FIND-MAX-CROSSING SUBARRAY
(A, low, mid, high):
```
left sum, sum = -∞, 0
for i in range(mid, low, -1):
    sum = sum + A[i]
    if sum > left-sum
        left-sum = sum
        max-left = i
right-sum, sum = 0
for j in range(mid+1, high):
    sum = sum + A[j]
    if sum > right-sum:
        right-sum = sum
        max-right = j
return (max-left, max-right,
        left-sum + right-sum)
```

9.1.1 Finds max elt.

4.1.2 Brute-force-max-subarray(A):

we assume the data is given s.t. the net diff. in input values is given.

c_1	1	max_sum = −∞
		~~sum = 0~~
c_2	$n+1$	for i in range(1,n):
c_3	n	temp_sum = 0
c_4	$n-i+2$	for j in range(i,n):
c_5	$n-i+1$	temp_sum = temp_sum + A(j)
c_6	$n-i+1$	if temp_sum > max_sum:
c_7	t_i	max_sum = temp_sum
c_8	t_i	low, high = i, j.
c_9	1	return (low, high, max_sum)

brought on day i-1

$\Theta(n^2)$

4.1.4. if sum is −ve, empty subarray

4.1.5 Find-max-subarray(A):

```
low_index = 1
high_index = 1
max_sum = A[1]
# initial max. subarray
  is A[1].
for i in range(2, n):
    if high_index == i-1:
        sum = max_sum + A[i]
        if sum > max_sum:
            max_sum = sum
            high_index = i
    if A[i] > max_sum:
        low_index = 1
        high_index = 1
        max_sum = A[i]
return (low_index, high_index,
        max_sum)
```

4.3.1 $T(1) = c$

$T(2) = c + 2$

$T(3) = c + 2 + 3$

$T(4) = c + 2 + 3 + 4$

$\vdots$

$T(n) = c + 2 + 3 + 4 + \dots + n$

Ignoring boundary condition.

$\Rightarrow$ guess is $T(n) = O(n^2)$

Assume claim is true for $n-1$ & n. Then,

$$T(n) \leq c(n-1)^2 + n$$
$$= cn^2 - 2cn + c + n$$
$$= cn^2 + n(1-2c) + c$$
$$\leq cn^2, \quad c > \frac{1}{2}, \; n \geq \frac{c}{2c-1}$$

4.3.2 We use the recurrence $T(n) = T(n/2) + 1$ to gen. a guess. This looks like recurrence we get in Binary tree search so try

$$T(n) = O(\lg n).$$

Assume it is true for $k \le n$ ..

$$
\begin{aligned}
\Rightarrow T(n) &= T(\lceil n/2 \rceil) + 1 && , T \text{ is inc.} \\
&\le T(n/2 + 1) + 1 && \text{bd. con! is positive} \\
&= T\left(\frac{n+2}{2}\right) + 1 \\
&\le c \lg\left(\frac{n+2}{2}\right) + 1 && \frac{n+2}{2} \le n \text{ if } n \ge 2 \\
&= c \lg(n+2) - c\lg 2 + 1
\end{aligned}
$$

$\leq c \lg(n+2)$, $c \geq 1$

Doesn't work.
Let's amend the guess.

Try $T(n) \leq \lg(n-2)$

$$\Rightarrow T(n) = T(\lceil n/2 \rceil) + 1$$
$$\leq \lg(\lceil n/2 \rceil - 2) + 1$$
$$\leq \lg\left(\frac{n}{2} - 1\right) + 1$$
$$= \lg\left(\frac{n-2}{2}\right) + 1$$
$$= \lg\left(\frac{n-2}{2}\right) + \lg 2$$
$$= \lg(n-2)$$

$$\Rightarrow T(n) = O(\lg(n-2)) = O(\lg n)$$

4.3.3 $T(n) = 2T\lfloor \frac{n}{2} \rfloor + n$

Assume that
$T(K) \geq c\,(K+2)\lg(K+2)$
for all $K \leq n$.

$$\Rightarrow T(n) = 2T\left\lfloor \frac{n}{2} \right\rfloor + n$$

$$\geq 2c\left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)\lg\left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right) + n$$

$$\geq 2c\left(\frac{n}{2} + 1\right)\lg\left(\frac{n}{2} + 1\right) + n$$

$$= c(n+2)\lg(n+2) + n - c(n+2)$$

$$* \geq c\,(n+2)\lg(n+2), \quad c \leq \frac{1}{3}$$

$$\Rightarrow T(n) \geq c\,(n+2)\lg(n+2), \quad n \geq n_0$$
$$= \Omega(n \lg n)$$

$$\Rightarrow T(n) = \Omega(n \lg n)$$

* For '1' to hold, we want

$$n - c(n+2) \geq 0$$

$$\Leftrightarrow \frac{n}{n+2} \geq c$$

$$\Leftrightarrow c \leq n/n+2$$

$n/n+2 \to 1$ and for $n=1$, $\frac{1}{1+2}$ it is $1/3$. So, let $c \leq \frac{1}{3}$.

4.3.4 $T(1) = 1$,

$T(n) = 2T\lfloor n/2 \rfloor + n$, $n \geq 1$.

