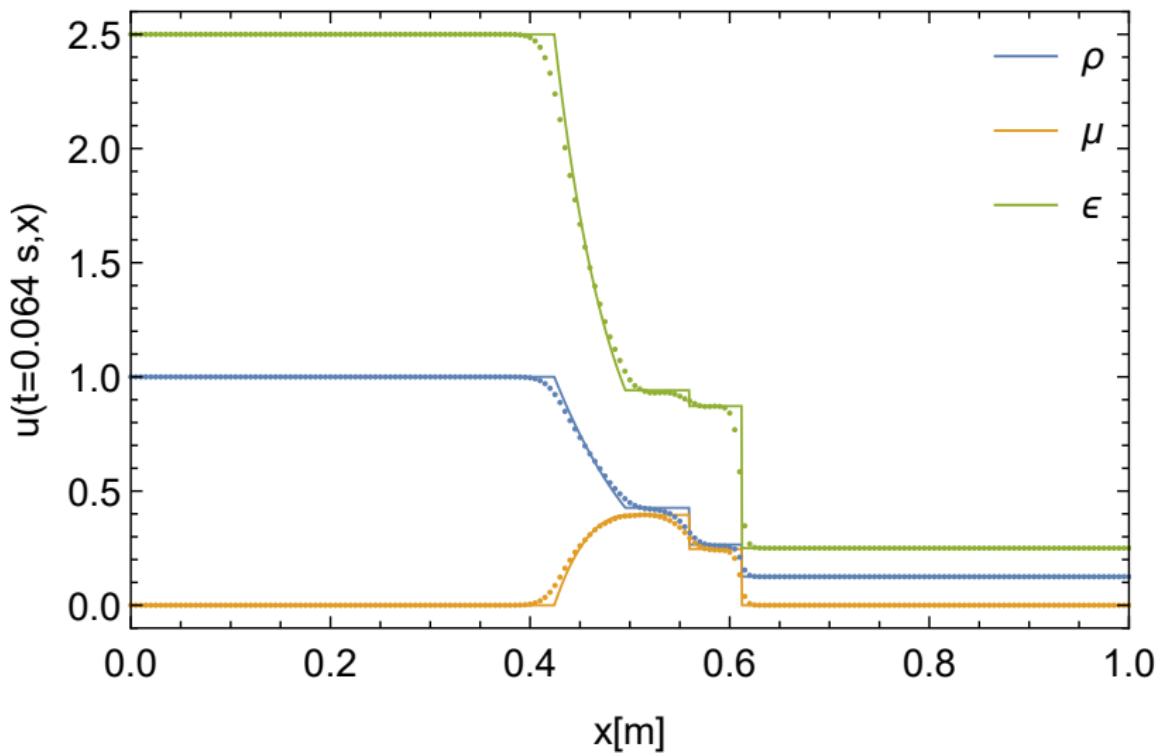


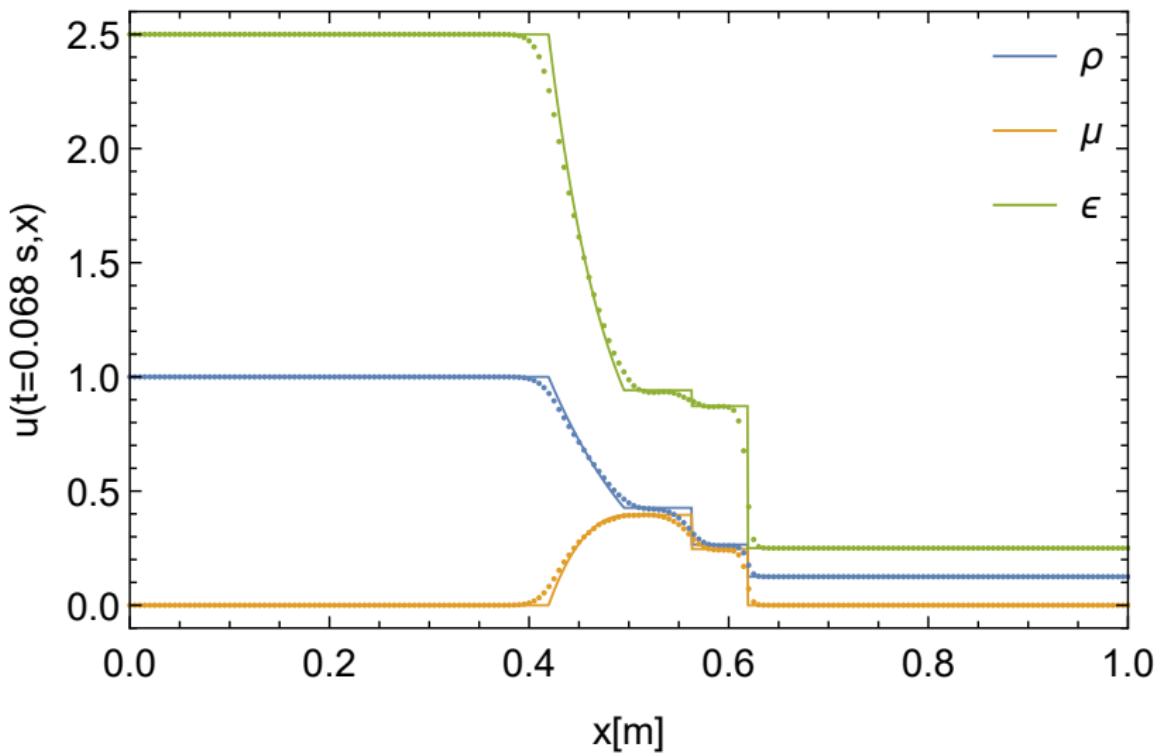


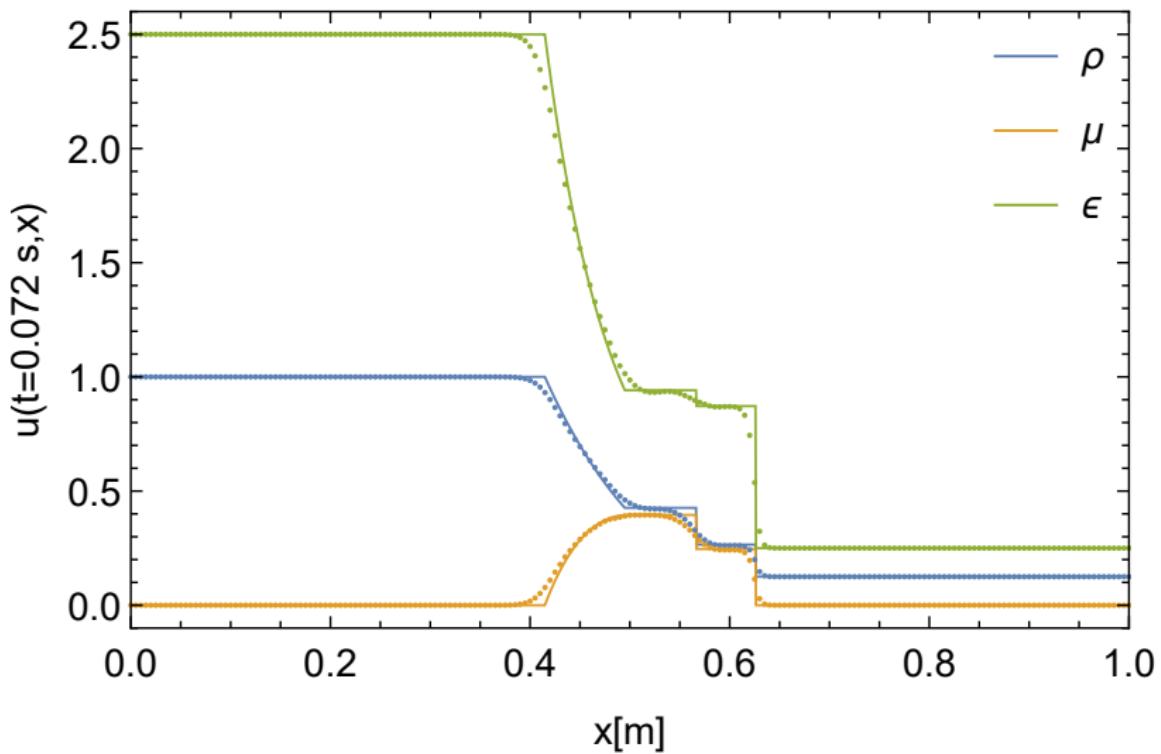


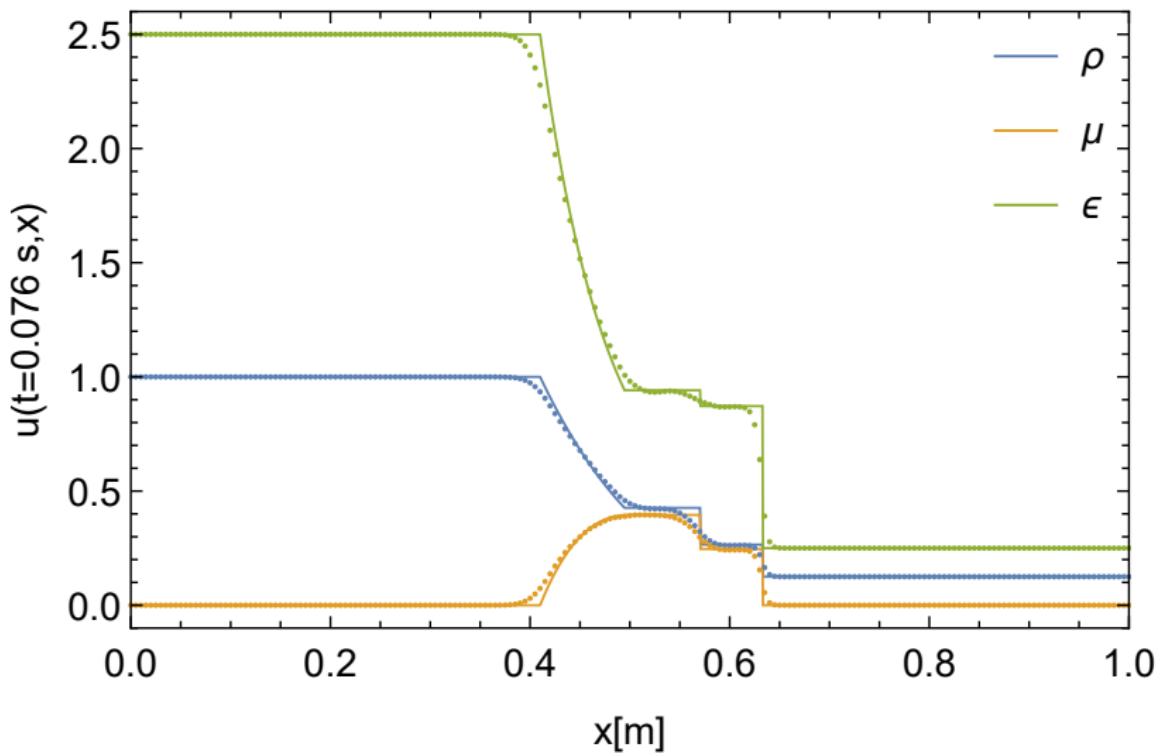


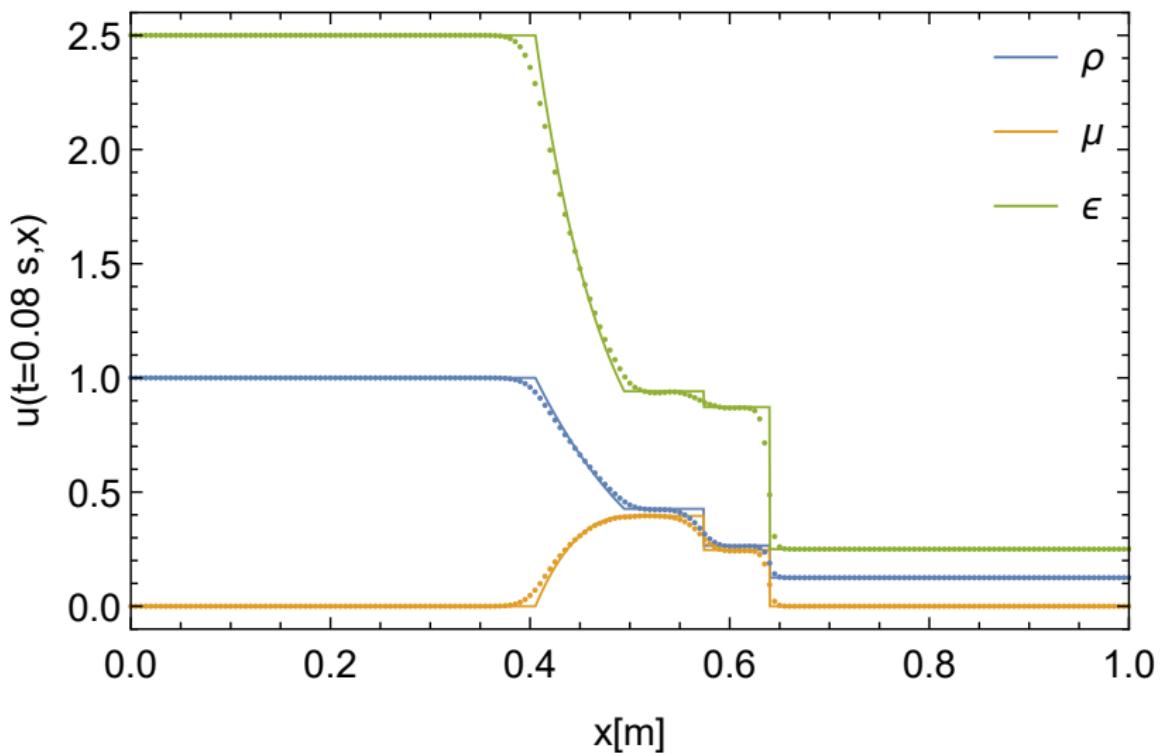


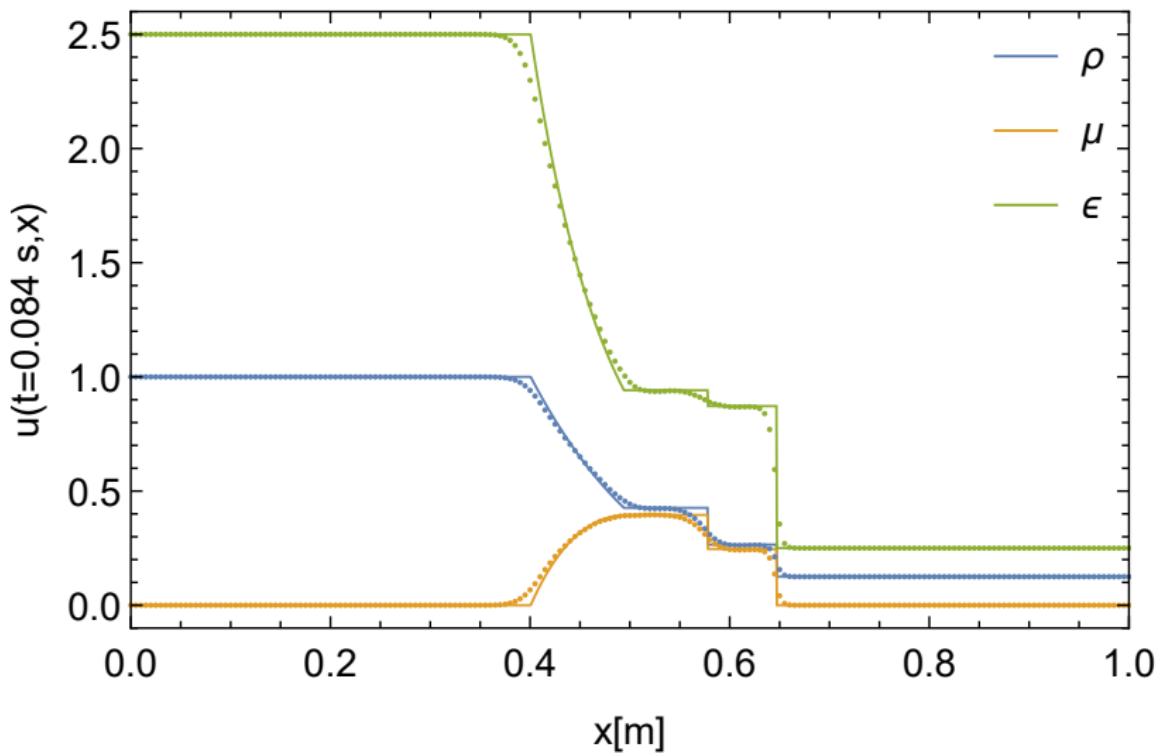


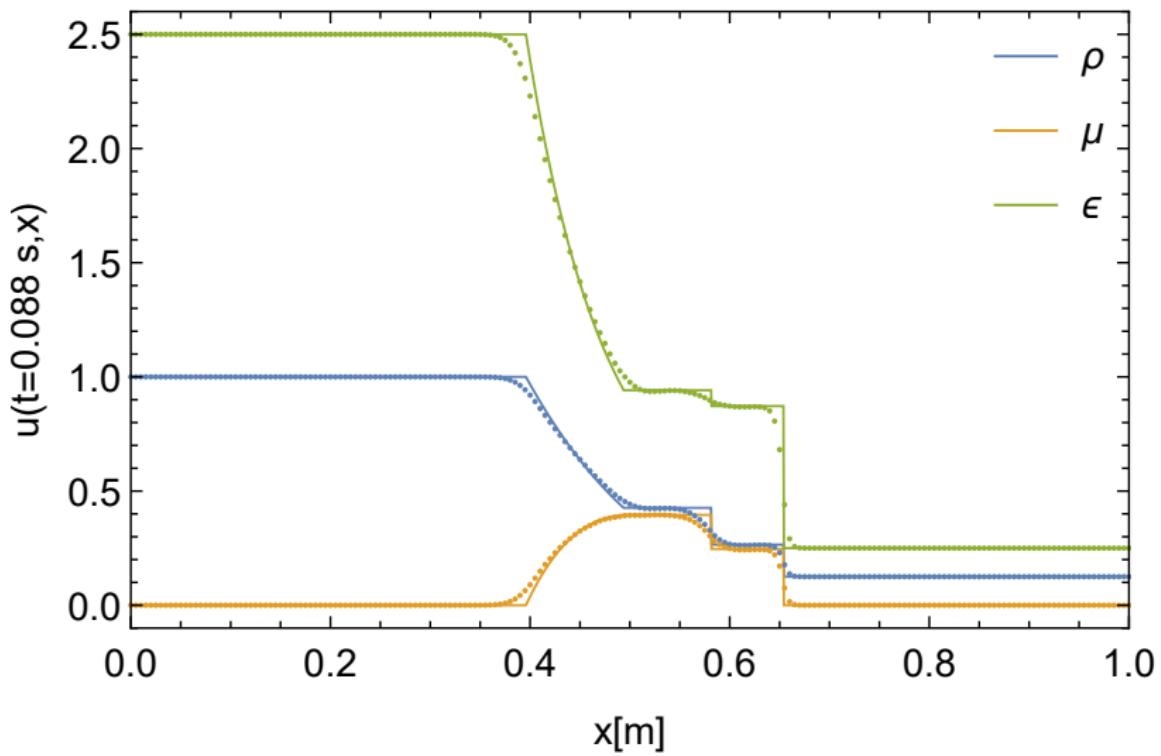


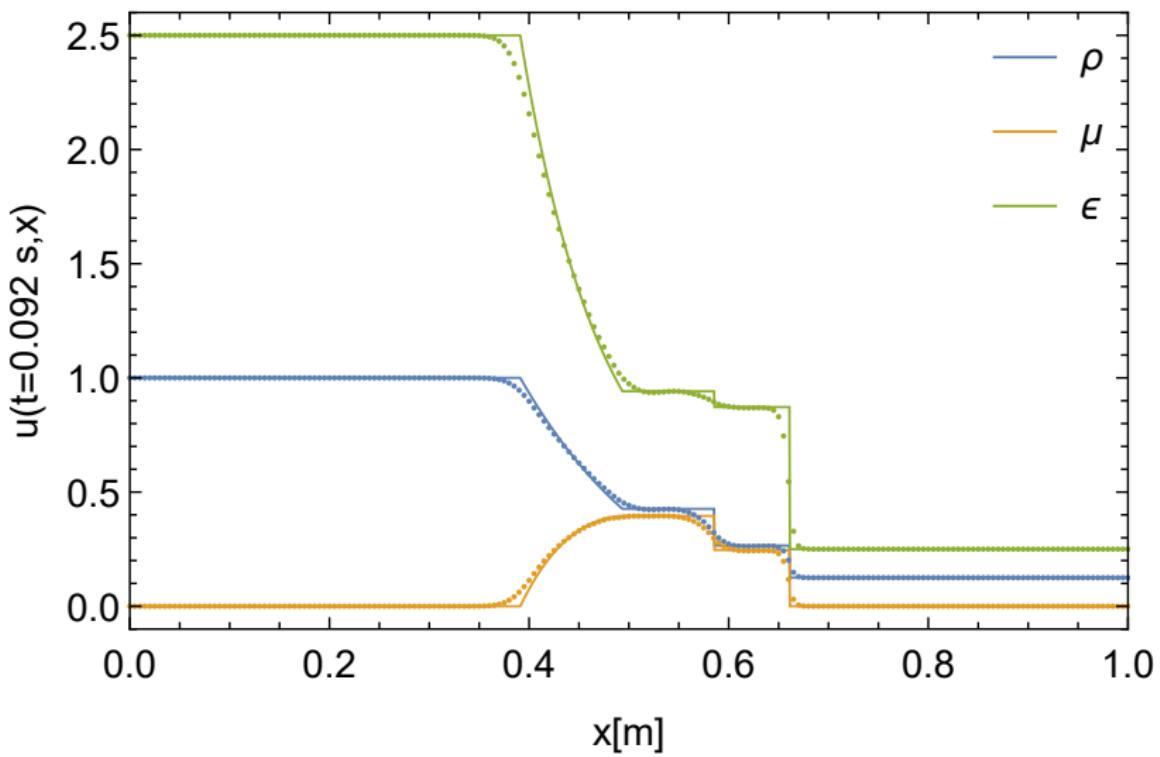


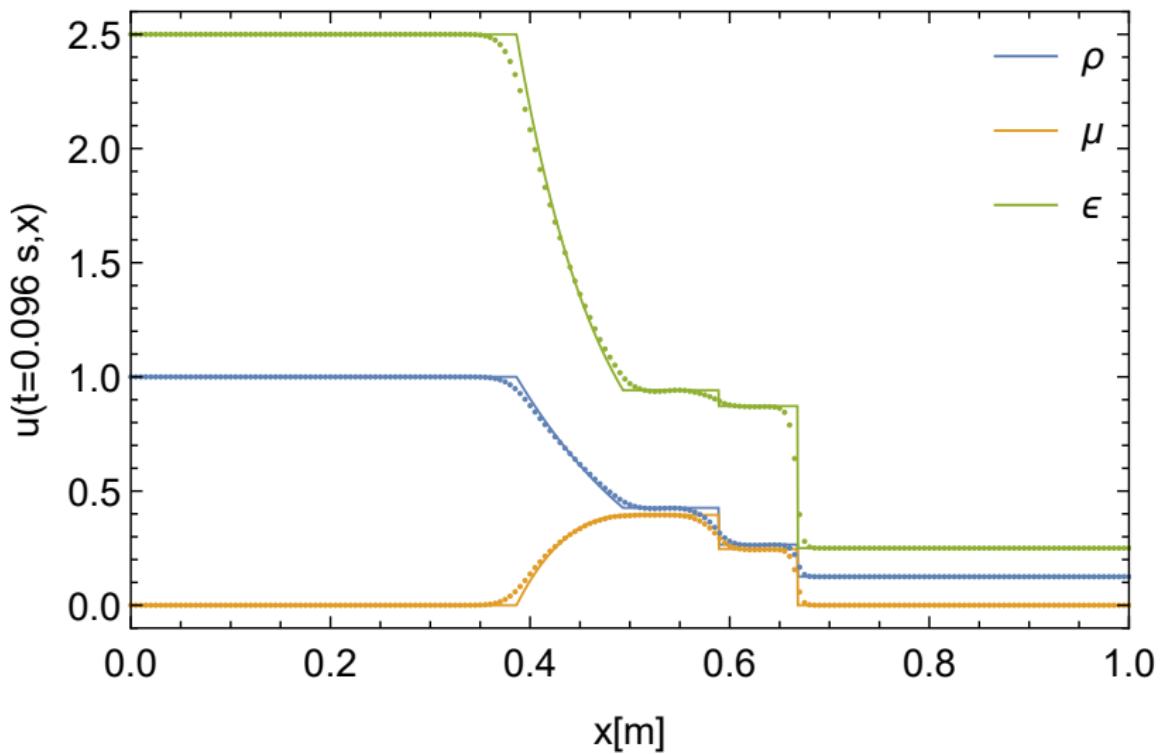


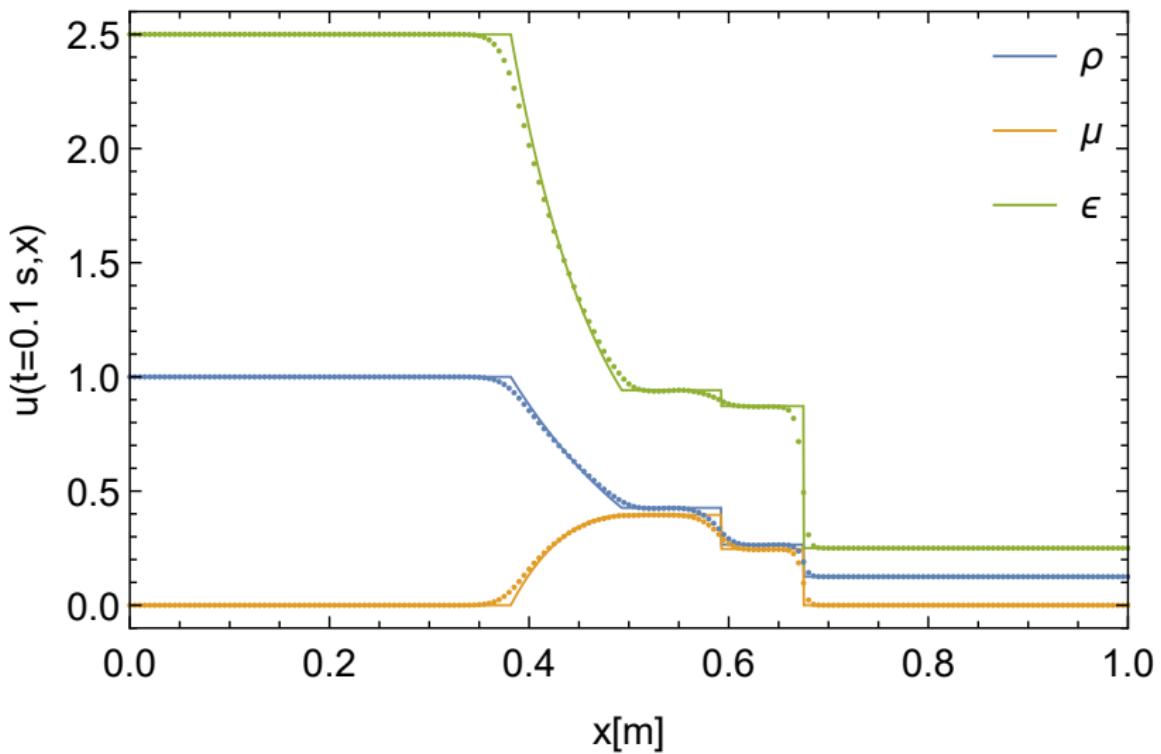


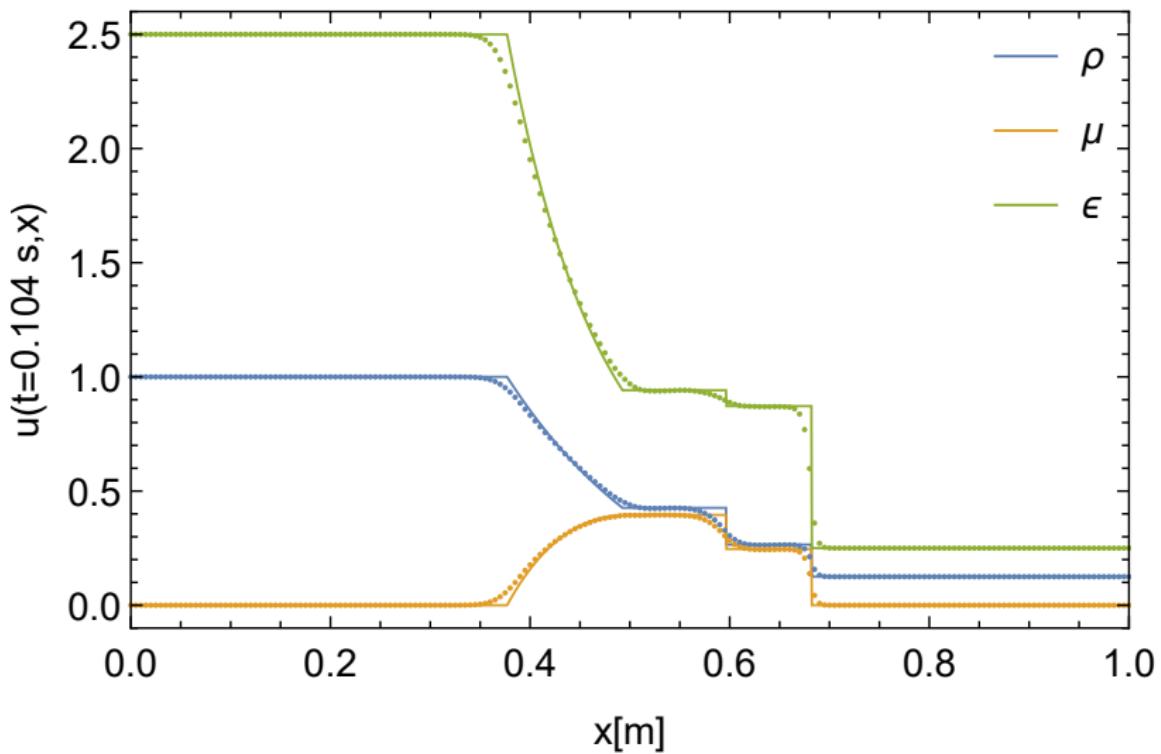


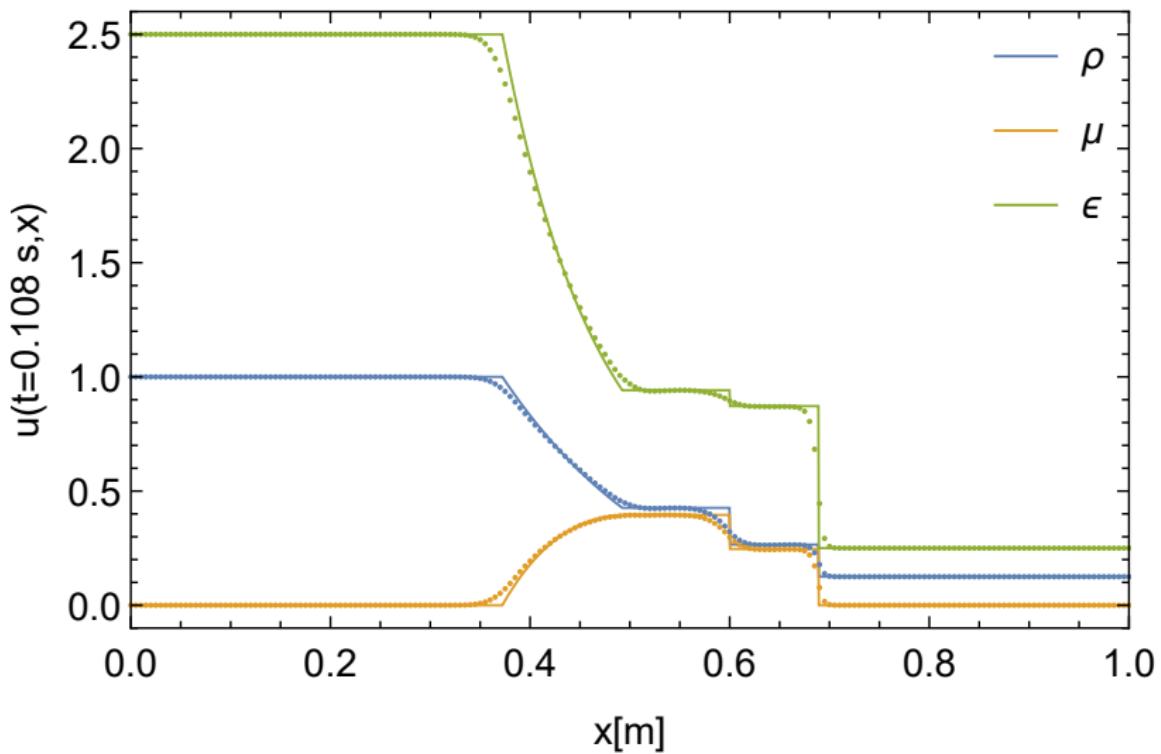


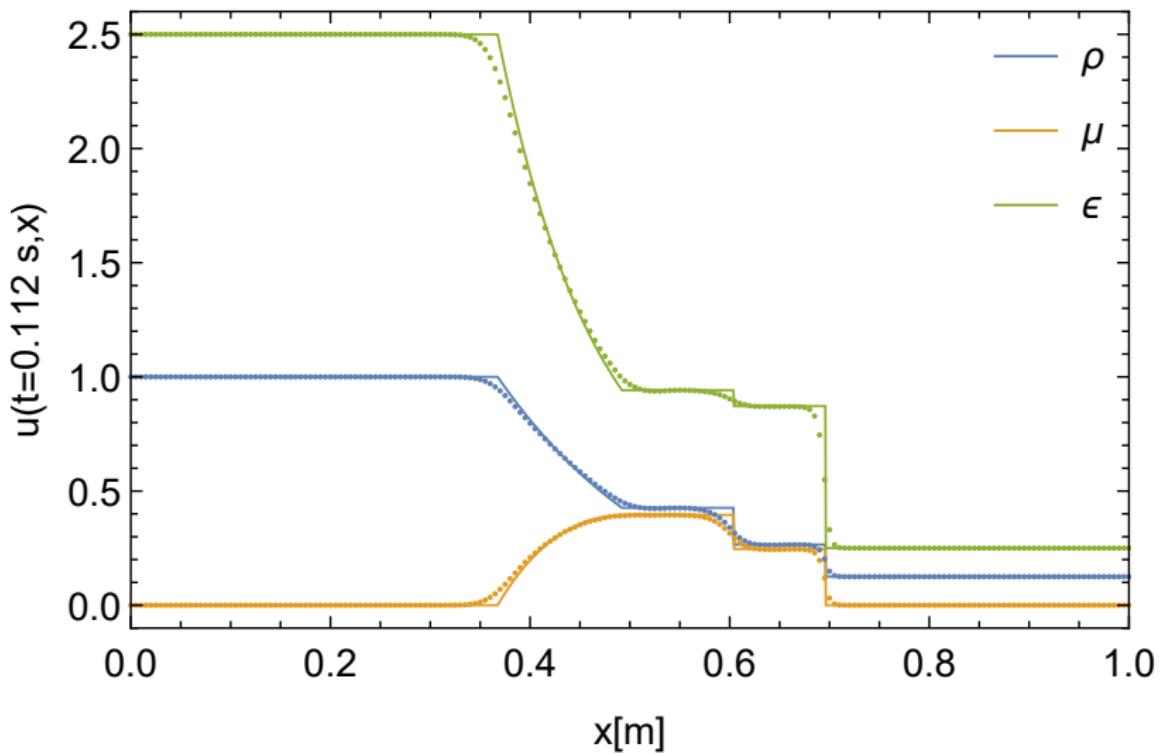


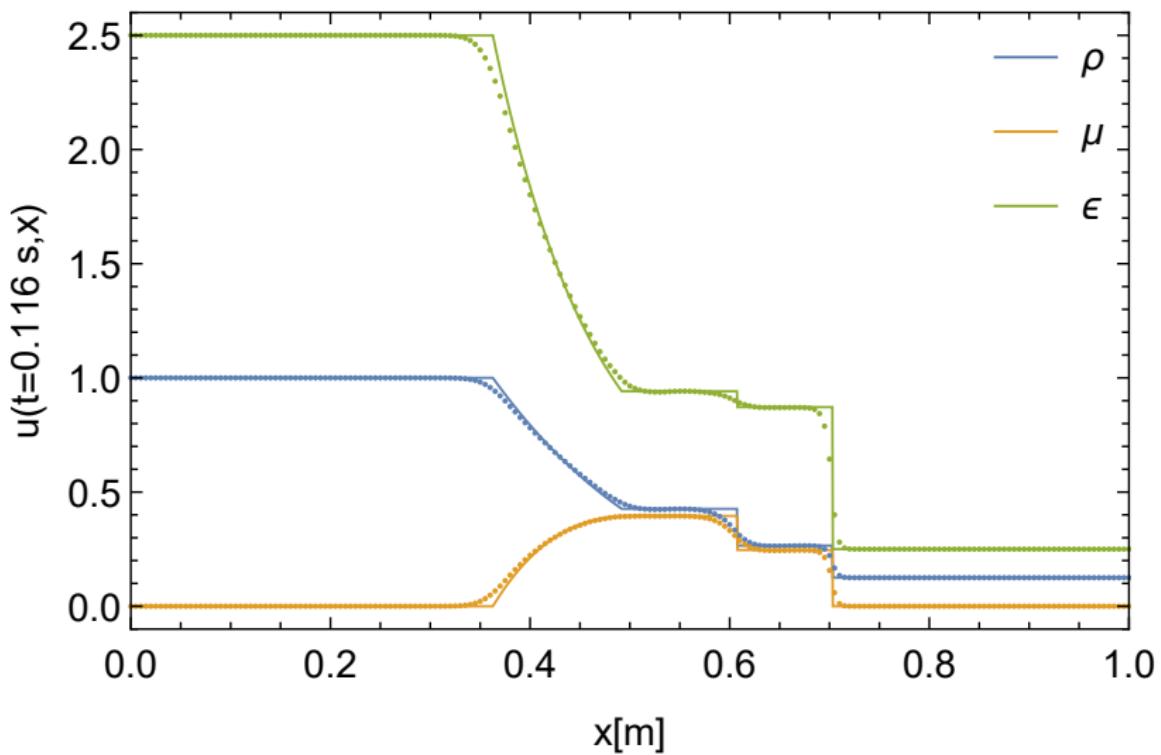


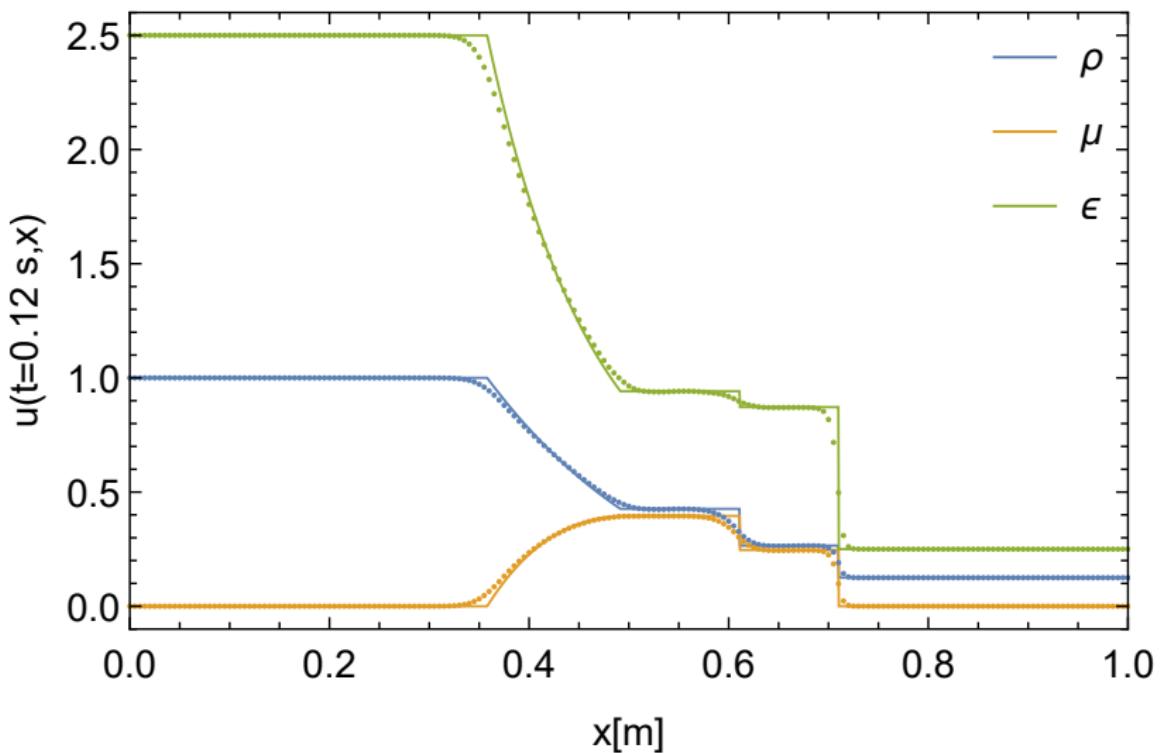


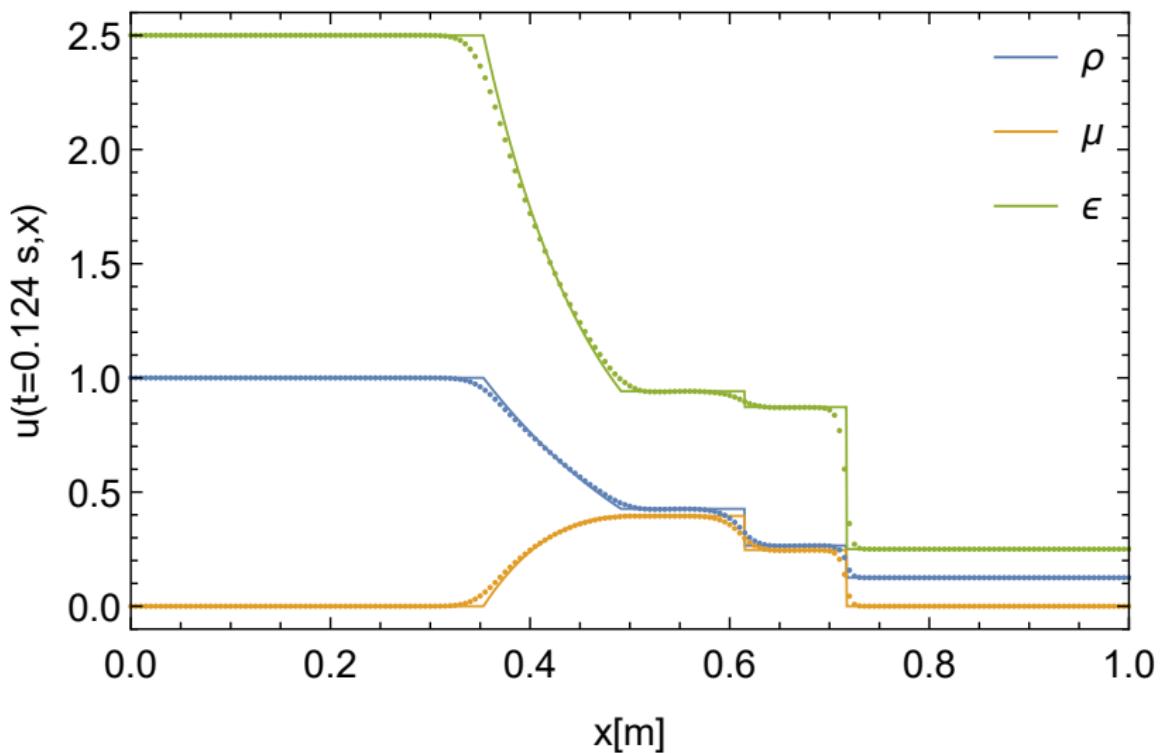


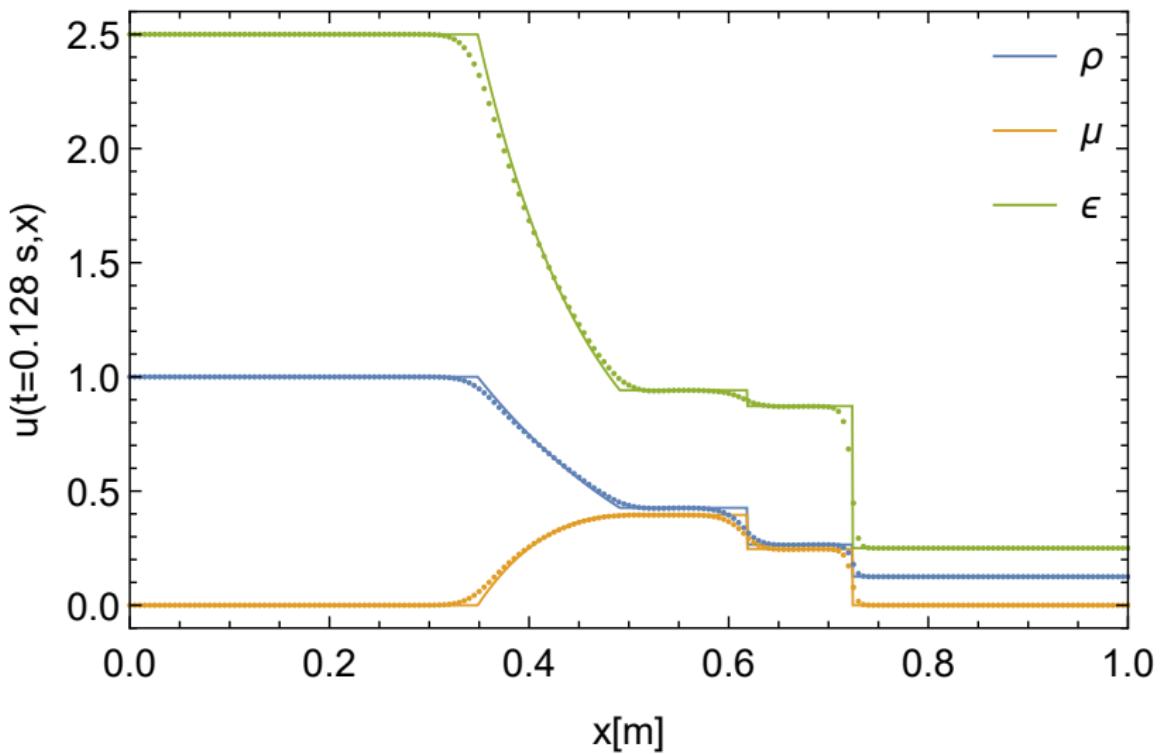


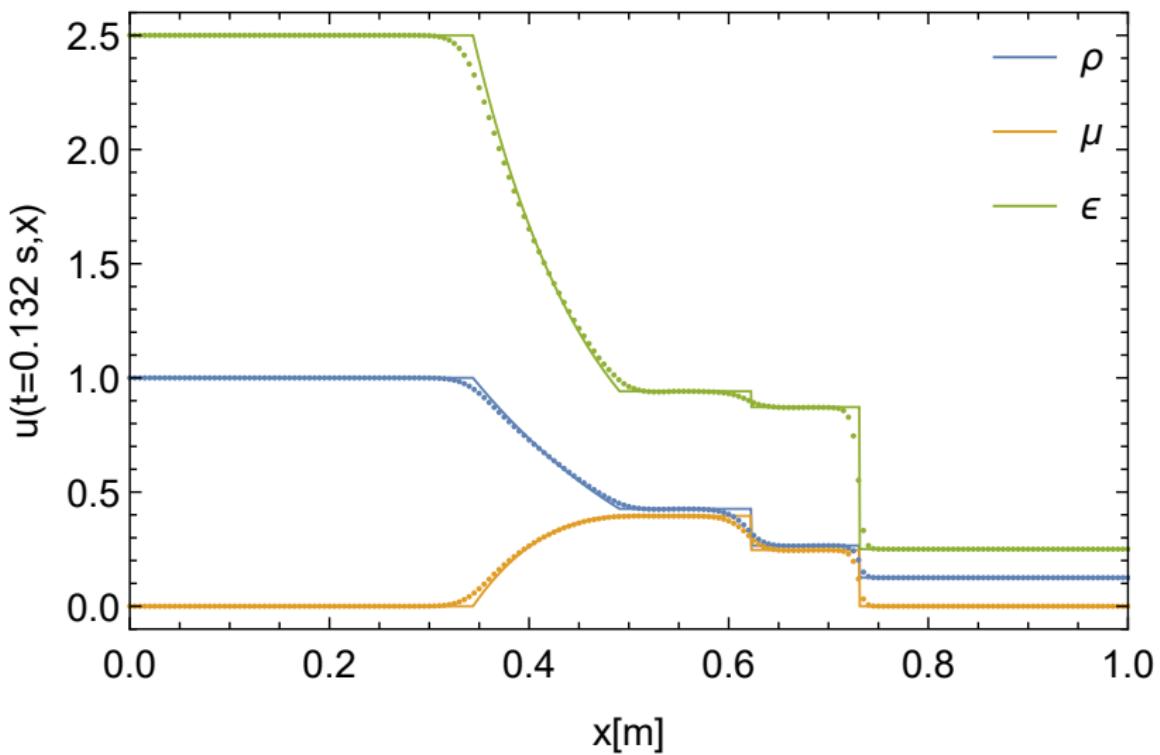


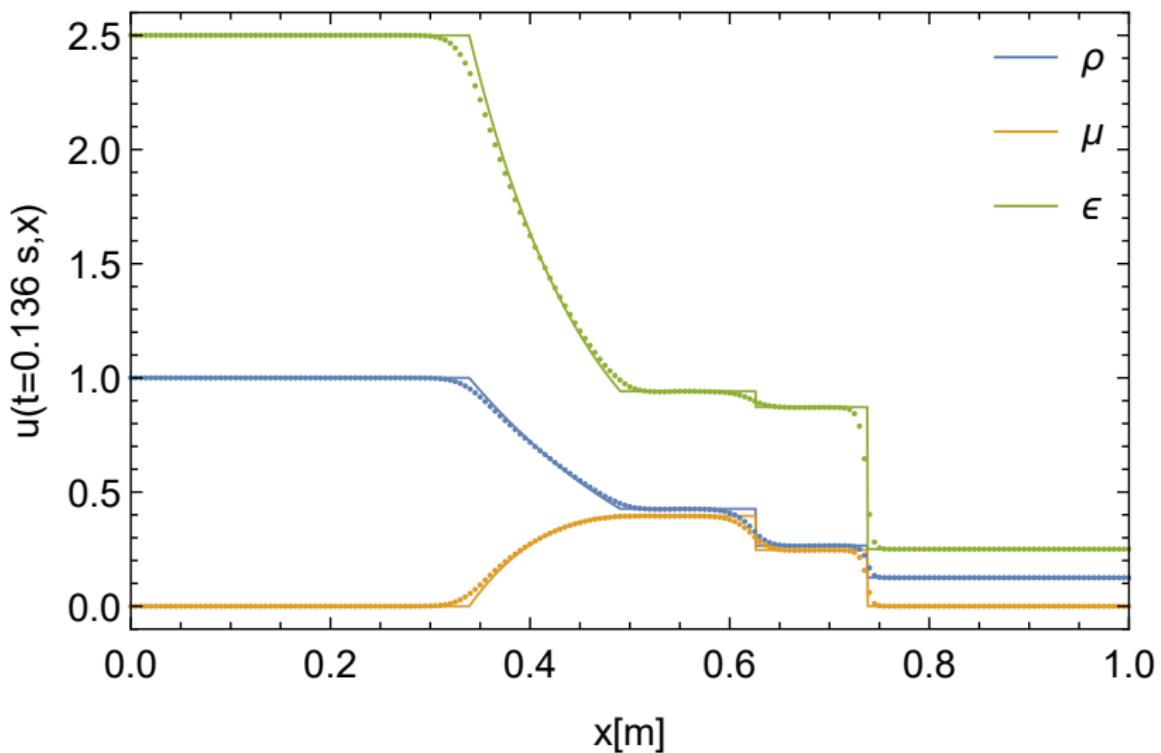


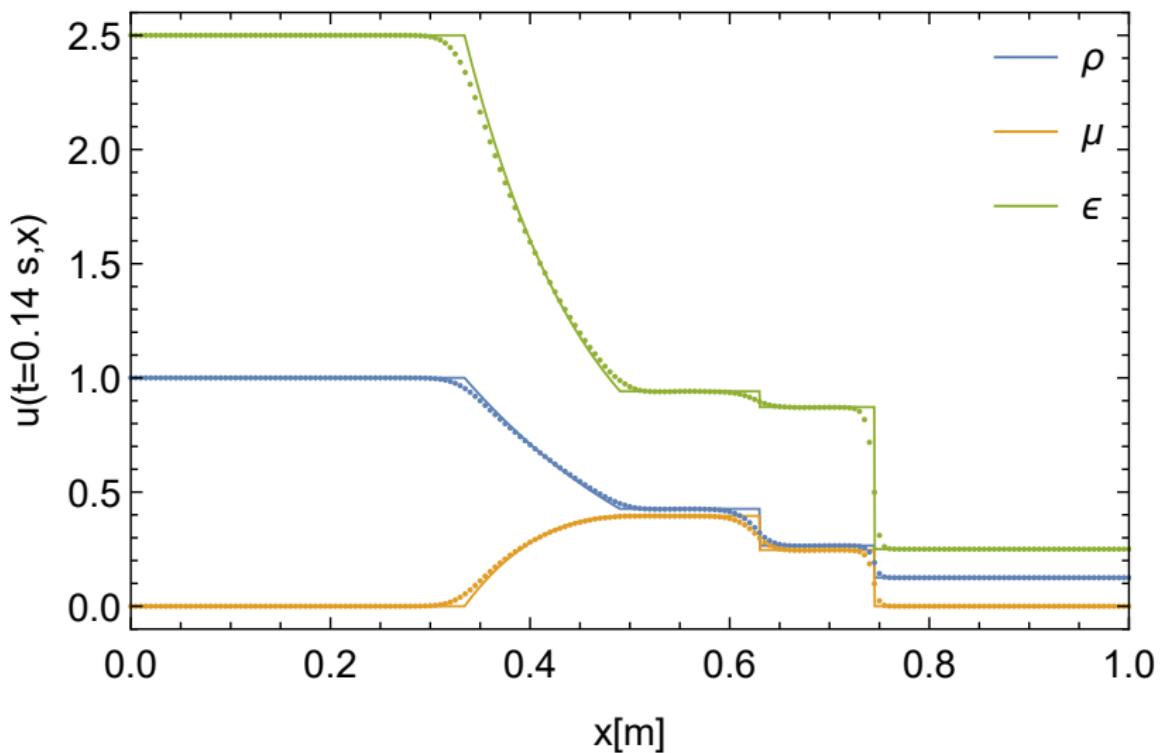


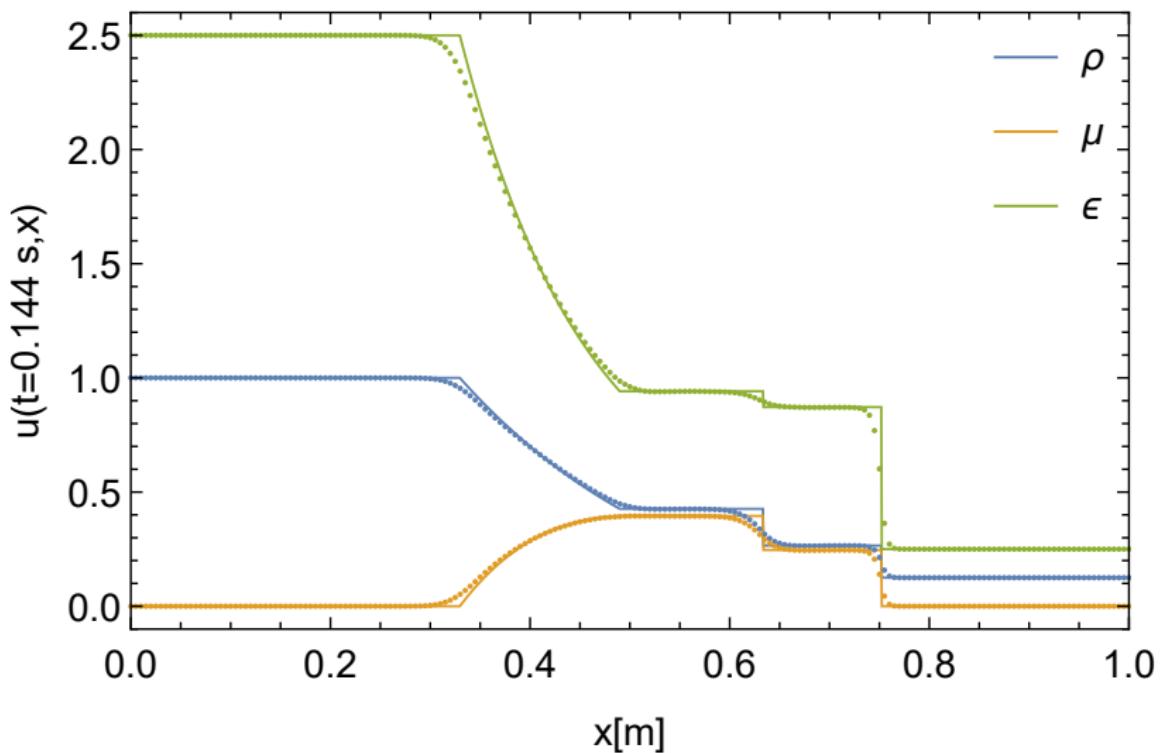


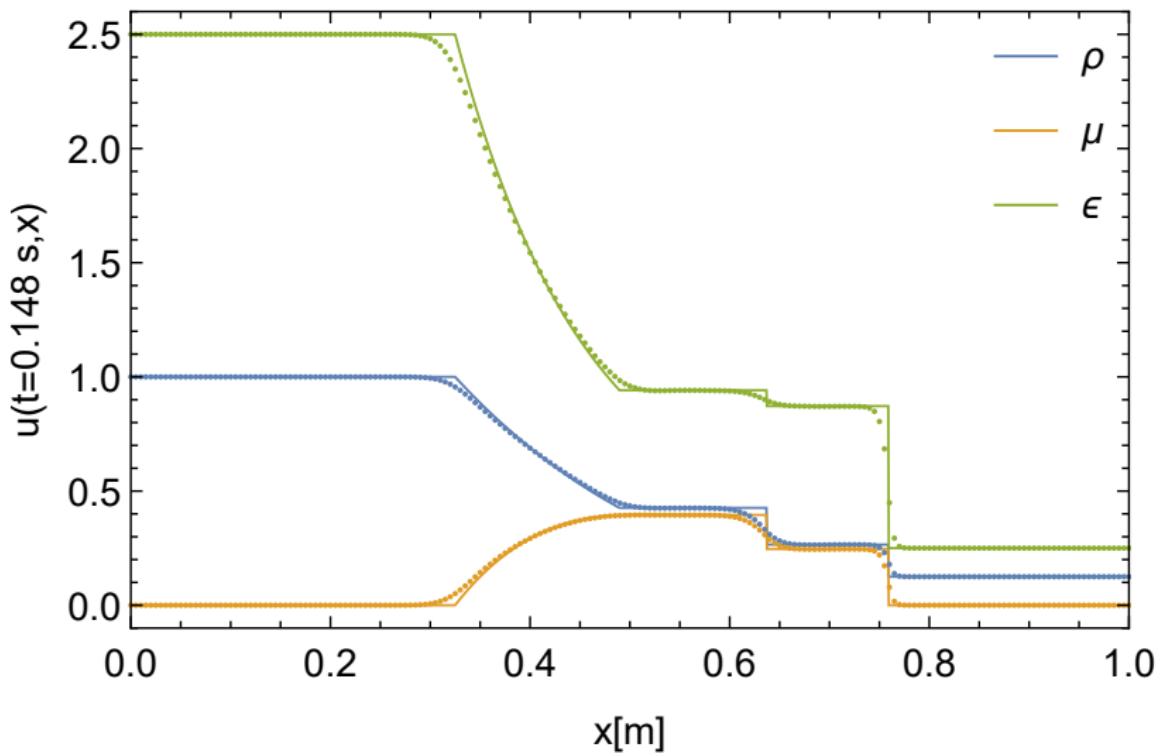


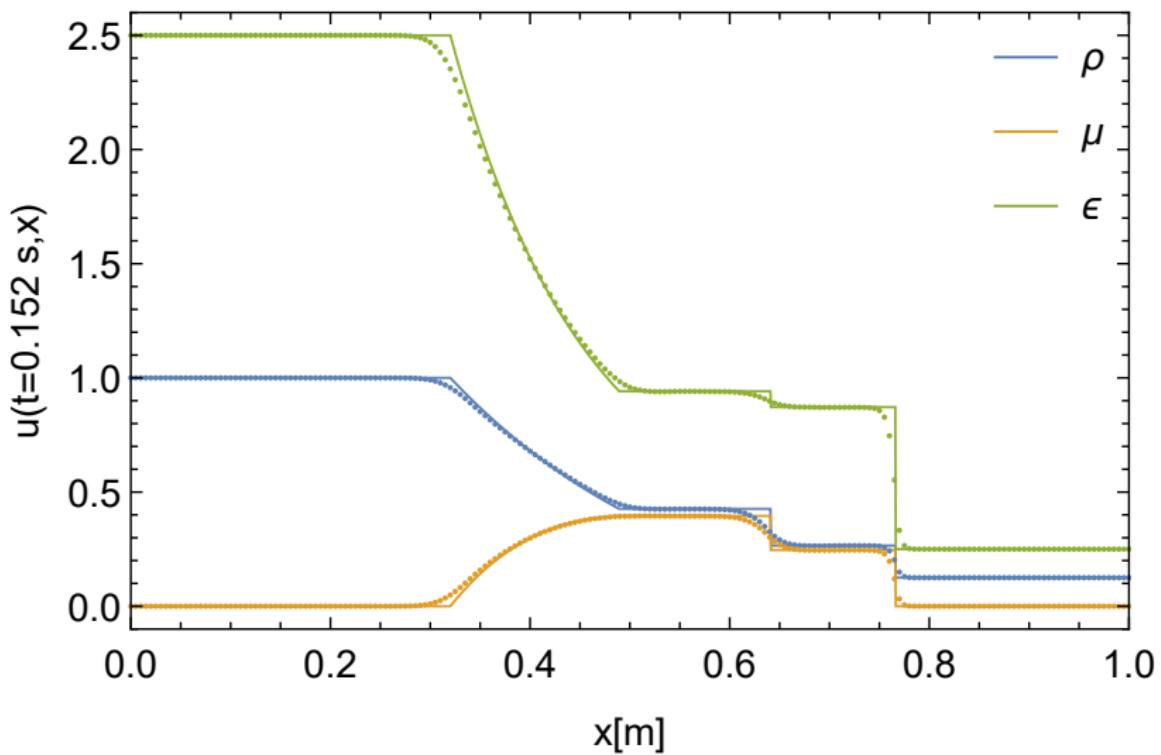


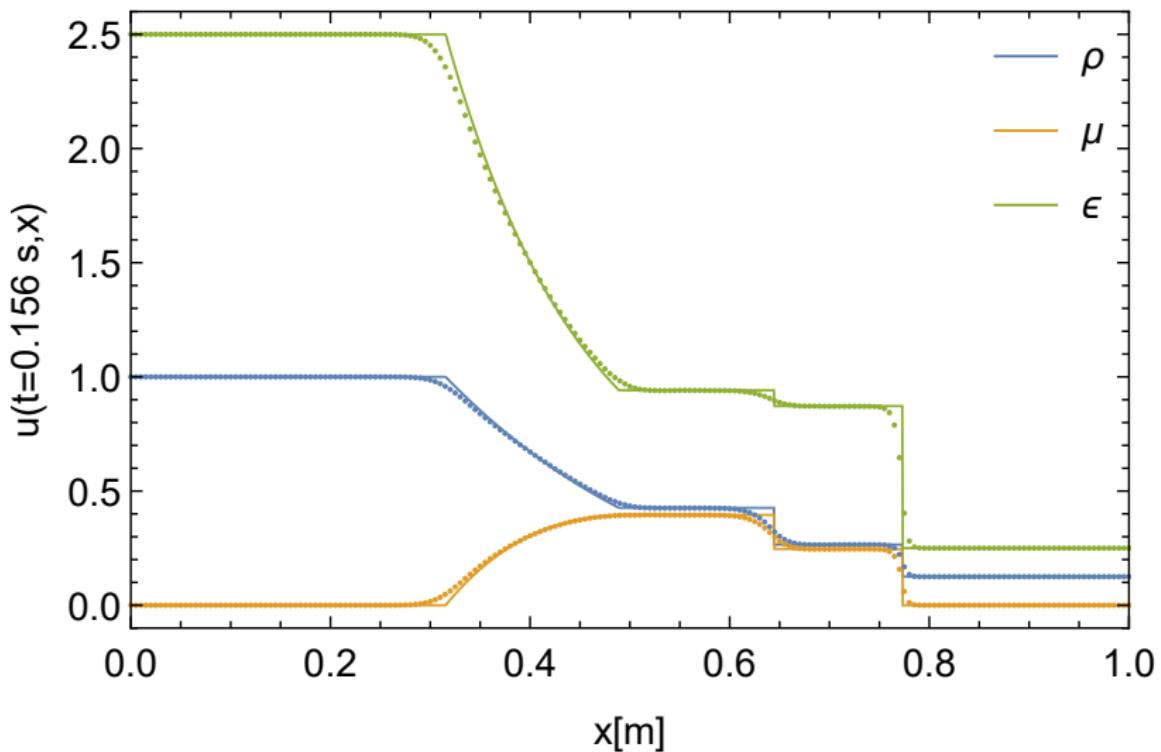


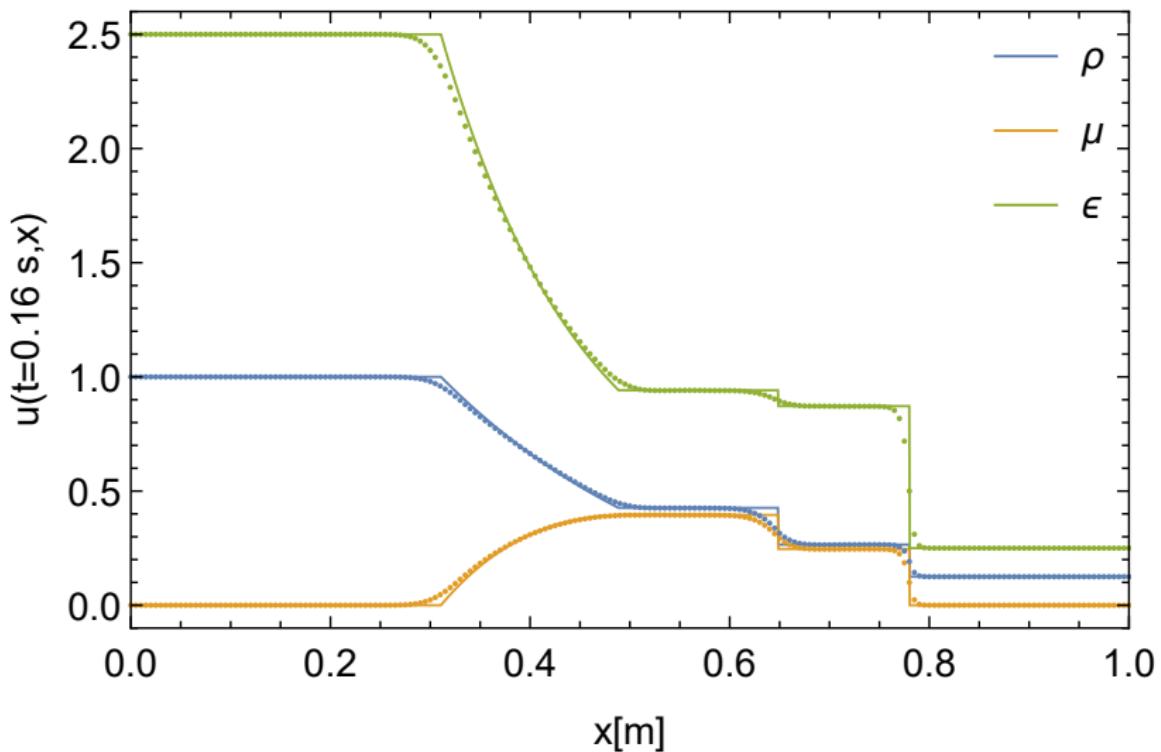


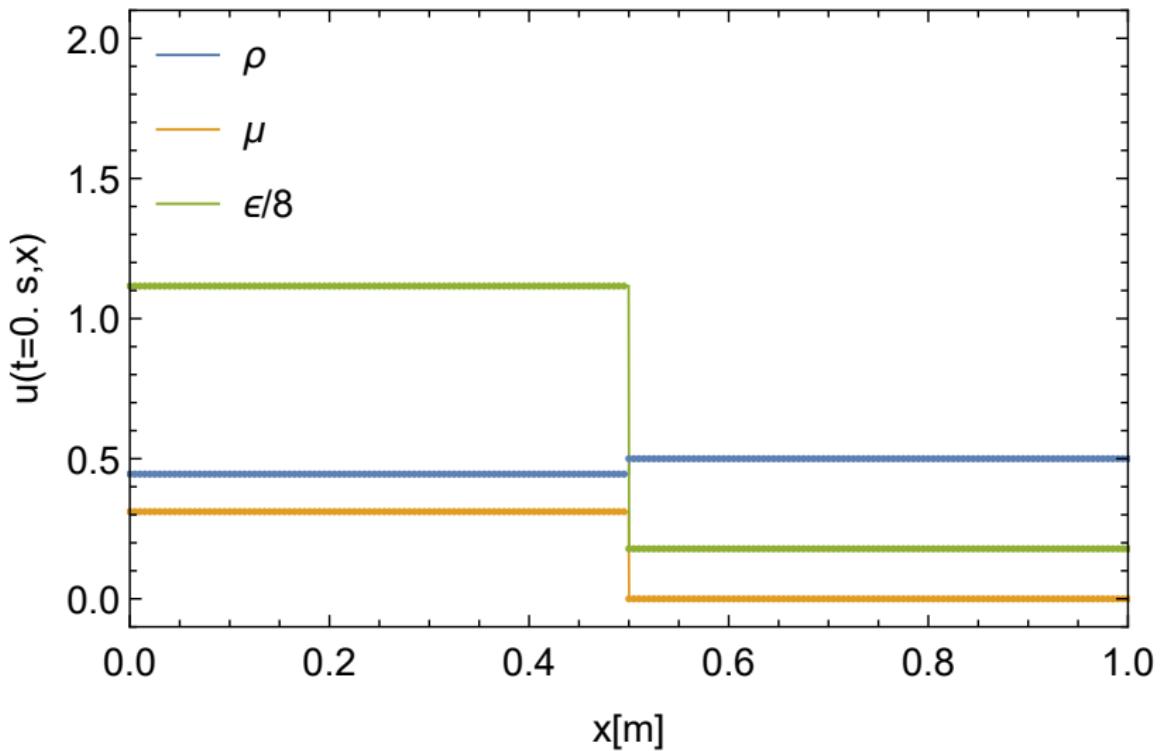


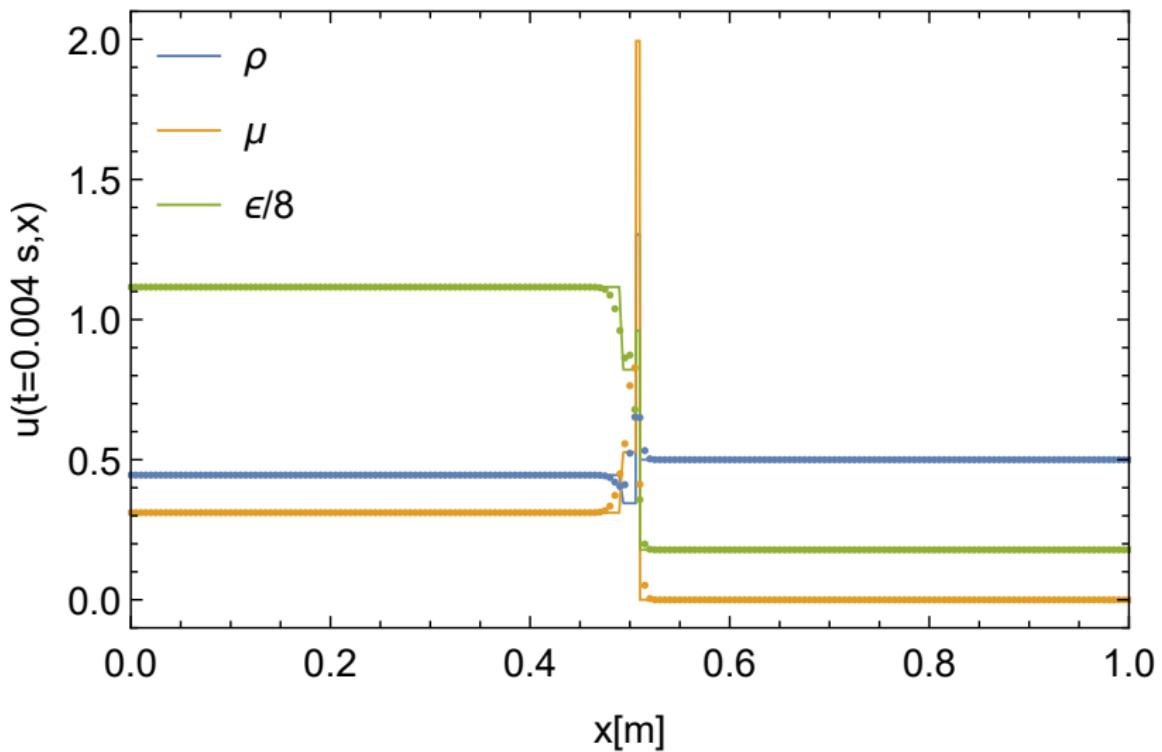


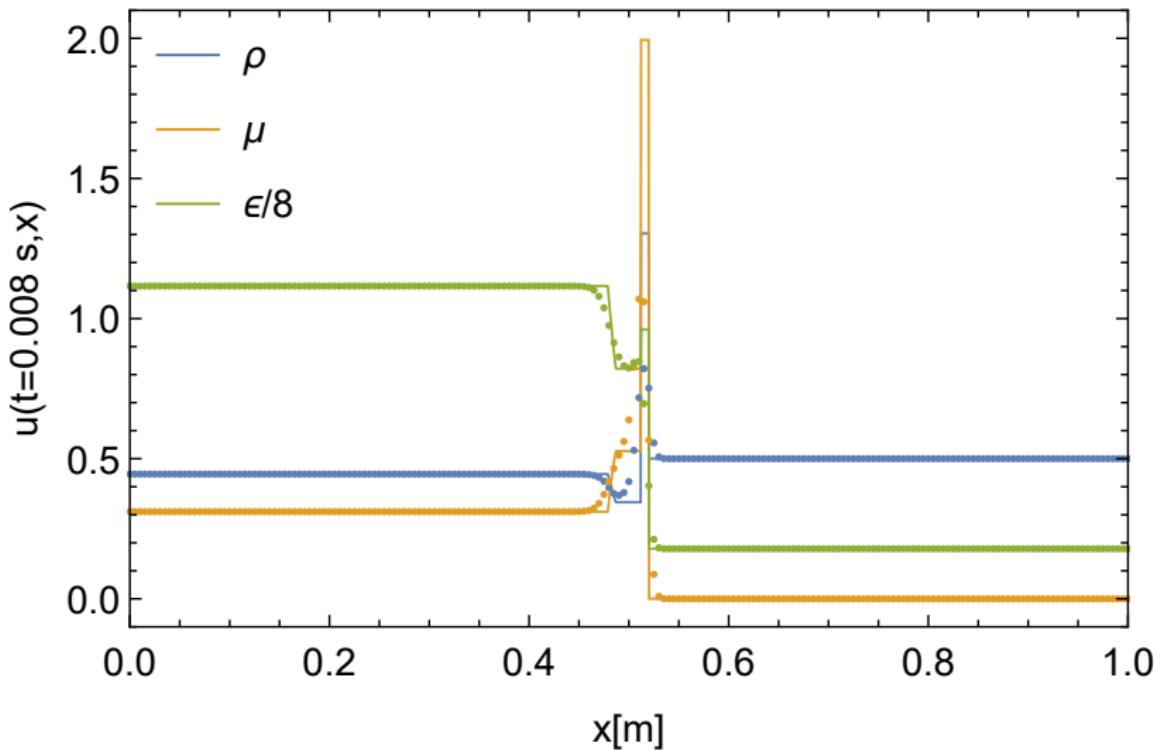


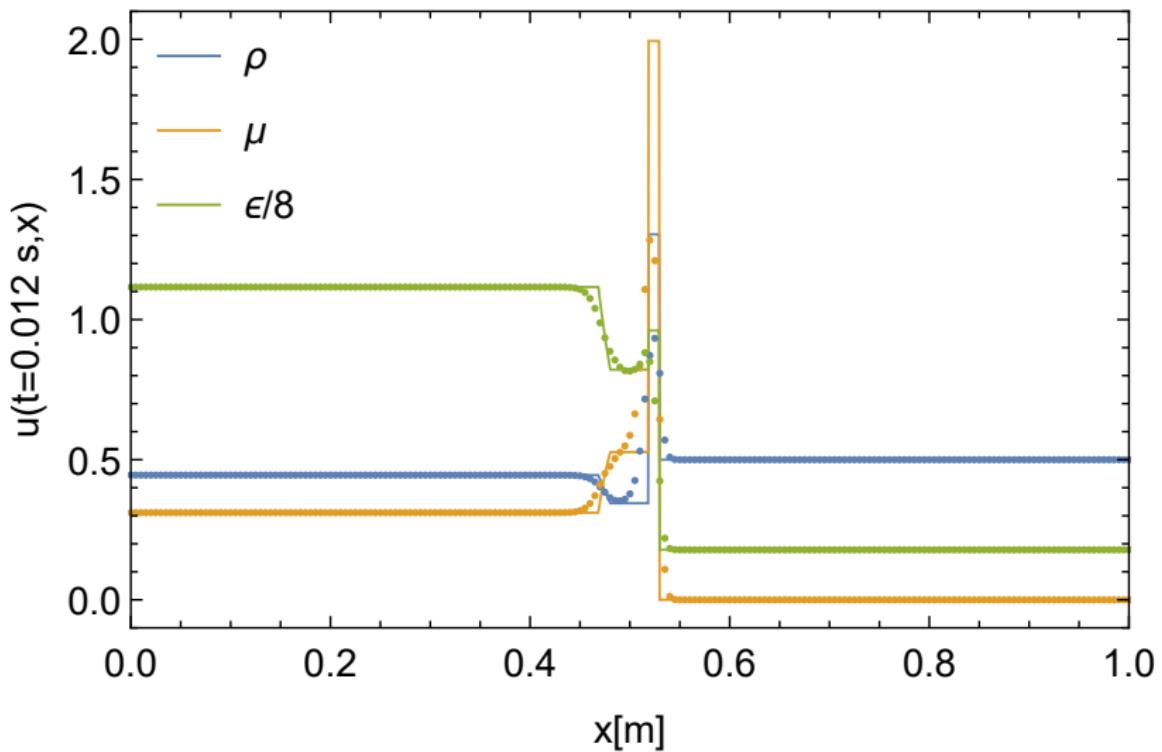


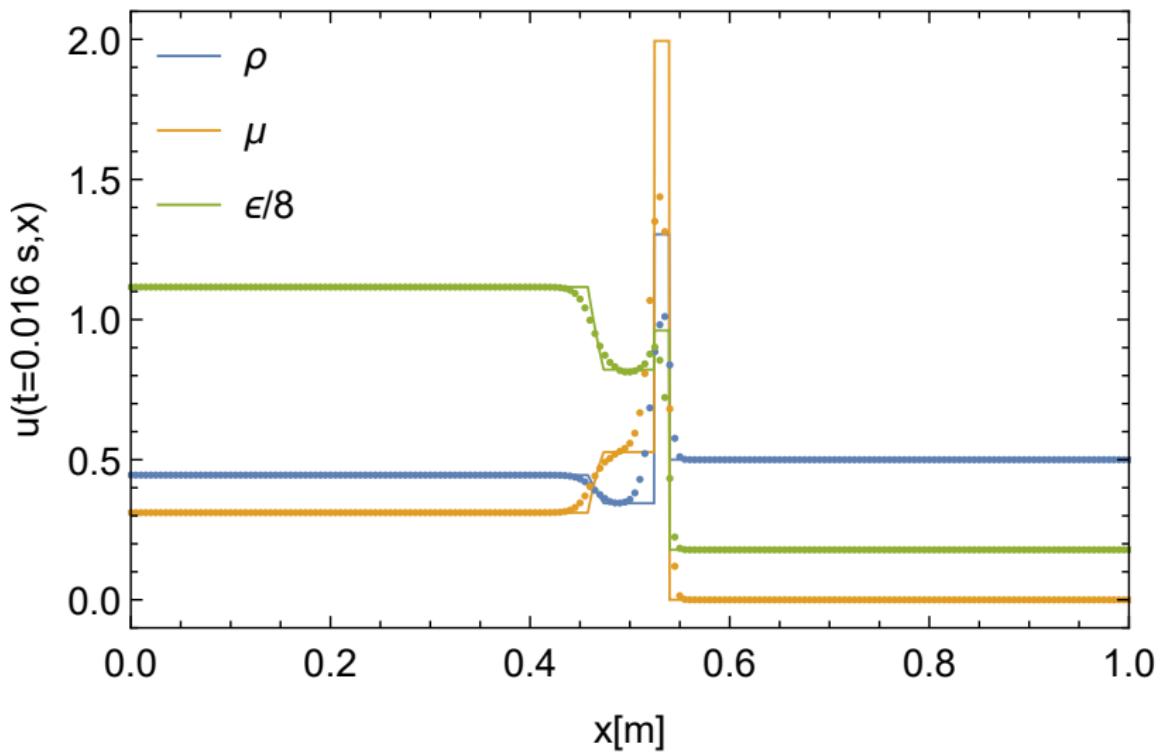


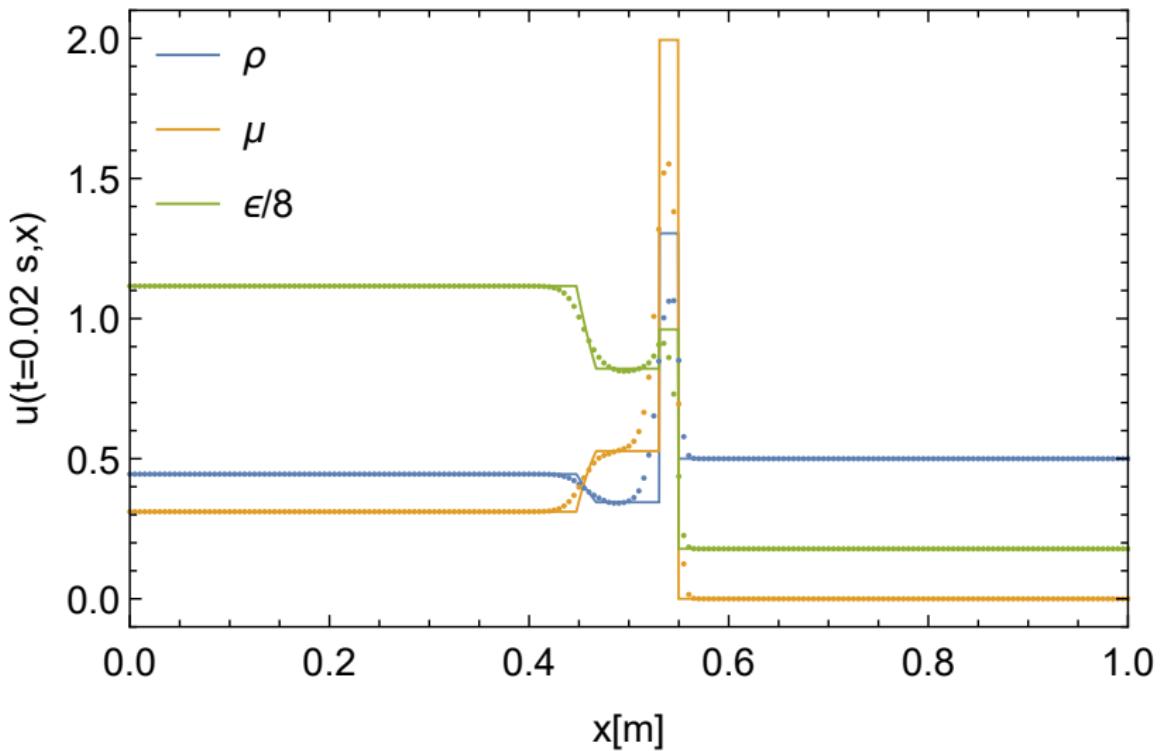


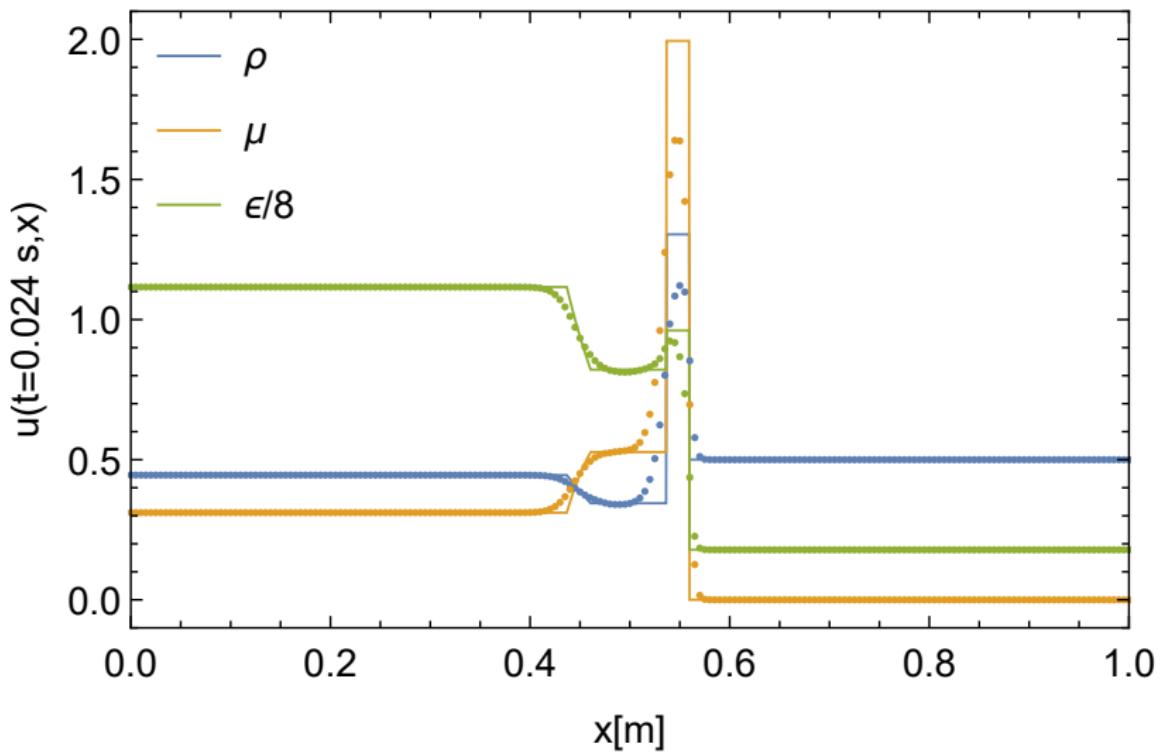


