3.1.1 Let $n_0 \in \mathbb{N}$ s.t. $f(n), g(n) \geq 0$. We find $c_1, c_2 > 0$ s.t.

$$c_1[f(n)+g(n)] \leq \max\{f(n), g(n)\} \leq c_2[f(n)+g(n)]$$

For $n \geq n_0$, note that

$$\max\{f(n), g(n)\} \leq f(n)+g(n),$$

so let $c_2 = 1$.

Recall that

$$\max\{f(n), g(n)\}$$

$$= \frac{f(n)+g(n)+|f(n)-g(n)|}{2}$$

$\Rightarrow$ For $n \geq n_0$,

$$\frac{f(n)+g(n)}{2} \leq \frac{f(n)+g(n)+|f(n)-g(n)|}{2}$$

$$= \max\{f(n), g(n)\}.$$

So let $c_1 = \frac{1}{2}$, $c_2 = 1$ for $n \geq n_0$.

3.1.2 We show that

$$(n+a)^b = \Theta(n^b),$$

$a, b \in \mathbb{R}$, $b > 0$.

First note that for $n_0 = |a|$, if $n \geq n_0$, then

$$(n+a)^b \geq 0$$

$\Rightarrow f(n) = (n+a)^b$ is asymptotically positive for $n \geq n_0$.

For $n \geq n_0$ we find $c_1, c_2 > 0$ s.t.

$$c_1 n^b \leq (n+a)^b \leq c_2 n^b$$

Now $n \geq |a| = n_0 \geq a$

$$\Rightarrow (n+a)^b \leq (n+n)^b$$

$$\Rightarrow (n+a)^b \leq 2^b n^b,$$

So take $\boxed{c_2 = 2^b}$

Let $n_1 = 2|a|$. Now

if $a > 0$, $n^b \leq (n+a)^b$

So WLOG let $a<0$.
Then,

$$\gamma n \leq n + a$$

$$\Rightarrow n(\gamma - 1) \leq a$$

$$\Rightarrow n(1-\gamma) \geqslant -a = |a|$$

For $\gamma = \frac{1}{2}$, we want

$$\frac{n}{2} \geqslant |a|$$

$$\Rightarrow n \geqslant 2|a|.$$

So let $n \geq n_1$

$\Rightarrow \quad \frac{n}{2} \leq (n+a)$

$\Rightarrow \underbrace{\left(\frac{1}{2}\right)^b}_{c_1} n^b \leq (n+a)^b,$

So let $c_1 = \left(\frac{1}{2}\right)^b,$

$c_2 = 2^b$

for $n \geq 2|a|$. If $a \geq 0$, $c_1 = 1$, $n \geq n_0 = |a|$

3.1.4 First note that $2^n \geq 0$ $\forall n \in \mathbb{N}$. Now,

$$2^{n+1} = 2 \cdot 2^n$$

For $c = 2 \Rightarrow 2^{n+1} \leq 2 \cdot 2^n$

$$\Rightarrow 2^{n+1} = O(2^n).$$

Now, $2^{2n} = (2^2)^n = 4^n$

$\Rightarrow$ If $\exists\, c > 0$ s.t.

$4^n \leq c2^n$ for $n \geq n_0$,

$n_0 \in \mathbb{N}$,

$\Rightarrow c \geq 2^n$, $n \geq n_0$, a contd.

$\Rightarrow 2^{2n} \neq O(2^n)$.

3.1.5 Clear.

3.1.6. Clear from 3.1.5.

3.1.7. If $f(n) \in o(g(n))$ &

$f(n) \in \omega(g(n))$,

then

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

$$= \infty$$

$$= \lim_{n \to \infty} \frac{f(n)}{g(n)}$$

But limits are unique.

3.1.8 $\Theta(g(n, m))$

$$= \{ f(n,m) : \exists c_1, c_2 > 0, \; n_0\ m_0 \in \mathbb{N} \text{ s.t. } 0 \leq c_1 g(n,m) \leq f(n,m) \leq c_2 g(n,m) \; \forall n \geq n_0 \; \& \; \forall m \geq m_0 \}$$

$\Omega(g(n, m))$

$$= \{ f(n,m) : \exists c > 0, \; n_0 \in \mathbb{N} \text{ s.t. } 0 \leq c\, g(n,m) \leq f(n,m) \; \forall n \geq n_0 \; \& \; \forall m \geq m_0 \}$$

We prove 3.6 + 3.7.
We prove 3.7 first.

