Network Analysis and Synthesis

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Dedicated to

My Father

Late Shri Ramsagar Singh

and

My Mother

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Preface

Overview

Network Analysis and Synthesis is one of the core subjects for students of Electrical Engineering, Electronics Engineering, Electronics and Telecommunication Engineering, Instrumentation Engineering, Biomedical Engineering and related branches of engineering in the third/fourth semester course of almost all universities in India. This subject is also one of the important topics of competitive examinations such as IAS, IES, etc. and examinations conducted by various public-sector undertakings in this field.

About the Book

Network Analysis and Synthesis is a complete solution for the undergraduate student who wishes to prepare for university and competitive examinations. This highly illustrative text makes the subject matter interesting and easy for students.

As per the course requirement, five major topics that are mandatory to be covered by any book on the subject are dc and ac circuits, transient analysis, network function, two-port networks and network synthesis. Generally, numerical problems are expected in university examinations in this subject. The weightage given to problems is more than 70-80% in examinations. This book attempts to cover almost all the topics and solved problems on these important topics. The simplistic presentation style sequentially takes the readers from basic to advance concepts and the crisp theory helps in ready understanding. Numerous solved examples and graded problems have been provided in this book, especially in important chapters from the examination viewpoint. Objective-type questions from various papers of competitive examinations are included in each chapter. This will help students in sharpening their knowledge about core concepts. Covering both analysis and synthesis of networks, the text provides a simple explanation about the concepts of electrical networks with brief theory and a large number of problems. Numerous examples and exercise problems have been included to help the reader develop an intuitive grasp of the contents.

Salient Features

- Span of coverage meets curriculum requirements
- Simplistic presentation and logical sequencing of topics
- Graded problem-solving approach
- Numerous solved examples from frequently asked examination questions
- Rich pedagogy

*** Illustrations: 1320**
*** Solved Examples: 1065**
*** Practice Problems: 340**

Chapter Organisation

The entire text is organised into 16 chapters, primarily based on the electrical and electronics engineering syllabi in BE/B.Tech undergraduate courses of universities around the country. The summary of the book can be organised in the following major parts:

- Circuit Elements and Network Analysis
- Network Theorems for dc and ac Circuits
- Resonance and Coupled Circuits

- Network Topology/Graph Theory
- Transient Analysis and Laplace Transforms
- Two-Port Networks
- Network Functions

Within this structure, a more in-depth and detailed breakdown can be obtained from the table of contents. The fluid flow of text dealing with different topics as well as illustrations, examples, questions and numerical problems are extremely relevant and appropriate for students as well as instructors. The main strength of the book thus is its strong focus towards students' readability and understanding with the scope for independent study and problem-solving skill development. I am confident that the depth of this book has been judiciously developed so that students not only treat this as a textbook, but, in addition, can also gather enough practical and theoretical knowledge to appear in national-level competitive examinations and interviews.

Web Supplements

The text is supplemented with an exhaustive Online Learning Center, which can be accessed at https://www.mhhe.com/singh/nas1

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Suggestions for improvements will always be welcome.

Ravish R. Singh

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1

Basic Network Concepts

1.1 INTRODUCTION

We know that like charges repel each other whereas unlike charges attract each other. To overcome this force of attraction, a certain amount of work or energy is required. When the charges are separated, it is said that a potential difference exists and the work or energy per unit charge utilised in this process is known as voltage or potential difference.

The phenomenon of transfer of charge from one point to another is termed current. Current (I) is defined as the rate of flow of electrons in a conductor. It is measured by the number of electrons that flow in unit time.

Energy is the total work done in the electric circuit. The rate at which the work is done in an electric circuit is called electric power. Energy is measured in joules (J) and power in watts (W).

1.2 RESISTANCE

Resistance is the property of a material due to which it opposes the flow of electric current through it.

Certain materials offer very little opposition to the flow of electric current and are called conductors, e.g., metals, acids and salt solutions. Certain materials offer very high resistance to the flow of electric current and are called insulators, e.g., mica, glass, rubber, Bakelite, etc.

The practical unit of resistance is **ohm** and is represented by the symbol Ω. A conductor is said to have resistance of one ohm if a potential difference of one volt across its terminals causes a current of one ampere to flow through it.

The resistance of a conductor depends on the following factors.

(i) It is directly proportional to its length.
(ii) It is inversely proportional to the area of cross section of the conductor.
(iii) It depends on the nature of the material.
(iv) It also depends on the temperature of the conductor.

Hence,

$$R \propto \frac{l}{A}$$

$$R = \rho \frac{l}{A}$$

where l is length of the conductor, A is the cross-sectional area and ρ is a constant known as specific resistance or resistivity of the material.

1. Power Dissipated in a Resistor We know that $v = R\,i$

When current flows through any resistor, power is absorbed by the resistor which is given by

$$p = v\,i$$

The power dissipated in the resistor is converted to heat which is given by

$$E = \int_0^t v\,i\,dt = \int_0^t R\,i\,i\,dt = i^2\,Rt$$

Example 1.1 *A 25 Ω resistor has a voltage of 150 sin 377 t. Find the corresponding current i and power p.*

Solution

$$R = 25\ \Omega$$

$$v = 150 \sin 377\,t$$

$$i = \frac{v}{R} = \frac{150 \sin 377\,t}{25} = 6 \sin 377\,t$$

$$p = vi = (150 \sin 377\,t)(6 \sin 377\,t) = 900 \sin^2 337\,t$$

Example 1.2 *A current waveform shown in Fig. 1.1 is applied to a 2 Ω resistor. Draw the voltage waveform.*

Fig. 1.1

Solution

For the resistor, $v = R\,i$

(a) For $0 < t < 1$, $v = R\,i = 2 \times 5 = 10$ V

(b) For $1 < t < 3$, $v = R\,i = 2 \times 0 = 0$

(c) For $3 < t < 4$, $v = R\,i = 2 \times 5 = 10$ V

The voltage waveform is shown in Fig. 1.2.

Fig. 1.2

1.3 INDUCTANCE

Inductance is the property of a coil that opposes any change in the amount of current flowing through it. If the current in the coil is increasing, the self-induced emf is set up in such a direction so as to oppose the rise of current. Similarly, if the current in the coil is decreasing, the self-induced emf will be in the same direction as the applied voltage.

Inductance is defined as the ratio of flux linkage to the current flowing through the coil. The practical unit of inductance is **henry** and is represented by the symbol H. A coil is said to have an inductance of one henry if a current of one ampere when flowing through it produces flux linkages of one weber-turn in it.

The inductance of an inductor depends on the following factors.

(i) It is directly proportional to the square of the number of turns.
(ii) It is directly proportional to the area of cross section.
(iii) It is inversely proportional to the length.
(iv) It depends on the absolute permeability of the magnetic material.

Hence,

$$L \propto \frac{N^2 A}{l}$$

$$L = \mu \frac{N^2 A}{l}$$

where l is the mean length, A is the cross-sectional area and μ is the absolute permeability of the magnetic material.

1. *Current–Voltage Relationships in an Inductor* We know that

$$v = L\frac{di}{dt}$$

Expressing inductor current as a function of voltage,

$$di = \frac{1}{L} v\, dt$$

Integrating both the sides,

$$\int_{i(0)}^{i(t)} di = \frac{l}{L}\int_0^t v\, dt$$

$$i(t) = \frac{1}{L}\int_0^t v\, dt + i(0)$$

The quantity $i(0)$ denotes the initial current through the inductor. When there is no initial current through the inductor,

$$i(t) = \frac{1}{L}\int_0^t v\, dt$$

2. *Energy Stored in an Inductor* Consider a coil of inductance L carrying a changing current I. When the current is changed from zero to a maximum value I, every change is opposed by the self-induced emf produced. To overcome this opposition, some energy is needed and this energy is stored in the magnetic field. The voltage v is given by

$$v = L\frac{di}{dt}$$

Energy supplied to the inductor during interval dt is given by

$$dE = v\, i\, dt = L\frac{di}{dt} i\, dt = L\, i\, dt$$

Hence, total energy supplied to the inductor when current is increased from 0 to I amperes is

$$E = \int_0^I dE = \int_0^I L\,i\,di = \frac{1}{2} L I^2$$

Example 1.3 *An inductance of 3 mH has a current $i = 5(1 - e^{-5000t})$. Find the corresponding voltage and the maximum stored energy.*

Solution $L = 3$ mH

$$i = 5(1 - e^{-5000t})$$

$$v = L\frac{di}{dt} = 3\times10^{-3}\frac{d}{dt}\left[5(1-e^{-5000t})\right] = 3\times10^{-3}(5\times5000\,e^{-5000t}) = 75\,e^{-5000t}$$

$$I_{\max} = i(\infty) = 5(1 - e^{-\infty}) = 5 \text{ A}$$

$$E_{\max} = \frac{1}{2} L I_{\max}^2 = \frac{1}{2}\times3\times10^{-3}\times(5)^2 = 37.5 \text{ mJ}$$

Example 1.4 *A current waveform is applied to a 2 H inductor. Draw the voltage waveform for Fig. 1.3.*

Fig. 1.3

Solution $v = L\dfrac{di}{dt}$

(a) For $0 < t < 2$,

$$v = 2\times\frac{(6-0)}{(2-0)} = 6 \text{ V}$$

(b) For $2 < t < 6$,

$$v = 2\times\frac{(6-6)}{(6-2)} = 0$$

(c) For $6 < t < 8$,

$$v = 2\times\frac{(0-6)}{(8-6)} = -6 \text{ V}$$

The voltage waveform is shown in Fig 1.4.

Fig. 1.4

Example 1.5 *A voltage waveform shown in Fig. 1.5 is applied across a 1 H inductor. Draw the current waveform.*

Fig. 1.5

Solution

$$i = \frac{1}{L}\int_0^t v\,dt + i(0)$$

(a) For $0 < t < 1$,

$$v = 1$$

$$i = \frac{1}{1}\int_0^t 1\,dt + i(0) = [t]_0^t + 0 = t$$

$$i(1) = 1 \text{ A}$$

(b) For $1 < t < 2$,

$$v = 0$$

$$i = \frac{1}{1}\int_1^t 0 \cdot dt + i(1) = 0 + 1 = 1 \text{ A}$$

$$i(2) = 1 \text{ A}$$

(c) For $2 < t < 3$,

$$v = 1$$

$$i = \frac{1}{1}\int_2^t 1\,dt + i(2) = [t]_2^t + 1 = t - 2 + 1 = t - 1$$

$$i(3) = 3 - 1 = 2 \text{ A}$$

The current waveform is shown in Fig. 1.6.

Fig. 1.6

Example 1.6 *A voltage waveform shown in Fig. 1.7 is applied across a 2 H inductor. Draw the current waveform.*

Fig. 1.7

Solution

$$i = \frac{1}{L}\int_0^t v(t) + i(0)$$

(a) For $0 < t < 2$,

$$v = 2$$

$$i = \frac{1}{2}\int_0^t 2\,dt + i(0) = 0.5[2t]_0^t + 0 = t$$

$$i(2) = 2 \text{ A}$$

(b) For $2 < t < 8$,

$$v = -\frac{3}{6}t + 3 = -0.5t + 3$$

$$i = \frac{1}{2}\int_2^t (-0.5t + 3)\,dt + i(2)$$

$$= 0.5\left[-0.5\frac{t^2}{2} + 3t\right]_2^t + 2 = 0.5[-0.25t^2 + 3t + 0.5\times 2 - 3\times 2] + 2 = 0.5[-0.25t^2 + 3t - 5] + 2$$

$$i(8) = 0.5[-0.25(64) + 3(8) - 5] + 2 = 3.5 \text{ A}$$

Fig. 1.8

The current waveform is shown in Fig. 1.8.

Example 1.7 *The following current waveform i(t) is passed through a series RL circuit with $R = 2\ \Omega$ and $L = 2$ mH. Find the voltage across each element and sketch the same.*

Fig. 1.9

Solution

(a) Voltage across the resistor of 2 Ω

$$v_R = Ri$$

Voltage across the resistor is a trapezoidal waveform with a peak value of 10 V.

(b) Voltage across the inductor of 2 mH

$$v_L = L\frac{di}{dt}$$

For $0 < t < 1$ ms,

$$v_L = 2\times10^{-3}\left(\frac{5-0}{1\times10^{-3}-0}\right) = 10 \text{ V}$$

For $1 \text{ ms} < t < 3$ ms,

$$v_L = 2\times10^{-3}\left(\frac{-5-5}{3\times10^{-3}-1\times10^{-3}}\right) = 0$$

For $3 \text{ ms} < t < 5$ ms,

$$v_L = 2\times10^{-3}\left(\frac{5-5}{5\times10^{-3}-3\times10^{-3}}\right) = -10 \text{ V}$$

For $5 \text{ ms} < t < 7$ ms,

$$v_L = 2\times10^{-3}\left(\frac{-5+5}{7\times10^{-3}-5\times10^{-3}}\right) = 0$$

For $7 \text{ ms} < t < 8$ ms,

$$v_L = 2\times10^{-3}\left(\frac{0+5}{8\times10^{-3}-7\times10^{-3}}\right) = 10 \text{ V}$$

The voltage waveforms are shown in Fig 1.10.

Fig. 1.10

1.4 CAPACITANCE

Capacitance is the property of a capacitor to store an electric charge when its plates are at different potentials. If Q coulombs of charge is given to one of the plates of a capacitor and if a potential difference of V volts is applied between the two plates then its capacitance is given by

$$C = \frac{Q}{V}$$

The practical unit of capacitance is **farad** and is represented by the symbol F. A capacitor is said to have capacitance of one farad if a charge of one coulomb is required to establish a potential difference of one volt between its plates.

The capacitance of a capacitor depends on the following factors.

(i) It is directly proportional to the area of the plates.
(ii) It is inversely proportional to the distance between two plates.
(iii) It depends on the absolute permittivity of the medium between the plates.

Hence,

$$C \propto \frac{A}{d}$$

$$C = \varepsilon\frac{A}{d}$$

where d is the distance between two plates, A is the cross-sectional area of the plates and ε is absolute permittivity of the medium between the plates.

1. *Current–Voltage Relationships in a Capacitor* The charge on a capacitor is given by

$$q = Cv$$

where q denotes the charge and v is the potential difference across the plates at any instant.
We know that

$$i = \frac{dq}{dt} = \frac{d}{dt}Cv = C\frac{dv}{dt}$$

Expressing capacitor voltage as a function of current,

$$dv = \frac{1}{C} i\, dt$$

Integrating both the sides,

$$\int_{v(0)}^{v(t)} dv = \frac{1}{C}\int_0^t i\, dt$$

$$v(t) = \frac{1}{C}\int_0^t i\, dt + v(0)$$

The quantity v (0) denotes the initial voltage across the capacitor. When there is no initial voltage on the capacitor,

$$v(t) = \frac{1}{C}\int_0^t i\, dt$$

2. *Energy Stored in a Capacitor* Let a capacitor of capacitance C farads be charged from a source of V volts. Then current i is given by

$$i = C\frac{dv}{dt}$$

Energy supplied to the capacitor during interval dt is given by

$$dE = v\, i\, dt = v\, C\frac{dv}{dt}dt$$

Hence, total energy supplied to the capacitor when potential difference is increased from 0 to V volts is

$$E = \int_0^V dE = \int_0^V C\, v\, dv = \frac{1}{2}CV^2$$

Example 1.8 *A voltage is defined by*

$$v(t) = \begin{cases} 0 & t < 0 \\ 2t & 0 < t < 2\,s \\ 4\,e^{-(t-2)} & t > 2\,s \end{cases}$$

and is applied to the $10\ \mu F$ *capacitor. Find i(t).*

Solution

$$i = C\frac{dv}{dt}$$

For $t < 0$,

$$i = 0$$

For $0 < t < 2\,s$,

$$v = 2t$$

$$i = 10 \times 10^{-6} \frac{d}{dt}(2t) = 20\ \mu\text{A}$$

For $t > 2\,s$,

$$v = 4\,e^{-(t-2)}$$

$$i = 10 \times 10^{-6} \frac{d}{dt}[4e^{-(t-2)}] = 10 \times 10^{-6}[-4e^{-(t-2)}] = -40\,e^{-(t-2)}\ \mu\text{A}$$

Example 1.9 *A voltage waveform shown in Fig. 1.11 is applied to the capacitor. Draw the current waveform.*

Fig. 1.11

Solution $$i = C\frac{dv}{dt}$$

(a) $0 < t < 1$
At $t = 0$, $v = 2$ V
At $t = 1$, $v = 2$ V

$$i = C\frac{dv}{dt} = 2\left(\frac{2-2}{1-0}\right) = 0$$

(b) $1 < t < 2$
At $t = 1$, $v = 2$ V
At $t = 2$, $v = 4$ V

$$i = C\frac{dv}{dt} = 2\left(\frac{4-2}{2-1}\right) = 4\ \text{A}$$

(c) $2 < t < 4$
At $t = 2$, $v = 4$ V
At $t = 4$, $v = -2$ V

$$i = C\frac{dv}{dt} = 2\left(\frac{-2-4}{4-2}\right) = 2\left(\frac{-6}{2}\right) = -6\ \text{A}$$

The current waveform is shown in Fig. 1.12.

Fig. 1.12

Example 1.10 *A voltage waveform shown in Fig. 1.13 is applied to the capacitor. Draw the current waveform.*

Fig. 1.13

Solution

(a) $0 < t < 2$

At $t = 0, \quad v = 0$ V

At $t = 2, \quad v = 50$ V

$$i = C\frac{dv}{dt} = 10^{-3}\left(\frac{50-0}{2-0}\right) = 10^{-3}\left(\frac{50}{2}\right) = 25\times10^{-3} \text{ A}$$

(b) $2 < t < 5$

At $t = 2, \quad v = 50$ V

At $t = 5, \quad v = -50$ V

$$i = C\frac{dv}{dt} = 10^{-3}\left(\frac{-50-50}{5-2}\right) = 10^{-3}\left(\frac{-100}{3}\right) = -33.33\times10^{-3} \text{ A}$$

(c) $5 < t < 6$

At $t = 5, \quad v = -50$ V

At $t = 6, \quad v = -50$ V

$$i = C\frac{dv}{dt} = 10^{-3}\left(\frac{-50+50}{6-5}\right) = 0$$

Fig. 1.14

The current waveform is shown in Fig. 1.14.

Example 1.11 *A current waveform shown in Fig. 1.15 is applied to a 0.1 F capacitor. Draw the voltage waveform.*

Fig. 1.15

Solution $$v = \frac{1}{C}\int_0^t i\,dt + v(0)$$

(a) For $0 < t < 1$,

$i = 2$

$$v = \frac{1}{0.1}\int_0^t 2\,dt + v(0) = 10[2t]_0^t + 0 = 20\,t$$

$v(1) = 20$ V

(b) For $1 < t < 2$,

$i = 1$

$$v = \frac{1}{0.1}\int_1^t 1\,dt + v(1) = 10[t]_1^t + 20 = 10(t-1) + 20$$

$v(2) = 10(2-1) + 20 = 30$ V

(c) For $2 < t < 4$,

$i = -2$

$$v = \frac{1}{0.1}\int_2^t -2\,dt + v(2) = 10[-2t]_2^t + 30 = 10(-2t+4) + 30$$

$v(4) = 10(-2\times 4 + 4) + 30 = -10$ V

The voltage waveform is shown in Fig. 1.16.

Fig. 1.16

Example 1.12 *A current with periodic waveform shown in Fig. 1.17 is applied to a series RLC circuit with $R = 1\ \Omega$, $L = 1$ mH and $C = 100\ \mu F$. Sketch the voltage across each element.*

Fig. 1.17

Solution

(a) Voltage across the resistor of 1 Ω

$$v_R = R\,i$$

Voltage across the resistor is a triangular waveform with peak value of 2 A and slope 2×10^3.

(b) Voltage across inductor of 1 mH

$$v_L = L\frac{di}{dt}$$

For $0 < t < 1$ ms,

$$v_L = 1\times 10^{-3}\left(\frac{2-0}{1\times 10^{-3} - 0}\right) = 2\text{ V}$$

For 1 ms $< t <$ 2 ms,

$$v_L = 1\times 10^{-3}\left(\frac{0-2}{2\times 10^{-3} - 1\times 10^{-3}}\right) = -2\text{ V}$$

Voltage across the inductor is a square waveform.

(c) Voltage across capacitor of 100 μF

$$v_C = \frac{1}{C}\int_0^t i\,dt + v(0)$$

For $0 < t < 1$ ms,

$$i = \frac{2}{10^{-3}}t = 2000t$$

$$v_C = \frac{1}{100\times10^{-6}}\int_0^t 2000t\,dt + 0 = 10^4\left(\frac{2000t^2}{2}\right) = 10^7\ t^2$$

$$v_C(1\text{ ms}) = 10^7(10^{-3})^2 = 10\text{ V}$$

For $1\text{ ms} < t < 2$ ms,

$$i = -\frac{2}{10^{-3}}t + 4 = -2000t + 4$$

$$v_C = \frac{1}{100\times10^{-6}}\int_{1\text{ ms}}^t (-2000t+4)dt + v(1\text{ ms}) = 10^4(-1000t^2 + 4t - 3\times10^{-3}) + 10$$

$$= 10^4(-1000t^2 + 4t) - 20$$

$$v_C(2\text{ ms}) = 10^4\left[-1000(2\times10^{-3})^2 + 4(2\times10^{-3})\right] - 20 = 20\text{ V}$$

Voltage across the capacitor keeps rising continuously. The voltage waveforms are shown in Fig. 1.18.

Fig. 1.18

Example 1.13 *Draw the waveform for i_R, i_L, i_C for the network shown in Fig. 1.19 (a) when it is excited by a voltage source having a waveform shown in Fig. 1.19 (b).*

Fig. 1.19

Solution

(a) Current through the resistor of 10 Ω

$$i_R = \frac{v}{R}$$

Current through the resistor is a rectangular waveform with a peak value of 1 A.

(b) Current through the inductor of 5 H

$$i_L = \frac{1}{L}\int_0^t v\,dt + i(0)$$

For $0 < t < 2$,

$$v = 10$$

$$i_L = \frac{1}{5}\int_0^t 10\,dt + i(0) = 0.2[10t]_0^t + 0 = 2t$$

$$i_L(2) = 2(2) = 4 \text{ A}$$

For $2 < t < 4$,

$$v = -10$$

$$i_L = \frac{1}{5}\int_2^t -10\,dt + i(2) = 0.2[-10t]_2^t + 4$$

$$= 0.2(-10t + 20) + 4 = -2t + 8$$

$$i_L(4) = -2(4) + 8 = 0$$

(c) Current through the capacitor of 2 F

$$i_C = C\frac{dv}{dt}$$

For $0 < t < 2$,

$$i_C = 2\left(\frac{10-10}{2-0}\right) = 0$$

For $2 < t < 4$,

$$i_C = 2\left(\frac{-10+10}{4-2}\right) = 0$$

The current waveforms are shown in Fig. 1.20.

Fig. 1.20

1.5 SOURCES

Source is a basic network element which supplies energy to the networks. There are two classes of sources, namely,

1. Independent sources
2. Dependent sources

1.5.1 Independent Sources

Output characteristics of an independent source are not dependent on any network variable such as a current or voltage. Its characteristics, however, may be time-varying. There are two types of independent sources:

1. Independent voltage source
2. Independent current source

1. ***Independent Voltage Source*** An independent voltage source is a two-terminal network element that establishes a specified voltage across its terminals. The value of this voltage at any instant is independent of the value or direction of the current that flows through it. The symbols for such voltage sources are shown in Fig. 1.21.

   The terminal voltage may be a constant, or it may be some specified function of time.

Fig. 1.21 *Symbols for independent voltage source*

2. ***Independent Current Source*** An independent current source is a two-terminal network element which produces a specified current. The value and direction of this current at any instant is independent of the value or direction of the voltage that appears across the terminals of the source. The symbols for such current sources are shown in Fig. 1.22.

   The output current may be a constant or it may be a function of time.

Fig. 1.22 *Symbols for independent current source*

1.5.2 Dependent Sources

If the voltage or current of a source depends in turn upon some other voltage or current, it is called as dependent or controlled source. The dependent sources are of four kinds, depending on whether the control variable is voltage or current and the controlled source is a voltage source or current source.

1. ***Voltage-Controlled Voltage Source (VCVS)*** A voltage-controlled voltage source is a four-terminal network component that establishes a voltage v_{cd} between two points c and d in the circuit that is proportional to a voltage v_{ab} between two points a and b.

