Trinity College Dublin
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The University of Dublin

EEU33E03: Probability and Statistics

Lecture 5: Probability

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Outline

Probability

Probability is a quantitative measure of how likely a random phenomenon is to occur.

A **random experiment** has an unknown outcome, but a well defined set of possible outcomes, S. The set S is called the **sample set** or **sample space** for the experiment.

The Law of Large Numbers

As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome, e.g., flipping a coin.

An **element** of the sample space, S, is called a **sample point (elementary event)**.

An **event** can be described by a collection of sample points, i.e., a subset of a sample space.

Example

For example, if we roll a die,

$$S = \{1, 2, 3, 4, 5, 6\}$$

i.e. there are 6 sample points.

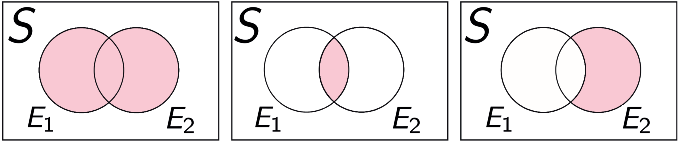
Let A be the event that the result is even. Then A can be described by the set $A = \{2, 4, 6\}$.

Any event can be represented by a subset of the sample set.

Combining Events

We often construct events by combining simpler events or we describe new events from combinations of existing events.

- The **union** of two events E_1 and E_2, denoted $E_1 \cup E_2$, is the set of outcomes that belong either to E_1, to E_2, or to both, i.e., "E_1 or E_2".
- The **intersection** of two events E_1 and E_2, denoted $E_1 \cap E_2$, is the set of outcomes that belong both to E_1 and to E_2, i.e., "E_1 and E_2".
- The **complement** of an event E, denoted E^C, is the set of outcomes that do not belong to E, i.e., E^C means "not E".

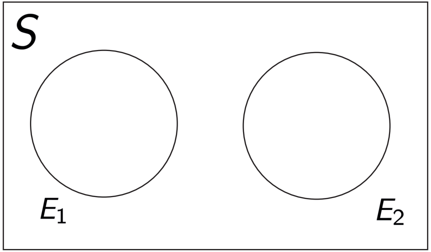


$E_1 \cup E_2$ $\quad$ $E_1 \cap E_2$ $\quad$ $E_2 \cap E_1^C$

Mutually Exclusive Events

The events E_1 and E_2 are said to be **mutually exclusive** if they have no outcomes in common.

More generally, a collection of events $E_1, E_2, \ldots, E_n$ is said to be mutually exclusive if no two of them have any outcomes in common.



Frequency Interpretation of Probability

We often use the letter P to stand for probability.

Suppose an experiment is repeated n times, where n is very large (effectively infinite), and event 'E' occurs r times. Then,

$$P(E) = \frac{r}{n} \tag{1}$$

It follows from this interpretation that,

a) $0 \leq P(E) \leq 1$.
b) $P(E) = 0$ if the event 'E' is impossible.
c) $P(E) = 1$ if the event 'E' is sure to happen.

Probabilities

- In many situations, the only way to estimate the probability of an event is to repeat the experiment many times and determine the proportion of times that the event occurs, e.g.,

  If it is desired to estimate the probability that a printed circuit board manufactured by a certain process is defective.

- In some cases, probabilities can be determined through knowledge of the physical nature of an experiment, e.g.,

  If it is known that the shape of a die is nearly a perfect cube and that its mass is distributed nearly uniformly, it may be assumed that each of the six faces is equally likely to land upward when the die is tossed.

Axioms of Probability

The probability theory is built based on three common sense rules, known as *axioms*.

Axioms of Probability

1. Given the sample space S, $P(S) = 1$.
2. For any event E, $0 \leq P(E) \leq 1$.
3. If E_1 and E_2 are mutually exclusive events, i.e., $(E_1 \cap E_2) = \varnothing$, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. And more generally, if $E_1, E_2, \ldots, E_k$, then $P(E_1 \cup \ldots \cup E_k) = P(E_1) + \ldots + P(E_k)$.

