DISTRIBUTIONS OF DISCRETE RVS

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Some material has been adopted from
the Pennsylvania State University online course STAT 414

Previous lecture

- Probability density function
- Cumulative distribution function
- Expected value and its properties

Today's lecture

- Moments of RV
- Discrete RV:
  - Binomial
  - Bernoulli
  - Geometric
  - Poisson

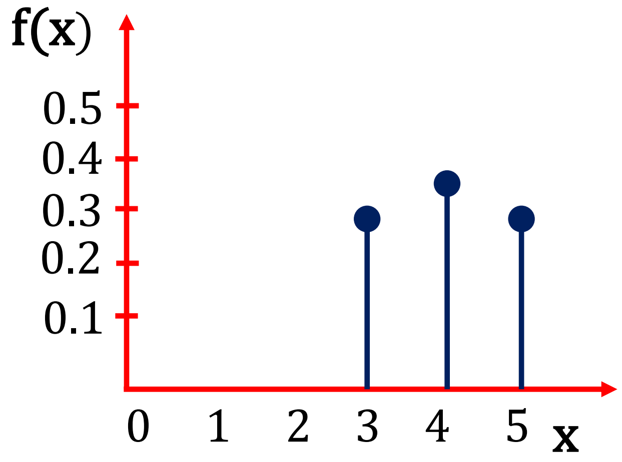
Example

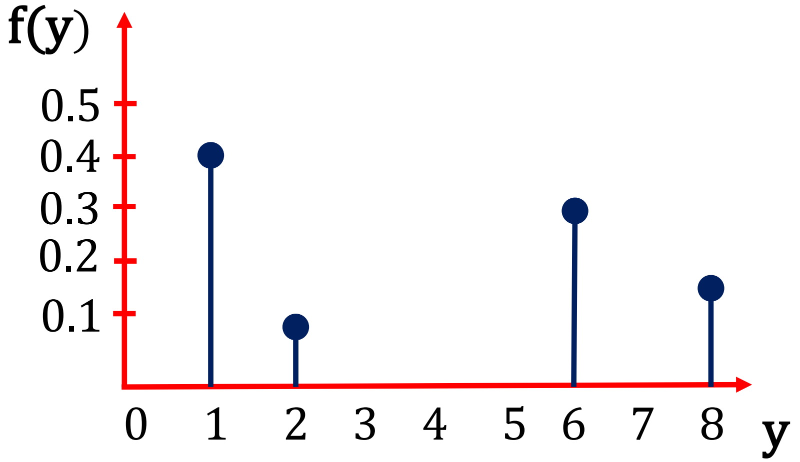
For two RVs with given PMFs, calculate the mean

x	3	4	5
$f(x)$	0.3	0.4	0.3

y	1	2	6	8
$f(y)$	0.4	0.1	0.3	0.2

Solution:





$$\mu_X = E(X) = \sum_{x \in S} x f(x)$$

$\mu_X = 3(0.3) + 4(0.4) + 5(0.3) = 4$

$\mu_Y = 1(0.4) + 2(0.1) + 6(0.3) + 8(0.2) = 4$

Moments of RV

n^{th} MOMENT of a RV

$E(X^n) = \sum_{x \in S} x^n f(x)$ for discrete RV

$E(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx$ for continuous RV

Central Moments of RV

n^{th} CENTRAL MOMENT of a RV

$E((X-\mu_X)^n) = \sum_{x\in S}(x-\mu_X)^n f(x)$ for discrete RV

$E((X-\mu_X)^n) = \int_{-\infty}^{\infty}(x-\mu_X)^n f(x)dx$ for continuous RV

- The central moment is the moment of the difference between the RV and its mean μ_X
- For n=1 the central moment is 0

Moments of RV

- A moment is a specific quantitative measure of the shape of a function
- For random variables they are expected values of powers of the RV
- The 0th moment is the total probability i.e. 1
- The 1st moment is the mean value of RV

$$E(X^n) = \sum_{x \in S} x^n f(x)$$

- The 2nd moment is the mean-square value of a RV
- The 2nd central moment is the variance
- The 3rd standardized moment is the skewness $\left(\gamma = \frac{E\left((X-\mu_X)^3\right)}{\sigma^3}\right)$
  - ✓ skewness is a measure of lopsidedness of the distribution
  - ✓ $\gamma = 0$ - symmetric
  - ✓ $\gamma > 0$ – skewed to the right
  - ✓ $\gamma < 0$ – skewed to the left