Assume $T(k) \leq c k \lg k + 1$, $k < n$. For $k = 1$, it's true

$$\Rightarrow T(n) \leq 2c\left\lfloor \frac{n}{2} \right\rfloor \lg \left\lfloor \frac{n}{2} \right\rfloor + 2 + n$$

$$\leq c n \lg\left(\frac{n}{2}\right) + n + 2$$

$$= c n \lg n + n - cn + 2$$

$$= c n \lg n + 1 + n - cn + 1$$

$$= c n \lg n + 1 + n(1 - c) + 1$$

$$\leq c n \lg n + 1, \quad c > 1, \quad n \geq \frac{1}{c-1}$$

$$\Rightarrow T(n) \leq c n \lg n + 1 \quad \forall n \geq n_0 \ (n \geq 1$$

$$\leq (c+1) n \lg n, \quad n \geq 2$$

$\Rightarrow T(n) = O(n \lg n)$

4.3.5 $T(n) = T\lceil n/2 \rceil + T\lfloor n/2 \rfloor + \Theta(n)$

We show that the solution to $S(n) = T\lceil n/2 \rceil + T\lfloor n/2 \rfloor$ is $\Theta(n \lg n)$.

We first show that $S(n) = O(n \lg n)$.

Assume that $T(k) \leq c(k-2)\lg(k-2)$ for $k < n$

$$\Rightarrow S(n) = T\lceil n/2 \rceil + T\lfloor n/2 \rfloor$$
$$\leq c\left(\left\lceil \frac{n}{2} \right\rceil - 2\right)\lg\left(\left\lceil \frac{n}{2} \right\rceil - 2\right) +$$
$$c\left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right)\lg\left(\lfloor n/2 \rfloor - 2\right)$$

$$\leq c\left(\left\lceil\frac{n}{2}\right\rceil-2\right)\lg\left(\left\lceil\frac{n}{2}\right\rceil-2\right)$$

$$+c\left(\left\lfloor\frac{n}{2}\right\rfloor-2\right)\lg\left(\left\lfloor\frac{n}{2}\right\rfloor-2\right)$$

$$\leq c\left(\left\lceil\frac{n}{2}\right\rceil+\left\lfloor\frac{n}{2}\right\rfloor-4\right)\lg\left(\left\lceil\frac{n}{2}\right\rceil-2\right)$$

$$=c(n-4)\lg\left(\left\lceil\frac{n}{2}\right\rceil-2\right)$$

$$\leq c(n-4)\lg\left(\frac{n}{2}+1-2\right)$$

$$\leq c(n-4)\lg\left(n/2-1\right)$$

$$=c(n-4)\lg\left(\frac{n-2}{2}\right)$$

$$\leq c(n-4)\lg(n-2)\leq c(n-2)\lg(n-2)$$

$$\Rightarrow S(n) = O(n \lg n)$$

Now asume $T(K) \geq (K+2)\lg(K+2)$, $K < n$.

$$\Rightarrow S(n) \geq c\left(\left\lceil \frac{n}{2} \right\rceil + 2\right) \lg \left(\left\lceil \frac{n}{2} \right\rceil + 2\right) + c\left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right) \lg \left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)$$

$$\geq c\left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor + 4\right) \log \left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)$$

$$= c(n+4) \lg \left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)$$

$$\geq c(n+4) \lg \left(\frac{n+1}{2}\right)$$

$$T(n) = S(n) + \Theta(n)$$

$$\Rightarrow T(n) \geq cn\lg(n) - c(n+4) + dn$$

$$= cn\lg(n) + n(d-c) - 4c$$

$$\geq cn\lg(n), \quad n \geq \frac{4c}{d-c}$$

$$d \neq c$$

For $c = d$,

$T(n) > c[(n+4)\lg(n+1) - 4]$

$> c(n+2)\lg(n+2)$ eventually

So result is true here as well $\Rightarrow T(n) = \Omega(n\lg n)$

So, we have $T(n) = \Omega(n\lg n)$

and $S(n) = O(n\lg n)$

$$\Rightarrow T(n) = O(n\lg n) + \Theta(n)$$

$$= O(n\lg n) + O(n)$$

$$= O(n\lg n + n)$$

$$= O(n\lg n) \Rightarrow T(n) = \Theta(n\lg n)$$

as $\frac{n\lg n + n}{n\lg n} = 1 + \frac{1}{\lg n}$ decreases to 1

$\Rightarrow n\lg n + n / n\lg n \leq c$ $\forall$ ($c > 1$) asymp.

4.3.6 WTS $T(n) \le c(n-a)\lg(n-a)$.
Assume this is true for $k<n$.

Note that we can make
$$\lfloor \tfrac{n}{2} \rfloor + 17 \le n$$

?

if $\lfloor \frac{n}{2} \rfloor + 17 \le \frac{n}{2} + 17 \le n$

and the last ineq. is true if $n \geq 34$.

So, $T(n) \le 2c\left(\lfloor \frac{n}{2} \rfloor + 17 - a\right) \times$
$$\lg\left(\lfloor \tfrac{n}{2} \rfloor + 17 - a\right) + n$$
$$\le 2c\left(\tfrac{n}{2} + 17 - a\right)\lg\left(\tfrac{n}{2} + 17 - a\right) + n$$