   The symbol for such a source is shown in Fig. 1.23.

   The (+) and (–) sign inside the diamond of the component symbol identifies the component as a voltage source.

Fig. 1.23 *Symbol for VCVS*

$$v_{cd} = \mu\, v_{ab}$$

The voltage v_{cd} depends upon the control voltage v_{ab} and the constant μ, a dimensionless constant called voltage gain.

2. *Voltage-Controlled Current Source (VCCS)* A voltage-controlled current source is a four-terminal network component that establishes a current i_{cd} in a branch of the circuit that is proportional to the voltage v_{ab} between two points a and b.

The symbol for such a source is shown in Fig. 1.24.

The arrow inside the diamond of the component symbol identifies the component as a current source.

Fig. 1.24 *Symbol for VCCS*

$$i_{cd} = g_m \, v_{ab}$$

The current i_{cd} depends only on the control voltage v_{ab} and the constant g_m, called the transconductance or mutual conductance. The constant g_m has dimension of ampere per volt or siemens (S).

3. *Current-Controlled Voltage Source (CCVS)* A current-controlled voltage source is a four-terminal network component that establishes a voltage v_{cd} between two points c and d in the circuit that is proportional to the current i_{ab} in some branch of the circuit.

The symbol for such a source is shown in Fig. 1.25.

Fig. 1.25 *Symbol for CCVS*

$$v_{cd} = r \, i_{ab}$$

The voltage v_{cd} depends only on the control current i_{ab} and the constant r called the transresistance or mutual resistance. The constant r has dimension of volt per ampere or ohm (Ω).

4. *Current-Controlled Current Source (CCCS)* A current-controlled current source is a four-terminal network component that establishes a current i_{cd} in one branch of a circuit that is proportional to the current i_{ab} in some branch of the network.

The symbol for such a source is shown in Fig. 1.26.

Fig. 1.26 *Symbol for CCCS*

$$i_{cd} = \beta \, i_{ab}$$

The current i_{cd} depends only on the control current i_{ab} and the dimensionless constant β, called the current gain.

1.6 SOME DEFINITIONS

1. *Network and Circuit* The interconnection of two or more circuit elements (viz., voltage sources, resistors, inductors and capacitors) is called an *electric network*. If the network contains at least one closed path, it is called an electric circuit. Every circuit is a network, but all networks are not circuits. Figure 1.27(a) shows a network which is not a circuit and Fig. 1.27(b) shows a network which is a circuit.

Fig. 1.27 *(a) Network which is not a circuit*
(b) Network which is a circuit

2. ***Linear and Non-linear Elements*** If the resistance, inductance or capacitance offered by an element does not change linearly with the change in applied voltage or circuit current, the element is termed as *linear element*. Such an element shows a linear relation between voltage and current as shown in Fig. 1.28. Ordinary resistors, capacitors and inductors are examples of linear elements.

   A non-linear circuit element is one in which the current does not change linearly with the change in applied voltage. A semiconductor diode operating in the curved region of characteristics as shown in Fig. 1.28 is common example of non-linear element.

   Other examples of non-linear elements are voltage-dependent resistor (VDR), voltage-dependent capacitor (varactor), temperature-dependent resistor (thermistor), light-dependent resistor (LDR), etc. Linear elements obey Ohm's law whereas non-linear elements do not obey Ohm's law.

Fig. 1.28 *V-I characteristics of linear and non-linear elements*

3. ***Active and Passive Elements*** An element which is a source of electrical signal or which is capable of increasing the level of signal energy is termed as *active element*. Batteries, BJTs, FETs or OP-AMPs are treated as active elements because these can be used for the amplification or generation of signals. All other circuit elements, such as resistors, capacitors, inductors, VDR, LDR, thermistors, etc., are termed *passive elements.* The behaviour of active elements cannot be described by Ohm's law.

4. ***Unilateral and Bilateral Elements*** If the magnitude of current flowing through a circuit element is affected when the polarity of the applied voltage is changed, the element is termed *unilateral element*. Consider the example of a semiconductor diode. Current flows through the diode only in one direction. Hence, it is called an unilateral element. Next, consider the example of a resistor. When the voltage is applied, current starts to flow. If we change the polarity of the applied voltage, the direction of the current is changed but its magnitude is not affected. Such an element is called a *bilateral element*.

5. ***Lumped and Distributed Elements*** A lumped element is the element which is separated physically, like resistors, inductors and capacitors. Distributed elements are those which are not separable for analysis purposes. Examples of distributed elements are transmission lines in which the resistance, inductance and capacitance are distributed along its length.

6. ***Active and Passive Networks*** A network which contains at least one active element such as an independent voltage or current source is an active network. A network which does not contain any active element is a passive network.

7. ***Time-invariant and Time-variant Networks*** A network is said to be time-invariant or fixed if its input–output relationship does not change with time. In other words, a network is said to time-invariant, if for any time shift in input, an identical time-shift occurs for output. In time-variant networks, the input–output relationship changes with time.

1.7 SERIES AND PARALLEL COMBINATIONS OF RESISTORS

Let R_1, R_2 and R_3 be the resistances of three resistors connected in series across a dc voltage source V as shown in Fig. 1.29. Let V_1, V_2 and V_3 be the voltages across resistances R_1, R_2 and R_3 respectively.

In series combination, the same current flows through each resistor but voltage across each resistor is different.

Fig. 1.29 *Series combination of resistors*

$$V = V_1 + V_2 + V_3$$
$$R_T I = R_1 I + R_2 I + R_3 I$$
$$R_T = R_1 + R_2 + R_3$$

Hence, when a number of resistors are connected in series, the equivalent resistance is the sum of all the individual resistance.

1. Voltage Division and Power in a Series Circuit

$$I = \frac{V}{R_1 + R_2 + R_3}$$
$$V_1 = R_1 I = \frac{R_1}{R_1 + R_2 + R_3} V$$
$$V_2 = R_2 I = \frac{R_2}{R_1 + R_2 + R_3} V$$
$$V_3 = R_3 I = \frac{R_3}{R_1 + R_2 + R_3} V$$

Total power
$$P_T = P_1 + P_2 + P_3$$
$$= I^2 R_1 + I^2 R_2 + I^2 R_3$$
$$= \frac{V_1^2}{R_1} + \frac{V_2^2}{R_2} + \frac{V_3^2}{R_3}$$

Fig. 1.30 *Parallel combination of resistors*

Figure 1.30 shows three resistors connected in parallel across a dc voltage source V. Let I_1, I_2 and I_3 be the current flowing through resistors R_1, R_2 and R_3 respectively.

In parallel combination, the voltage across each resistor is same but current through each resistor is different.

$$I = I_1 + I_2 + I_3$$
$$\frac{V}{R_T} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
$$R_T = \frac{R_1 R_2 R_3}{R_2 R_3 + R_3 R_1 + R_1 R_2}$$

Hence, when a number of resistors are connected in parallel, the reciprocal of the equivalent resistance is equal to the sum of reciprocals of individual resistances.

2. Current Division and Power in a Parallel Circuit

$$V = R_T I = R_1 I_1 = R_2 I_2 = R_3 I_3$$
$$I_1 = \frac{V}{R_1} = \frac{R_T I}{R_1} = \frac{R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$
$$I_2 = \frac{V}{R_2} = \frac{R_T I}{R_2} = \frac{R_1 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$
$$I_3 = \frac{V}{R_3} = \frac{R_T I}{R_3} = \frac{R_1 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1} I$$

Total power

$$P_T = P_1 + P_2 + P_3$$
$$= I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3$$
$$= \frac{V^2}{R_1} + \frac{V^2}{R_2} + \frac{V^3}{R_3}$$

Note: For two branch circuits,

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$
$$V = R_T I = R_1 I_1 = R_2 I_2$$
$$I_1 = \frac{V}{R_1} = \frac{R_T I}{R_1} = \frac{R_2}{R_1 + R_2} I$$
$$I_2 = \frac{V}{R_2} = \frac{R_T I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

Example 1.14 *Find an equivalent resistance between A and B in the network of Fig. 1.31.*

Fig. 1.31

Solution Marking all the junctions and redrawing the network (Fig. 1.32),

Fig. 1.32

$$R_{AB} = 10 \,||\, 10 \,||\, 10 = 3.33\ \Omega$$

Example 1.15 *Find an equivalent resistance between A and B in the network of Fig. 1.33.*

Fig. 1.33

Solution Marking all the junctions and redrawing the network (Fig. 1.34),

Fig. 1.34

$$R_{AB} = 8 \parallel 6 \parallel 10 \parallel 12 = 2.11\ \Omega$$

Example 1.16 *Find the equivalent resistance between A and B in the network of Fig. 1.35.*

Fig. 1.35

Solution Marking all the junctions and redrawing the network (Fig. 1.36),

Fig. 1.36

$$R_{AB} = 22.5\ \Omega$$

Example 1.17 *What is the resistance between the terminals A and B in network of Fig. 1.37 when the potential difference between C and D is zero?*

Fig. 1.37

Solution When the potential difference between C and D is zero, the resistor connected between C and D is shorted. Hence, the points C and D are at same potential.

Redrawing the network (Fig. 1.38),

Simplifying the network (Fig. 1.39),

Fig. 1.38

Fig. 1.39

$$R_{AB} = \frac{R}{2} + \frac{R}{2} = R\ \Omega$$

Example 1.18 *Determine the current delivered by the source in the network of Fig. 1.40.*

Fig. 1.40

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.41).

Fig. 1.41

$$I = \frac{30}{1.06} = 28.3 \text{ A}$$

Example 1.19 *Find the current delivered by the source in the network of Fig. 1.42.*

Fig. 1.42

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.43).

Fig. 1.43

$$I = \frac{50}{0.88} = 56.82 \text{ A}$$

Example 1.20 *Three equal resistors of 30 Ω each are connected in parallel across a 120 V dc supply as shown in Fig. 1.44. What is the current through each of them (a) if one of the resistors burns out, or (b) if one of the resistors gets shorted?*

Fig. 1.44

Solution

(a) If one of the resistors burns out, it will act as an open circuit (Fig. 1.45).

$$I_3 = 0$$

$$I_1 = I_2 = \frac{120}{30} = 4 \text{ A}$$

Fig. 1.45

(b) If one of the resistors gets shorted, the effective resistance becomes zero (Fig. 1.46).

$$I_1 = I_2 = 0$$

Fig. 1.46

Example 1.21 *A lamp rated at 100 V, 75 W is to be connected across a 230 V supply. Find the value of resistance to be connected in series with the lamp. Also find the power loss occuring in the resistor.*

Solution The network is shown in Fig. 1.47.

$$V_1 = 100 \text{ V}, \quad P_1 = 75 \text{ W}, \quad V = 230 \text{ V}$$

Fig. 1.47

(a) Value of resistance

Rated current of the lamp

$$I = \frac{P_1}{V_1} = \frac{75}{100} = 0.75 \text{ A}$$

Lamp will operate normally on 230 V supply if the current flowing through the lamp remains the rated current, i.e., 0.75 A.

Voltage across the resistor R

$$V_2 = 230 - 100 = 130 \text{ V}$$

$$R = \frac{130}{0.75} = 173.33 \ \Omega$$

(b) Power loss occurring in the resistor

$$P_2 = \frac{V_2^2}{R} = \frac{(130)^2}{173.33} = 97.5 \text{ W}$$

Example 1.22 *A 100 V, 60 W lamp is connected in series with a 100 V, 100 W lamp and the combination is connected across 200 V mains. Find the value of the resistance that should be connected in parallel with the first lamp so that each lamp may get the rated current at rated voltage.*

Solution The network is shown in Fig. 1.48.

$$V_1 = 100 \text{ V}$$
$$P_1 = 60 \text{ W}$$
$$V_2 = 100 \text{ V}$$
$$P_2 = 100 \text{ W}$$
$$V = 200 \text{ V}$$

Fig. 1.48

Rated current of the lamp L_1

$$I_1 = \frac{P_1}{V_1} = \frac{60}{100} = 0.6 \text{ A}$$

Rated current of the lamp L_2

$$I_2 = \frac{P_2}{V_2} = \frac{100}{100} = 1 \text{ A}$$

Let R be the value of resistance that should be connected in parallel with the lamp L_1 so that rated current flows through the lamp L_1.
Current through resistor R

$$I = 1 - 0.6 = 0.4 \text{ A}$$

$$R = \frac{V_1}{I} = \frac{100}{0.4} = 250\ \Omega$$

Example 1.23 *A 100 V, 60 W lamp is connected in series with a 100 V, 100 W lamp across 200 V supply. What will be the current drawn by the lamps? What will be the power consumed by each lamp and will such a combination work?*

Solution The network is shown in Fig. 1.49.

$$V_1 = 100 \text{ V}, \qquad P_1 = 60 \text{ W}$$
$$V_2 = 100 \text{ V}, \qquad P_2 = 100 \text{ W}$$
$$V = 200 \text{ V}$$

Fig. 1.49

(a) Current drawn by the lamps
Resistance of the lamp L_1

$$R_1 = \frac{V_1^2}{P_1} = \frac{(100)^2}{60} = 166.67\ \Omega$$

Resistance of the lamp L_2

$$R_2 = \frac{V_2^2}{P_2} = \frac{(100)^2}{100} = 100\ \Omega$$

Current drawn by the lamps

$$I = \frac{V}{R_1 + R_2} = \frac{200}{166.67 + 100} = 0.75 \text{ A}$$

(b) Power consumed by the lamps

Power consumed by the lamp $L_1 = I^2 R_1 = (0.75)^2 \times 166.67 = 93.75$ W

Power consumed by the lamp $L_2 = I^2 R_2 = (0.75)^2 \times 100 = 56.25$ W

If a 100 V, 60 W lamp draws a power of 93.75 W, its filament will be overheated and will burn out. Hence, such a combination will not work.

Example 1.24 *A resistor R is connected in series with two lamps of 12 V, 9 W across a 300 V supply. Find the value of R so that both the lamps operate at rated conditions. If one of the lamps is short circuited, find the current through the circuit and the power dissipited in each lamp.*

Solution The network is shown in Fig. 1.50.

$$V_1 = V_2 = 12 \text{ V}$$

$$P_1 = P_2 = 9 \text{ W}$$

$$\text{V} = 300 \text{ V}$$

Fig. 1.50

(a) Value of R

Rated current of lamps L_1 and L_2

$$I = \frac{P_1}{V_1} = \frac{P_2}{V_2} = \frac{9}{12} = 0.75 \text{ A}$$

Resistance of the lamps L_1 and L_2

$$R_1 = R_2 = \frac{V_1^2}{P_1} = \frac{V_2^2}{P_2} = \frac{(12)^2}{9} = 16\ \Omega$$

Voltage across the resistor R

$$V_3 = 300 - 12 - 12 = 276 \text{ V}$$

$$R = \frac{V_3}{I} = \frac{276}{0.75} = 368\ \Omega$$

(b) Current through the circuit if the lamp L_2 is short circuited

If the lamp L_2 is short circuited, resistance across terminals of L_2 is zero. If the new current is I' then

$$300 = I'(16 + 368)$$

$$I' = 0.78 \text{ A}$$

(c) Power dissipated in each lamp if the lamp L_2 is short circuited

Power dissipated in the lamp $L_1 = (I')^2 R_1 = (0.78)^2 \times 16 = 9.73$ W

Power dissipated in the lamp $L_2 = 0$

Example 1.25 *A dc arc has voltage/current relation expressed by* $V = 44 + \frac{30}{I}$. *It is connected in series with a resistor across a 100 V supply. If the voltage across the arc and the resistor are equal, find the ohmic value of the resistor.*

Solution The network is shown in Fig 1.51.

Fig. 1.51

$$V = 100\text{ V}$$
$$V_1 = V_2$$
$$V = V_1 + V_2 = 100$$
$$V_1 = V_2 = 50$$

For a dc arc,

$$V_1 = 44 + \frac{30}{I}$$
$$50 = 44 + \frac{30}{I}$$
$$I = 5\text{ A}$$

Ohmic value of the resistor

$$R = \frac{V_2}{I} = \frac{50}{5} = 10\ \Omega$$

Example 1.26 *Find the voltage V_{AB} in the network as shown in Fig. 1.52.*

Fig. 1.52

Solution The resistors of 10 Ω and 20 Ω are in series. Similarly, the resistors of 30 Ω and 40 Ω are in series. By voltage-division rule,

$$V_A = 10 \times \frac{20}{10+20} = 6.67\text{ V}$$
$$V_B = 10 \times \frac{40}{30+40} = 5.71\text{ V}$$
$$V_{AB} = V_A - V_B = 6.67 - 5.71 = 0.96\text{ V}$$

1.8 SERIES AND PARALLEL COMBINATION OF INDUCTORS

Let L_1, L_2 and L_3 be the inductances of three inductors connected in series across an ac voltage source v as shown in Fig. 1.53. Let v_1, v_2 and v_3 be the voltages across inductances L_1, L_2 and L_3 respectively.

In series combination, the same current flows through each inductor but the voltage across each inductor is different.

Fig. 1.53 *Series connection of inductors*

$$v = v_1 + v_1 + v_3$$
$$L_T \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$
$$L_T = L_1 + L_2 + L_3$$

Hence, when a number of inductors are connected in series, the equivalent inductance is the sum of all the individual inductances.

Figure 1.54 shows three inductors connected in parallel across an ac voltage source v. Let i_1, i_2 and i_3 be the current through each inductance L_1, L_2 and L_3 respectively.

In parallel combination, the voltage across each inductor is same but the current through each inductor is different.

Fig. 1.54 *Parallel connection of inductors*

$$i = i_1 + i_2 + i_3$$

$$\frac{1}{L_T}\int v\,dt = \frac{1}{L_1}\int v\,dt + \frac{1}{L_2}\int v\,dt + \frac{1}{L_3}\int v\,dt$$

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

$$L_T = \frac{L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1}$$

Hence, when a number of inductors are connected in parallel, the reciprocal of the equivalent inductance is equal to the sum of reciprocals of individual inductances.

Example 1.27 *Find an equivalent inductance between terminals A and B in the network of Fig. 1.55.*

Fig. 1.55

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.56).

Fig. 1.56

$$L_{AB} = 7 \text{ H}$$

Example 1.28 *Find an equivalent inductance of the network shown in Fig 1.57.*

Fig. 1.57

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.58).

Fig. 1.58

$L_{AB} = 11.83$ H

1.9 SERIES AND PARALLEL COMBINATION OF CAPACITORS

Let C_1, C_2 and C_3 be the capacitances of three capacitors connected in series across an ac voltage source v as shown in Fig 1.59. Let v_1, v_2 and v_3 be the voltages across capacitances C_1, C_2 and C_3 respectively.

Fig. 1.59 *Series combination of capacitors*

In series combination, the charge on each capacitor is same but voltage across each capacitor is different.

$$v = v_1 + v_2 + v_3$$

$$\frac{1}{C_T}\int i\,dt = \frac{1}{C_1}\int i\,dt + \frac{1}{C_2}\int i\,dt + \frac{1}{C_3}\int i\,dt$$

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_T = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_3C_1}$$

Hence, when a number of capacitors are connected in series, the reciprocal of the equivalent capacitance is equal to the sum of reciprocals of individual capacitances.

1. Voltage Division in a Series Circuit

$$Q = C_TV = C_1V_1 = C_2V_2 = C_3V_3$$

$$V_1 = \frac{Q}{C_1} = \frac{C_TV}{C_1} = \frac{C_2\,C_3}{C_1C_2 + C_2C_3 + C_3C_1}V$$

$$V_2 = \frac{Q}{C_2} = \frac{C_TV}{C_2} = \frac{C_1\,C_3}{C_1C_2 + C_2C_3 + C_3C_1}V$$

$$V_3 = \frac{Q}{C_3} = \frac{C_TV}{C_3} = \frac{C_1\,C_2}{C_1C_2 + C_2C_3 + C_3C_1}V$$

Figure 1.60 shows three capacitors connected in parallel across an ac voltage source v. Let i_1, i_2 and i_3 be the current through each capacitance C_1, C_2 and C_3 respectively.

In parallel combination, the voltage across each capacitor is same but current through each capacitor is different.

$$i = i_1 + i_2 + i_3$$

$$C_T \frac{dv}{dt} = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

$$C_T = C_1 + C_2 + C_3$$

Hence, when a number of capacitors are connected in parallel, the equivalent capacitance is the sum of all the individual capacitance.

Fig. 1.60 *Parallel combination of capacitors*

Example 1.29 *What is the equivalent capacitance between terminals A and B in the network of Fig. 1.61.*

Fig. 1.61

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.62).

Fig. 1.62

$$C_{AB} = 0.75\ \mu\text{F}$$

Example 1.30 *Find the equivalent capacitance for the network of Fig. 1.63.*

Fig. 1.63

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.64).

Fig. 1.64

$C_{AB} = 2.05\ \mu F$

Example 1.31 *What is equivalent capacitance between terminals A and B in the network of Fig. 1.65?*

Fig. 1.65

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.66).

Fig. 1.66

$$C_{AB} = 2.85\ \mu F$$

Example 1.32 *What is equivalent capacitance between terminals A and B in the network of Fig. 1.67?*

Fig. 1.67

Solution The network can be simplified by series–parallel reduction technique (Fig. 1.67).

Fig. 1.68

$$C_{AB} = 11.25\ \mu F$$

Example 1.33 *A combination of four capacitors is shown in Fig. 1.69. Find the value of C to obtain an equivalent capacitance of 0.5 μF.*

Fig. 1.69

Solution

$$C_{eq} = \frac{(C+0.6)(0.2+0.8)}{C+0.6+0.2+0.8} = 0.5$$

$$C + 0.6 = 0.5C + 0.8$$

$$0.5C = 0.2$$

$$C = 0.4\ \mu\text{F}$$

1.10 STAR-DELTA TRANSFORMATION

When a circuit cannot be simplified by normal series–parallel reduction technique, the star-delta transformation can be used.

Figure 1.70 (a) shows three resistors R_A, R_B and R_C connected in delta.

Figure 1.70 (b) shows three resistors R_1, R_2 and R_3 connected in star.

Fig. 1.70 *(a) Delta networks (b) Star Network*

These two networks will be electrically equivalent if the resistance as measured between any pair of terminals is the same in both the arrangements.

1.10.1 Delta to Star Transformation

Referring to delta network shown in Fig. 1.70 (a), the resistance between terminals 1 and 2 $= R_C \parallel (R_A + R_B) = \dfrac{R_C(R_A + R_B)}{R_A + R_B + R_C}$

Referring to the star network shown in Fig. 1.70 (b), the resistance between terminals 1 and 2 = $R_1 + R_2$. Since the two networks are electrically equivalent,

$$R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \quad ...(1.1)$$

Similarly,

$$R_2 + R_3 = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C} \quad ...(1.2)$$

and

$$R_3 + R_1 = \frac{R_B(R_A + R_C)}{R_A + R_B + R_C} \quad ...(1.3)$$

Subtracting Eq. (1.2) from Eq. (1.1),

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \quad ...(1.4)$$

Adding Eq. (1.4) and Eq. (1.3),

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

Similarly,

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

Thus, star resistor connected to a terminal is equal to the product of the two delta resistors connected to the same terminal divided by the sum of the delta resistors.

1.10.2 Star to Delta Transformation

Multiplying the above equations,

$$R_1 R_2 = \frac{R_A R_B R_C^2}{(R_A + R_B + R_C)^2} \quad ...(1.5)$$

$$R_2 R_3 = \frac{R_A^2 R_B R_C}{(R_A + R_B + R_C)^2} \quad ...(1.6)$$

$$R_3 R_1 = \frac{R_A R_B^2 R_C}{(R_A + R_B + R_C)^2} \quad ...(1.7)$$

Adding Eqs (1.5), (1.6) and (1.7),

$$R_1 R_2 + R_2 R_3 + R_3 R_1 = \frac{R_A R_B R_C (R_A + R_B + R_C)}{(R_A + R_B + R_C)^2} = \frac{R_A R_B R_C}{R_A + R_B + R_C}$$

$$= R_A R_1 = R_B R_2 = R_C R_3$$

Hence,

$$R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} = R_2 + R_3 + \frac{R_2 R_3}{R_1}$$

$$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} = R_1 + R_3 + \frac{R_3 R_1}{R_2}$$

$$R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} = R_1 + R_2 + \frac{R_1 R_2}{R_3}$$

Thus, delta resistor connected between the two terminals is the sum of two star resistors connected to the same terminals plus the product of the two resistors divided by the remaining third star resistor.