By DA, we have

$a = bq + r, \quad 0 \le r \le b-1$

$\Rightarrow \frac{a}{b} = q + \frac{r}{b}$

$\Rightarrow \left\lfloor \frac{a}{b} \right\rfloor = q + \left\lfloor \frac{r}{b} \right\rfloor$

$= q, \quad \left\lfloor \frac{r}{b} \right\rfloor = 0$

If $q < \frac{a-(b-1)}{b}$, then

$$a = bq + r$$

$$< b \cdot \left(\frac{a-(b-1)}{b}\right) + r$$

$$= a - (b-1) + r$$

$$\leq a - (b-1) + (b-1)$$

$$= a \Rightarrow a < a,$$

a cont.

$$\Rightarrow q = \lfloor a/b \rfloor \geq \frac{a-(b-1)}{b}$$

Similarly,

$$\left\lceil \frac{a}{b} \right\rceil = q + \left\lceil \frac{r}{b} \right\rceil$$

$$= q + 1, \quad \left\lceil \frac{r}{b} \right\rceil = 1$$

If $q + 1 > \frac{a + b - 1}{b}$

$$\Rightarrow q > \frac{a+b-1}{b} - 1$$

$$= \frac{a+b-1}{b} - \frac{b}{b}$$

$$= \frac{a-1}{b}$$

$$\Rightarrow a = bq + r$$

$$> b \cdot \frac{a-1}{b} + r$$

$$= a - 1 + r$$

$$\Rightarrow 0 > r - 1$$

$$\Rightarrow r < 1$$

$$\Rightarrow r = 0 \Rightarrow b \mid a.$$

so if $b \nmid a$,

$$q + 1 = \left\lceil \frac{a}{b} \right\rceil \le \frac{a+b-1}{b}$$

So let $b|a$. Then
$a = bq$,

$$\frac{a}{b} = q$$

$$\Rightarrow \left\lceil \frac{a}{b} \right\rceil = q \checkmark$$

$$\begin{aligned}\frac{a+b-1}{b} &= \frac{a}{b} + \frac{b-1}{b}\\ &= q + \frac{b-1}{b}\\ &= q + 1 - \frac{1}{b}\\ &\geq q = \left\lceil \frac{a}{b} \right\rceil\end{aligned}$$

$\Rightarrow$ we get the result

in this case as well.

We now prove 3.5 & 3.7. First note

$$\left\lfloor \frac{x}{a} \right\rfloor \leq \frac{x}{a} \leq \left\lceil \frac{x}{a} \right\rceil$$

$$\Rightarrow \frac{\lfloor x/a \rfloor}{b} \leq \frac{x}{ab} \leq \frac{\lceil x/a \rceil}{b}$$

$$\Rightarrow \frac{x}{ab} \leq \frac{\lceil x/a \rceil}{b} \leq \left\lceil \frac{\lceil x/a \rceil}{b} \right\rceil$$

$$\Rightarrow \boxed{\left\lceil \frac{x}{ab} \right\rceil \leq \left\lceil \frac{\lceil x/a \rceil}{b} \right\rceil}$$

Similarly,

$$\left\lfloor \frac{\lfloor x/a \rfloor}{b} \right\rfloor \le \frac{\lfloor x/a \rfloor}{b} \le \frac{x}{ab}$$

$$\Rightarrow \boxed{\left\lfloor \frac{\lfloor x/a \rfloor}{b} \right\rfloor \le \left\lfloor \frac{x}{ab} \right\rfloor}$$

We now prove:

$$\left\lfloor \frac{\lfloor x/a \rfloor}{b} \right\rfloor \geq \left\lfloor \frac{x}{ab} \right\rfloor$$

$$\left\lceil \frac{\lceil x/a \rceil}{b} \right\rceil \le \left\lceil \frac{x}{ab} \right\rceil$$

Let $x > -1$, $f(x) = \ln(1+x)$.