$$= c(n+34-2a)\lg\left(\frac{n+34-2a}{2}\right)+n$$

$$\leq c(n+34-2a)\lg(n+34-2a)+n$$
$$- c(n+34-2a).$$

Now $n+34-2a \leq n-a$

$\iff$ $34 \leq a \iff a \geq 34$

$$\Rightarrow T(n) \leq c(n-a)\lg(n-a)$$
$$-c(n+34-2a)+n$$

Now $n - c(n+34-2a) \leq 0$

$\iff$ $n(1-c) + c(\underbrace{34-2a}_{\leq 0 \text{ as } a \geq 34}) \leq 0$

So let $c > 1$

$$\Rightarrow T(n) \leq c(n-a)\lg(n-a)$$

$$\Rightarrow T(n) = O((n-a)\lg(n-a))$$
$$= O(n \lg n).$$

4.3.7 Assume $T(k) \leq c\,k^{\log_3 4}$, $k < n$.

$$\Rightarrow T(n) = 4T\left(\frac{n}{3}\right) + n$$
$$\leq 4c\left(\left(\frac{n}{3}\right)^{\log_3 4}\right) + n$$
$$= \frac{4c\,n^{\log_3 4}}{3^{\log_3 4}} + n \leq c\,n^{\log_3 4}$$

not true ever unless $c < 0$

Try $T(k) \leq c\,k^{\log_3 4} - 3n$, $\log_3 4 > 1$

$$\Rightarrow T(n) \leq \frac{\cancel{4}\,n^{\log_3 4}}{\cancel{3^{\log_3 4}}} - \frac{n}{3} = n^{\log_3 4} - n$$
$$= O(n^{\log_3 4})$$

4.3.8 Assume that

$$T(k) \leq ck^2 \quad , k < n$$

$$\Rightarrow T(n) \leq 4c\left(\frac{n}{2}\right)^2 + n$$

$$= \frac{4cn^2}{4} + n$$

$$= cn^2 + n \not\leq cn^2$$

Consider

$$T(k) \leq ck^2 - k \quad k < n$$

$$\Rightarrow T(n) \leq 4T(n/2) + n$$

$$\leq 4\left(c\left(\frac{n}{2}\right)^2 - \frac{n}{2}\right) + n$$

$$= cn^2 - 2n + n$$

$$= cn^2 - n$$

$n \geq n_0$

$$\Rightarrow T(n) \leq cn^2 - n = O(n^2) \Rightarrow T(n) = O(n^2)$$

4.3.9 $T(n) = 3T(\sqrt{n}) + \lg n$

Let $m = \lg n \Rightarrow n = 2^m$

$\Rightarrow T(2^m) = 3T(2^{m/2}) + m$

Let $S(m) = T(2^m)$

$\Rightarrow S(m) = 3S(\frac{m}{2}) + m$

Influenced by 4.2.7, we claim that

$S(m) = \Theta(m^{\lg_2 3})$

Assume for $k < m$

$S(k) = ck^{\lg_2 3} - 2k$

$$\Rightarrow S(m) = 3\,S(m/2) + m$$

$$\leq \cancel{3}\left(c\left(\frac{m}{\cancel{2}}\right)^{\lg 3} - \frac{2m}{2}\right) + m$$

$$= c\, m^{\lg 3} - 2m$$

$$\Rightarrow S(m) \leq c\, m^{\lg 3} - 2m,\ m \geq m_0$$

$$\leq c\, m^{\lg 3}$$

$$\Rightarrow S(m) = O\left(m^{\lg 3}\right)$$

We now show that

$$S(m) = \Omega\left(m^{\lg 3}\right)$$

For $k < n$, assume that

$$S(K) \geq c\, K^{\lg 3}$$

$$\Rightarrow S(m) \geq \cancel{3}\, \frac{m^{\lg 3}}{\cancel{2}^{\lg 3}} + m \geq m^{\lg 3}$$

$$\Rightarrow S(m) = \Omega(m^{\lg 3})$$

$$\Rightarrow S(m) = \Theta(m^{\lg 3})$$

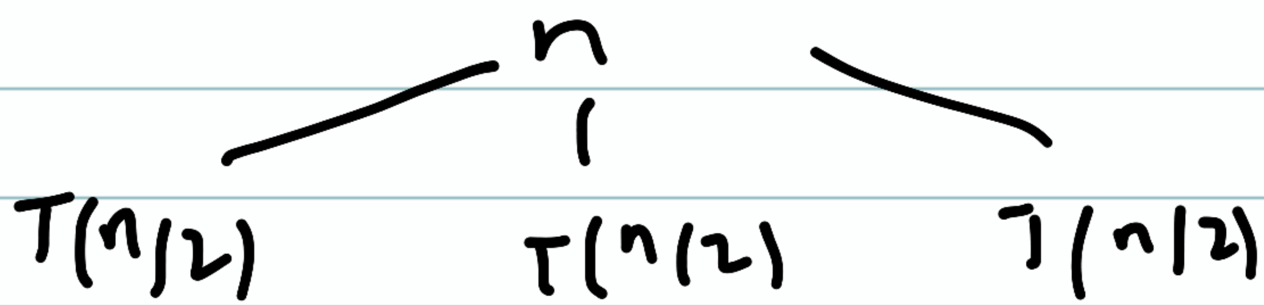
$$\Rightarrow \underbrace{T(2^m)}_{T(n)} = S(m) = \Theta(m^{\lg 3})$$