These axioms imply the following results.

$$P(E^C) = 1 - P(E), \tag{2}$$

$$P(\varnothing) = 0. \tag{3}$$

Probability:
Example

Example 1: The following table presents probabilities for the number of times that a certain computer system will crash in the course of a week. Let A be the event that there are more than two crashes during the week, and let B be the event that the system crashes at least once. Find a sample space. Then find the subsets of the sample space that correspond to the events A and B. Then find $P(A)$ and $P(B)$.

Number of Crashes	Probability
0	0.60
1	0.30
2	0.05
3	0.04
4	0.01

Probability:
Example

Solution: The sample space for the experiment is the set $S = \{0, 1, 2, 3, 4\}$. The events are $A = \{3, 4\}$ and $B = \{1, 2, 3, 4\}$.

Probability:
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$$P(A) = P(3 \text{ crashes or } 4 \text{ crashes happen}) = P(3 \text{ crashes}) + \mathrm{P}(4 \text{ crashes})$$
$$0.04 + 0.01 = 0.05.$$

Probability:
Example

Solution: The sample space for the experiment is the set $S = \{0, 1, 2, 3, 4\}$. The events are $A = \{3, 4\}$ and $B = \{1, 2, 3, 4\}$. To find $P(A)$, notice that A is the event that either 3 crashes happen or 4 crashes happen. The events "3 crashes happen" and "4 crashes happen" are mutually exclusive. Therefore, using Axiom 3,

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$$0.04 + 0.01 = 0.05.$$

As B^C is the event that no crashes happen. Therefore, using (2),

$$P(B) = 1 - P(B^C) = 1 - P(0 \text{ crashes}) = 1 - 0.6 = 0.4.$$

Sample Spaces with Equally Likely Outcomes

When the model of **equally likely outcomes** is assumed, the probabilities are chosen to be equal.

If S is a sample space containing N equally likely outcomes, and if E is an event containing k outcomes, then

$$P(E) = \frac{k}{N}. \tag{4}$$

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $\frac{1}{N}$.

Probability

Example

Example 2: An extrusion die is used to produce aluminum rods. Specifications are given for the length and the diameter of the rods. For each rod, the length is classified as too short, too long, or OK, and the diameter is classified as too thin, too thick, or OK. In a population of 1000 rods, the number of rods in each class is as follows:

	Diameter		
Length	**Too Thin**	**OK**	**Too Thick**
Too Short	10	3	5
OK	38	900	4
Too Long	2	25	13

A rod is sampled at random from this population. What is the probability that it is too short?

Probability
Example

	Diameter		
Length	**Too Thin**	**OK**	**Too Thick**
Too Short	10	3	5
OK	38	900	4
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Solution: We can think of each of the 1000 rods as an outcome in a sample space. Each of the 1000 outcomes is equally likely. We'll solve the problem by counting the number of outcomes that correspond to the event. The number of rods that are too short is $10 + 3 + 5 = 18$. Since the total number of rods is 1000,

$$P(\text{too short}) = \frac{18}{1000} = 0.018.$$

The Addition Rule

If E_1 and E_2 are **mutually exclusive** events, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Example 3: Refer to Example 2, If a rod is sampled at random, what is the probability that it is either too short or too thick?

	Diameter		
Length	**Too Thin**	**OK**	**Too Thick**
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	Diameter		
Length	**Too Thin**	**OK**	**Too Thick**
Too Short	10	3	5
OK	38	900	4
Too Long	2	25	13

Solution1:

Of the 1000 outcomes, the number that are either too short or too thick is $10 + 3 + 5 + 4 + 13 = 35$. Thus,

$$P(\text{too short or too thick}) = \frac{35}{1000}.$$

The Addition Rule

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Solution1:

Of the 1000 outcomes, the number that are either too short or too thick is $10 + 3 + 5 + 4 + 13 = 35$. Thus,

$$P(\text{too short or too thick}) = \frac{35}{1000}.$$

Solution2:

$P(\text{too short}) = \frac{18}{1000}$, $P(\text{too thick}) = \frac{22}{1000}$, and $P(\text{too short and too thick}) = \frac{5}{1000}$. Hence,

$$P(\text{too short or too thick}) = P(\text{too short}) + P(\text{too thick}) - P(\text{too short and too thick})$$

$$= \frac{18}{1000} + \frac{22}{1000} - \frac{5}{1000}.$$

The Addition Rule

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The probability of union of the events E_1 and E_2 is

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2). \tag{5}$$

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Proof.