Note: (1) When three equal resistors are connected in delta (Fig. 1.71), the equivalent star resistance is given by

$$R_Y = \frac{R_\Delta R_\Delta}{R_\Delta + R_\Delta + R_\Delta} = \frac{R_\Delta}{3}$$

$$R_\Delta = 3R_Y$$

Fig. 1.71 *Equivalent star resistance for three equal delta resistors*

(2) Star-delta transformation can also be applied to network containing inductors and capacitors.

Example 1.34 *Find an equivalent resistance between A and B in the network of Fig. 1.72.*

Fig. 1.72

Solution Converting the two delta networks formed by resistors of 4.5 Ω, 3 Ω and 7.5 Ω into equivalent star networks (Fig. 1.73 and Fig. 1.74),

Fig. 1.73

$$R_1 = R_6 = \frac{4.5 \times 7.5}{4.5 + 7.5 + 3} = 2.25\ \Omega$$

$$R_2 = R_5 = \frac{7.5 \times 3}{4.5 + 7.5 + 3} = 1.5\ \Omega$$

$$R_3 = R_4 = \frac{4.5 \times 3}{4.5 + 7.5 + 3} = 0.9\ \Omega$$

Fig. 1.74

Simplifying the network (Fig. 1.75),

Fig. 1.75

$$R_{AB} = 7.45\ \Omega$$

Example 1.35 *Find an equivalent resistance between A and B in the network of Fig. 1.76.*

Fig. 1.76

Solution Redrawing the network (Fig. 1.77),

Fig. 1.77

Converting the delta network formed by three resistors of 10 Ω into an equivalent star network (Fig. 1.78 and Fig. 1.79),

Fig. 1.78

Fig. 1.79

$$R_1 = R_2 = R_3 = \frac{10 \times 10}{10+10+10} = \frac{10}{3}\ \Omega$$

Simplifying the network (Fig. 1.80),

Fig. 1.80

$$R_{AB} = 10\ \Omega$$

Example 1.36 *Find an equivalent resistance between A and B in the network of Fig. 1.81.*

Fig. 1.81

Solution Converting the star network formed by resistors of 3 Ω, 4 Ω and 6 Ω into an equivalent delta network (Fig. 1.82 and Fig. 1.83),

Fig. 1.82

Fig. 1.83

$$R_1 = 6 + 4 + \frac{6 \times 4}{3} = 18\ \Omega$$

$$R_2 = 6 + 3 + \frac{6 \times 3}{4} = 13.5\ \Omega$$

$$R_3 = 4 + 3 + \frac{4 \times 3}{6} = 9\ \Omega$$

Simplifying the network (Fig. 1.84),

$$R_{AB} = 6 \parallel (1.35 + 0.9)$$

$$= 6 \parallel 2.25$$

$$= 1.64\ \Omega$$

Fig. 1.84

Example 1.37 *Find an equivalent resistance between A and N by solving outer delta ABC in Fig. 1.85.*

Fig. 1.85

Solution Converting outer delta *ABC* into a star network (Fig. 1.86),

$$R_Y = \frac{12 \times 12}{12+12+12} = 4\ \Omega$$

Fig. 1.86

Simplifying the network (Fig. 1.87),

Fig. 1.87

$$R_{AN} = 2.22\ \Omega$$

Example 1.38 *Find an equivalent resistance between A and B in the network of Fig. 1.88.*

Fig. 1.88

Solution The resistors of 2 Ω and 4 Ω and the resistors of 4 Ω and 11 Ω are connected in series (Fig. 1.89).

Fig. 1.89

Converting the two outer delta networks into equivalent star networks (Fig. 1.90),

$$R_{Y_1} = \frac{6\times 6}{6+6+6} = 2\ \Omega$$

$$R_{Y_2} = \frac{15\times 15}{15+15+15} = 5\ \Omega$$

Fig. 1.90

Simplifying the network (Fig. 1.91),

Fig. 1.91

$$R_{AB} = 23\ \Omega$$

Example 1.39 *Find an equivalent resistance between A and B in the network of Fig. 1.92.*

Fig. 1.92

Solution Drawing the resistor of 30 Ω from outside (Fig. 1.93),

Fig. 1.93

Converting the delta network formed by resistors of 20 Ω, 25 Ω and 35 Ω into an equivalent star network (Fig. 1.94),

$$R_1 = \frac{20 \times 35}{20 + 35 + 25} = 8.75\ \Omega$$

$$R_2 = \frac{20 \times 25}{20 + 35 + 25} = 6.25\ \Omega$$

$$R_3 = \frac{35 \times 25}{20 + 35 + 25} = 10.94\ \Omega$$

Fig. 1.94

Redrawing the network (Fig. 1.95),

Fig. 1.95

Simplifying the network (Fig. 1.96),

(a)

(b)

(c)

Fig. 1.96 *Continued*

Fig. 1.96

$$R_{AB} = 32.36\ \Omega$$

Example 1.40 *Find an equivalent resistance between A and B in the network of Fig. 1.97.*

Fig. 1.97

Solution The resistors of 5 Ω and 25 Ω and the resistors of 10 Ω and 5 Ω are connected in series (Fig. 1.98).

Fig. 1.98

Converting the delta network formed by the resistors of 20 Ω, 5 Ω and 15 Ω and 15 Ω into an equivalent star network (Fig. 1.99),

$$R_1 = \frac{20 \times 5}{20+5+15} = 2.5\ \Omega$$

$$R_2 = \frac{20 \times 15}{20+5+15} = 7.5\ \Omega$$

$$R_3 = \frac{5 \times 15}{20+5+15} = 1.875\ \Omega$$

Fig. 1.99

Redrawing the network (Fig. 1.100),

Fig. 1.100

Converting the delta network formed by the resistors of 3.875 Ω, 37.5 Ω and 30 Ω into an equivalent star network (Fig. 1.101),

$$R_4 = \frac{3.875 \times 37.5}{3.875 + 37.5 + 30} = 2.04\ \Omega$$

$$R_5 = \frac{3.875 \times 30}{3.875 + 37.5 + 30} = 1.63\ \Omega$$

$$R_6 = \frac{37.5 \times 30}{3.875 + 37.5 + 30} = 15.76\ \Omega$$

Fig. 1.101

Simplifying the network (Fig. 1.102),

Fig. 1.102

$$R_{AB} = 23.52\ \Omega$$

Example 1.41 *Find an equivalent resistance between A and B in the network of Fig. 1.103.*

Fig. 1.103

Solution Converting the star network formed by the resistors of 3 Ω, 5 Ω and 8 Ω into an equivalent delta network (Fig. 1.104 and Fig. 1.105),

Fig. 1.104

Fig. 1.105

$$R_1 = 3 + 5 + \frac{3 \times 5}{8} = 9.875\ \Omega$$

$$R_2 = 3 + 8 + \frac{3 \times 8}{5} = 15.8\ \Omega$$

$$R_3 = 5 + 8 + \frac{5 \times 8}{3} = 26.33\ \Omega$$

The resistors of 15.8 Ω and 5 Ω and the resistors of 26.33 Ω and 4 Ω are connected in parallel (Fig. 1.106),

Fig. 1.106

Converting the delta network into a star network (Fig. 1.107),

Fig. 1.107

$$R_4 = \frac{3.8 \times 9.875}{3.8 + 9.875 + 3.47} = 2.19\ \Omega$$

$$R_5 = \frac{3.8 \times 3.47}{3.8 + 9.875 + 3.47} = 0.77\ \Omega$$

$$R_6 = \frac{3.47 \times 9.875}{3.8 + 9.875 + 3.47} = 2\ \Omega$$

Simplifying the network (Fig. 1.108),

Fig. 1.108

$$R_{AB} = 4.23\ \Omega$$

Example 1.42 *Find an equivalent resistance between A and B in the network of Fig. 1.109.*

Fig. 1.109

Solution Converting the star network formed by the resistors of 3 Ω, 4 Ω and 5 Ω into an equivalent delta network (Fig. 1.110),

$$R_1 = 5+4+\frac{5\times 4}{3} = 15.67\ \Omega$$

$$R_2 = 3+4+\frac{3\times 4}{5} = 9.4\ \Omega$$

$$R_3 = 5+3+\frac{5\times 3}{4} = 11.75\ \Omega$$

Fig. 1.110

Similarly, converting the star network formed by the resistors of 4 Ω, 6 Ω and 8 Ω into an equivalent delta network (Fig. 1.111),

$$R_4 = 6+8+\frac{6\times 8}{4} = 26\ \Omega$$

$$R_5 = 4+8+\frac{4\times 8}{6} = 17.33\ \Omega$$

$$R_6 = 6+4+\frac{6\times 4}{8} = 13\ \Omega$$

Fig. 1.111

These two delta networks are connected in parallel between points *A* and *B* (Fig. 1.112).

Fig. 1.112

Fig. 1.113

The resistors of 9.4 Ω and 17.33 Ω are in parallel with a short. Hence, the equivalent resistance of this combination becomes zero.

Simplifying the parallel networks (Fig. 1.113),

$$R_{AB} = 6.17 \parallel 9.78 = 3.78\ \Omega$$

Example 1.43 *Determine the current supplied by the battery in the network of Fig. 1.114.*

Fig. 1.114

Solution Converting the delta network formed by resistors of 6 Ω, 6 Ω and 6 Ω into an equivalent star network (Fig. 1.115),

Fig. 1.115

Simplifying the network (Fig. 1.116),

Fig. 1.116

$$I = \frac{50}{5+2} = 7.14 \text{ A}$$

Example 1.44 *Calculate the current flowing the 10 Ω resistor in Fig. 1.117.*

Fig. 1.117

Solution Between terminals *A* and *B* resistors of 8 Ω and 4 Ω are connected in series. Similarly, between terminals *F* and *E*, resistors of 17 Ω and 13 Ω are connected in series (Fig. 1.118),

Fig. 1.118

Converting delta *ABC* and *DEF* into an equivalent star network (Fig. 1.119),

Fig. 1.119

Simplifying the network (Fig. 1.120),

Fig. 1.120

$$I = \frac{180}{4+16+10} = 6 \text{ A}$$

By current division rule,

$$I' = I_{24\,\Omega} = I_{10\,\Omega} = 6 \times \frac{48}{24+48} = 4 \text{ A}$$

Example 1.45 *Find the current supplied by the battery in the network of Fig. 1.121.*

Fig. 1.121

Solution Converting the star network formed by resistors of 40 Ω, 20 Ω and 50 Ω into an equivalent delta network (Fig. 1.122 and Fig. 1.123),

$$R_1 = 40 + 20 + \frac{40 \times 20}{50} = 76\ \Omega$$

$$R_2 = 40 + 50 + \frac{40 \times 50}{20} = 190\ \Omega$$

$$R_3 = 20 + 50 + \frac{20 \times 50}{40} = 95\ \Omega$$

Fig. 1.122

Fig. 1.123

The resistors of 190 Ω and 10 Ω and the resistors of 95 Ω and 30 Ω are connected in parallel (Fig. 1.124).

Fig. 1.124

Simplifying the network (Fig. 1.125),

$$I = \frac{15}{22.67 + 5} = 0.542 \text{ A}$$

Fig. 1.125

Example 1.46 *Find an equivalent inductance between terminals A and B in the network of Fig. 1.126.*

Fig. 1.126

Solution Converting the star network formed by inductors of 2 H, 2 H and 2 H into an equivalent delta network (Fig. 1.127),

Fig. 1.127

Simplifying the network (Fig. 1.128),

Fig. 1.128

$$L_{AB} = 2 \text{ H}$$

Example 1.47 *Determine the capacitance between terminals A and B in the network of Fig. 1.129.*

Fig. 1.129

Solution Converting delta network formed by capacitors of 2 μF, 1 μF and 3 μF into an equivalent star network [Fig. 1.130 (a)],

$$C_1 = 1 + 2 + \frac{1 \times 2}{3} = 3.67\ \mu\text{F}$$

$$C_2 = 1 + 3 + \frac{1 \times 3}{2} = 5.5\ \mu\text{F}$$

$$C_3 = 2 + 3 + \frac{2 \times 3}{1} = 11\ \mu\text{F}$$

(a)

Simplifying the network (Fig. 1.130),

Fig. 1.130

$$C_{AB} = 2.21\ \mu\text{F}$$

Example 1.48 *If the combined capacitance of network shown is 5 μF, find capacitance C in the network of Fig. 1.31.*

Fig. 1.131

Solution The capacitors of 1 μF and 1 μF and the capacitors of 1 μF and 3 μF are in parallel (Fig. 1.132),

Fig. 1.132

Replacing star network formed by capacitors of 4 μF, 4 μF and 2 μF into an equivalent delta network (Fig. 1.133),

$$C_1 = \frac{4 \times 4}{4+4+2} = 1.6\ \mu\text{F}$$

$$C_2 = C_3 = \frac{4 \times 2}{4+4+2} = 0.8\ \mu\text{F}$$

Fig. 1.133

Simplifying the network (Fig. 1.134),

Fig. 1.134

$$1.575 + 0.8 + \frac{3C}{3+C} = 5$$

$$\frac{3C}{3+C} = 2.625$$

$$C = 21\ \mu\text{F}$$

1.11 SOURCE TRANSFORMATION

A voltage source with a series resistor can be converted into a equivalent current source with a parallel resistor. Conversely, a current source with a parallel resistor can be converted into a voltage source with a series resistor as shown in Fig. 1.135.

Fig. 1.135 *Source transformation*

Source transformation can be applied to dependent sources as well. The controlling variable, however must not be tampered with any way since the operation of the controlled sources depends on it.

Example 1.49 *Replace the given network of Fig. 1.136 with a single current source and a resistor.*

Fig. 1.136

Solution Since the resistor of 5 Ω is connected in parallel with the voltage source of 20 V it becomes redundant. Converting parallel combination of current source and resistor into equivalent voltage source and resistor (Fig. 1.137),

Fig. 1.137

By source transformation (Fig. 1.138),

Fig. 1.138

Example 1.50 *Reduce the network shown in Fig. 1.139 into a single source and a single resistor between terminals A and B.*

Fig. 1.139

Solution Converting all voltage sources into equivalent current sources (Fig. 1.140),

Fig. 1.140

Adding the current sources and simplifying the network (Fig. 1.141),

Fig. 1.141

Converting the current sources into equivalent voltage sources (Fig. 1.142),

Fig. 1.142

Example 1.51 *Replace the circuit between A and B in Fig. 1.143 with a voltage source in series with a single resistor.*

Fig. 1.143

Solution Converting the series combination of voltage source of 20 V and a resistor of 5 Ω into equivalent parallel combination of current source and resistor (Fig. 1.144),

Fig. 1.144

Adding the two current sources and simplifying the circuit (Fig. 1.145),

Fig. 1.145

By source transformation (Fig. 1.146),

Fig. 1.146

Example 1.52 *Find the power delivered by the 50 V source in the network of Fig. 1.147.*

Fig. 1.147

Solution Converting the series combination of voltage source of 10 V and resistor of 3 Ω into equivalent current source and resistor (Fig. 1.148),

Fig. 1.148

Adding the two current sources and simplifying the network (Fig. 1.149),

Fig. 1.149

By source transformation (Fig. 1.150),

Fig. 1.150

$$I = \frac{50-16}{5+1.2} = 5.48 \text{ A}$$

Power delivered by the 50 V source = 50 × 5.48 = 274 W

Example 1.53 *Find the current in the 4 Ω resistor shown in network of Fig. 1.151.*

Fig. 1.151

Solution Converting the parallel combination of the current source of 5 A and the resistor of 2 Ω into an equivalent series combination of voltage source and resistor (Fig. 1.152),

Fig. 1.152

Adding two voltage sources (Fig. 1.153),

Fig. 1.153

Again by source transformation (Fig. 1.154),

Fig. 1.154

Adding two current sources (Fig. 1.155),

Fig. 1.155

By current-division rule,

$$I_{4\,\Omega} = 4 \times \frac{2}{2+4} = 1.33 \text{ A}$$

Example 1.54 *Find the voltage across the 4 Ω resistor shown in network of Fig. 1.156.*

Fig. 1.156

Solution Converting the series combination of the voltage source of 6 V and the resistor of 3 Ω into equivalent current source and resistor (Fig. 1.157),

Fig. 1.157

By series–parallel reduction technique (Fig. 1.158),

Fig. 1.158

By source transformation (Fig. 1.159),

Fig. 1.159 *(continued)*

Fig. 1.159

$$I = \frac{16}{4+1+4} = 1.78 \text{ A}$$

Voltage across the 4 Ω resistor $= 4I = 4 \times 1.78 = 7.12$ V

Example 1.55 *Find the voltage at Node 2 of the network shown in Fig. 1.160.*

Fig. 1.160

Solution We cannot change the network between nodes 1 and 2 since the controlling current I, for the controlled source, is in the resistor between these nodes. Applying source transformation to series combination of controlled source and the 100 Ω resistor (Fig. 1.161),

Fig. 1.161

Applying KVL to the mesh,

$$15 - 50I - 50I - 5I = 0$$

$$I = \frac{15}{105} = 0.143 \text{ A}$$

Voltage at Node 2 $= 15 - 50I = 15 - 50 \times 0.143 = 7.86$ V

1.12 SOURCE SHIFTING

Source shifting is the simplification technique used when there is no resistor in series with a voltage source or a resistor in parallel with a current source.

Example 1.56 *Calculate the voltage across the 6 Ω resistor in the network of Fig. 1.162 using source-shifting technique.*

Fig. 1.162

Solution Adding a voltage source of 18 V to the network and connecting to Node 2 (Fig. 1.163), we have

Fig. 1.163

Since nodes 1 and 2 are maintained at the same voltage by the sources, the connection between nodes 1 and 2 is removed. Now the two voltage sources have resistors in series and source transformation can be applied (Fig. 1.164).

Fig. 1.164

Simplifying the network (Fig. 1.165),

Fig. 1.165

Applying KCL at the node,

$$\frac{V_a - 18}{3} + \frac{V_a - 5.985}{2.33} + \frac{V_a}{6} = 0$$

$$V_a = 9.23 \text{ V}$$

Exercises

1.1 Find the resistance between terminals A and B in the network of Fig. 1.166.

Fig. 1.166

[4 Ω]

1.2 Find the resistance between terminals A and B in the network of Fig. 1.167.

Fig. 1.167

[2.67 Ω]

1.3 Find the equivalent resistance between terminals A and B in the network of Fig. 1.168.

Fig. 1.168

[8 Ω]

1.4 What is the equivalent resistance between terminals A and B of the networks shown in Fig. 1.169?

Fig. 1.169

[(a) 0 (b) 0]

1.5 Find the equivalent resistance between terminals A and B in the network of Fig. 1.170.

Fig. 1.170

[4 Ω]

1.6 Find the equivalent resistance between terminals A and B in the network of Fig. 1.171.

Fig. 1.171

[4 Ω]

1.7 Find the equivalent resistance between terminals A and B in the network of Fig. 1.172.

Fig. 1.172

[3 Ω]

1.8 Find the equivalent resistance between A and B in the network of Fig. 1.173.

Fig. 1.173

[5 Ω]

1.9 Find the equivalent resistance between A and B in the network of Fig. 1.174.

Fig. 1.174

[25 Ω]

1.10 Find the equivalent resistance between A and B in the network of Fig. 1.175.

Fig. 1.175

$\left[\frac{4}{7}R\right]$

1.11 Find the equivalent resistance between A and B in the network of Fig. 1.176.

Fig. 1.176

[17 Ω]

1.12 Find R_{AB} by solving the outer delta (X-B-Y) only in the network of Fig. 1.177.

Fig. 1.177

[1.41 Ω]

1.13 Find the equivalent resistance between A and B in the network of Fig. 1.178.

Fig. 1.178

[2.625 Ω]

1.14 Find the equivalent resistance between the terminals A and B in the network of Fig. 1.179.

Fig. 1.179

[3.5 Ω]

1.15 Find the equivalent resistance between terminals A and B in the network of Fig. 1.180.

Fig. 1.180

[1.82 Ω]

1.16 Find the equivalent resistance between the terminals A and B in the network of Fig. 1.181.

Fig. 1.181

[10.32 Ω]

1.17 Find the equivalent resistance between A and B in the network of Fig. 1.182.

Fig. 1.182

[4.59 Ω]

1.18 Find the equivalent resistance between *A* and *B* in the network of Fig. 1.183.

Fig. 1.183

[6.24 Ω]

1.19 Determine the current *I* in the network of Fig. 1.184.

Fig. 1.184

[8.59 A]

1.20 Find the voltage between terminals *A* and *B* in the network of Fig. 1.185.

Fig. 1.185

[0.56 V]

1.21 Determine the power supplied to the network in the network of Fig. 1.186.

Fig. 1.186

[4705.88 Ω]

1.22 Replace the given network with a single voltage source and a resistor.

Fig. 1.187

[8.6 V, 0.43 Ω]

1.23 Use source transformation to simplify the network until two elements remain to the left of terminals *A* and *B*.

Fig. 1.188

[88.42 V, 7.92 k Ω]

1.24 Determine the voltage V_x in the network of Fig. 1.189 by source-shifting technique.

Fig. 1.189

[1.129 V]

Objective-Type Questions

1.1 A network contains linear resistors and ideal voltage sources. If values of all the resistors are doubled then the voltage across each resistor is

(a) halved
(b) doubled
(c) increased by four times
(d) not changed

1.2 Four resistances 80 Ω, 50 Ω, 25 Ω, and R are connected in parallel. Current through 25 Ω resistor is 4 A. Total current of the supply is 10 A. The value of R will be

(a) 66.66 Ω (b) 40.25 Ω
(c) 36.36 Ω (d) 76.56 Ω

1.3 A delta connected network with its wye-equivalent is shown in Fig.190. The resistances R_1, R_2 and R_3 (in ohms) are respectively.

Fig. 1.190

(a) 1.5, 3, 9 (b) 3, 9, 1.5
(c) 9, 3, 1.5 (d) 3, 1.5, 9

1.4 If each branch of a delta network has resistance $\sqrt{3}\ R$, then each branch of the equivalent wye network has resistance

(a) $\frac{R}{\sqrt{3}}$ (b) 3 R
(c) $3\sqrt{3}$ R (d) $\frac{R}{3}$

1.5 Viewed from the terminal AB, the network of Fig.1.191 can be reduced to an equivalent network of a single voltage source in series with a single resistor with the following parameters

Fig. 1.191

(a) 5 V source in series with a 10 Ω resistor
(b) 1 V source in series with a 2.4 Ω resistor
(c) 15 V source in series with a 2.4 Ω resistor
(d) 1 V source in series with a 10 Ω resistor

1.6 Consider the star network shown in Fig. 192. The resistance between terminals A and B with C open is 6 Ω, between terminals B and C with A open is 11 Ω and between terminals C and A with B open is 9 Ω, then resistacne R_A, R_B and R_C will be

Fig. 1.192

(a) $R_A = 4\ \Omega,\quad R_B = 2\ \Omega,\quad R_C = 5\ \Omega$
(b) $R_A = 2\ \Omega,\quad R_B = 4\ \Omega,\quad R_C = 7\ \Omega$
(c) $R_A = 3\ \Omega,\quad R_B = 3\ \Omega,\quad R_C = 4\ \Omega$
(d) $R_A = 5\ \Omega,\quad R_B = 1\ \Omega,\quad R_C = 10\ \Omega$

1.7 A 10 V battery with an internal resistance of 1 Ω is connected across a nonlinear load whose *V-I* characteristic is given by $7I = V^2 + 2V$. The current delivered by the battery is

(a) 0 (b) 10 A
(c) 5 A (d) 8 A

1.8 If the length of a wire of resistance R is uniformly stretched to n times its original value, its new resistance is

(a) nR (b) $\frac{R}{n}$

(c) n^2R (d) $\frac{R}{n^2}$

1.9 All the resistances in Fig. 1.193 are 1 Ω each. The value of I will be

Fig. 1.193

(a) $\frac{1}{15}$ A (b) $\frac{2}{15}$ A

(c) $\frac{4}{15}$ A (d) $\frac{8}{15}$ A

1.10 The current waveform in a pure resistor at 10 Ω is shown in Fig. 1.194. Power dissipated in the resistor is

Fig. 1.194

(a) 7.29 W (b) 52.4 W
(c) 135 W (d) 270 W

1.11 Two wires A and B of the same material and length L and $2L$ have radius r and $2r$ respectively. The ratio of their specific resistance will be

(a) 1 : 1 (b) 1 : 2
(c) 1 : 4 (d) 1 : 8

Answers to Objective-Type Questions

1.1. (d)	1.2. (c)	1.3. (d)	1.4. (a)	1.5. (b)	1.6. (b)	1.7. (c)
1.8. (c)	1.9. (d)	1.10. (d)	1.11. (b)			

2

Elementary Network Theorems

2.1 INTRODUCTION

In Chapter 1, we have studied basic network concepts. In network analysis, we have to find currents and voltages in various parts of networks. In this chapter, we will study elementary network theorems like Kirchhoff's laws, mesh analysis and node analysis. These methods are applicable to all types of networks. The first step in analyzing networks is to apply Ohm's law and Kirchhoff's laws. The second step is the solving of these equations by mathematical tools.