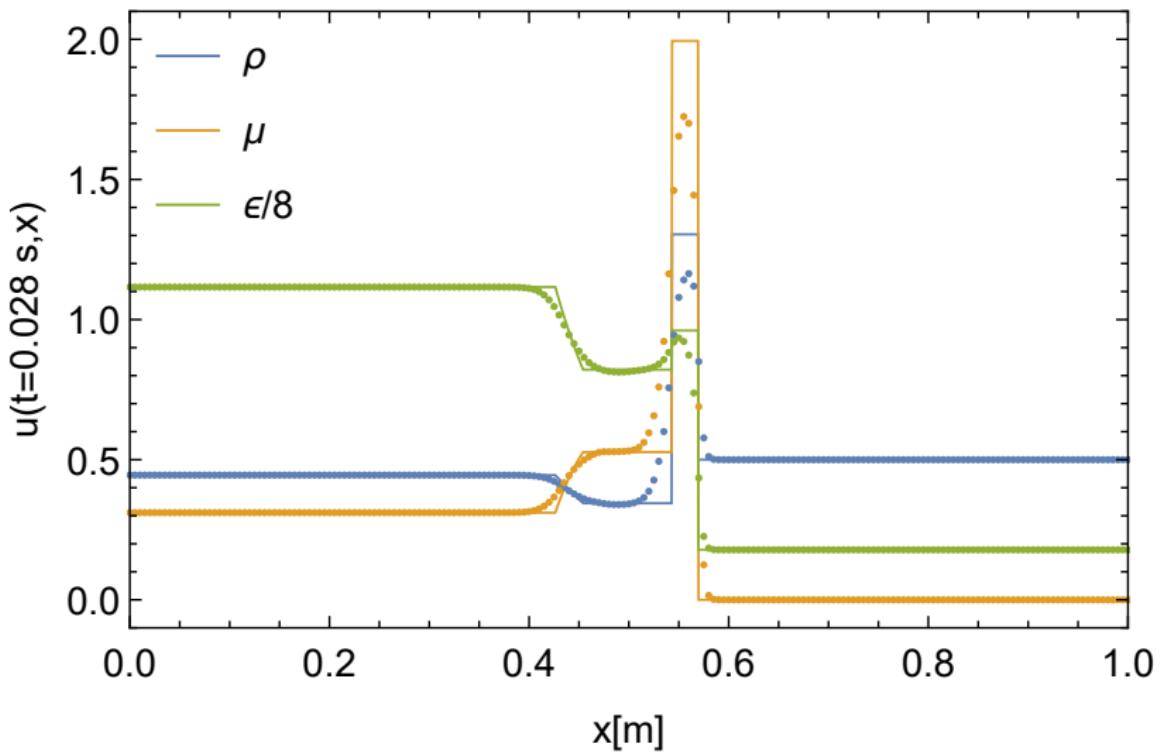


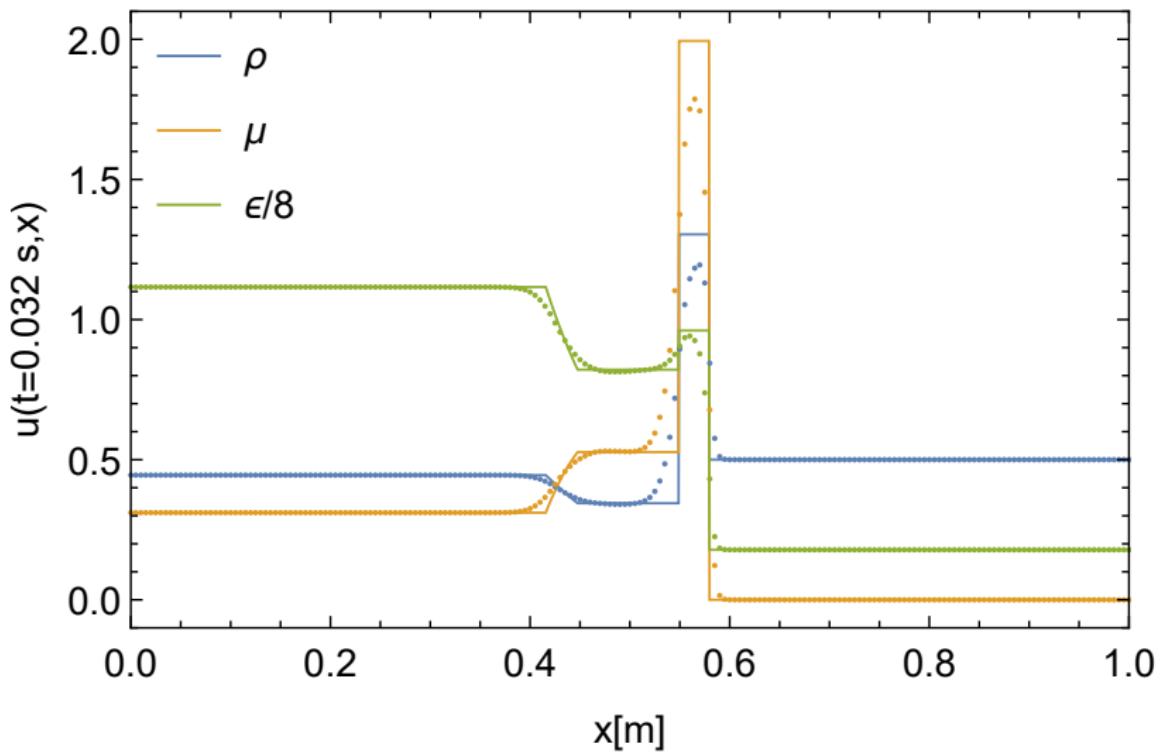


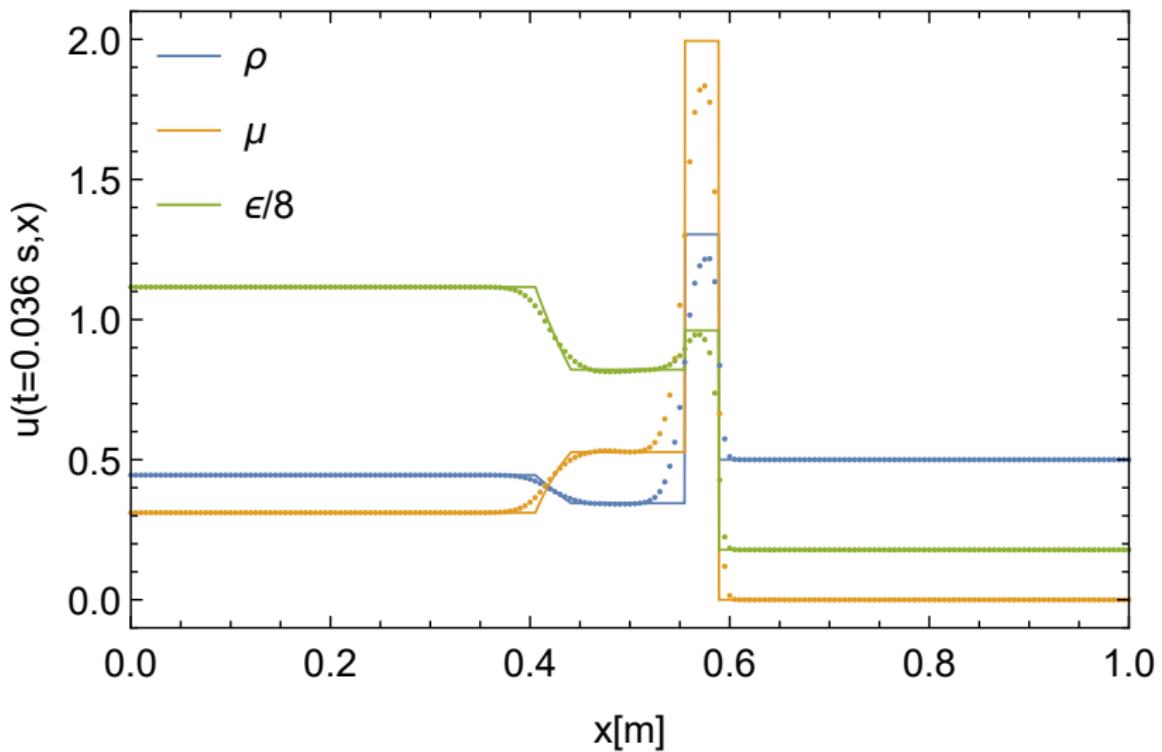


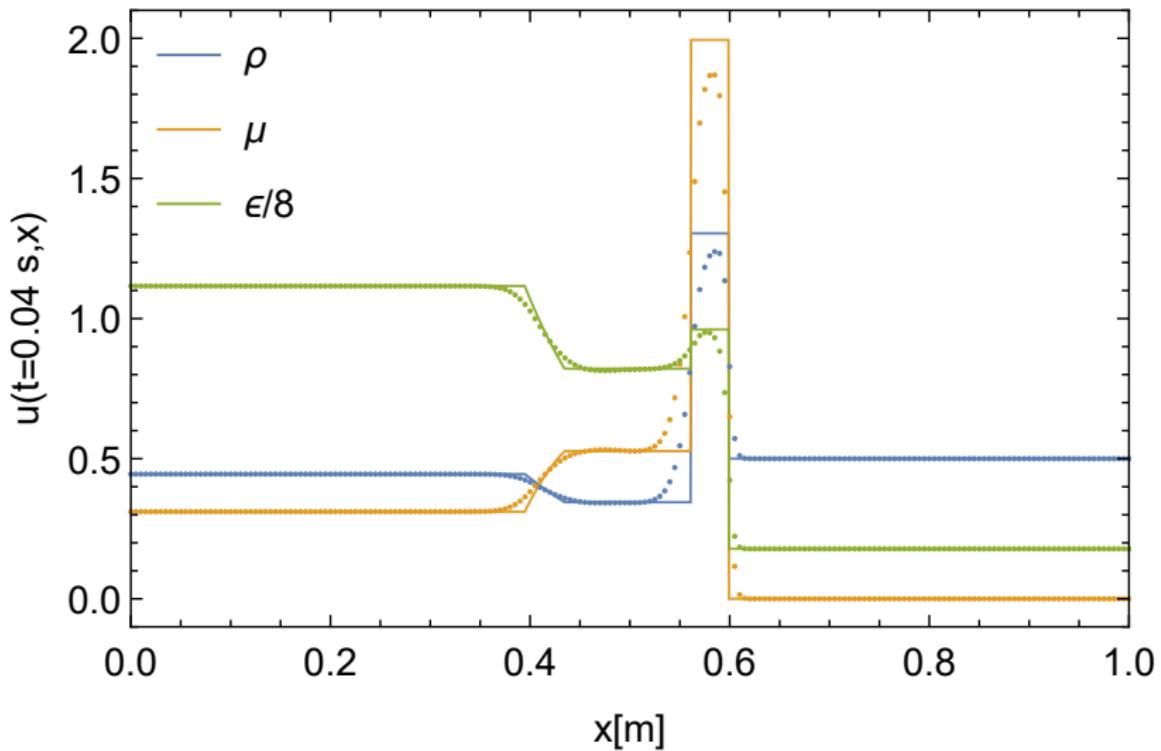


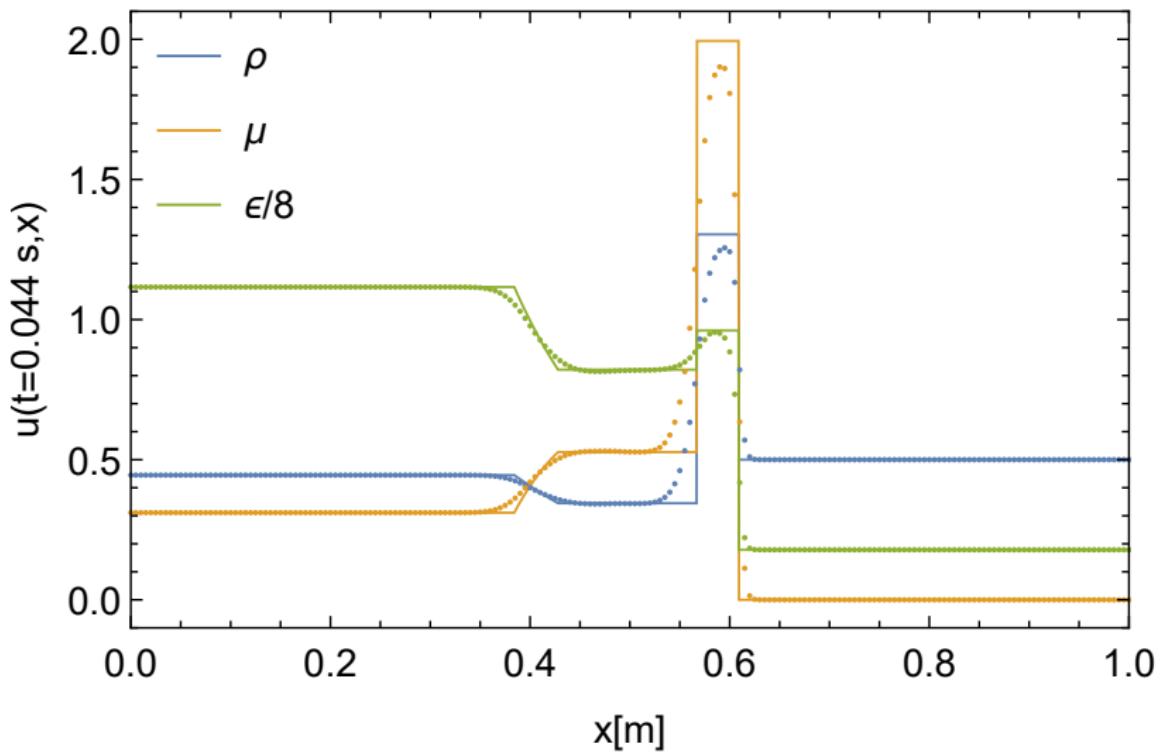


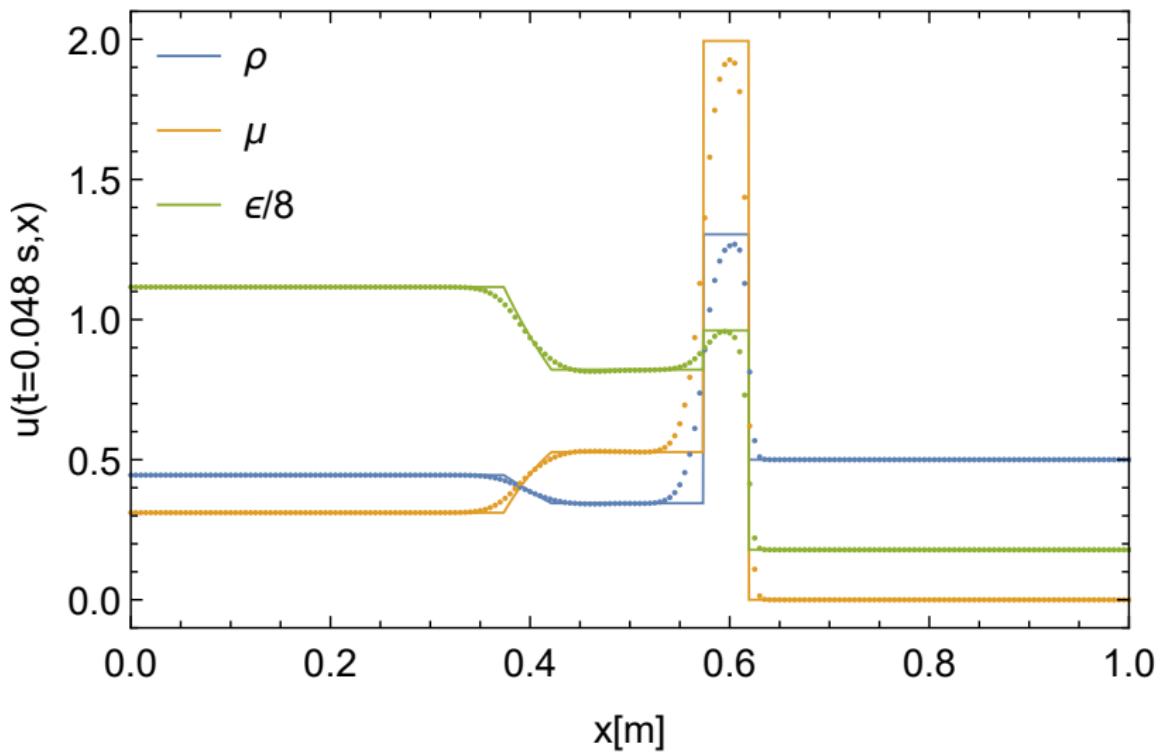


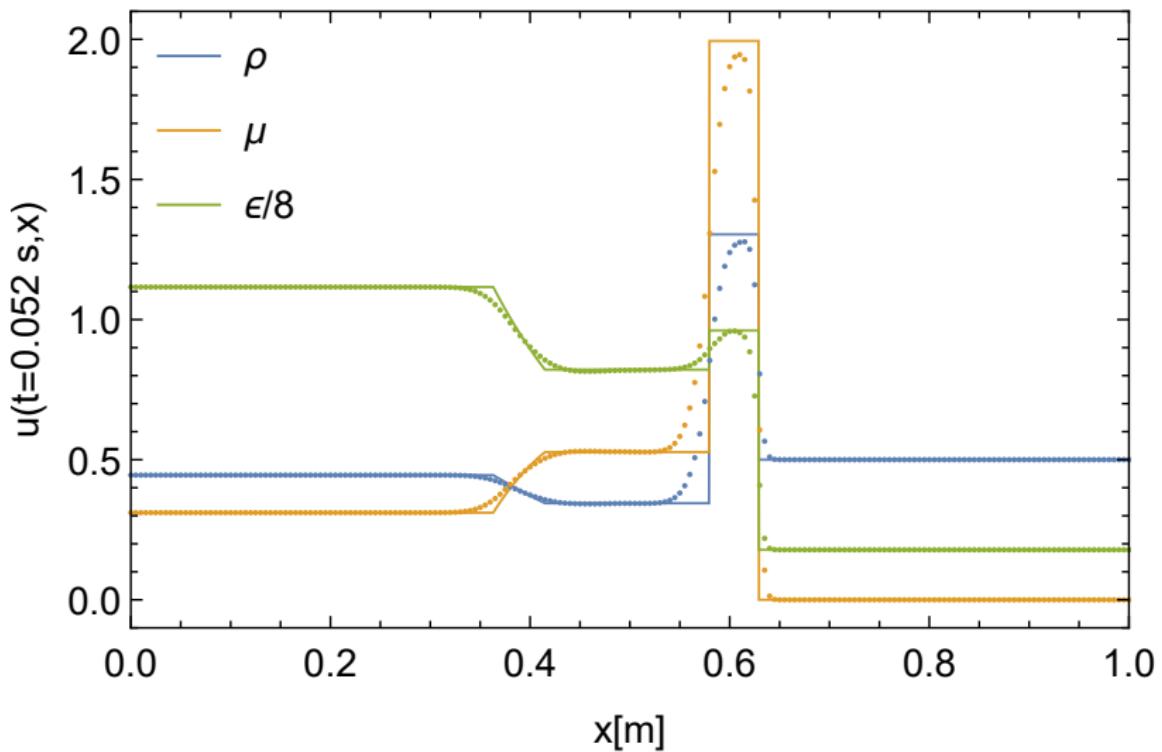


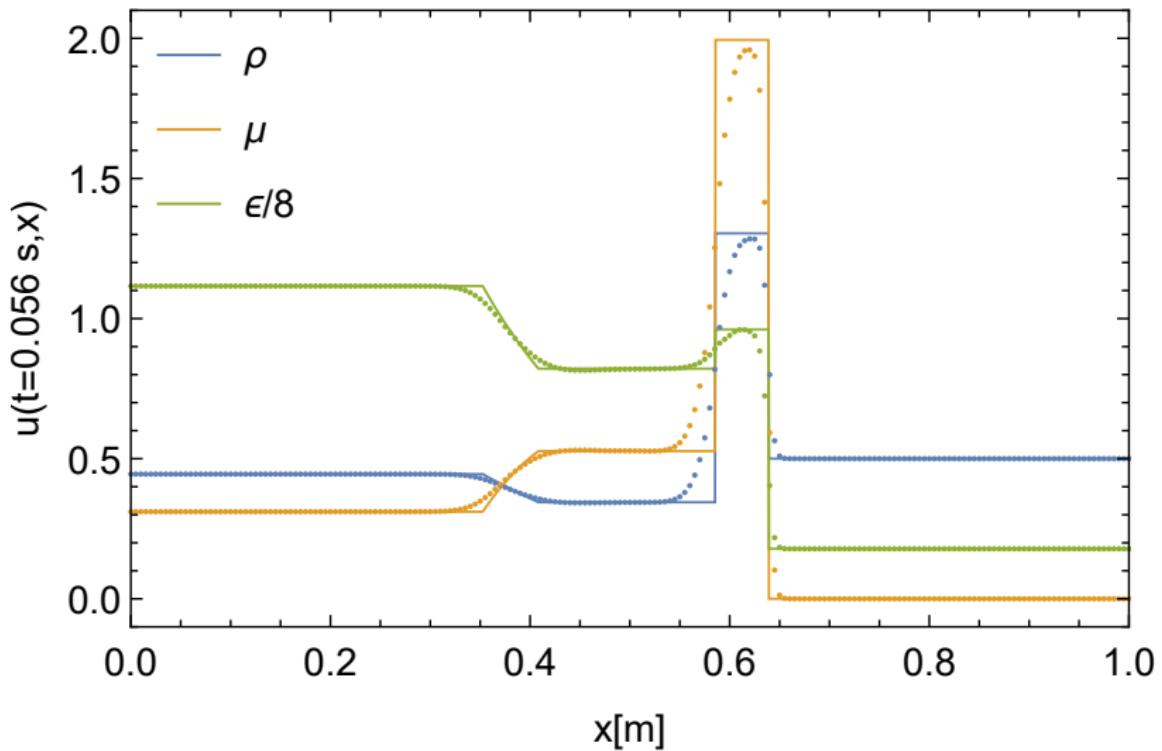


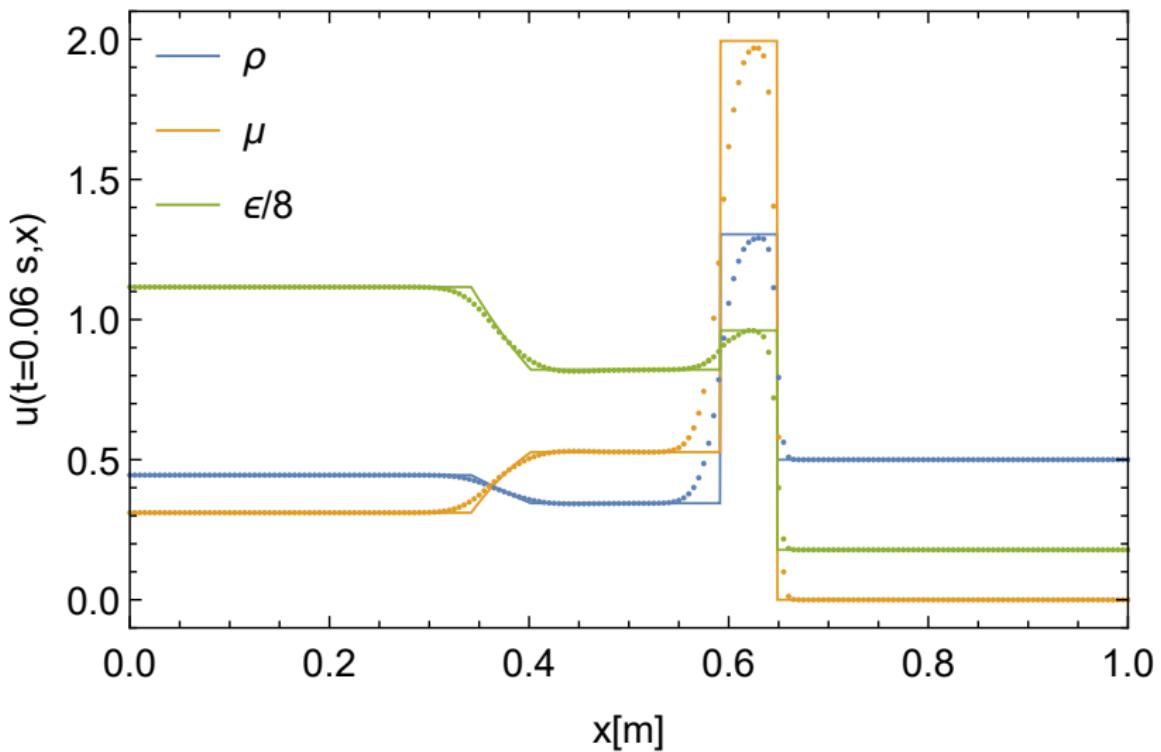


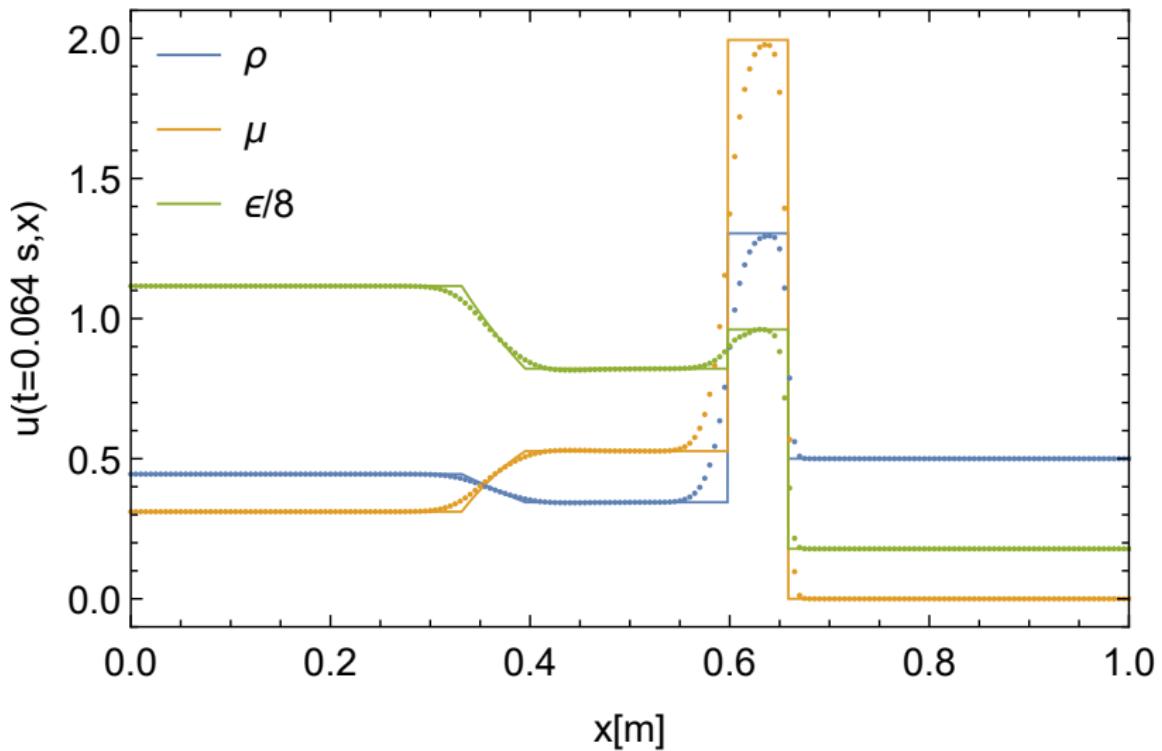


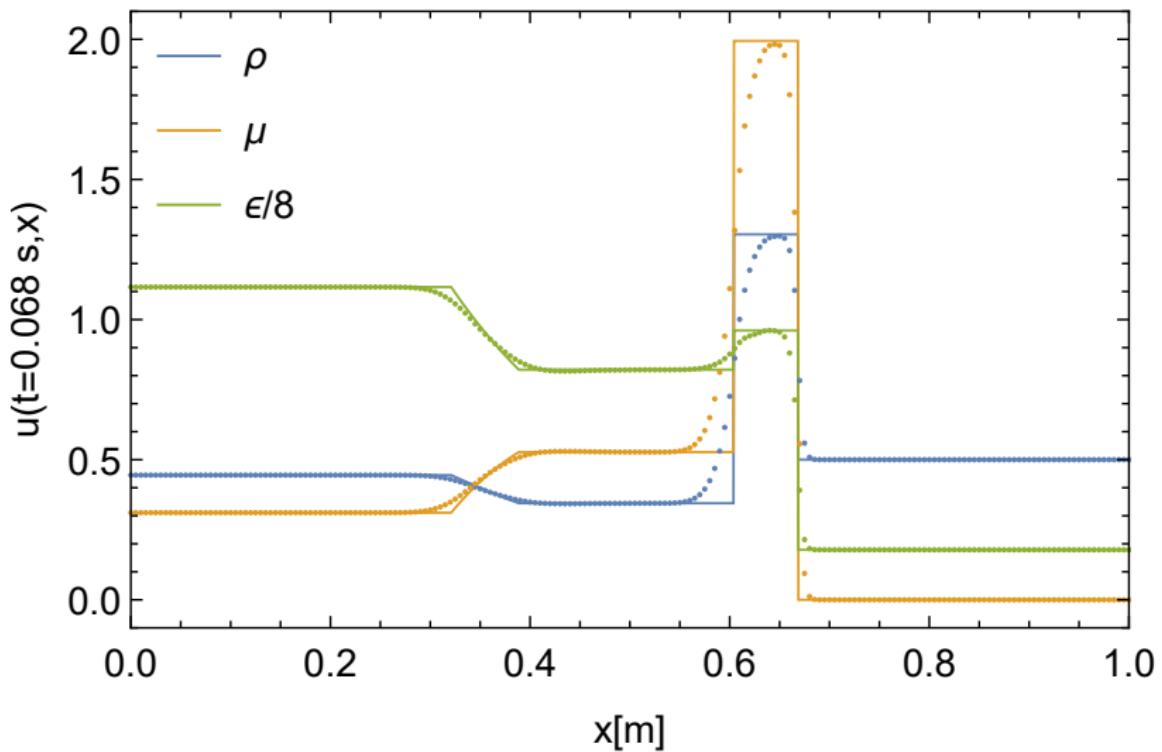


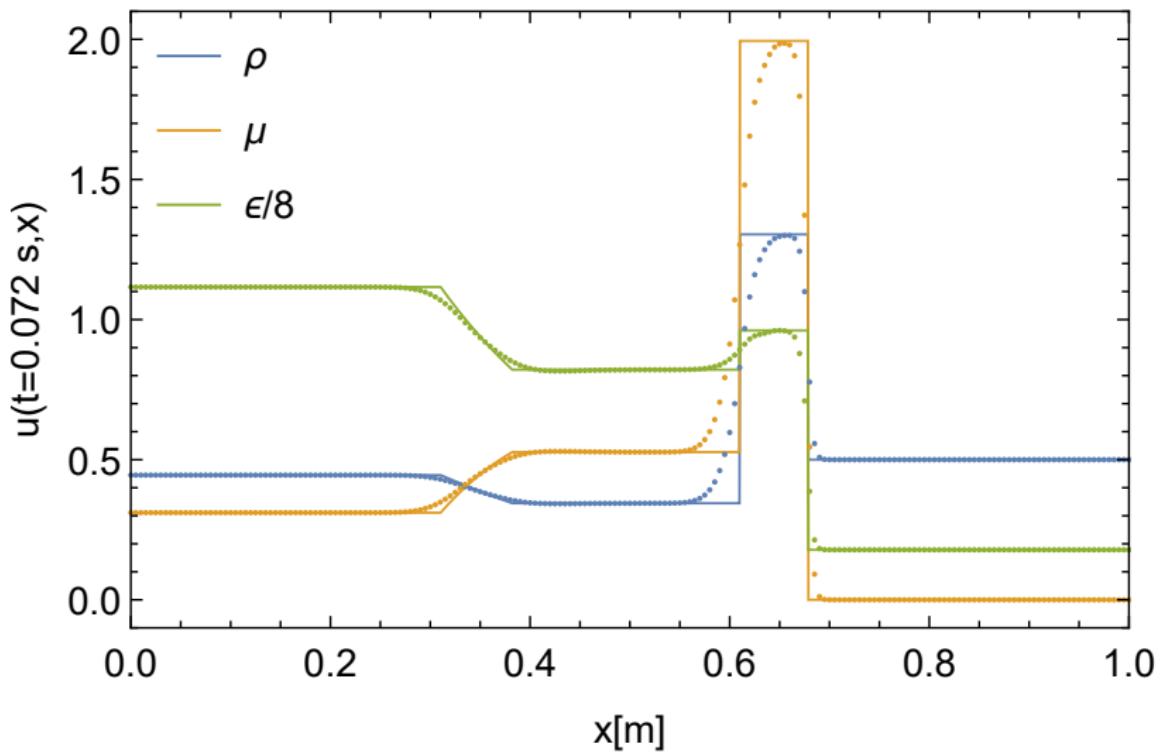


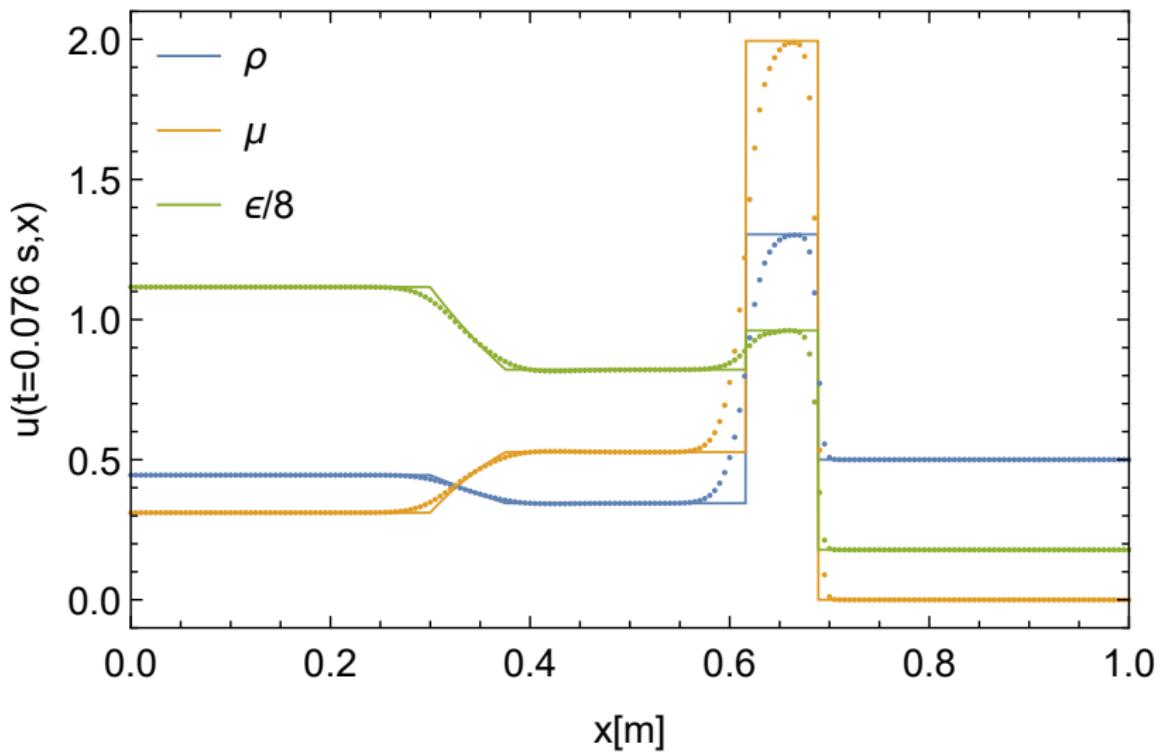


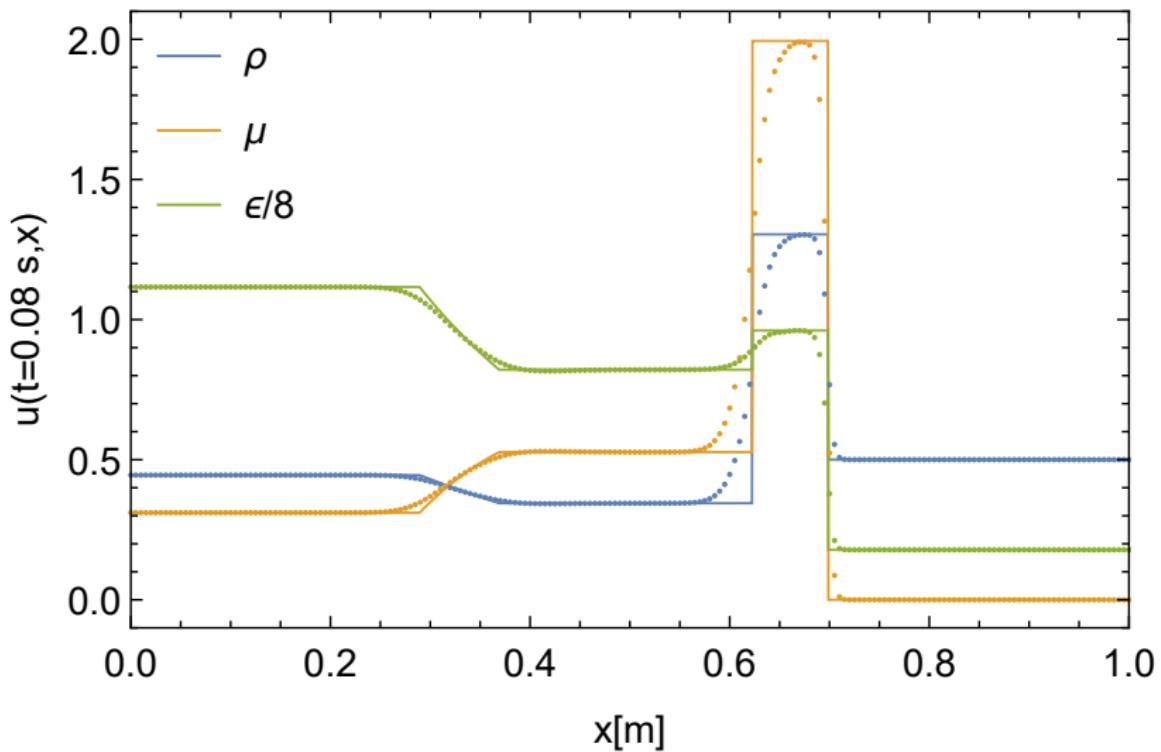


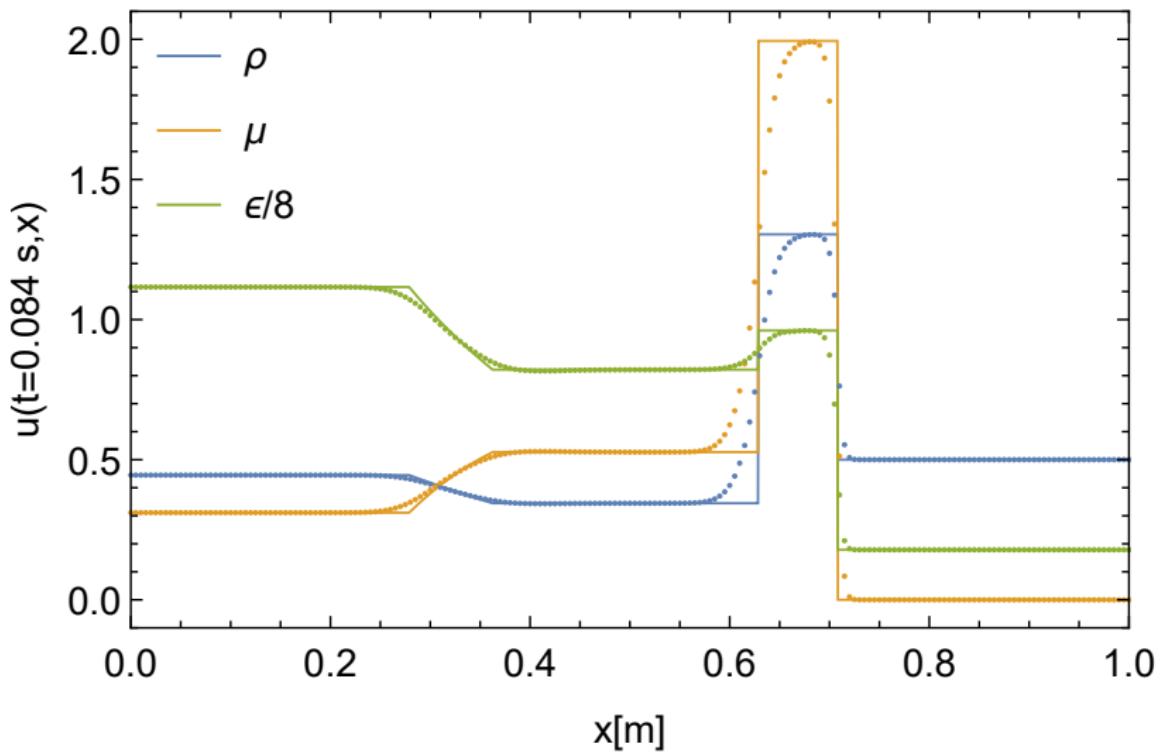


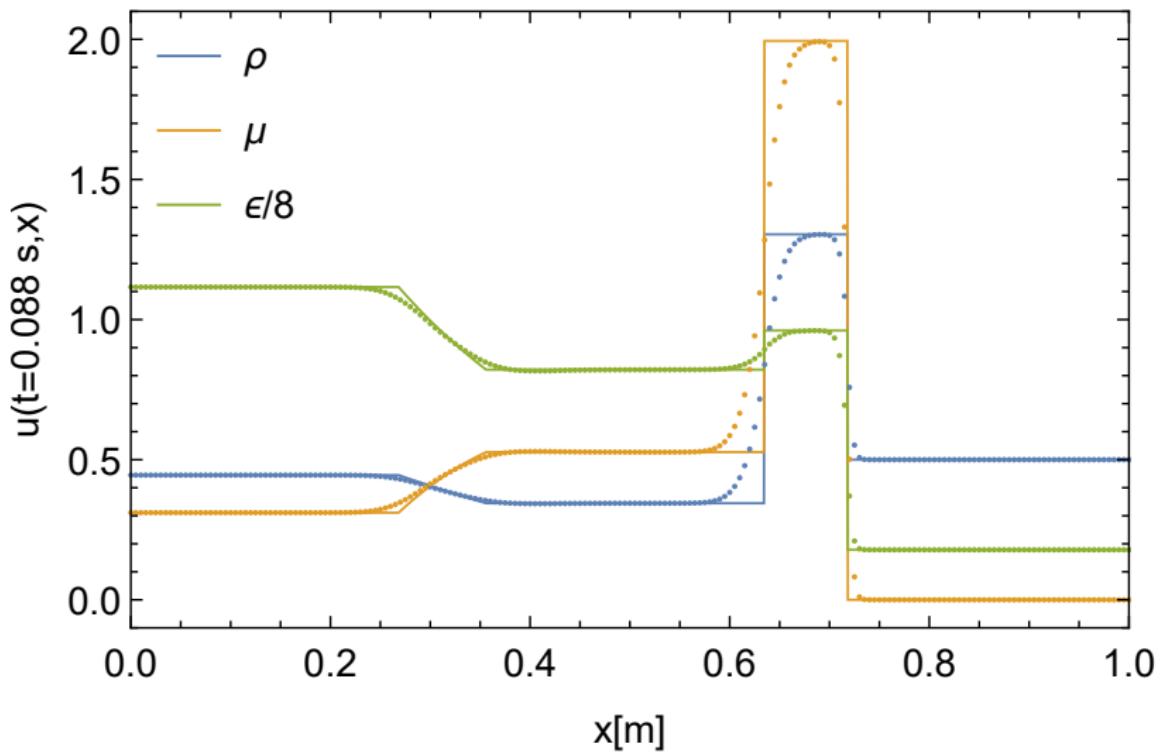


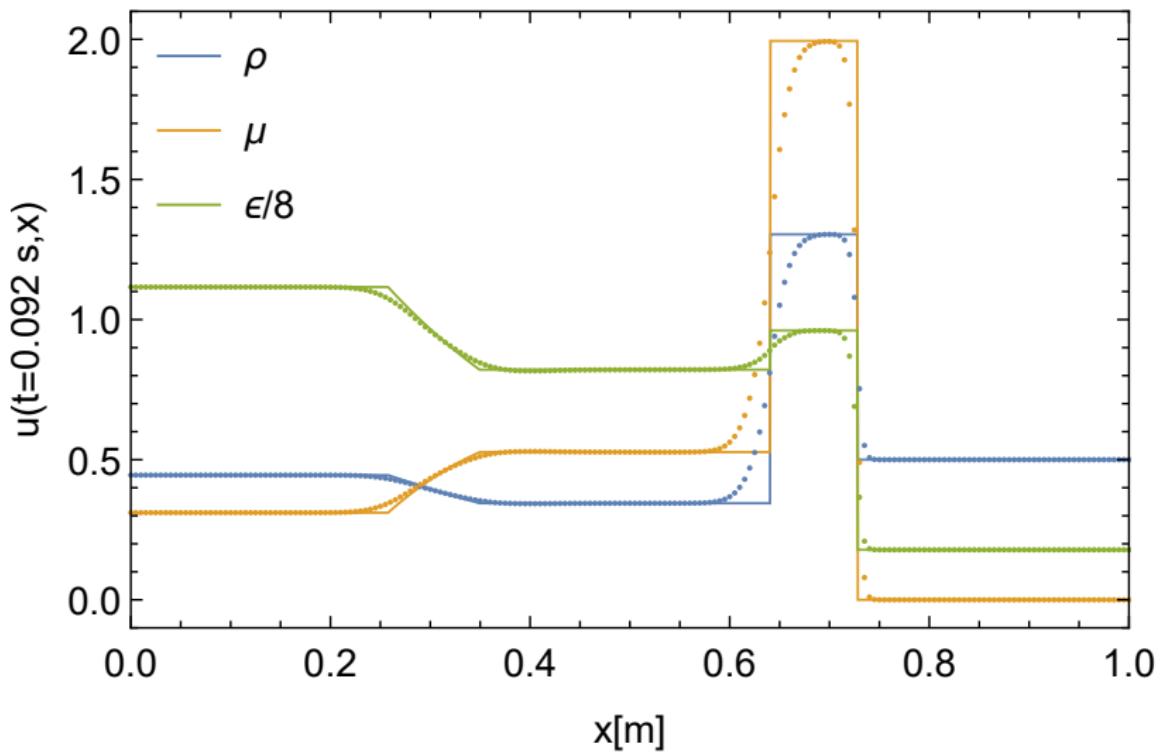


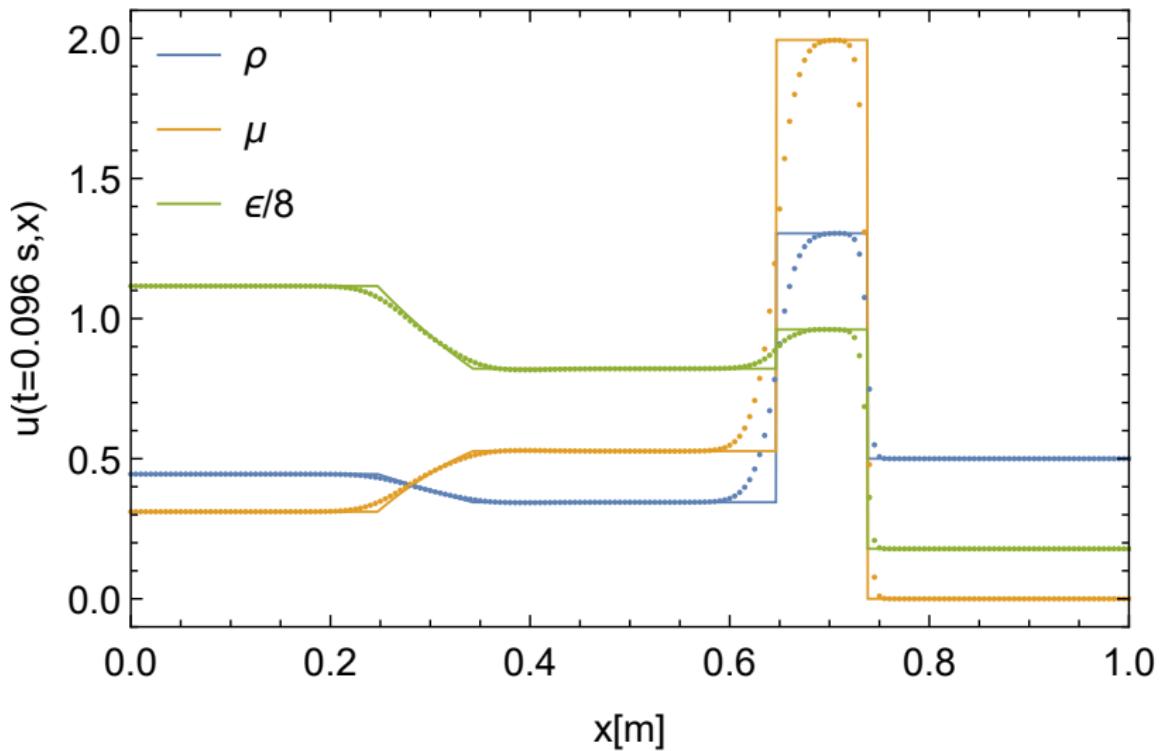


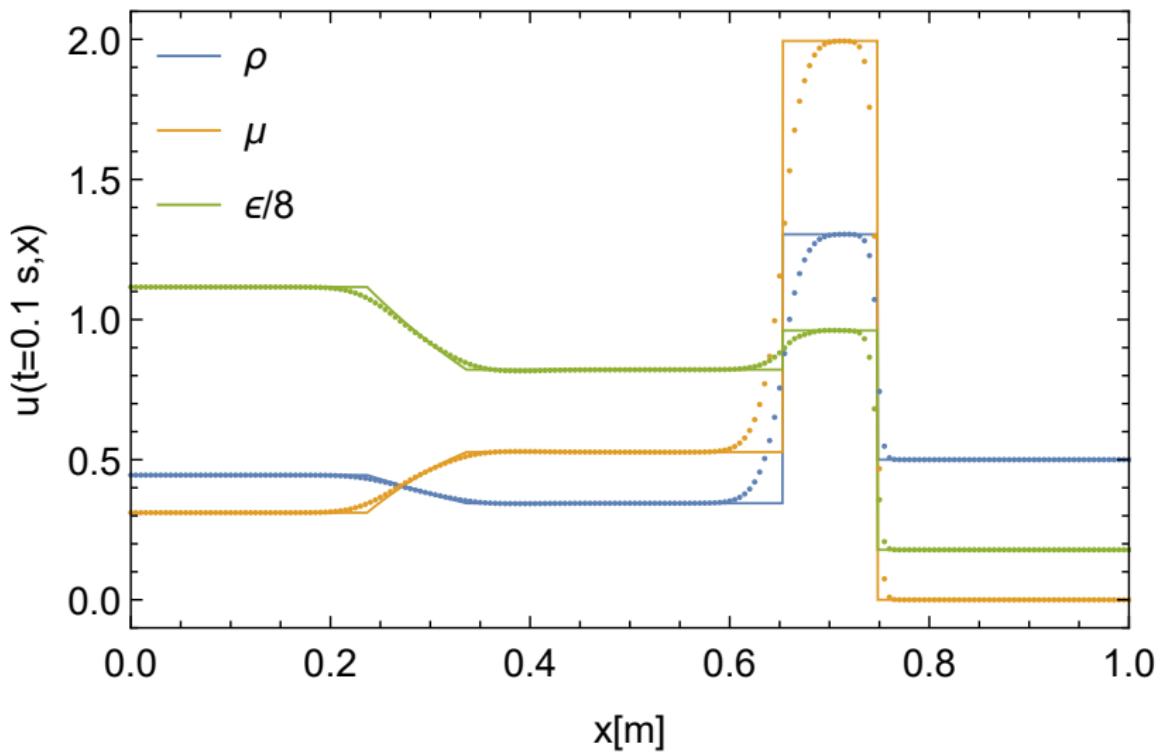


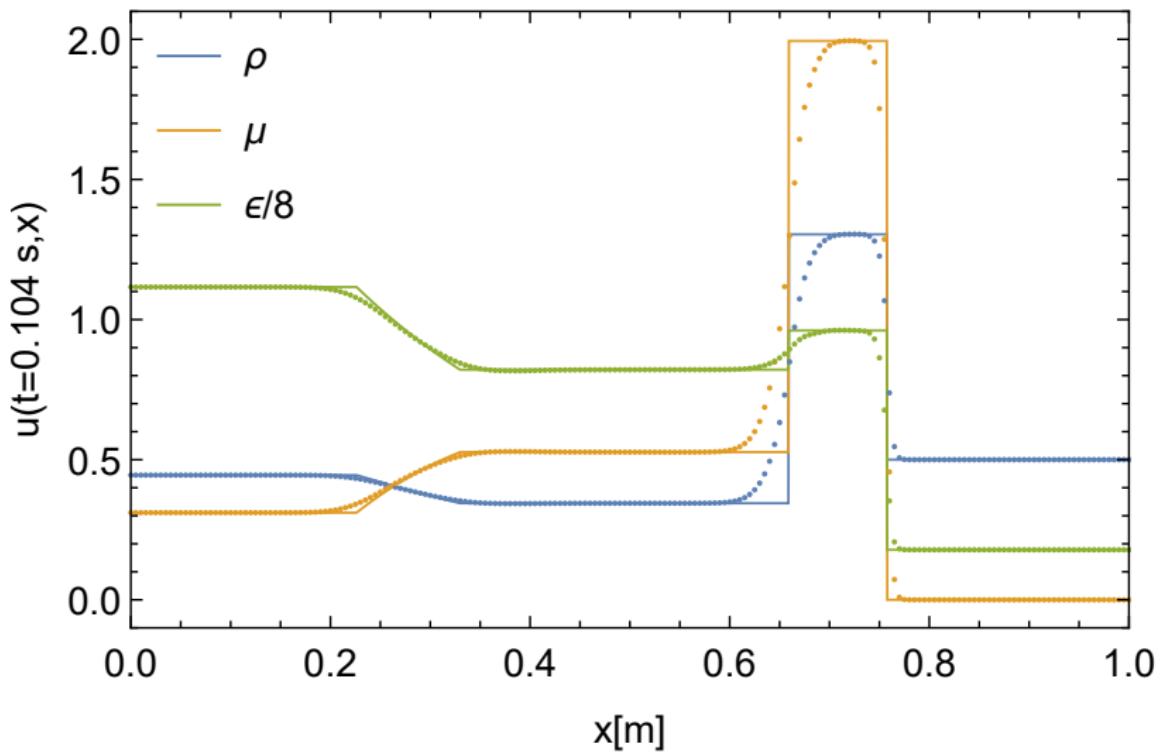


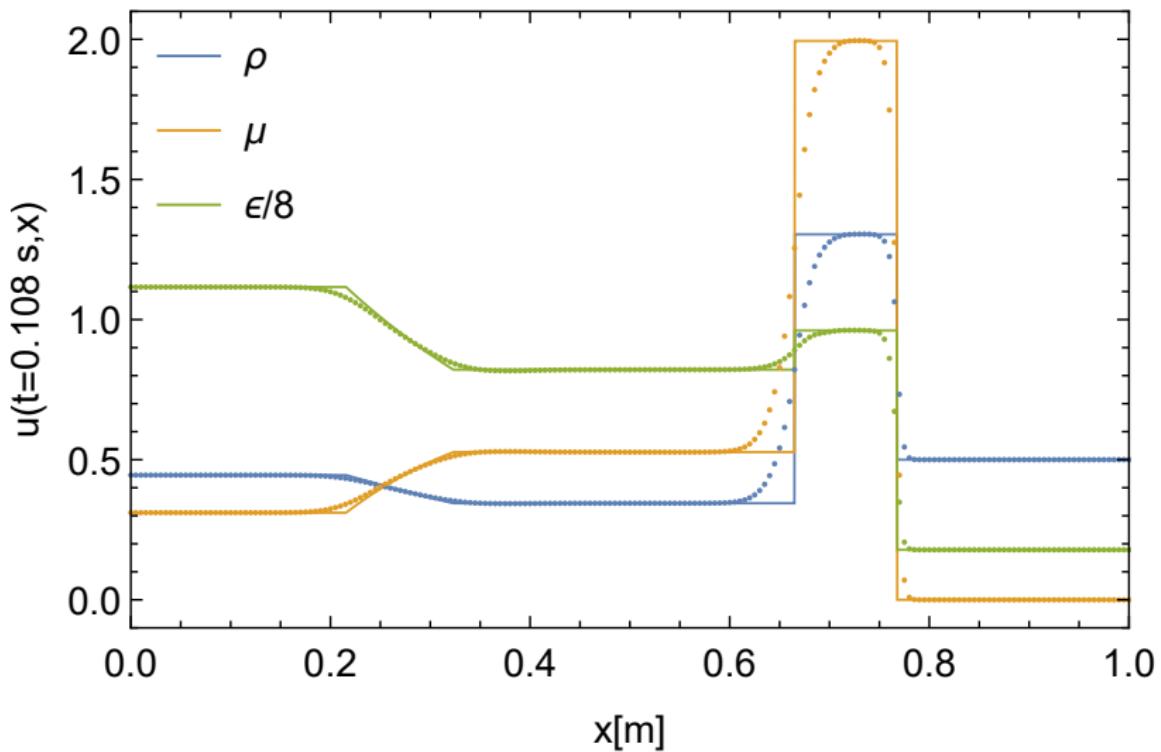


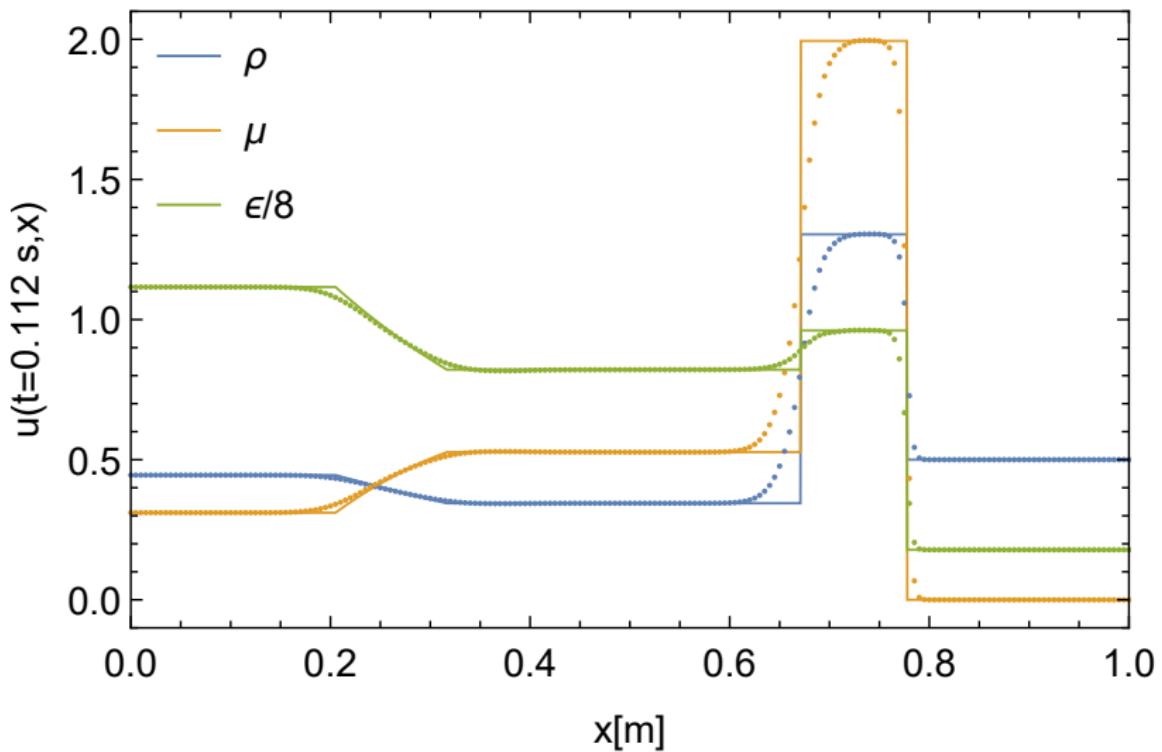


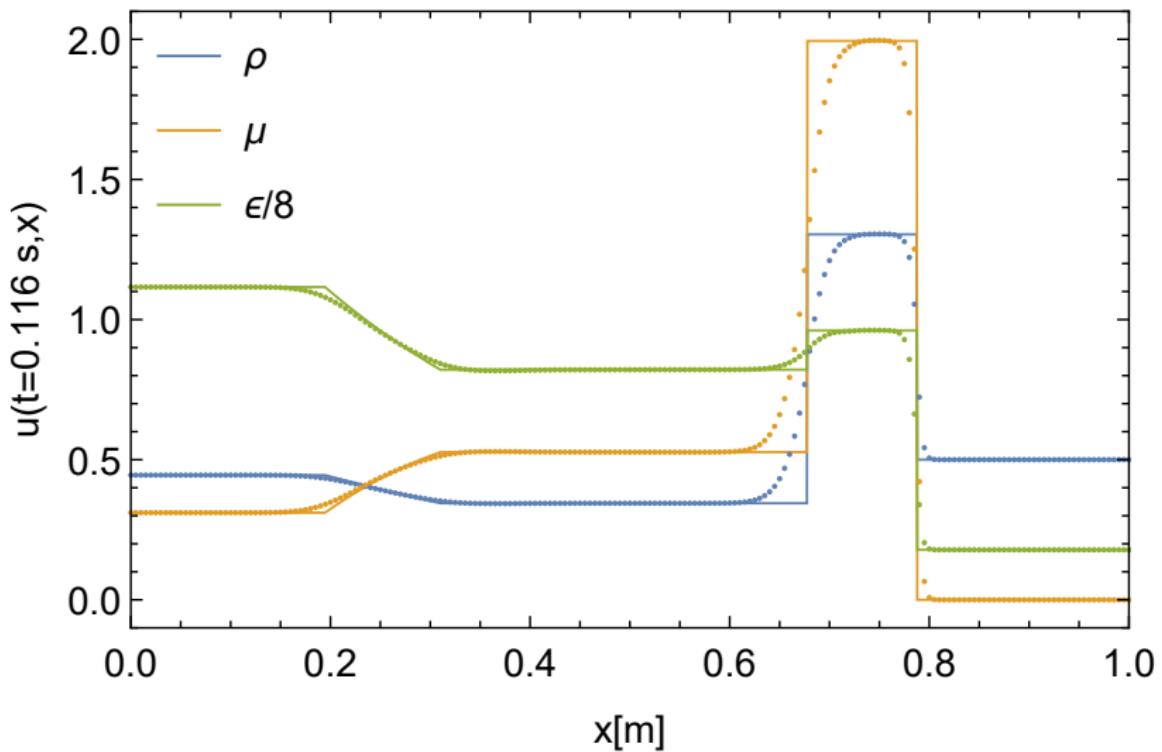


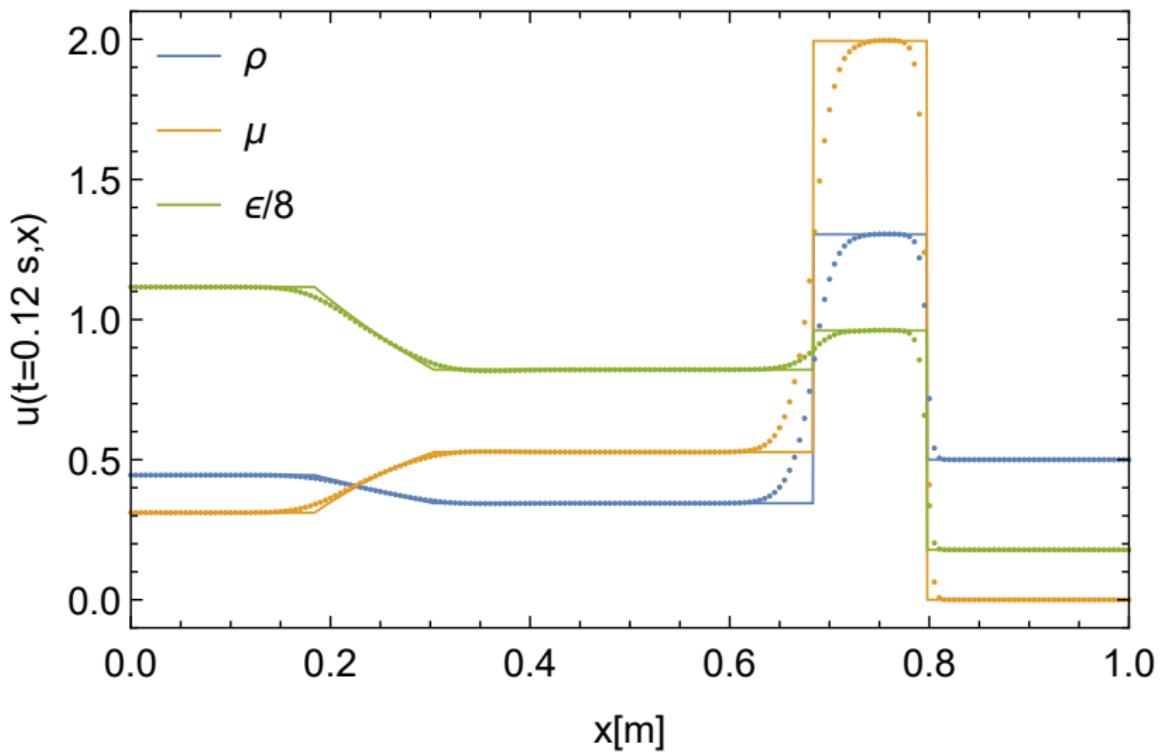


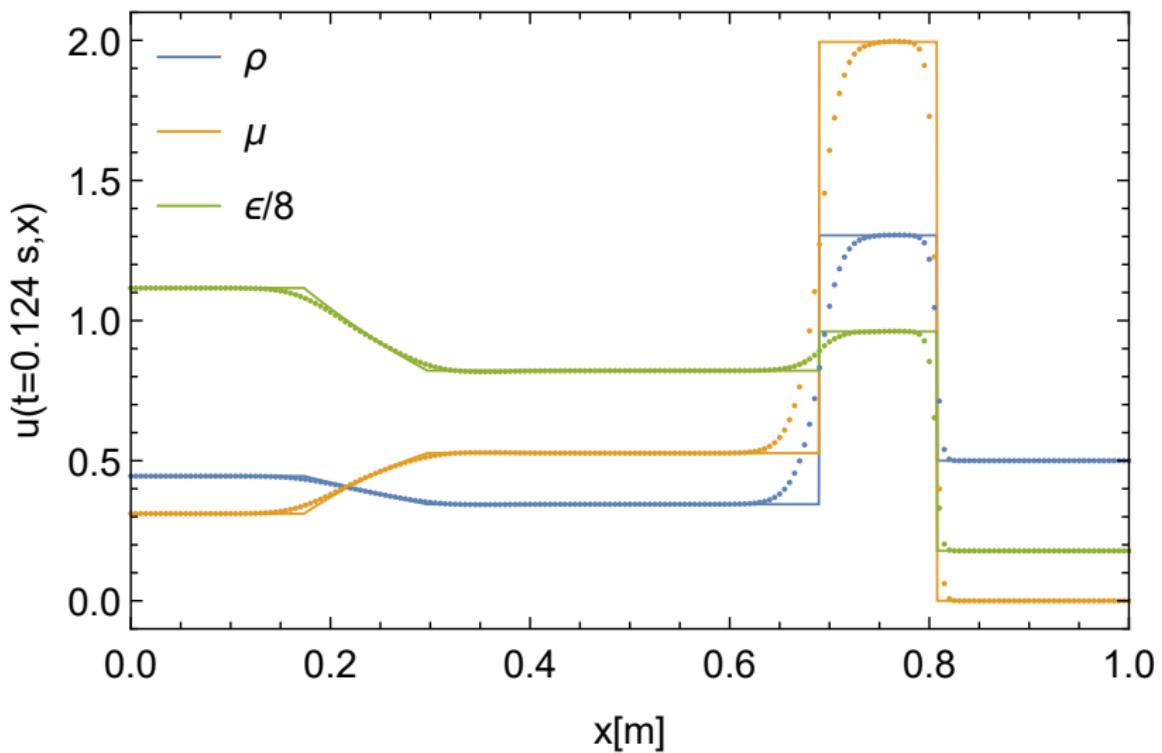


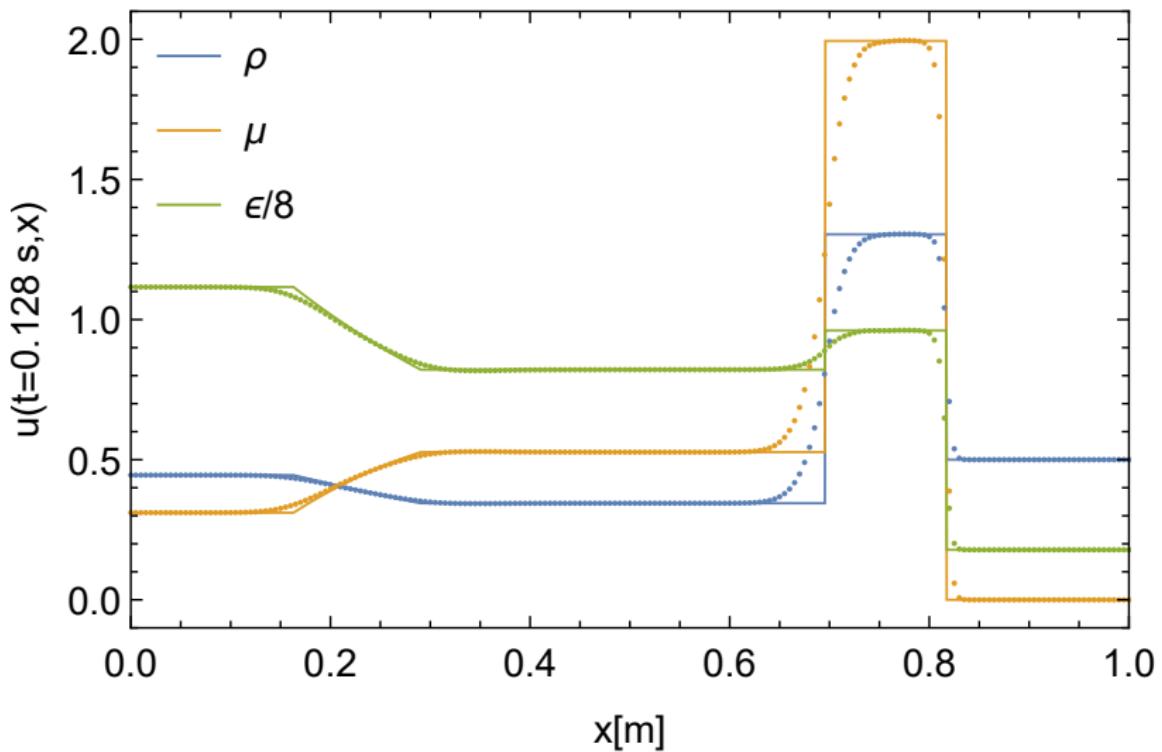


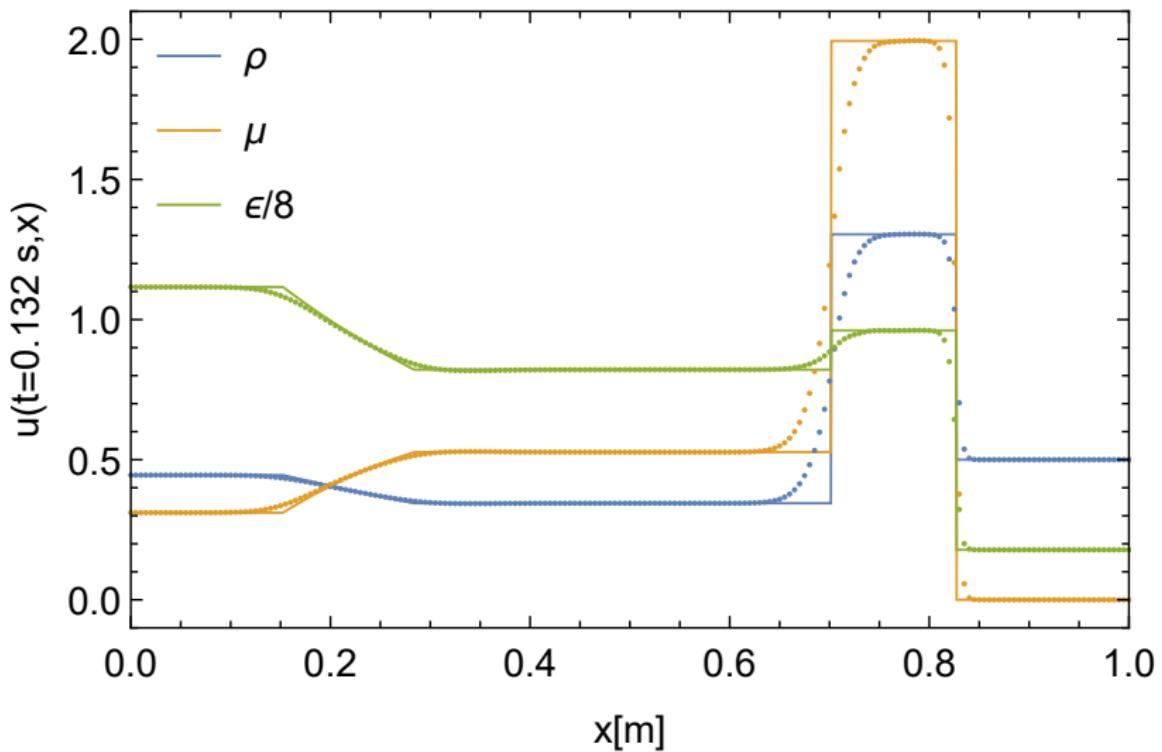


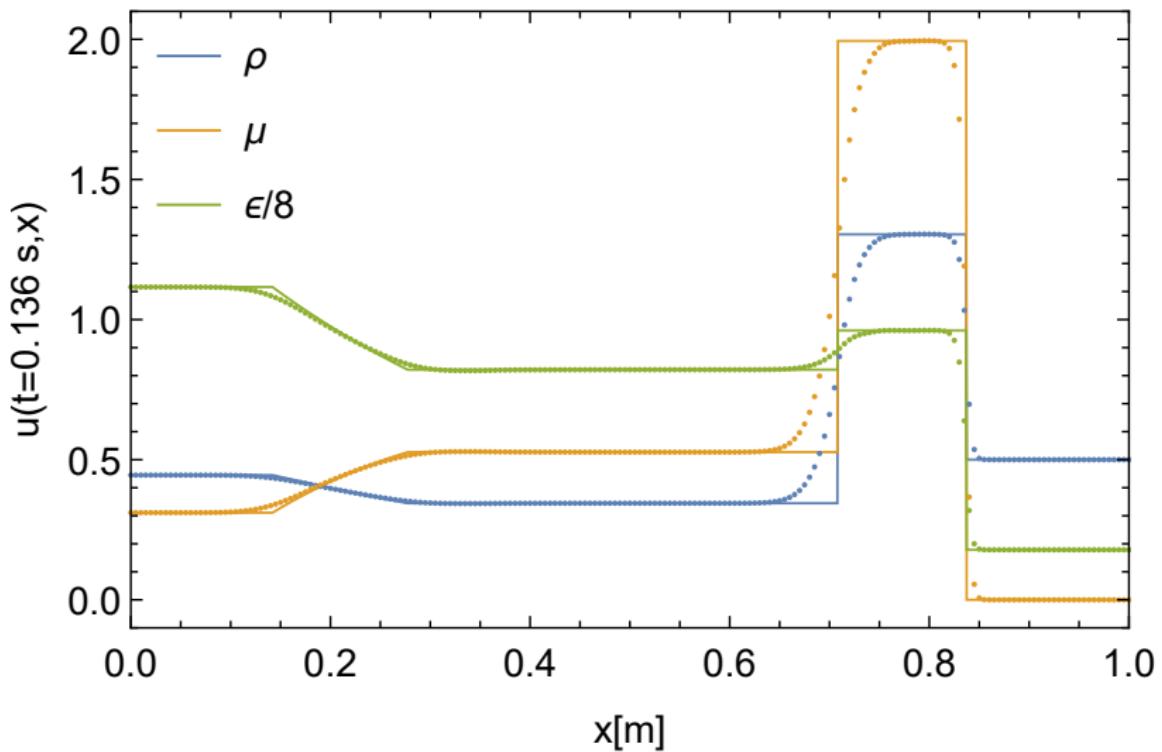


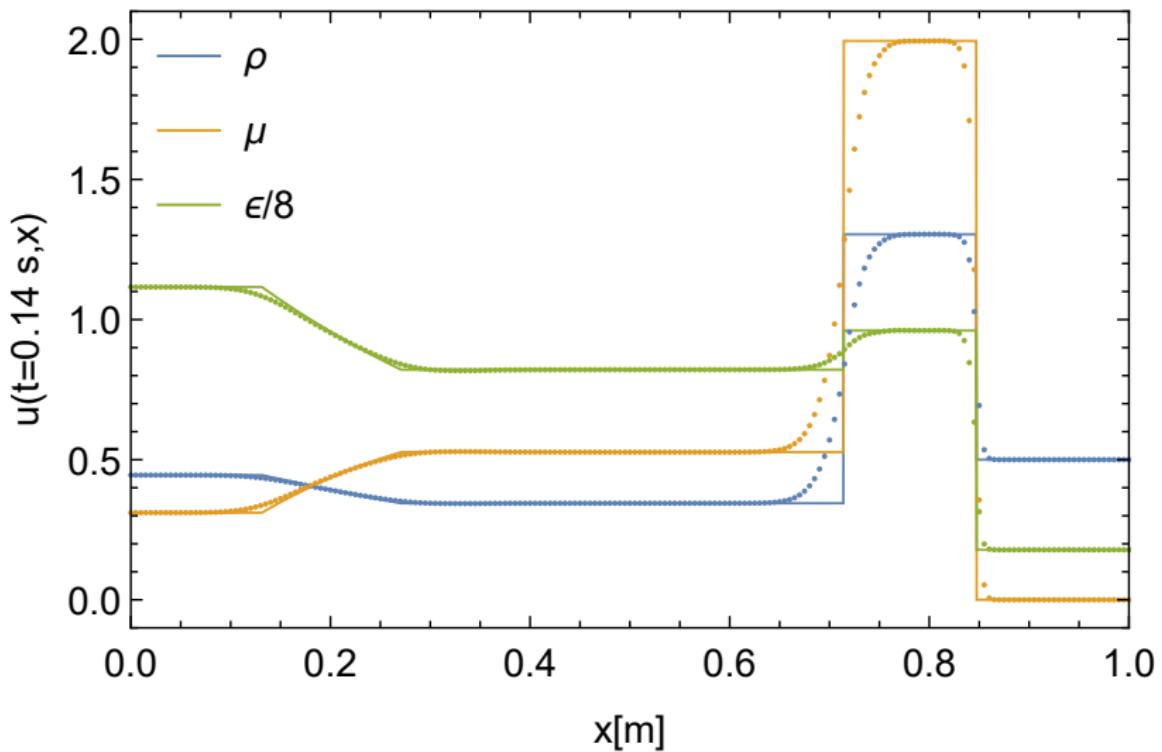


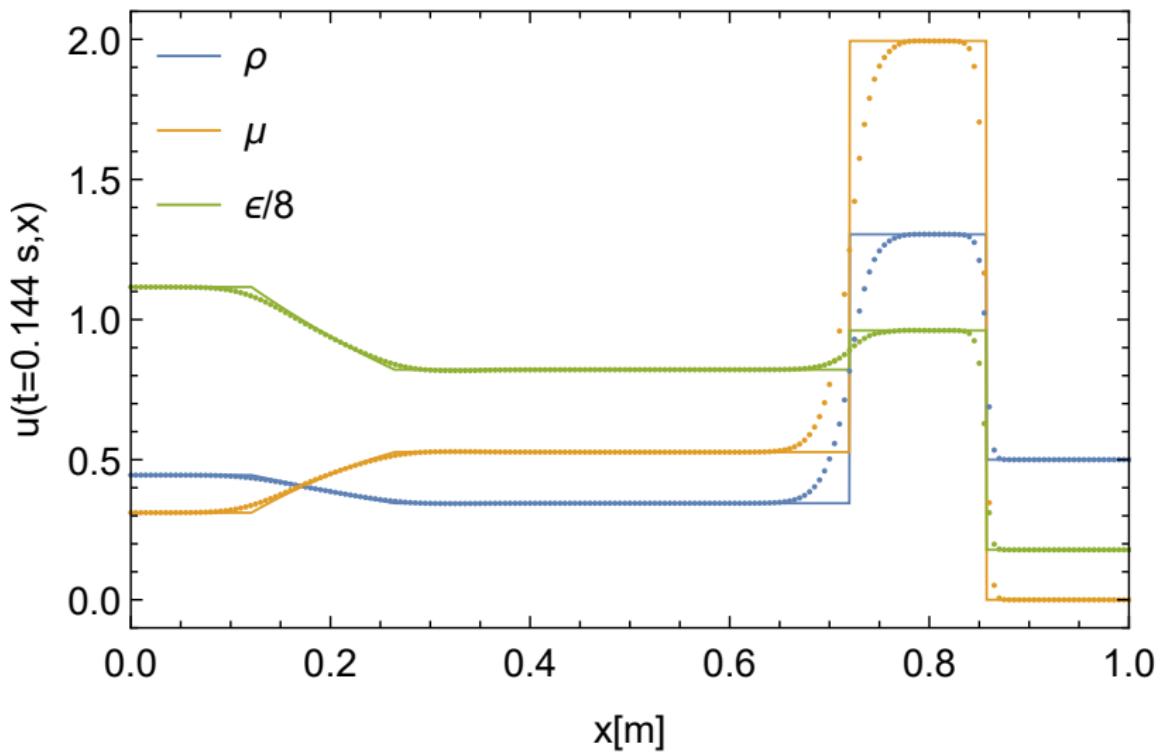


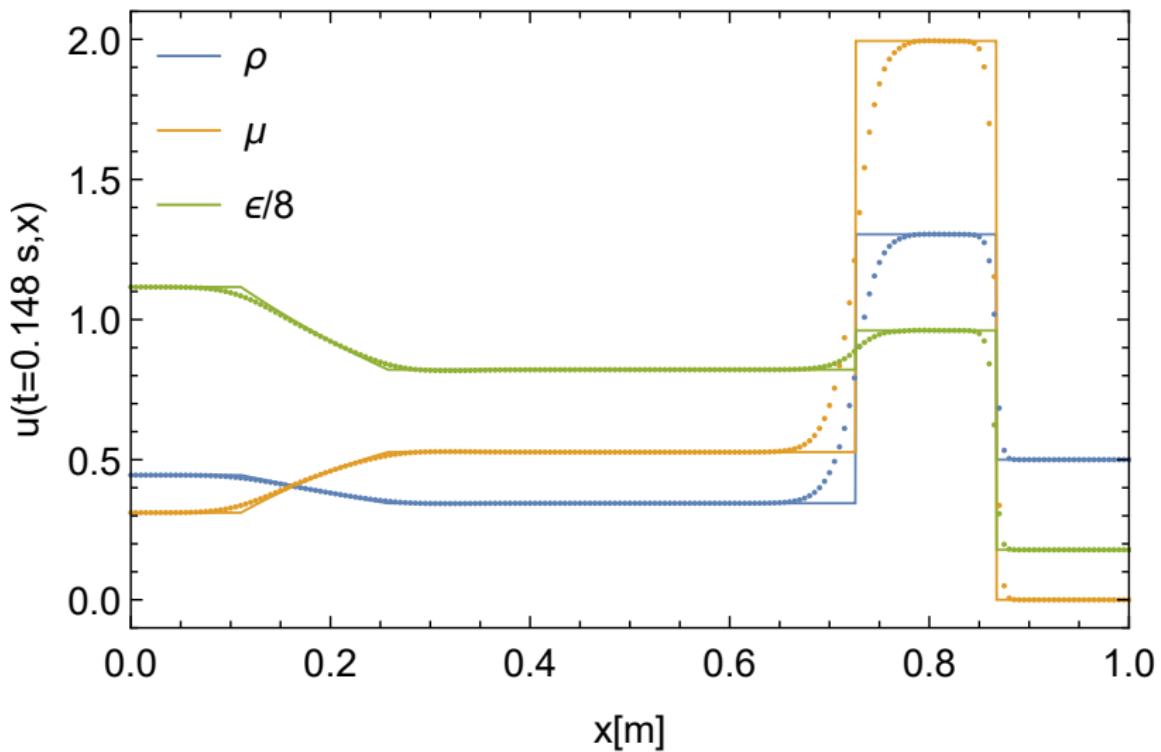


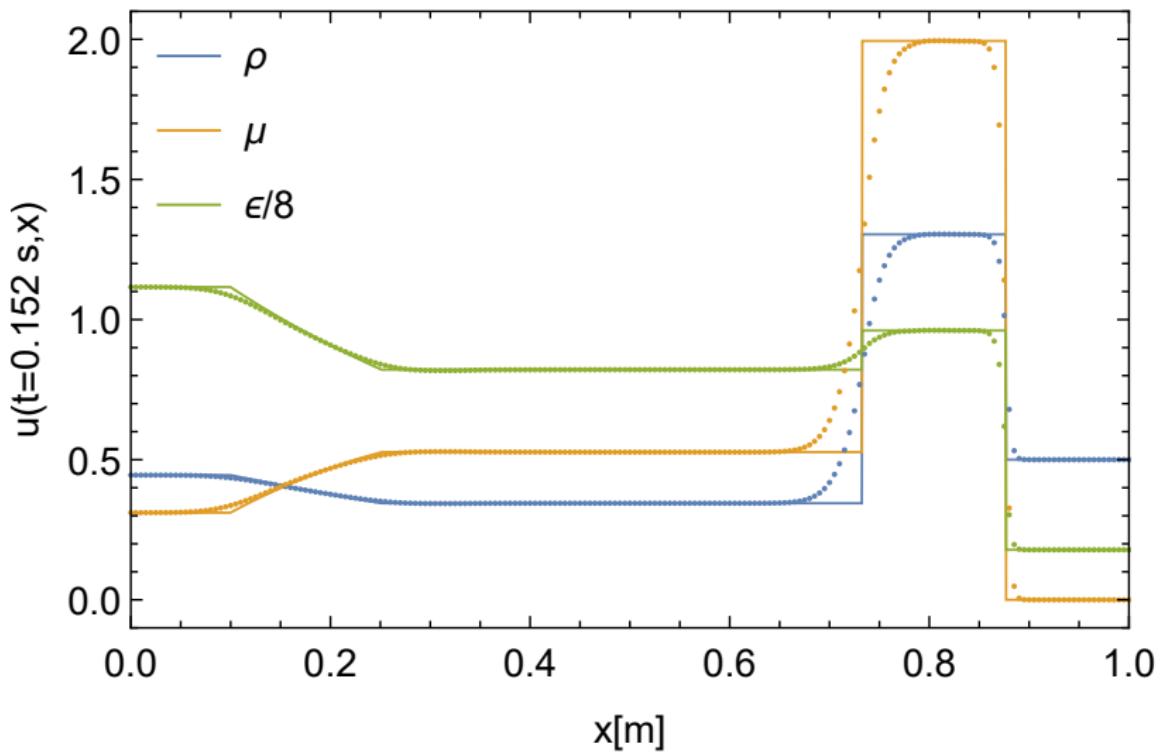