Note first that $E_1 \cup E_2$ can be decomposed into two *disjoint* events E_1 and $E_1^C \cap E_2$ and thus,

$$P(E_1 \cup E_2) = P(E_1) + P(E_1^C \cap E_2). \tag{6}$$

Similarly, E_2 is union of the two disjoint events $E_1 \cap E_2$ and $E_1^C \cap E_2$. Hence, $P(E_2) = P(E_1 \cap E_2) + P(E_1^C \cap E_2)$ which can be rearranges as $P(E_1^C \cap E_2) = P(E_2) - P(E_1 \cap E_2)$ and substituted into (6) to obtain (5). □

Counting Methods

When computing probabilities, it is sometimes necessary to determine the number of outcomes in a sample space.

Example 4: A certain make of automobile is available in any of three colors: red, blue, or green, and comes with either a large or small engine. In how many ways can a buyer choose a car?

	Red	Blue	Green
Large	Red, Large	Blue, Large	Green, Large
Small	Red, Small	Blue, Small	Green, Small

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the total number of ways to perform the two operations is $n_1 \times n_2$.

Counting Methods

Example 5 (Counting with Repetition): How many unique passwords can be made with 4 digits?

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$\underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = (10)^4$

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Example 6 (Counting without Repetition): Three members from a 20-member committee are to be randomly selected to serve as chair, vice-chair, and secretary. The first person selected is the chair, the second is the vice-chair, and the third is the secretary. How many different committee structures are possible?

Counting Methods

Example 5 (Counting with Repetition): How many unique passwords can be made with 4 digits?

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Solution:

$\underline{20} \times \underline{19} \times \underline{18} = 6840$

Counting Methods: Permutations

A **permutation** is an ordering of a collection of objects.

The fundamental principle of counting can be used to determine the number of permutations of any set of objects.

For example, there are six permutations of the letters A, B, C: ABC, ACB, BAC, BCA, CAB, and CBA.

Therefore, the number of permutations for n distinct objects can be obtained as

$$n(n-1)\cdots(3)(2)(1). \tag{7}$$

Definition

For any positive integer $n \geq 0$, the **factorial symbol**, $n!$, is defined as follows:

$$n! = n(n-1)\cdots(3)(2)(1). \qquad 0! = 1, \quad \text{and} \quad 1! = 1.$$

Counting Methods: Permutations

In **Example 6**, we found the number of ways that we can randomly choose three members from a 20-member committee to serve as chair, vice-chair, and secretary.

The reasoning that we used to solve this problem can be generalized to the case where we want to choose k objects in an **ordered** arrangement from n different objects as

$$\begin{aligned}(n)(n-1)\cdots(n-k+1) &= \frac{n(n-1)\cdots(n-k+1)(n-k)\cdots(2)(1)}{(n-k)\cdots(2)(1)} \\ &= \frac{n!}{(n-k)!}\end{aligned}$$

The number of permutations of k objects chosen from a group of n objects is

$$\frac{n!}{(n-k)!} \tag{8}$$

Counting Methods: Combinations

In some cases, when choosing a set of objects from a larger set, we **do not** care about the ordering of the chosen objects

Each distinct group of objects that can be selected, without regard to order, is called a **combination**.

Number of Combinations of k Objects Chosen from n Objects:
The number of different arrangements of k objects chosen from n objects, in which

1. The order is not important
2. The n objects are distinct, and
3. Repetition of objects is not allowed,

is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{9}$$

Counting Methods: Combinations

Sometimes we are interested in counting the number of ordered sequences for objects that are not all different.