For $x > 0$,

3.17 in text

$$\frac{f(x) - f(0)}{x - 0} = \frac{\ln(1+x)}{x}$$

By MVT, $\exists\ c \in (0, x)$

s.t. $\dfrac{\ln(1+x)}{x} = f'(c)$.

$$f'(c) = \frac{1}{1+c} \leq 1$$

$$\Rightarrow \ln(1+x) \leq x, \quad x > 0$$

For $-1<x<0$, we use the series. By Alternating Series Test, If we truncate the series at x, the remainder is $-$ve

$$\Rightarrow \ln(1+x) \leq x$$

for $-1<x<0$

$$\Rightarrow \boxed{\ln(1+x) \leq x, \quad x>-1}$$

Now let $x > 0$ First note

$$f'(x) = \frac{1}{1+x}$$

$$g'(x) = \frac{1}{(1+x)^2}, \quad g(x) = \frac{x}{1+x}$$

and $f(0) = \ln(1)$

$$= 0 = g(0)$$

For $x > 0$, $(1+x)^2 > (1+x)$

$\Rightarrow f'(x) > g'(x) \quad \forall x > 0$

$$\Rightarrow \frac{x}{1+x} \leq \ln(1+x).$$

For $-1<x<0$, note

$$\ln(1+x) = \int_x^0 \frac{1}{1+s}\,ds$$

$$= \int_x^0 \sum_{k=0}^{\infty} (-1)^k s^k\,ds$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_x^0 s^k\,ds$$

$$\geq \sum_{k=0}^{\infty} (-1)^k x^k \int_x^0 ds$$

$$= x \sum_{k=0}^{\infty} (-1)^k x^k = \frac{x}{1+x}$$

$\Rightarrow$ For $x > -1$,

$$\boxed{\ln(1+x) \geqslant \frac{x}{1+x}}$$

3.2.2 We prove tha

$$a^{\underbrace{\log_b c}_{r}} = c^{\underbrace{\log_b a}_{t}}$$

We have $r, t \in \mathbb{R}$ s.t.

$$b^r = c, \quad b^t = a$$

Now $a^r = b^{tr}$
$= (b^r)^t$
$= c^t \Rightarrow$ the result.

Other identities in eq. 4 follow similarly!

3.2.3 $$\frac{n!}{n^n} = \frac{n\,(n-1)\,(n-2)\cdots 1}{n\;\;n\;\;n\cdots n}$$

$$= \frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{1}{n}$$

$$= \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n!}{n^n} =$$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdot\lim_{n\to\infty}\left(1-\frac{2}{n}\right)$$

$$\cdots\lim_{n\to\infty}\frac{1}{n} = 1\cdot 1\cdots 0$$

$$= 0$$

$$\Rightarrow \boxed{n! = o(n^n)}$$

"$n!, << n^n$"

To compute

$$\lim_{n \to \infty} \frac{n!}{2^n},$$

we apply the ratio test. We have if

$$a_n = n!/2^n,$$

then,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n+1}} \frac{2^n}{n!}$$

$$= \frac{n+1}{2}$$

$$\Rightarrow \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = +\infty$$

By the extended version/limsup version, we have

$$\lim_{n \to \infty} \frac{n!}{2^n} = +\infty$$

$\Rightarrow$ $\boxed{n! = \omega(2^n)}$

'$n! >> 2^n$'

Since $n! = o(n^n)$,
we have that $\forall c > 0$,
$\exists n_0 \in \mathbb{N}$ s.t.

$$n! < c\, n^n \quad \forall n \geq n_0.$$

Let $c = 1$, $n_0 \in \mathbb{N}$ s.t.

$$n! < n^n \quad \forall n \geq n_0.$$

As lg is strictly inc.