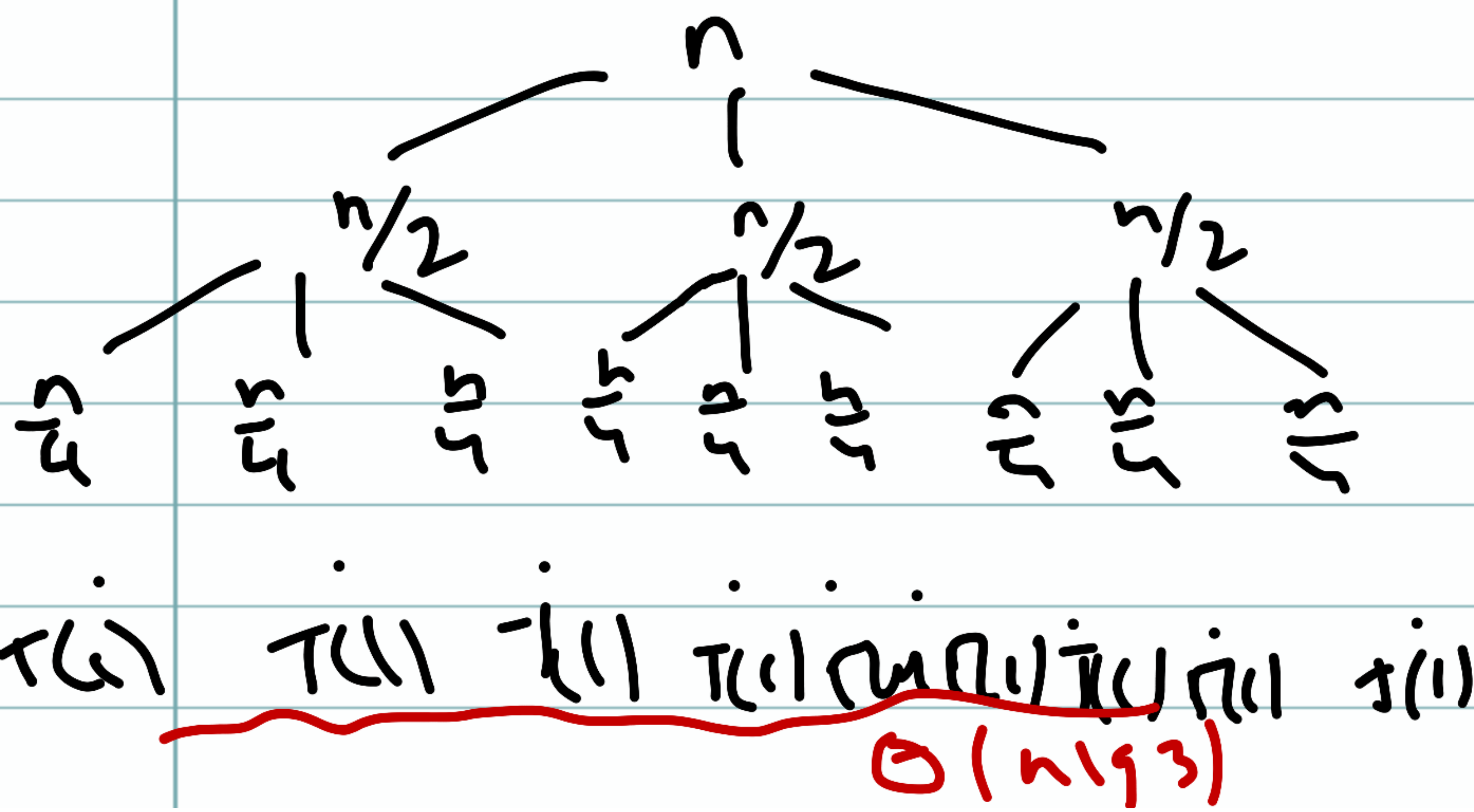
$$\Rightarrow T(n) = \Theta((\lg n)^{\lg 3})$$

4.4.1) $T(n) = 3T(n/2) + n$

Let $n = 2^k$

$T(n)$





There are $k+1$ steps. At the i^{th} step, there are 3^i costs, where each individual cost is $\frac{n}{2^i}$.

Bottom level has $3^{\lg n} = n^{\lg 3}$ with cost $T(1)$.

$$\Rightarrow T(n) = n \sum_{i=0}^{\lg n - 1} \left(\frac{3}{2}\right)^i + \Theta(n^{\lg 3})$$

$$= n \frac{(3/2)^{\lg n} - 1}{(3/2) - 1} + \Theta(n^{\lg 3})$$

$$= 2n\left[(3/2)^{\lg n} - 1\right] + \Theta(n^{\lg 3})$$

$$= 2n^{\lg 3} - 2n + \Theta(n^{\lg 3})$$
$$= \Theta(n^{\lg 3}) - 2n$$
$$\leq O(n^{\lg 3})$$

We prove this by subs.
Assume

$$T(k) \leq c\,n^{\lg 3} - 2k$$

$k \geq n_0,\ k < n$

$$\Rightarrow T(n) \leq 3T(n/2) + n, \quad T \text{ is inc.}$$
$$\leq \cancel{3}\left[c\frac{n^{\lg 3}}{\cancel{2^{\lg 3}}} - 6\frac{n}{2}\right] + n$$
$$= c\,n^{\lg 3} - 3n + n$$
$$= c\,n^{\lg 3} - 2n \quad \boxtimes$$

4.4.2 Tr. 1

$n = 2^k$

n^2

$T(n/2)$

There $\lg n + 1$ leaves.

n^2

$T(n/2)$

At the i^{th} leaf cost is $\frac{n^2}{(2^i)^2}$

$n^2/(2^2)^0$

$n^2/2^2$

$T(n/4)$

$\vdots$

$T(1) = O(1)$ 1 leaf.