2.2 KIRCHHOFF'S LAWS

The entire study of electric network analysis is based mainly on Kirchhoff's laws. But before discussing this, it is essential to familiarise ourselves with the following terms:

Node A node is a junction where two or more network elements are connected together.

Branch An element or number of elements connected between two nodes constitute a branch.

Loop A loop is any closed part of the circuit.

Mesh A mesh is the most elementary form of a loop and cannot be further divided into other loops. All meshes are loops but all loops are not meshes.

1. ***Kirchhoff's Current Law (KCL)*** The algebraic sum of currents meeting at a junction or node in an electric circuit is zero.

   Consider five conductors, carrying currents I_1, I_2, I_3, I_4 and I_5 meeting at a point O as shown in Fig. 2.1. Assuming the incoming currents to be positive and outgoing currents negative, we have

$$I_1 + (-I_2) + I_3 + (-I_4) + I_5 = 0$$
$$I_1 - I_2 + I_3 - I_4 + I_5 = 0$$
$$I_1 + I_3 + I_5 = I_2 + I_4$$

Fig. 2.1 *Kirchhoff's current law*

   Thus, the above law can also be stated as the sum of currents flowing towards any junction in an electric circuit is equal to the sum of the currents flowing away from that junction.

2. ***Kirchhoff's Voltage Law (KVL)*** The algebraic sum of all the voltages in any closed circuit or mesh or loop is zero.

   If we start from any point in a closed circuit and go back to that point, after going round the circuit, there is no increase or decrease in potential at that point. This means that the sum of emfs and the sum of voltage drops or rises meeting on the way is zero.

3. *Determination of Sign* A rise in potential can be assumed to be positive while a fall in potential can be considered negative. The reverse is also possible and both conventions will give the same result.

(i) If we go from the positive terminal of the battery or source to the negative terminal, there is a fall in potential and so the emf should be assigned a negative sign (Fig. 2.2a). If we go from the negative terminal of the battery or source to the positive terminal, there is a rise in potential and so the emf should be given a positive sign (Fig. 2.2b).

(a) Fall in potential (b) Rise in potential

Fig. 2.2 *Sign convention*

(ii) When current flows through a resistor, there is a voltage drop across it. If we go through the resistor in the same direction as the current, there is a fall in the potential and so the sign of this voltage drop is negative (Fig. 2.3a). If we go opposite to the direction of the current flow, there is a rise in potential and hence, this voltage drop should be given a positive sign (Fig. 2.3b).

(a) Fall in potential (b) Rise in potential

Fig. 2.3 *Sign convention*

Example 2.1 *In Fig. 2.4, the voltage drop across the 15 Ω resistor is 30 V, having the polarity indicated. Find the value of R.*

Fig. 2.4

Solution Current through the 15 Ω resistor

$$I = \frac{30}{15} = 2 \text{ A}$$

Current through the 5 Ω resistor $= 5 + 2 = 7$ A

Applying KVL to the closed path,

$$-5(7) - R(2) + 100 - 30 = 0$$

$$-35 - 2R + 100 - 30 = 0$$

$$R = 17.5\ \Omega$$

Example 2.2 *Determine the currents I_1, I_2 and I_3 in Fig. 2.5.*

Fig. 2.5

Solution Assigning currents to all the branches (Fig. 2.6),

Fig. 2.6

From Fig. 2.6,

$$I_1 = I_1 - I_2 + 9 + I_3 + 4$$

$$I_2 - I_3 = 13 \qquad \ldots\text{(i)}$$

Also,

$$-12\,I_1 - 8(I_1 - I_2) = 0$$

$$-20\,I_1 + 8\,I_2 = 0 \qquad \ldots\text{(ii)}$$

and

$$-12\,I_1 - 16\,I_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 4 \text{ A}$$

$$I_2 = 10 \text{ A}$$

$$I_3 = -3 \text{ A}$$

Example 2.3 *Find currents in all the branches of the network shown in Fig. 2.7.*

Fig. 2.7

Solution Let

$$I_{AF} = x$$

Then

$$I_{FE} = x - 30$$

$$I_{ED} = x + 40$$

$$I_{DC} = x - 80$$

$$I_{CB} = x - 20$$

$$I_{BA} = x - 80$$

Fig. 2.8

Applying KVL to the closed path *AFEDCBA* (Fig. 2.8),

$$-0.02x - 0.01(x-30) - 0.01(x+40) - 0.03(x-80) - 0.01(x-20) - 0.02(x-80) = 0$$

$$x = 41 \text{ A}$$

$$I_{AF} = 41 \text{ A}$$

$$I_{FE} = 41 - 30 = 11 \text{ A}$$

$$I_{ED} = 41 + 40 = 81 \text{ A}$$

$$I_{DC} = 41 - 80 = -39 \text{ A}$$

$$I_{CD} = 39 \text{ A}$$

$$I_{CB} = 41 - 20 = 21 \text{ A}$$

$$I_{BA} = 41 - 80 = -39 \text{ A}$$

$$I_{AB} = 39 \text{ A}$$

Example 2.4 *Find currents in all the branches of the network shown in Fig. 2.9.*

Fig. 2.9

Solution Assigning currents to all the branches (Fig. 2.10),
Applying KVL to the closed path *OBAO*,

$$-2(1-x) - 3y + 1(x) = 0$$

$$3x - 3y = 2 \qquad \ldots\text{(i)}$$

Applying KVL to the closed path *ABCA*,

$$3y - 4(1-x-y) + 5(x+y) = 0$$

$$9x + 12y = 4 \qquad \ldots\text{(ii)}$$

Fig. 2.10

Solving Eqs (i) and (ii),

$$x = 0.57 \text{ A}$$
$$y = -0.095 \text{ A}$$
$$I_{OA} = 0.57 \text{ A}$$
$$I_{OB} = 1 - 0.57 = 0.43 \text{ A}$$
$$I_{AB} = 0.095 \text{ A}$$
$$I_{AC} = 0.57 - 0.095 = 0.475 \text{ A}$$
$$I_{BC} = 1 - 0.57 + 0.095 = 0.525 \text{ A}$$

Example 2.5 *What is the potential difference between points x and y in the network shown in Fig. 2.11?*

Fig. 2.11

Solution

$$I_1 = \frac{2}{2+3} = 0.4 \text{ A}$$
$$I_2 = \frac{4}{3+5} = 0.5 \text{ A}$$

Potential difference between points x and $y = V_{xy} = V_x - V_y$

Writing KVL equation for the path x to y,

$$V_x + 3I_1 + 4 - 3I_2 - V_y = 0$$
$$V_x + 3(0.4) + 4 - 3(0.5) - V_y = 0$$
$$V_x - V_y = -3.7$$
$$V_{xy} = -3.7 \text{ V}$$

Example 2.6 *Find the voltage between points A and B in Fig. 2.12.*

Fig. 2.12

Solution

$$I_1 = \frac{20}{10+5} = 1.33 \text{ A}$$

$$I_2 = \frac{15}{4+6} = 1.5 \text{ A}$$

From Fig. 2.13,

Voltage between points A and $B = V_{AB} = V_A - V_B$

Writing KVL equation for the path A to B,

$$V_A - 5I_1 - 5 - 15 + 6I_2 - V_B = 0$$

$$V_A - 5(1.33) - 5 - 15 + 6(1.5) - V_B = 0$$

$$V_A - V_B = 17.65$$

$$V_{AB} = 17.65 \text{ V}$$

Fig. 2.13

Example 2.7 *Determine the potential difference V_{AB} for the given network in Fig. 2.14.*

Fig. 2.14

Solution The resistor of 3 Ω is connected across a short circuit. Hence, it gets shorted (Fig. 2.15).

Fig. 2.15

$$I_1 = \frac{5}{2} = 2.5 \text{ A}$$

$$I_2 = 2 \text{ A}$$

Potential difference $V_{AB} = V_A - V_B$

Writing KVL equation for the path A to B,

$$V_A - 2I_1 + 8 - 5I_2 - V_B = 0$$

$$V_A - 2(2.5) + 8 - 5(2) - V_B = 0$$

$$V_A - V_B = 7$$
$$V_{AB} = 7\,\text{V}$$

Example 2.8 *Find the voltage of the point A w.r.t. B in Fig. 2.16.*

Fig. 2.16

Solution

$$I_1 = \frac{10}{5+3} = 1.25\text{ A}$$
$$I_2 = 5\text{ A}$$

Applying KVL to the path from A to B,

$$V_A - 3I_1 - 8 + 3I_2 - V_B = 0$$
$$V_A - 3(1.25) - 8 + 3(5) - V_B = 0$$
$$V_A - V_B = -3.25$$
$$V_{AB} = -3.25\text{ V}$$

Example 2.9 *In Fig. 2.17, what values must R_1 and R_2 have*

(a) when $I_1 = 4$ A and $I_2 = 6$ A both charging?
(b) when $I_1 = 2$ A discharging and $I_2 = 20$ A charging?
(c) when $I_1 = 0$?

Fig. 2.17

Solution Applying KVL to the closed path *abcfa*,

$$110 - 2(I_1 + I_2) - R_1 I_1 - 80 = 0$$
$$110 - 2I_1 - 2I_2 - R_1 I_1 - 80 = 0$$
$$(2 + R_1)I_1 + 2I_2 = 30 \qquad \text{...(i)}$$

Applying KVL to the closed path *fcdef*,

$$80 + R_1 I_1 - R_2 I_2 - 50 = 0$$
$$R_1 I_1 - R_2 I_2 = -30 \qquad \text{...(ii)}$$

Case (a) $I_1 = 4$ A and $I_2 = 6$ A both charging

i.e. $I_1 = 4$ A and $I_2 = 6$ A

Substituting I_1 and I_2 in Eq. (i),

$$(2 + R_1)\,4 + 2\,(6) = 30$$
$$R_1 = 2.5\ \Omega$$

Substituting R_1, I_1 and I_2 in Eq. (ii),

$$2.5\,(4) - R_2\,(6) = -30$$
$$R_2 = 6.67\ \Omega$$

Case (b) $I_1 = 2$ A discharging and $I_2 = 20$ A charging

i.e. $I_1 = -2$ A and $I_2 = 20$ A

Substituting I_1 and I_2 in Eq. (i),

$$(2 + R_1)(-2) + 2\,(20) = 30$$
$$R_1 = 3\ \Omega$$

Substituting R_1, I_1 and I_2 in Eq. (ii),

$$3\,(-2) - R_2\,(20) = -30$$
$$R_2 = 1.2\ \Omega$$

Case (c) $I_1 = 0$

Substituting in Eq. (i)

$$(2 + R_1)\,(0) + 2\,I_2 = 30$$
$$I_2 = 15\ \text{A}$$

Substituting I_1 and I_2 in Eq. (ii),

$$0 - 15\,R_2 = -30$$
$$R_2 = 2\ \Omega$$

Example 2.10 *In Fig. 2.18, find I_1 and I_2 when (a) R = 2.3 Ω, (b) R = 0.5 Ω, and (c) for what values of R is I_1 = 0?*

Fig. 2.18

Solution Applying KVL to closed path *abcda*,

$$130 - 0.2\,(I_1 + I_2) - 0.2\,I_1 - 110 = 0$$
$$0.4\,I_1 + 0.2\,I_2 = 20 \qquad \ldots\text{(i)}$$

Applying KVL to the closed path *dcefd*,

$$110 + 0.2\,I_1 - R\,I_2 = 0$$
$$0.2\,I_1 - R\,I_2 = -110 \qquad \ldots\text{(ii)}$$

Case (a) $R = 2.3\ \Omega$

Substituting R in Eq. (ii),

$$0.2\,I_1 - 2.3\,I_2 = -110 \qquad \ldots\text{(iii)}$$

Solving Eqs (i) and (iii),

$$I_1 = 25\text{ A}$$
$$I_2 = 50\text{ A}$$

Case (b) $R = 0.5\ \Omega$

Substituting R in Eq. (i),

$$0.2\,I_1 - 0.5\,I_2 = -110 \qquad \ldots\text{(iv)}$$

Solving Eqs (i) and (iv),

$$I_1 = -50\text{ A}$$
$$I_2 = 200\text{ A}$$

Case (c) $I_1 = 0$

Substituting I_1 in Eq. (i),

$$0.2\,I_2 = 20$$
$$I_2 = 100\text{ A}$$

Substituting I_1 and I_2 in Eq. (ii),

$$0.2(0) - R(100) = -110$$
$$R = 1.1\ \Omega$$

Example 2.11 *In Fig. 2.19, find the value of R.*

Fig. 2.19

Solution Assigning currents to all the branches (Fig. 2.20),

Applying KVL to the closed path *abcda*,

$$80 - 10\,I - 14(I - 3) = 0$$
$$I = 5.08\text{ A}$$

Applying KVL to the closed path *dcefd*,

$$14(I - 3) - 3R = 0$$
$$14(5.08 - 3) - 3R = 0$$
$$R = 9.71\ \Omega$$

Fig. 2.20

Example 2.12 *Determine current drawn by the ammeter shown in Fig. 2.21.*

Fig. 2.21

Solution Assigning currents to all the branches (Fig. 2.22),

Applying KVL to the closed path *abcda*,

$$-5(I_1+I_2)+9-10(I_1+I_2)-30I_1=0$$

$$45I_1+15I_2=9 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *dcefd*,

$$30I_1-5I_2=0 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_2=0.4 \text{ A}$$

Current drawn by ammeter = 0.4 A

Fig. 2.22

Example 2.13 *Find branch currents in the various branches of Fig. 2.23.*

Fig. 2.23

Solution Assigning currents to various branches (Fig. 2.24),

Applying KVL to the closed path *abcda*,

$$2-0.1I_1-10I_1-5(I_1-I_2)=0$$

$$15.1I_1-5I_2=2 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *dcefd*,

$$5(I_1-I_2)-20I_2-0.2I_2-4=0$$

$$5I_1-25.2I_2=4 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1=0.086 \text{ A}$$

$$I_2=-0.142 \text{ A}$$

Fig. 2.24

Example 2.14 *In Fig. 2.25, find the value of R and current flowing through it when the current is zero in the branch OA.*

Fig. 2.25

Solution Assigning currents to all branches (Fig. 2.26),
Applying KVL to the closed path *OACO*,

$$480\,I_3 - 1.5(I_1 - I_3) + R(I_2 + I_3) = 0$$

But current in the branch *OA* is zero,
i.e.

$$I_3 = 0$$

$$-1.5\,I_1 + R\,I_2 = 0 \qquad \ldots\text{(i)}$$

Fig. 2.26

Applying KVL to the closed path *BOCB*,

$$-4\,I_2 - R(I_2 + I_3) - 2(I_1 + I_2) + 10 = 0$$

But $\qquad I_3 = 0$

$$-2\,I_1 - (6 + R)\,I_2 = -10 \qquad \ldots\text{(ii)}$$

Applying KVL to the closed path *BOAB*,

$$-4\,I_2 + 480\,I_3 + I_1 = 0$$

But

$$I_3 = 0$$

$$-4\,I_2 + I_1 = 0$$

$$I_1 = 4\,I_2 \qquad \ldots\text{(iii)}$$

Substituting I_1 in Eq. (i) and (ii),

$$-6\,I_2 + R\,I_2 = 0 \qquad \ldots\text{(iv)}$$

and

$$-14\,I_2 - R\,I_2 = -10 \qquad \ldots\text{(v)}$$

From Eq. (iv) and (v),

$$I_2 = 0.5 \text{ A}$$

Substituting I_2 in Eq. (iv),

$$-6(0.5) + R(0.5) = 0$$

$$R = 6\ \Omega$$

Current in branch $\quad OC = I_2 + I_3 = 0.5$ A

Example 2.15 *In Fig. 2.27, find the current supplied by the battery.*

Fig. 2.27

Solution Assigning currents to all the branches (Fig. 2.28),
Applying KVL to the closed path *OABEDO*,

$$50 - 2(I_1 + I_2) - 6I_1 - 4(I_1 - I_3) = 0$$

$$12I_1 + 2I_2 - 4I_3 = 50 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *BCEB*,

$$-2I_2 + 5I_3 + 6I_1 = 0$$

$$6I_1 - 2I_2 + 5I_3 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to the closed path *ECDE*,

$$-5I_3 - 3(I_2 + I_3) + 4(I_1 - I_3) = 0$$

$$4I_1 - 3I_2 - 12I_3 = 0 \quad \ldots\text{(iii)}$$

Fig. 2.28

Solving Eqs (i), (ii) and (iii),

$$I_1 = 2.817 \text{ A}$$

$$I_2 = 6.647 \text{ A}$$

$$I_3 = -0.723 \text{ A}$$

Current supplied by the battery $= I_1 + I_2 = 2.817 + 6.647 = 9.464$ A

Example 2.16 *In Fig.2.29, find the current flowing through the 2 Ω resistor.*

Fig. 2.29

Solution Assigning currents to all the branches (Fig. 2.30),
Applying KVL to the closed path *OABFGO*,

$$20 - 16I_1 - 2I_3 = 0$$
$$16I_1 + 2I_3 = 20 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *BCFB*,

$$-16I_2 + 32(I_1 - I_3) + 16I_1 = 0$$
$$48I_1 - 16I_1 + 32I_3 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to the closed path *GFCDEG*,

$$2I_3 - 32(I_1 - I_3) - 20 = 0$$
$$-32I_1 + 34I_3 = 20 \quad \ldots\text{(iii)}$$

Fig. 2.30

Solving Eqs (i), (ii) and (iii),

$$I_1 = 1.05 \text{ A}$$
$$I_2 = 6.32 \text{ A}$$
$$I_3 = 1.58 \text{ A}$$

Current through the 2 Ω resistor = $I_3 = 1.58$ A

Example 2.17 *In Fig. 2.31, find the current through the 4 Ω resistor.*

Fig. 2.31

Solution Assigning currents to all the branches (Fig. 2.32),
Applying KVL to the closed path *ABEDA*,

$$-2I_2 - 12(I_2 - I_3) + 1(I_1 - I_2) + 12 = 0$$
$$I_1 - 15I_2 + 12I_3 = -12 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *BCFEB*,

$$-2I_3 - 10 + 3(I_1 - I_3) + 12(I_2 - I_3) = 0$$
$$3I_1 + 12I_2 - 17I_3 = 10 \quad \ldots \text{(ii)}$$

Applying KVL to the closed path *DEFHGD*,

$$-1(I_1 - I_2) - 3(I_1 - I_3) - 4I_1 + 24 = 0$$
$$-8I_1 + I_2 + 3I_3 = -24 \quad \ldots\text{(iii)}$$

Fig. 2.32

Solving Eqs (i), (ii) and (iii),

$$I_1 = 4.11 \text{ A}$$
$$I_1 = 2.72 \text{ A}$$
$$I_3 = 2.06 \text{ A}$$

Current through the 4 Ω resistor = $I_1 = 4.11$ A

Example 2.18 *In Fig. 2.33, find the current through the 10 Ω resistor.*

Fig. 2.33

Solution Assigning currents to all the branches (Fig. 2.34),
Applying KVL to the closed path *ABGHA*,

$$-5(I_1 + I_2) - 15I_1 + 4 = 0$$
$$-20I_1 - 5I_2 = -4 \qquad \ldots\text{(i)}$$

Fig. 2.34

Applying KVL to the closed path *BCFGB*,

$$-10I_2 - 8I_3 + 15I_1 = 0$$
$$15I_1 - 10I_2 - 8I_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to the closed path *CDEFC*,

$$-12(I_2 - I_3) - 6 + 8I_3 = 0$$
$$-12I_2 + 20I_3 = 6 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.19 \text{ A}$$
$$I_2 = 0.032 \text{ A}$$
$$I_3 = 0.32 \text{ A}$$

Current through the 10 Ω resistor = $I_2 = 0.032$ A

Example 2.19 *In Fig. 2.35, determine the current supplied by each battery.*

Fig. 2.35

Solution Assigning currents to all the branches (Fig. 2.36),
Applying KVL to the closed path *ADEA*,

$$-40I_2 + 8 - 1I_1 = 0$$

$$I_1 + 40I_2 = 8 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *ABDA*,

$$-20(I_1 - I_2) - 10(I_1 - I_2 + I_3) + 40I_2 = 0$$

$$-30I_1 + 70I_2 - 10I_3 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to the closed path *BCDB*,

$$2I_3 - 12 + 10(I_1 - I_2 + I_3) = 0$$

$$10I_1 - 10I_2 + 12I_3 = 12 \quad \ldots\text{(iii)}$$

Fig. 2.36

Solving Eqs (i), (ii), and (iii),

$$I_1 = 0.1005 \text{ A}$$

Current supplied by the 8 V battery = $I_1 = 0.1005$ A $\quad I_2 = 0.197$ A

Current supplied by the 12 V battery = $I_3 = 1.081$ A $\quad I_3 = 1.081$ A

Example 2.20 *In Fig. 2.37, find the value of the unknown resistance R such that 2 A current flows through it.*

Fig. 2.37

Solution Assigning currents to all the branches (Fig. 2.38),
Applying KVL to the closed path *ABCDEA*,

$$-2R + 4(I_1 - I_2 - 2) + 2(I_1 - 2) = 0$$

$$6I_1 - 4I_2 - 2R = 12 \quad \ldots\text{(i)}$$

Applying KVL to the closed path *HEDGH*,

$$10 - 2(I_1 - 2) - 3I_2 = 0$$

$$2I_1 + 3I_2 = 14 \quad \ldots\text{(ii)}$$

Applying KVL to the closed path *GDCFG*,

$$3I_2 - 4(I_1 - I_2 - 2) - 5(I_1 - I_2) = 0$$

$$9I_1 - 12I_2 = 8 \quad \ldots\text{(iii)}$$

Fig. 2.38

Solving Eqs (i), (ii) and (iii),

$$I_1 = 3.76 \text{ A}$$

$$I_2 = 2.16 \text{ A}$$

$$R = 0.98\ \Omega$$

Unknown resistance $R = 0.98\ \Omega$

Example 2.21 *In Fig. 2.39, find the current delivered by the 12 V battery.*

Fig. 2.39

Solution Assigning currents to all the branches (Fig. 2.40),
Applying KVL to the closed path *ABCDEA*,

$$3(I_1 - I_2) - 2I_2 - 5(I_2 - I_3) = 0$$
$$3I_1 - 10I_2 + 5I_3 = 0 \qquad \ldots\text{(i)}$$

Applying KVL to the closed path *HEDGH*,

$$4(I_1 - I_3) + 5(I_2 - I_3) - 4I_3 = 0$$
$$4I_1 + 5I_2 - 13I_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to the closed path *GDCFG*,

$$4I_3 + 2I_2 - 12 + 4I_1 = 0$$
$$4I_1 + 2I_2 + 4I_3 = 12 \qquad \ldots\text{(iii)}$$

Fig. 2.40

Solving Eqs (i), (ii), and (iii),

$$I_1 = 1.66 \text{ A}$$
$$I_2 = 0.93 \text{ A}$$
$$I_3 = 0.87 \text{ A}$$

Current delivered by the 12 V battery $= I_1 = 1.66$ A

EXAMPLES WITH DEPENDENT SOURCES

Example 2.22 *In the network of Fig. 2.41, find I_1, I_2 and V.*

Fig. 2.41

Solution Applying KCL at the node,

$$2+4V=\frac{V}{3}+\frac{V}{6}$$

$$\frac{V}{3}+\frac{V}{6}-4V=2$$

$$-\frac{7}{2}V=2$$

$$V=-\frac{4}{7}\text{ V}$$

$$I_1=\frac{V}{3}=-\frac{4}{21}\text{ A}$$

$$I_2=\frac{V}{6}=-\frac{2}{21}\text{ A}$$

Example 2.23 *Find voltages V_1 and V_2 in the network of Fig. 2.42.*

Fig. 2.42

Solution Applying KVL to the loop,

$$6-2I+4I-4I=0$$

$$I=3\text{ A}$$

From Fig. 2.42,

$$V_1=2I=2(3)=6\text{ V}$$
$$V_2=4I=4(3)=12\text{ V}$$

Example 2.24 *Find the power delivered by the dependent source in the network of Fig. 2.43.*

Fig. 2.43

From Fig. 2.43,

$$V_x=0.5I$$

Applying KVL to the loop,

$$30-1I+3V_x-0.5I-3I=0$$
$$30-I+3(0.5I)-0.5I-3I=0$$

$$30 - 3I = 0$$
$$I = 10 \text{ A}$$
$$V_x = 0.5(10) = 5 \text{ V}$$

Power supplied by the dependent source $= (3V_x)(I) = 3 \times 5 \times 10 = 150$ W

Example 2.25 *Find the current I_2 in the network of Fig. 2.44.*

Fig. 2.44

Solution Applying KVL to the left loop,

$$-4 + 8 - V_1 = 0$$
$$V_1 = 4 \text{ V}$$

Applying KCL to the right part,

$$10\,V_1 + I_2 = 0$$
$$10(4) + I_2 = 0$$
$$I_2 = -40 \text{ A}$$

Example 2.26 *Find the voltage V_x in the network of Fig. 2.45.*

Fig. 2.45

Solution Applying KCL at the node,

$$2 + \frac{1}{6}V_y = \frac{V_x}{9} \qquad \ldots\text{(i)}$$

From Fig. 2.45,

$$V_y = 3\left(\frac{V_x}{9}\right) = \frac{V_x}{3} \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$2 + \frac{1}{6}\left(\frac{V_x}{3}\right) = \frac{V_x}{9}$$
$$2 + \frac{V_x}{18} - \frac{V_x}{9} = 0$$
$$V_x = 36 \text{ V}$$

Example 2.27 *In the network of Fig. 2.46, find the current I_1 and power dissipated in the 500 Ω resistor.*

Fig. 2.46

Solution Assigning currents to all the branches as shown in Fig. 2.47.