σ – standard deviation

Variance of RV

Variance of a RV - 2nd central moment of RV

$\sigma_X^2 = \mathrm{E}\big((\mathrm{X} - \mu_X)^2\big) = \sum_{\mathrm{x} \in \mathrm{S}} (\mathrm{x} - \mu_X)^2\, \mathrm{f}(\mathrm{x})$ for discrete RV

$\sigma_X^2 = \mathrm{E}\big((\mathrm{X} - \mu_X)^2\big) = \int_{-\infty}^{\infty} (\mathrm{x} - \mu_X)^2\, \mathrm{f}(\mathrm{x})\mathrm{dx}$ for continuous RV

- The square root of variance is called ***standard deviation:***

$$\sigma = \sqrt{\sigma_X^2}$$

Variance of RV

Variance of a RV

$$\sigma_X^2 = \mathrm{E}(\mathrm{X}^2) - {\mu_X}^2$$

Proof:

$$\sigma_X^2 = \mathrm{E}\left((\mathrm{X} - \mu_X)^2\right) = \mathrm{E}(\mathrm{X}^2) - 2E(X)\mu_X + {\mu_X}^2 =$$

$$= \mathrm{E}(\mathrm{X}^2) - 2{\mu_X}^2 + {\mu_X}^2$$

$$= \mathrm{E}(\mathrm{X}^2) - {\mu_X}^2$$

μ_X is a constant!

Variance of RV

Theorem

For X with μ_X and σ_X^2, the mean, variance and std. dev. of RV $Y = aX + b$ is:

$$\mu_Y = a\mu_X + b$$

$$\sigma_Y^2 = a^2\sigma_X^2$$

$$\sigma_Y = |a|\sigma_X$$

Variance of RV

- Proof:

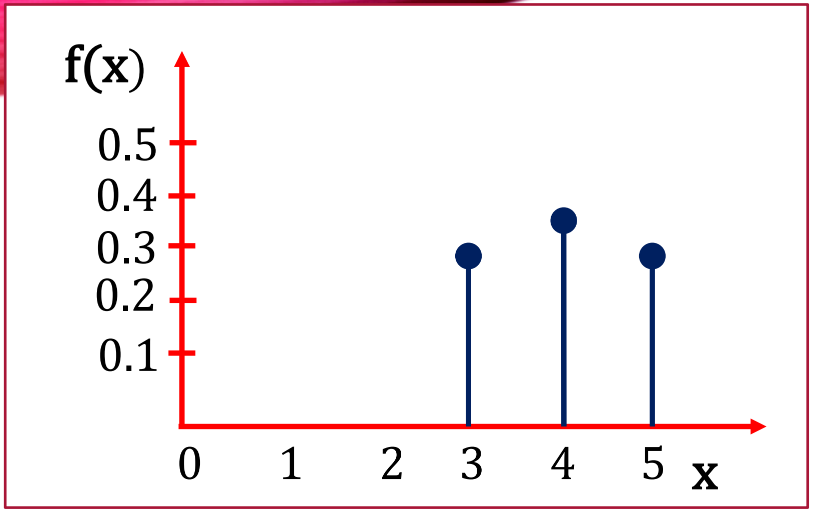
$$E(aX + b) = aE(X) + b = a\mu_X + b$$

$$\sigma_X^2 = \mathrm{E}([(\mathrm{aX} + \mathrm{b}) - (a\mu_X + b)]^2) =$$

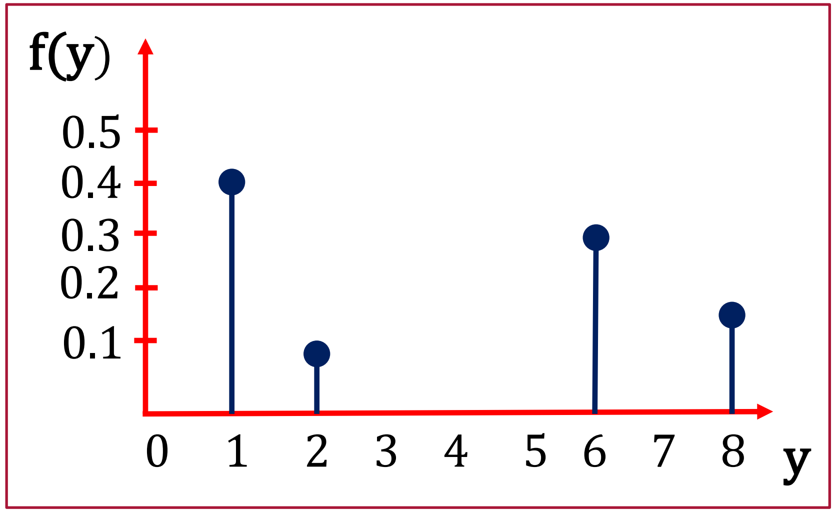
$$= \mathrm{E}(\mathrm{a}^2 (\mathrm{X} - \mu_X)^2) = \mathrm{a}^2\mathrm{E}((\mathrm{X} - \mu_X)^2 = \mathrm{a}^2\sigma_X^2$$

$$\sigma = \sqrt{\sigma_X^2} = |a|\sigma_X$$

Variance - Example



$$\mu_X = \mu_Y = 4$$



$$\text{Variance: } \sigma_X^2 = \mathrm{E}\big((\mathrm{X} - \mu_X)^2\big) \quad or \quad \sigma_X^2 = \mathrm{E}(\mathrm{X}^2) - {\mu_X}^2$$

$$\sigma_X^2 = 3^2\frac{3}{10} + 4^2\frac{4}{10} + 5^2\frac{3}{10} - 4^2$$
$$= 2.7 + 6.4 + 7.5 - 16 = 0.6$$

$\sigma_X = 0.77$

$$\sigma_Y^2 = 1^2\frac{4}{10} + 2^2\frac{1}{10} + 6^2\frac{3}{10} + 8^2\frac{2}{10} - 4^2$$
$$= 0.4 + 0.4 + 10.8 + 12.8 - 16 = 8.4$$

$\sigma_Y = 2.9$

Y has a higher variance than X i.e. there is much larger spread of the values of Y

Permutations

- Permutations - the number of different arrangements, in which items can be placed (the order of the items matters)
- If there are n items to be arranged and we choose r of those items at a time ($r \leq n$), the number of permutations of n items chosen r at a time (${}_nP_r$) is calculated in a following way:
  - ✓ 1st item can be chosen from n items, 2nd out of (n-1), 3rd out of (n-2), rth out of (n-(r-1))
  - ✓ Total no. of of permutations of n items taken r at a time, is

$$ {}_nP_r = n(n-1)\ldots(n-r+1) = \frac{n(n-1)\ldots(n-r+1)(n-r)\ldots 2\cdot 1}{(n-r)(n-r-1)\ldots 2\cdot 1} $$

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Combinations

- Combinations are similar to permutations, but they take no account of order
- The no. of permutations is larger than the no. of combinations, by the no. of ways the chosen items can be arranged
- Since r items can be arrange in r! ways, the no. of combinations equals:

$$ {}_{n}C_{r} = \frac{{}_{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!} $$

- ${}_{n}C_{r}$ gives the number of equally likely ways of choosing r items from a group of n distinguishable items.

Some “important” RVs

- When certain conditions are met, we can derive a general formula for the PMF/PDF of a random variable X
- We can then use that formula to calculate probabilities concerning X rather than resorting to first principles
- Or we can use cumulative probability tables that others have created

Discrete RV: Binomial

RV has a binomial distribution if:

1. An experiment, or trial, is performed in exactly the same way *n* times.
2. Each of the *n* trials has only two possible outcomes. One of the outcomes is called a "success," while the other is called a "failure."
3. The *n* trials are independent.
4. The probability of success, denoted *p*, is the same for each trial:

$$p(x=0)= p$$

$$p(X=1)=1\text{-}p$$

5. The random variable *X* = the number of successes in the *n* trials.

Binomial RV – Example 1

At a match of team A and B the fans of team A constitute 80% of the spectators, while the remaining 20% are the fans of team B. We randomly choose 3 fans. What is the probability that we choose:

a) 0 fans of team A
b) 1 fan of team A
c) 2 fans of team A
d) 3 fans of team A?