$n = 2^k$
$k = \lg n$

$$\Rightarrow T(n) = n^2 \sum_{i=0}^{\lg n - 1} \frac{1}{(2^i)^2} + O(1)$$

$$= n^2 \sum_{i=0}^{\lg n - 1} \left(\frac{1}{4}\right)^i + O(1)$$

$$= n^2 \frac{1 - (1/4)^{\lg n}}{1 - 1/4} + O(1)$$

$$= \frac{3n^2}{4}\left(1 - \left(\frac{1}{4}\right)^{\lg n}\right) + O(1)$$

$$\leq 3n^2/4 + O(1)$$

$\Rightarrow$ Guess: $T(n) = O(n^2)$. Let

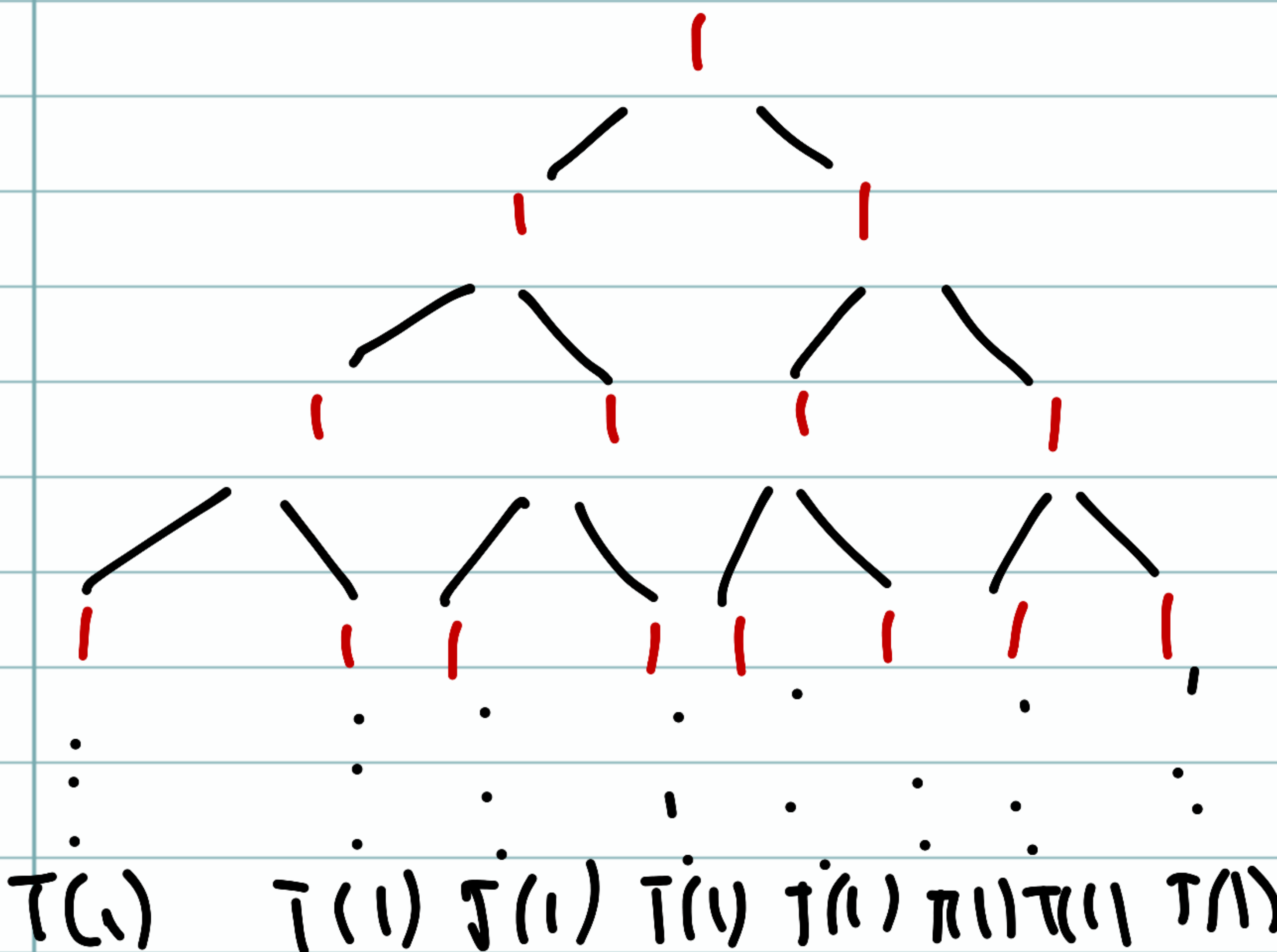
$T(k) \leq cn^2$, $k \geq n_0$, $k < n$

$$\Rightarrow T(n) \leq c\left(\frac{n}{2}\right)^2 + n^2$$

$$= \frac{cn^2}{4} + n^2$$

$$= \left(\frac{c}{4} + 1\right) n^2, \quad c \geq \frac{4}{3}.$$

4.4.4 $T(n) = 2T(n-1) + 1$



There are n-levels. The n^{th}-level has cost $O(1)$ with 2^n leaves

The i^{th} level, $i = 0, \dots, n-1$, has 2^i leaves with cost $\underline{1}$.