$\Rightarrow \lg n! < \lg n^n$

$= n \lg n, \; n \geq n_0$

$\Rightarrow \lg n! \leq n \lg n, \; n \geq n_0.$

$\Rightarrow \lg n! = O(n \lg n).$

We anaylze the seq.

$$a_n = \frac{n^{n/2}}{n!}$$

We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{\frac{n+1}{2}}}{(n+1)!} \cdot \frac{n!}{n^{n/2}}$$

$$= \frac{(n+1)^{\frac{n-1}{2}}}{n^{n/2}}$$

$$= \sqrt{\frac{(n+1)^{n-1}}{n^n}}$$

$$= \sqrt{\left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n+1}}$$

$$\to \sqrt{e \cdot 0} = 0.$$

$\Rightarrow \exists\, n_1 \in \mathbb{N}$ s.t.

$$\frac{n^{\frac{n}{2}}}{n!} < 1 \,, \quad n \geq n_1$$

$$\Rightarrow n^{n/2} < n!$$

$$\Rightarrow \frac{n}{2} \lg n < \lg n!$$

$$\Rightarrow \frac{1}{2} n \lg n \leq \lg n \,, \quad n \geq n_1$$

Taking $m = \max\{n_0, n_1\}$,

We get

$$\frac{n \lg n}{2} \le \lg n! \le n \lg n$$

for $n \ge m$

$$\Rightarrow \boxed{\lg n! = \Theta(n \lg n)}$$

`$\lg n! \sim n \lg n$'

3.2.4 We saw hints.

Note that $f(n) = O(n^{k})$

$\Longleftrightarrow \lg f(n) = O(k \lg n)$

- $\lg n! = \Theta(n \lg n)$
- $\lceil \lg n \rceil = \Theta(\lg n)$

- $f(n) = \lceil \lg n \rceil !$

$\lg f(n) = \lg \lceil \lg n \rceil !$

$\Theta(\lg(f(n))) = \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil)$

$= \Theta(\lceil \lg n \rceil)$

$\Theta(\lg \lceil \lg n \rceil)$

$= \Theta(\lg n) \times \Theta(\lg(\lceil \lg n \rceil))$

$= \Theta(\lg n)\,\Theta(\lg \lg n)$

limit $= \Theta(\lg n)\,\Theta(\lg n)$

comp. $= \Theta(\lg^2 n)$

test $= \omega(\lg n)$ So no.

$\Rightarrow \lceil \lg n \rceil !$ isn't polynomially bd.

• $f(n) = \lceil \lg \lg n \rceil !$

$\lg f(n) = \lg \lceil \lg \lg n \rceil !$

$\Theta(\lg f(n)) = \Theta(\lceil \lg \lg n \rceil)$

$\Theta(\lg \lceil \lg \lg n \rceil)$

$= \Theta(\lg \lg n)\, \Theta(\lg \lceil \lg \lg n \rceil)$

$= \Theta(\lg \lg n)\, \Theta(\lg \lg \lg n)$

$= \Theta(\lg \lg n \; \lg \lg \lg n)$

Let $a_n = \dfrac{\lg \lg n \; \lg \lg \lg n}{\lg n}$

We compute $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n}$.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\lg\lg n+1 \ \lg\lg\lg n+1}{\lg n+1} \times}{\frac{\lg n}{\lg\lg n \ \lg\lg\lg n}}$$

$$= \frac{\lg n}{\lg n+1} \cdot \frac{\lg\lg n+1}{\lg\lg n} \cdot \frac{\lg\lg\lg n+1}{\lg\lg\lg n}$$

- $\frac{\lg n}{\lg n+1} \to 1$ by LH

- $\frac{\lg\lg n+1}{\lg\lg n} \to \frac{\frac{1}{\lg n+1 \ n+1}}{\frac{1}{\lg n \ n}}$

$$= \frac{n \lg n}{n+1 \ \lg n+1} \to 1$$

- $\dfrac{\lg\lg\lg n+1}{\lg\lg\lg n}$

This proves a stronger result

$$\rightarrow \frac{\dfrac{1}{\lg\lg n+1 \;\; \lg n+1 \;\; n+1}}{\dfrac{1}{\lg\lg n \;\; \lg n \;\; n}}$$

$$= \frac{\lg\lg n \;\; \lg n \;\; n}{\lg\lg n+1 \;\; \lg n+1 \;\; n+1}$$

$\rightarrow 1.$

$\Rightarrow$ $\boxed{\Theta(f(n) = \Theta(\lg n)}$

$\Rightarrow$ J Doly sd.