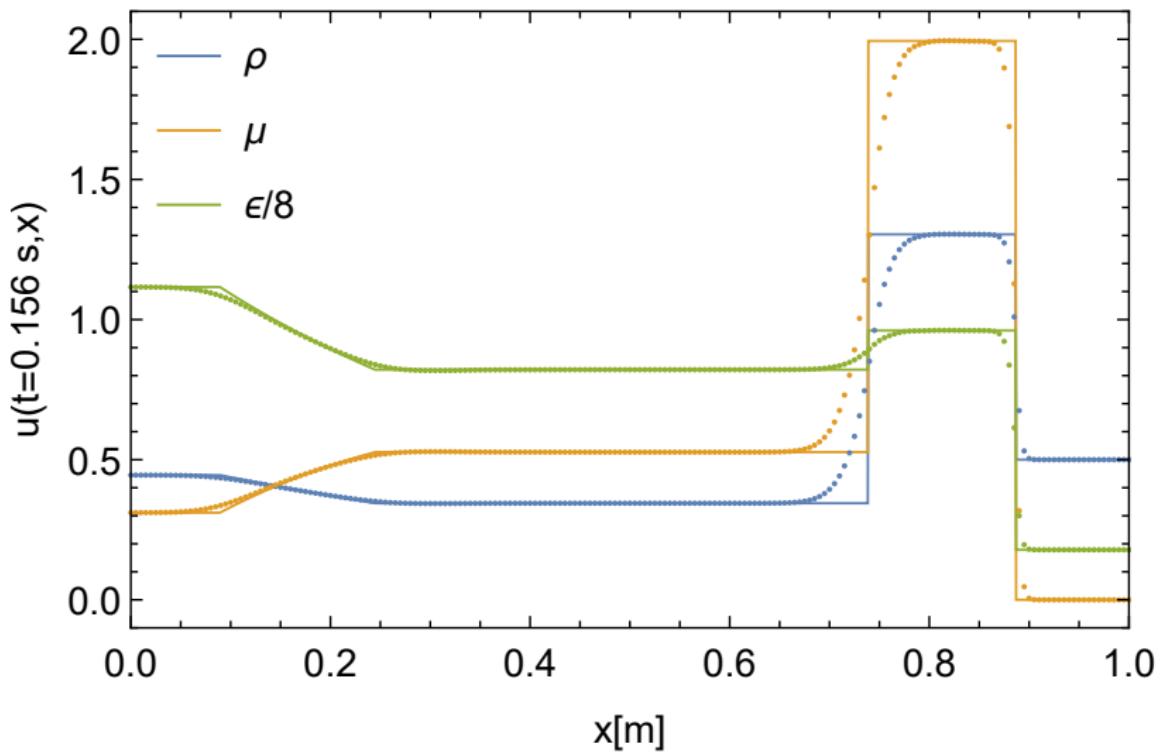


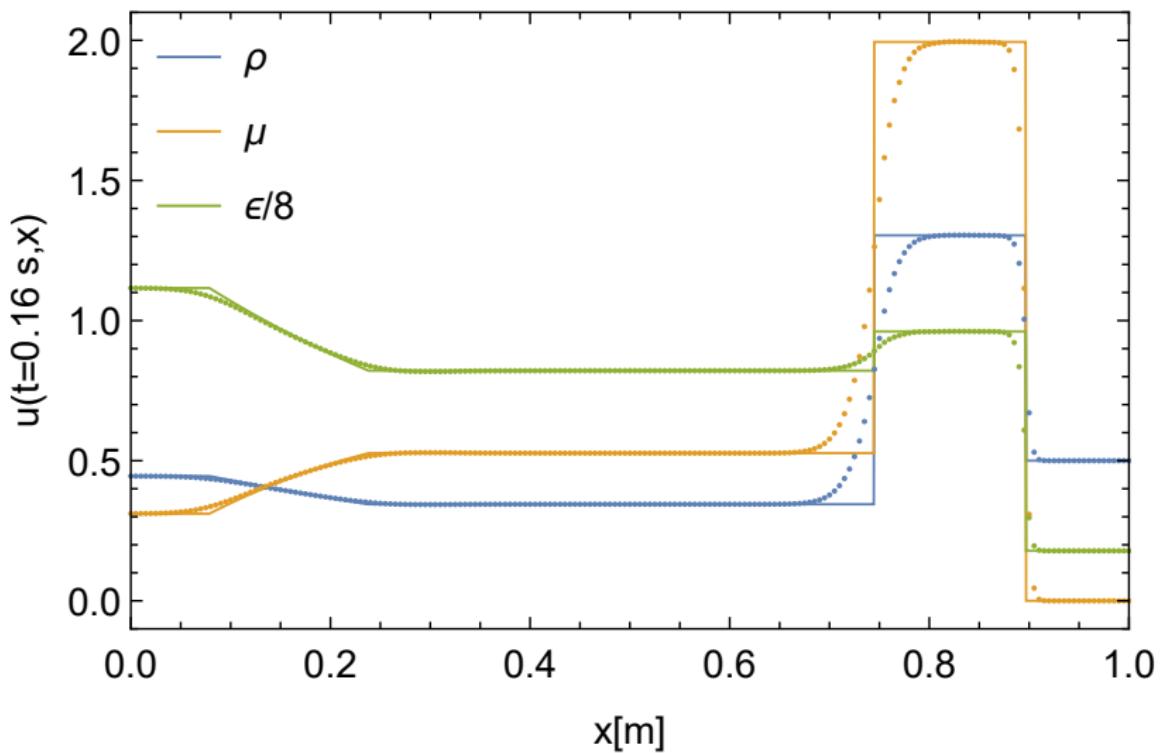


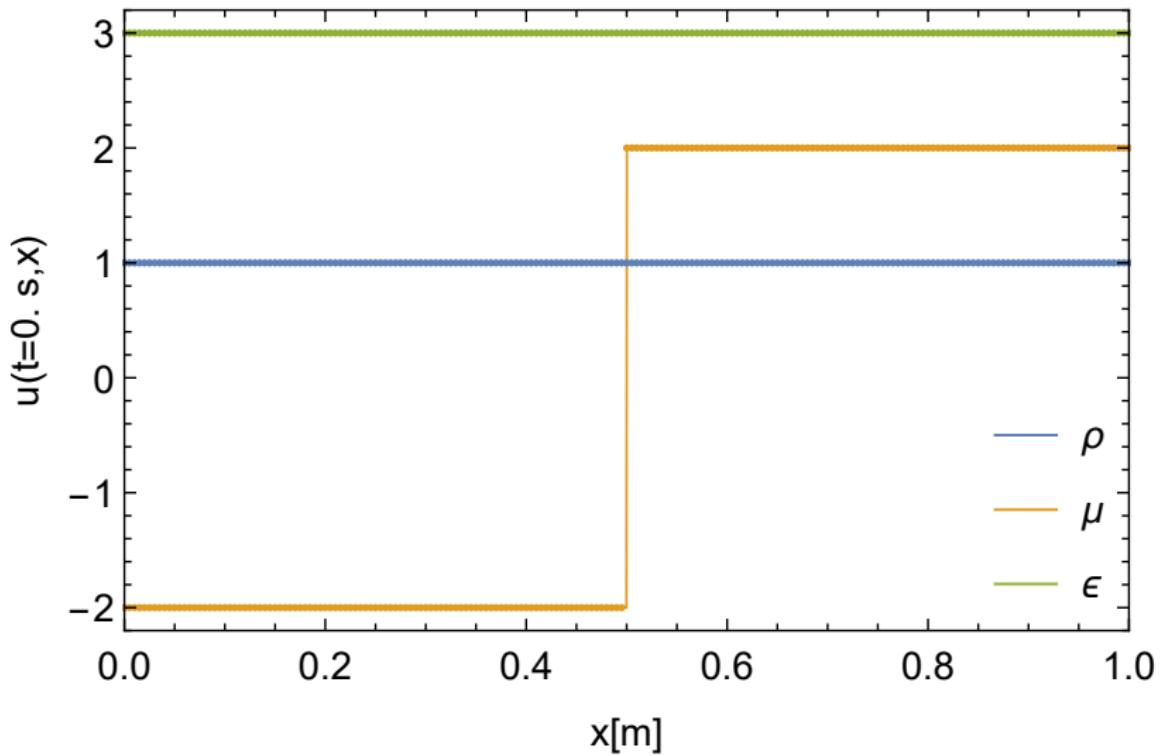


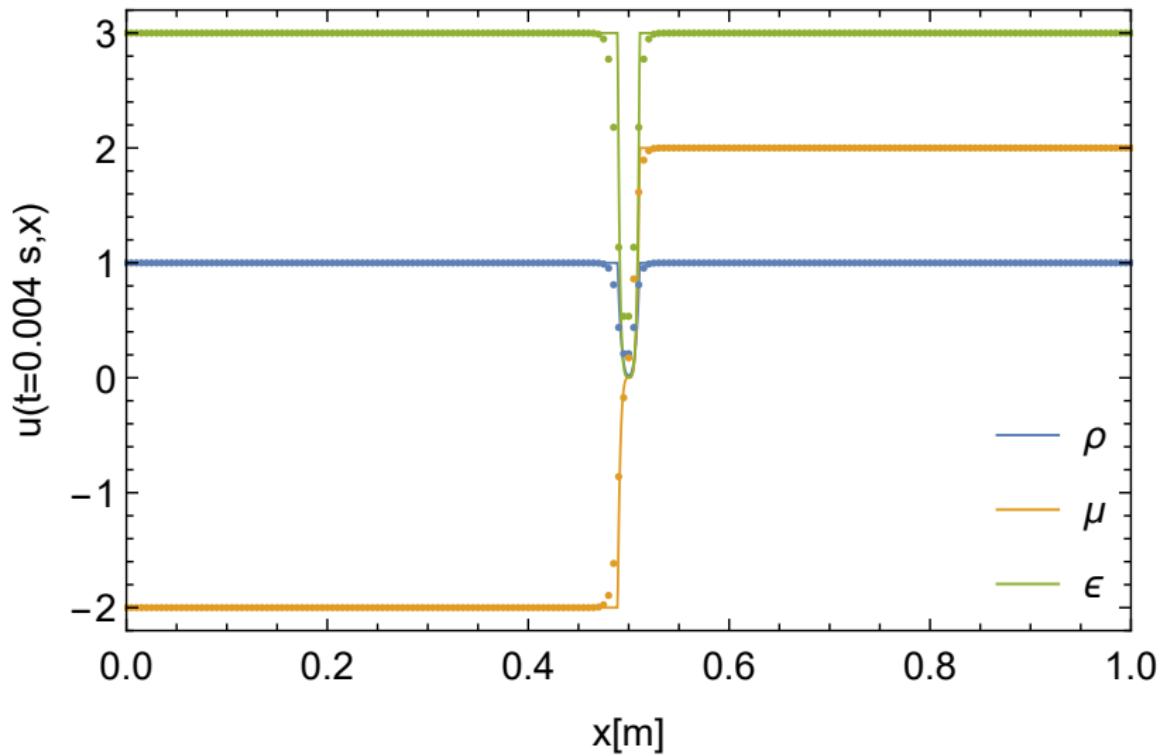


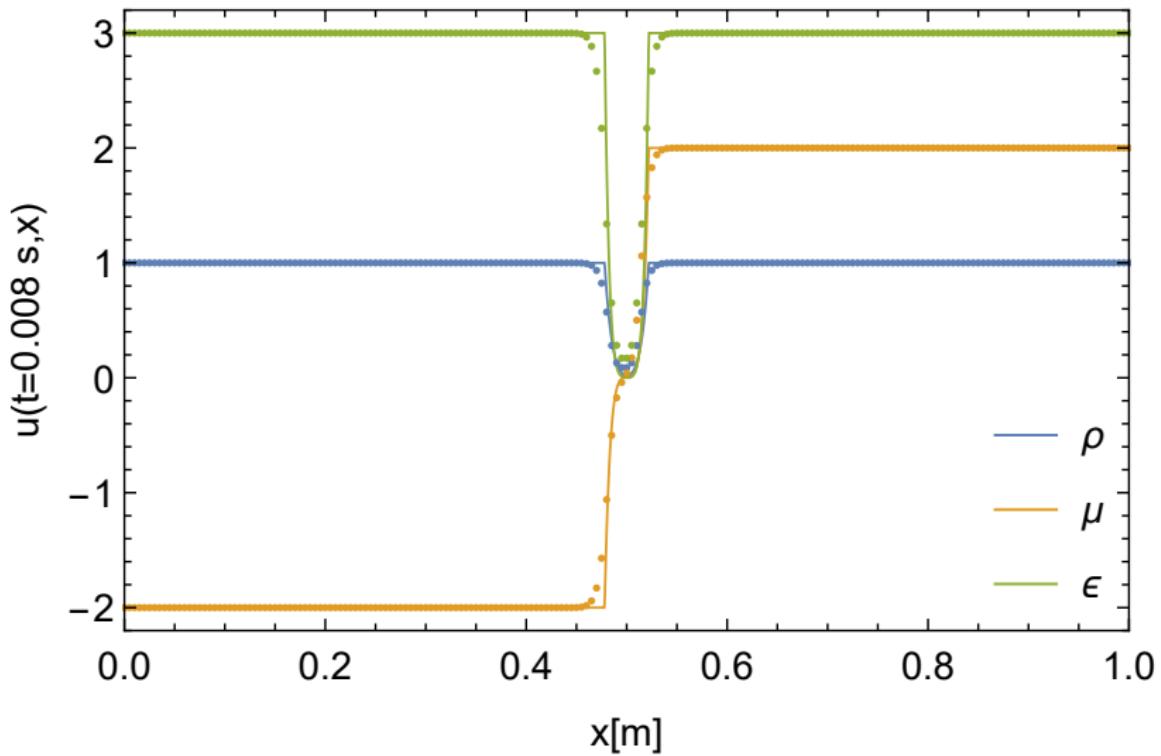


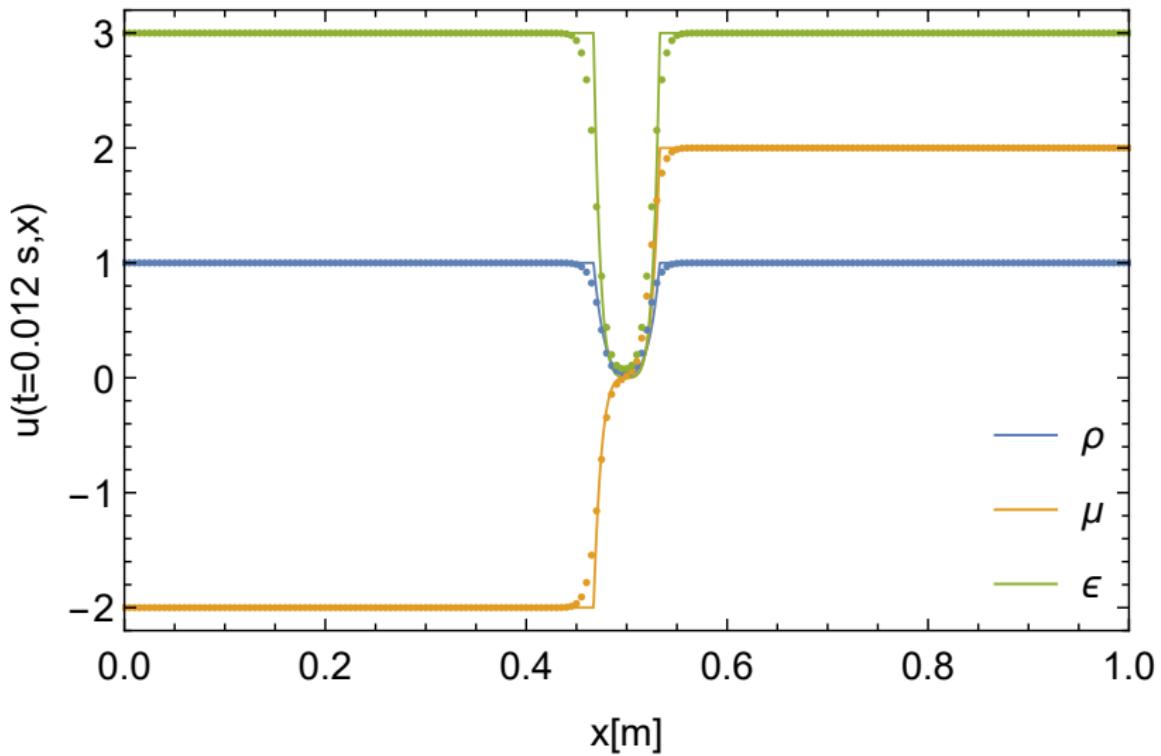


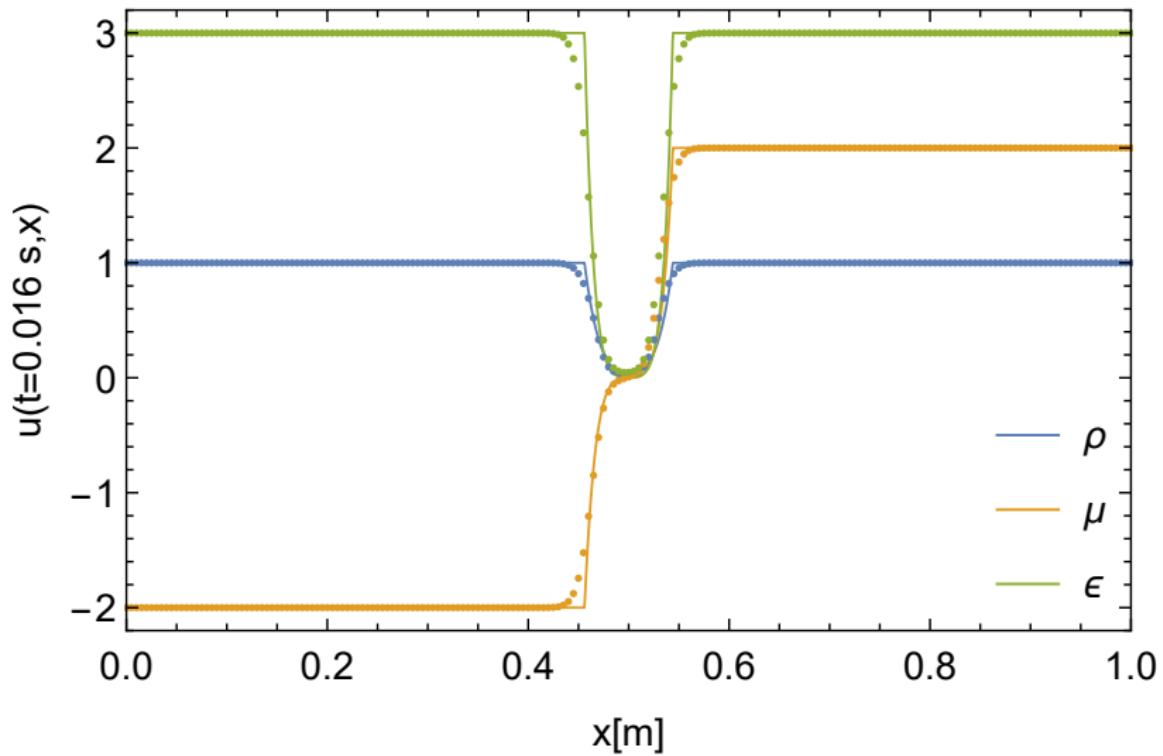


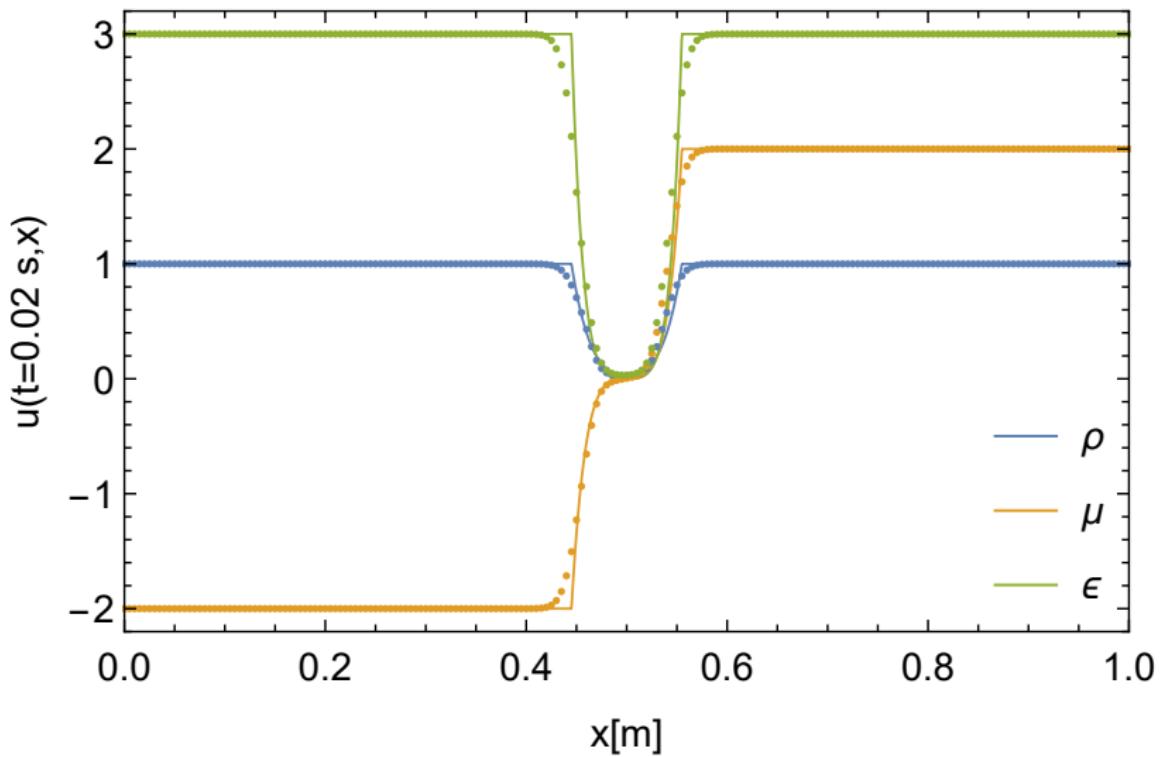


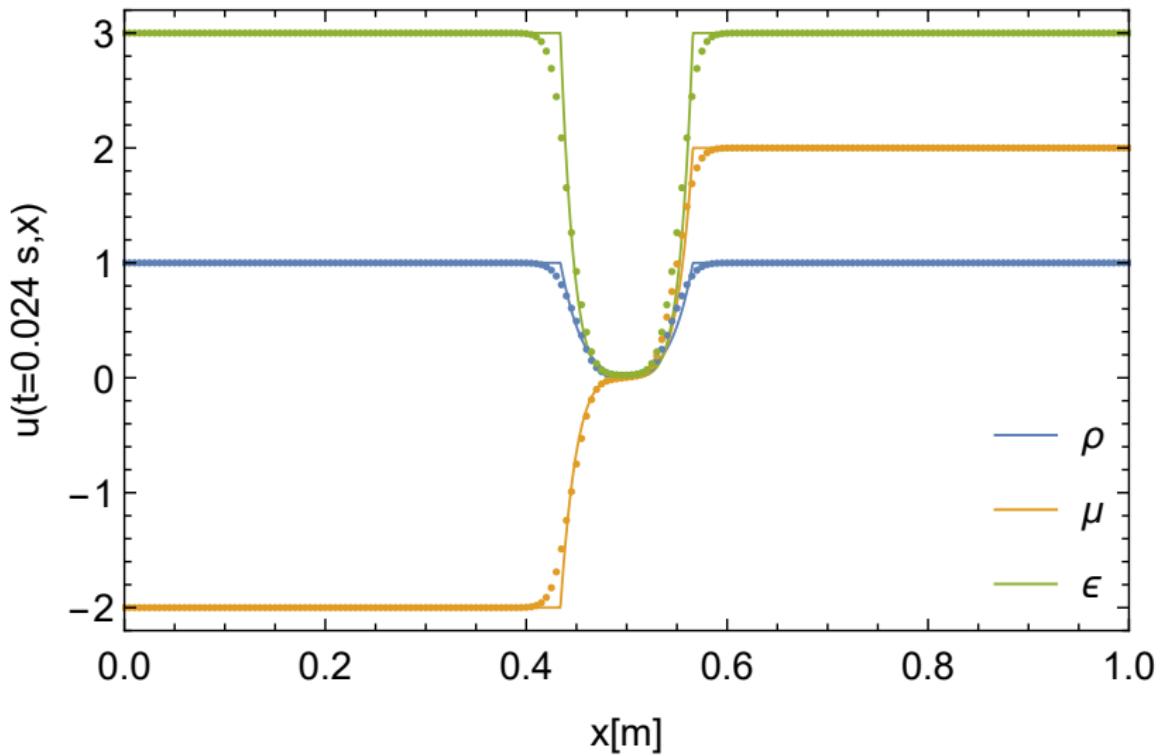


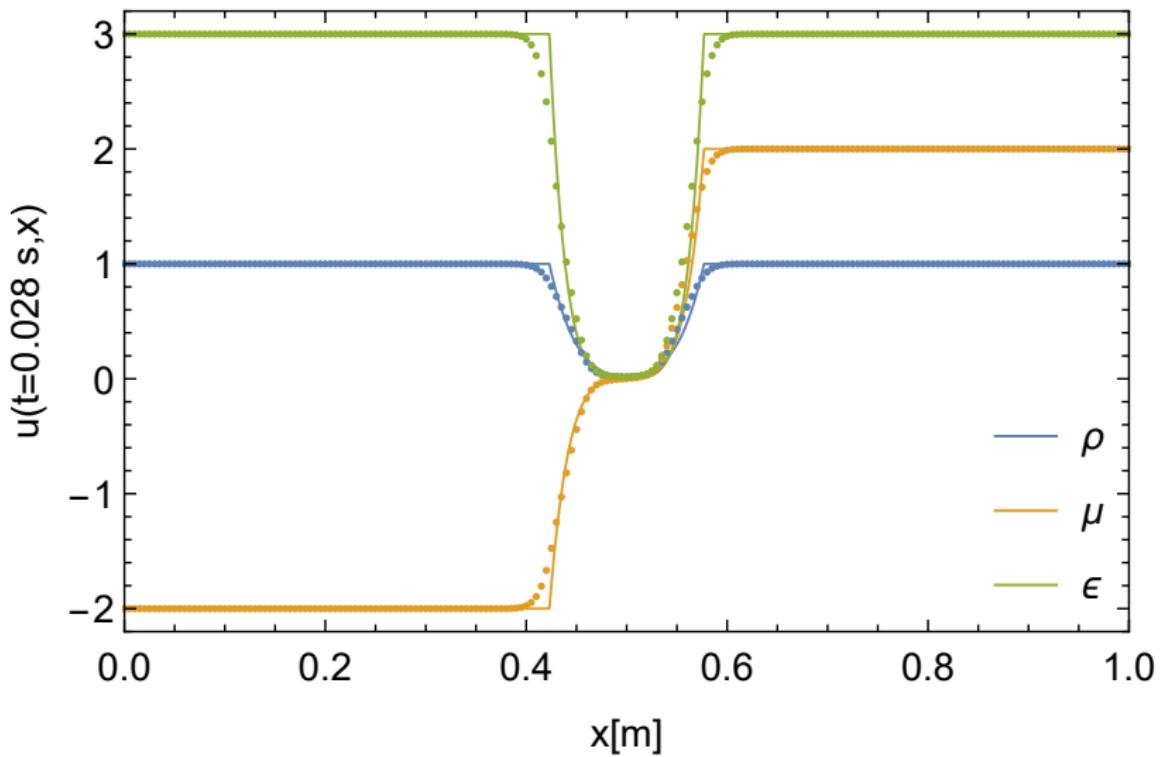


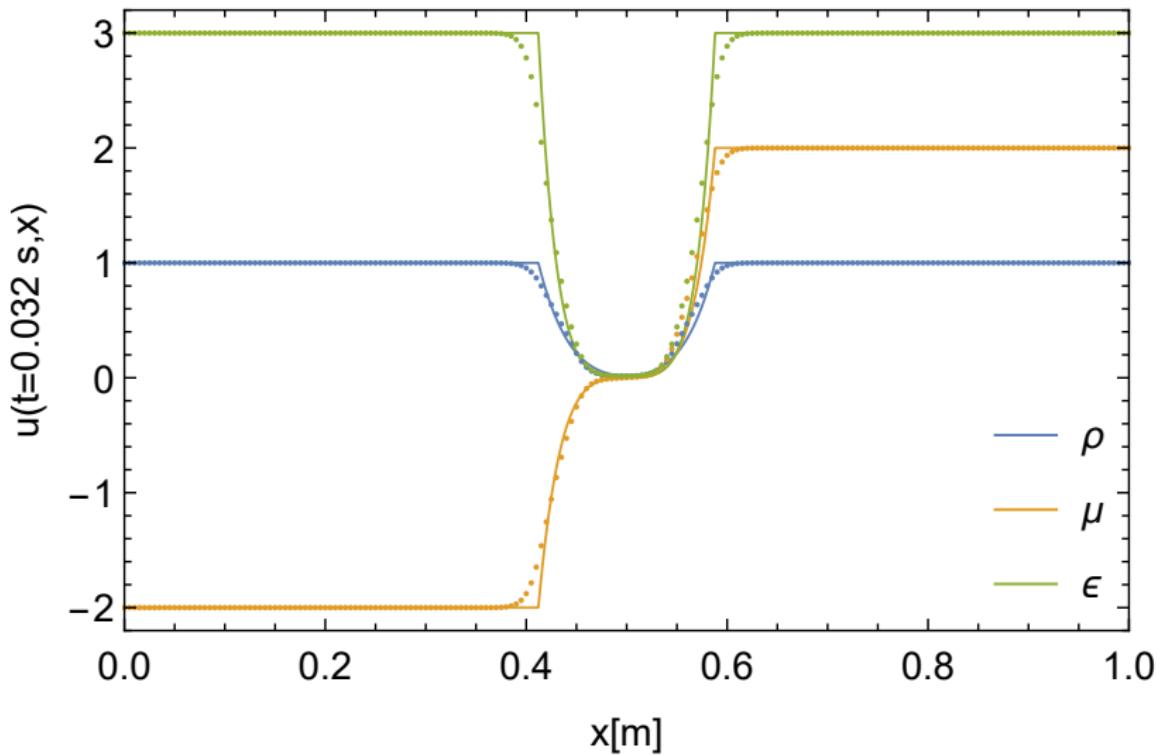


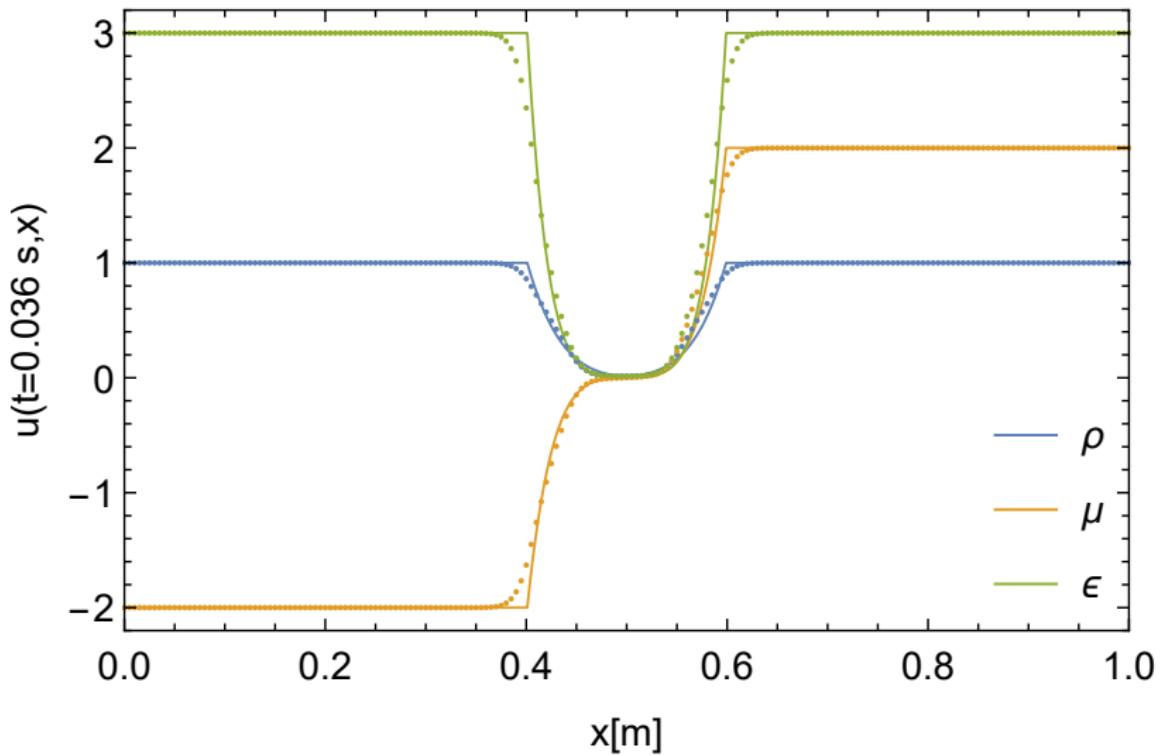


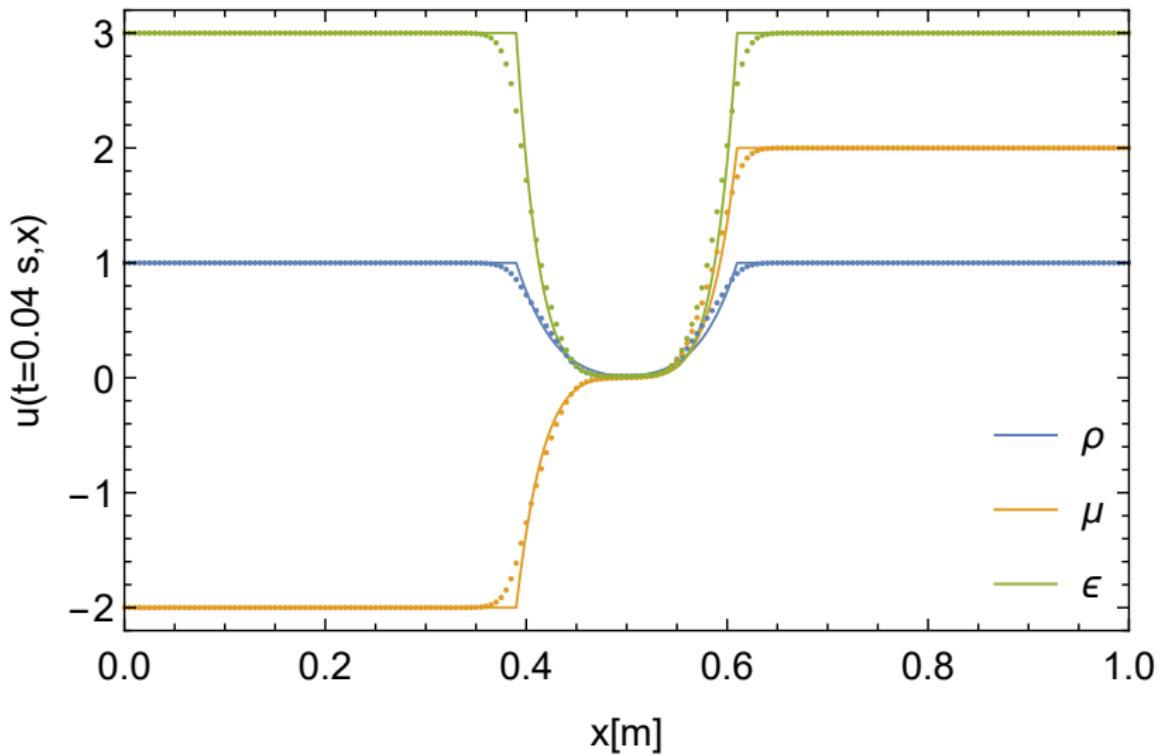


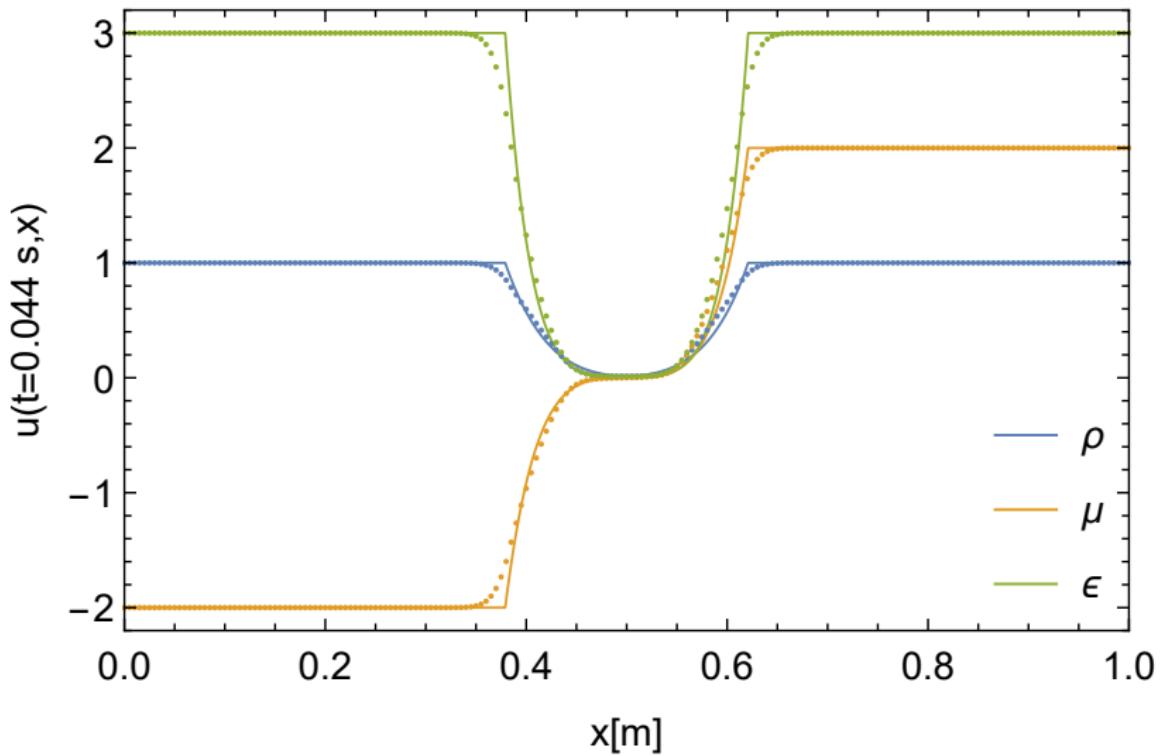


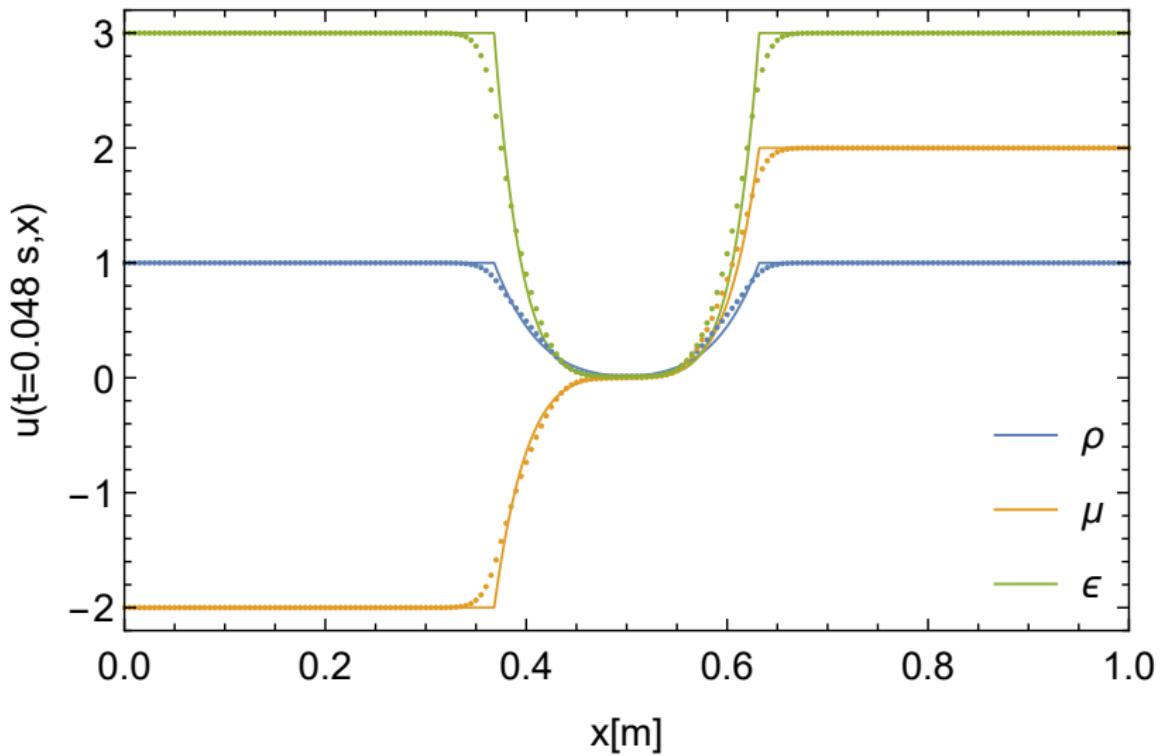


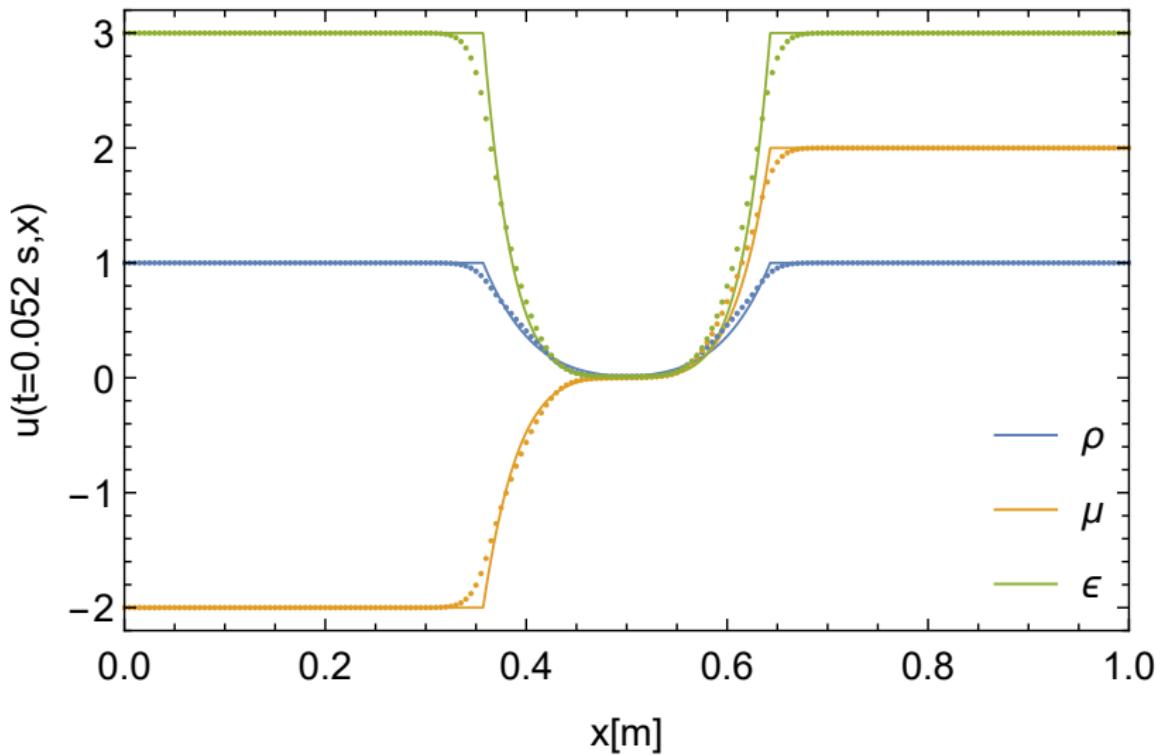


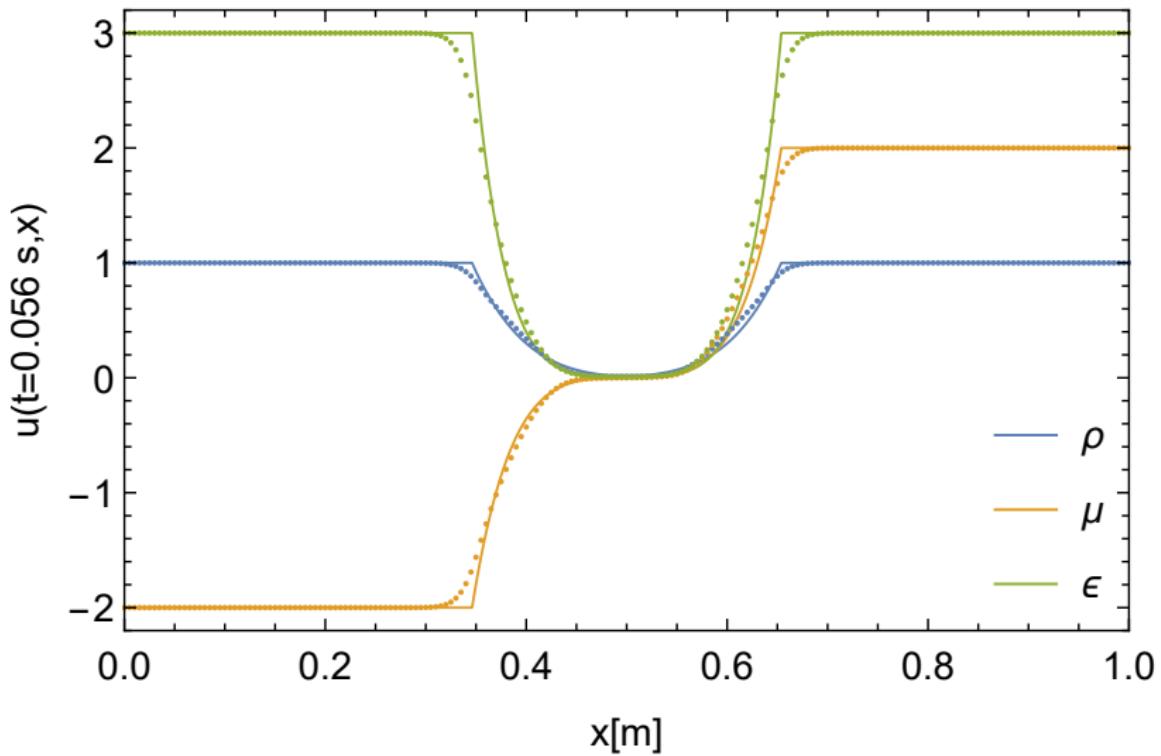


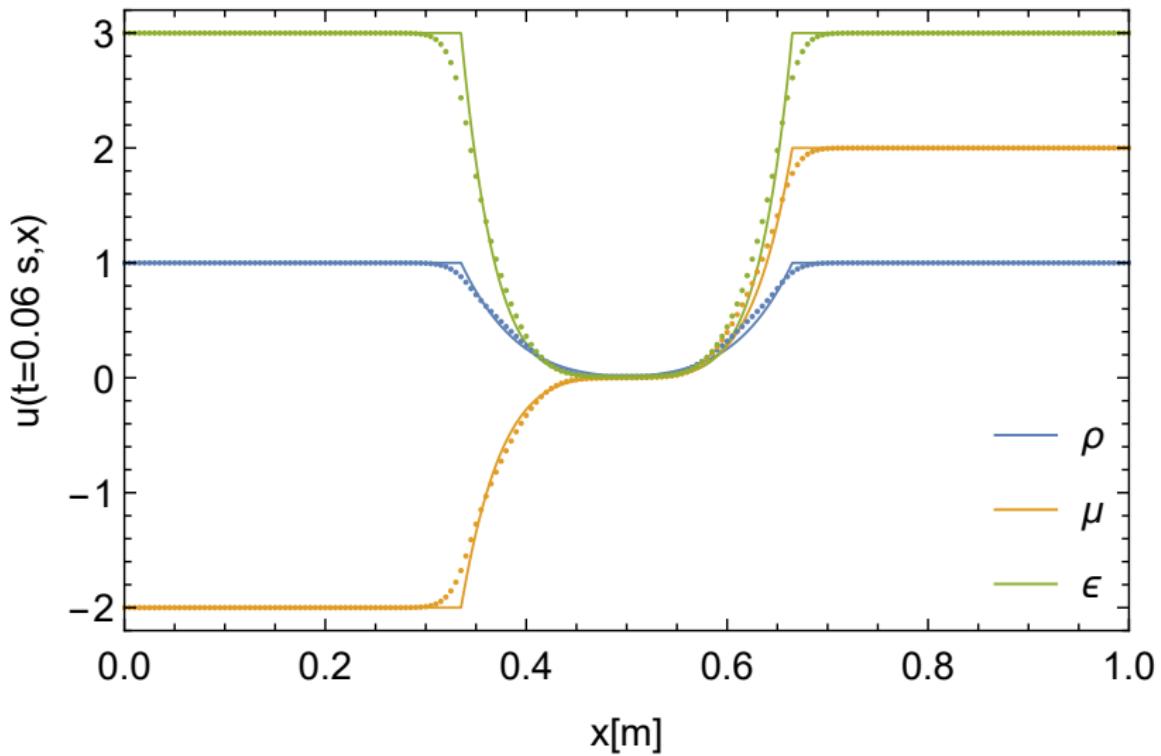


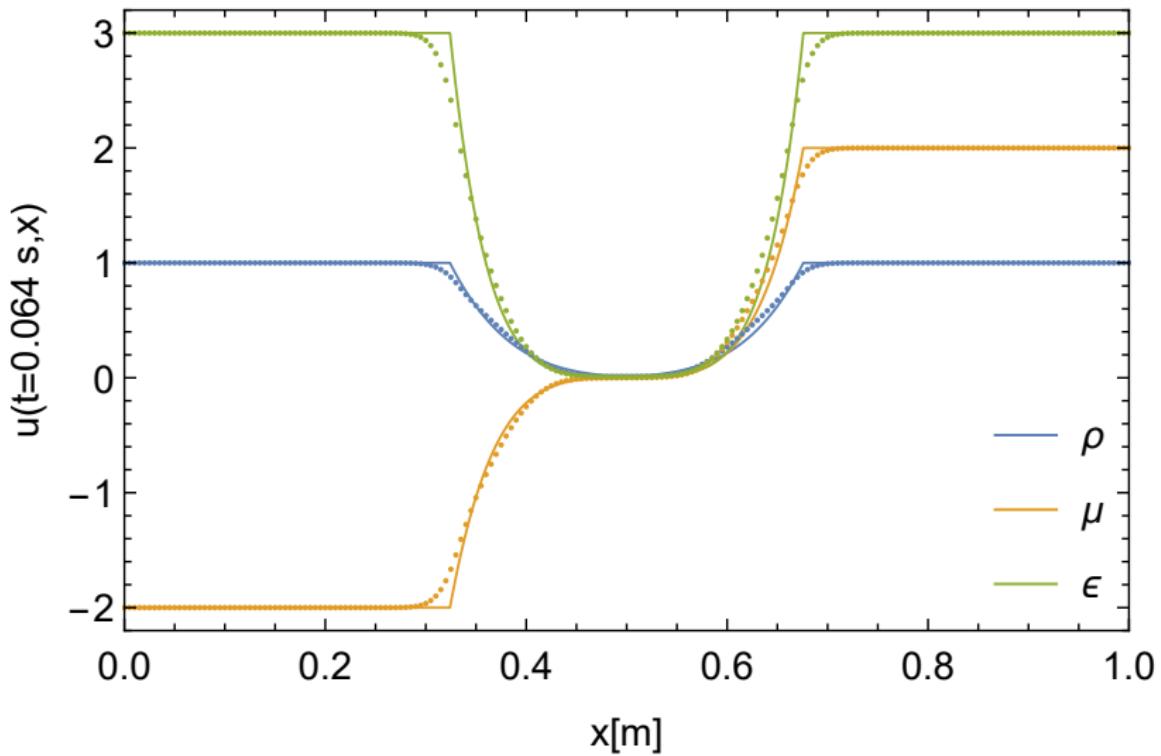


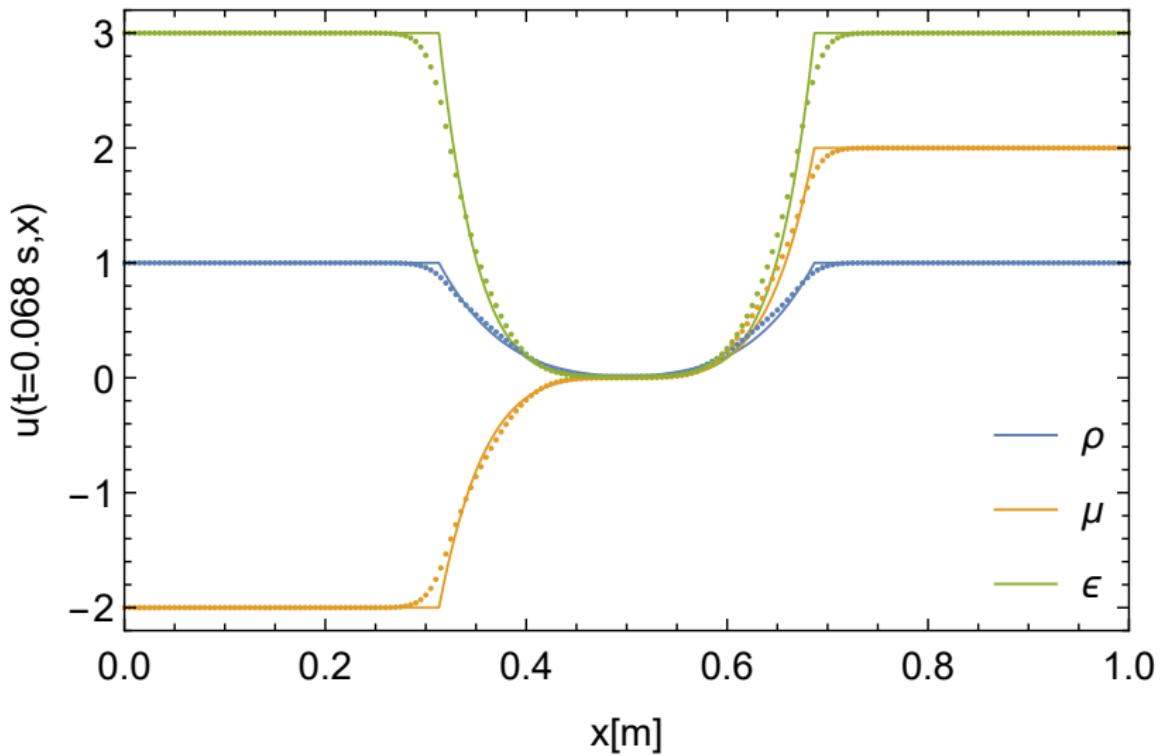


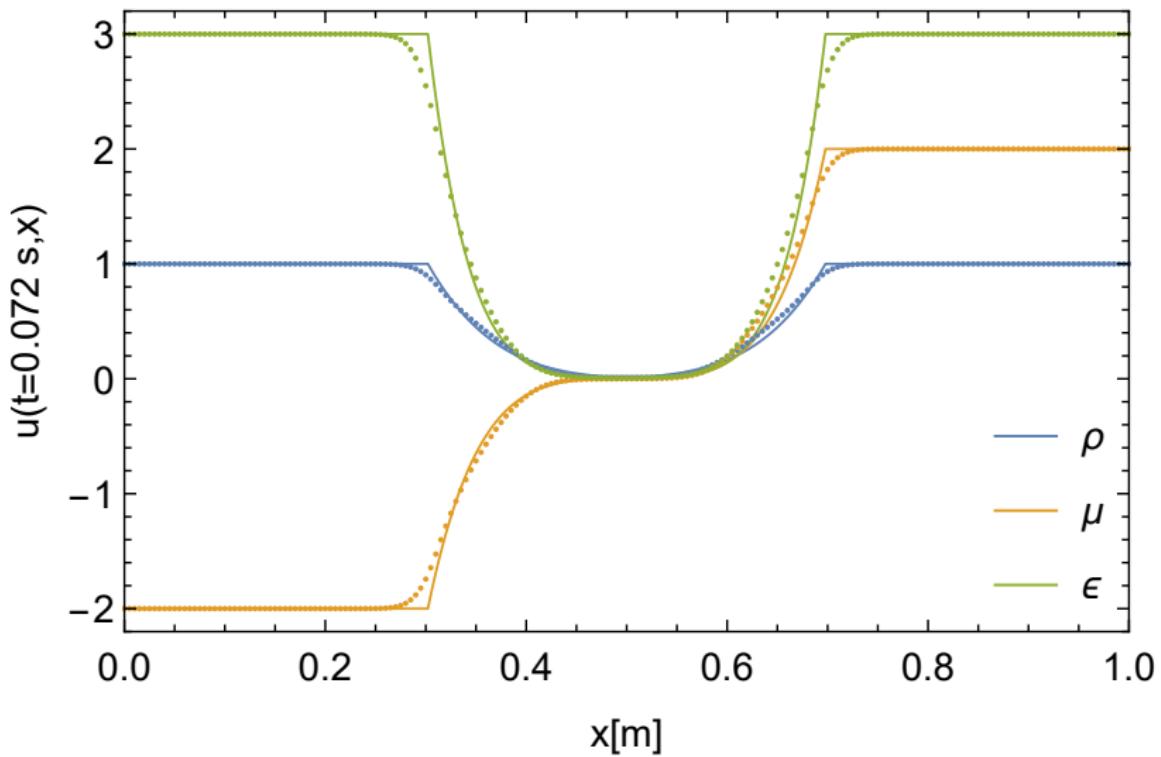


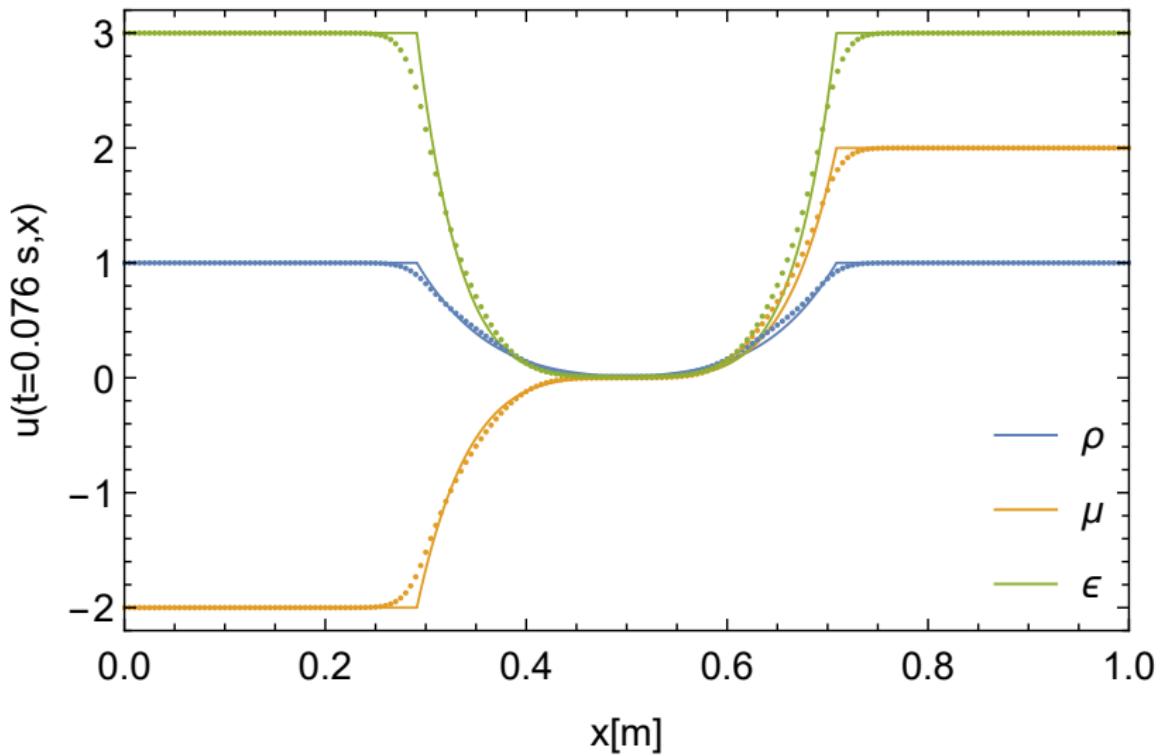


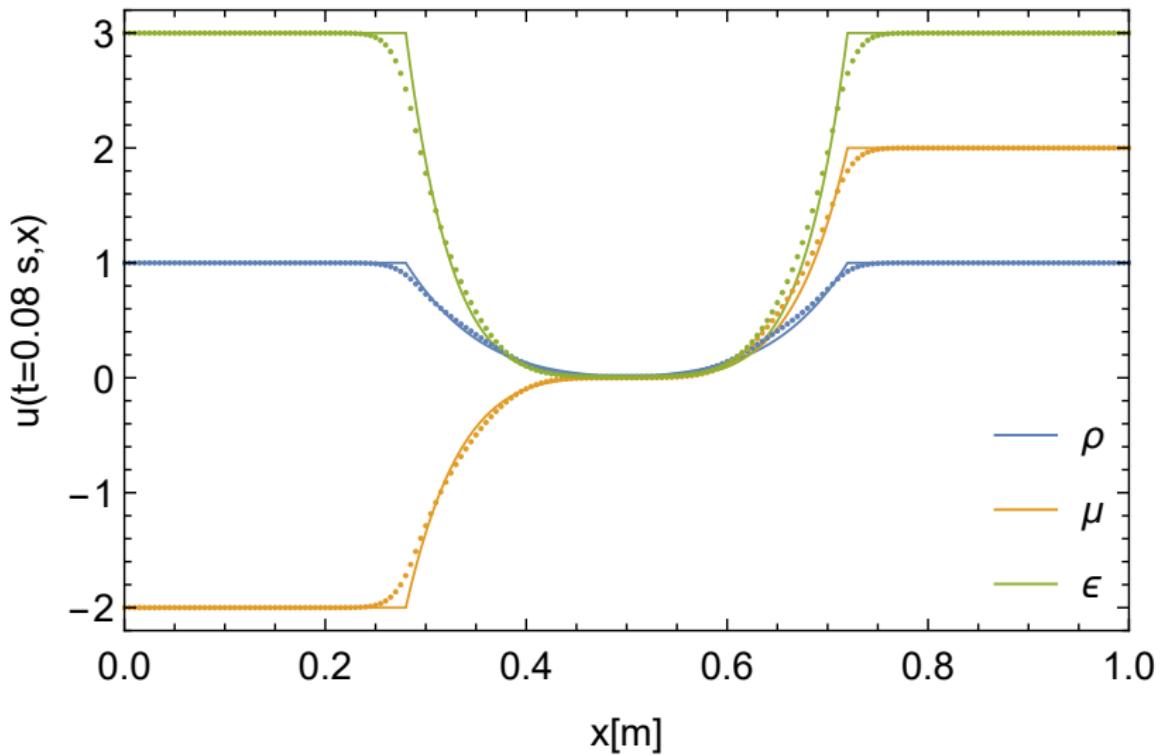


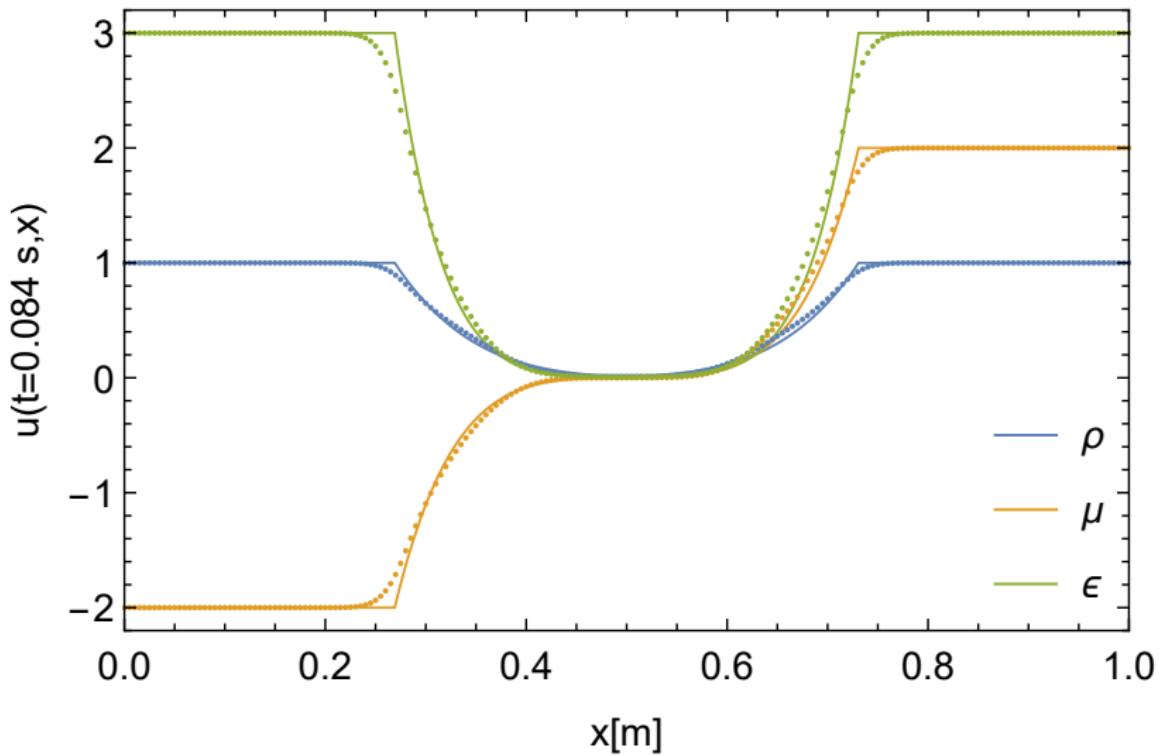


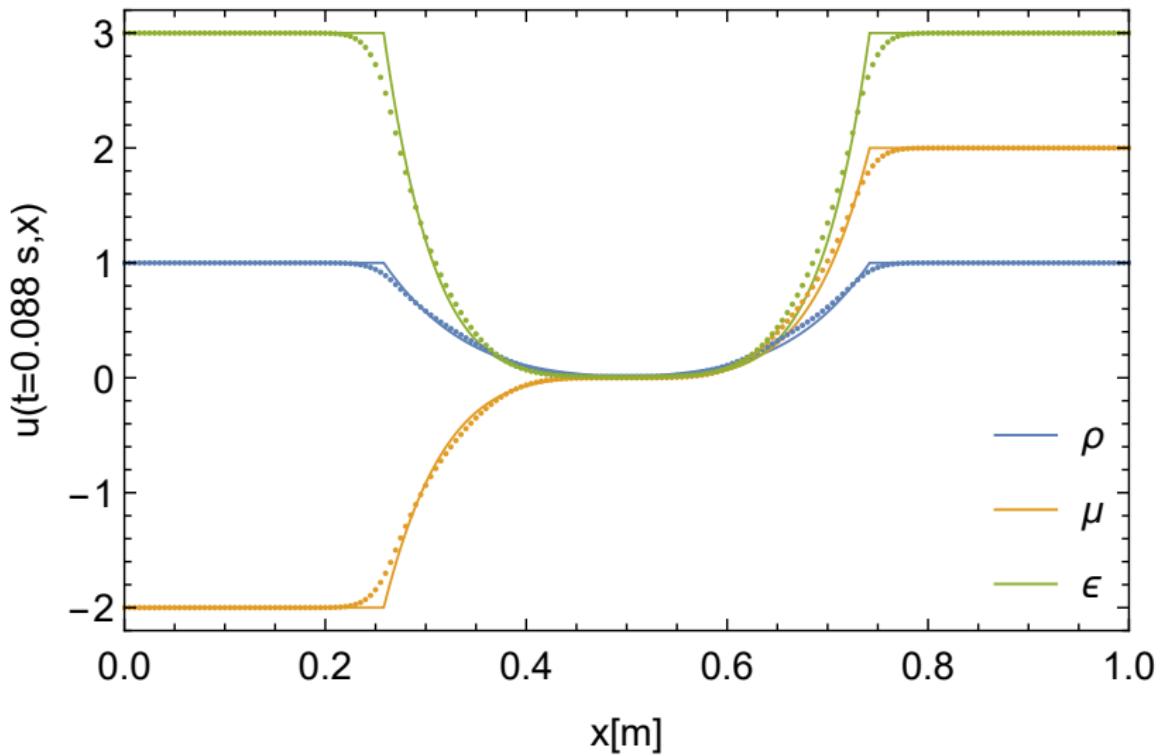


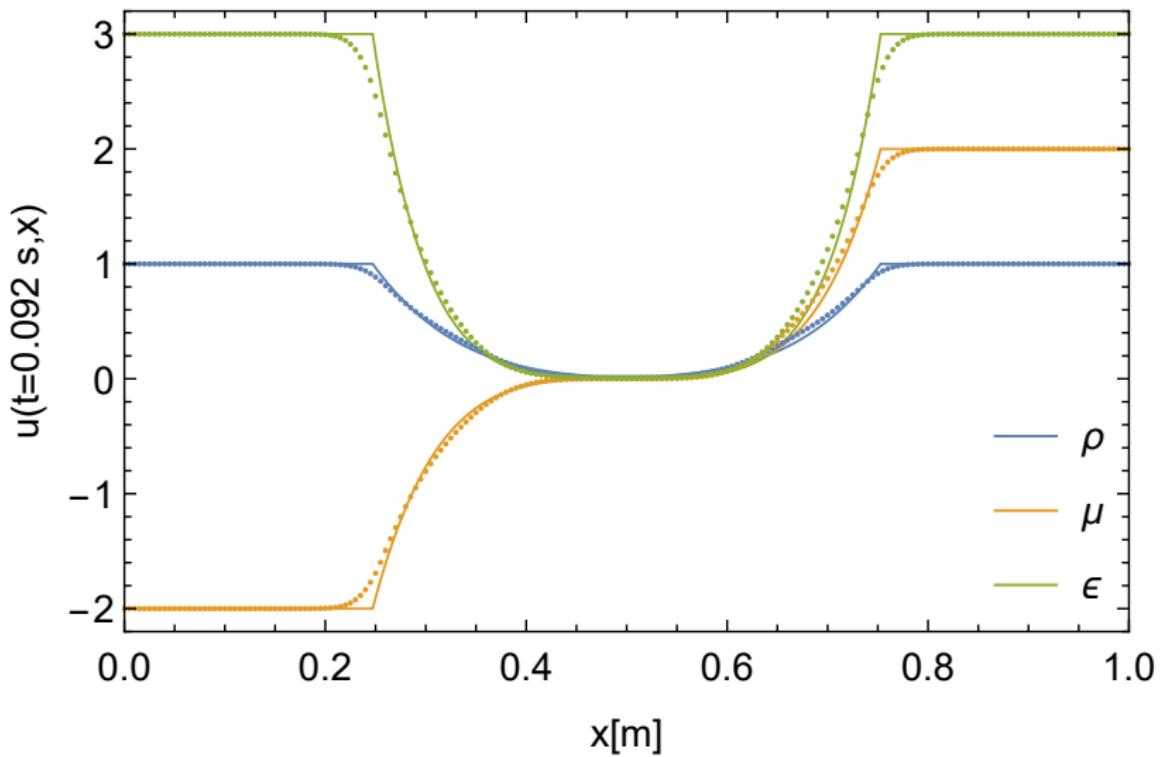


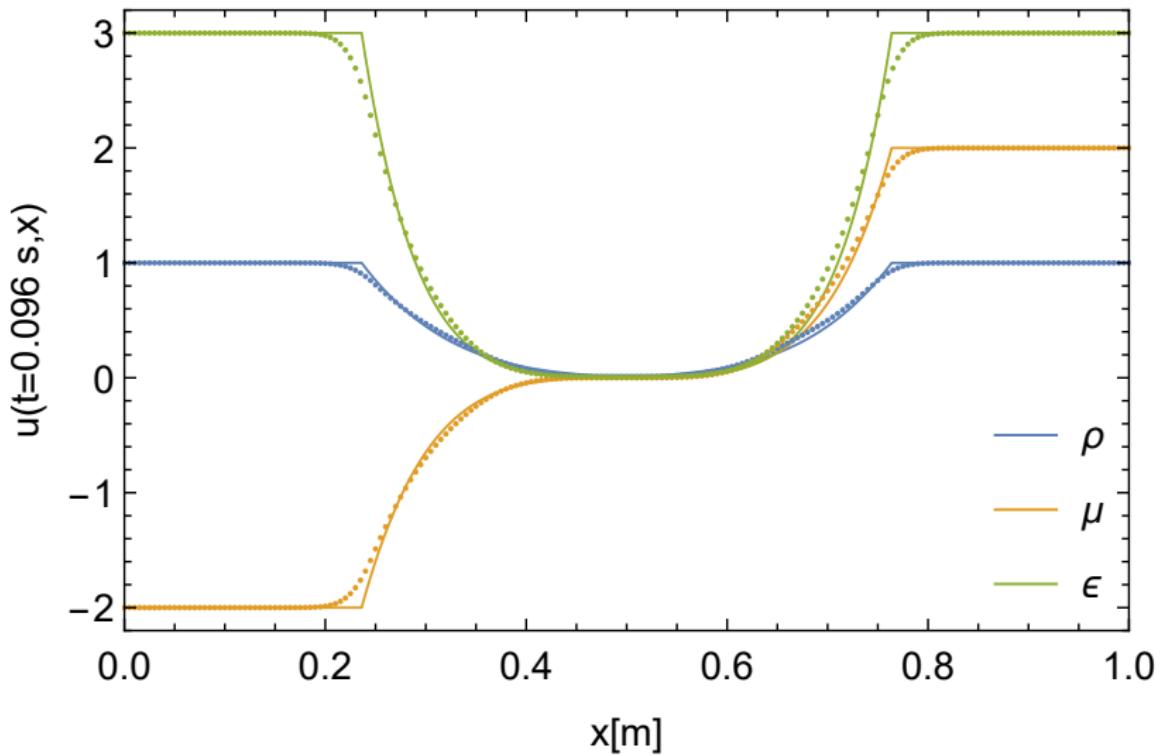


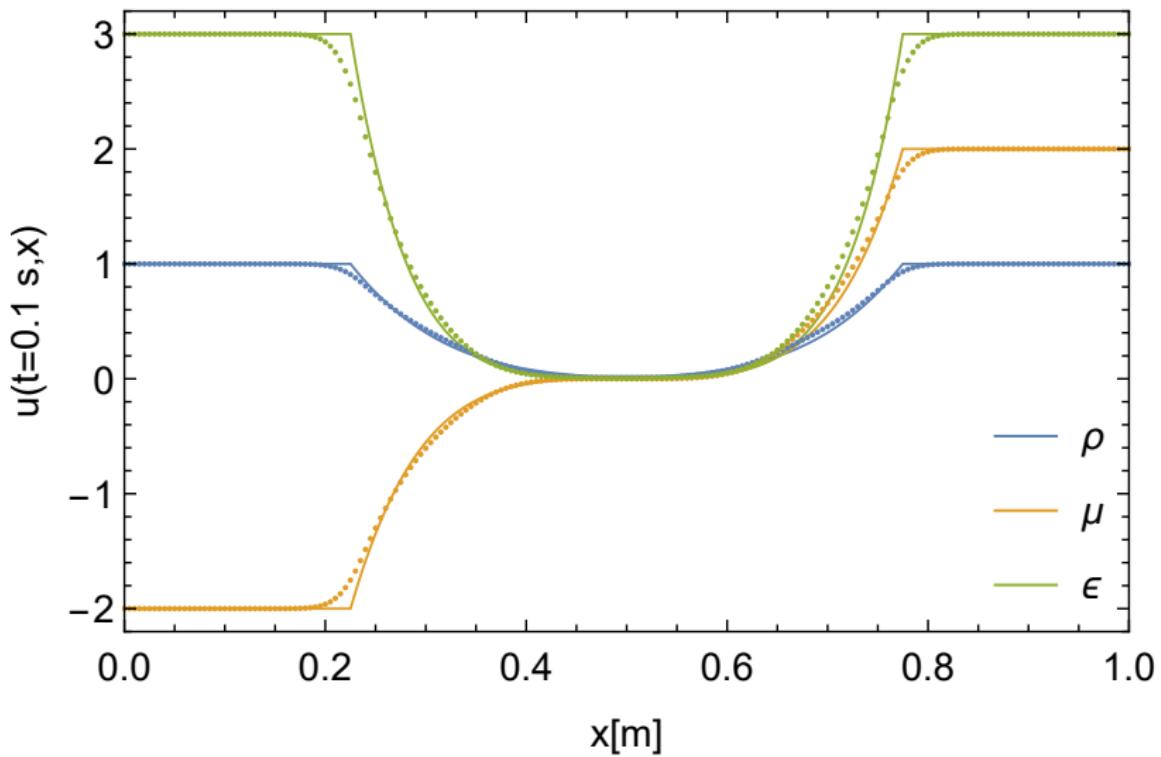


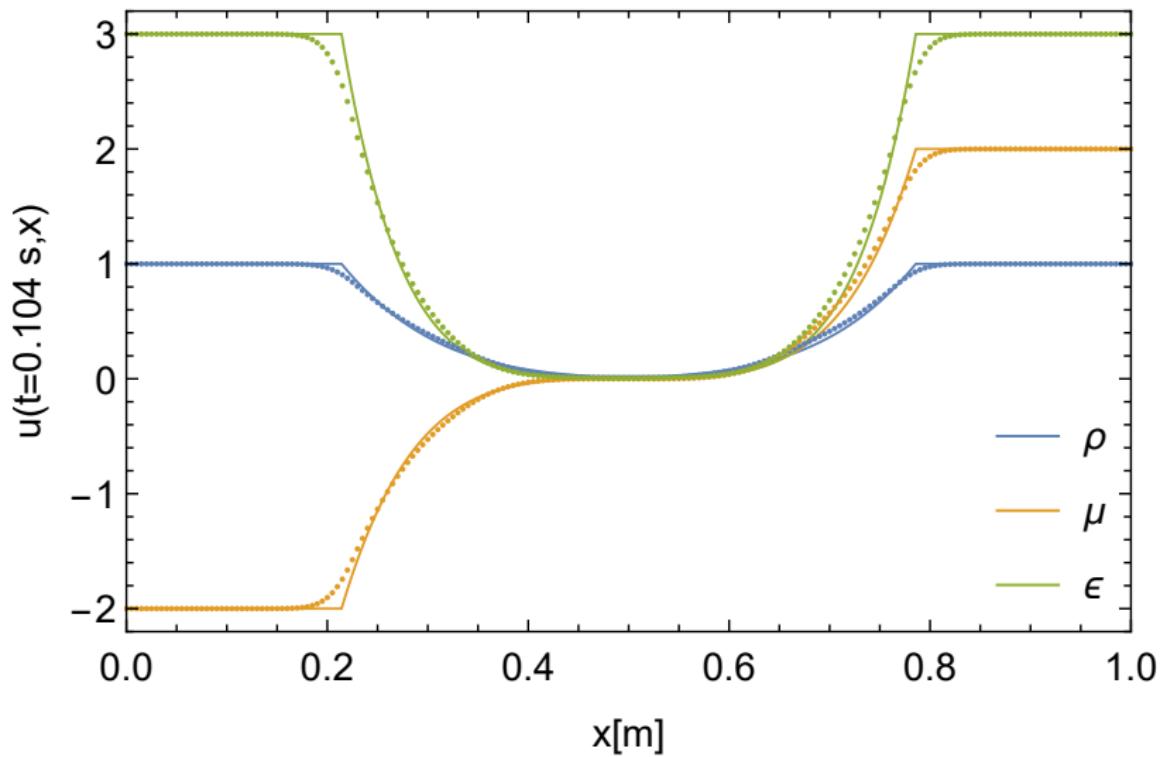


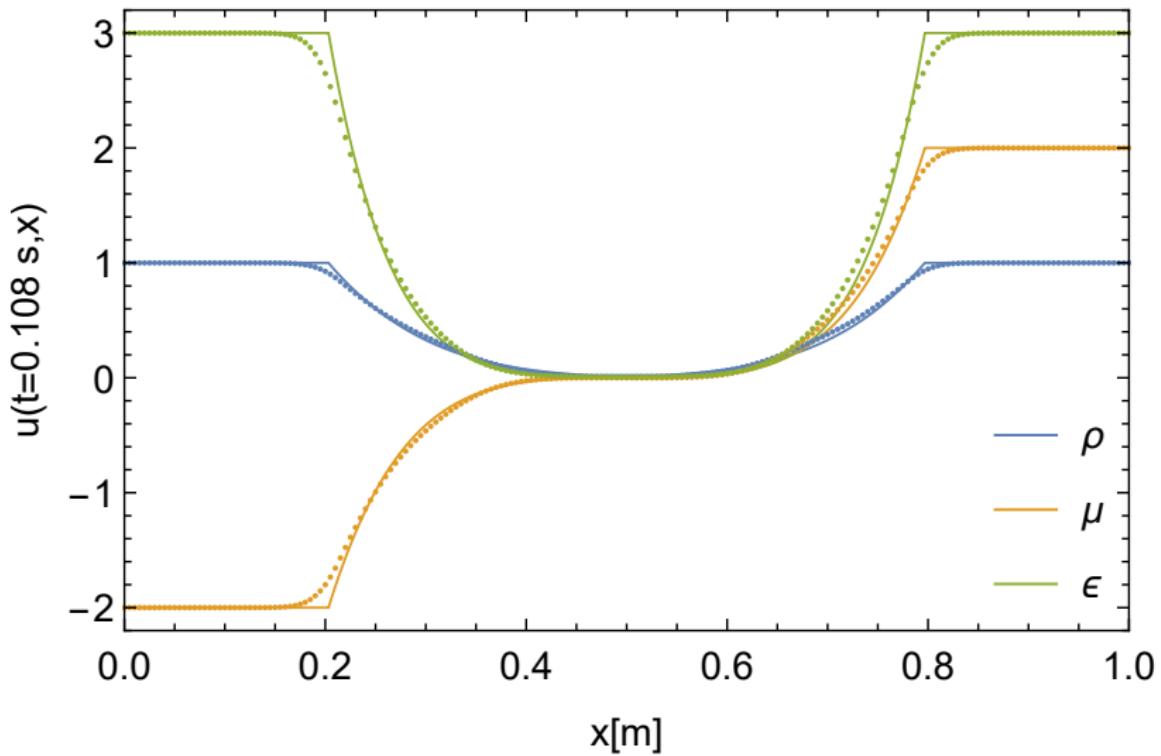


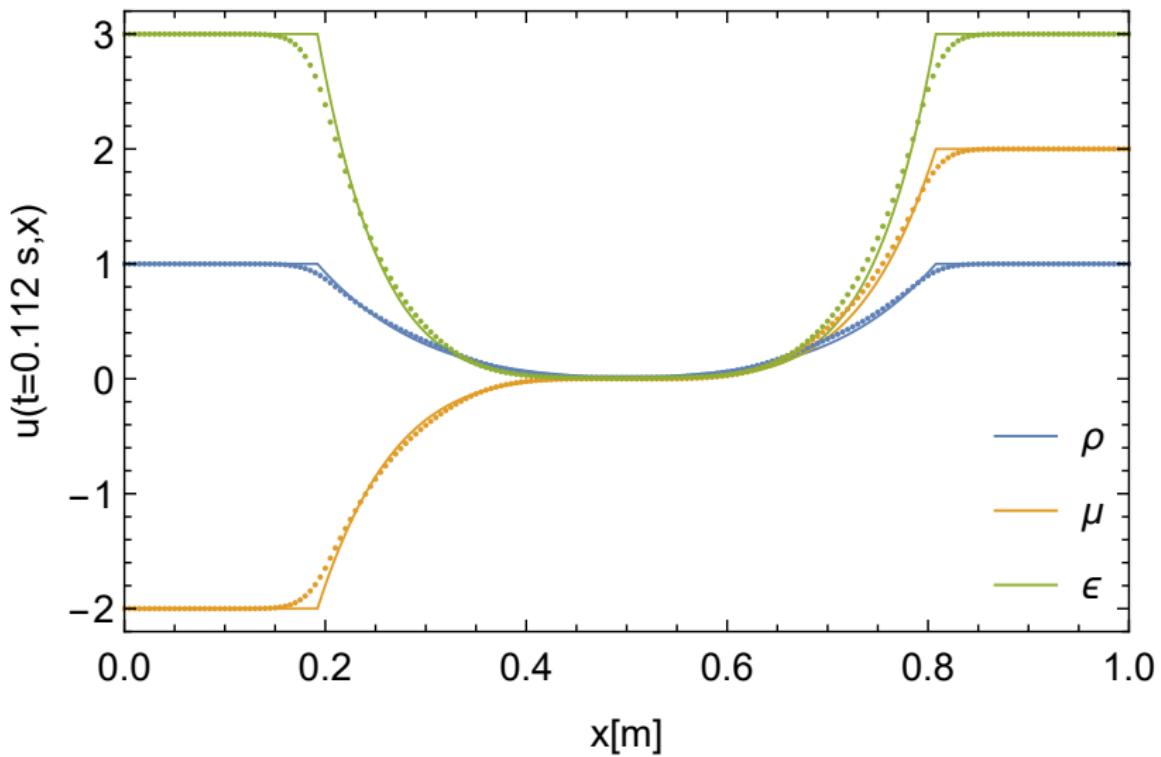


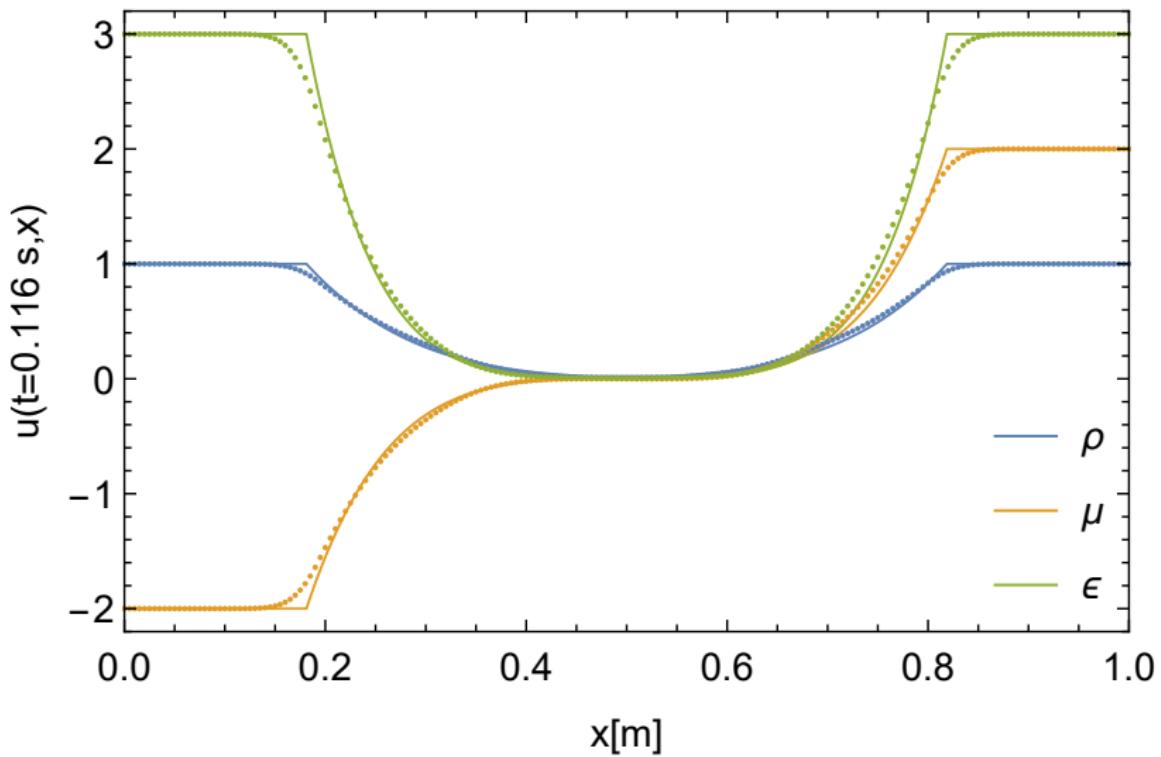


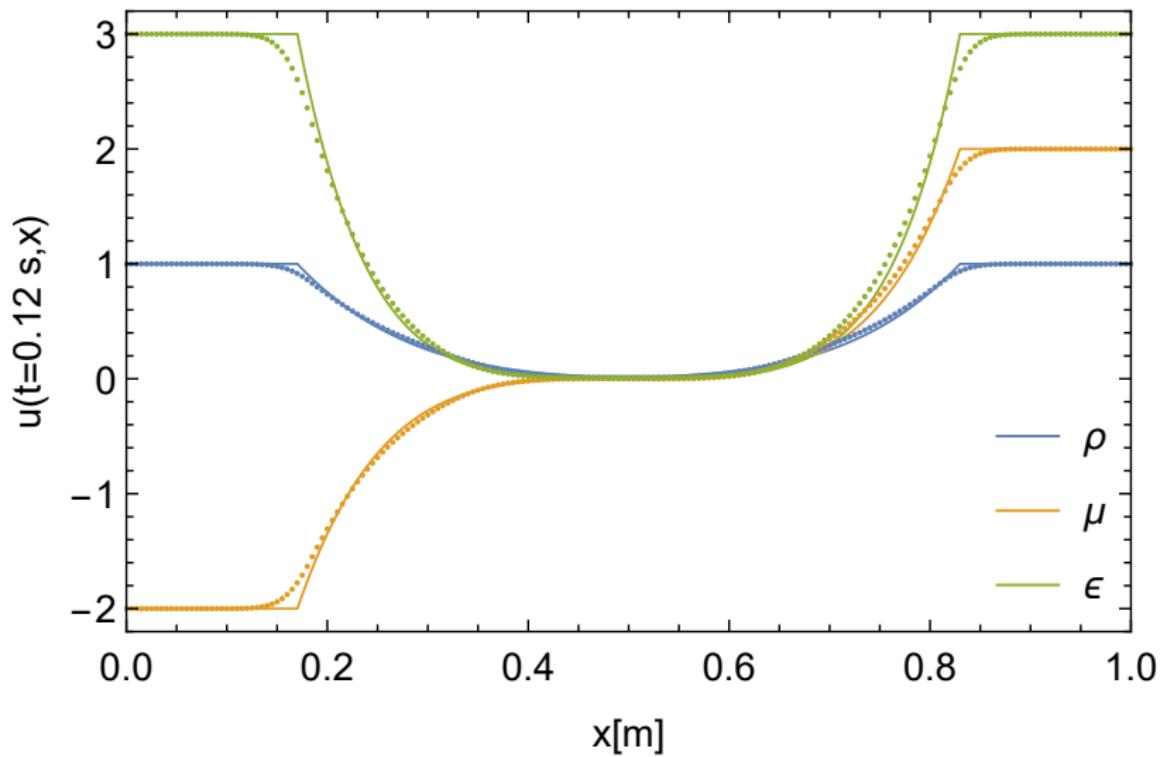