Fig. 2.47

Applying KVL to the closed loop *abcda*,

$$50 - 300\,I_1 - 500\,(0.6\,I_1) = 0$$

$$I_1 = 0.083 \text{ A}$$

Power dissipated in the 500 Ω resistor = $500\,(0.6\,I_1)^2 = 500\,(0.6 \times 0.083)^2 = 1.24$ W

Example 2.28 *Find the current I in the network shown in Fig. 2.48.*

Fig. 2.48

Solution Assigning currents in all the branches as shown in Fig. 2.49.

Fig. 2.49

From Fig. 2.49,

$$V_x = 2(I - 2) \qquad \ldots\text{(i)}$$

Applying KVL to the closed loop *fecdf*,

$$3V_x - 5I - 4 - 2(I - 2) = 0$$

$$3[2(I - 2)] - 5I - 4 - 2(I - 2) = 0$$

$$6I - 12 - 5I - 4 - 2I + 4 = 0$$
$$-I = 12$$
$$I = -12 \text{ A}$$

2.3 MESH ANALYSIS

A mesh is defined as a loop which does not contain any other loops within it. Mesh analysis is applicable only for planar networks. A network is said to be planar if it can be drawn on a plane surface without crossovers. In this method, the currents in different meshes are assigned continuous paths so that they do not split at a junction into branch currents. If a network has a large number of voltage sources, it is useful to use mesh analysis. Basically, this analysis consists of writing mesh equations by Kirchhoff's voltage law in terms of unknown mesh currents.

Steps to be Followed in Mesh Analysis

1. Identify the mesh, assign a direction to it and assign an unknown current in each mesh.
2. Assign the polarities for voltage across the branches.
3. Apply KVL around the mesh and use Ohm's law to express the branch voltages in terms of unknown mesh currents and the resistance.
4. Solve the simultaneous equations for unknown mesh currents.

Consider the network shown in Fig. 2.50 which has three meshes. Let the mesh currents for the three meshes be I_1, I_2, and I_3 and all the three mesh currents may be assumed to flow in the clockwise direction. The choice of direction for any mesh current is arbitrary.

Fig. 2.50

Applying KVL to Mesh 1,

$$V_1 - R_1(I_1 - I_2) - R_2(I_1 - I_3) = 0$$
$$(R_1 + R_2)I_1 - R_1 I_2 - R_2 I_3 = V_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - R_3 I_2 - R_4(I_2 - I_3) - R_1(I_2 - I_1) = 0$$
$$-R_1 I_1 + (R_1 + R_3 + R_4) I_2 - R_4 I_3 = V_2 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-R_2(I_3 - I_1) - R_4(I_3 - I_2) - R_5 I_3 + V_3 = 0$$
$$-R_2 I_1 - R_4 I_2 + (R_2 + R_4 + R_5) I_3 = V_3 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii), and (iii) in matrix form,

$$\begin{bmatrix} R_1 + R_2 & -R_1 & -R_2 \\ -R_1 & R_1 + R_3 + R_4 & -R_4 \\ -R_2 & -R_4 & R_2 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

In general,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where, R_{11} = Self-resistance or sum of all the resistance of mesh 1
$R_{12} = R_{21}$ = Mutual resistance or sum of all the resistances common to meshes 1 and 2
$R_{13} = R_{31}$ = Mutual resistance or sum of all the resistances common to meshes 1 and 3
R_{22} = Self-resistance or sum of all the resistance of mesh 2
$R_{23} = R_{32}$ = Mutual resistance or sum of all the resistances common to meshes 2 and 3
R_{33} = Self-resistance or sum of all the resistance of mesh 3

If the directions of the currents passing through the common resistance are the same, the mutual resistance will have a positive sign, and if the direction of the currents passing through common resistance are opposite then the mutual resistance will have a negative sign. If each mesh current is assumed to flow in the clockwise direction then all self-resistances will always be positive and all mutual resistances will always be negative.

The voltages V_1, V_2 and V_3 represent the algebraic sum of all the voltages in meshes 1, 2 and 3 respectively. While going along the current, if we go from negative terminal of the battery to the positive terminal then its emf is taken as positive. Otherwise, it is taken as negative.

Example 2.29 *Find the current through the 5 Ω resistor is shown in Fig. 2.51.*

Fig. 2.51

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.52.
Applying KVL to Mesh 1,

$$10 - 1I_1 - 3(I_1 - I_2) - 6(I_1 - I_3) = 0$$
$$10I_1 - 3I_2 - 6I_3 = 10 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 5I_2 - 5 = 0$$
$$-3I_1 + 10I_2 = -5 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) + 5 - 4I_3 + 20 = 0$$
$$-6I_1 + 10I_3 = 25 \qquad \ldots\text{(iii)}$$

Fig. 2.52

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 10 & -3 & -6 \\ -3 & 10 & 0 \\ -6 & 0 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 25 \end{bmatrix}$$

We can write matrix equation directly from Fig. 2.52,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where R_{11} = Self-resistance of Mesh 1 = 1 + 3 + 6 = 10 Ω

R_{12} = Mutual resistance common to meshes 1 and 2 = –3 Ω

Here, negative sign indicates that the current through common resistance are in opposite direction.

R_{13} = Mutual resistance common to meshes 1 and 3 = –6 Ω

Similarly,

$$R_{21} = -3\ \Omega$$
$$R_{22} = 3+2+5 = 10\ \Omega$$
$$R_{23} = 0$$

$$R_{31} = -6\ \Omega$$
$$R_{32} = 0$$
$$R_{33} = 6+4 = 10\ \Omega$$

For voltage matrix,

$$V_1 = 10\text{ V}$$
$$V_2 = -5\text{ V}$$
V_3 = algebraic sum of all the voltages in mesh 3 = 5 + 20 = 25 V

Solving Eqs (i), (ii) and (iii),

$$I_1 = 4.27\text{ A}$$
$$I_2 = 0.78\text{ A}$$
$$I_3 = 5.06\text{ A}$$
$$I_{5\,\Omega} = I_2 = 0.78\text{ A}$$

Example 2.30 *Find the current through the 2 Ω resistor of the network shown in Fig. 2.53.*

Fig. 2.53

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.54,

Applying KVL to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$
$$7I_1 - I_2 = 10 \qquad \ldots\text{(i)}$$

Fig. 2.54

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$
$$-I_1 + 6I_2 - 3I_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-3(I_3 - I_2) - 10I_3 - 20 = 0$$
$$-3I_2 + 13I_3 = -20 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 1.34\text{ A}$$
$$I_2 = -0.62\text{ A}$$

$$I_3 = -1.68 \text{ A}$$
$$I_{2\,\Omega} = I_2 = -0.62 \text{ A}$$

Example 2.31 *Determine the current through the 5 Ω resistor of the network shown in Fig. 2.55.*

Fig. 2.55

Solution Assigning clockwise currents in three meshes as shown in Fig. 2.56.
Applying KVL to Mesh 1,

$$8 - 1(I_1 - I_2) - 2(I_1 - I_3) = 0$$
$$3I_1 - I_2 - 2I_3 = 8 \qquad \ldots\text{(i)}$$

Fig. 2.56

Applying KVL to Mesh 2,

$$10 - 4I_2 - 3(I_2 - I_3) - 1(I_2 - I_1) = 0$$
$$-I_1 + 8I_2 - 3I_3 = 10 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - 3(I_3 - I_2) - 5I_3 + 12 = 0$$
$$-2I_1 - 3I_2 + 10I_3 = 12 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$I_1 = 6.01 \text{ A}$$
$$I_2 = 3.27 \text{ A}$$
$$I_3 = 3.38 \text{ A}$$
$$I_{5\,\Omega} = I_3 = 3.38 \text{ A}$$

Example 2.32 *Find the current supplied by the battery of the network shown in Fig. 2.57.*

Fig. 2.57

Solution

Applying KVL to Mesh 1,

$$4-3I_1-1(I_1-I_2)-4(I_1-I_3)=0$$
$$8I_1-I_2-4I_3=4 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-2I_2-5(I_2-I_3)-1(I_2-I_1)=0$$
$$-I_1+8I_2-5I_3=0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-6I_3-4(I_3-I_1)-5(I_3-I_2)=0$$
$$-4I_1-5I_2+15I_3=0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=0.66 \text{ A}$$
$$I_2=0.24 \text{ A}$$
$$I_3=0.26 \text{ A}$$

Current supplied by the battery $= I_1 = 0.66$ A.

Example 2.33 *Find the current through the 4 Ω resistor in the network of Fig. 2.58.*

Fig. 2.58

Solution Applying KVL to Mesh 1,

$$8-2I_1-2(I_1-I_2)-2(I_1-I_3)=0$$
$$6I_1-2I_2-2I_3=8 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-2(I_2-I_1)-2I_2-4(I_2-I_3)-6=0$$
$$-2I_1+8I_2-4I_3=-6 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3-I_1)+6-4(I_3-I_2)-2I_3=0$$
$$-2I_1-4I_2+8I_3=6 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=2 \text{ A}$$
$$I_2=0.5 \text{ A}$$
$$I_3=1.5 \text{ A}$$
$$I_{4\Omega}=I_3-I_2=1.5-0.5=1 \text{ A}$$

Example 2.34 *Determine the voltage V which causes the current I_1 to be zero in the network of Fig. 2.59.*

Fig. 2.59

Solution Applying KVL to Mesh 1,

$$20 - 6I_1 - 2(I_1 - I_2) - 5(I_1 - I_3) - V = 0$$

$$V + 13I_1 - 2I_2 - 5I_3 = 20 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 3I_2 - 1(I_2 - I_3) = 0$$

$$2I_1 - 6I_2 + I_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-1(I_3 - I_2) - 4I_3 + V - 5(I_3 - I_1) = 0$$

$$V + 5I_1 + I_2 - 10I_3 = 0 \qquad \ldots\text{(iii)}$$

Putting $I_1 = 0$ in Eqs (i), (ii) and (iii),

$$V - 2I_2 - 5I_3 = 20$$

$$-6I_2 + I_3 = 0$$

$$V + I_2 - 10I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (i), (ii) and (iii),

$$V = 43.7 \text{ V}$$

Example 2.35 *Find the current through the 2 Ω resistor in the network of Fig. 2.60.*

Fig. 2.60

Solution Mesh 1 contains a current source of 6 A. Hence, we can write current equation for mesh 1. Since direction of current source and mesh current I_1 are same,

$$I_1 = 6 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$36 - 12(I_2 - I_1) - 6(I_2 - I_3) = 0$$

$$36 - 12(I_2 - 6) - 6I_2 + 6I_3 = 0$$

$$18I_2 - 6I_3 = 108 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_2) - 3I_3 - 2I_3 - 9 = 0$$

$$6I_2 - 11I_3 = 9 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_3 = 3\text{ A}$$

$$I_{2\Omega} = I_3 = 3\text{ A}$$

Example 2.36 *Determine the mesh currents I_1, I_2 and I_3 in the network of Fig. 2.61.*

Fig. 2.61

Solution Applying KVL to Mesh 1,

$$-30 - 6I_1 - 15(I_1 - I_2) = 0$$

$$21I_1 - 15I_2 = -30 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_3) - 15(I_2 - I_1) + 50 - 5I_2 = 0$$

$$-15I_1 + 30I_2 - 10I_3 = 50 \qquad \ldots\text{(ii)}$$

For Mesh 3,

$$I_3 = 1 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0$$

$$I_2 = 2\text{ A}$$

$$I_3 = 1\text{ A}$$

Example 2.37 *Find the current through the 5 Ω resistor in the network of Fig. 2.62.*

Fig. 2.62

Solution Writing current equations for Meshes 1, 2 and 4,

$$I_1 = 4 \quad \ldots\text{(i)}$$

$$I_2 = 3 \quad \ldots\text{(ii)}$$

$$I_4 = -3 \quad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-5(I_3 - I_1) - 2(I_3 - I_2) - 2(I_3 - I_4) - 2 = 0 \quad \ldots\text{(iv)}$$

Substituting Eqs (i), (ii) and (iii) in Eq. (iv),

$$-5(I_3 - 4) - 2(I_3 - 3) - 2(I_3 + 3) - 2 = 0$$

$$I_3 = 2 \text{ A}$$

$$I_{5\Omega} = I_1 - I_3 = 4 - 2 = 2 \text{ A}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.38 *Obtain the branch currents in the network shown in Fig. 2.63.*

Fig. 2.63

Solution Assigning clockwise currents in two meshes as shown in Fig. 2.64,
From Fig. 2.64,

$$I_A = I_1 \quad \ldots\text{(i)}$$

$$I_B = I_2 \quad \ldots\text{(ii)}$$

Fig. 2.64

Applying KVL to Mesh 1,

$$5 - 5I_1 - 10I_B - 10(I_1 - I_2) - 5I_A = 0$$

$$5 - 5I_1 - 10I_2 - 10I_1 + 10I_2 - 5I_1 = 0$$

$$-20I_1 = -5$$

$$I_1 = \frac{1}{4} = 0.25 \text{ A} \quad \ldots\text{(iii)}$$

Applying KVL to Mesh 2,

$$5I_A - 10(I_2 - I_1) - 5I_2 - 10 = 0$$

$$5I_1 - 10I_2 + 10I_1 - 5I_2 = 10$$

$$15I_1 - 15I_2 = 10 \quad \ldots\text{(iv)}$$

Putting $I_1 = 0.25$ A in Eq. (iv),

$$15(0.25) - 15I_2 = 10$$

$$I_2 = -0.416 \text{ A}$$

Example 2.39 *Find the mesh currents in the network shown in Fig. 2.65.*

Fig. 2.65

Solution Assigning clockwise currents in the two meshes as shown in Fig. 2.66,

Fig. 2.66

From Fig. 2.66, …(i)

$$V_1 = -5I_1$$

$$V_2 = 2I_2 \quad \text{…(ii)}$$

Applying KVL to Mesh 1,

$$-5 - 5I_1 - 2V_2 - 4I_1 - 1(I_1 - I_2) + 2V_1 = 0$$

$$-5 - 5I_1 - 2(2I_2) - 4I_1 - I_1 + I_2 + 2(-5I_1) = 0$$

$$20I_1 + 3I_2 = -5 \quad \text{…(iii)}$$

Applying KVL to Mesh 2,

$$-2V_1 - 1(I_2 - I_1) - 2I_2 - 10 = 0$$

$$-2(-5I_1) - I_2 + I_1 - 2I_2 = 10$$

$$11I_1 - 3I_2 = 10 \quad \text{…(iv)}$$

Solving Eqs (iii) and (iv),

$$I_1 = 0.161 \text{ A}$$

$$I_2 = -2.742 \text{ A}$$

Example 2.40 *Find currents I_x and I_y of the network shown in the Fig. 2.67.*

Fig. 2.67

Solution Assigning clockwise currents in the two Meshes as shown in Fig. 2.68.

From Fig. 2.68,

$$I_y = I_1$$
$$I_x = I_1 - I_2$$

Fig. 2.68

Applying KVL to Mesh 1,

$$-5 - 5I_1 - 2I_x - 4I_1 - 1(I_1 - I_2) + 2I_y = 0$$
$$-5 - 5I_1 - 2(I_1 - I_2) - 4I_1 - I_1 + I_2 + 2I_1 = 0$$
$$-5 - 5I_1 - 2I_1 + 2I_2 - 4I_1 - I_1 + I_2 + 2I_1 = 0$$
$$-10I_1 + 3I_2 = 5 \qquad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-2I_y - 1(I_2 - I_1) - 2I_2 - 10 = 0$$
$$-2I_1 - I_2 + I_1 - 2I_2 = 10$$
$$-I_1 - 3I_2 = 10 \qquad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$I_1 = -\frac{15}{11} = -1.364 \text{ A}$$
$$I_2 = -2.878 \text{ A}$$
$$I_y = -1.364 \text{ A}$$
$$I_x = I_1 - I_2 = -1.364 + 2.878 = 1.514 \text{ A}$$

Example 2.41 *Find the currents in the three meshes of the network shown in Fig. 2.69.*

Fig. 2.69

Solution Assigning clockwise currents in the three meshes is shown in Fig. 2.70.

Fig. 2.70

From Fig. 2.70,

$$I_x = I_1 \quad \text{...(i)}$$
$$I_y = I_2 - I_3 \quad \text{...(ii)}$$

Applying KVL to Mesh 1,

$$5 - 1I_1 - I_y - 1(I_1 - I_2) = 0$$
$$5 - I_1 - (I_2 - I_3) - (I_1 - I_2) = 0$$
$$-2I_1 + I_3 = -5 \quad \text{...(iii)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + I_y - 1I_2 - I_x - 1(I_2 - I_3) = 0$$
$$-(I_2 - I_1) + (I_2 - I_3) - I_2 - I_1 - (I_2 - I_3) = 0$$
$$-2I_2 = 0 \quad \text{...(iv)}$$

For Mesh 3,

$$I_3 = -1 \quad \text{...(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = 2 \text{ A}$$
$$I_2 = 0$$
$$I_3 = -1 \text{ A}$$

Example 2.42 *For the network shown in Fig. 2.71, find the power supplied by the dependent voltage source.*

Fig. 2.71

Solution Assigning clockwise currents in three Meshes as shown in Fig. 2.72.

Fig. 2.72

From Fig. 2.72,

$$V_1 - 20(I_1 - I_3) - 0.4\,V_1 = 0$$

$$0.6\,V_1 = 20\,I_1 - 20\,I_3$$

$$V_1 = 33.33\,I_1 - 33.33\,I_3 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 5 \qquad \ldots\text{(ii)}$$

For Mesh 2,

$$I_2 = -0.01\,V_1 = -0.01(33.33\,I_1 - 33.33\,I_3)$$

$$0.33\,I_1 + I_2 - 0.33\,I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-50\,I_3 - 30(I_3 - I_2) - 20(I_3 - I_1) = 0$$

$$-20\,I_1 - 30\,I_2 + 100\,I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 5 \text{ A}$$

$$I_2 = -1.47 \text{ A}$$

$$I_3 = 0.56 \text{ A}$$

$$V_1 = 33.33\,I_1 - 33.33\,I_3 = 33.33(5) - 33.33(0.56) = 148 \text{ V}$$

Power supplied by the dependent voltage source = 0.4 $V_1(I_1 - I_2)$ = 0.4 (148)(5 + 1.47) = 383.02 W

Example 2.43 *Find the voltage V_x in the network shown in Fig. 2.73.*

Fig. 2.73

Solution Assigning clockwise currents in the three meshes as shown in Fig. 2.74.

Fig. 2.74

From Fig. 2.74,

$$V_x = 16.67\, I_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$-30 - 16.67\, I_1 - 33.33(I_1 - I_3) - 25(I_1 - I_2) - 10 = 0$$

$$-30 - 16.67\, I_1 - 33.33\, I_1 + 33.33\, I_3 - 25\, I_1 + 25\, I_2 - 10 = 0$$

$$-75\, I_1 + 25\, I_2 + 33.33\, I_3 = 40 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$10 - 25(I_2 - I_1) + 2\, V_x = 0$$

$$10 - 25(I_2 - I_1) + 2(16.67\, I_1) = 0$$

$$10 - 25\, I_2 + 25\, I_1 + 33.34\, I_1 = 0$$

$$58.34\, I_1 - 25\, I_2 = -10 \qquad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = 0.45 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -0.9 \text{ A}$$

$$I_2 = -1.7 \text{ A}$$

$$I_3 = 0.45 \text{ A}$$

$$V_x = 16.67\, I_1 = 16.67\,(-0.9) = -15 \text{ V} \qquad \ldots\text{(v)}$$

Example 2.44 *For the network shown in Fig. 2.75, find the mesh currents I_1, I_2 and I_3.*

Fig. 2.75

Solution From Fig. 2.75,

$$V_x = 3(I_1 - I_2) \qquad \ldots\text{(i)}$$

Writing current equation for the two current sources,

$$I_3 = 15 \qquad \ldots\text{(ii)}$$

and

$$0.111\, V_x = I_1 - I_3$$

$$0.111\,[3\,(I_1 - I_2) = I_1 - I_3$$

$$0.333\, I_1 - 0.333\, I_2 - I_1 + I_3 = 0$$

$$-0.667\, I_1 - 0.333\, I_2 + I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 1(I_2 - I_3) - 2I_2 = 0$$

$$-3I_1 + 6I_2 - I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 17 \text{ A}$$

$$I_2 = 11 \text{ A}$$

$$I_3 = 15 \text{ A}$$

Example 2.45 *For the network shown in Fig. 2.76, find the magnitude of V_0 and the current supplied by it, given that power loss in R_L = 2 Ω resistor is 18 W.*

Fig. 2.76

Solution Assigning clockwise currents in meshes is shown in Fig. 2.77.

Fig. 2.77

From Fig. 2.77,

$$V_x = 5I_1 \qquad \ldots\text{(i)}$$

Also,

$$I_4^2 R_L = 18$$

$$I_4^2(2) = 18$$

$$I_4 = 3 \text{ A} \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 1,

$$V_0 - 5I_1 - 2(I_1 - I_2) = 0$$

$$7I_1 - 2I_2 = V_0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 4I_2 = 0$$

$$-2I_1 + 6I_2 = 0 \qquad \ldots\text{(iv)}$$

For Mesh 3,

$$I_3 = 2V_x = 2(5I_1) = 10I_1$$

$$10I_1 - I_3 = 0 \qquad \ldots\text{(v)}$$

Applying KVL to Mesh 4,

$$-2(I_4 - I_3) - 5I_4 - 2I_4 = 0$$
$$-2I_3 + 9I_4 = 0$$
$$-2I_3 + 9(3) = 0$$
$$I_3 = 13.5 \text{ A}$$

From Eq. (v),

$$I_1 = \frac{I_3}{10} = \frac{13.5}{10} = 1.35 \text{ A}$$

From Eq. (iv),

$$-2(1.35) + 6I_2 = 0$$
$$I_2 = 0.45 \text{ A}$$

From Eq. (iii),

$$7(1.35) - 2(0.45) = V_0$$
$$V_0 = 8.55 \text{ V}$$

Current supplied by voltage source $V_0 = I_1 = 1.35$ A

Example 2.46 *In the network shown in Fig. 2.78, find voltage V_2 such that $V_x = 0$.*

Fig. 2.78

Solution Assigning clockwise currents in four meshes as shown in Fig. 2.79.