Example 7: Assume that in a class of 12 students, a project is assigned in which the students will work in groups. Three groups are to be formed, consisting of 5, 4, and 3 students. Calculate the number of ways in which the groups can be formed.

Counting Methods: Combinations

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Example 7: Assume that in a class of 12 students, a project is assigned in which the students will work in groups. Three groups are to be formed, consisting of 5, 4, and 3 students. Calculate the number of ways in which the groups can be formed.

Solution:

The process of dividing the class into 3 groups is performed sequentially, as follows.

1. First, we choose 5 students from the total of 12 students that can be done in $\binom{12}{5} = \frac{12!}{5!7!}$ different ways.
2. Second, we choose 4 students from the remaining $7 = 12 - 5$ students to form the second group. This can be done in $\binom{7}{4} = \frac{7!}{4!3!}$.
3. Third, we choose 3 students from the remaining $3 = 7 - 4$ students to form the second group. This can be done in $\binom{3}{3} = \frac{3!}{3!0!} = 1$ way.

Counting Methods: Combinations

Solution (Continued):
The third step above is not necessary as group 3, automatically consists of 3 students once the groups 1 and 2 are formed.

Notice that the numerator in the final answer is the factorial of the total group size, while the denominator is the product of the factorials of the sizes of the groups chosen from it. This holds in general.

The number of ways of dividing a group of n objects into groups of $k_1, \ldots, k_\ell$ objects, wherek $k_1 + \ldots + k_\ell = n$, is

$$\frac{n!}{k_1! \ldots k_\ell!} \tag{10}$$

Conditional Probability

That the sample space, S, contains all the possible outcomes of an experiment.

Sometimes, we obtain some additional information about an experiment telling us that the outcome comes from a subset of S.

Thus, the probability of an event is based on the outcomes in a subset of S.

A probability that is based on a part of a sample space is called a **conditional probability**.

In **Example 2**, we discussed a population of 1000 aluminium rods where each rod was classified as too short, too long, or OK, and the diameter was classified as too thin, too thick, or OK. We calculated $P(\text{too short})$ considering the entire sample space. This probability is called **unconditional probability**.

Conditional Probability

Now, in **Example 2**, assume that a rod is sampled, and its length is measured and found to meet the specification. What is the probability that the diameter also meets the specification?

	Diameter		
Length	**Too Thin**	**OK**	**Too Thick**
Too Short	10	3	5
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The knowledge that the length meets the specification reduces the sample space to 942 rods.

Of the 942 rods in this sample space, 900 of them meet the diameter specification. Thus, $P(\text{diameter OK } \text{given} \text{ length OK}) = 900/942$.

Hence, the **conditional probability** that the rod meets the diameter specification **given** that it meets the length specification is represented as

$$P(\text{diameter OK} \mid \text{length OK}) = 900/942 \tag{11}$$

Conditional Probability

Example 8: Compute the conditional probability $P(\text{diameter OK} \mid \text{length too long})$. Is this the same as the unconditional probability $P(\text{diameter OK})$?

	Diameter		
Length	**Too Thin**	**OK**	**Too Thick**
Too Short	10	3	5
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Too Long	2	25	13

Solution:

Of the 40 outcomes, 25 meet the diameter specification, thus, $P(\text{diameter OK} \mid \text{length too long}) = \frac{25}{40}$.

The unconditional probability is $P(\text{diameter OK}) = \frac{928}{1000}$.

The denominator of the conditional probability, $\frac{25}{40}$, represents the number of outcomes that satisfy the condition that the rod is too long. While the numerator, 25, represents the number of outcomes that satisfy both the condition that the rod is too long and that its diameter is OK.

Conditional Probability

Solution (Continued):

$$P(\text{diameter OK} \mid \text{length too long}) = \frac{25/1000}{40/1000} = \frac{P(\text{diameter OK and length too long})}{P(\text{length too long})}$$

Based on this reasoning, we can construct a definition for the conditional probability that holds for any sample space.