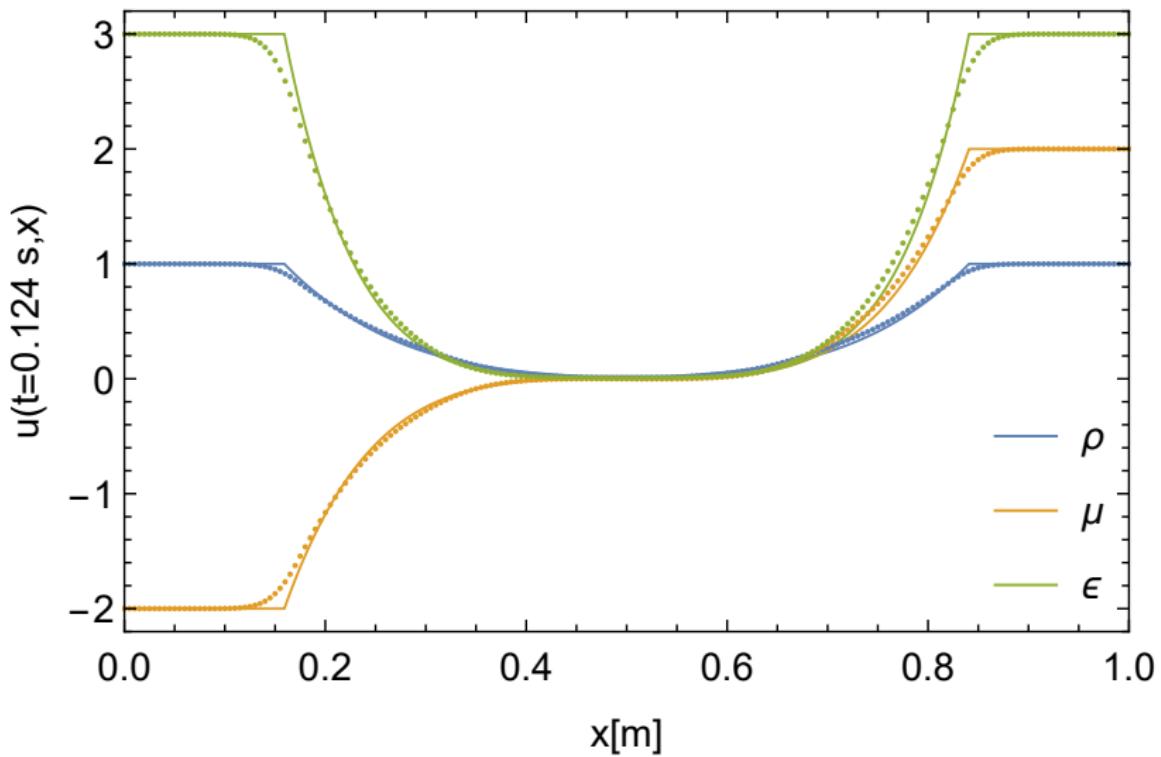


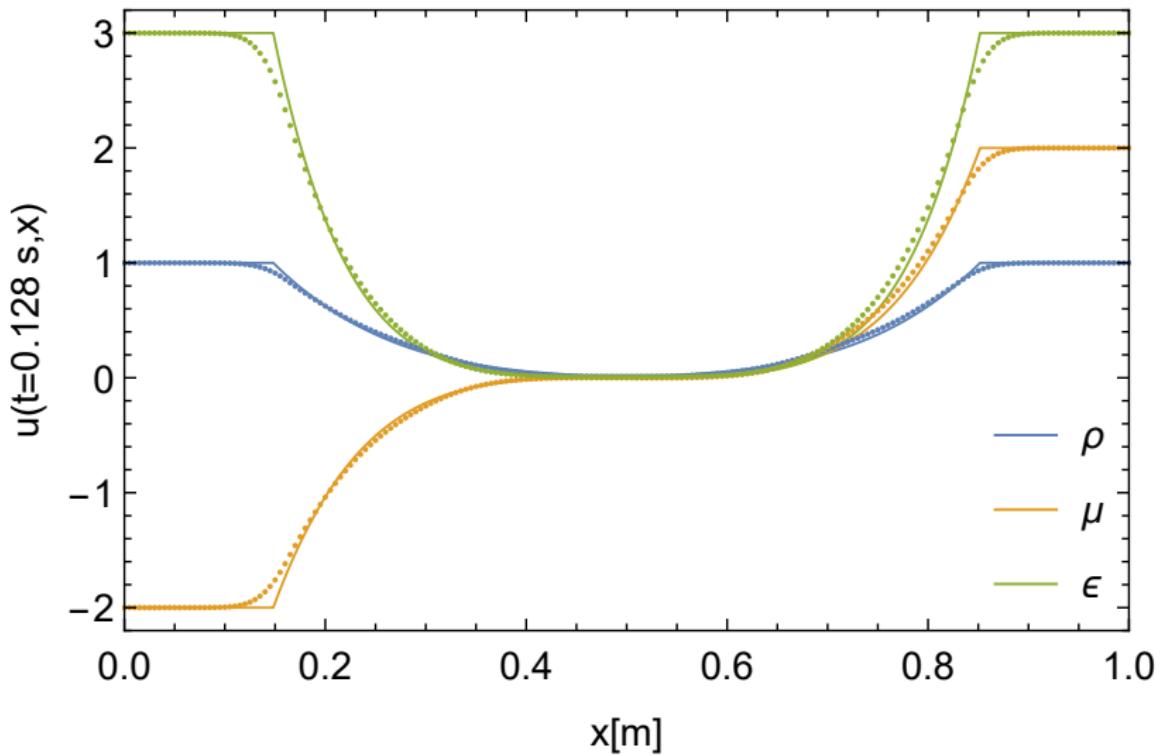


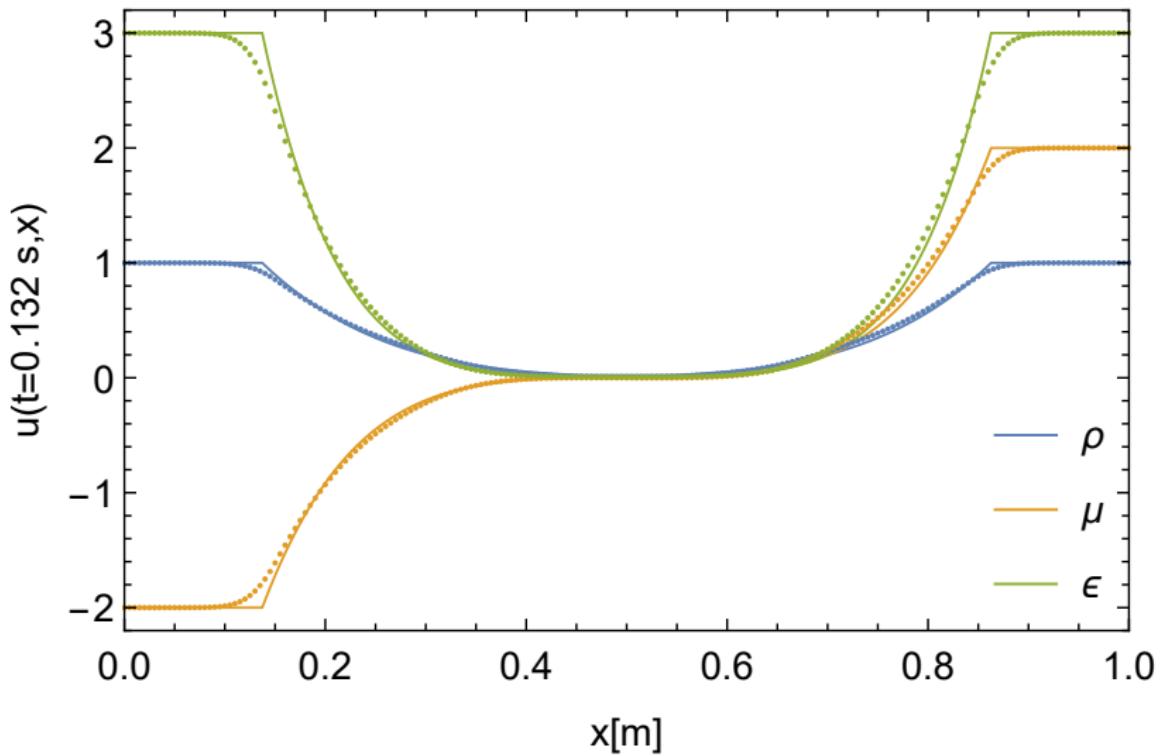


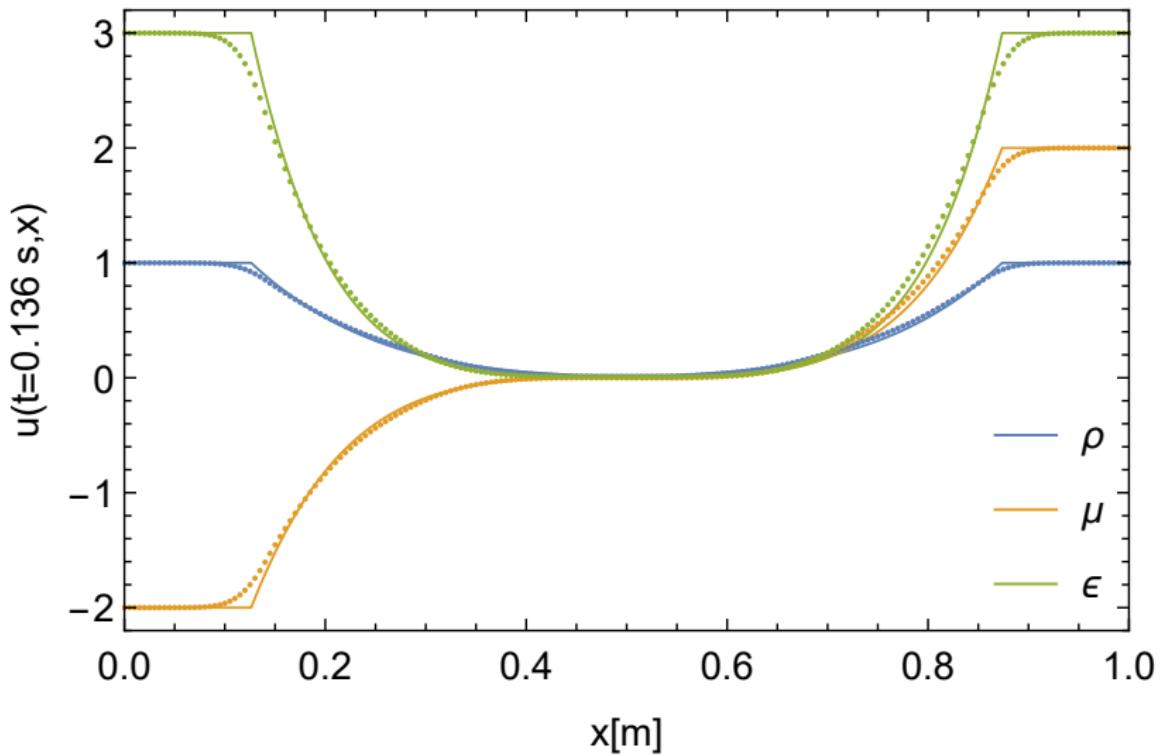


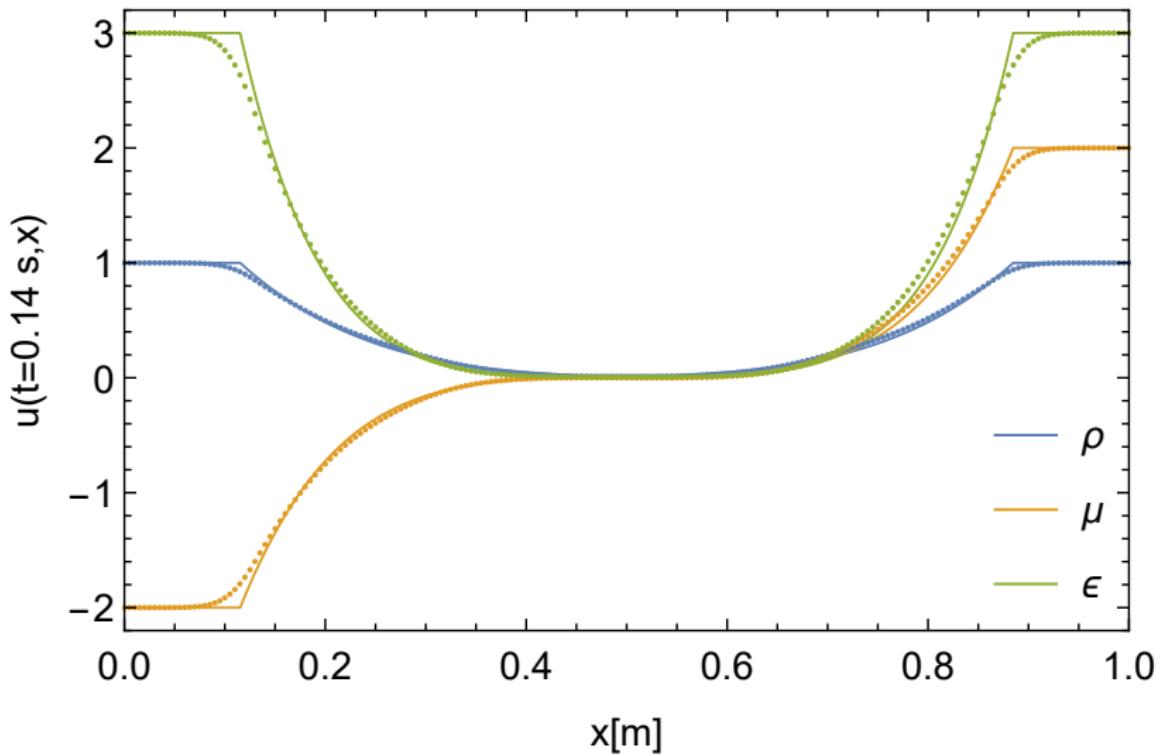


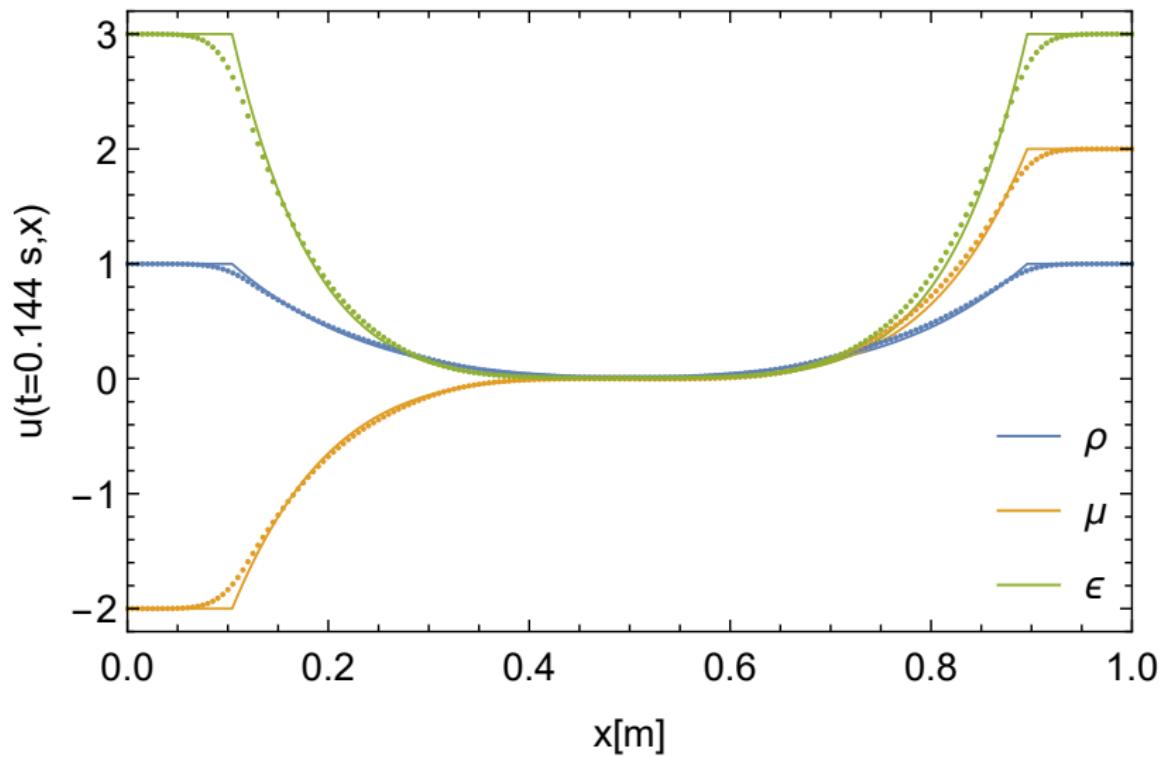


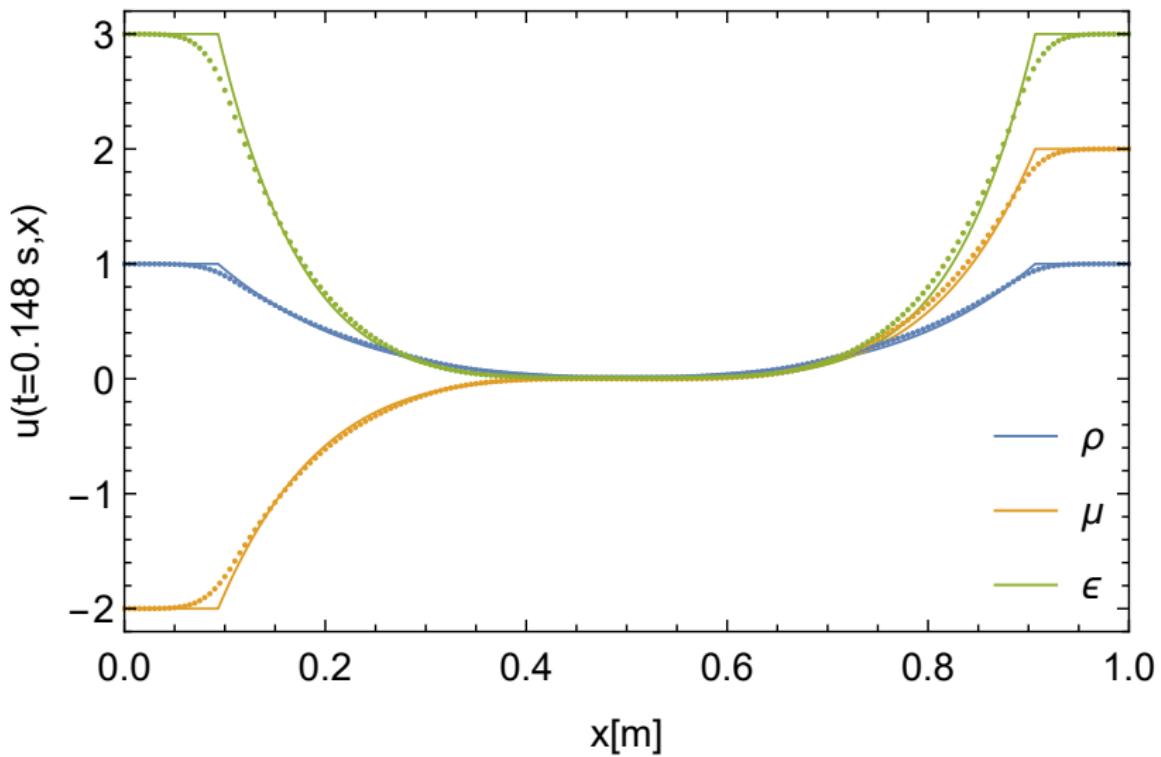


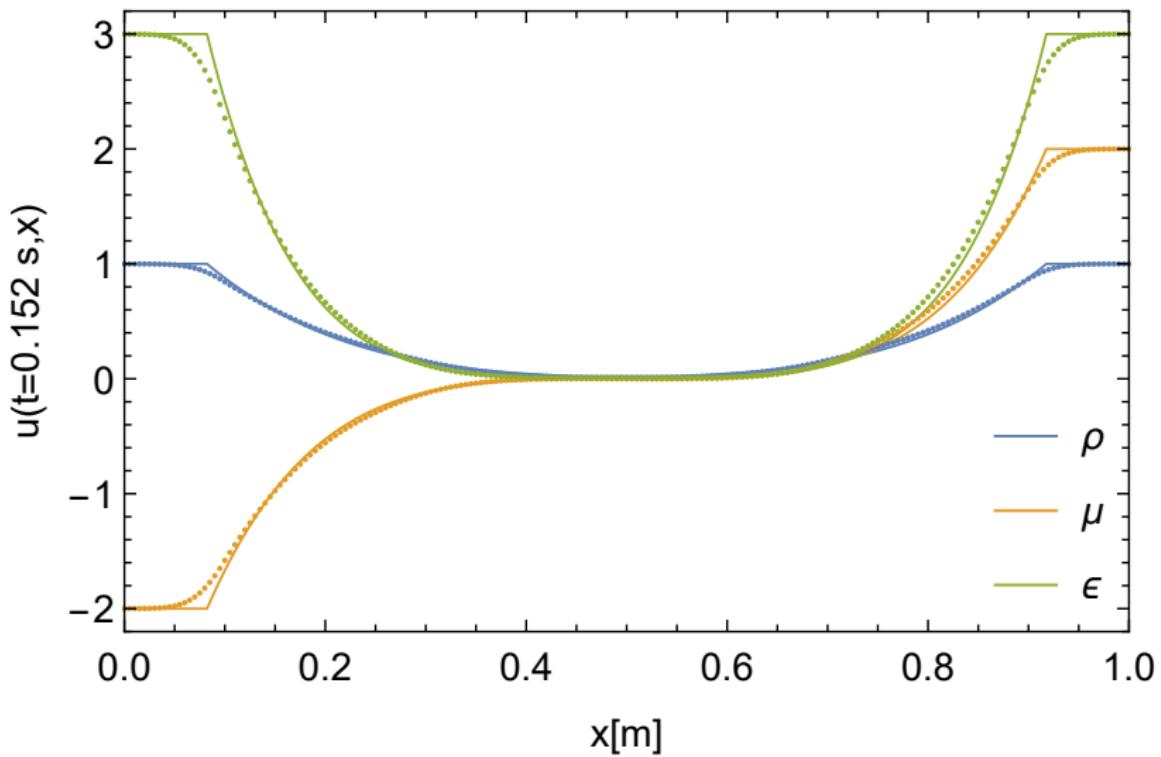


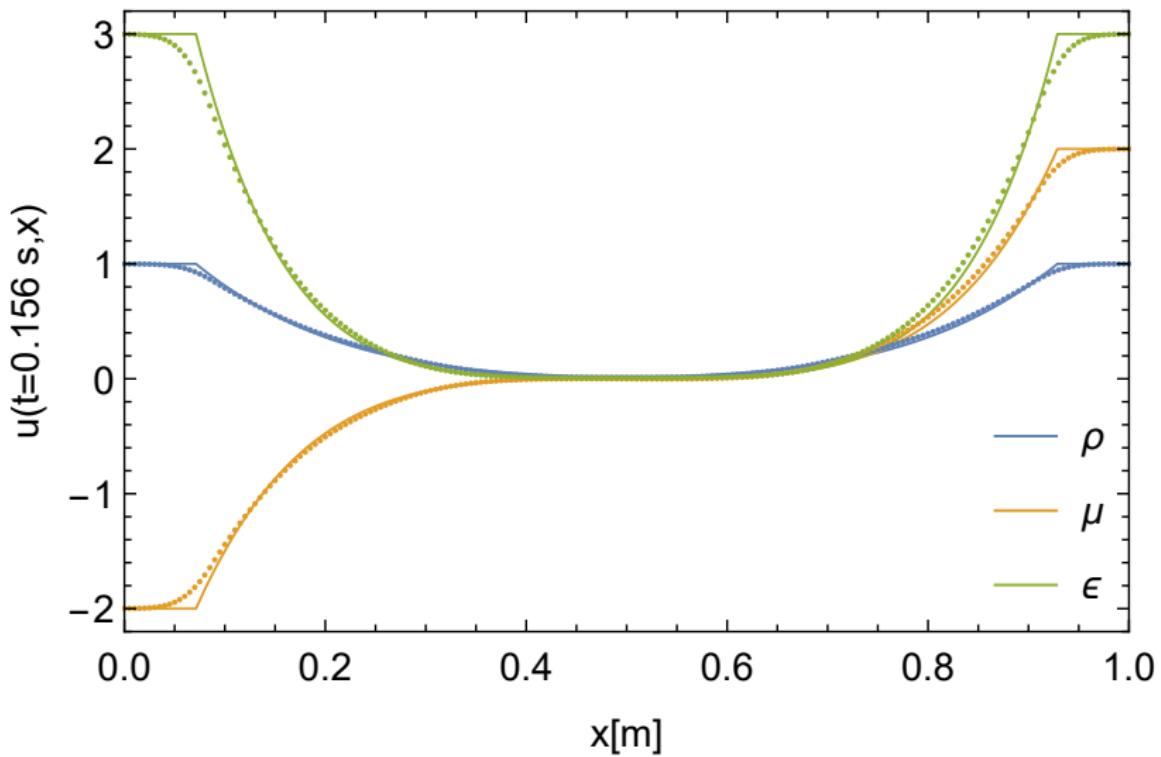


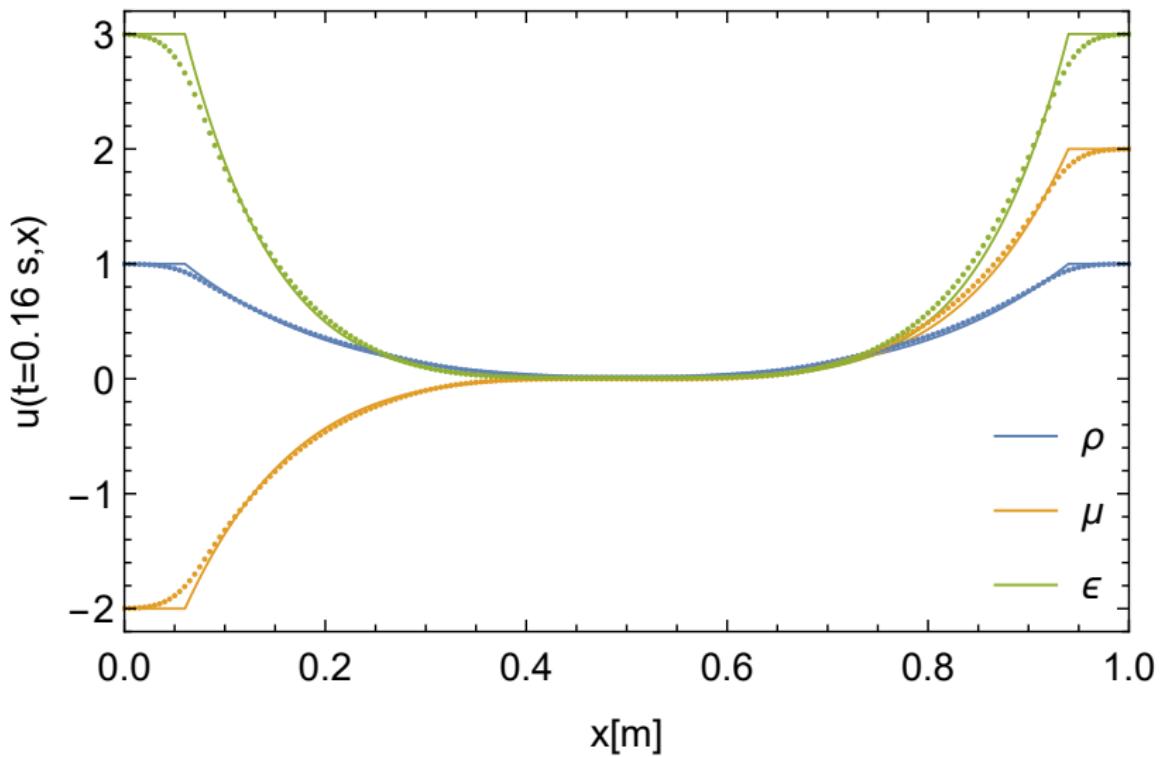












# O(N) model - LPA flow equation $d > 0$

$$\partial_t u(t, x) + c_d \frac{e^{-dt}}{1 + e^{2t} u(t, x)} = -\frac{c_d}{N-1} \frac{e^{-dt}}{1 + e^{2t}(u(t, x) + 2x \partial_x u(t, x))} \quad (12)$$

- ▶ LPA flow equation of the O(N) model in  $d > 0$  dimensions using  $d$ -dim. Litim regulator with

$$t \equiv -\log k/\Lambda \quad (13)$$

$$x \equiv \frac{\rho}{(N-1)\Lambda^{d-2}} \quad (14)$$

$$u(t, x) \equiv \frac{\partial U(t, \rho)}{\Lambda^2 \partial \rho} \quad (15)$$

$$c_d \equiv \frac{2^{-d} \pi^{-d/2}}{\Gamma(1+d/2)} \quad (16)$$

# $O(\infty)$ model - $d = 3$ initial conditions

- ▶ Riemann problem (restored phase for  $O(\infty)$ ,  $d=3$ )

$$u(t=0, x) = \begin{cases} 0.0 & 0.02 < x < 0.05 \\ 0.1 & \text{otherwise} \end{cases} \quad (\text{GW.2a})$$

- ▶ Quartic potential (restored phase for  $O(\infty)$ ,  $d=3$ )

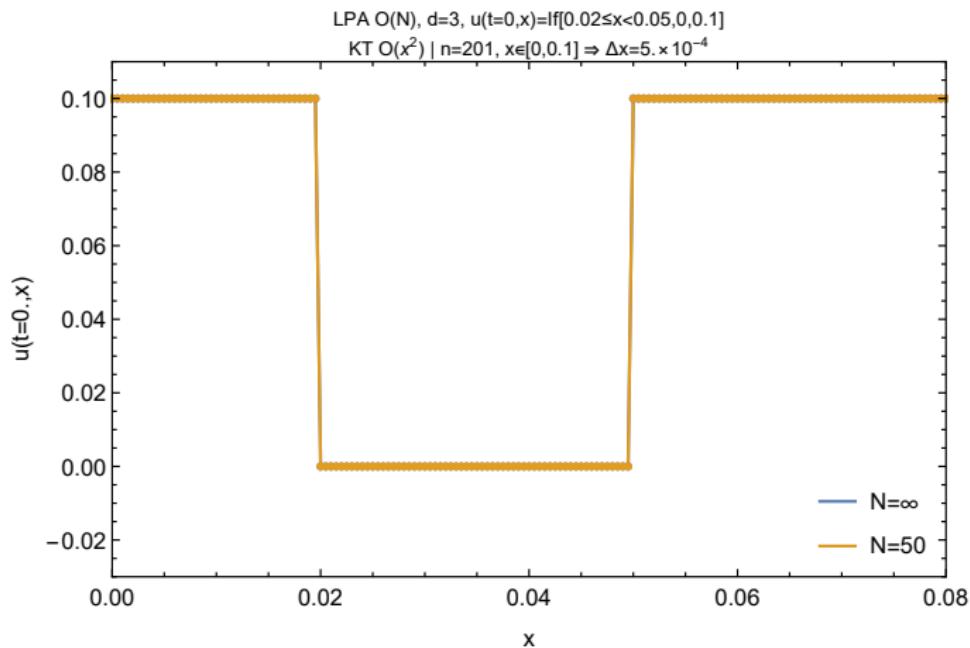
$$u(t=0, x) = 1.0x - 0.1 \quad (\text{GW.3a})$$