From Fig. 2.79,

$$V_x = 20(I_3 - I_4) \qquad \ldots\text{(i)}$$

Writing current equations for Meshes 1 and 2,

$$I_1 = 2 \qquad \ldots\text{(ii)}$$
$$I_2 = 3 \qquad \ldots\text{(iii)}$$

Fig. 2.79

Applying KVL to Mesh 3,

$$24 - 10(I_3 - I_1) - 20(I_3 - I_4) = 0$$
$$24 - 10(I_3 - 2) - 20(I_3 - I_4) = 0$$
$$-30I_3 + 20I_4 = -44 \qquad \ldots\text{(iv)}$$

Applying KVL to Mesh 4,

$$-20(I_4 - I_3) - 5(I_4 - I_2) + V_2 = 0$$
$$-20(I_4 - I_3) - 5(I_4 - 3) + V_2 = 0$$
$$20I_3 - 25I_4 = -V_2 - 15 \qquad \ldots\text{(v)}$$

But

$$V_x = 0$$
$$20(I_3 - I_4) = 0$$
$$I_3 = I_4$$

From Eq. (iv),

$$-30\,I_3 + 20\,I_3 = -44$$
$$I_3 = 4.4 \text{ A}$$
$$I_4 = 4.4 \text{ A}$$

From Eq. (v),

$$20\,(4.4) - 25\,(4.4) = -\,V_2 - 15$$
$$V_2 = 7 \text{ V}$$

2.4 SUPERMESH ANALYSIS

Meshes that share a current source with other meshes, none of which contains a current source in the outer loop, form a supermesh. A path around a supermesh doesn't pass through a current source. A path around each mesh contained within a supermesh passes through a current source. The total number of equations required for a supermesh is equal to the number of meshes contained in the supermesh. A supermesh requires one mesh current equation, that is, a KVL equation. The remaining mesh current equations are KCL equations.

Example 2.47 *Find the current through the 10 Ω resistor of the network shown in Fig. 2.80.*

Fig. 2.80

Solution Applying KVL to Mesh 1,

$$2 - 1\,I_1 - 10\,(I_1 - I_2) = 0$$
$$11\,I_1 - 10\,I_2 = 2 \qquad \ldots\text{(i)}$$

Since meshes 2 and 3 contain a current source of 4 A, these two meshes will form a supermesh. A supermesh is formed by two adjacent meshes that have a common current source. The direction of the current source of 4 A and current $(I_3 - I_2)$ are same, i.e., in the upward direction.
Writing current equation to the supermesh,1

$$I_3 - I_2 = 4 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-10\,(I_2 - I_1) - 5\,I_2 - 15\,I_3 = 0$$
$$10\,I_1 - 15\,I_2 - 15\,I_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = -2.35 \text{ A}$$
$$I_2 = -2.78 \text{ A}$$
$$I_3 = 1.22 \text{ A} \qquad \ldots\text{(iv)}$$

Current through the 10 Ω resistor $= I_1 - I_2 = -(2.35) - (-2.78) = 0.43$ A

Example 2.48 *Find the current in the 3 Ω resistor of the network shown in Fig. 2.81.*

Fig. 2.81

Solution Meshes 1 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_3 = 7 \qquad \ldots\text{(i)}$$

Applying KVL to the outer path of the supermesh,

$$7 - 1(I_1 - I_2) - 3(I_3 - I_2) - 1I_3 = 0$$
$$-I_1 + 4I_2 - 4I_3 = -7 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$
$$I_1 - 6I_2 + 3I_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 9 \text{ A}$$
$$I_2 = 2.5 \text{ A}$$
$$I_3 = 2 \text{ A}$$

Current through the 3 Ω resistor = $I_2 - I_3 = 2.5 - 2 = 0.5$ A

Example 2.49 *Find the current in the 5 Ω resistor of the network shown in Fig. 2.82.*

Fig. 2.82

Solution Applying KVL to Mesh 1,

$$50 - 10(I_1 - I_2) - 5(I_1 - I_3) = 0$$

$$15I_1 - 10I_2 - 5I_3 = 50 \quad \ldots\text{(i)}$$

Meshes 2 and 3 will form a supermesh as these two meshes share a common current source of 2 A. Writing current equation for the supermesh,

$$I_2 - I_3 = 2 \quad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-10(I_2 - I_1) - 2I_2 - 1I_3 - 5(I_3 - I_1) = 0$$

$$-15I_1 + 12I_2 + 6I_3 = 0 \quad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 20 \text{ A}$$

$$I_2 = 17.33 \text{ A}$$

$$I_3 = 15.33 \text{ A}$$

Current through the 5 Ω resistor $= I_1 - I_3 = 20 - 15.33 = 4.67$ A

Example 2.50 *Determine the power delivered by the voltage source and the current in the 10 Ω resistor of the network shown in Fig. 2.83.*

Fig. 2.83

Solution Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = 3 \quad \ldots\text{(i)}$$

Applying KVL to the outer path of the supermesh,

$$50 - 5I_1 - 5I_2 - 10(I_2 - I_3) - 1(I_1 - I_3) = 0$$

$$-6I_1 - 15I_2 + 11I_3 = -50 \quad \ldots\text{(ii)}$$

For Mesh 3,

$$I_3 = 10 \quad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 9.76 \text{ A}$$

$$I_2 = 6.76 \text{ A}$$

$$I_3 = 10 \text{ A}$$

Power delivered by the voltage source $= 50\, I_1 = 50 \times 9.76 = 488$ W

$$I_{10\,\Omega} = I_3 - I_2 = 10 - 6.76 = 3.24 \text{ A}$$

Example 2.51 *For the network shown in Fig. 2.84, find current through the 8 Ω resistor.*

Fig. 2.84

Writing current equations for Meshes 1 and 4,

$$I_1 = -3 \quad \text{...(i)}$$

$$I_4 = -12 \quad \text{...(ii)}$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 7 \quad \text{...(iii)}$$

Applying KVL to the outer path of the supermesh,

$$5 - 4(I_2 - I_1) - 6I_2 - 8(I_3 - I_4) + 10 = 0$$

$$5 - 4(I_2 + 3) - 6I_2 - 8(I_3 + 12) + 10 = 0$$

$$-10I_2 - 8I_3 = 93 \quad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$I_2 = -8.28 \text{ A}$$

$$I_3 = -1.28 \text{ A}$$

$$I_{8\Omega} = I_3 - I_4 = -1.28 + 12 = 10.72 \text{ A}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.52 *In the network of Fig. 2.85, find currents I_1 and I_2.*

Fig. 2.85

Solution From Fig. 2.85,

$$-10 - 8I_1 - V_0 = 0$$

$$V_0 = -10 - 8I_1 \quad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equations for the supermesh,

$$I_2 - I_1 = -3 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-10 - 8I_1 - 3V_0 - 10I_2 = 0$$

$$-10 - 8I_1 - 3(-10 - 8I_1) - 10I_2 = 0$$

$$16I_1 - 10I_2 = -20 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -8.33 \text{ A}$$

$$I_2 = -11.33 \text{ A}$$

Example 2.53 *In the network of Fig. 2.86, find the current through the 3 Ω resistor.*

Fig. 2.86

Solution Assigning clockwise currents in two meshes as shown in Fig. 2.87.

From Fig. 2.87,

$$V_x = -2I_1 \qquad \ldots\text{(i)}$$

Fig. 2.87

Meshes 1 and 2 will form a supermesh.

Writing current equations for the supermesh,

$$I_2 - I_1 = 2 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-2I_1 - 4 - 3I_2 - 5V_x = 0$$

$$-2I_1 - 4 - 3I_2 - 5(-2I_1) = 0$$

$$8I_1 - 3I_2 = 4 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$

$$I_2 = 4 \text{ A}$$

$$I_{3\Omega} = I_2 = 4 \text{ A}$$

Example 2.54 *Find the currents I_1 and I_2 at the network shown in Fig. 2.88.*

Fig. 2.88

Solution From Fig. 2.88,

$$V_1 = 2(I_1 - I_2)$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 0.5\ V_1 = 0.5 \times 2(I_1 - I_2) = I_1 - I_2$$
$$I_3 = I_1$$

Applying KVL to outer path of the supermesh,

$$-2(I_2 - I_1) - 10\,I_3 - 6\,I_2 = 0$$
$$-2\,I_2 + 2\,I_1 - 10\,I_1 - 6\,I_2 = 0$$
$$I_1 = -I_2$$

Applying KVL to Mesh 1,

$$110 - 14\,I_1 - 4\,I_1 - 2(I_1 - I_2) = 0$$
$$110 - 20\,I_1 + 2\,I_2 = 0$$
$$110 + 20\,I_1 + 2\,I_2 = 0$$
$$I_2 = -5\ \text{A}$$
$$I_1 = -I_2 = 5\ \text{A}$$

Example 2.55 *For the network of Fig. 2.89, find current through the 8 Ω resistor.*

Fig. 2.89

Solution Assigning clockwise currents to the three meshes as shown in Fig. 2.90.
From Fig. 2.90,

$$I_x = -I_1 \qquad \ldots\text{(i)}$$
$$I_y = I_2 - I_3 \qquad \ldots\text{(ii)}$$

Fig. 2.90

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 0.5\,I_x = 0.5\,(-I_1)$$
$$-0.5\,I_1 + I_2 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to the outer path of the supermesh,

$$50 - 10(I_1 - I_3) - 6(I_2 - I_3) - 8\,I_2 - 52 = 0$$
$$-10\,I_1 - 14\,I_2 + 16\,I_3 = 2 \qquad \ldots\text{(iv)}$$

Applying KVL to Mesh 3,

$$-5\,I_y - 6\,(I_3 - I_2) - 10\,(I_3 - I_1) = 0$$

$$-5\,(I_2 - I_3) - 6\,(I_3 - I_2) - 10\,(I_3 - I_1) = 0$$

$$10\,I_1 + I_2 - 11\,I_3 = 0 \qquad \ldots\text{(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = -1.56 \text{ A}$$

$$I_2 = -0.58 \text{ A}$$

$$I_3 = -1.11 \text{ A}$$

$$I_{8\,\Omega} = I_2 = -0.58 \text{ A}$$

Example 2.56 *For the network shown in Fig. 2.91, find the current through the 10 Ω resistor.*

Fig. 2.91

Solution Meshes 1, 2 and 3 will form a supermesh.
Writing current equations for the supermesh,

$$I_1 - I_2 = 2 \qquad \ldots\text{(i)}$$

$$I_3 - I_2 = 2\,I_1 \qquad \ldots\text{(ii)}$$

and

$$2\,I_1 + I_2 - I_3 = 0$$

Applying KVL to the outer path of the supermesh,

$$15 - 10\,I_1 - 20 - 5\,I_2 - 4I_3 + 40 = 0$$

$$10\,I_1 + 5\,I_2 + 4\,I_3 = 35 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 1.96 \text{ A}$$

$$I_2 = -0.04 \text{ A}$$

$$I_3 = 3.89 \text{ A}$$

$$I_{10\,\Omega} = I_1 = 1.96 \text{ A}$$

Example 2.57 *In the network shown in Fig. 2.92, find the power delivered by the 4 V source and voltage across the 2 Ω resistor.*

Fig. 2.92

Solution From Fig. 2.92,

$$V_2 = 2\,I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 5 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$4 - 5\,I_2 - 2\,I_1 - 6\,I_1 - 4\,(I_1 - I_3) - 1\,(I_2 - I_3) = 0$$

$$-12\,I_1 - 6\,I_2 + 5\,I_3 = -4 \qquad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = \frac{V_2}{2} = \frac{2\,I_1}{2} = I_1$$

$$I_1 - I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -2 \text{ A}$$

$$I_2 = 3 \text{ A}$$

$$I_3 = -2 \text{ A}$$

Power delivered by the 4 V source $= 4\,I_2 = 4\,(3) = 12$ W

$$V_{2\,\Omega} = 2\,I_1 = 2\,(-2) = -4 \text{ V}$$

Example 2.58 *Find currents I_1, I_2, I_3, I_4 of the network shown in Fig. 2.93.*

Fig. 2.93

From Fig. 2.93,

$$V_x = \frac{1}{5}(I_2 - I_1) \qquad \ldots\text{(i)}$$

For Mesh 4,

$$I_4 = 40 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 1,

$$-6 - \frac{1}{10}I_1 - \frac{1}{5}(I_1 - I_2) - \frac{1}{6}(I_1 - I_4) = 0$$

$$-6 - \frac{1}{10}I_1 - \frac{1}{5}I_1 + \frac{1}{5}I_2 - \frac{1}{6}I_1 + \frac{1}{6}(40) = 0$$

$$-\frac{7}{15}I_1 + \frac{1}{5}I_2 = -\frac{2}{3} \qquad \ldots\text{(iii)}$$

Mesh 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$I_3 - I_2 = 5\,V_x = 5\left[\frac{1}{5}(I_2 - I_1)\right] = I_2 - I_1$$

$$I_1 - 2I_2 + I_3 = 0 \qquad \ldots\text{(iv)}$$

Applying KVL to the outer path of the supermesh,

$$-\frac{1}{5}(I_2 - I_1) - \frac{1}{20}I_2 - \frac{1}{15}I_3 - \frac{1}{2}(I_3 - I_4) = 0$$

$$-\frac{1}{5}I_2 + \frac{1}{5}I_1 - \frac{1}{20}I_2 - \frac{1}{15}I_3 - \frac{1}{2}I_3 + \frac{1}{2}(40) = 0$$

$$\frac{1}{5}I_1 - \frac{1}{4}I_2 - \frac{17}{30}I_3 = -20 \qquad \ldots\text{(v)}$$

Solving Eqs (iii), (iv) and (v),

$$I_1 = 10 \text{ A}$$
$$I_2 = 20 \text{ A}$$
$$I_3 = 30 \text{ A}$$
$$I_4 = 40 \text{ A}$$

2.5 NODE ANALYSIS

Node analysis is based on Kirchhoff's current law which states that the algebraic sum of currents meeting at a point is zero. Every junction where two or more branches meet is regarded as a node. One of the nodes in the network is taken as *reference node* or *datum node*. If there are n nodes in any network, the number of simultaneous equations to be solved will be $(n - 1)$.

Steps to be followed in Node Analysis

1. Assuming that a network has n nodes, assign a reference node and the reference directions, and assign a current and a voltage name for each branch and node respectively.
2. Apply KCL at each node except for the reference node and apply Ohm's law to the branch currents.
3. Solve the simultaneous equations for the unknown node voltages.
4. Using these voltages, find any branch currents required.

Example 2.59 *Calculate the current through 2 Ω resistor for the network shown in Fig. 2.94.*

Fig. 2.94

Solution Assume that the currents are moving away from the nodes.

Applying KCL at Node A,

$$\frac{V_A - 20}{1} + \frac{V_A}{1} + \frac{V_A - V_B}{0.5} = 0$$

$$\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{0.5}\right) V_A - \frac{1}{0.5} V_B = \frac{20}{1}$$

$$4\,V_A - 2\,V_B = 20 \qquad \ldots\text{(i)}$$

Applying KCL at Node B,

$$\frac{V_B - V_A}{0.5} + \frac{V_B}{2} + \frac{V_B - 20}{1} = 0$$

$$-\frac{1}{0.5} V_A \left(\frac{1}{0.5} + \frac{1}{2} + \frac{1}{1}\right) V_B = \frac{20}{1}$$

$$-2\,V_A + 3.5\,V_B = 20 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_A = 11 \text{ V}$$

$$V_B = 12 \text{ V}$$

Current through the 2 Ω resistor $= \dfrac{V_B}{2} = \dfrac{12}{2} = 6$ A

Example 2.60 *Find the voltage at nodes 1 and 2 for the network shown in Fig. 2.95.*

Fig. 2.95

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$1 = \frac{V_1}{2} + \frac{V_1 - V_2}{2}$$

$$\left(\frac{1}{2} + \frac{1}{2}\right) V_1 - \frac{1}{2} V_2 = 1$$

$$V_1 - 0.5\,V_2 = 1 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$2 = \frac{V_2}{1} + \frac{V_2 - V_1}{2}$$

$$-\frac{1}{2} V_1 + \left(1 + \frac{1}{2}\right) V_2 = 2$$

$$-0.5\,V_1 + 1.5\,V_2 = 4 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_1 = 2 \text{ V}$$
$$V_2 = 2 \text{ V}$$

Example 2.61 *Find the current in the 100 Ω resistor for the network shown in Fig. 2.96.*

Fig. 2.96

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 60}{20} + \frac{V_1 - V_2}{30} = 1$$

$$\left(\frac{1}{20} + \frac{1}{30}\right) V_1 - \frac{1}{30} V_2 = \frac{60}{20} + 1$$

$$0.083\, V_1 - 0.033\, V_2 = 4 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{30} + \frac{V_2 - 40}{50} + \frac{V_2}{100} = 0$$

$$-\frac{1}{30} V_1 + \left(\frac{1}{30} + \frac{1}{50} + \frac{1}{100}\right) V_2 = \frac{40}{50}$$

$$-0.033\, V_1 + 0.063\, V_2 = 0.8 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_1 = 67.25 \text{ V}$$
$$V_2 = 48 \text{ V}$$

Current through the 100 Ω resistor $= \dfrac{V_2}{100} = \dfrac{48}{100} = 0.48$ A

Example 2.62 *Find V_A and V_B for the network shown in Fig. 2.97.*

Fig. 2.97

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node A,

$$2 = \frac{V_A}{2} + \frac{V_A - 1}{2} + \frac{V_A - V_B}{1}$$

$$\left(\frac{1}{2} + \frac{1}{2} + 1\right) V_A - V_B = 2 + \frac{1}{2}$$

$$2V_A - V_B = 2.5 \qquad \ldots\text{(i)}$$

Applying KCL at Node B,

$$\frac{V_B - V_A}{1} + \frac{V_B - 2}{2} = 1 \qquad \ldots\text{(ii)}$$

$$-V_A + \left(1 + \frac{1}{2}\right) V_B = 1 + \frac{2}{2}$$

$$-V_A + 1.5\, V_B = 2$$

Solving Eqs (i) and (ii),

$$V_A = 2.875 \text{ V}$$

$$V_B = 3.25 \text{ V}$$

Example 2.63 *Find currents I_1, I_2 and I_3 for the network shown in Fig. 2.98.*

Fig. 2.98

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1}{2} + \frac{V_1 - 25}{5} + \frac{V_1 - V_2}{10} = 0$$

$$\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right) V_1 - \frac{1}{10} V_2 = \frac{25}{5}$$

$$0.8\, V_1 - 0.1\, V_2 = 5 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{10} + \frac{V_2}{4} + \frac{V_2 - (-50)}{2} = 0$$

$$-\frac{1}{10} V_1 + \left(\frac{1}{10} + \frac{1}{4} + \frac{1}{2}\right) V_2 = -\frac{50}{2}$$

$$-0.1\, V_1 + 0.85\, V_2 = -25 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_1 = 2.61\,\text{V}$$
$$V_2 = -29.1\,\text{V}$$
$$I_1 = -\frac{V_1}{2} = -\frac{2.61}{2} = -1.31\ \text{A}$$
$$I_2 = \frac{V_1 - V_2}{10} = \frac{2.61-(-2.91)}{10} = 3.17\,\text{A}$$
$$I_3 = \frac{V_2 + 50}{2} = \frac{-29.1+50}{2} = 10.45\ \text{A}$$

Example 2.64 *Find currents I_1, I_2 and I_3 and voltages V_a and V_b for the network shown in Fig. 2.99.*

Fig. 2.99

Solution Applying KCL at Node *a*,

$$I_1 = 30 + I_2$$
$$\frac{120 - V_a}{0.2} = 30 + \frac{V_a - V_b}{0.3}$$
$$36 - 0.3\,V_a = 1.8 + 0.2\,V_a - 0.2\,V_b$$
$$0.5\,V_a - 0.2\,V_b = 34.2 \qquad \ldots\text{(i)}$$

Applying KCL at Node *b*,

$$I_2 + I_3 = 20$$
$$\frac{V_a - V_b}{0.3} + \frac{110 - V_b}{0.1} = 20$$
$$\frac{0.1\,V_a - 0.1\,V_b + 33 - 0.3\,V_b}{0.03} = 20$$
$$0.1\,V_a - 0.4\,V_b = -32.4 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_a = 112\ \text{V}$$
$$V_b = 109\ \text{V}$$
$$I_1 = \frac{120 - V_a}{0.2} = \frac{120-112}{0.2} = 40\ \text{A}$$
$$I_2 = \frac{V_a - V_b}{0.3} = \frac{112-109}{0.3} = 10\ \text{A}$$
$$I_3 = \frac{110 - V_b}{0.1} = \frac{110-109}{0.1} = 10\ \text{A}$$

Example 2.65 *Calculate the current through the 5 Ω resistor for the network shown in Fig. 2.100.*

Fig. 2.100

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$4+\frac{V_1}{2}+\frac{V_1-V_2}{3}=0$$

$$\left(\frac{1}{2}+\frac{1}{3}\right)V_1-\frac{1}{3}V_2=4$$

$$0.83\,V_1-0.33\,V_2=-4 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{3}+\frac{V_2-(-20)}{2}+\frac{V_2-V_3}{5}=0$$

$$-\frac{1}{3}V_1+\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{5}\right)V_2-\frac{1}{5}V_3=-\frac{20}{2}$$

$$-0.33\,V_1+1.03\,V_2-0.2\,V_3=-10 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{5}+\frac{V_3}{4}=8$$

$$-\frac{1}{5}V_2+\left(\frac{1}{5}+\frac{1}{4}\right)V_3=8$$

$$-0.2\,V_2+0.45\,V_3=8 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=-8.76\text{ V}$$
$$V_2=-9.92\text{ V}$$
$$V_3=13.37\text{ V}$$

Current through the 5 Ω resistor $=\dfrac{V_3-V_2}{5}=\dfrac{13.37-(-9.92)}{5}=4.66$ A

Example 2.66 *Find the voltage across the 5 Ω resistor for the network shown in Fig. 2.101.*

Fig. 2.101

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1-12}{4}+\frac{V_1-V_2}{2}+\frac{V_1-V_3}{4}+9=0$$

$$\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right)V_1-\frac{1}{2}V_2-\frac{1}{4}V_3=-9+\frac{12}{4}$$

$$V_1-0.5\,V_2-0.25\,V_3=-6 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{2}+\frac{V_2}{100}+\frac{V_2-V_3}{5}=0$$

$$-\frac{1}{2}V_1+\left(\frac{1}{2}+\frac{1}{100}+\frac{1}{5}\right)V_2-\frac{1}{5}V_3=0$$

$$-0.5\,V_1+0.71\,V_2-0.2\,V_3=0 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{5}+\frac{V_3}{20}+\frac{V_3-V_1}{4}=9$$

$$-\frac{1}{4}V_1-\frac{1}{5}V_2+\left(\frac{1}{5}+\frac{1}{20}+\frac{1}{4}\right)V_3=9$$

$$-0.25\,V_1-0.2\,V_2+0.5\,V_3=9 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=6.35\text{ V}$$
$$V_2=11.76\text{ V}$$
$$V_3=25.88\text{ V}$$

Voltages across the 5 Ω resistor $=V_3-V_2=25.88-11.76=14.12$ V

Example 2.67 *Determine the current through the 5 Ω resistor for the network shown in Fig. 2.102.*

Fig. 2.102

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1}{4}+\frac{V_1-V_2}{2}+\frac{V_1-36-V_3}{4}=3$$

$$\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right)V_1-\frac{1}{2}V_2-\frac{1}{4}V_3=3+\frac{36}{4}$$

$$V_1-0.5\,V_2-0.25\,V_3=12 \qquad \cdots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{2}+\frac{V_2}{100}+\frac{V_2-V_3}{5}=0$$

$$-\frac{1}{2}V_1+\left(\frac{1}{2}+\frac{1}{100}+\frac{1}{5}\right)V_2-\frac{1}{5}V_3=0$$

$$-0.5\,V_1+0.71\,V_2-0.2\,V_3=0 \qquad \cdots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_2}{5}+\frac{V_3}{20}+\frac{V_3-(-36)-V_1}{4}=0$$

$$-\frac{1}{4}V_1-\frac{1}{5}V_2+\left(\frac{1}{5}+\frac{1}{20}+\frac{1}{4}\right)V_3=-9$$

$$-0.25\,V_1-0.2\,V_2+0.5\,V_3=-9 \qquad \cdots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=13.41\text{ V}$$
$$V_2=7.06\text{ V}$$
$$V_3=-8.47\text{ V}$$

Current through the 5 Ω resistor $=\dfrac{V_2-V_3}{5}=\dfrac{7.06-(-8.47)}{5}=3.11\text{ A}$

Example 2.68 *Find the voltage drop across the 5 Ω resistor in the network shown in Fig. 2.103.*

Fig 2.103

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$1+\frac{V_1-V_2}{5}+\frac{V_1-V_3}{4}=0$$

$$\left(\frac{1}{5}+\frac{1}{4}\right)V_1-\frac{1}{5}V_2-\frac{1}{4}V_3=-1$$

$$0.45\,V_1-0.2\,V_2-0.25\,V_3=-1 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{5}+\frac{V_2}{1}=2$$

$$-\frac{1}{5}V_1+\left(\frac{1}{5}+1\right)V_2=2$$

$$-0.2\,V_1+1.2\,V_2=2 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3}{2}+\frac{V_3-V_1}{4}+2=0$$

$$-\frac{1}{4}V_1+\left(\frac{1}{2}+\frac{1}{4}\right)V_3=-2$$

$$-0.25\,V_1+0.75\,V_3=-2 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=-4\text{ V}$$

$$V_2=1\text{ V}$$

$$V_3=-4\text{ V}$$

$$V_{5\,\Omega}=V_2-V_1=1-(-4)=5\text{ V}$$

Example 2.69 *Find the power dissipated in the 6 Ω resistor for the network shown in Fig. 2.104.*

Fig. 2.104

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1-20}{3}+\frac{V_1-V_2}{1}+\frac{V_1-V_3}{2}=0$$

$$\left(\frac{1}{3}+1+\frac{1}{2}\right)V_1-V_2-\frac{1}{2}V_3=\frac{20}{3}$$

$$1.83\,V_1-V_2-0.5\,V_3=6.67 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2-V_1}{1}+\frac{V_2-V_3}{6}=5$$

$$-V_1+\left(1+\frac{1}{6}\right)V_2-\frac{1}{6}V_3=5$$

$$-V_1+1.17\,V_2-0.17\,V_3=5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3-V_1}{2}+\frac{V_3}{5}+\frac{V_3-V_2}{6}=0$$

$$-\frac{1}{2}V_1-\frac{1}{6}V_2+\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{6}\right)V_3=0$$

$$-0.5\,V_1-0.17\,V_2+0.87\,V_3=0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1=23.82\text{ V}$$
$$V_2=27.4\text{ V}$$
$$V_3=19.04\text{ V}$$

$$I_{6\,\Omega}=\frac{V_2-V_3}{6}=\frac{27.4-19.04}{6}=1.39\text{ A}$$

Power dissipated in the 6 Ω resistor $=(1.39)^2\times 6=11.59$ W

Example 2.70 *Find the voltage V in the network shown in Fig. 2.105 which makes the current in the 10 Ω resistor zero.*

Fig. 2.105

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - V}{3} + \frac{V_1}{2} + \frac{V_1 - V_2}{10} = 0$$

$$\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{10}\right) V_1 - \frac{1}{10} V_2 - \frac{1}{3} V = 0$$

$$0.93\, V_1 - 0.1\, V_2 - 0.33\, V = 0 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{10} + \frac{V_2}{5} + \frac{V_2 - 50}{7} = 0$$

$$-\frac{1}{10} V_1 + \left(\frac{1}{10} + \frac{1}{5} + \frac{1}{7}\right) V_2 = \frac{50}{7}$$

$$-0.1\, V_1 + 0.44\, V_2 = 7.14 \qquad \ldots\text{(ii)}$$

$$I_{10\,\Omega} = \frac{V_1 - V_2}{10} = 0$$

$$V_1 - V_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V = 52.82 \text{ V}$$

Example 2.71 *Find V_1 and V_2 for the network shown in Fig. 2.106.*

Fig. 2.106

Solution Assume that the currents are moving away from the nodes.