$$\Rightarrow T(n) = \sum_{i=0}^{n-1} 2^i + \Theta(2^n)$$

$$= 2^n - 1 + \Theta(2^n)$$

$$\leq 2^n + \Theta(2^n)$$

$$= \Theta(2^n)$$

Guess: $T(n) = O(2^n)$

Assume $T(k) \leq c\,2^k - d$

$k \geq n_0$, $k < n$

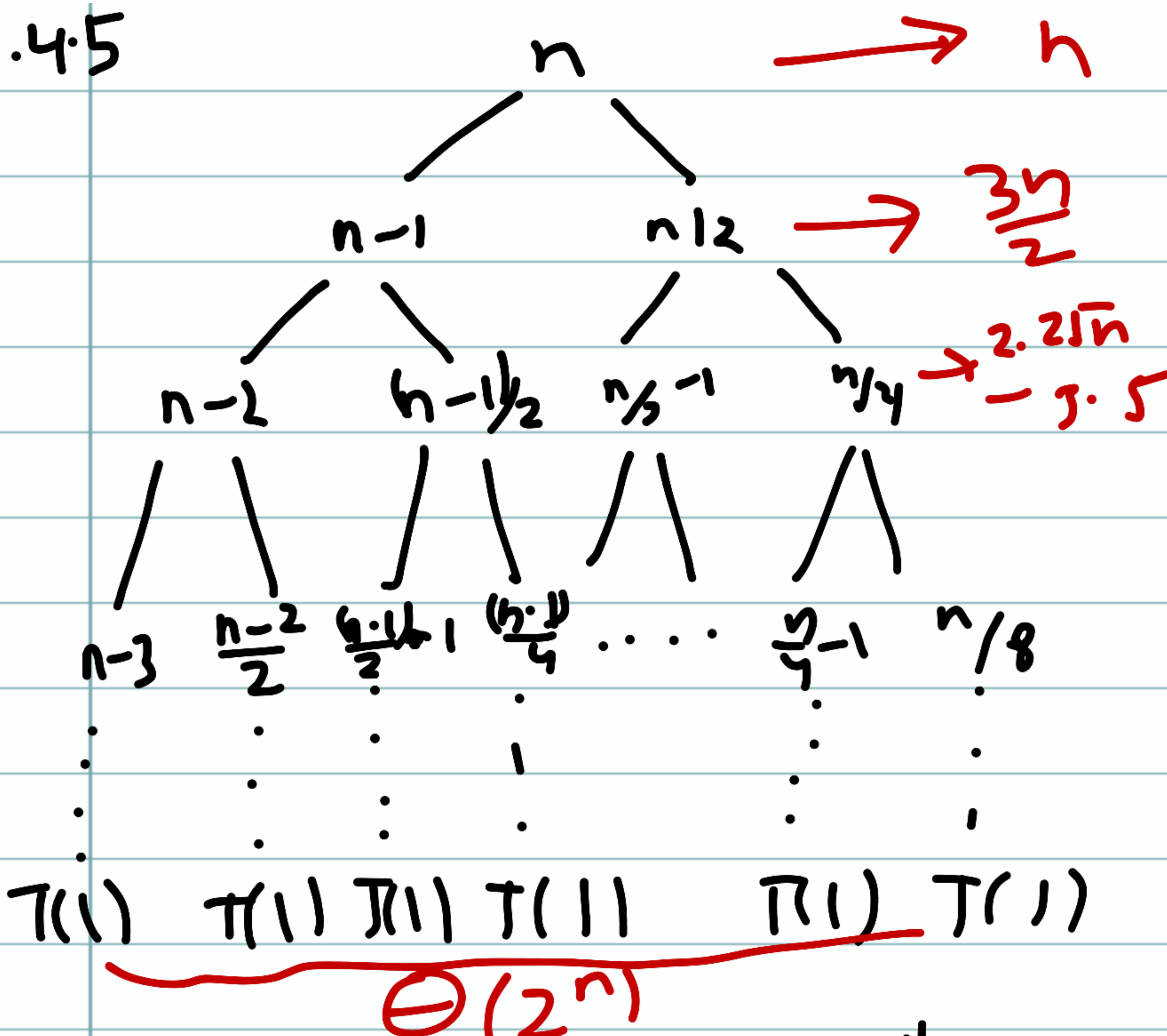
$$\Rightarrow T(n) \leq 2c\,2^{n-1} - 2d + 1$$

$$\leq c\,2^n - 2d + 1$$

$$= c\,2^n - 2d + 1$$

$$\leq c\,2^n, \quad d \geq 1. \quad \square$$

4.4.5



There are, asymptotically, at most n levels as the longest path is

$i=0,\ldots,n-1$ $\quad n \to n-1 \to n-2 \to \cdots \to T(1)$

At the i^{th} level there are 2^i levels; asymptotically, the worst-case cost of each leaf is at the i^{th} level is $n-i$

Summing up this gives the worst case cost at the i^{th} level to be $2^i(n-i)$

Now of course some leafs are empty but we find out the worst case cost, which seems to be

$$\sum_{i=0}^{n-1} 2^i(n-i) + \Theta(2^n)$$

$$\leq n \cdot \sum_{i=0}^{n-1} 2^i + \Theta(2^n) = \Theta(2^n)$$

We show that $T(n) = O(2^n)$

Let

$T(k) \leq c2^n - n$ for $k \geq n_0$, $k < n$.

$$\Rightarrow T(n) \leq c2^{n-1} - (n-1) + c2^{n/2} - (n/2) + n$$

$$\leq c2^{n-1} + c2^{n/2} - (n-1) - (n/2) + n$$

for $n \geq 2$ $\leq c2^{n-1} + c2^{n-1} - (n-1) - (n/2) + 1$

$= c2^n - (n-1) - (n/2) + 1$

for $n \geq 2$ $\leq c2^n$ ▨

4.4.1 Shortest path down to the root of the tree is in K-steps, where

$$\frac{n}{3^k} = 1$$

$\iff k = \log_3 n$. Till then, no other leaf has cost zero. The minimal cost associated with each of the preceding i-levels, from $i = 0, \ldots, \log_3 n$, is $c\left(\frac{2}{3}\right)^i$. At the k^{th} level, keep it 1.