3.2.5 As lg is mon. increasing

$i \geq 0$

$$\lg(\lg^* n) = \min\{\lg i : \lg^{(i)} n \leq 1\}$$

$$\lg^*(\lg n) = \min\{i \geq 0 : \lg^{(i)} \lg n \leq 1\}$$

$$= \min\{i \geq 0 : \lg^{(i+1)} n \leq 1\}$$

$$= \lg^* n - 1$$

$$= \Theta(\lg^* n)$$

$$= \omega(\lg \lg^* n)$$

$\Rightarrow$ $\lg^* \lg n$ is asymp. bigger

3.2.6 Clear

3.2.7 We have

$$\frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}\right]$$

$$= 1$$

$$= F_1$$

Clearly, $\frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = 0$

Assume that

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} \quad \forall k \leq n$$

Now

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \left[\phi^n + \phi^{n-1} - \left(\hat{\phi}^n + \hat{\phi}^{n-1} \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\phi^{n-1}(1+\phi) - \hat{\phi}^{n-1}(1+\hat{\phi}) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\phi^{n-1}\phi^2 - \hat{\phi}^{n-1}\hat{\phi}^2 \right]$$

$$= \frac{1}{\sqrt{5}} \left[\phi^{n+1} - \hat{\phi}^{n+1} \right]. \blacksquare$$

3.2.8 Let $f(n,k) = k \ln k$
$g(n,k) = n$

Now $k \ln k = \Theta(n)$

$\Rightarrow \exists\, c_1, c_2 > 0;\ n_0, k_0 \in \mathbb{N}$
s.t.

$$c_1 n \le k \ln k \le c_2 n,$$

when $n \geq n_0$ and $k \geq k_0$

$$\Rightarrow \frac{c_1 n}{\ln n} \le \frac{k \ln k}{\ln n} \le \frac{c_2 n}{\ln n}$$

for $n \geq n_0$ and $k \geq k_0$

$\Rightarrow k \ln k / \ln n = \Theta(n / \ln n)$

$$\Rightarrow \quad n \ln n = \Theta\left(k \frac{\ln k}{\ln n}\right)$$

Thus it suffices to show that

$$\frac{k \ln k}{\ln n} = \Theta(k)$$

We do this by the limit comparison test.

Let $a_{n,k} = \dfrac{k \ln k}{\ln n}$

$$b_{n,k} = k$$

$$\Rightarrow c_{n,k} = \frac{a_{n,k}}{b_{n,k}} = \frac{\ln k}{\ln n}$$

Hence,

$$c_{n,k} = (c_1)_n \, (c_2)_k$$

$$\Rightarrow \frac{c_{n+1,k+1}}{c_{n,k}} = \frac{\ln n}{\ln n+1} \; \frac{\ln k+1}{\ln k}$$

$$\longrightarrow 1$$

as $k \to \infty$ & $n \to \infty$ indep.

$$\Rightarrow \frac{k \ln k}{\ln n} = \Theta(k),$$ proving the claim.

PROBLEMS

1a. Let $k \geq d$. We find $c > 0$ s.t.

$$a_0 + a_1 n^1 + \cdots + a_d n^d \leq c n^k$$

$$\Longleftrightarrow \frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \cdots + \frac{a_d}{n^{k-d}} \leq c$$

Now $n^{k-x} \geq 1$, $0 \leq x \leq d$

$$\Longrightarrow \frac{a_x}{n^{k-x}} \leq a_x, \quad 0 \leq x \leq d$$

$$\Longrightarrow \frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \cdots + \frac{a_d}{n^{k-d}} \leq \cdots$$

$a_0 + a_1 + \cdots + a_d$

$\leq a_d + |a_1| + \cdots + |a_0|$

$:= c$

$\Rightarrow \quad p(n) = O(n^k)$

6. Now let $k \leq d$.

we find $c > 0$ s.t.

$c n^k \leq a_0 + a_1 n^1 + \cdots + a_d n^d$

Note that

$$\frac{a_0 + a_1 n^1 + \cdots + a_d n^d}{n^k} \to +\infty$$

So take $c > 0$, $n_0 \in \mathbb{N}$ s.t.

