High Dimensional Statistics

Homework 2

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1 Properties of ℓ_q-Ball

(a) Consider a vector $\theta \in B_q(R_q)$ therefore for this vectore we know

$$\sum_{j=1}^{d} |\theta_j|^q \leq R_q.$$

Now for a $t \in [0,1]$ we have $|t\theta_j|^q = |t|^q |\theta_j|^q \leq |\theta_j|^q$ as we know $|t|^q \leq 1$. Therefore we have

$$\sum_{j=1}^{d} |t\theta_j|^q \leq \sum_{j=1}^{d} |\theta_j|^q \leq R_q,$$

which means that $t\theta \in B_q(R_q)$ and therefore the ℓ_q-ball is a star-shaped set.

(b) We know $\alpha > 1/q$ and as $q \in (0,1]$ we can say that $\alpha q > 1$. Now for a vector $\theta \in B_{w(\alpha)}(C)$ we have:

$$|\theta|_{(j)} \leq Cj^{-\alpha} \implies |\theta|_{(j)}^q \leq C^q j^{-\alpha q} \leq C^q$$

Therefore we have $\sum_{j=1}^{d} |\theta|_{(j)}^q \leq dC^q$.

2 Welch Bound

Using the provide hint we can subsititute the $\mathrm{tr}(G) = n$ and $\mathrm{rank}(G) \leq m$ in the inequality as follows:

$$\mathrm{tr}(G)^2 \leq \mathrm{rank}(G) \cdot \mathrm{tr}(G^2) \implies n^2 \leq m \cdot \mathrm{tr}(G^2) \implies \mathrm{tr}(G^2) \geq \frac{n^2}{m}.$$

Now for the $\mathrm{tr}(G^2)$ we have:

$$\begin{aligned}
\mathrm{tr}(G^2) &= \sum_{i=1}^{n} (G^2)_{ii} \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} G_{ij} G_{ji} \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (G_{ij})^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle a_i, a_j \rangle^2 \\
&= \sum_{i=1}^{n} \langle a_i, a_i \rangle^2 + \sum_{i \neq j} \langle a_i, a_j \rangle^2 \\
&= \sum_{i=1}^{n} 1^2 + \sum_{i \neq j} \langle a_i, a_j \rangle^2 \\
&= n + \sum_{i \neq j} \langle a_i, a_j \rangle^2 \\
&\leq n + \sum_{i \neq j} \mu(A)^2 \\
&= n + n(n-1)\mu(A)^2.
\end{aligned}$$

Now by substituting this back in the previous inequality we have:

$$\frac{n^2}{m} \leq n + n(n-1)\mu(A)^2 \implies \frac{n}{m} - 1 \leq (n-1)\mu(A)^2 \implies \mu(A)^2 \geq \frac{n-m}{m(n-1)}.$$

3 Mutual Coherence

(a) For the upper bound we can use the Cauchy-Schwarz inequality as follows:

$$|\psi_i^T \phi_j| \leq \|\psi_i\|_2 \|\phi_j\|_2 = 1 \cdot 1 = 1 \implies \max_{1 \leq i,j \leq n} |\psi_i^T \phi_j| = \mu(\psi, \phi) \leq 1.$$

For the lower bound we can use the fact that transforming with an orthogonal matrix does not change the inner product. In other words for a vecotr $v \in \mathbb{R}^n$ we have:

$$\|\psi^T v\|_2^2 = (\psi^T v)^T (\psi^T v) = v^T \psi \psi^T v = v^T I v = \|v\|_2^2.$$

Now let's considre $v = \phi_j$ for some $j \in \{1, 2, \ldots, n\}$. Therefore we have:

$$\|\phi_j\|_2^2 = \|\psi^T \phi_j\|_2^2 = \sum_{i=1}^{n} \left(\psi_i^T \phi_j\right)^2 \leq \sum_{i=1}^{n} \mu(\psi, \phi)^2 = n\mu(\psi, \phi)^2.$$