Applying KCL at Node a,

$$\frac{V_a - 80}{50} + \frac{V_a - V_b}{10} + 2 = 0$$

$$\left(\frac{1}{50} + \frac{1}{10}\right) V_a - \frac{1}{10} V_b = \frac{80}{50} - 2$$

$$0.12\, V_a - 0.1\, V_b = -0.4 \qquad \ldots\text{(i)}$$

Applying KCL at Node b,

$$\frac{V_b - V_a}{10} + \frac{V_b}{50} + \frac{V_b - V_c}{20} = 0$$

$$-\frac{1}{10} V_a + \left(\frac{1}{10} + \frac{1}{50} + \frac{1}{20}\right) V_b - \frac{1}{20} V_c = 0$$

$$-0.1\, V_a + 0.17\, V_b - 0.05\, V_c = 0 \qquad \ldots\text{(ii)}$$

Node c is directly connected to a voltage source of 20 V. Hence, we can write voltage equation at Node c.

$$V_c = 20 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$V_a = 3.08 \text{ V}$$

$$V_b = 7.69 \text{ V}$$

$$V_1 = V_a - V_b = 3.08 - 7.69 = -4.61 \text{ V}$$

$$V_2 = V_b - V_c = 7.69 - 20 = -12.31 \text{ V}$$

Example 2.72 *Find the voltage across the 100 Ω resistor for the network shown in Fig. 2.107.*

Fig. 2.107

Solution

Node A is directly connected to a voltage source of 20 V. Hence, we can write voltage equation at Node A.

$$V_A = 60 \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the nodes.

Applying KCL at Node B,

$$\frac{V_B - V_A}{20} + \frac{V_B - V_C}{20} + \frac{V_B}{20} = 0.6$$

$$-\frac{1}{20} V_A + \left(\frac{1}{20} + \frac{1}{20} + \frac{1}{20}\right) V_B - \frac{1}{20} V_C = 0.6$$

$$-0.05\, V_A + 0.15\, V_B - 0.05\, V_C = 0.6 \qquad \ldots\text{(ii)}$$

Applying KCL at Node C,

$$\frac{V_C - V_A}{50} + \frac{V_C - V_B}{20} + \frac{V_C - 12}{50} + \frac{V_C}{100} = 0$$

$$-\frac{1}{50}V_A - \frac{1}{20}V_B + \left(\frac{1}{50} + \frac{1}{20} + \frac{1}{50} + \frac{1}{100}\right)V_C = \frac{12}{50}$$

$$-0.02\,V_A - 0.05\,V_B + 0.1\,V_C = 0.24 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$V_C = 31.68 \text{ V}$$

Voltages across the 100 Ω resistor = 31.68 V

EXAMPLES WITH DEPENDENT SOURCES

Example 2.73 *Find the voltage across the 5 Ω resistor in the network shown in Fig. 2.108.*

Fig. 2.108

Solution From Fig. 2.108,

$$I_1 = \frac{V_1 - 50}{20 + 10} = \frac{V_1 - 50}{30} \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the node.
Applying KCL at Node 1,

$$2 = \frac{V_1}{5} + \frac{V_1 + 30\,I_1}{10} + \frac{V_1 - 50}{30}$$

$$2 = \frac{V_1}{5} + \frac{V_1 + 30\left(\dfrac{V_1 - 50}{30}\right)}{10} + \frac{V_1 - 50}{30}$$

$$2 = \frac{V_1}{5} + \frac{2V_1 - 50}{10} + \frac{V_1 - 50}{30} \qquad \ldots\text{(ii)}$$

Solving Eq. (ii),

$$V_1 = 20 \text{ V}$$

Voltage across the 5 Ω resistor = 20 V

Example 2.74 *For the network shown in Fig. 2.109, find the voltage V_x.*

Fig. 2.109

Solution From Fig. 2.109,

$$I_y = \frac{V_x}{100} \qquad \ldots(i)$$

Assume that the currents are moving away from the node.
Applying KCL at Node x,

$$25\,I_y + 0.6 = \frac{V_x}{100} + \frac{V_x}{50} + \frac{V_x - 0.2\,V_x}{40}$$

$$25\left(\frac{V_x}{100}\right) + 0.6 = \frac{V_x}{100} + \frac{V_x}{50} + \frac{0.8\,V_x}{40}$$

$$\left(\frac{1}{4} - \frac{1}{100} - \frac{1}{50} - \frac{0.8}{40}\right)V_x = -0.6$$

$$0.2\,V_x = -0.6$$

$$V_x = -3 \text{ V}$$

Example 2.75 *For the network shown in Fig. 2.110, find voltages V_1 and V_2.*

Fig. 2.110

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$2 = \frac{V_1 - 20}{20} + \frac{V_1 - 0.5\,V_1 - V_2}{5}$$

$$\left(\frac{1}{20} + \frac{1}{5} - \frac{0.5}{5}\right)V_1 - \frac{1}{5}V_2 = 3$$

$$0.15\,V_1 - 0.2\,V_2 = 3 \qquad \ldots(i)$$

Applying KCL at Node 2,

$$\frac{V_2 + 0.5\,V_1 - V_1}{5} + \frac{V_2}{2} + \frac{V_2 - 40}{10} = 4$$

$$\left(\frac{0.5}{5} - \frac{1}{5}\right)V_1 + \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{10}\right)V_2 = 4 + \frac{40}{10}$$

$$-0.1\,V_1 + 0.8\,V_2 = 8 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$V_1 = 40 \text{ V}$$

$$V_2 = 15 \text{ V}$$

Example 2.76 *Determine the voltages V_1 and V_2 in the network of Fig. 2.111.*

Fig. 2.111

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$\frac{V_1 - 2}{0.5} + \frac{V_1}{0.25} + \frac{V_1 - V_2}{1} = 0$$

$$\left(\frac{1}{0.5} + \frac{1}{0.25} + 1\right)V_1 - V_2 = \frac{2}{0.5}$$

$$7\,V_7 - V_2 = 4 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{1} + V_1 = 0$$

$$2\,V_2 = 0$$

$$V_2 = 0 \qquad \ldots\text{(ii)}$$

From Eq. (i),

$$V_1 = \frac{4}{7} \text{ V}$$

Example 2.77 *In the network of Fig. 2.112, find the node voltages V_1, V_2 and V_3.*

Fig. 2.112

Solution From Fig. 2.112,

$$V_x = V_2 - V_1 \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$2 = \frac{V_1}{2} + \frac{V_1 - V_2}{1}$$

$$\left(\frac{1}{2} + 1\right) V_1 - V_2 = 2$$

$$1.5\, V_1 - V_2 = 2 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{2} + \frac{V_2 - V_3}{1} = 0$$

$$-V_1 + \left(1 + \frac{1}{2} + 1\right) V_2 - V_3 = 0$$

$$-V_1 + 2.5\, V_2 - V_3 = 0 \qquad \ldots\text{(iii)}$$

At Node 3,

$$V_3 - 4\, V_x = 2$$

$$V_3 - 4(V_2 - V_1) = 2$$

$$4\, V_1 - 4\, V_2 + V_3 = 2 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$V_1 = -1.33 \text{ V}$$

$$V_2 = -4 \text{ V}$$

$$V_3 = -8.67 \text{ V}$$

Example 2.78 *For the network shown in Fig. 2.113, find the node voltages V_1 and V_2.*

Fig. 2.113

Solution From Fig. 2.113,

$$I_x = \frac{V_1 - V_2}{0.5} \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$3\,V_2 + 1 = \frac{V_1}{1} + \frac{V_1 - V_2}{0.5} + 3$$

$$\left(1 + \frac{1}{0.5}\right)V_1 - \left(3 + \frac{1}{0.5}\right)V_2 = -2$$

$$3\,V_1 - 5\,V_2 = -2 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$3 + 2 = \frac{V_2 - V_1}{0.5} + \frac{V_2}{0.25} + 2\,I_x$$

$$5 = \frac{V_2 - V_1}{0.5} + \frac{V_2}{0.25} + 2\left(\frac{V_1 - V_2}{0.5}\right)$$

$$\left(-\frac{1}{0.5} + \frac{2}{0.5}\right)V_1 + \left(\frac{1}{0.5} + \frac{1}{0.25} - \frac{2}{0.5}\right)V_2 = 5$$

$$2\,V_1 + 2\,V_2 = 5 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1 = 1.31\text{ V}$$
$$V_2 = 1.19\text{ V}$$

Example 2.79 *Find voltages V_1 and V_2 in the network shown in Fig. 2.114.*

Fig. 2.114

Solution From Fig. 2.114,

$$I_1 = \frac{V_1 - V_2}{2} \qquad \ldots\text{(i)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$5 = \frac{V_1}{1} + \frac{V_1 - V_2}{2} + V_1$$

$$\left(1 + \frac{1}{2} + 1\right) V_1 - \frac{1}{2} V_2 = 5$$

$$2.5\,V_1 - 0.5\,V_2 = 5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{1} + \frac{V_2}{1} = 2I_1 + V_1$$

$$V_2 - V_1 + V_2 = 2\left(\frac{V_1 - V_2}{2}\right) + V_1$$

$$3\,V_1 = 3\,V_2$$

$$V_1 = V_2 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1 = 2.5 \text{ V}$$

$$V_2 = 2.5 \text{ V}$$

Example 2.80 *Find the power supplied by the 10 V source in the network shown in Fig. 2.115.*

Fig. 2.115

Solution Assume that the currents are moving away from the nodes.
From Fig. 2.115,

$$V_3 = V_1 + 10 - V_2 \qquad \ldots\text{(i)}$$

Applying KCL at Node 1,

$$10 + \frac{V_1}{2} + \frac{V_1 + 10 - V_2}{4} + \frac{V_1 - V_2}{2} = 0$$

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right)V_1 - \left(\frac{1}{4} + \frac{1}{2}\right)V_2 = -10 - \frac{10}{4}$$

$$1.25\,V_1 - 0.75\,V_2 = -12.5 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - 10 - V_1}{4} + \frac{V_2 - V_1}{2} + \frac{V_2}{1} = 4\,V_3$$

$$\frac{V_2 - 10 - V_1}{4} + \frac{V_2 - V_1}{2} + \frac{V_2}{1} = 4(V_1 + 10 - V_2)$$

$$\left(-\frac{1}{4} - \frac{1}{2} - 4\right)V_1 + \left(\frac{1}{4} + \frac{1}{2} + 1 + 4\right)V_2 = \frac{10}{4} + 40$$

$$-4.75\,V_1 + 5.75\,V_2 = 42.5 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$V_1 = -11.03 \text{ V}$$

$$V_2 = -1.72 \text{ V}$$

$$I_{10\text{ V}} = \frac{V_1 + 10 - V_2}{4} = \frac{-11.03 + 10 - (-1.72)}{4} = 0.173 \text{ A}$$

Power supplied by the 10 V source = 10 × 0.173 = 1.73 W

Example 2.81 *For the network shown in Fig. 2.116, find voltages V_1 and V_2.*

Fig. 2.116

Solution From Fig. 2.116,

$$V_x = V_1 - V_2 \qquad \ldots\text{(i)}$$

$$I_y = \frac{V_2}{40} \qquad \ldots\text{(ii)}$$

Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$0.4 = \frac{V_1}{100} + \frac{V_1 - V_2}{20} + 0.03\,V_x$$

$$0.4 = \frac{V_1}{100} + \frac{V_1 - V_2}{20} + 0.03(V_1 - V_2)$$

$$\left(\frac{1}{100} + \frac{1}{20} + 0.03\right)V_1 - \left(\frac{1}{20} + 0.03\right)V_2 = 0.4$$

$$0.09\,V_1 - 0.08\,V_2 = 0.4 \qquad \ldots\text{(iii)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{20} + \frac{V_2}{40} + \frac{V_2 - V_3}{40} = 0$$

$$-\frac{1}{20}V_1 + \left(\frac{1}{20} + \frac{1}{40} + \frac{1}{40}\right)V_2 - \frac{1}{40}V_3 = 0$$

$$-0.05\,V_1 + 0.1\,V_2 - 0.025\,V_3 = 0 \qquad \ldots\text{(iv)}$$

For Node 3,

$$V_3 = 80\,I_y = 80\left(\frac{V_2}{40}\right) = 2\,V_2$$

$$2\,V_2 - V_3 = 0 \qquad \ldots\text{(v)}$$

Solving Eqs (iii), (iv) and (v),

$$V_1 = 40 \text{ V}$$
$$V_2 = 40 \text{ V}$$
$$V_3 = 80 \text{ V}$$

Example 2.82 *Find voltages V_a, V_b and V_c in the network shown in Fig. 2.117.*

Fig. 2.117

Solution From Fig. 2.117,

$$I_1 = \frac{V_a - V_c}{2}$$

Assume that the currents are moving away from the nodes.

Applying KCL at Node a,

$$4 = \frac{V_a}{1} + \frac{V_a - V_c}{2} + \frac{V_a - 2 - V_b}{2}$$

$$\left(1 + \frac{1}{2} + \frac{1}{2}\right) V_a - \frac{1}{2} V_b - \frac{1}{2} V_c = 5$$

$$2\,V_a - 0.5\,V_b - 0.5\,V_c = 5 \qquad \ldots\text{(i)}$$

Applying KCL at Node b,

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = 2I_1$$

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = 2\left(\frac{V_a - V_c}{2}\right)$$

$$\frac{V_b + 2 - V_a}{2} + \frac{V_b - V_c}{3} = V_a - V_c$$

$$\left(-\frac{1}{2} - 1\right) V_a + \left(\frac{1}{2} + \frac{1}{3}\right) V_b + \left(1 - \frac{1}{3}\right) V_c = -1$$

$$-1.5\,V_a + 0.83\,V_b + 0.67\,V_c = -1 \qquad \ldots\text{(ii)}$$

Applying KCL at Node c,

$$\frac{V_c - V_b}{3} + \frac{V_c}{5} = I_1$$

$$\frac{V_c - V_b}{3} + \frac{V_c}{5} = \frac{V_a - V_c}{2}$$

$$-\frac{1}{2} V_a - \frac{1}{3} V_b + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{2}\right) V_c = 0$$

$$-0.5\,V_a - 0.33\,V_b + 1.033\,V_c = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii), and (iii),

$$V_a = 4.303 \text{ V}$$
$$V_b = 3.88 \text{ V}$$
$$V_c = 3.33 \text{ V}$$

2.6 SUPERNODE ANALYSIS

Nodes that are connected to each other by voltage sources, but not to the reference node by a path of voltage sources, form a *supernode*. A supernode requires one node voltage equation, that is, a KCL equation. The remaining node voltage equations are KVL equations.

Example 2.83 *Determine the current in the 5 Ω resistor for the network shown in Fig. 2.118.*

Fig. 2.118

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$10 = \frac{V_1}{3} + \frac{V_1 - V_2}{2}$$

$$\left(\frac{1}{3} + \frac{1}{2}\right) V_1 - \frac{1}{2} V_2 = 10$$

$$0.83\, V_1 - 0.5\, V_2 = 10 \qquad \ldots\text{(i)}$$

Nodes 2 and 3 will form a supernode.
Writing voltage equation for the supernode,

$$V_2 - V_3 = 20 \qquad \ldots\text{(ii)}$$

Applying KCL at the supernode,

$$\frac{V_2 - V_1}{2} + \frac{V_2}{1} + \frac{V_3 - 10}{5} + \frac{V_3}{2} = 0$$

$$-\frac{1}{2} V_1 + \left(\frac{1}{2} + 1\right) V_2 + \left(\frac{1}{5} + \frac{1}{2}\right) V_3 = 2$$

$$-0.5\, V_1 + 1.5\, V_2 + 0.7\, V_3 = 2 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 19.04 \text{ V}$$
$$V_2 = 11.6 \text{ V}$$
$$V_3 = -8.4 \text{ V}$$

$$I_{5\,\Omega} = \frac{V_3 - 10}{5} = \frac{-8.4 - 10}{5} = -3.68 \text{ A}$$

Example 2.84 *Find the power delivered by the 5 A current source in the network shown in Fig. 2.119.*

Fig. 2.119

Solution Assume that the currents are moving away from the nodes.
Nodes 1 and 2 will form a supernode.
Writing voltage equation for the supernode,

$$V_1 - V_2 = 10 \qquad \text{...(i)}$$

Applying KCL at the supernode,

$$2 + \frac{V_1 - V_3}{3} + \frac{V_2}{5} + \frac{V_2 - V_3}{1} = 5$$

$$\frac{1}{3}V_1 + \left(\frac{1}{5}+1\right)V_2 - \left(\frac{1}{3}+1\right)V_3 = 3$$

$$0.33\,V_1 + 1.2\,V_2 - 1.33\,V_3 = 3 \qquad \text{...(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{3} + \frac{V_3 - V_2}{1} + \frac{V_3}{2} = 0$$

$$-\frac{1}{3}V_1 - V_2 + \left(\frac{1}{3}+1+\frac{1}{2}\right)V_3 = 0$$

$$-0.33\,V_1 - V_2 + 1.83\,V_3 = 0 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 13.72 \text{ V}$$

$$V_2 = 3.72 \text{ V}$$

$$V_3 = 4.51 \text{ V}$$

Power delivered by the 5 A source = 5 V_2 = 5 × 3.72 = 18.6 W

Example 2.85 *In the network of Fig. 2.120, find the node voltages V_1, V_2 and V_3.*

Fig. 2.120

Solution Assume that the currents are moving away from the nodes.
Applying KCL at Node 1,

$$4 = \frac{V_1 - V_2}{0.33} + \frac{V_1 - V_3}{0.5}$$

$$\left(\frac{1}{0.33} + \frac{1}{0.5}\right)V_1 - \frac{1}{0.33}V_2 - \frac{1}{0.5}V_3 = 4$$

$$5.03\,V_1 - 3.03\,V_2 - 2\,V_3 = 4 \qquad \ldots\text{(i)}$$

Nodes 2 and 3 will form a supernode.
Writing voltage equation for the supernode,

$$V_3 - V_2 = 5 \qquad \ldots\text{(ii)}$$

Applying KCL at the supernode,

$$\frac{V_2 - V_1}{0.33} + \frac{V_2}{0.2} + \frac{V_3}{1} + \frac{V_3 - V_1}{0.5} = 0$$

$$\left(-\frac{1}{0.33} - \frac{1}{0.5}\right)V_1 + \left(\frac{1}{0.33} + \frac{1}{0.2}\right)V_2 + \left(1 + \frac{1}{0.5}\right)V_3 = 0$$

$$-5.03\,V_1 + 8.03\,V_2 + 3\,V_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 2.62 \text{ V}$$
$$V_2 = -0.17 \text{ V}$$
$$V_3 = 4.83 \text{ V}$$

EXAMPLES WITH DEPENDENT SOURCES

Example 2.86 *For the network shown in Fig. 2.121, determine the voltage V_x.*

Fig. 2.121

Solution From Fig. 2.121,

$$V_x = V_2 - V_3$$

Assume that the currents are moving away from the nodes.
Node 1 and 2 will form a supernode.
Writing voltage equations for the supernode,

$$V_1 - V_2 = 6 \qquad \ldots\text{(i)}$$

Applying KCL at the supernode,

$$2 = \frac{V_1+5}{4} + \frac{V_2 - 2V_x}{10} + \frac{V_2 - 8 - V_3}{7} + \frac{V_2 - V_3}{11}$$

$$2 = \frac{V_1+5}{4} + \frac{V_2 - 2(V_2 - V_3)}{10} + \frac{V_2 - 8 - V_3}{7} + \frac{V_2 - V_3}{11}$$

$$\frac{1}{4}V_1 + \left(\frac{1}{10} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11}\right)V_2 + \left(\frac{1}{5} - \frac{1}{7} - \frac{1}{11}\right)V_3 = 2 - \frac{5}{4} + \frac{8}{7}$$

$$0.25\,V_1 + 0.133\,V_2 - 0.033\,V_3 = 1.89 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3 - V_2}{11} + \frac{V_3 + 8 - V_2}{7} + 12 = 0$$

$$\left(-\frac{1}{11} - \frac{1}{7}\right)V_2 + \left(\frac{1}{11} + \frac{1}{7}\right)V_3 = -12 - \frac{8}{7}$$

$$-0.233\,V_2 + 0.233\,V_3 = -13.14 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$V_1 = 1.8 \text{ V}$$
$$V_2 = -4.2 \text{ V}$$
$$V_3 = -60.6 \text{ V}$$