$\Rightarrow$ The worst-case cost seems to be

$$c \sum_{i=0}^{\log_3 n - 1} \left(\frac{2}{3}\right)^i + \Theta\left(2^{\log_3 n}\right)$$

$$= \frac{1-(2/3)^{\log_3 n}}{1-2/3} + \Theta(n^{\log_3 2})$$

$$= 3\left[1 - \frac{2^{\log_3 n}}{n}\right] + \Theta(n^{\log_3 2})$$

$$= 3\left[1 - n^{\log_3 2 - 1}\right] + \Theta(n^{\log_3 2})$$

$$= 3\left[1 - n^{\log_3 2 - \log_3 3}\right] + \Theta(n^{\log_3 2})$$

$$= 3\left[1 - n^{\log_3 (\frac{2}{3})}\right] + \Theta(n^{\log_3 2})$$

$$= \Theta(n^{\log_3 2})$$

$$= \Theta(n \lg n \quad)$$

as if $c_n = \frac{n \lg n}{n^{\log_3 2}}$

$= \frac{n \lg n}{n^{1/\lg 3}}$

We have

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)\lg(n+1)}{(n+1)^{1/\lg 3}} \cdot \frac{n^{1/\lg 3}}{n \lg(n)}$$

$\rightarrow 1$

This suggests that

$T(n) = \Omega(n \lg n)$

We try and prove it.

Let

$$T(k) \geqslant ck\lg k$$

$k \geqslant n_0,\ k < n$

$$\Rightarrow T(n) = T(n/3) + T(2n/3) + n$$

$$\geqslant c\left(\frac{n}{3}\right)\lg\left(\frac{n}{3}\right) + c\left(\frac{2n}{3}\right)\lg\left(\frac{2n}{3}\right) + n$$

$\geqslant \lg(n)$

$$= c\left(\frac{n}{3}\right)\lg(n) + c\left(\frac{2n}{3}\right)\lg(2n)$$

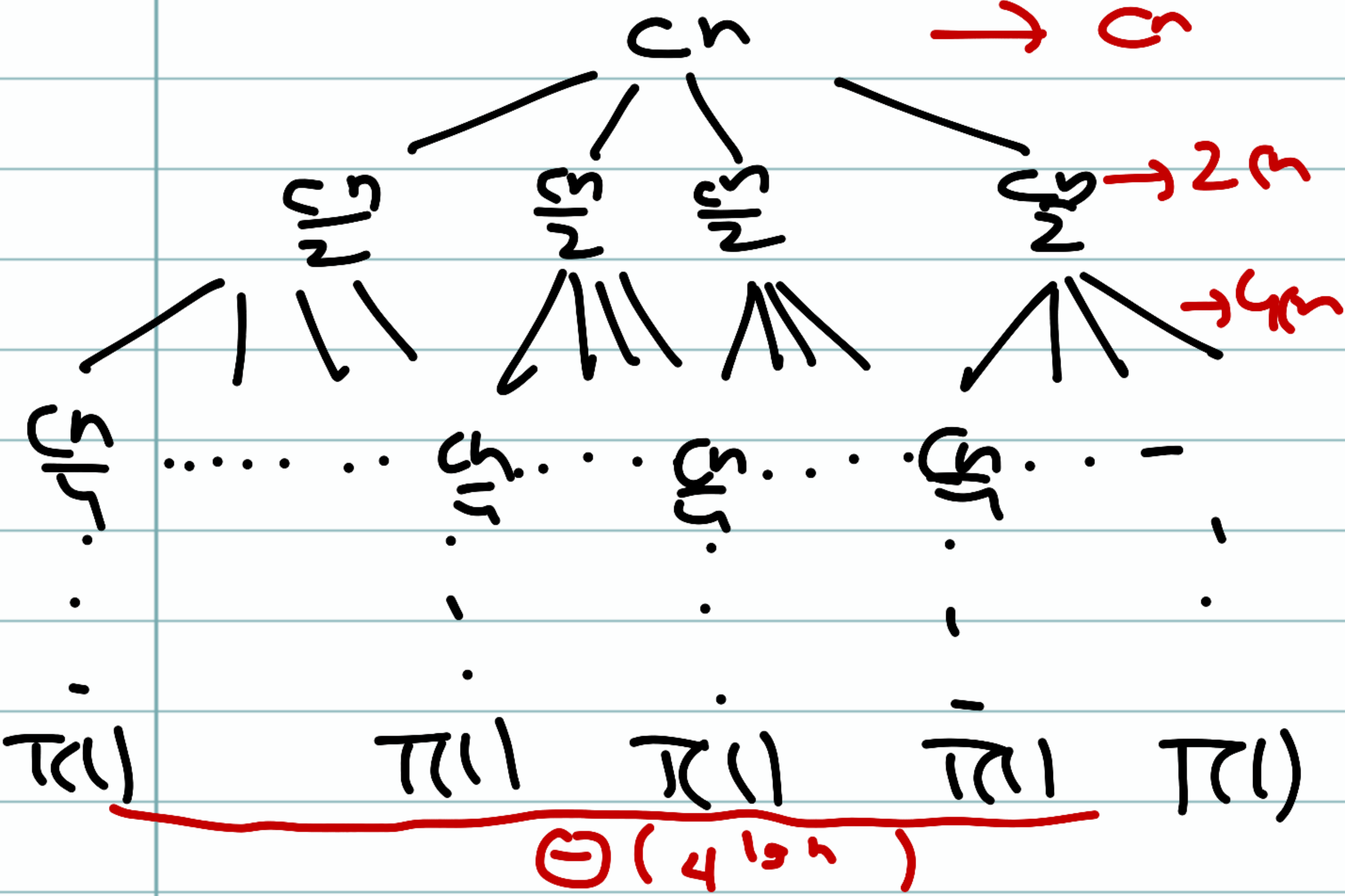
$$+ n - c\left(\frac{n}{3}\right)\lg 3 - c\left(\frac{2n}{3}\right)\lg 3$$