Therefore we have $1 \leq n\mu(\psi, \phi)^2 \implies \mu(\psi, \phi) \geq \frac{1}{\sqrt{n}}$. Thus we've shown that:

$$\frac{1}{\sqrt{n}} \leq \mu(\psi, \phi) \leq 1.$$

(b) Let's set $\mu = \mu(\psi, \phi)$ for simplicity. As we know $\psi = I$ we have:

$$\max_{i,j} |\psi_i^T \phi_j| = \max_{i,j} |\phi_{ij}|$$

Therefore to construct a bound on the μ we need to bound the $\max |\phi_{ij}|$. Let's define $Z_{ij} = \sqrt{n}\phi_{ij}$, we have:

$$\phi_{ij} \sim (0, \frac{1}{n}) \implies Z_{ij} = \sqrt{n}\phi_{ij} \sim \mathcal{N}(0, 1).$$

Now for the mu we have:

$$\begin{aligned}
\mathbb{P}\left(\mu > f(n)\right) &= \mathbb{P}\left(\max_{i,j} |\phi_{ij}| > f(n)\right) \\
&= \mathbb{P}\left(\max_{i,j} |Z_{ij}| > \sqrt{n} f(n)\right) \\
&\leq \sum_{i,j} \mathbb{P}\left(|Z_{ij}| > \sqrt{n} f(n)\right) \quad \text{(by union bound)} \\
&= n^2 \mathbb{P}\left(|Z| > \sqrt{n} f(n)\right) \\
&\leq n^2 2 \exp\left(-\frac{n f(n)^2}{2}\right) \quad \text{(by Gaussian tail bound)} \\
&= 2 \exp\left(2 \log n - \frac{n f(n)^2}{2}\right).
\end{aligned}$$

We need this bound to go to zero as n grows, therefore we need:

$$2 \log n - \frac{n f(n)^2}{2} \to -\infty \implies f(n) \geq \sqrt{\frac{4 \log n}{n}}.$$

Therefore we can set $f(n) = c\sqrt{\frac{\log n}{n}}$ for some constant $c > 2$ so that with high probability we would have $\mu(\psi, \phi) \leq c\sqrt{\frac{\log n}{n}}$.

(c) Same as Problem 2

4 Relationship between RIC and Incoherence

(a) From the definition for $\delta_2(A)$ we have:

$$\begin{aligned}
\delta_2(A) &= \max_{S\subset\{1,2,\dots,d\},|S|\leq 2} \|A_S^T A_S - I_2\|_{op} \\
&= \max_{i\neq j} \|A_{\{i,j\}}^T A_{\{i,j\}} - I_2\|_{op} \\
&= \max_{i\neq j} \left\| \begin{bmatrix} 1 & \langle a_i, a_j\rangle \\ \langle a_j, a_i\rangle & 1 \end{bmatrix} - I_2 \right\|_{op} \\
&= \max_{i\neq j} \left\| \begin{bmatrix} 0 & \langle a_i, a_j\rangle \\ \langle a_j, a_i\rangle & 0 \end{bmatrix} \right\|_{op}
\end{aligned}$$

Now the operation norm of the matrix is equal to its largest eigenvalue in absolute value. The eigenvalues of the above matrix are $\pm|\langle a_i, a_j\rangle|$. Therefore we have:

$$\delta_2(A) = \max_{i\neq j} |\langle a_i, a_j\rangle| = \mu(A)$$

(b) Let $S \subset [n]$ be a subset of indices with cardinality $|S| = k$. Consider the Gram matrix of the subset of columns, $G_S = A_S^\top A_S$. The restricted isometry constant δ_k is defined by the deviation of eigenvalues of G_S from 1. Equivalently, it is the spectral radius (largest absolute eigenvalue) of the matrix $M_S = A_S^\top A_S - I_k$. The entries of the $k \times k$ matrix M_S are given by:

$$(M_S)_{ij} = \begin{cases} \langle a_i, a_j\rangle & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

where $i, j \in S$. According to the Gershgorin Circle Theorem, every eigenvalue λ of M_S lies in at least one disc centered at the diagonal entries (which are 0) with a radius equal to the sum of the absolute values of the off-diagonal entries in that row. Therefore, for any eigenvalue λ of M_S:

$$|\lambda| \leq \max_{i\in S} \sum_{j\in S, j\neq i} |(M_S)_{ij}| = \max_{i\in S} \sum_{j\in S, j\neq i} |\langle a_i, a_j\rangle|.$$

Since $\delta_k = \sup_{S:|S|=k} \|M_S\|_2$ is the maximum spectral radius over all such sets, we have:

$$\delta_k \leq \max_{S:|S|=k} \left(\max_{i\in S} \sum_{j\in S, j\neq i} |\langle a_i, a_j\rangle| \right).$$