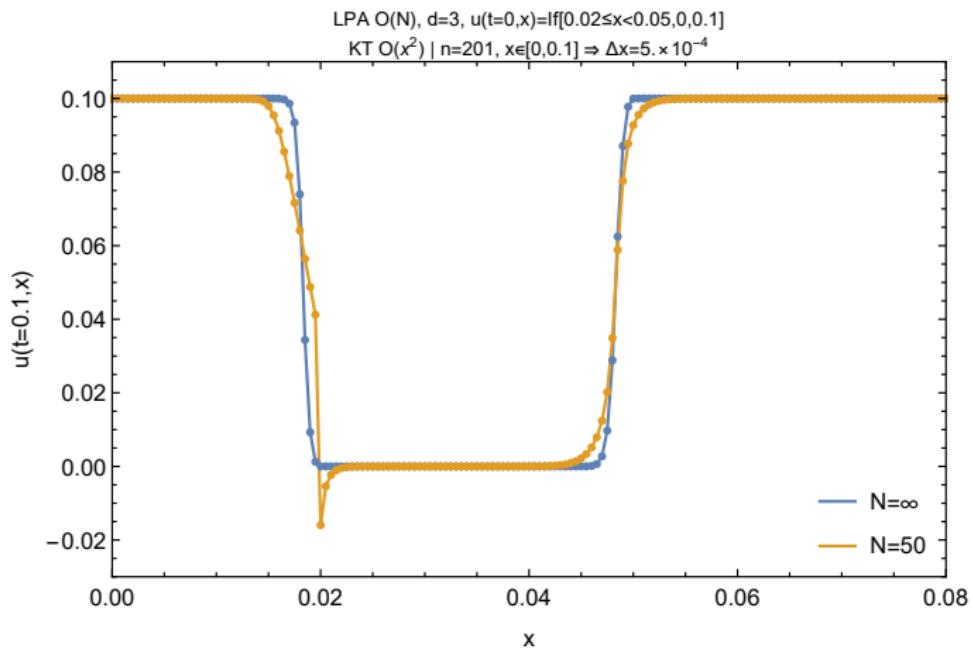
- ▶ Sextic potential (restored phase for  $O(\infty)$ ,  $d=3$ )

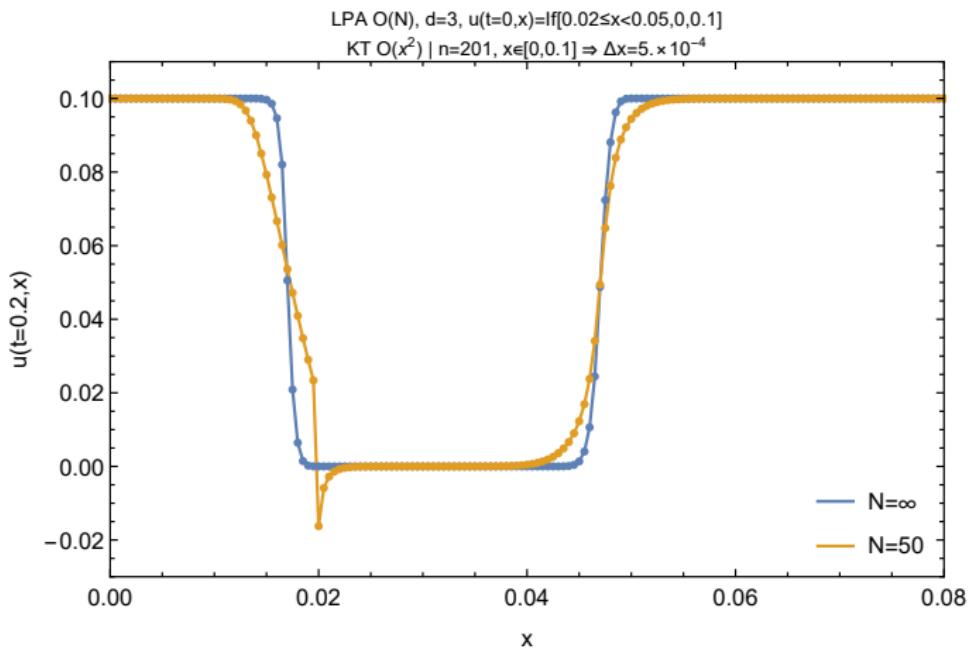
$$u(t=0, x) = 1.0x^2 - 0.103x + 0.0024 \quad (\text{GW.8c})$$

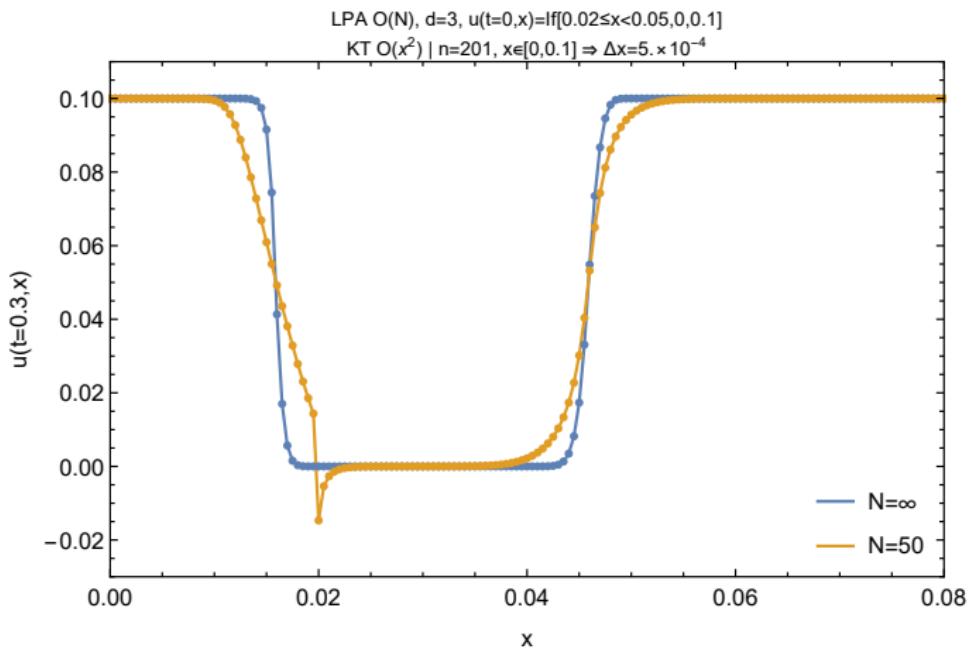
- ▶ Sextic potential (broken phase for  $O(\infty)$ ,  $d=3$ )

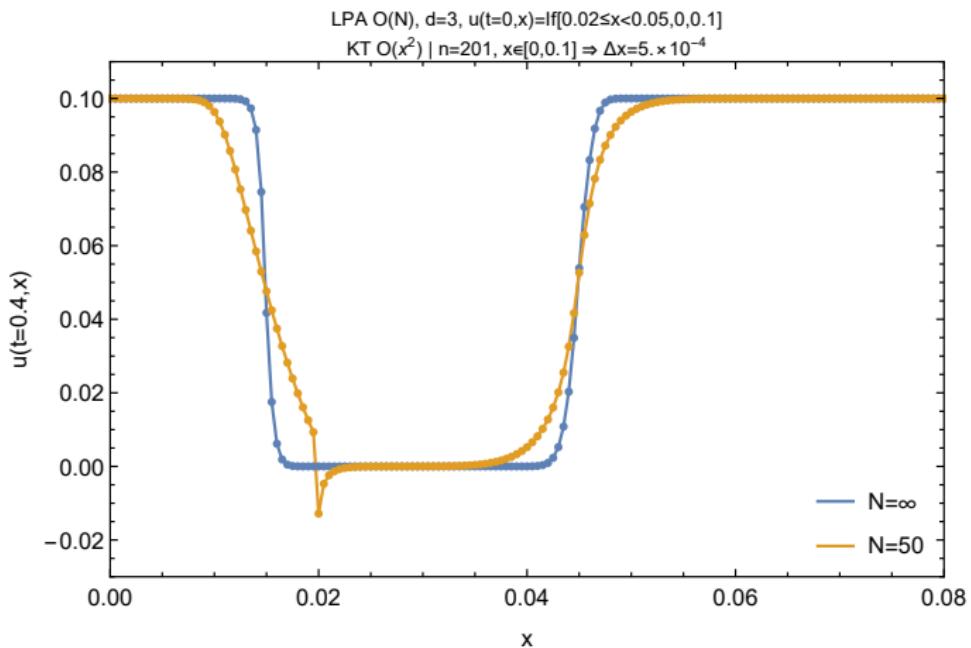
$$u(t=0, x) = 1.0x^2 - 0.105x + 0.0024 \quad (\text{GW.8d})$$

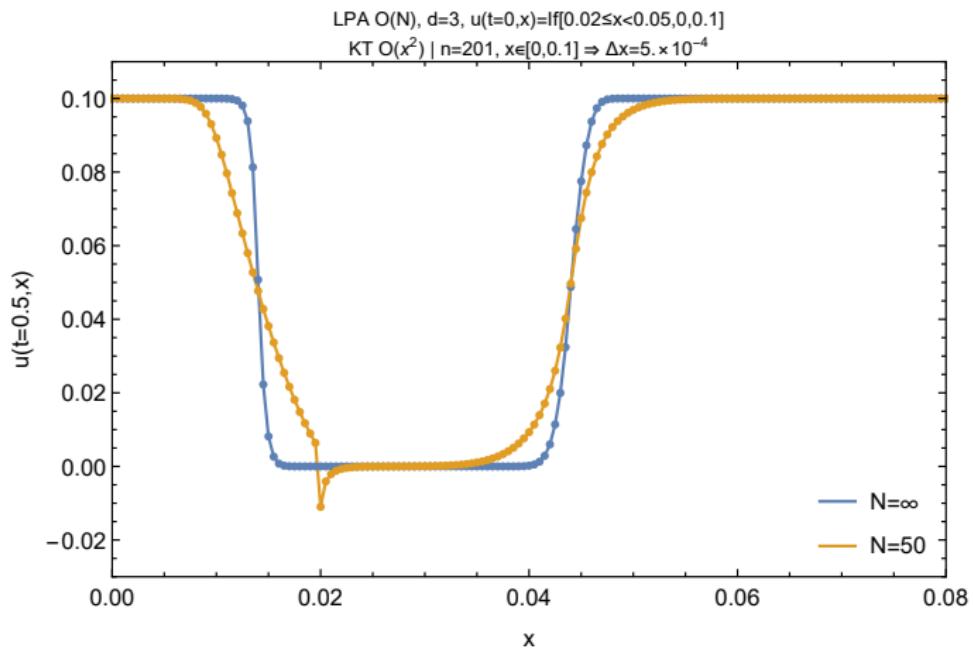




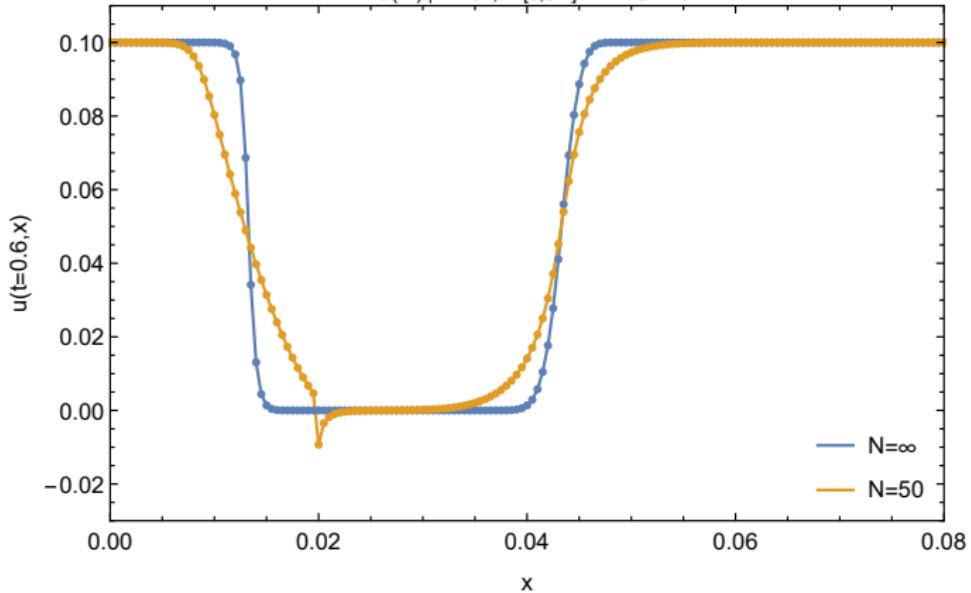


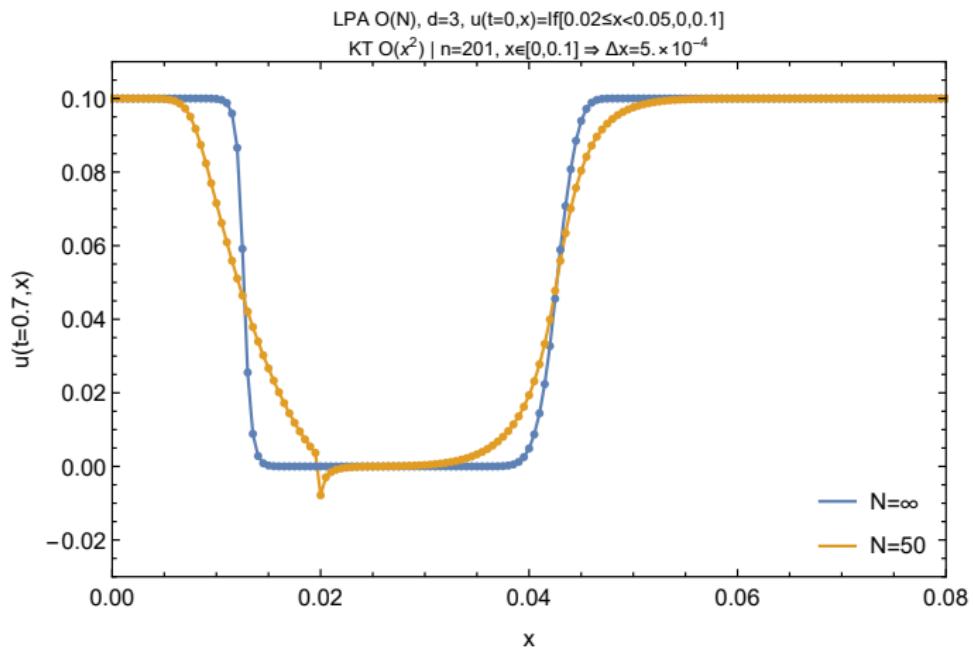


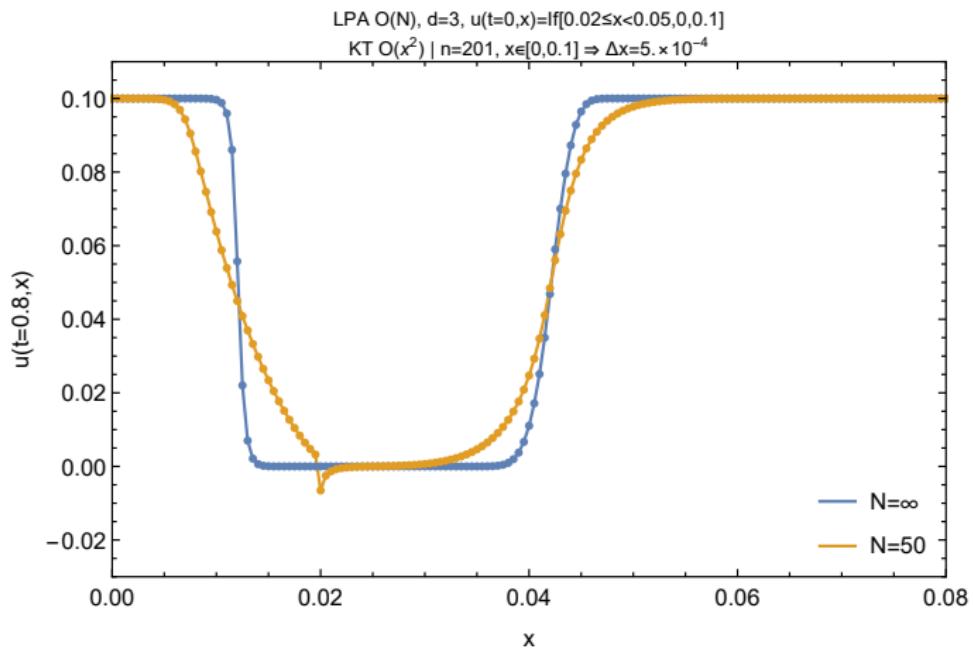




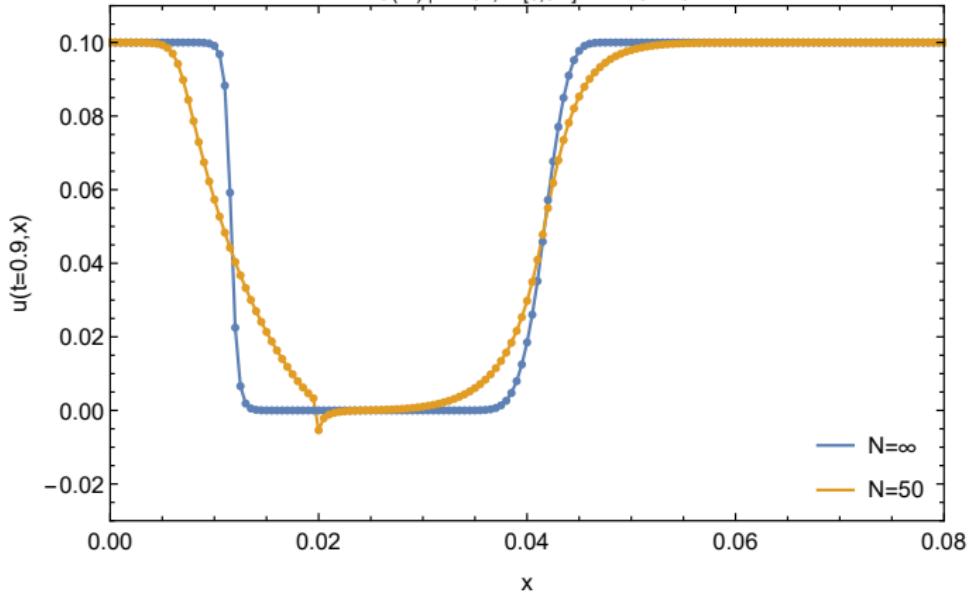
LPA  $O(N)$ ,  $d=3$ ,  $u(t=0,x)=\text{If}[0.02 \leq x < 0.05, 0, 0.1]$   
KT  $O(x^2)$  |  $n=201$ ,  $x \in [0, 0.1] \Rightarrow \Delta x = 5 \cdot 10^{-4}$



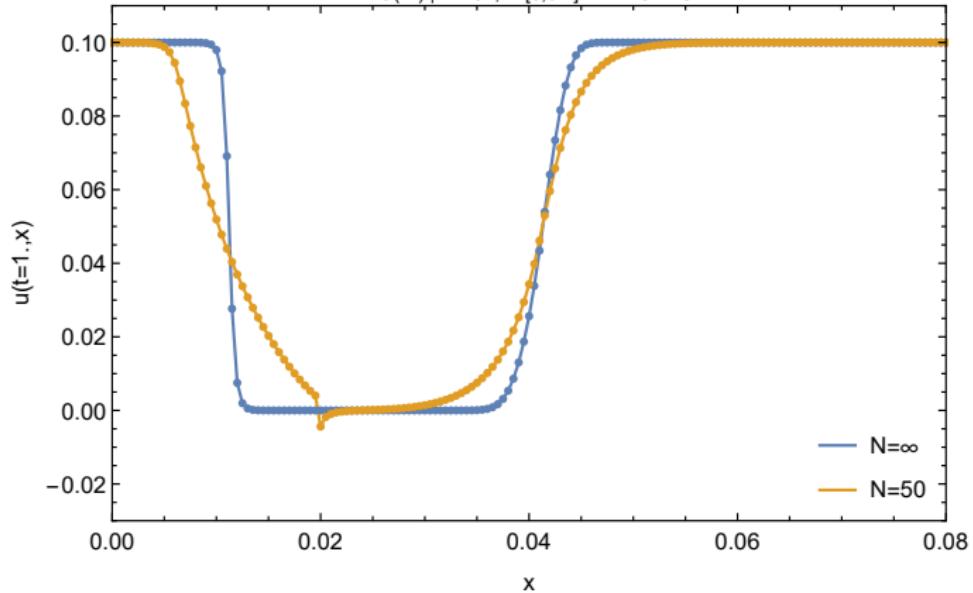




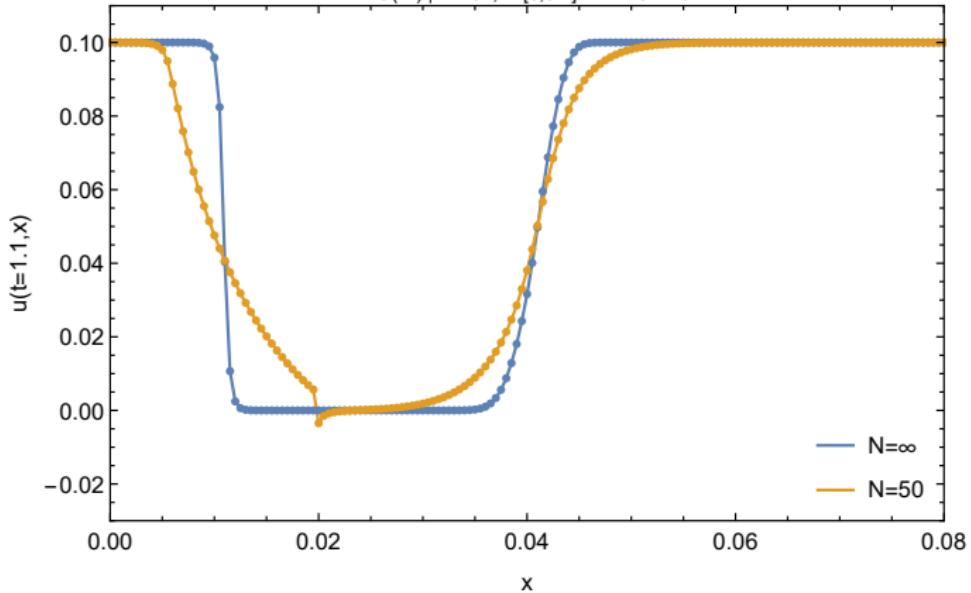
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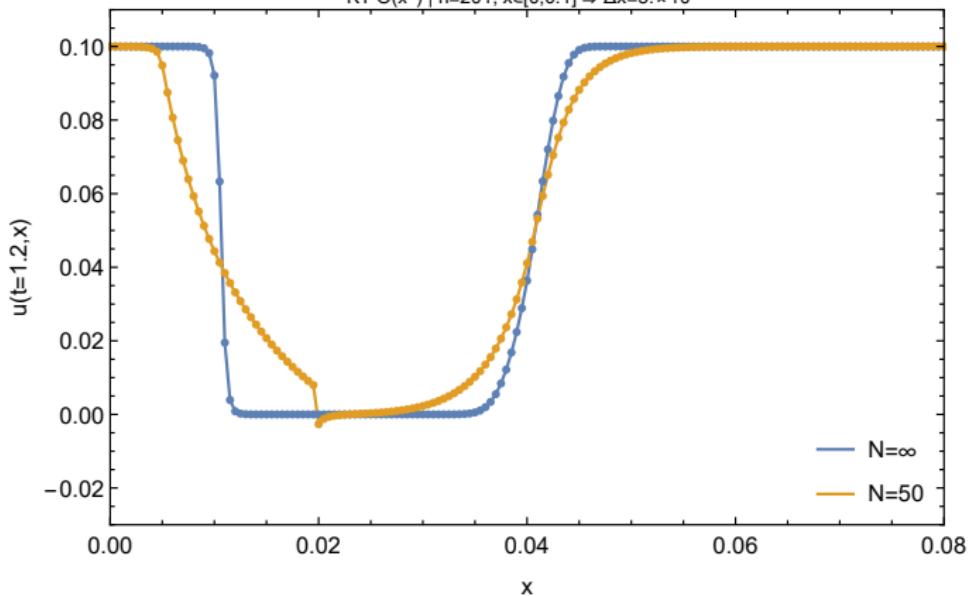
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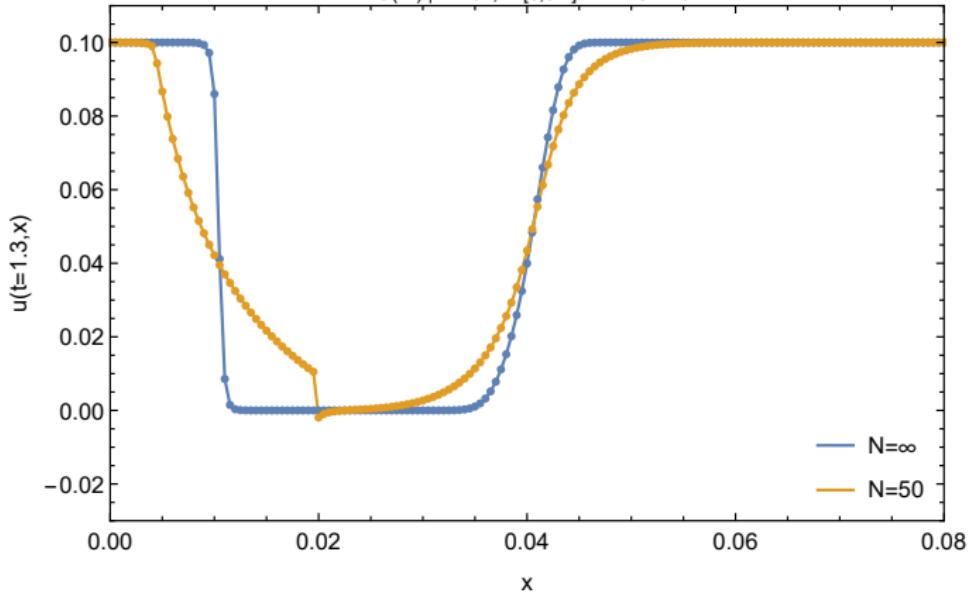
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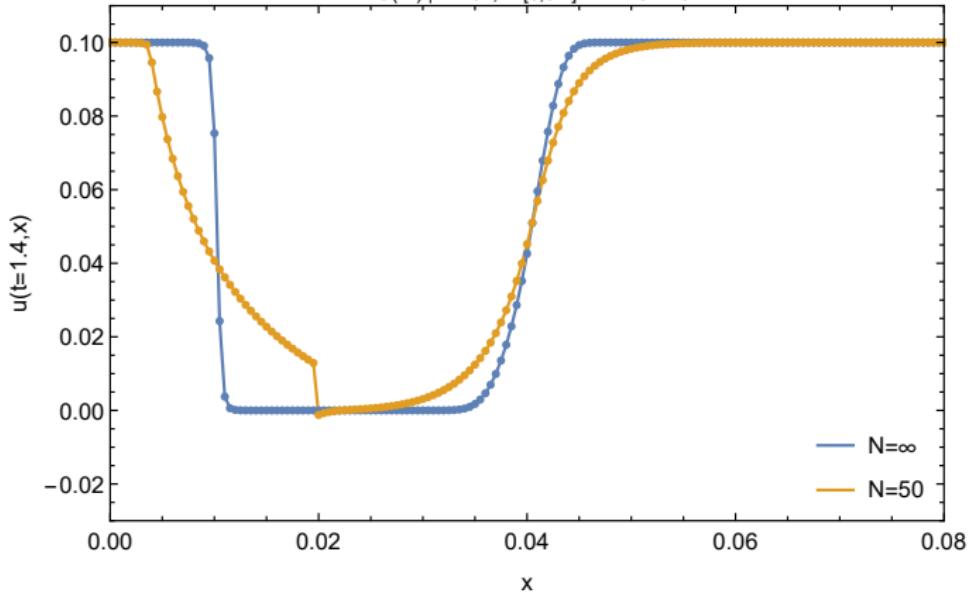
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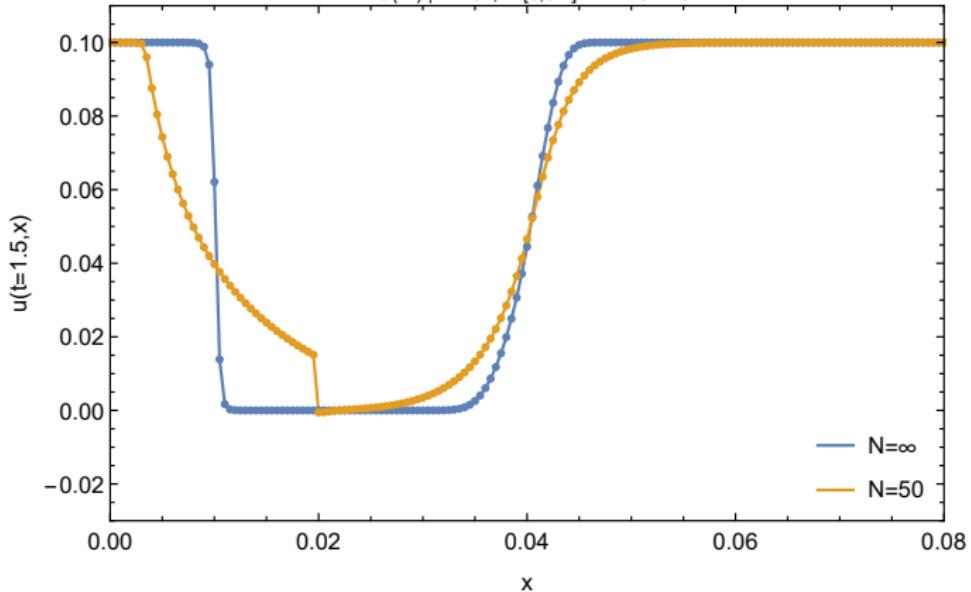
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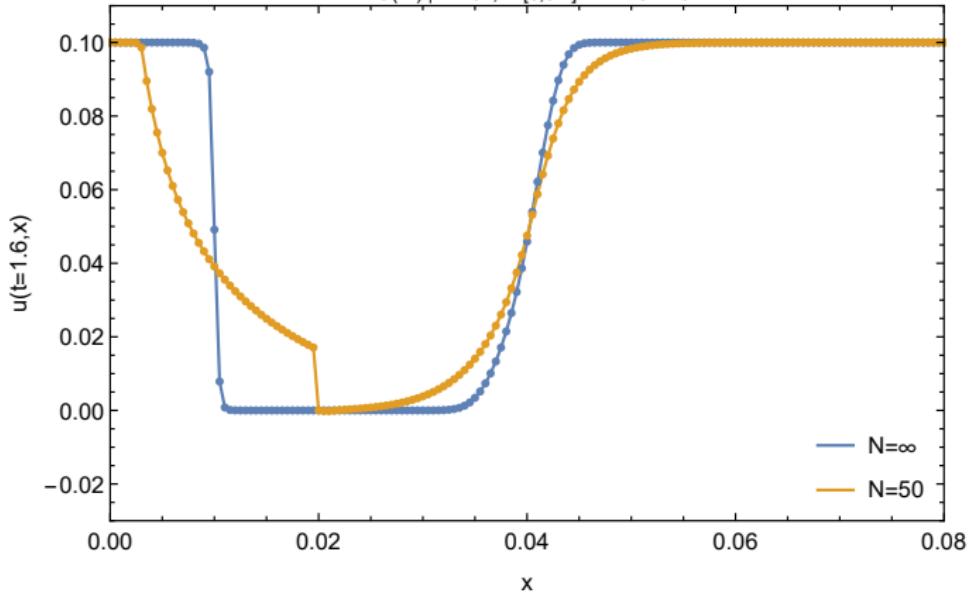