The sample space is: {AAA, AAB, ABB, BBB, ABA, BAA, BAB, BBA}

Binomial RV – Example 1

a) There is only 1 possibility for choosing 0 fans of team A: X={BBB},

b) There 3 possibilities of choosing 1 A fan:

{BBA, BAB, ABB}

a) There 3 possibilities of choosing 2 A fans

{ABA, AAB, BAA}

d) There is only 1 possibility for choosing 3 fans of team A: {AAA}

The no. of possibilities is equal to ${}_3C_r = \frac{{}_3P_r}{r!} = \frac{3!}{r!(3-r)!}$, where r=0, 1, 2 and 3

Binomial RV – Example 1

a) The probability of choosing 0 fans of team A is

$p(X=0)= (0.2)(0.2)(0.2)=1(0.2)^3\,(0.8)^0$ since: $p(B)=20\%$ and $p(A)=0.8$

b) The probability of choosing 1 A fan is:

$p(X=1)=P(BBA) + P(BAB) + P(ABB)= 3(0.8)(0.2)(0.2)=3(0.8)^1(0.2)^2$

c) The probability of choosing 2 A fans is:

$p(X=2)=P(ABA) + P(AAB) + P(BAA)= 3(0.8)(0.8)(0.2)=3(0.8)^2(0.2)^1$

d) The probability of choosing 3 A fans is:

$p(X=2)= P(AAA) = 1(0.8)(0.8)(0.8)=1(0.8)^3(0.2)^0$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\binom{n}{k}$is also know as binomial coefficients

PMF of Binomial RV

PMF of Binomial RV is

$$f(X) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- n is a sample size
- x is the no. of successes
- The mean and variance of binomial RV are:

$$\mu_X = np$$

$$\sigma_X^2 = np(1-p)$$

Binomial RV – Example 2

Weighted coin (70% chance of getting a head) is tossed 100 times in the exactly the same way. If X equals the number of heads tossed, is X a binomial RV?

Solution:

1. The coin is tossed in exactly the same way 100 times✓
2. Each toss results in either a head (success) or a tail (failure) ✓
3. One toss doesn't affect the outcome of another toss. The trials are independent. ✓
4. The probability of getting a head is 0.70 for each toss of the coin. ✓
5. X equals the number of heads (successes). ✓

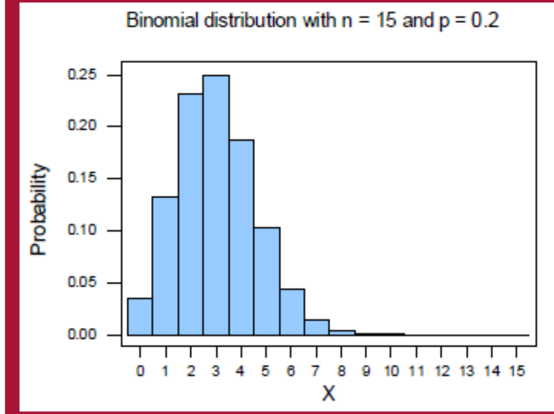
Yes, X is a binomial RV

Binomial RV – Example 3

X represents the # of bit errors in a block of n bits, if errors are independent and occur with probability p

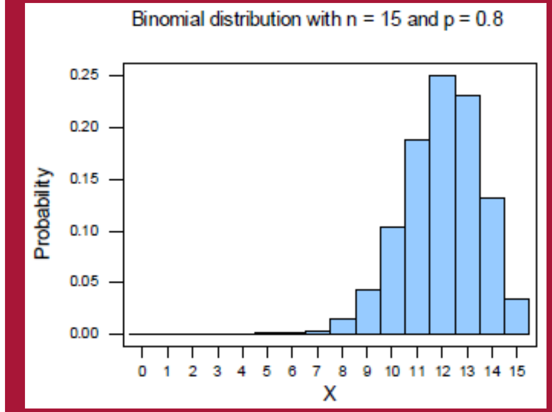
Effect of n and p on Binomial Distribution