Definition

Let E_1 and E_2 be events with $P(E_2) \neq 0$. The conditional probability of E_1 given E_2 is

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}. \tag{12}$$

Independent Events

Sometimes the knowledge that one event has occurred does not change the probability that another event occurs.

Thus, conditional and unconditional probabilities are the same, and such events are said to be **independent**. This implies that $P(E_1 \cap E_2) = P(E_1)P(E_2)$.

Definition

Two events E_1 and E_2 are independent, if the probability of each event remains the same whether or not the other one occurs.
If $P(E_1) \neq 0$ and $P(E_2) \neq 0$, then E_1 and E_2 are independent if

$$P(E_1|E_2) = P(E_1|E_2^C) = P(E_1) \quad \text{or,} \quad P(E_2|E_1) = P(E_2|E_1^C) = P(E_2). \tag{13}$$

If either $P(E_1) = 0$ or $P(E_2) = 0$, then E_1 and E_2 are independent.

Independent Events

Definition

Events $E_1, E_2, \ldots, E_n$ are independent if the probability of each remains the same no matter which of the others occur.
Events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_i | E_{i_1} \cap \cdots \cap E_{i_k}) = P(E_i), \tag{14}$$

and

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k}). \tag{15}$$

Exercise: Find $P(\text{too long})$ and $P(\text{too long} \mid \text{too thin})$ in Example 8. Are these probabilities different? Comment on the independence of the events E_1, the rod being too long, and E_2, the rod being too thin.

The Multiplication Rule

Sometimes we know $P(E_1|E_2)$ or $P(E_2|E_1)$, and we wish to find $P(E_1 \cap E_2)$. Using (12), given that $P(E_1) \neq 0$ and $P(E_2) \neq 0$,

$$P(E_1 \cap E_2) = P(E_2)P(E_1|E_2) = P(E_1)P(E_2|E_1). \tag{16}$$

If events E_1 and E_2 are independent,

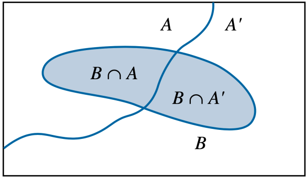
$$P(E_1 \cap E_2) = P(E_1)P(E_2). \tag{17}$$

This result can be extended to any number of independent events $E_1, E_2, \ldots, E_n$, i.e.,

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2) \cdots P(E_n). \tag{18}$$

The Law of Total Probability

Consider the sample space in the figure on the right that contains the *mutually exclusive* events A and A'.



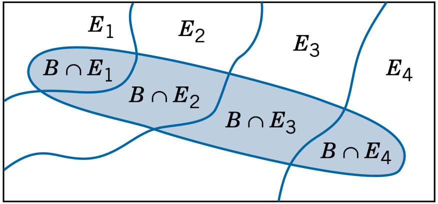
From this figure, it is clear that the events $B \cap A$ and $B \cap A'$ are also mutually exclusive and their union forms the event B. Therefore,

$$
\begin{aligned}
P(B) = P((B \cap A) \cup (B \cap A')) &= P(B \cap A) + P(B \cap A') \\
&= P(B|A)P(A) + P(B|A')P(A').
\end{aligned}
$$

The Law of Total Probability

Given the mutually exclusive sets, $E_1, E_2, \ldots, E_n$, this result can be generalized as

$$\begin{aligned} P(B) =& P(B \cap E_1) + \cdots + P(B \cap E_n) \\ =& \sum_{i=1}^{n} P(B|E_i)P(E_i). \end{aligned}$$



Bayes' Theorem

From the definition of conditional probability,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A). \tag{19}$$

Considering the conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and (19),

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0. \tag{20}$$

Using this result, we can find $P(A|B)$ in terms of $P(B|A)$. Furthermore, using the rule of total probability, based on the Bayes' Theorem, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}. \tag{21}$$

Bayes' Theorem

Theorem (Bayes' Theorem)

If $E_1, E_2, \ldots, E_n$ *are* n *mutually exclusive and exhaustive events and* B *is any event,*

$$P(E_k|B) = \frac{P(B|E_k)P(E_k)}{\sum_{i=1}^{n} P(B|E_i)P(E_i)}, \tag{22}$$

for $P(B) \neq 0$.