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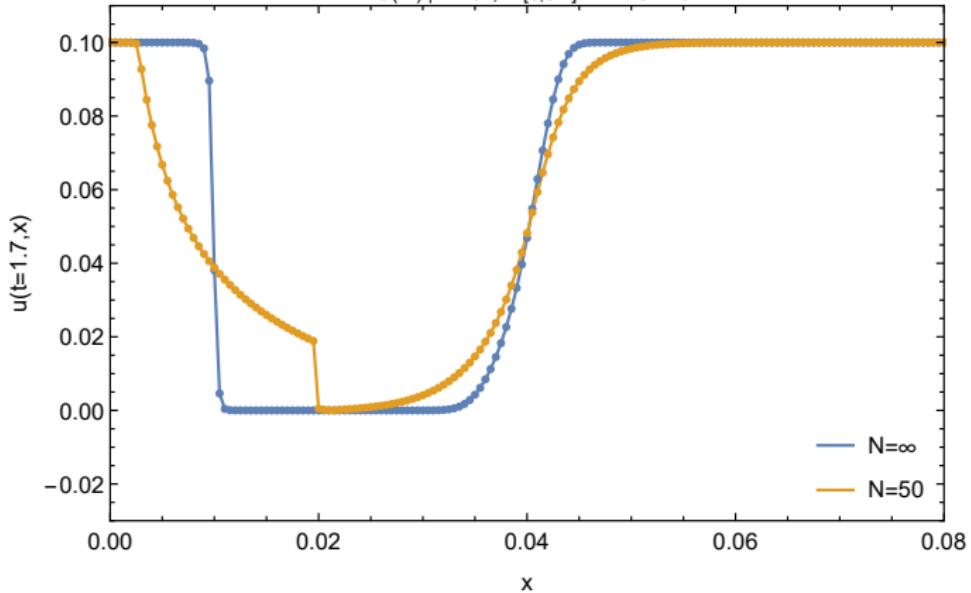
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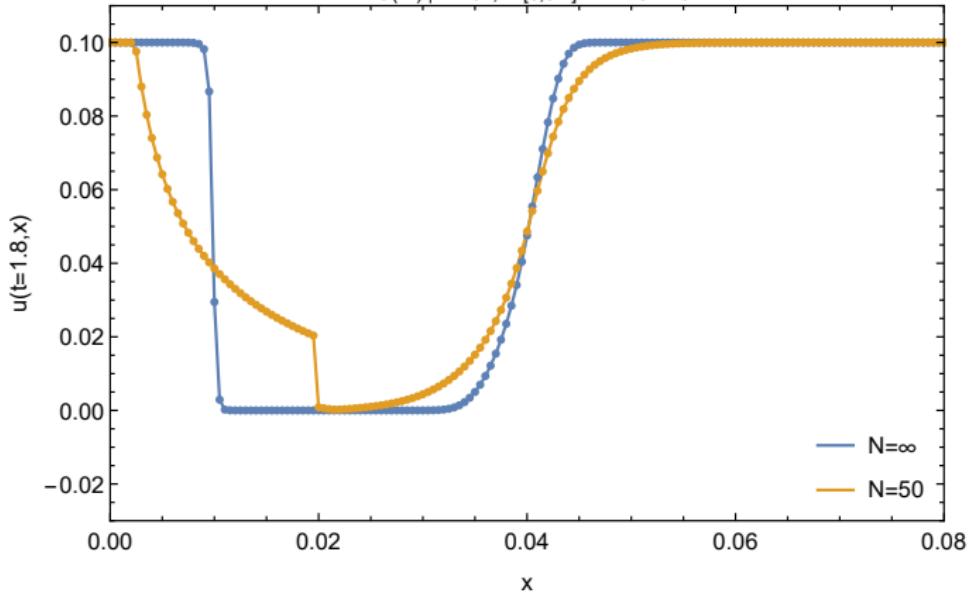
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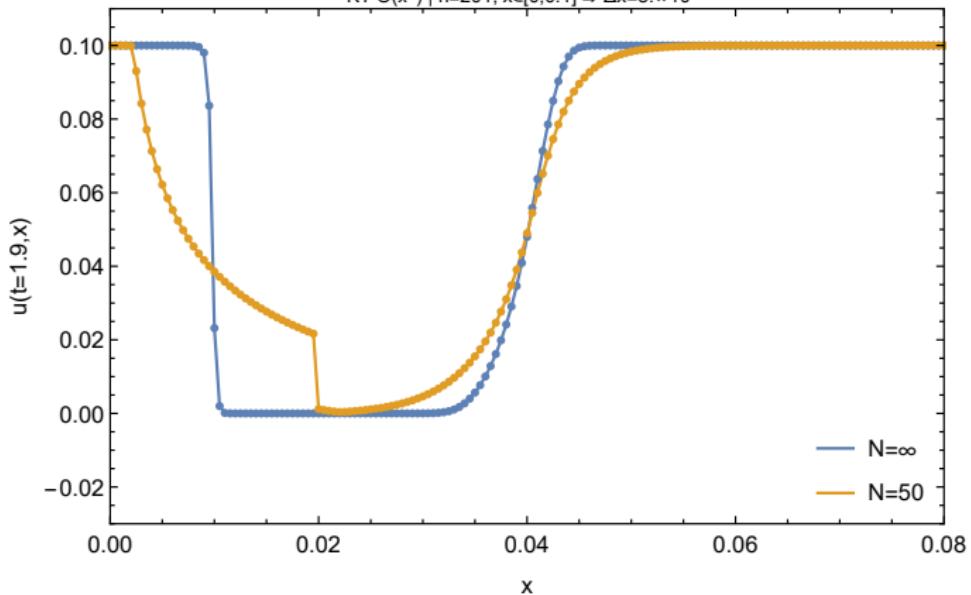
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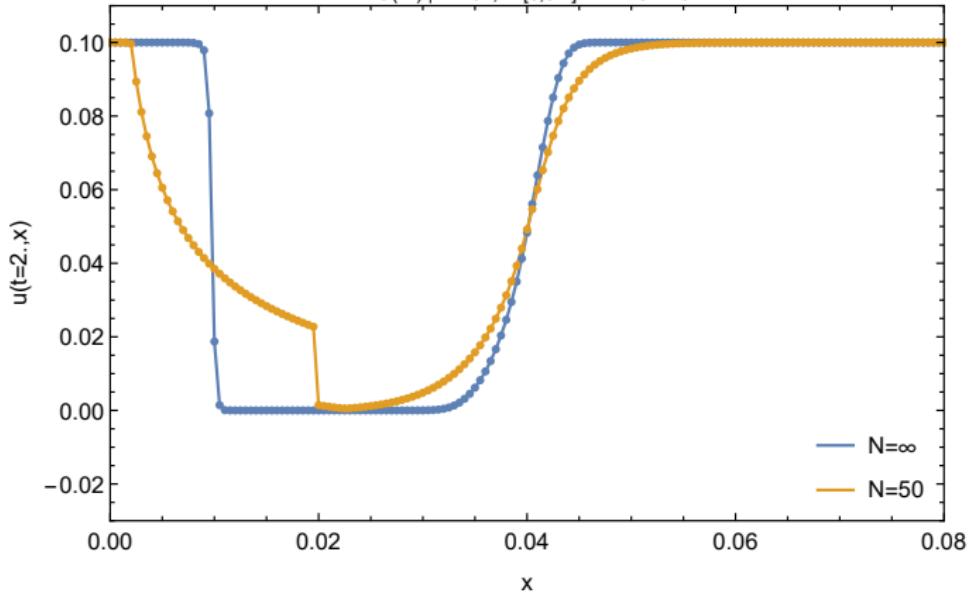
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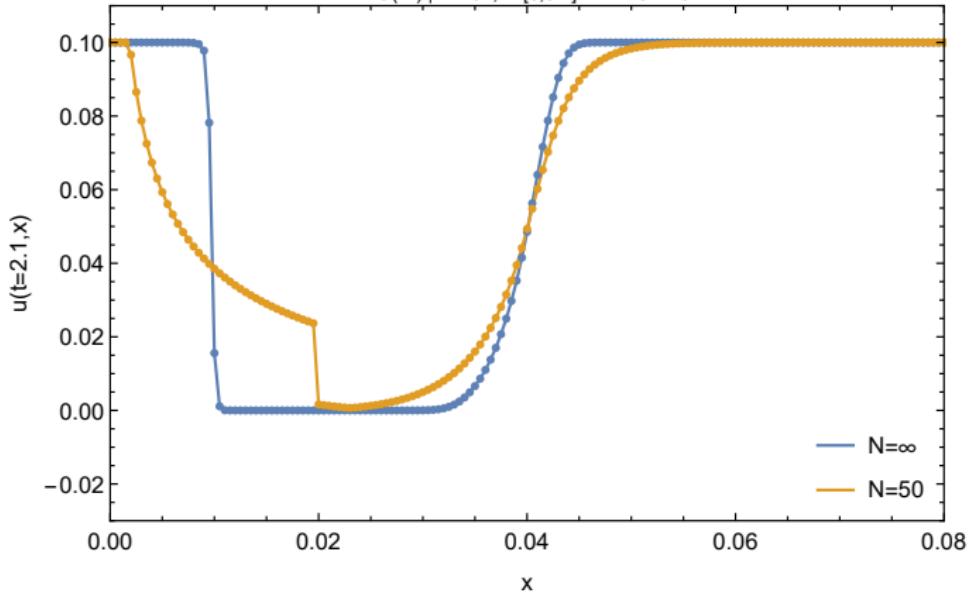
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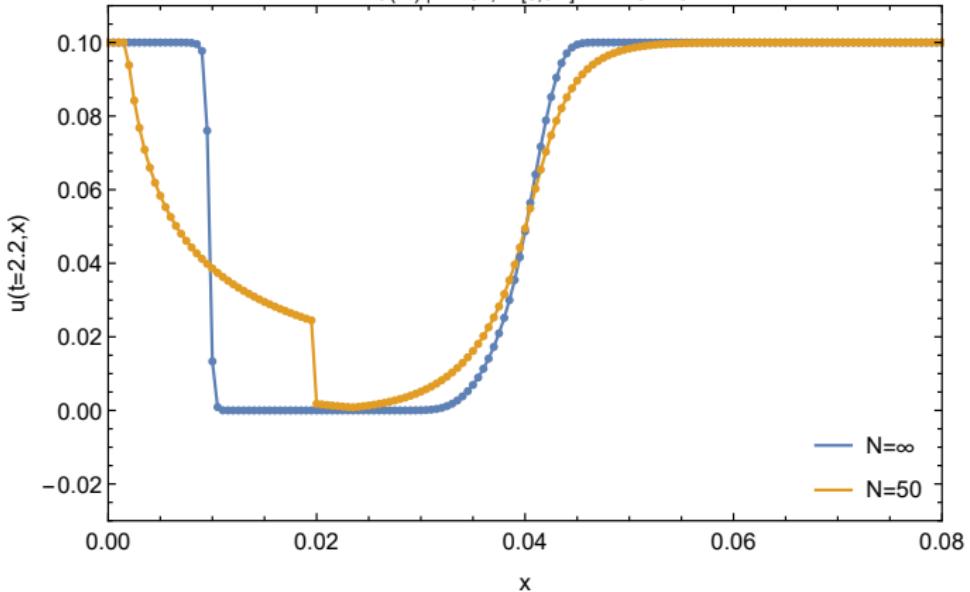
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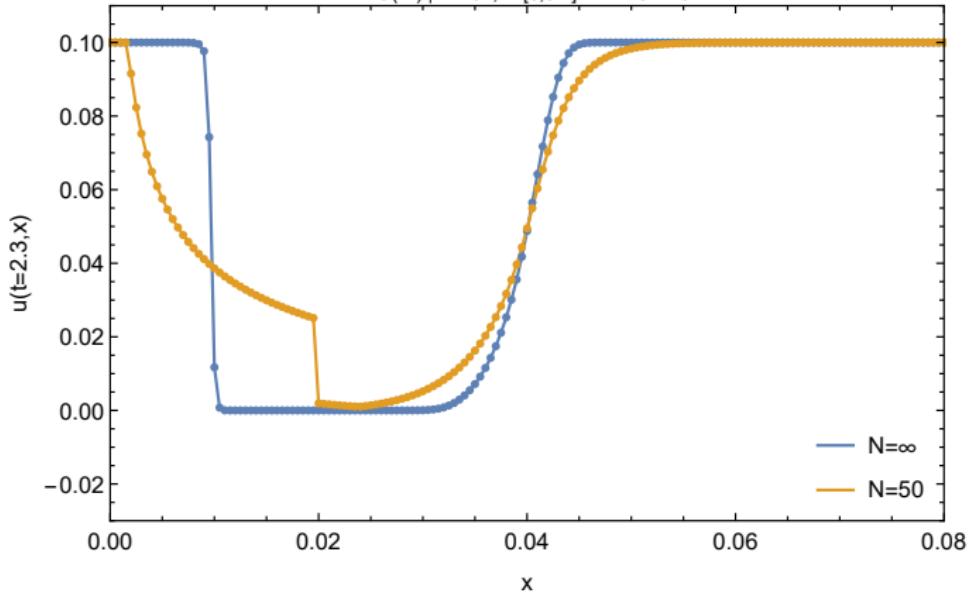
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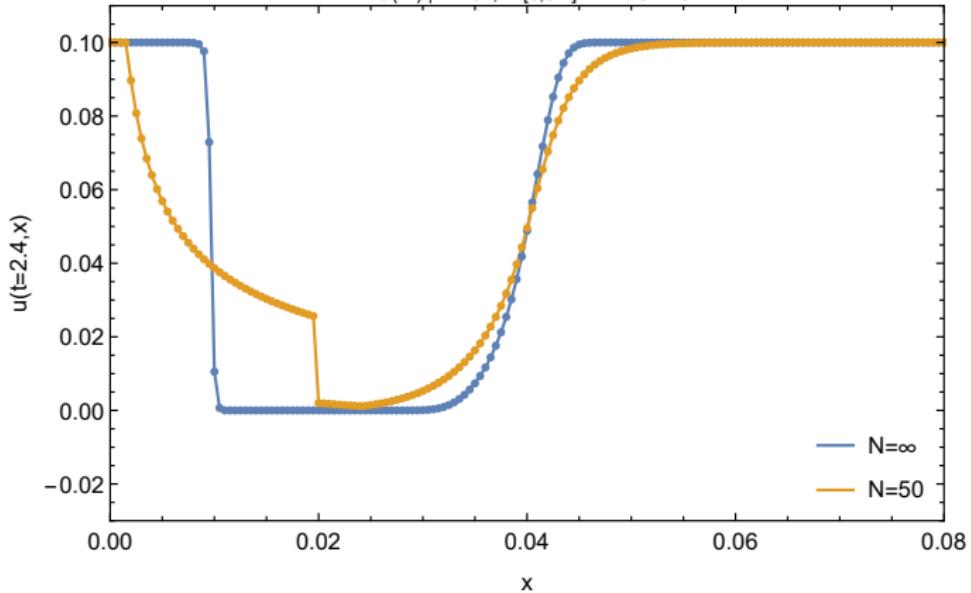
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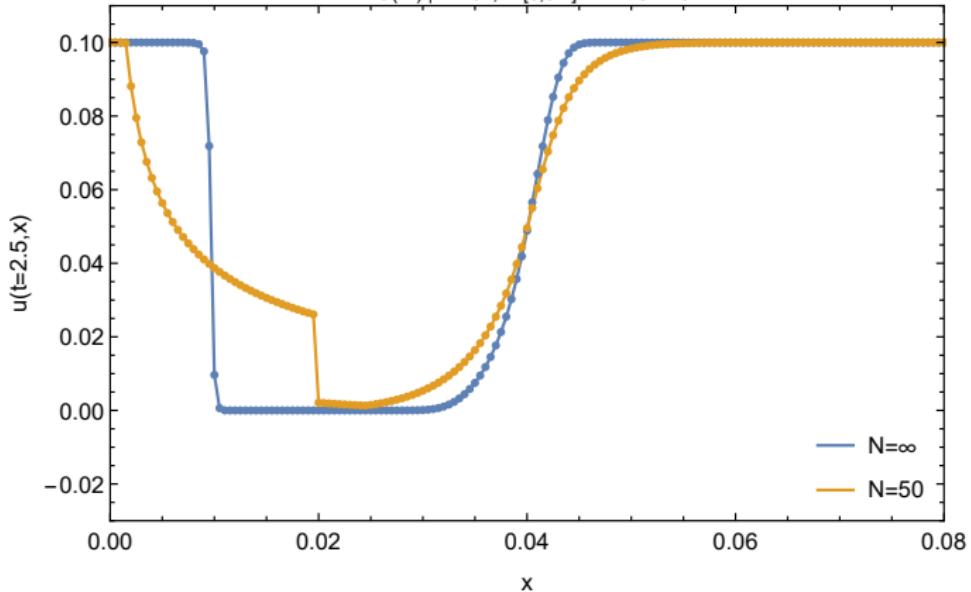
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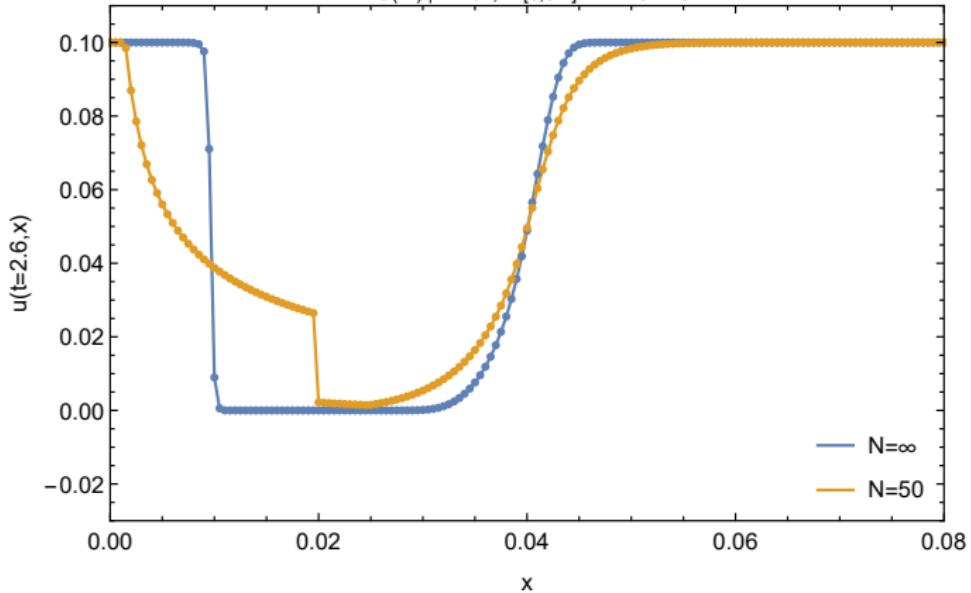
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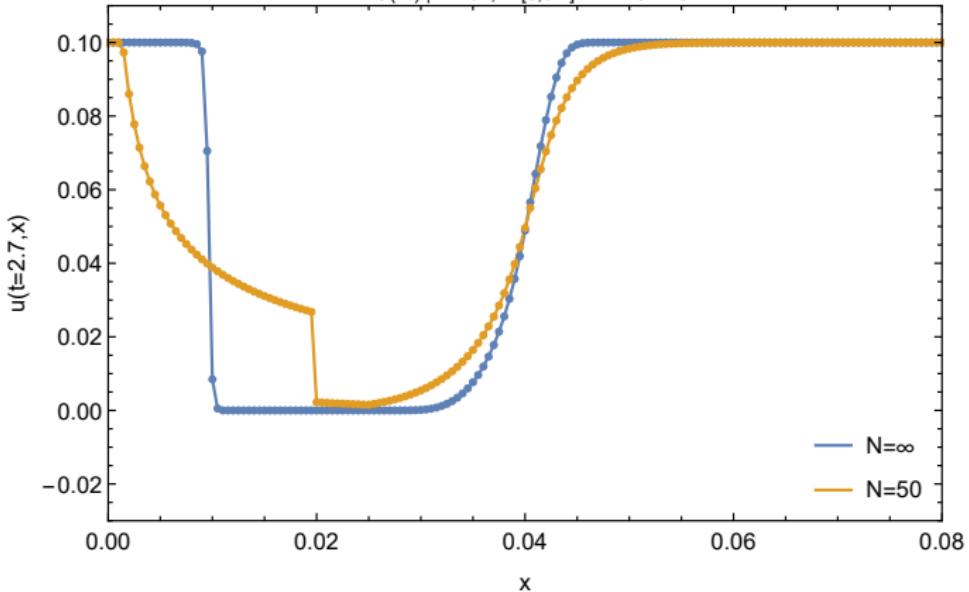
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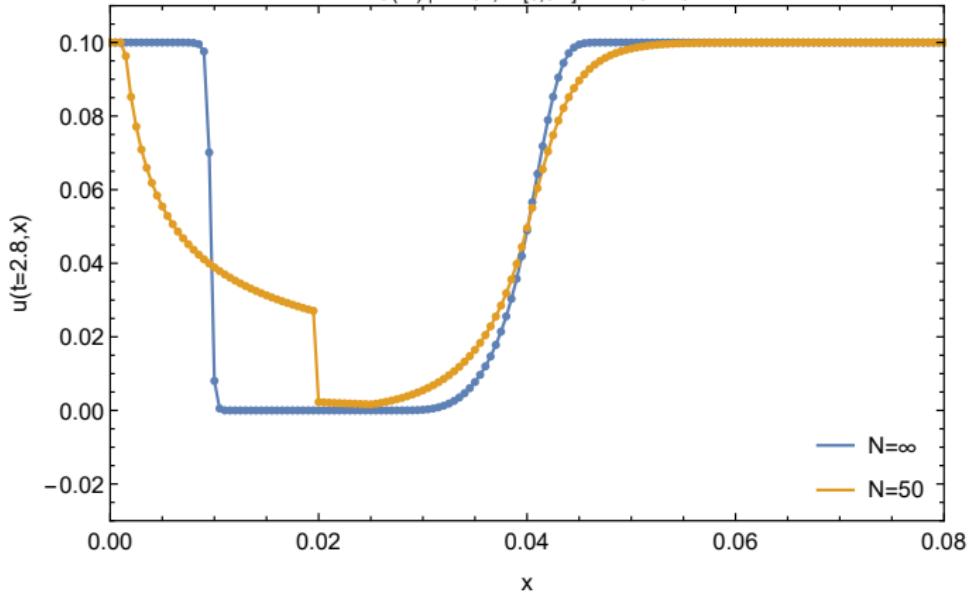
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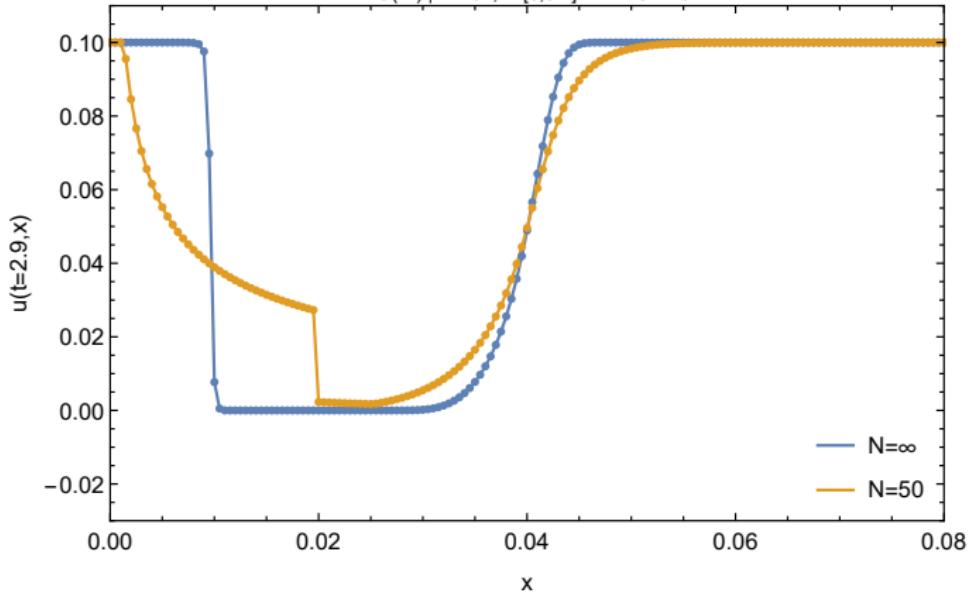
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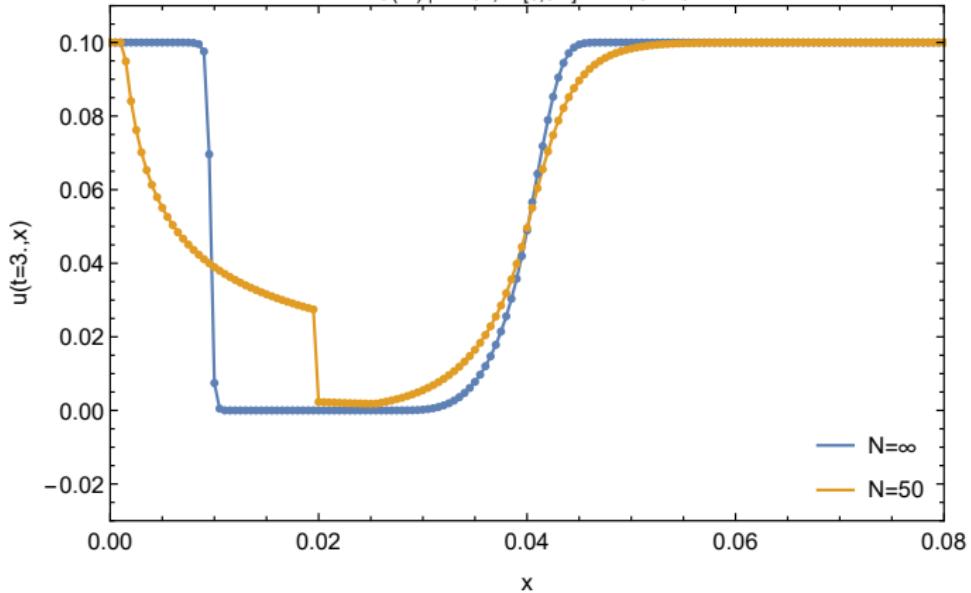
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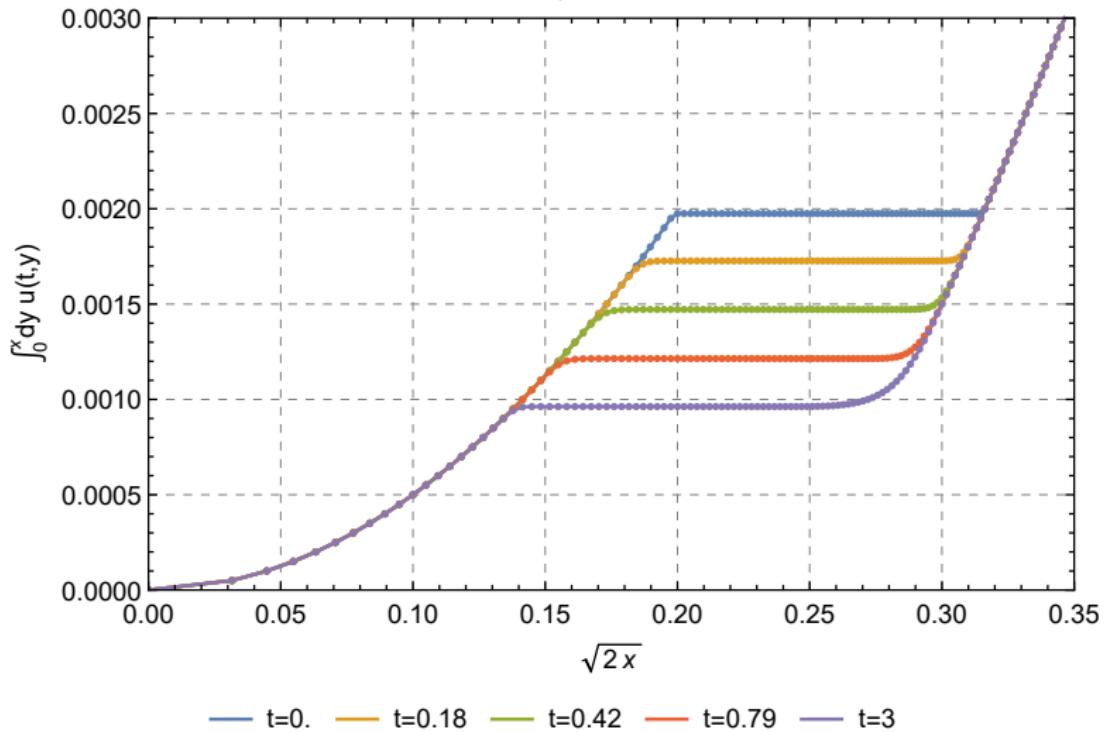