$$\geqslant cn\lg n + n - cn\lg 3$$

$$\geqslant cn\lg n,\quad c \leq 1/\lg 3 \sim 0.63$$

4.47 We draw the tree for $T(n) = 4T(n/2) + cn$

$n = 2^K$



The tree has $K = \lg n$ levels.
The approximate cost is

$$cn \sum_{i=0}^{\lg n - 1} 2^i + \Theta(4^{\lg n})$$

$= cn(2^{\lg n} - 1) + \Theta(4^{\lg n})$

$= cn(n-1) + \Theta(n^{\lg 4})$

$= cn(n-1) + \Theta(n^2)$

$= \Theta(n^2).$

By subs. we prove that
$T(n) = \Theta(n^2)$!

Assume that,

$$T(k) \le ck^2 - dk ,$$

$k \ge n_0$, $k < n$.

$$\Rightarrow T(n) \le 4c\lfloor n/2 \rfloor^2 - 4d\lfloor \tfrac{n}{2} \rfloor + cn$$
$$\le cn^2 - 4d\lfloor n/2 \rfloor + cn$$

Now we want

$$-4d\lfloor n/2 \rfloor + cn \le -dn$$

$$\iff cn \le 4d\lfloor n/2 \rfloor + dn$$

Let $d = c > 0$. Then this is eq. to

$$cn \le c[4\lfloor n/2 \rfloor + n]$$ ✓

Now let

$$T(k) \ge c(k+2)^2 + ck$$

$k \ge n_0$, $k < n$.

$$\Rightarrow T(k) = 4T\lfloor n/2 \rfloor + cn$$
$$\ge 4c\left(\left\lfloor \tfrac{n}{2} \right\rfloor + 2\right)^2 + 4c\left\lfloor \tfrac{n}{2} \right\rfloor + cn$$

$$\geq 4c\left(\frac{n}{2}-1+2\right)^2+4c\lfloor n/2\rfloor+cn$$
$$= 4c\left(\frac{n}{2}+1\right)^2+4c\lfloor n/2\rfloor+cn$$
$$= 4c\left(\frac{n+2}{2}\right)^2+4c\lfloor n/2\rfloor+cn$$
$$= c(n+2)^2+4c\lfloor n/2\rfloor+cn$$
$$\geq c(n+2)^2+cn$$

$$\Rightarrow T(n)=\Omega\left(c(n+2)^2+cn\right)$$
$$=\Omega\left(cn^2\right)$$

$$\Rightarrow T(n)=\Theta(n^2).$$

4.4.8 Not we can't go below a. The lowest level is a.

$$cn + T(a)$$
$$|$$
$$c(n-a) + T(a)$$
$$|$$
$$c(n-2a) + T(a)$$
$$|$$
$$\vdots$$
$$T(a)$$

Assume we start at $n = Ka$, $K \geq 2$.
There are K-levels, with the K-th level having cost $T(a)$

and the first $k-1$ tuples have cost $c(k-i)a + \lceil 1/a \rceil$, $i = 0, \ldots, (k-2)$

$\Rightarrow$ The approx. cost is

$$ca \sum_{i=0}^{k-2} (k-i) + O(1)$$

$$= ca[2 + \cdots + k] + O(1)$$

$$= ca\left[\frac{k(k+1)}{2} - 1\right] + O(1)$$

$$= \Theta\left[\frac{ca\,k(k+1)}{2}\right] + O(1)$$

$$= \Theta\left[\frac{c}{2} \cdot nc \cdot \left(\frac{n}{a} + 1\right)\right]$$

$= \Theta(n^2)$. We prove this using subs. now

Assume that,

$$T(k) \le cn^2$$

$k \geq n_0$, $k < n$

$$\Rightarrow T(n) = T(n-a) + T(a) + cn$$
$$\le c(n-a)^2 + T(a) + cn$$
$$= cn^2 - 2acn + a^2c$$
$$+ T(a) + cn$$
$$\le cn^2$$

$$\Leftrightarrow a^2c + cn + T(a) \le 2acn$$
$$\Leftrightarrow a^2c + T(a) \le cn(2a-1)$$
$$\Leftrightarrow n \geq \frac{a^2c + T(a)}{2a-1}, \text{ so}$$

this is true asymptotically.

Assume that,

$$T(k) \geq c k^2$$

$\forall, n_0, k < n$

$$\Rightarrow T(n) = T(n-a) + T(a) + cn$$

$$\geq c(n-a)^2 + T(a) + cn$$

X Try by replacing $T(a)$ by ca.