The Law of Total Probability: Example

Example 9: Customers who purchase a certain make of car can order an engine in any of three small, medium and large sizes. The percentage of sold cars with different engine sizes together with the failure rate in emissions test within two years of purchase are presented in the table below. What is the probability that a randomly chosen car will fail an emissions test within two years?

Engine Size	Percentage of All Cars Sold	Emission Test Failure Rate
Small	45%	10%
Medium	35%	12%
Large	20%	15%

The Law of Total Probability: Example

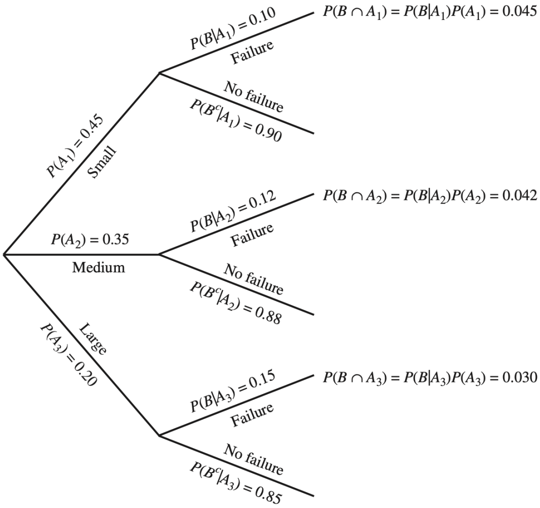
Solution: Let B denote the event that a car fails an emissions test, and A_1, A_2, and A_3 denote the events that a car has a small, medium and large engine size, respectively. Thus, $P(A_1) = 0.45$, $P(A_2) = 0.35$, and $P(A_3) = 0.2$.

The probabilities that a car fails a test given different engine sizes are stated in the problem, i.e., $P(B|A_1) = 0.1$, $P(B|A_2) = 0.12$, and $P(B|A_3) = 0.15$. Thus, using law of total probability, we have

$$P(B) = \sum_{i=1}^{3} P(B|A_i)P(A_i) = (0.10)(0.45) + (0.12)(0.35) + (0.15)(0.20) = 0.117.$$

The Law of Total Probability: Example

Alternative Solution: Sometimes problems like this example are solved using **tree diagrams**.



Bayes' Theorem: Example

Example 10: Based on the data that was provided in Example 9, consider that a record for a failed emissions test is chosen at random. What is the probability that it is for a car with a small engine?

Bayes' Theorem: Example

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Solution: The event B represents that a car fails an emissions test, and also the events A_1, A_2, and A_3 as defined in the solution to the Example 9. In this example, we wish to find $P(A_1|B)$. Using the Bayes' theorem, and $P(B|A_1) = 0.1$, $P(B|A_2) = 0.12$, $P(B|A_3) = 0.15$, we have

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

$$\frac{(0.10)(0.45)}{(0.10)(0.45) + (0.12)(0.35) + (0.15)(0.20)} = 0.385.$$

Application to Reliability Analysis

Reliability analysis is the branch of engineering concerned with estimating the *failure rates* of systems.

Example 11: A system contains two components, A and B, connected in series as shown in the following diagram. The system will function only if both components function. The probability that A and B function is given by $P(A) = 0.96$, and $P(B) = 0.92$, respectively. Assume that A and B function independently. Find the probability that the system functions.

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Solution: $P(\text{system functions}) = P(A \cap B) = P(A)P(B) = (0.96)(0.92) = 0.8832$

Application to Reliability Analysis

Exercise: Assuming that the two components, A and B, in Example 11 were connected in parallel, find the probability that the system functions.