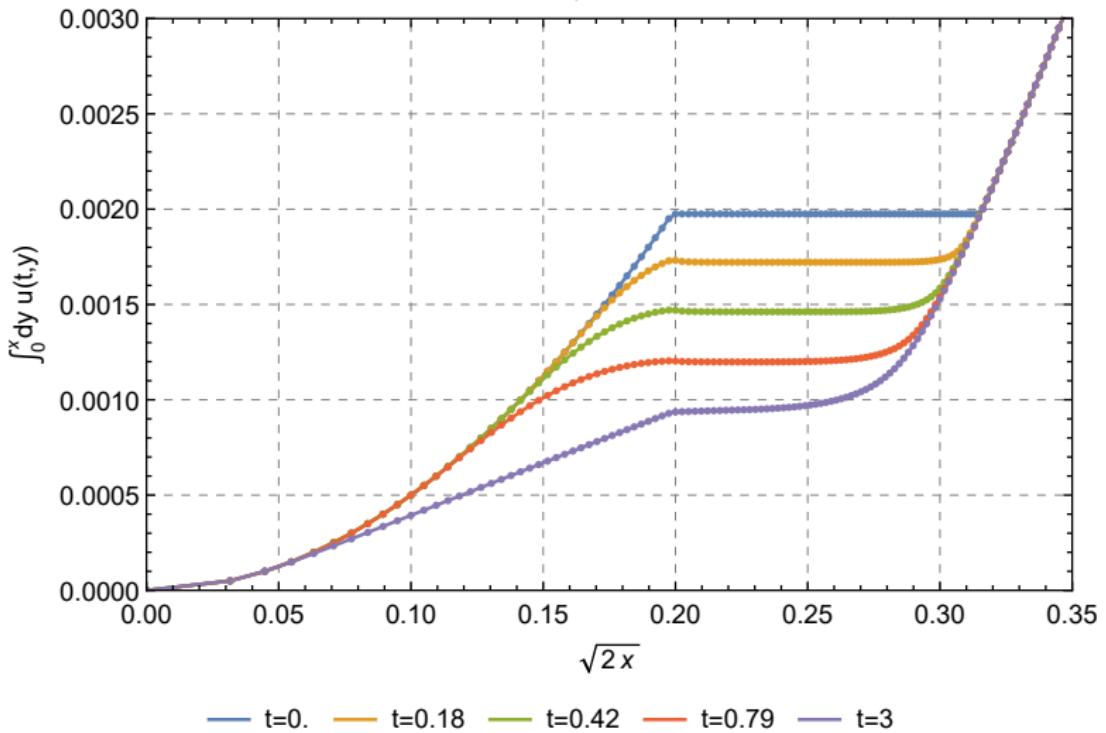
LPA  $O(N)$ ,  $d=3$ ,  $u(t=0,x)=\text{If}[0.02 \leq x < 0.05, 0, 0.1]$   
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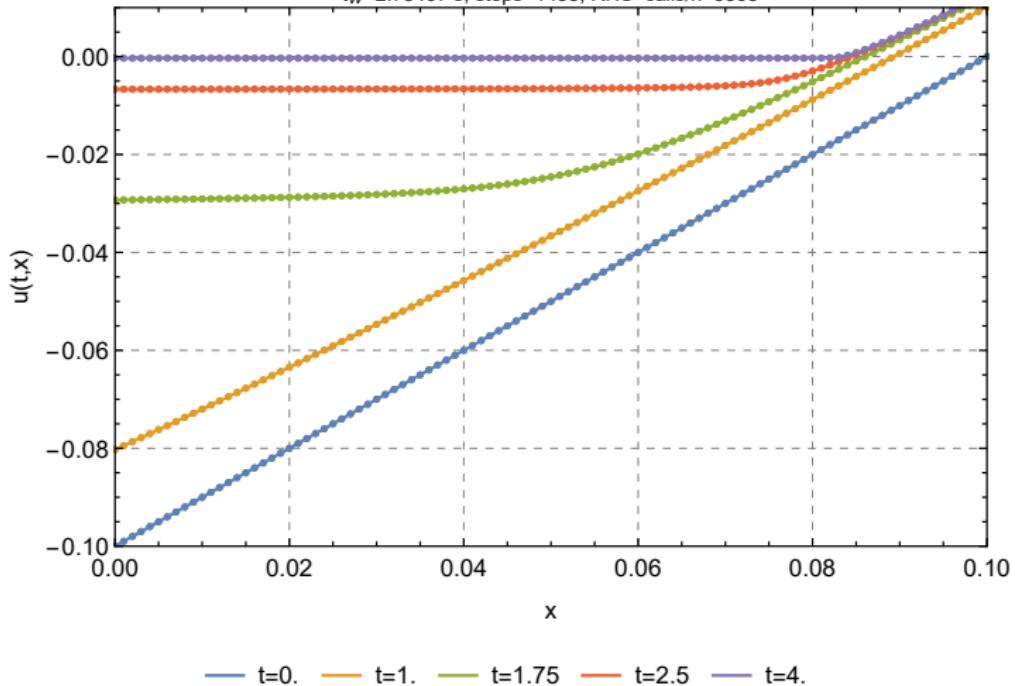
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LPA  $O(\infty)$ , d=3,  $u(t=0,x) = \text{lf}[0.02 \leq x < 0.05, 0, 0.1]$ KT  $O(x^2)$  | n=201,  $x \in [0, 0.1] \Rightarrow \Delta x = 5 \times 10^{-4}$  $t_W = 3.93594$  s, steps=1601, RHS-calls/n=3710

LPA O(50), d=3,  $u(t=0,x) = \text{lf}[0.02 \leq x < 0.05, 0, 0.1]$ KT O( $x^2$ ) | n=201,  $x \in [0, 0.1] \Rightarrow \Delta x = 5 \times 10^{-4}$  $t_W = 1.43539 \times 10^1$  s, steps=4959, RHS-calls/n=12492

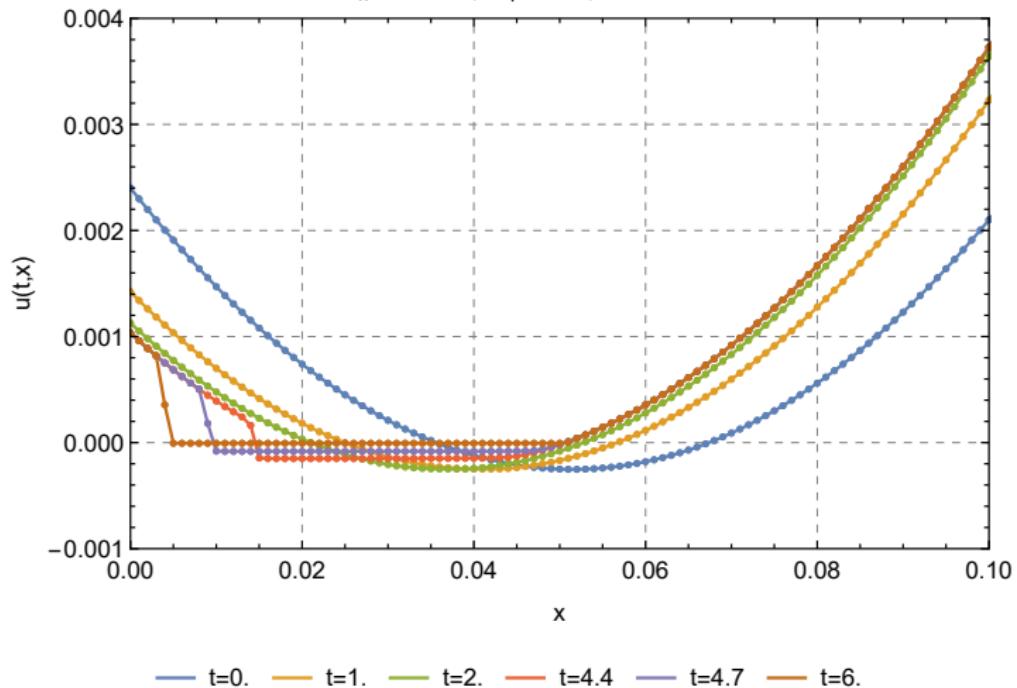
LPA  $O(\infty)$ , d=3,  $v(t=0, x) = 1 \cdot x - 0.1$   
KT  $O(x^2)$  | n=151,  $x \in [0, 0.15] \Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 2.78407$  s, steps=1453, RHS-calls/n=3305

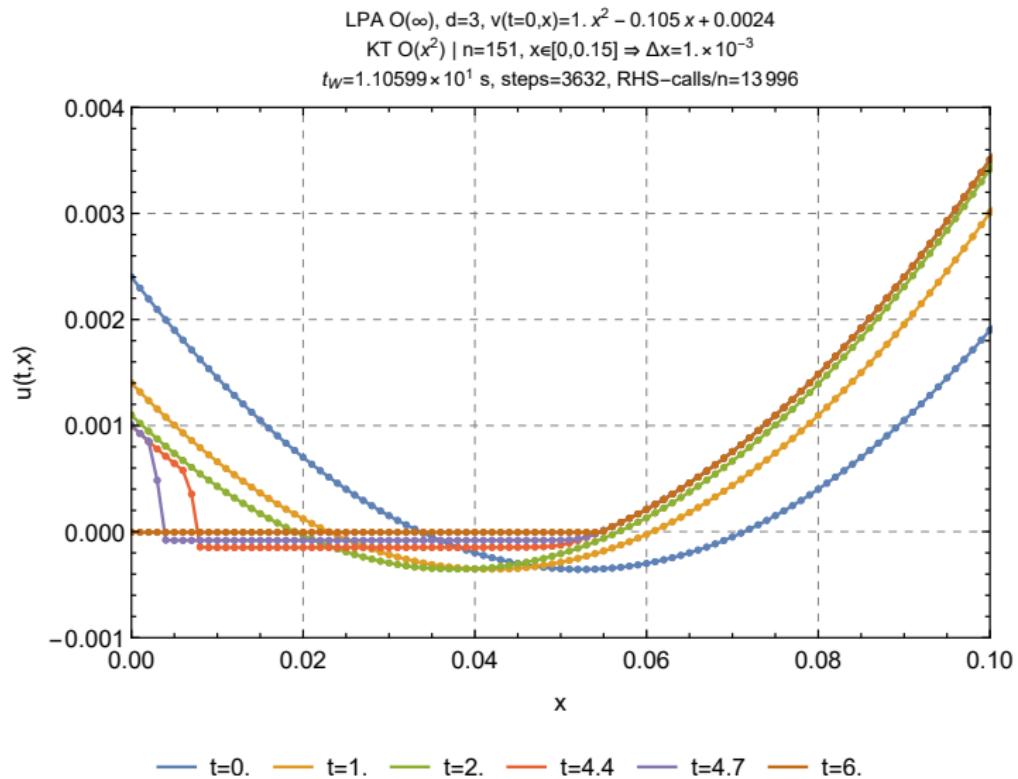


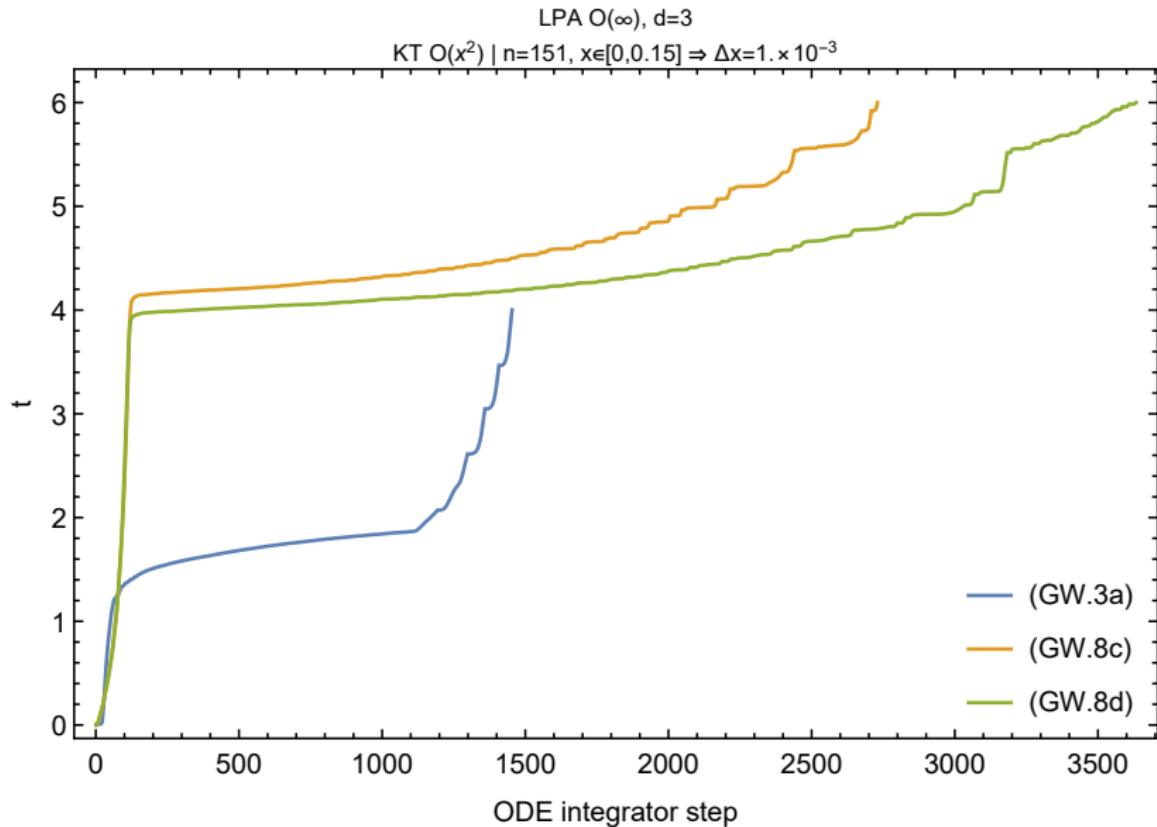
LPA  $O(\infty)$ , d=3,  $v(t=0,x)=1 \cdot x^2 - 0.103 x + 0.0024$

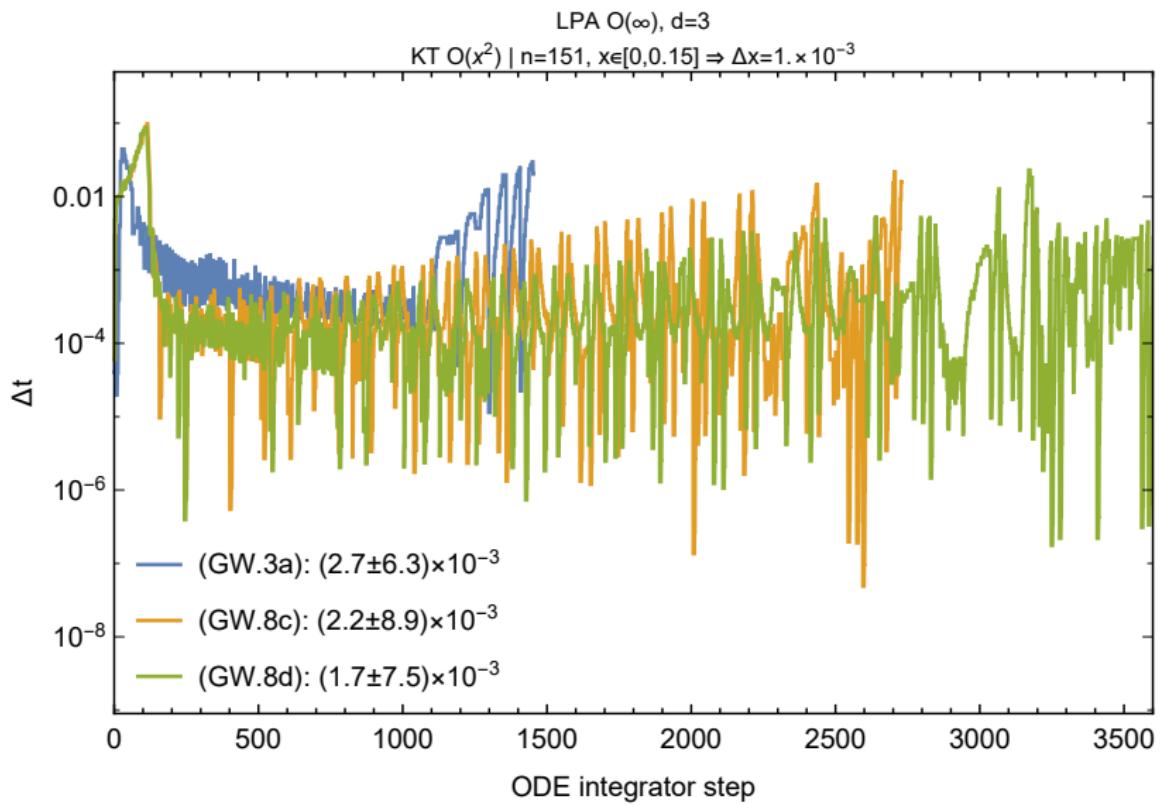
KT  $O(x^2)$  | n=151,  $x \in [0, 0.15] \Rightarrow \Delta x = 1 \cdot 10^{-3}$

$t_W = 8.52115$  s, steps=2729, RHS-calls/n=9978

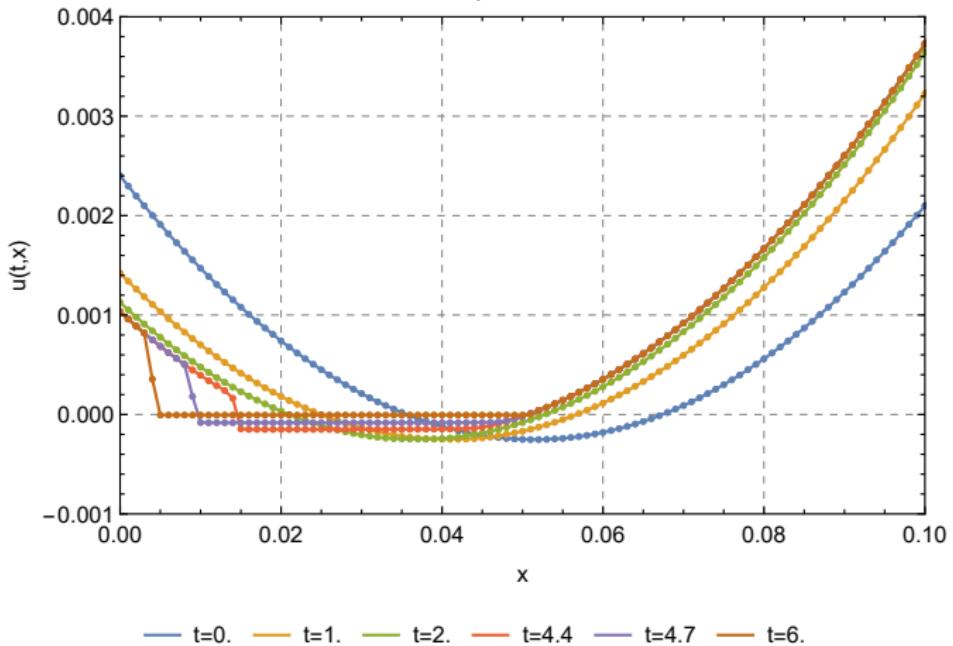




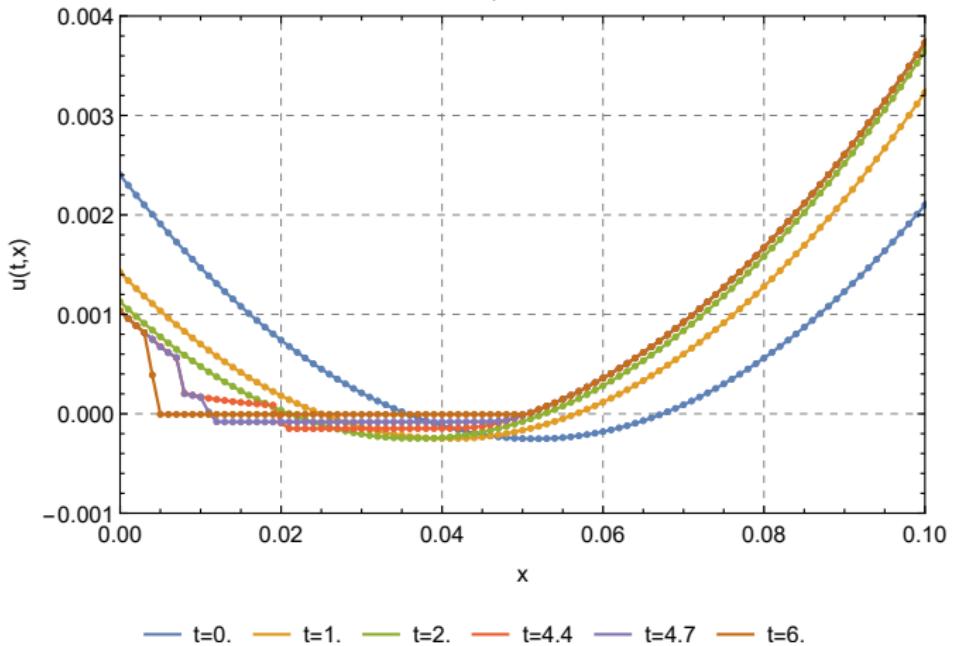




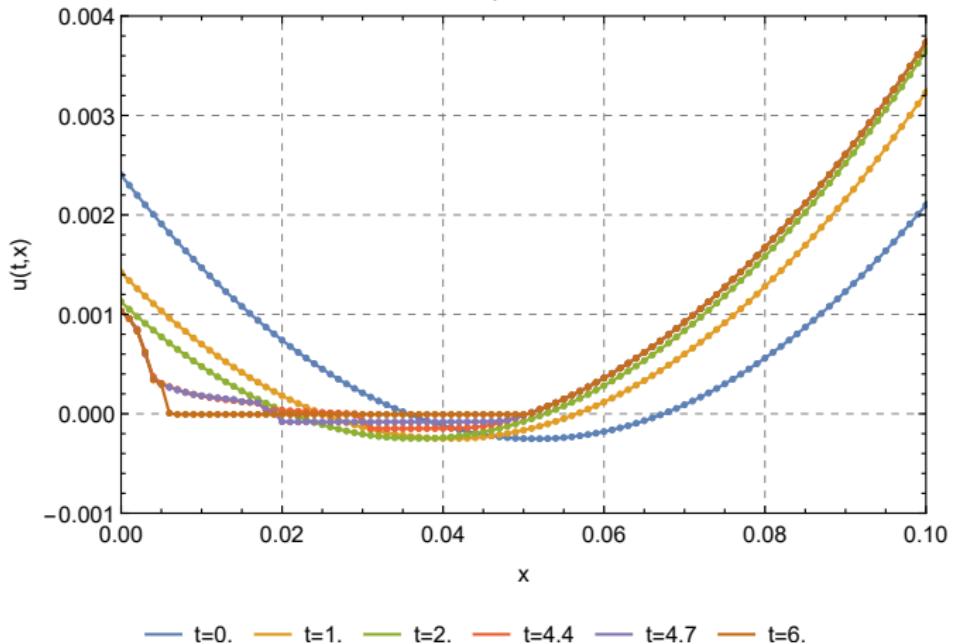
LPA O( $\infty$ ), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103x + 0.0024$   
 KT O( $x^2$ ) | n=151, x $\in$ [0,0.15]  $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 8.16544$  s, steps=2729, RHS-calls/n=9978



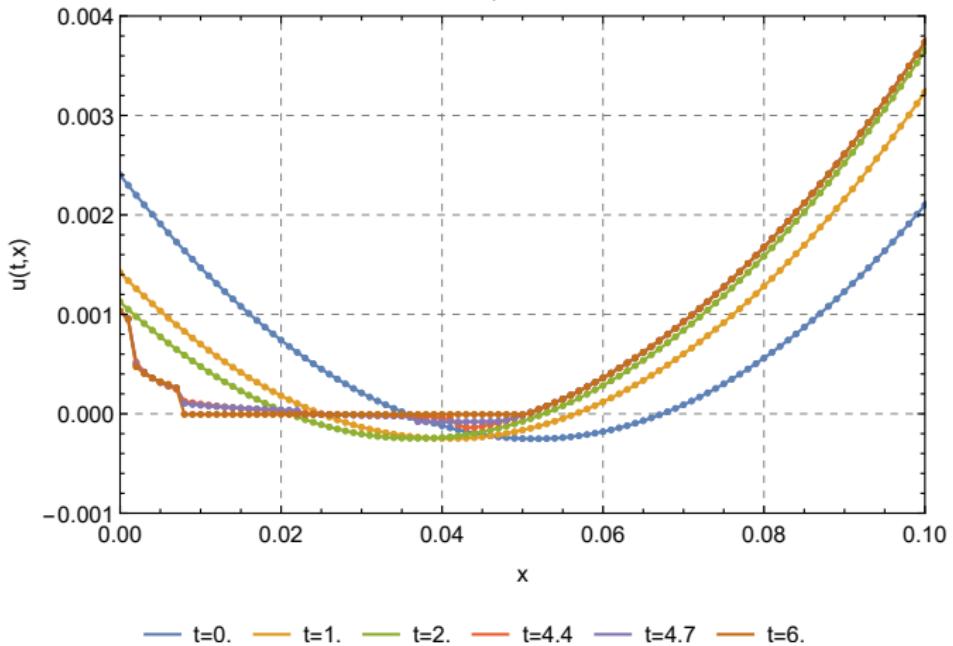
LPA O(2000), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
 KT O( $x^2$ ) | n=151,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 1.65594 \times 10^1$  s, steps=4431, RHS-calls/n=18721



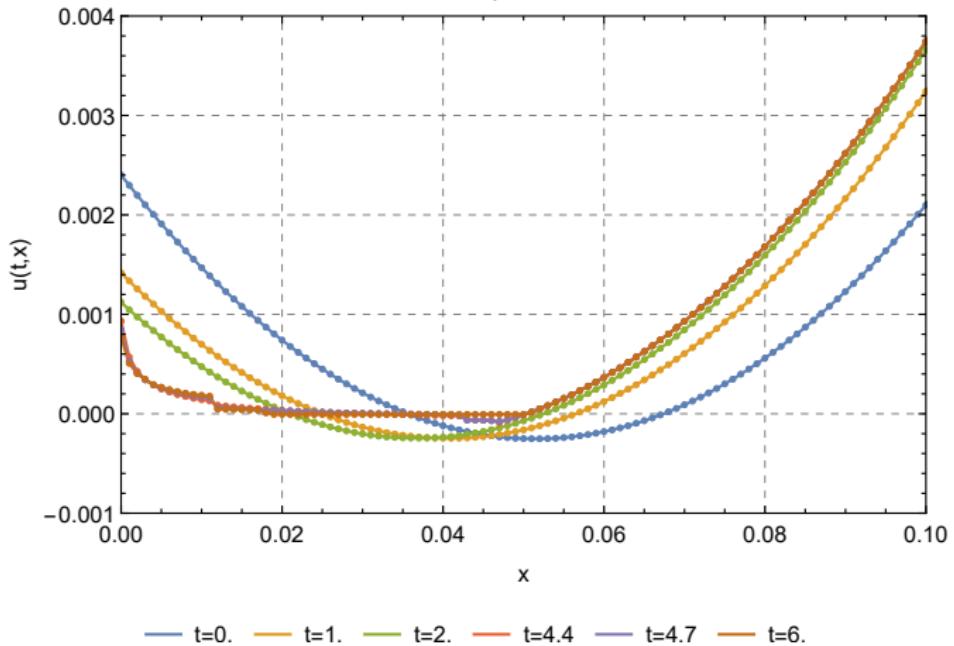
LPA O(1500), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
 KT O( $x^2$ ) | n=151,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 1.96345 \times 10^1$  s, steps=5381, RHS-calls/n=24 766



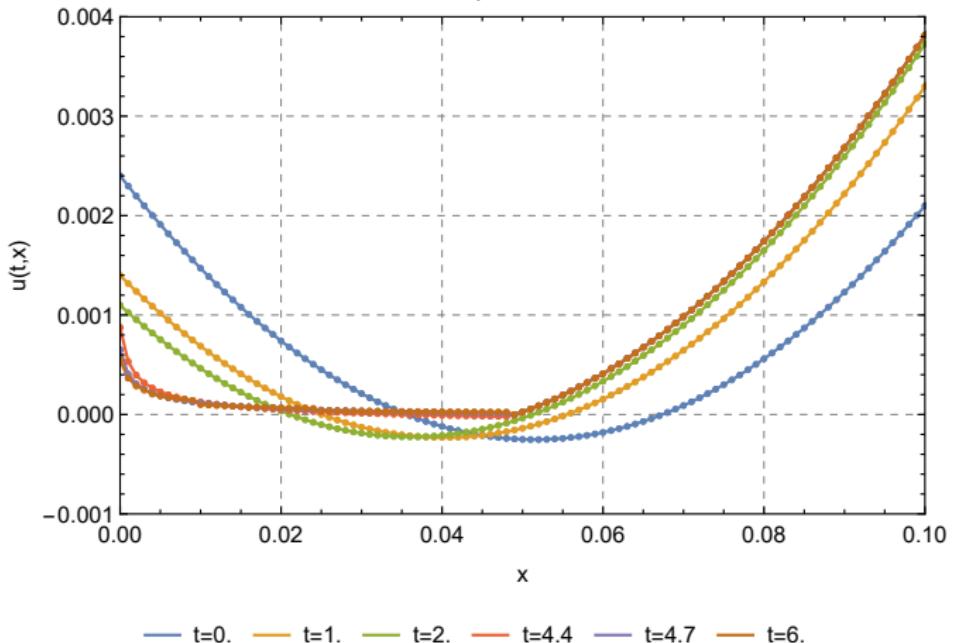
LPA  $O(1000)$ ,  $d=3$ ,  $v(t=0,x)=1 \cdot x^2 - 0.103 x + 0.0024$   
 KT  $O(x^2)$  |  $n=151$ ,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 1.86091 \times 10^1$  s, steps=4990, RHS-calls/n=23 552



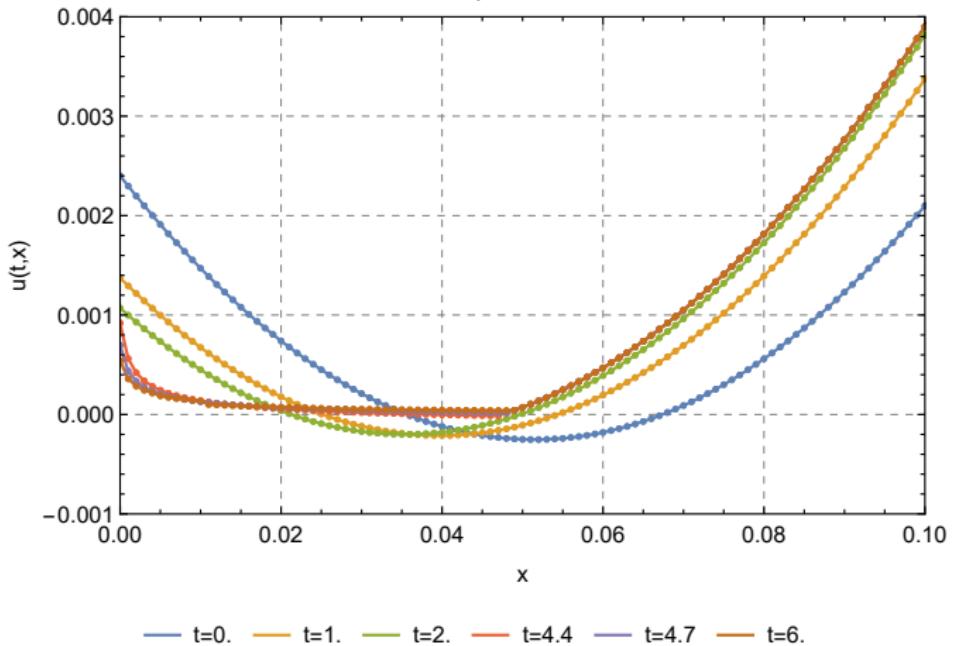
LPA O(500), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
KT O( $x^2$ ) | n=151,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 1.57667 \times 10^1$  s, steps=3985, RHS-calls/n=18999



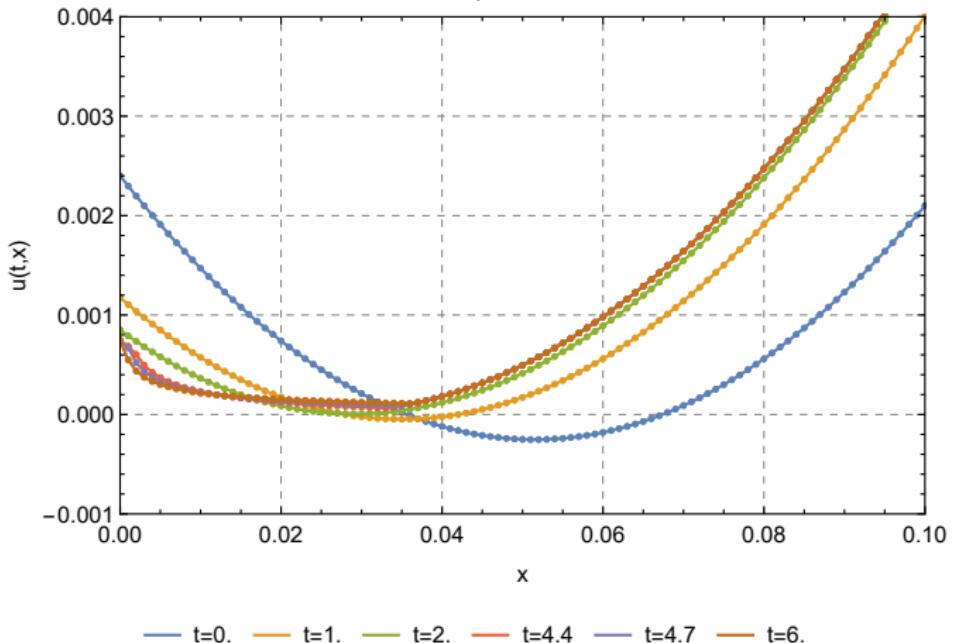
LPA O(100), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
KT O( $x^2$ ) | n=151,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 6.19885$  s, steps=1843, RHS-calls/n=7732



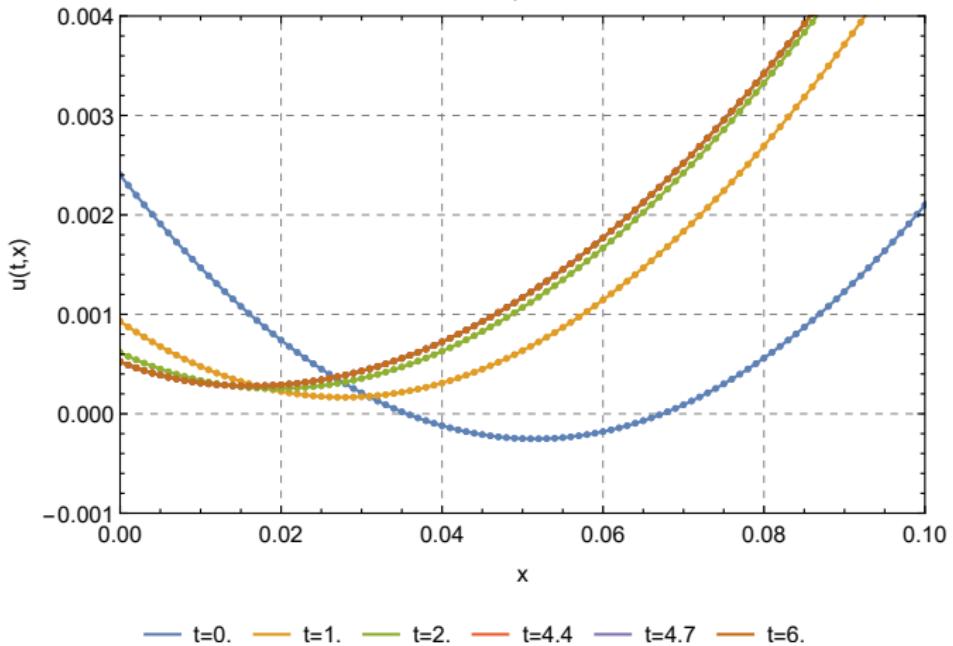
LPA O(50), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
KT O( $x^2$ ) | n=151,  $x \in [0, 0.15] \Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 5.94463$  s, steps=1735, RHS-calls/n=6806



LPA O(10), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
 KT O( $x^2$ ) | n=151,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 3.57829$  s, steps=1152, RHS-calls/n=4044



LPA O(5), d=3, v(t=0,x)= $1 \cdot x^2 - 0.103 x + 0.0024$   
 KT O( $x^2$ ) | n=151,  $x \in [0, 0.15]$   $\Rightarrow \Delta x = 1 \cdot 10^{-3}$   
 $t_W = 9.31835 \times 10^{-1}$  s, steps=442, RHS-calls/n=917



# O(N) model in d=0 - Exact solution

- ▶ A zero dimensional field-theory of  $N$  scalars  $\phi_a$ :

$$S = U(\phi_a \phi^a / 2) = \frac{\lambda_1}{2} \phi_a \phi^a + \frac{\lambda_2}{8} (\phi_a \phi^a)^2 + \frac{\lambda_3}{24} (\phi_a \phi^a)^3 + \dots \quad (17)$$

$$= U(\rho), \quad \rho \equiv \phi_a \phi^a / 2 \quad (18)$$

- The physical minimum is always at  $\langle \phi_a \phi^a \rangle = 0$
- All  $n$ -Point functions can be computed by solving one-dimensional integrals: e.g.:

$$\Gamma^{(2)} = \frac{NR_{N-1}}{R_{N+1}}, \quad R_N = \int_0^\infty dy y^N \exp \left[ -U(y^2/2) \right] \quad (19)$$

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for further details see e.g.: J. Keitel and L. Bartosch, J. Phys. **A45** (2012), arXiv: 1109.3013 [cond-mat.stat-mech]

- ▶ The Wetterich eq. in  $d = 0$  is a partial differential equation:

$$\frac{dU_t(\rho)}{dt} = \frac{N-1}{R_t + U'_t(\rho)} \frac{dR_t}{dt} + \frac{1}{R_t + U'_t(\rho) + 2\rho U''_t(\rho)} \frac{dR_t}{dt} \quad (20)$$

with  $t = 0$  and  $R_0 = \infty$  in the UV and  $t = 1$  and  $R_1 = 0$  in the IR.