The inner sum in inequality, $\sum_{j\in S, j\neq i} |\langle a_i, a_j\rangle|$, represents the sum of absolute correlations between a specific a_i and $k-1$ other distinct vectors in S. Recalling the definition of the Babel function (or 1-coherence parameter) $\mu_1(m)$:

$$\mu_1(m) = \max_{i\notin\Lambda, |\Lambda|=m} \sum_{j\in\Lambda} |\langle a_i, a_j\rangle|.$$

Since the sum in our expression involves exactly $k-1$ other vectors, it is bounded by definition by $\mu_1(k-1)$. Therefore:

$$\delta_k \leq \mu_1(k-1).$$

Consider the definition of $\mu_1(k-1)$ again:

$$\mu_1(k-1) = \max_{i,\Lambda} \sum_{j\in\Lambda} |\langle a_i, a_j\rangle|, \quad \text{where } |\Lambda| = k-1.$$

We know that the pairwise incoherence parameter μ is the maximum absolute inner product between any two distinct columns: $\mu = \max_{u\neq v} |\langle a_u, a_v\rangle|$. Thus, for every term in the sum, we have $|\langle a_i, a_j\rangle| \leq \mu$. Substituting this upper bound into the sum:

$$\sum_{j\in\Lambda} |\langle a_i, a_j\rangle| \leq \sum_{j\in\Lambda} \mu = (k-1)\mu.$$

Since this holds for the maximum sum, we conclude:

$$\mu_1(k-1) \leq (k-1)\mu.$$

5 Restricted Isometry Property

(a) We first show that for a fixed support set T with $|T| = s$, the random matrix A_T preserves the norm of vectors with high probability. Let $x \in \mathbb{R}^s$ be a unit vector ($||x||_2 = 1$). Since the entries of A are i.i.d. $\mathcal{N}(0, 1/m)$, the vector Ax follows a multivariate Gaussian distribution, and $m||Ax||_2^2$ follows a Chi-squared distribution with m degrees of freedom (χ^2_m).

Using concentration inequalities for Chi-squared variables (or sub-exponential variables), for any $t > 0$, we have:

$$\mathbb{P}\left(\left|||Ax||_2^2 - 1\right| \geq t\right) \leq 2\exp(-c_0 m t^2)$$

Here we first select our vector x and then observe the randomness over the matrix A. However, we want this bound for the x that maximizes the deviation. This makes the vectore x dependent on A, and we can not simply use this bound for all x. We need to use a covering argument to uniformize this bound over all x in the unit sphere. Let $\mathcal{N}_\epsilon$ be an ϵ-net of the unit sphere S^{s-1}. From standard covering arguments, we know that:

$$|\mathcal{N}_\epsilon| \leq \left(1 + \frac{2}{\epsilon}\right)^s$$

Using the union bound over all $x \in \mathcal{N}_\epsilon$ and setting $t = \delta/2$, we can write:

$$\mathbb{P}\left(\sup_{x \in \mathcal{N}_\epsilon} \left|||Ax||_2^2 - 1\right| \geq \delta/2\right) \leq \sum_{x \in \mathcal{N}_\epsilon} \mathbb{P}\left(\left|||Ax||_2^2 - 1\right| \geq \delta/2\right)$$

$$\leq \left(1 + \frac{2}{\epsilon}\right)^s 2\exp(-c_0 m (\delta/2)^2)$$

By choosing ϵ appropriately (e.g., $\epsilon < c\delta$), the norm preservation on the net implies norm preservation on the entire sphere with constant δ. Thus, for a fixed support T:

$$\mathbb{P}(\delta_T > \delta) \leq 2\exp\left(s\ln(C/\delta) - c' m\delta^2\right)$$

Now we need to ensure this holds for all possible support sets of size s. There are $\binom{n}{s}$ such subspaces. Using the standard bound for the binomial coefficient $\binom{n}{s} \leq (en/s)^s$, we apply the union bound again:

$$\mathbb{P}(\delta_s > \delta) = \mathbb{P}\left(\max_{|T|=s} \delta_T > \delta\right)$$

$$\leq \sum_{|T|=s} \mathbb{P}(\delta_T > \delta)$$

$$\leq \left(\frac{en}{s}\right)^s 2\exp(s\ln(C/\delta) - c' m\delta^2)$$

$$= 2\exp\left(s\ln\left(\frac{en}{s}\right) + s\ln(C/\delta) - c' m\delta^2\right)$$