$a_0 + a_1 n^1 + \cdots + a_d n^d \geq c n^k$,

$n \geq n_0$

c. Apply b and c. <u>OR</u>

$$\lim_{n \to \infty} \sum_{i=0}^{d} a_i n^i / n^d$$

$$= a^d + \lim_{n \to \infty} \sum_{i=0}^{d-1} \frac{a_i}{n^{d-i}}$$

$$= a^d$$

$$\Rightarrow p(n) = \Theta(n^k),\ k = d$$

d. We have

$$\lim_{n\to\infty} \sum_{i=0}^{d} a_i n^i / n^k$$

$$= \lim_{n\to\infty} \sum_{i=0}^{d} \frac{a_i}{n^{k-i}} \to 0$$

$$\Rightarrow p(n) = o(n^k), \quad k > d$$

e. We have

$$\lim_{n\to\infty} \sum_{i=0}^{d} \frac{a_i n^i}{n^k} \to +\infty,$$

$$\Rightarrow p(n) = \omega(n^k), \quad k < d$$

3.2

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^{ϵ}	✓	✓	✗	✗	✗
n^k	c^n	✓	✓	✗	✗	✗
$\sqrt{n}$	$n^{\sin n}$	✗	✗	✗	✗	✗
2^n	$2^{n/2}$	✗	✗	✓	✓	✗
$n^{\lg c}$	$c^{\lg n}$	✓	✗	✓	✗	✓
$\lg n!$	$\lg n^n$	✓	✗	✓	✗	✓

* $o \Rightarrow O$; $\omega \Rightarrow \Omega$

3.4 a. No. Let $f(n) = \lg n$,
$g(n) = n$

$$\frac{\lg n}{n} \to 0 \Rightarrow \lg n = o(n)$$
$$\Rightarrow \lg n = O(n)$$

But $n \neq O\lg(n)$

Assume that

$$n = O(\lg n)$$

Then $\exists\, c > 0,\ n_0 \in \mathbb{N}$
s.t.

$$n \le c \lg n, \quad n \ge n_0$$

$$\Rightarrow \frac{n}{\lg n} \le c, \quad n \ge n_0$$

But $\lim_{n \to \infty} \frac{n}{\lg n} = +\infty$,

yielding a contradiction

b. Clearly,

$$\min\{f(n)+g(n)\} \leq f(n)+g(n)$$

$$\Rightarrow \min\{f(n), g(n)\} = O(f(n)+g(n)).$$

But $\min\{f(n), g(n)\} \neq O(f(n) \cdot g(n))$ in general.

Let $f(n) = g(n) = n$. Then $\min\{f(n), g(n)\} = n$ and $\nexists\, c > 0$ s.t.

$$cn^2 \leq n \text{ for } n \geq n_0,$$

$n_0 \in \mathbb{N}$.

c&d

e. No. Let $f(n) = \frac{1}{n}$.

Then if $\exists\, c > 0,\ n_0 \in \mathbb{N}$

s.t.

$$\frac{1}{n} \le \underbrace{c}_{\frac{1}{d}} \frac{1}{n^2},\quad n \geq n_0$$

$$\Rightarrow\quad n \geq d n^2$$

$$\Rightarrow\quad 1 \geq d n$$

$$\Rightarrow\quad n \le \frac{1}{d},\quad n \geq n_0.$$

f. True, as

$0 \leq f \leq c\, g \iff$

$0 \leq c^{-1} f \leq g$.

g. No. Let $f(n) = 2^n$.
Then

$$2^{n/2} = O(2^n)$$

but $2^{\frac{n}{2}} \neq \Omega(2^n)$

h. Let $g \in \Theta(f)$. Then

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0.$$

Then as f is asymp. positive, we have,

$$\lim_{n\to\infty} \frac{f(n)+g(n)}{f(n)}$$

$$= \lim_{n\to\infty} 1 + \frac{g(n)}{f(n)} = 1$$

So by the limit comp. test

$$f+g = \Theta(f)$$

$$\Rightarrow f + o(f) = \Theta(f).$$