- Various regulators can be constructed e.g.:

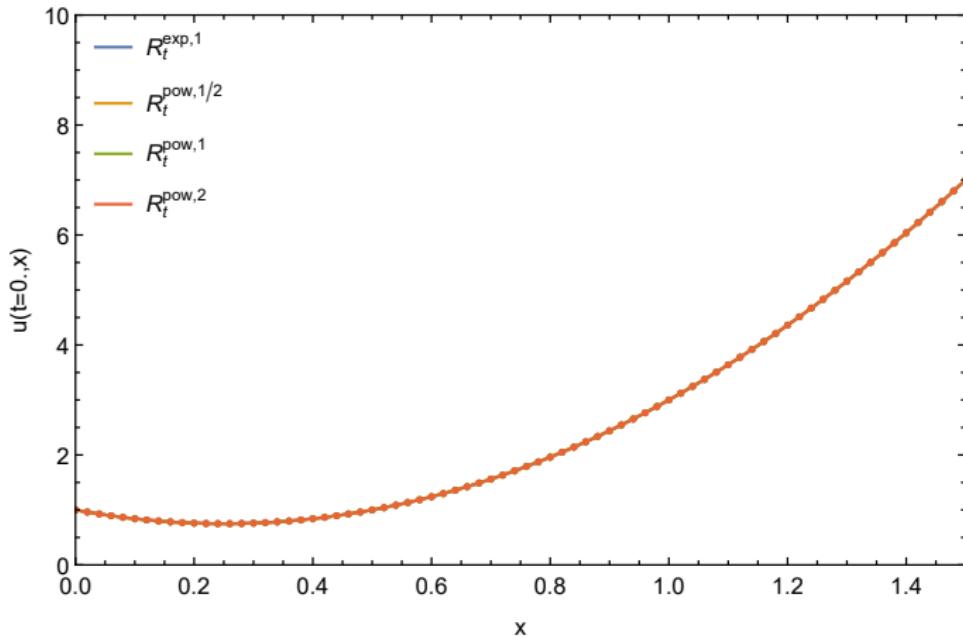
$$R_t^{\text{exp},m} = \exp[1/t^m - 1] - 1 \quad \text{with } m > 0 \quad (21)$$

$$R_t^{\text{pow},m} = 1/t^m - 1, \quad \text{with } m > 0 \quad (22)$$

- The exact flow eqs. can be solved numerically and results can be confronted with  $n$ -point functions from exact integration

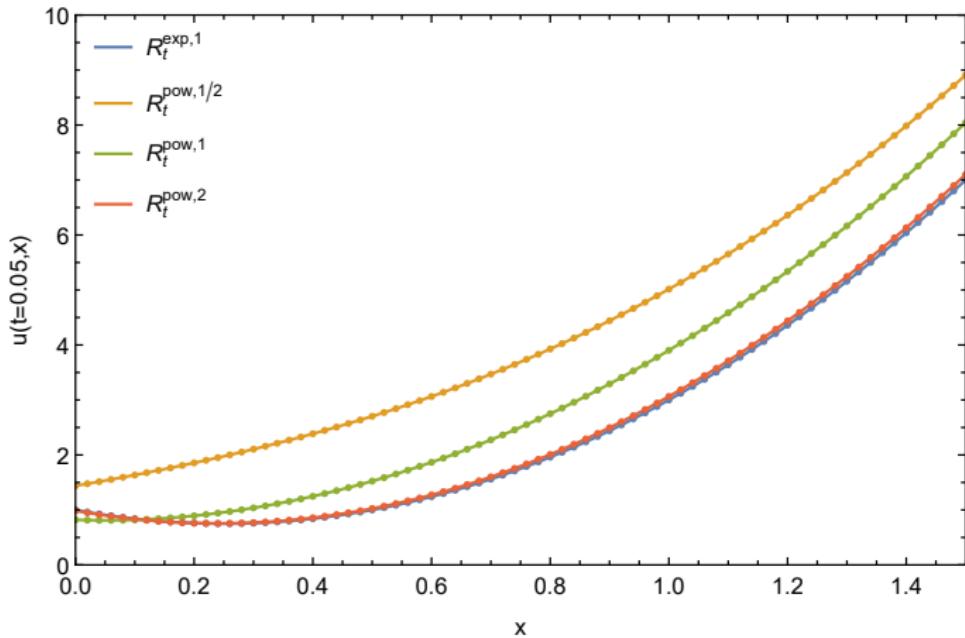
# O(N) model in d=0 - Regulator trajectories

$O(4), d=0, u(t=0, x) = 1 + (-2 + 4x)x$   
KT  $O(x^2) \mid n=201, x \in [0, 4] \Rightarrow \Delta x = 2 \times 10^{-2}$

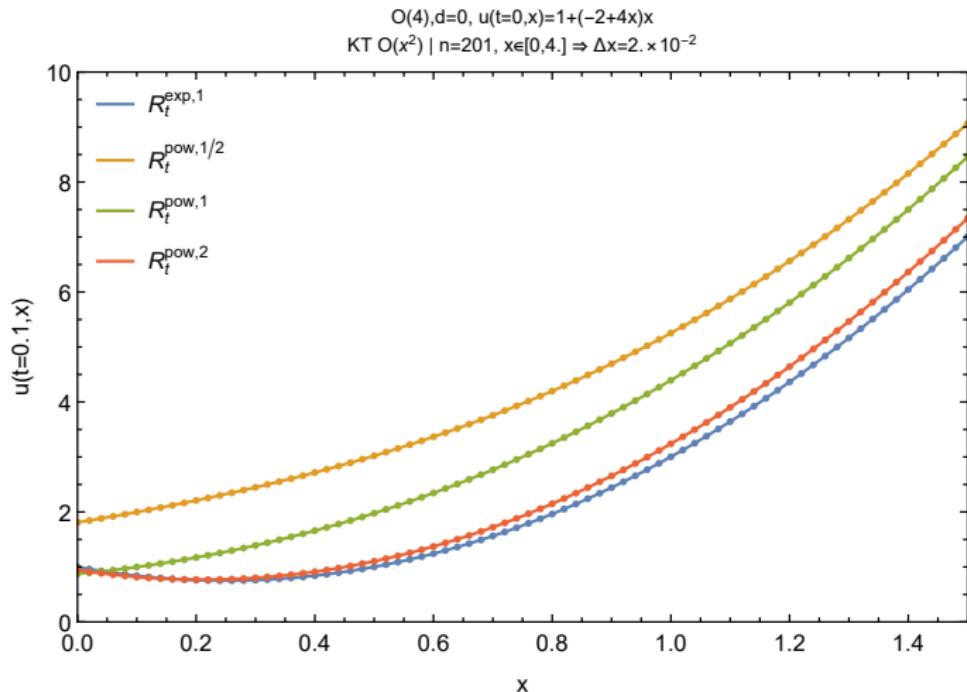


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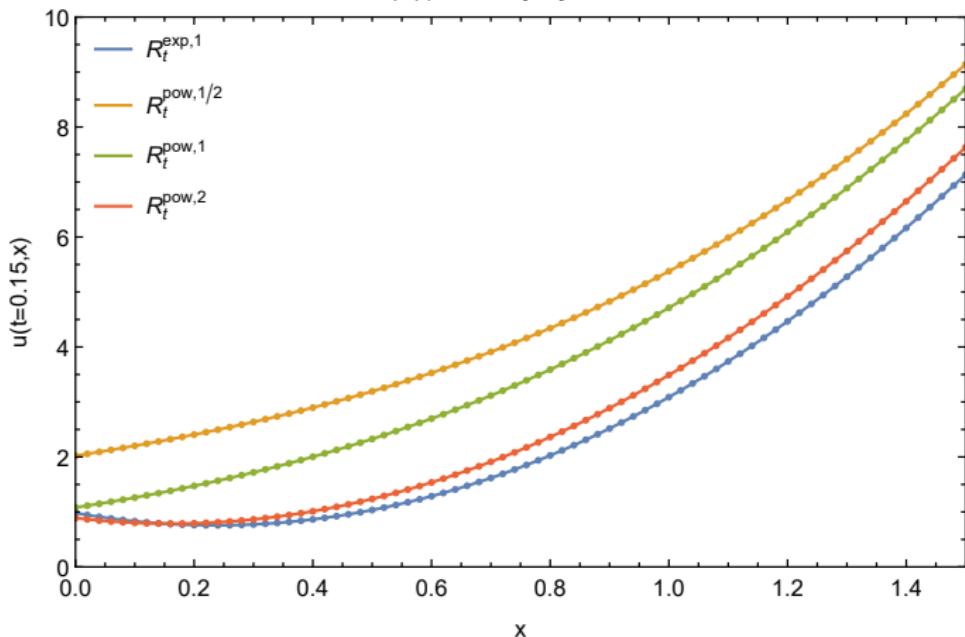


# O(N) model in d=0 - Regulator trajectories



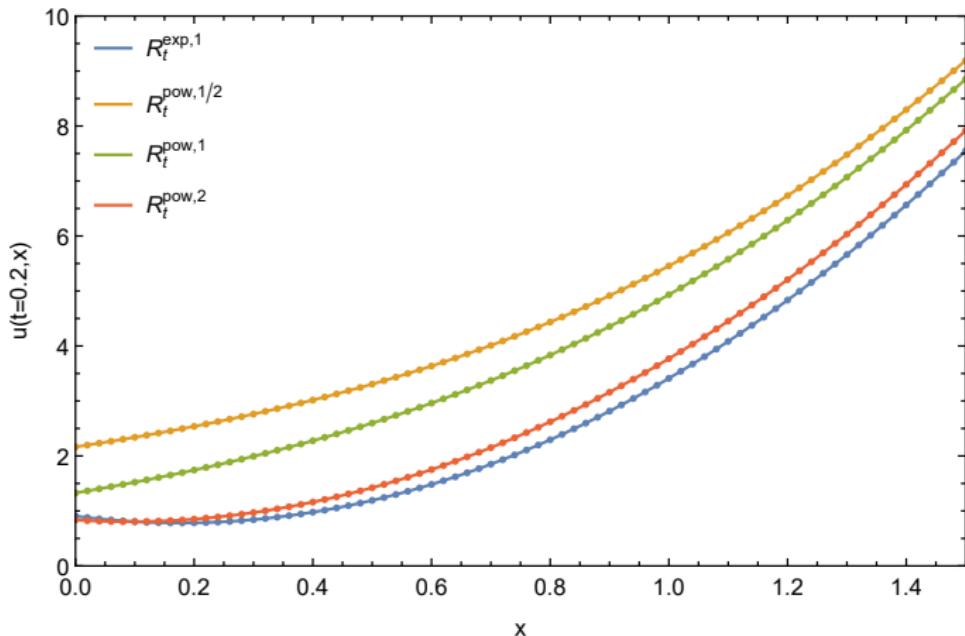
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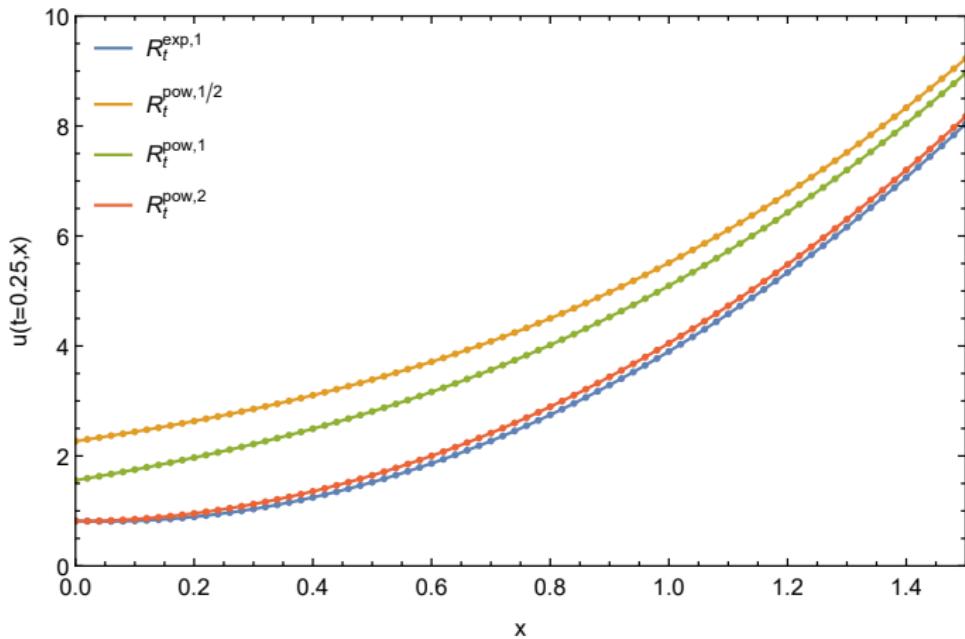


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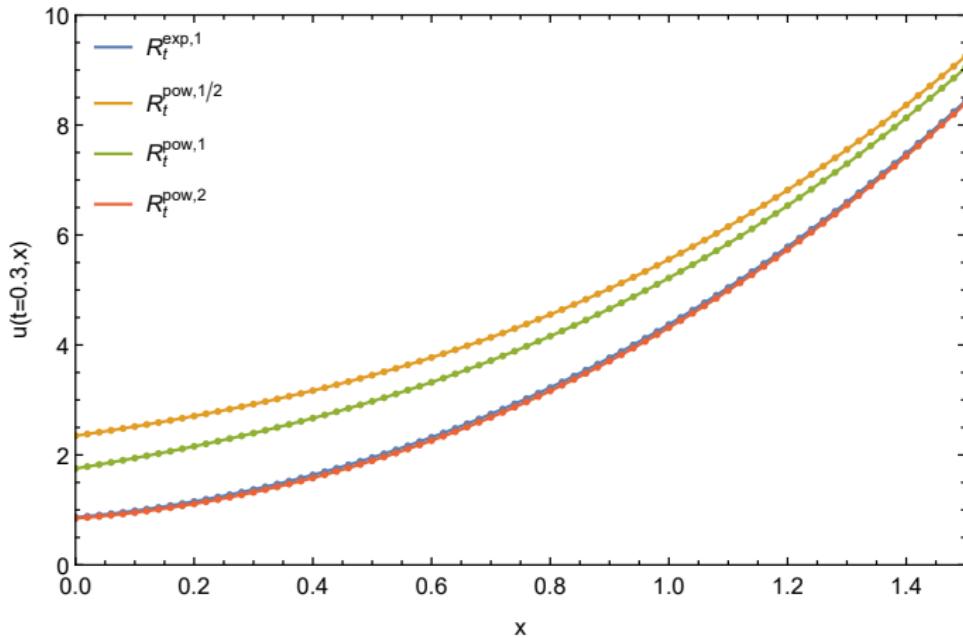


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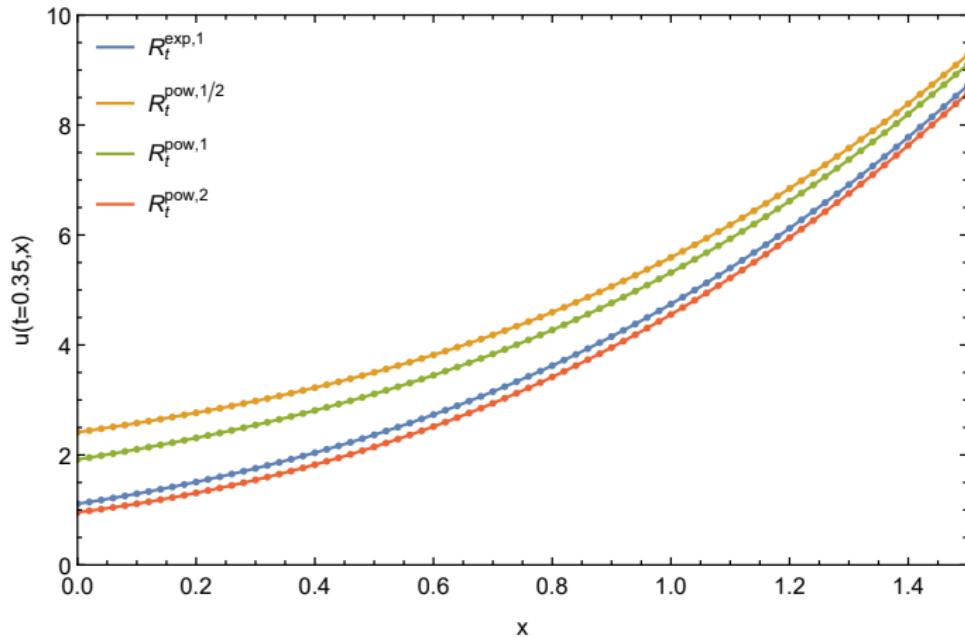


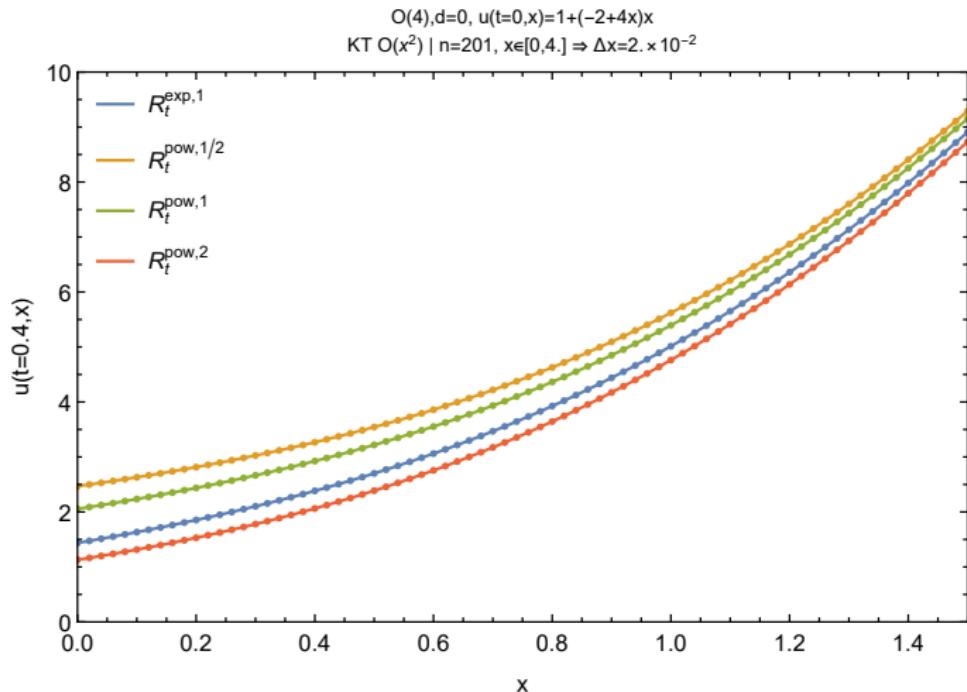
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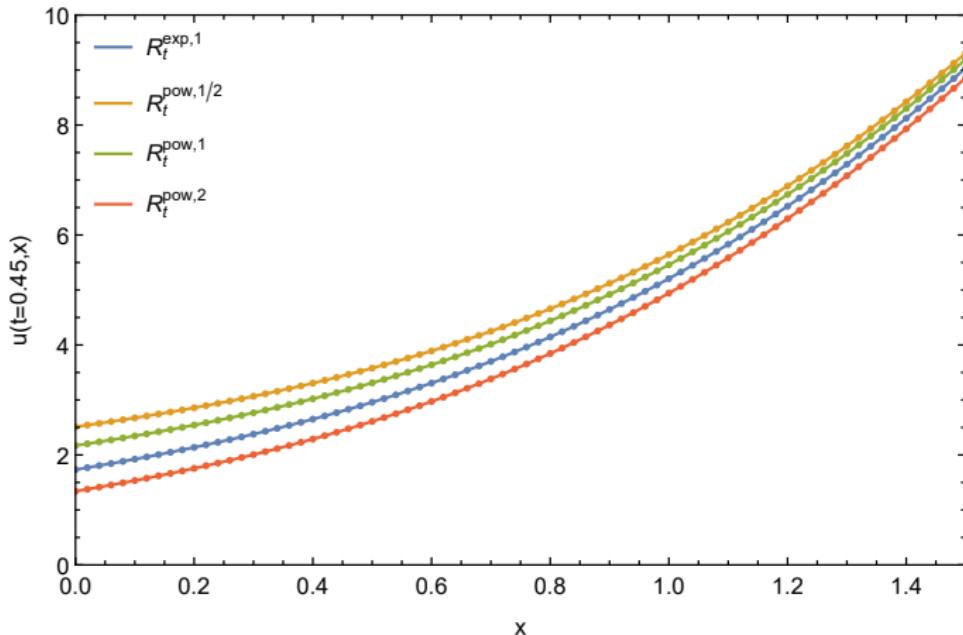
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 KT  $O(x^2) \mid n=201, x \in [0, 4] \Rightarrow \Delta x = 2 \cdot 10^{-2}$





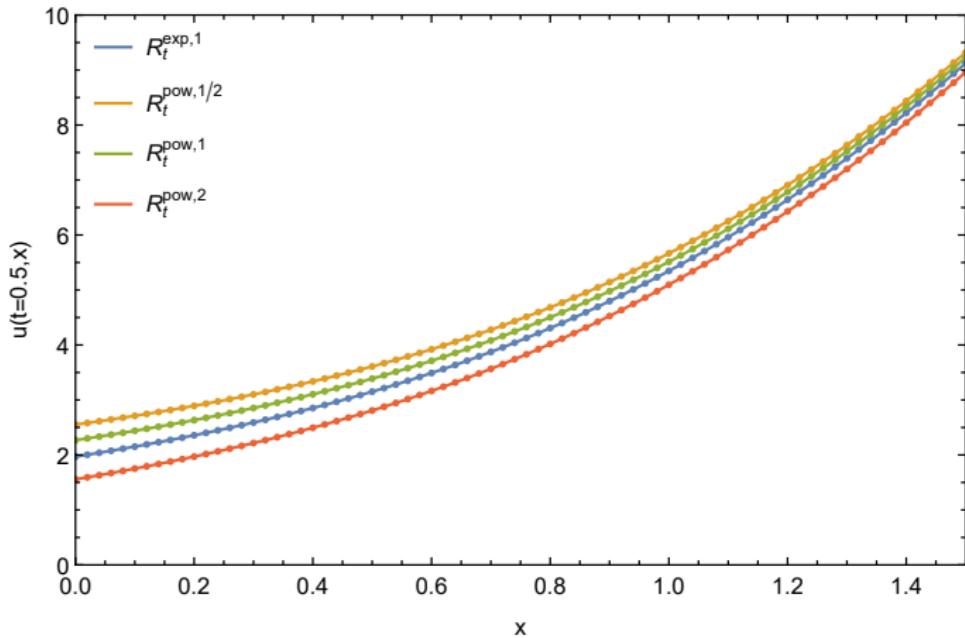
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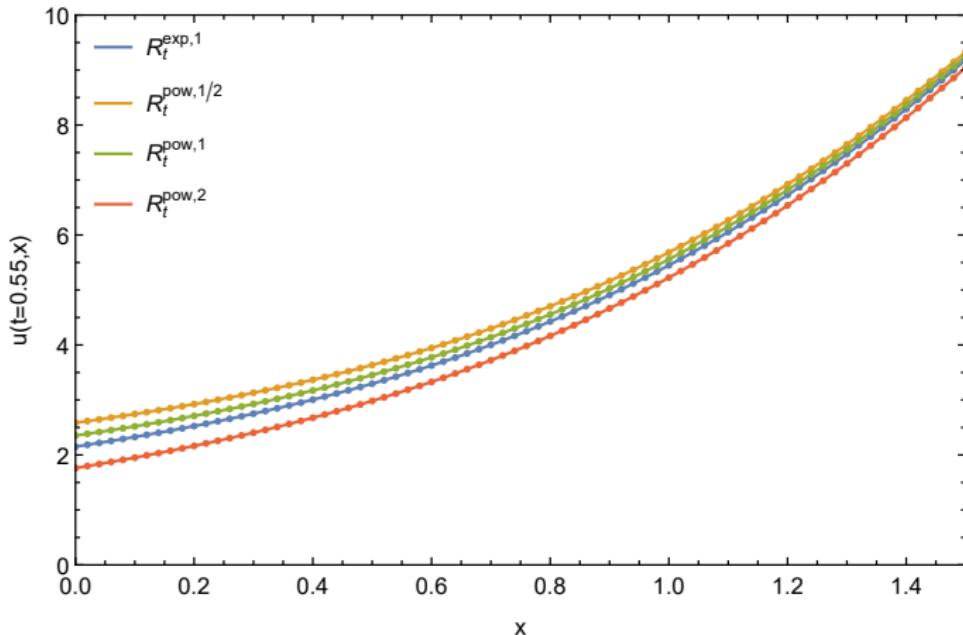


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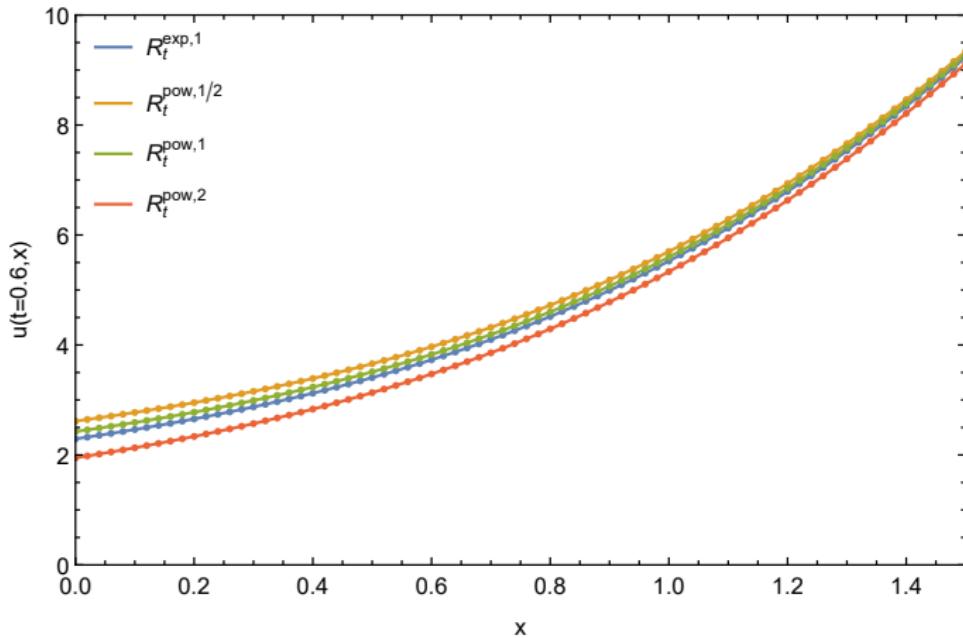


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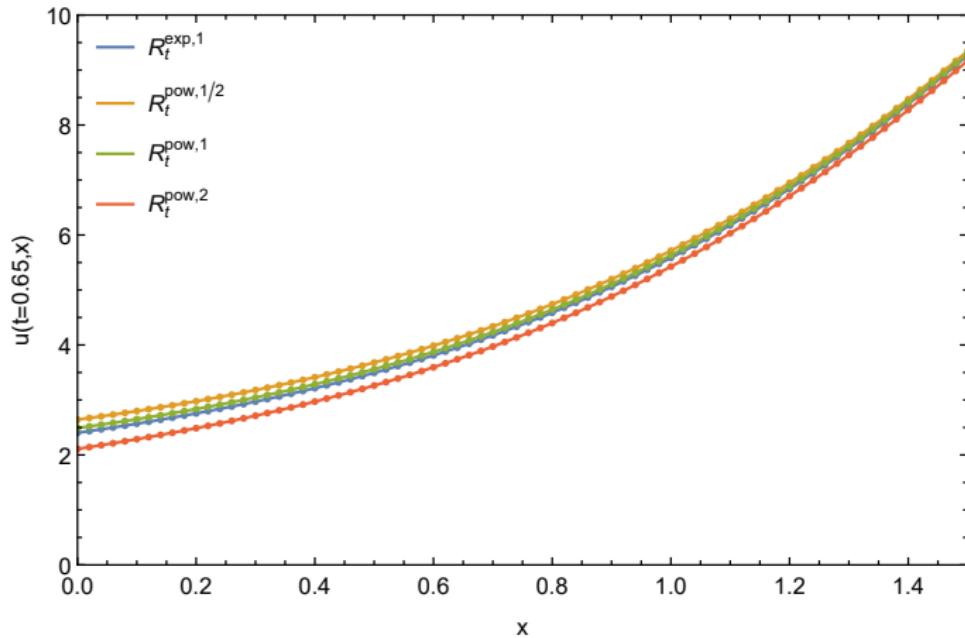


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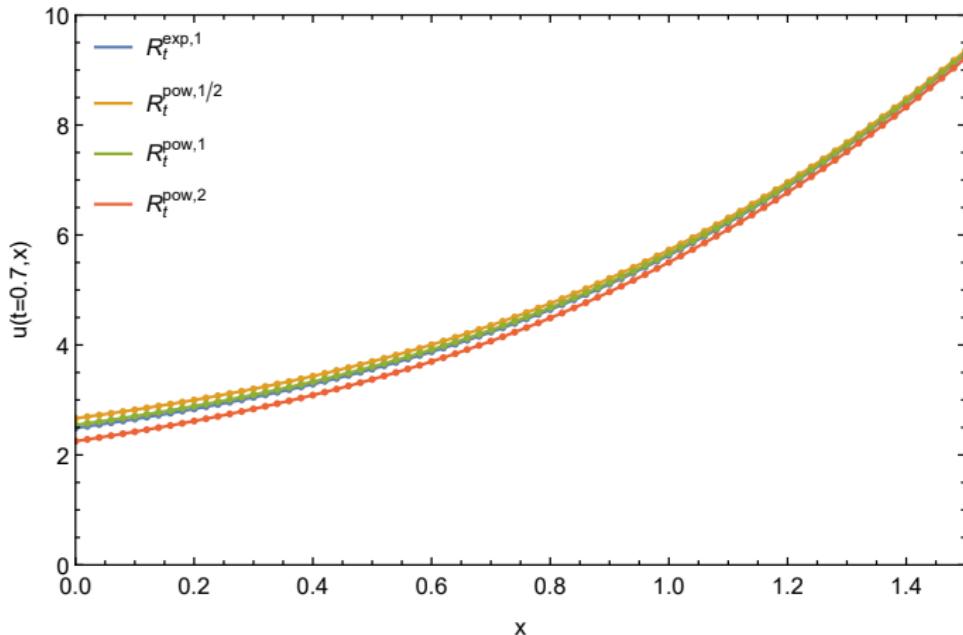


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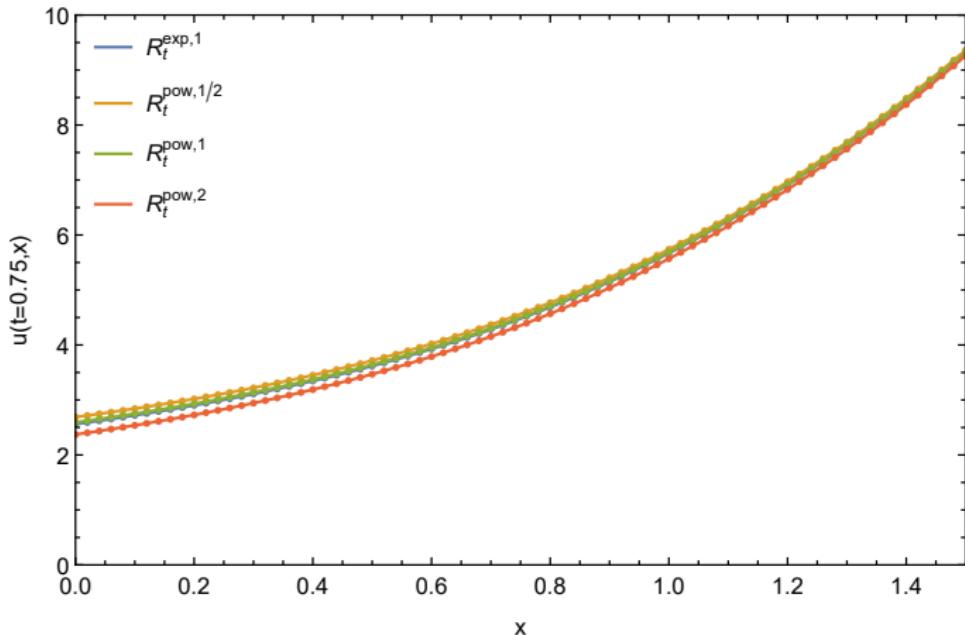


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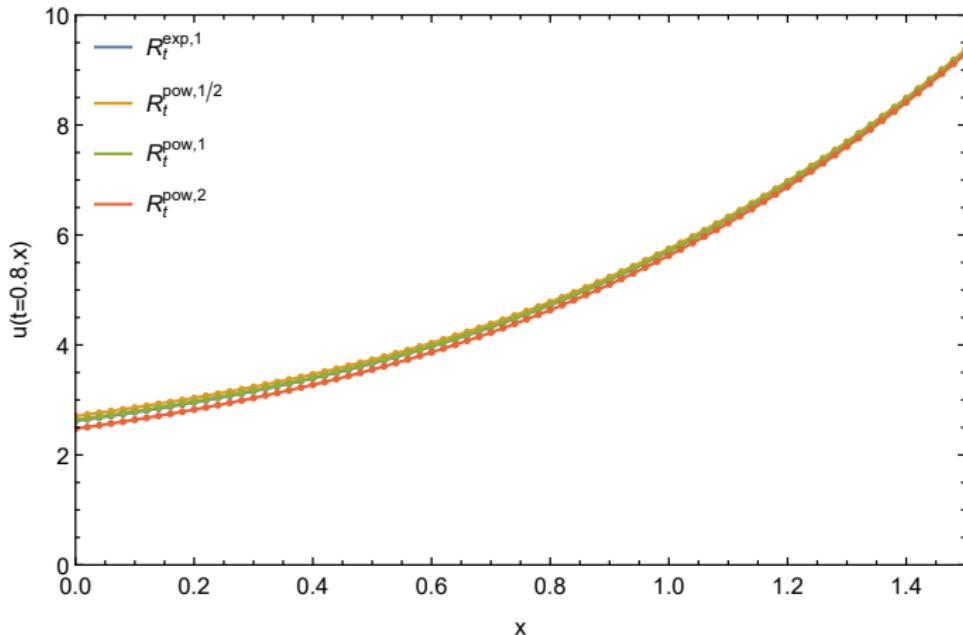


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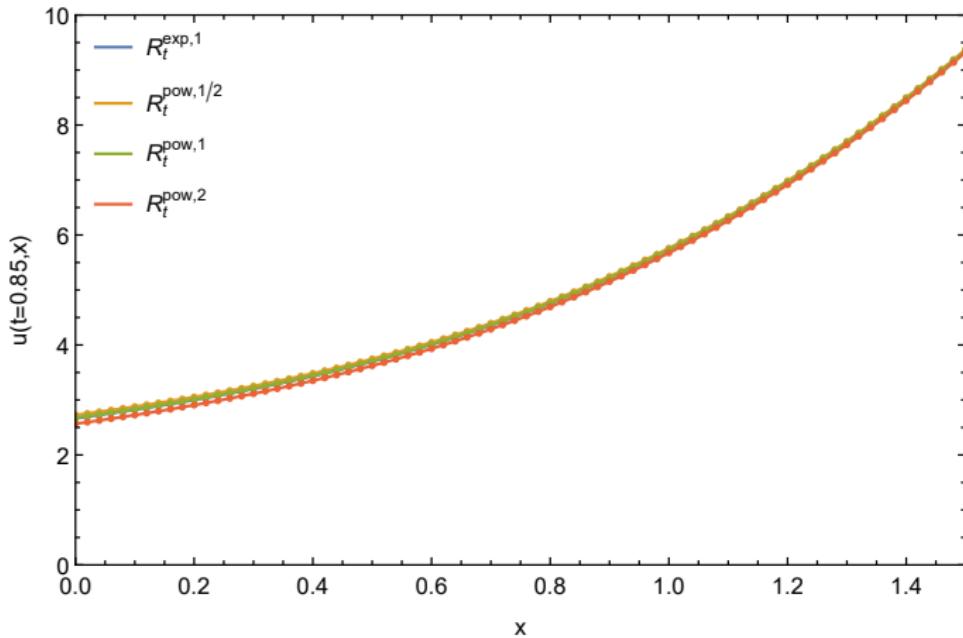
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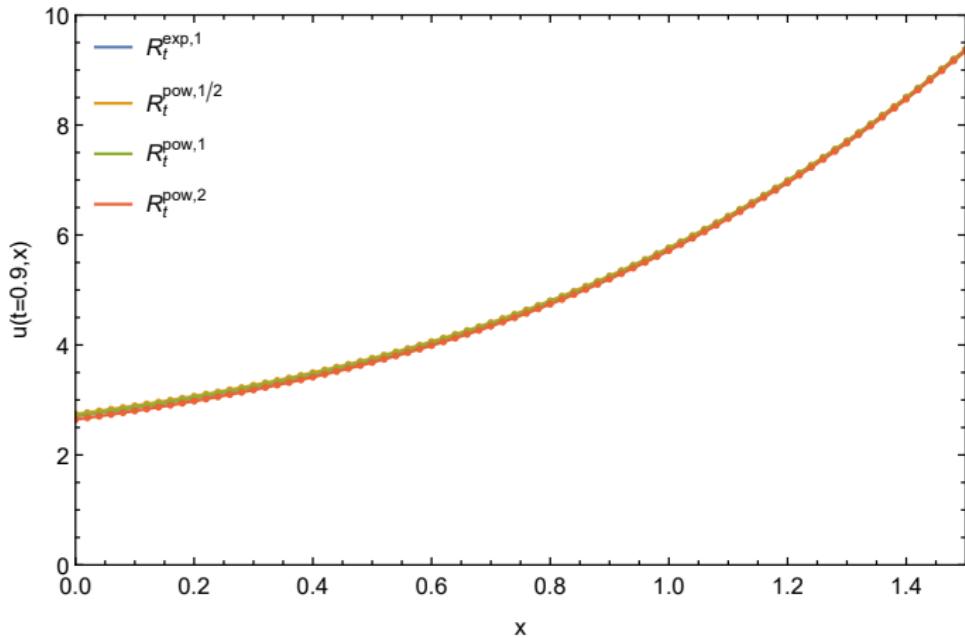
# O(N) model in d=0 - Regulator trajectories

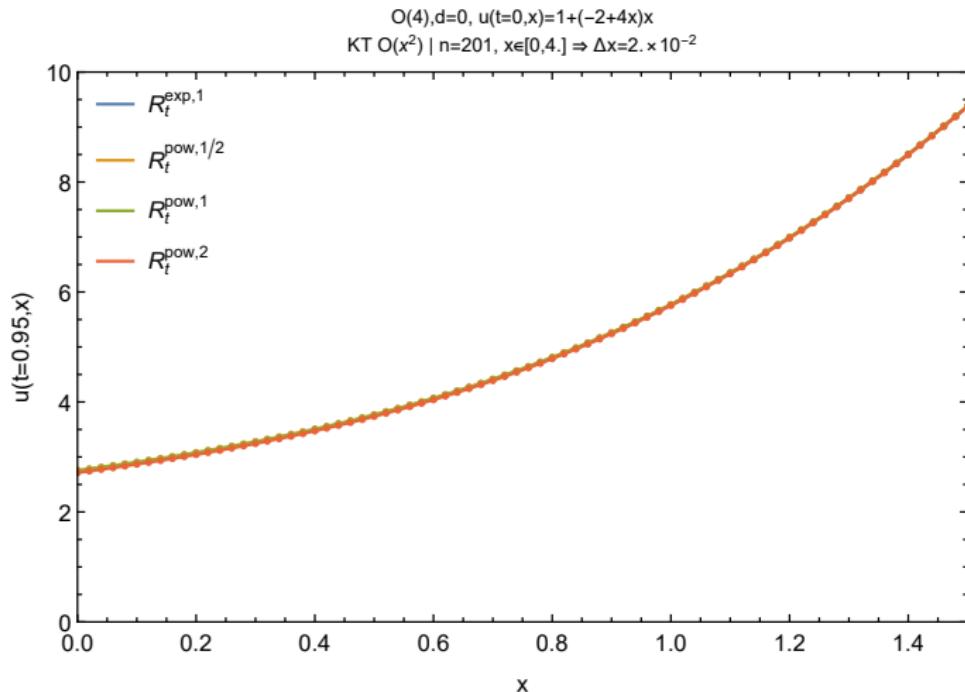
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# O(N) model in d=0 - Regulator trajectories

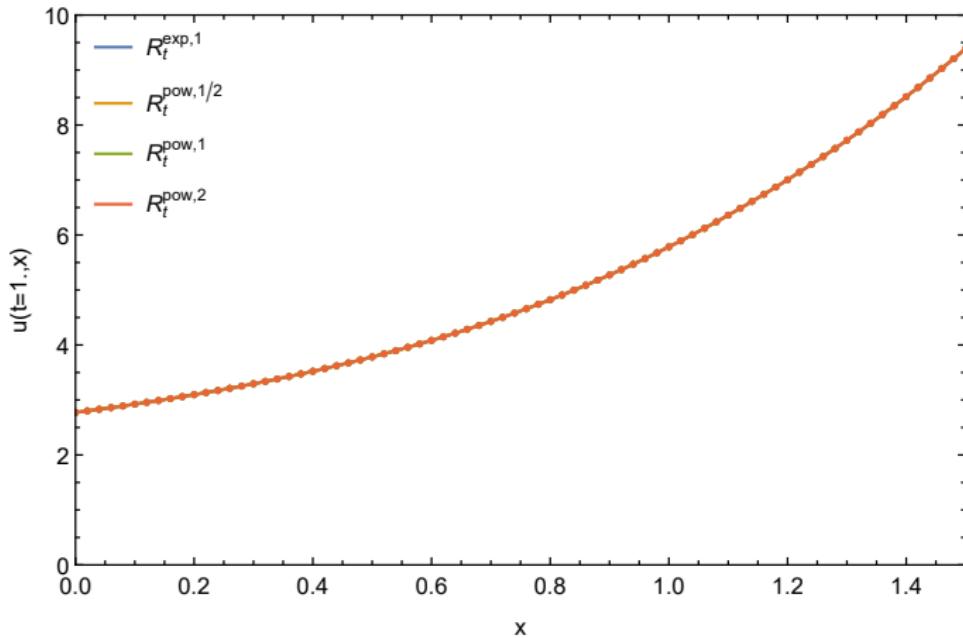
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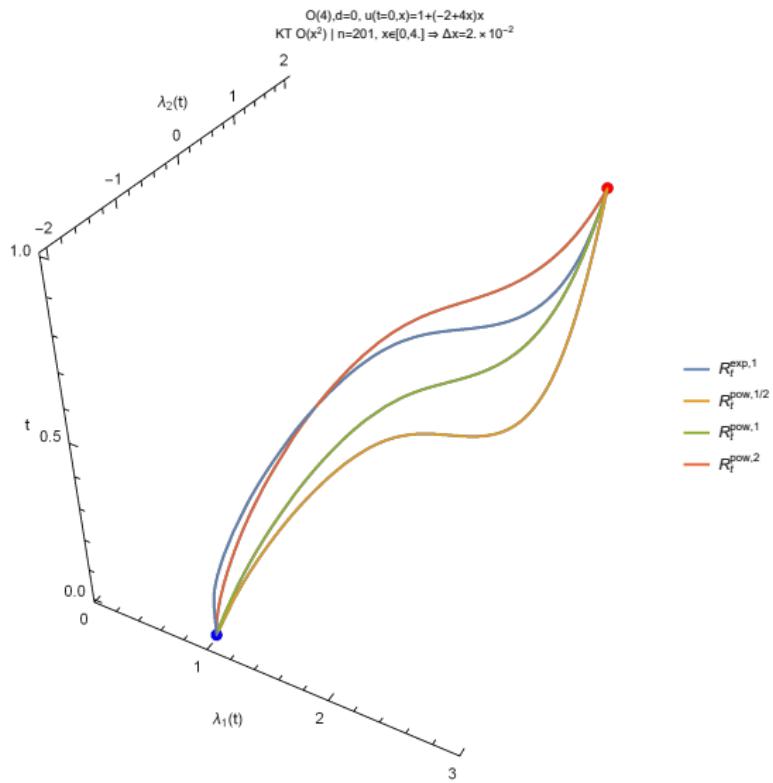


# O(N) model in d=0 - Regulator trajectories

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# O(N) model in d=0 - $\lambda_1(t)$ and $\lambda_2(t)$



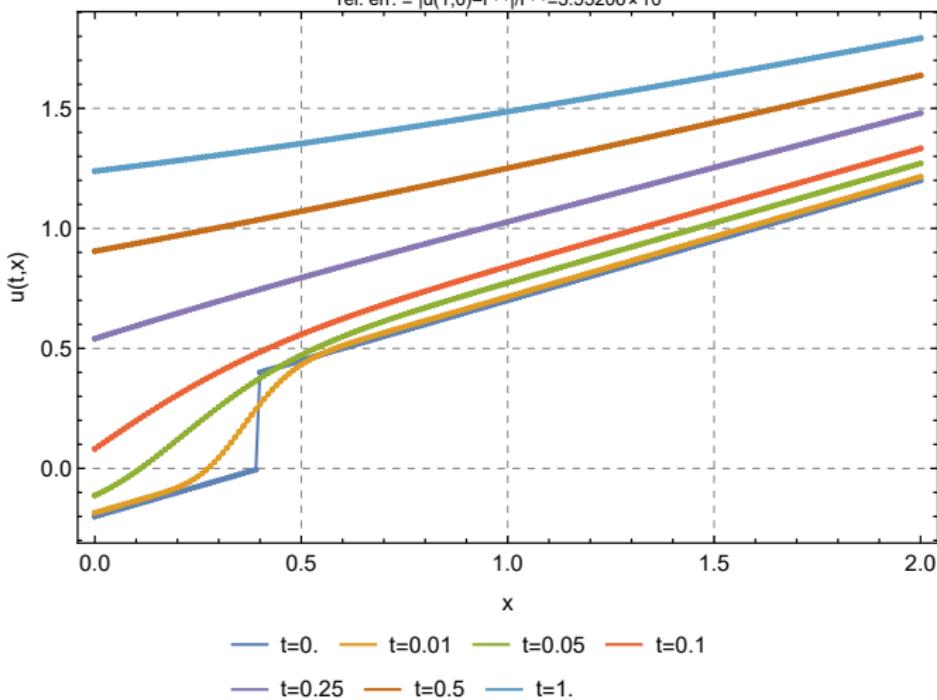
# O(4) model in d=0 - Discontinuous IC

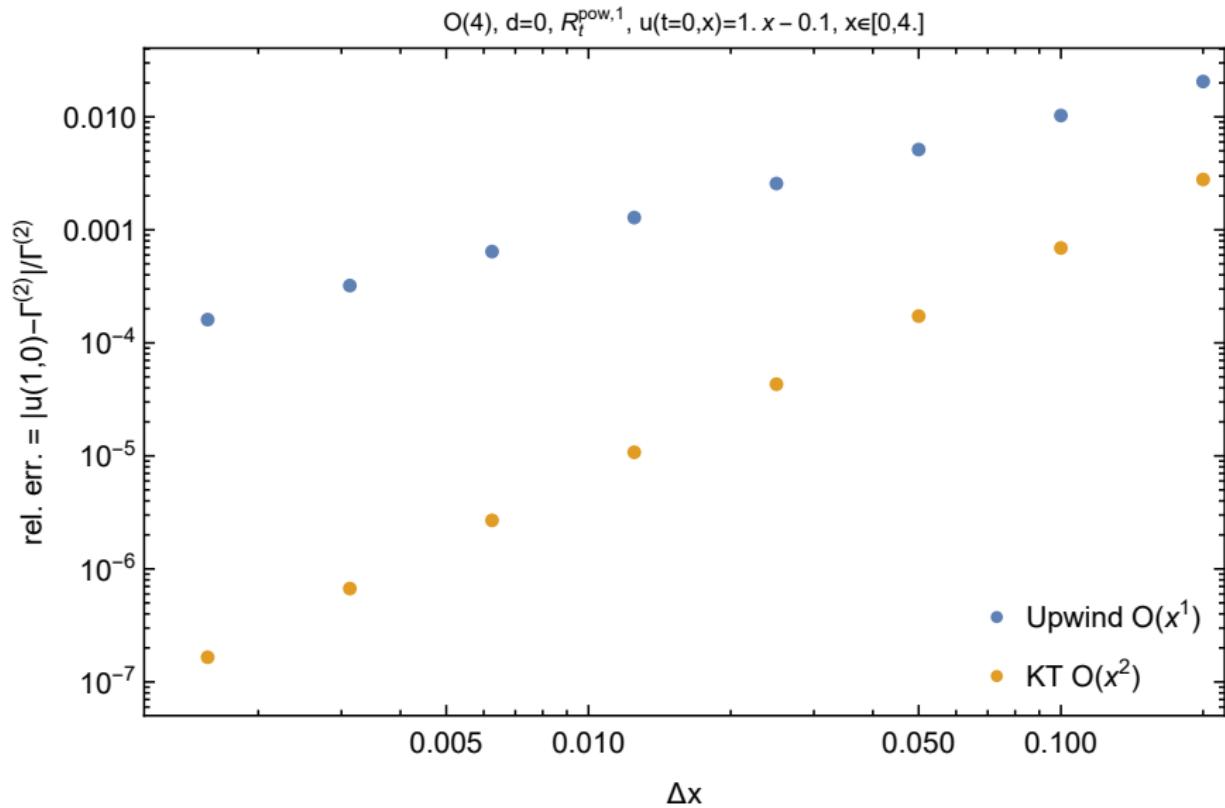
$O(4)$ ,  $d=0$ ,  $R_t^{\text{pow},1}$   $u(t=0,x)=0.5x+\text{lf}[x<0.4,-0.2,0.2]$

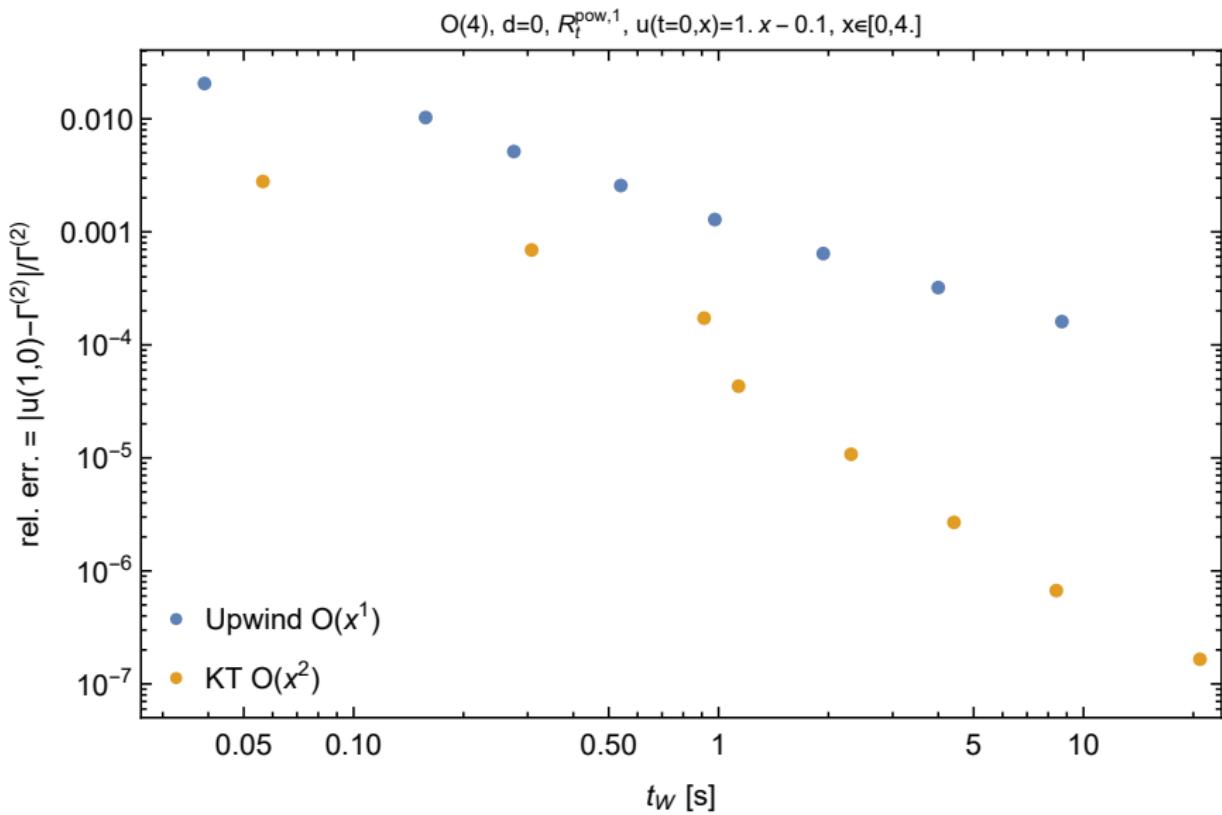
KT  $O(x^2)$  |  $n=201$ ,  $x \in [0,2]$   $\Rightarrow \Delta x = 1 \times 10^{-2}$

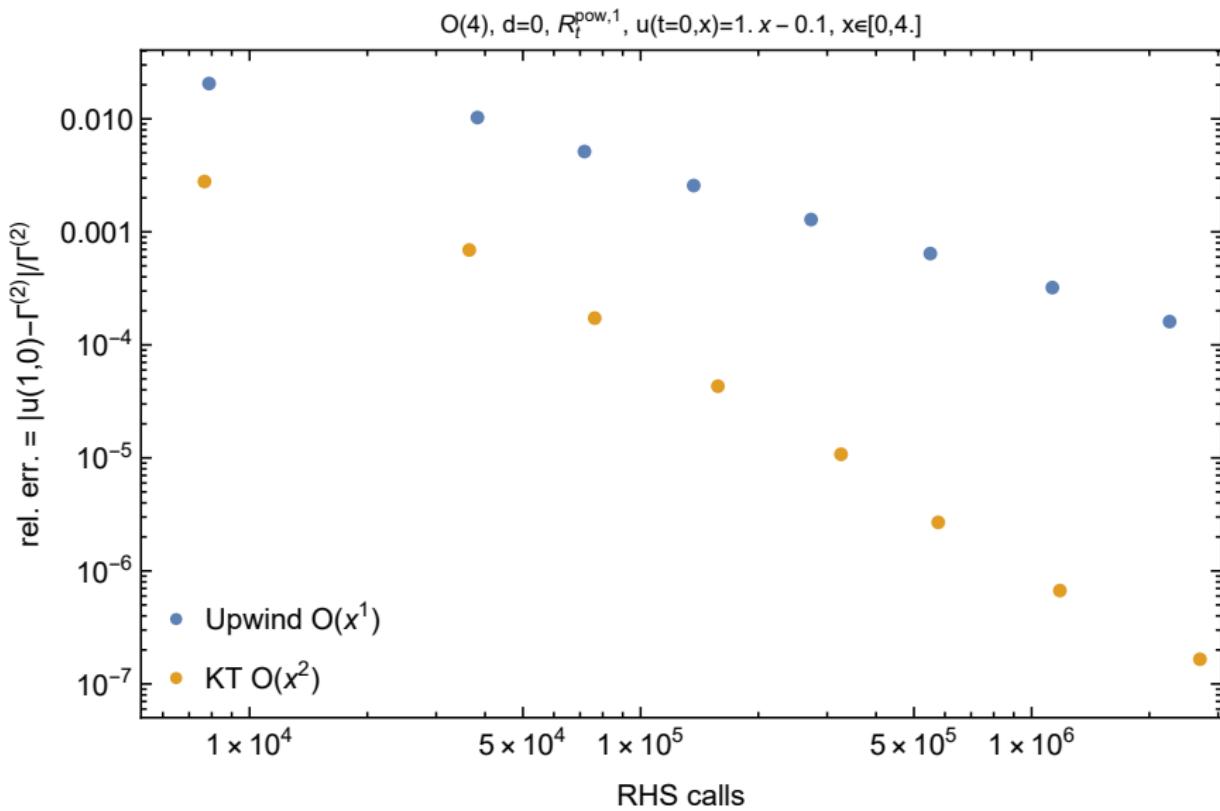
$t_w=1.25177 \times 10^1$  s, steps=3288, RHS-calls/n=9026

rel. err. =  $|u(1,0)-\Gamma^{(2)}|/\Gamma^{(2)}=3.55206 \times 10^{-4}$









- ▶ The second order KT-scheme in its semi-discrete form (4)
  - Robust and competitive spatial discretisation scheme
  - Easy to implement/extend
  - First applications to FRG LPA flow eqs. look very promising
- ▶ O(N) model in zero dimensions
  - Well suited for numerical test (exact solutions, non-trivial flow eqs.)
  - Very instructive toy model for various aspects of FRG and QFTs in general
- ▶ TODO:
  - Finite volume/KT scheme beyond LPA
  - Finite volume/KT scheme at finite  $T$  and  $\mu$
  - ...