To have this probability bounded by $2\exp(-c_2 m)$, we require the exponent to be dominated by the negative m term. Specifically, we need:

$$c' m\delta^2 \geq s\ln\left(\frac{en}{s}\right) + s\ln(C/\delta) + c_2 m$$

Ignoring constants and lower order terms, this inequality holds provided that:

$$m \geq c_1 \delta^{-2} s \log\left(\frac{en}{s}\right).$$

(b) We discuss the optimality of this bound. The known lower bound for any matrix satisfying the RIP of order s is $m \geq Cs\log(n/s)$. Since our derived sufficient condition matches this lower bound (up to constant factors), Gaussian matrices are order-optimal for compressed sensing. The logarithmic factor $\log(n/s)$ has a specific geometric interpretation regarding the set of sparse vectors.

  (a) **Union of Subspaces:** The set of s-sparse vectors is not a linear subspace, but rather a union of $\binom{n}{s}$ subspaces.

  (b) **Encoding Complexity:** To identify an s-sparse vector, one must identify both its values (requiring s degrees of freedom) and its support. The number of bits required to encode the support location is approximately $\log_2 \binom{n}{s} \approx s\log(n/s)$.

6 Covering and Packing Numbers

(a) We know N is a maximal ϵ-seperated set, therefore there is no element in K that we can add to N such that the set remains ϵ-seperated. This means that for every $x \in K$ there exists at least one $z \in N$ such that $d(x, z) \leq \epsilon$, otherwise we could add x to N and contradict the maximality of N. Therefore N is also an ϵ-net of K.

(b) Let's say M is a maximal ϵ-seperated set of K, from part (a) we know M is also an ϵ-net of K. From the definiton we know $N(K, d, \epsilon)$ is the size of the smallest ϵ-net of K, therefore we have:

$$N(K, d, \epsilon) \leq |M| = P(K, d, \epsilon).$$

To show the lower bound, we show there exist a one-to-one mapping from S to C where S is a maximal 2ϵ-seperated set of K and C is a minimal ϵ-net of K. For every element $x \in S$ we can find a $z \in C$ such that $d(x, z) \leq \epsilon$ as we know C is an ϵ-net of K. Now let's assume two different elements $x_1, x_2 \in S$ map to the same $z \in C$. Therefore using the triangle inequality we have:

$$d(x_1, x_2) \leq d(x_1, z) + d(z, x_2) \leq \epsilon + \epsilon = 2\epsilon,$$

which contradicts the fact that S is a 2ϵ-seperated set. Therefore every element in S maps to a different element in C and we have $|S| \leq |C|$ or equivalently $P(K, d, 2\epsilon) \leq N(K, d, \epsilon)$. Thus we've shown:

$$P(K, d, 2\epsilon) \leq N(K, d, \epsilon) \leq P(K, d, \epsilon).$$

(c) To show the lower bound, we show the constructed set with $k = C(K, d, \epsilon)$ is an ϵ-net of K. Letting $C(K, d, \epsilon) = k$ means we have access to a set $Z = \{z_1, \ldots, z_{2^k}\}$ elements that can be used to specify elements in K with ϵ-accuracy. In other words for every $x \in K$ there exists a $z \in Z$ such that $d(x, z) \leq \epsilon$. Therefore Z is an ϵ-net of K and from the definition we have:

$$N(K, d, \epsilon) \leq |Z| = 2^{C(K,d,\epsilon)} \implies \log_2 N(K, d, \epsilon) \leq C(K, d, \epsilon).$$

To show the upper bound, we set M to be the minimal $\epsilon/2$-net of K, therefore for each $x \in K$ there exists a $z \in M$ such that $d(x, z) \leq \epsilon/2$. Therefore we can specify elements in K with ϵ-accuracy by specifying the closest element in M to that element. From the definition we know $N(K, d, \epsilon/2) = |M|$, therefore we can specify elements in K with ϵ-accuracy using $\lceil \log_2 N(K, d, \epsilon/2) \rceil$ bits. Thus we have:

$$C(K, d, \epsilon) \leq \lceil \log_2 N(K, d, \epsilon/2) \rceil.$$