Small p, small n



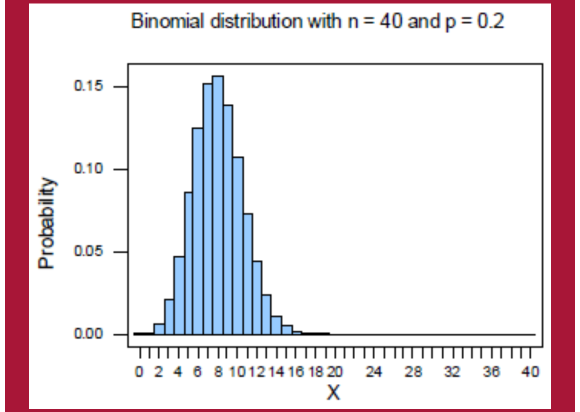
The distribution is skewed right

Large p, small n



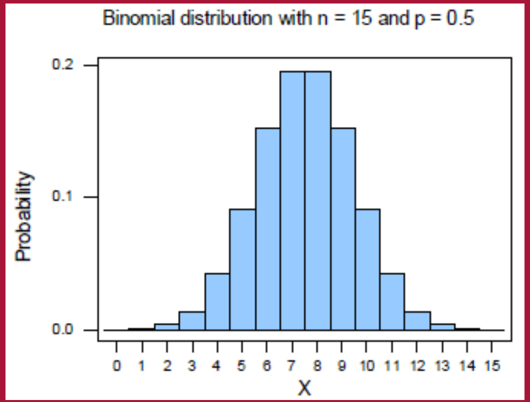
The distribution is skewed left

Small p, large n



The distribution => symmetry

$p = \frac{1}{2}$, n sml or lrg



The distribution is symmetric

Discrete RV: Bernoulli RV

- Special case of binomial RV with n=1

$$p(x=0)= p$$

$$p(X=1)=1-p$$

- **A Bernoulli trial**
  1. there are two possible outcomes: "success" or "failure”
  2. the trials are independent
  3. p, the probability of success, remains the same from trial to trial

Geometric RV – Example

A receiver detects a stream of bits. If the probability of receiving bit in error is p, what is the probability of receiving 10 correct bits, before the 1st occurrence of an error?

Solution:

What is the probability that 11th bit will be erroneous:

we are looking for a following bit sequence: C1, C2, C3 … C10, E1

$$P(X=11)=\underbrace{(1-p)(1-p)\dots(1-p)}_{(1-p)^{10}}\cdot p = (1-p)^{10}\,p = (1-p)^{11-1}p$$

Discrete RV: Geometric

- For Bernoulli trials, let X denote the number of trials until the first success.
- Then, the X has a geometric distribution with the PMF:

PMF of Geometric RV is

$$f(X) = (1-p)^{x-1}p$$

Discrete RV: Poisson

- Probability distribution, of a random events happening at a constant average rate
- Probability of any possible no. of events, in a given time interval or area of space
- The outcomes must occur randomly and independent of each other
- Used when:
  - ✓ the number of discrete occurrences is much larger than the average number of occurrences in a given interval of time or space
  - ✓ counting the occurrences is possible, but not of the corresponding non-occurrences (no. of thunders can be counted, but no. of times thunder was not heard not)
- Can also be used as a convenient approximation to the binomial distribution in some circumstances

Discrete RV: Poisson

- Let the discrete random variable X denote the number of times an event occurs in an interval of time (or space).
- Then X may be a Poisson random variable with x = 0, 1, 2, … with the PMF:

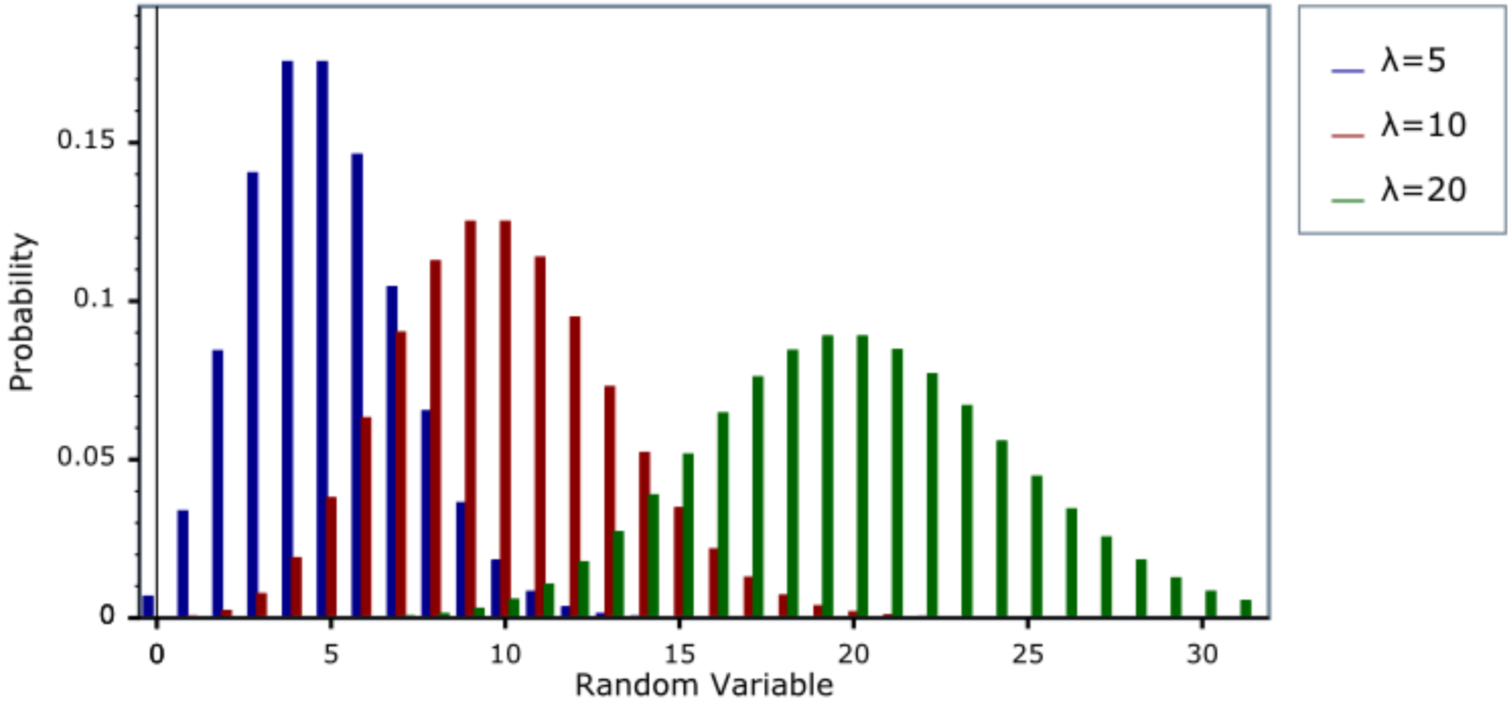
PMF of Poisson RV is

$$f(X) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad e=2.71828...$$

- x – no. of successes,
- λ is the MEAN <u>and</u> the VARIANCE of X and $\lambda > 0$

Discrete RV: Poisson

- For small values of λ, the PMF is skewed to the right, for large λ - symmetric
- Valid for x=0, 1, 2…∞



Example: If X has a Poisson distribution with λ=3 (Po(3)), what is P(5)?

$$f(X) = \frac{\lambda^x e^{-\lambda}}{x!} \Rightarrow P(5) = \frac{3^5 e^{-3}}{5!} = 0.101$$

Poisson Distribution - Example

In a binary transmission the error occurrence has a Poisson distribution with an average of 2 errors per frame.

1. What is the probability that there will be 3 errors in a 2 frames
2. What is the probability that a given frame will have more than 2 errors? i.e. X~ Po(2) what is P(X≥2)?

Solution:

1. λ=2 errors/frame, thus λ=4 errors/2 frames, X~ Po(4)

$$\mathrm{P}(X=3)=\frac{4^3e^{-4}}{3!}=\frac{64\cdot e^{-4}}{6}=10.6e^{-4}=10.6\cdot 0.018=0.194=19\%$$

Poisson Distribution - Example

2. This could be calculated by adding P(X=2)+P(X=3)… infinite no of additions

   Alternatively, we can use the property: P(A)=1-P($\bar{A}$)

$$\mathrm{P}(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$$

$$\mathrm{P}(X \geq 2) = 1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!} =$$

$$= 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2} = 0.594 = 5.9\%$$

Poisson RVs - Examples

- Let X equal the number of typos on a printed page (example of an interval of space).
- Let X equal the number of cars passing through the given intersection in one minute (example of an interval of time).
- Let X equal the number of customers at an ATM in 10-minute intervals.
- Let X equal the number of students arriving during office hours.

<u>Poisson is a good approximation for binomial RVs when:</u>

<u>n is large and p small, with λ = np</u>

(i.e. $n \geq 100$ and $p \leq 0.10$, allows to avoid calculating factorials of large no's).

Today's lecture

- Moments of RV
- Discrete RV:
  - Binomial
  - Bernoulli
  - Geometric
  - Poisson