Example 2.87 *Find the node voltages in the network shown in Fig. 2.122.*

Fig. 2.122

Solution From Fig. 2.122,

$$I_x = \frac{V_2 - V_1}{5} \qquad \ldots\text{(i)}$$

For Node 4,

$$V_4 = 40 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 1,

$$6+\frac{V_1}{10}+\frac{V_1-V_2}{5}+\frac{V_1-V_4}{6}=0$$

$$6+\frac{V_1}{10}+\frac{V_1-V_2}{5}+\frac{V_1-40}{6}=0$$

$$\left(\frac{1}{10}+\frac{1}{5}+\frac{1}{6}\right)V_1-\frac{1}{5}V_2=\frac{40}{6}-6$$

$$\frac{7}{15}V_1-\frac{1}{5}V_2=\frac{2}{3} \qquad \ldots\text{(iii)}$$

Nodes 2 and 3 will form a supernode,
Writing voltage equation for the supernode,

$$V_3-V_2=5\,I_x=5\left(\frac{V_2-V_1}{5}\right)=V_2-V_1$$

$$V_1-2V_2+V_3=0 \qquad \ldots\text{(iv)}$$

Applying KCL to the supernode,

$$\frac{V_2-V_1}{5}+\frac{V_2}{20}+\frac{V_3}{15}+\frac{V_3-V_4}{2}=0$$

$$\frac{V_2-V_1}{5}+\frac{V_2}{20}+\frac{V_3}{15}+\frac{V_3-40}{2}=0$$

$$-\frac{1}{5}V_1+\left(\frac{1}{5}+\frac{1}{20}\right)V_2+\left(\frac{1}{15}+\frac{1}{2}\right)V_3=20$$

$$-\frac{1}{5}V_1+\frac{1}{4}V_2+\frac{17}{30}V_3=20 \qquad \ldots\text{(v)}$$

Solving Eqs (iii), (iv) and (v),

$$V_1=10\text{ V}$$
$$V_2=20\text{ V}$$
$$V_3=30\text{ V}$$
$$V_4=40\text{ V}$$

Example 2.88 *Find the node voltages in the network shown in Fig. 2.123.*

Fig. 2.123

Solution Selecting the central node as reference node,

$$V_1 = -12\text{ V} \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{V_2 - V_1}{0.5} + \frac{V_2 - V_3}{2} = 14$$

$$-\frac{1}{0.5}V_1 + \left(\frac{1}{0.5} + \frac{1}{2}\right)V_2 - \frac{1}{2}V_3 = 14$$

$$-2\,V_1 + 2.5\,V_2 - 0.5\,V_3 = 14 \qquad \ldots\text{(ii)}$$

Nodes 3 and 4 will form a supernode,
Writing voltage equation for the supernode,

$$V_3 - V_4 = 0.2V_y = 0.2(V_4 - V_1)$$

$$0.2V_1 + V_3 - 1.2V_4 = 0 \qquad \ldots\text{(iii)}$$

Applying KCL to the supernode,

$$\frac{V_3 - V_2}{2} - 0.5V_x + \frac{V_4}{1} + \frac{V_4 - V_1}{2.5} = 0$$

$$\frac{V_3 - V_2}{2} - 0.5(V_2 - V_1) + V_4 + \frac{V_4 - V_1}{2.5} = 0$$

$$\left(0.5 - \frac{1}{2.5}\right)V_1 - \left(\frac{1}{2} + 0.5\right)V_2 + \frac{1}{2}V_3 + \left(1 + \frac{1}{2.5}\right)V_4 = 0$$

$$0.1\,V_1 - V_2 + 0.5\,V_3 + 1.4\,V_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (i), (ii), (iii) and (iv),

$$V_1 = -12\text{ V}$$
$$V_2 = -4\text{ V}$$
$$V_3 = 0$$
$$V_4 = -2\text{ V}$$

Exercises

KIRCHHOFF'S LAWS

2.1 Find I_x and V_x in the network shown in Fig. 2.124.

Fig. 2.124

[−5 A, −15 V]

2.2 Find V_1 and V_2 in the network shown in Fig. 2.125.

Fig. 2.125

[2 V, 5 V]

2.3 Find the values of unknown currents in the network shown in Fig. 2.126.

Fig. 2.126

$[I_1 = 2\text{ A}, I_2 = 2\text{ A}, I_3 = 4\text{ A}, I_4 = 10\text{ A}]$

2.4 Find the current in the branch *XY* as shown in Fig. 2.127.

Fig. 2.127

[40 A]

2.5 Find I and V_{AB} for the network as shown in Fig. 2.128.

Fig. 2.128

[3 A, 19 V]

2.6 In the network shown in Fig. 2.129, find the voltage between points *A* and *B*.

Fig. 2.129

[5 V]

2.7 In the network shown in Fig. 2.130, find the voltage between points *A* and *B*.

Fig. 2.130

[30 V]

MESH ANALYSIS

2.8 Find the current through the 10 Ω resistor in the network shown in Fig. 2.131.

Fig. 2.131

[0.68 A]

2.9 Find the current through the 20 Ω resistor in the network shown in Fig. 2.132.

Fig. 2.132

[1.46 A]

2.10 Find the current through the 10 Ω resistor in the network shown in Fig. 2.133.

Fig. 2.133

[0.37 A]

2.11 Find the current through the 1 Ω resistor in the network shown in Fig. 2.134.

Fig. 2.134

[0.95 A]

2.12 Find the current through the 4 Ω resistor in the network shown in Fig. 2.135.

Fig. 2.135

[1.33 A]

2.13 Find currents I_x and I_y in the network shown in Fig. 2.136.

Fig. 2.136

[0.5 A, 0.1 A]

2.14 In the network shown in Fig. 2.137, find V_3 if element A is a

(i) short circuit
(ii) 5 Ω resistor
(iii) 20 V independent voltage source, positive reference on the right
(iv) dependent voltage source of 1.5 i_1, with positive reference on the right
(v) dependent current source 5 i_1, arrow directed to the right

Fig. 2.137

[69.4 V, 72.38 V, 73.68 V, 70.71 V, 97.39 V]

2.15 Find currents I_1, I_2, and I_3 in the network shown in Fig. 2.138.

Fig. 2.138

[15 A, 11 A, 17 A]

2.16 Find currents I_x in the network shown in Fig. 2.139.

Fig. 2.139

[8.33 A]

2.17 Find currents I_1 in the network shown in Fig. 2.140.

Fig. 2.140

[–12 A]

NODE ANALYSIS

2.18 Find the current I_x in the network shown in Fig. 2.141.

Fig. 2.141

[−0.93 A]

2.19 Find V_A and V_B in the network shown in Fig. 2.142.

Fig. 2.142

[2.88 V, 3.25 V]

2.20 Find the current through the 6 Ω resistor in the network shown in Fig. 2.143.

Fig. 2.143

[2.04 A]

2.21 Calculate the current through the 10 Ω resistor in the network shown in Fig. 2.144.

Fig. 2.144

[1.62 A]

2.22 Find the current through the branch *ab* in the network shown in Fig. 2.145.

Fig. 2.145

[0.038 A]

2.23 Find the current through the 4 Ω resistor in the network shown in Fig. 2.146.

Fig. 2.146

[1.34 A]

2.24 Find the current through the 4 Ω resistor in the network shown in Fig. 2.147.

Fig. 2.147

[1 A]

2.25 Find the voltage V_x in the network shown in Fig. 2.148.

Fig. 2.148

[–4.31 V]

2.26 Find the currents V_x in the network shown in Fig. 2.149.

Fig. 2.149

[2.09 V]

2.27 Find the voltage V_x in the network shown in Fig. 2.150.

Fig. 2.150

[6.2 V]

2.28 Determine V_1 in the network shown in Fig. 2.151.

Fig. 2.151

[140 V]

2.29 Find the voltage V_y in the network shown in Fig. 2.152.

Fig. 2.152

[–10 V]

2.30 Find the voltage V_2 in the network shown in Fig. 2.153.

Fig. 2.153

[25.9 V]

Objective–Type Questions

2.1 Two electrical sub-networks N_1 and N_2 are connected through three resistors as shown in Fig. 2.154. The voltages across the 5 Ω resistor and 1 Ω resistor are given to be 10 V and 5 V respectively. Then the voltage across the 15 Ω resistor is

(a) –105 V (b) 105 V
(c) –15 V (d) 15 V

Fig. 2.154

2.2 The nodal method of circuit analysis is based on
(a) KVL and Ohm's law
(b) KCL and Ohm's law
(c) KCL and KVL
(d) KCL, KVL and Ohm's law

2.3 The voltage across terminals a and b in Fig. 2.155 is

Fig. 2.155

(a) 0.5 V (b) 3 V
(c) 3.5 V (d) 4 V

2.4 The voltage V_0 in Fig. 2.156 is

Fig. 2.156

(a) 48 V (b) 24 V
(c) 36 V (d) 28 V

2.5 The dependent current source shown in Fig. 2.157.

Fig. 2.157

(a) delivers 80 W
(b) absorbs 80 W
(c) delivers 40 W
(d) absorbs 40 W

2.6 If $V = 4$ in Fig. 2.158, the value of I_s is given by

Fig. 2.158

(a) 6 A (b) 2.5 A
(c) 12 A (d) none of these

2.7 The value of V_x, V_y and V_z in Fig. 2.159 shown are

Fig. 2.159

(a) –6, 3, –3 (b) –6, –3, 1
(c) 6, 3, 3 (d) 6, 1, 3

2.8 The circuit shown in Fig. 2.160 is equivalent to a load of

Fig. 2.160

(a) $\frac{4}{3}\,\Omega$ (b) $\frac{8}{3}\,\Omega$
(b) 4 Ω (d) 2 Ω

2.9 In the network shown in Fig. 2.161, the effective resistance faced by the voltage source is

Fig. 2.161

(a) 4 Ω (b) 3 Ω

(c) 2 Ω (d) 1 Ω

2.10 A network contains only an independent current source and resistors. If the values of all resistors are doubled, the value of the node voltages will

(a) become half
(b) remain unchanged
(c) become double
(d) none of these

Answers to Objective-Type Questions

2.1. (a)	2.2. (b)	2.3. (c)	2.4. (d)	2.5. (a)	2.6. (d)	2.7. (a)
2.8. (a)	2.9. (d)	2.10. (b)	2.11. (b)	2.12. (b)	2.13. (c)	2.14. (b)

3

Network Theorems (Application to dc Networks)

3.1 INTRODUCTION

In Chapter 2, we have studied elementary network theorems like Kirchhoff's laws, mesh analysis and node analysis. There are some other methods also to analyse circuits. In this chapter, we will study superposition theorem, Thevenin's theorem, Norton's theorem, maximum power transfer theorem, reciprocity theorem, Millman's theorem, Tellegen's theorem, substitution theorem and compensation theorem. We can find currents and voltages in various parts of the circuits with these methods.

3.2 SUPERPOSITION THEOREM

It states that '*in a linear network containing more than one independent source and dependent source, the resultant current in any element is the algebraic sum of the currents that would be produced by each independent source acting alone, all the other independent sources being represented meanwhile by their respective internal resistances.*'

The independent voltage sources are represented by their internal resistances if given or simply with zero resistances, i.e., short circuits if internal resistances are not mentioned. The independent current sources are represented by infinite resistances, i.e., open circuits.

The dependent sources are not sources but dissipative components—hence they are active at all times. A dependent source has zero value only when its control voltage or current is zero.

A linear network is one whose parameters are constant, i.e., they do not change with voltage and current.

Explanation Consider the network shown in Fig. 3.1. Suppose we have to find current I_4 through resistor R_4.

Fig. 3.1 *Network to illustrate superposition theorem*

The current flowing through resistor R_4 due to constant voltage source V is found to be say I_4' (with proper direction), representing constant current source with infinite resistance, i.e., open circuit.

The current flowing through resistor R_4 due to constant current source I is found to be say I_4'' (with proper direction), representing the constant voltage source with zero resistance or short circuit.

The resultant current I_4 through resistor R_4 is found by superposition theorem.

$$I_4 = I_4' + I_4''$$

Fig. 3.2 *When voltage source V is acting alone*

Fig. 3.3 *When current source I is acting alone*

Steps to be followed in Superposition Theorem

1. Find the current through the resistance when only one independent source is acting, replacing all other independent sources by respective internal resistances.
2. Find the current through the resistance for each of the independent sources.
3. Find the resultant current through the resistance by the superposition theorem considering magnitude and direction of each current.

Example 3.1 *Find the current through the 2 Ω resistor in Fig. 3.4.*

Fig. 3.4

Solution

Step I When the 40 V source is acting alone (Fig. 3.5)

Fig. 3.5

By series parallel reduction technique (Fig. 3.6),

$$I = \frac{40}{5+1.67} = 6 \text{ A}$$

From Fig. 3.5, by current-division rule,

$$I' = 6 \times \frac{10}{10+2} = 5 \text{ A } (\rightarrow)$$

Fig. 3.6

Step II When the 20 V source is acting alone (Fig. 3.7)

Fig. 3.7

By series–parallel reduction technique (Fig. 3.8),

$$I = \frac{20}{5+1.67} = 3 \text{ A}$$

Fig. 3.8

From Fig. 3.7, by current-division rule,

$$I'' = 3 \times \frac{10}{10+2} = 2.5 \text{ A}(\leftarrow) = -2.5 \text{ A} \quad (\rightarrow)$$

Step III When the 10 V source is acting alone (Fig. 3.9)

Fig. 3.9

By series–parallel reduction technique (Fig. 3.10),

$$I''' = \frac{10}{3.33+2} = 1.88 \text{ A}(\rightarrow)$$

Fig. 3.10

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 5 - 2.5 + 1.88 = 4.38 \text{ A} \quad (\rightarrow)$$

Example 3.2 *Find the current through the 1 Ω resistor in Fig. 3.11.*

Fig. 3.11

Solution

Step I When the 100 V source is acting alone (Fig. 3.12)

Fig. 3.12

By series–parallel reduction technique (Fig. 3.13),

$$I = \frac{100}{4+0.67} = 21.41 \text{ A}$$

Fig. 3.13

From Fig. 3.12, by current-division rule,

$$I' = 21.41 \times \frac{2}{2+1} = 14.27 \text{ A}(\downarrow)$$

Step II When the 50 V source is acting alone (Fig. 3.14)

Fig. 3.14

By series–parallel reduction technique (Fig. 3.15),

$$I'' = \frac{50}{1+1.33} = 21.46 \text{ A}(\uparrow) = -21.46 \text{ A}(\downarrow)$$

Fig. 3.15

Step III When the 40 V source is acting alone (Fig. 3.16)

Fig. 3.16

By series–parallel reduction technique (Fig. 3.17),

$$I = \frac{40}{0.8+2} = 14.29 \text{ A}$$

From Fig. 3.16, by current-division rule,

$$I''' = 14.29 \times \frac{4}{4+1} = 11.43 \text{ A}(\downarrow)$$

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 14.27 - 21.46 + 11.43 = 4.24 \text{ A}(\downarrow)$$

Fig. 3.17

Example 3.3 *Find the current through the 8 Ω resistor in Fig. 3.18.*

Fig. 3.18

Solution

Step I When the 4 V source is acting alone (Fig. 3.19)

Fig. 3.19

By series–parallel reduction technique (Fig. 3.20),

$$I = \frac{4}{5+7.45} = 0.32 \text{ A}$$

From Fig. 3.20(b), by current-division rule,

$$I_1 = 0.32 \times \frac{15}{15+14.8} = 0.16 \text{ A}$$

From Fig. 3.19, by current-division rule,

$$I' = 0.16 \times \frac{12}{12+8} = 0.096 \text{ A}(\downarrow)$$

Fig. 3.20

Step II When the 6 V source is acting alone (Fig. 3.21)

Fig. 3.21

By series–parallel reduction technique (Fig. 3.22),

$$I = \frac{6}{12+5.06} = 0.35 \text{ A}$$

From Fig. 3.22(b), by current division rule,

$$I'' = 0.35 \times \frac{13.75}{13.75+8} = 0.22 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 0.096 + 0.22 = 0.316 \text{ A}(\downarrow)$$

Fig. 3.22

Example 3.4 *Find the current through the 4 Ω resistor in Fig. 3.23.*

Fig. 3.23

Solution

Step I When the 40 V source is acting alone (Fig. 3.24)

Fig. 3.24

By series–parallel reduction technique (Fig. 3.25),

Fig. 3.25

$$I = \frac{40}{12 + 2.92} = 2.68 \text{ A}$$

From Fig. 3.25(a), by current-division rule,

$$I' = 2.68 \times \frac{5}{5+7} = 1.12 \text{ A}(\rightarrow) = -1.12 \text{ A } (\leftarrow)$$

Step II When the 8 A source is acting alone (Fig. 3.26)

Fig. 3.26

By series–parallel reduction technique (Fig. 3.27),

Fig. 3.27

$$I'' = 8 \times \frac{3}{7.53 + 3} = 2.28 \text{ A}(\leftarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = -1.12 + 2.28 = 1.16 \text{ A } (\leftarrow)$$

Example 3.5 *Find the current in the 10 Ω resistor in Fig. 3.28.*

Fig. 3.28

Solution

Step I When the 10 V source is acting alone (Fig. 3.29)

Fig. 3.29

By series–parallel reduction technique (Fig. 3.30),

Fig. 3.30

$$I = \frac{10}{1+4.12} = 1.95 \text{ A}$$

From Fig. 3.30 (a), by current-division rule,

$$I' = 1.95 \times \frac{7}{7+10} = 0.8 \text{ A}(\downarrow)$$

Step II When the 4 A source is acting alone (Fig. 3.31)

Fig. 3.31

By series–parallel reduction technique (Fig. 3.32),

Fig. 3.32

$$I = 4 \times \frac{5}{2.91+5} = 2.53 \text{ A}$$

From Fig. 3.31, by current-division rule,

$$I'' = 2.53 \times \frac{1}{1+10} - 0.23 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 0.8 + 0.23 = 1.03 \text{ A}(\downarrow)$$

Example 3.6 *Find the current through the 8 Ω resistor in Fig. 3.33.*

Fig. 3.33

Solution

Step I When the 5 A source is acting alone (Fig. 3.34)

$$I' = 5 \times \frac{12}{12+8+30} = 1.2 \text{ A } (\rightarrow)$$

Fig. 3.34

Step II When the 25 A source is acting alone (Fig. 3.35)

$$I'' = 25 \times \frac{30}{30+12+8} = 15 \text{ A } (\rightarrow)$$

Fig. 3.35

Step III By superposition theorem,

$$I = I' + I'' = 1.2 + 15 = 16.2 \text{ A}(\rightarrow)$$

Example 3.7 *Find the current through the 4 Ω resistor in Fig. 3.36.*

Fig. 3.36

Solution

Step I When the 5 A source is acting alone (Fig. 3.37)

Fig. 3.37

By series–parallel reduction technique (Fig. 3.38),

Fig. 3.38

$$I' = 5 \times \frac{8.73}{8.73+4} = 3.43 \text{ A}(\downarrow)$$

Step II When the 20 V source is acting alone (Fig. 3.39)

Fig. 3.39

By series–parallel reduction technique (Fig. 3.40),

Fig. 3.40

$$I = \frac{20}{5+3.75} = 2.29 \text{ A}$$

From Fig. 3.40 (a), by current-division rule,

$$I'' = 2.29 \times \frac{6}{6+10} = 0.86 \text{ A}(\downarrow)$$

Step III By superposition theorem

$$I = I' + I'' = 3.43 + 0.86 = 4.29 \text{ A } (\downarrow)$$

Example 3.8 *Find the current through the 3 Ω resistor in Fig. 3.41.*

Fig. 3.41

Solution

Step I When the 5 A source is acting alone (Fig. 3.42)

Fig. 3.42

By series–parallel reduction technique (Fig. 3.43),

$$I' = 5 \times \frac{15}{15+2+3} = 3.75 \text{ A}(\downarrow)$$

Fig. 3.43

Step II When the 20 V source is acting alone (Fig. 3.44)

Fig. 3.44

By series–parallel reduction technique (Fig. 3.45),

Fig. 3.45

$$I = \frac{20}{3.33} = 6 \text{ A}$$

From Fig. 3.45(a), by current-division rule,

$$I'' = 6 \times \frac{4}{20+4} = 1 \text{ A}(\uparrow) = -1 \text{ A}(\downarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.75 - 1 = 2.75 \text{ A } (\downarrow)$$

Example 3.9 *Find the current in the 1 Ω resistors in Fig. 3.46.*

Fig. 3.46

Solution

Step I When the 4 V source is acting alone (Fig. 3.47)

$$I' = \frac{4}{2+1} = 1.33 \text{ A}(\downarrow)$$

Fig. 3.47

Step II When the 3 A source is acting alone (Fig. 3.48)
By current-division rule,

$$I'' = 3 \times \frac{2}{1+2} = 2 \text{ A } (\downarrow)$$

Fig. 3.48

Step III When the 1 A source is acting alone (Fig. 3.49)

Fig. 3.49

Redrawing the network (Fig. 3.50),
By current-division rule,

$$I''' = 1 \times \frac{2}{2+1} = 0.66 \text{ A}(\downarrow)$$

Fig. 3.50

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 1.33 + 2 + 0.66 = 4 \text{ A } (\downarrow)$$

Example 3.10 *Find the voltage V_{AB} in Fig. 3.51.*

Fig. 3.51

Solution

Step I When the 6 V source is acting alone (Fig. 3.52)

$$V'_{AB} = 6 \text{ V}$$

Fig. 3.52

Step II When the 10 V source is acting alone (Fig. 3.53)
Since the resistor of 5 Ω is shorted, the voltage across it is zero.

$$V''_{AB} = 10 \text{ V}$$

Fig. 3.53

Step III When the 5 A source is acting alone (Fig. 3.54)
Due to short circuit in both the parts,

$$V'''_{AB} = 0$$

Step IV By superposition theorem,

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB} = 6 + 10 + 0 = 16 \text{ V}$$

Fig. 3.54

Example 3.11 *Find the current through the 5 Ω resistor in Fig. 3.55.*

Fig. 3.55

Solution

Step I When the 24 V source is acting alone (Fig. 3.56)

Fig. 3.56

By series–parallel reduction technique (Fig. 3.57),

$$I' = \frac{24}{5+6.67} = 2.06\ \text{A}(\rightarrow) = -2.06\ \text{A}(\leftarrow)$$

Fig. 3.57

Step II When the 2 A source is acting alone
By series–parallel reduction technique (Fig. 3.58),

Fig. 3.58

By current-division rule,

$$I'' = 2 \times \frac{6.67}{5+6.67} = 1.14\ \text{A}(\leftarrow)$$

Step III When the 36 V source is acting alone (Fig. 3.59)
By series–parallel reduction technique,

Fig. 3.59

$$I = \frac{36}{10+4} = 2.57 \text{ A}$$

By current-division rule,

$$I''' = 2.57 \times \frac{20}{20+5} = 2.06 \text{ A}(\leftarrow)$$

Step IV By superposition theorem,

$$I = I' + I'' + I''' = -2.06 + 1.14 + 2.06 = 1.14 \text{ A } (\leftarrow)$$

Example 13.12 *Find the current through the 4 Ω resistor in Fig. 3.60.*

Fig. 3.60

Solution

Step I When the 5 A source is acting alone (Fig. 3.61)
By current-division rule,

$$I' = 5 \times \frac{2}{2+4} = 1.67 \text{ A}(\downarrow)$$

Fig. 3.61

Step II When the 2 A source is acting alone (Fig. 3.62)
By current-division rule,

$$I'' = 2 \times \frac{2}{2+4} = 0.67 \text{ A}(\downarrow)$$

Fig. 3.62

Step III When the 6 V source is acting alone (Fig. 3.63)
Applying KVL to the mesh,

$$-2I''' - 6 - 4I''' = 0$$

$$I''' = -1 \text{ A } (\downarrow)$$

Fig. 3.63

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 1.67 + 0.67 - 1 = 1.34 \text{ A } (\downarrow)$$

EXAMPLES WITH DEPENDENT SOURCES

Example 3.13 *Find the current through the 6 Ω resistor in Fig. 3.64.*

Fig. 3.64

Solution

Step I When the 15 V source is acting alone (Fig 3.65)
From Fig. 3.65,

$$I' = I_1 \qquad \ldots\text{(i)}$$

Fig. 3.65

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3I' = 3I_1$$

$$4I_1 - I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$15 - 6I_1 - 8I_2 = 0$$

$$6I_1 + 8I_2 = 15 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.39\text{A}$$
$$I_2 = 1.59\text{A}$$
$$I' = I_1 = 0.39 \text{ A } (\rightarrow)$$

Step II When the 10 V source is acting alone (Fig 3.66)
From Fig. 3.66,

$$I'' = I_1 \qquad \ldots\text{(i)}$$

Fig. 3.66

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3I'' = 3I_1$$

$$4I_1 - I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-6I_1 - 8I_2 + 10 = 0$$

$$6I_1 + 8I_2 = 10 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.26 \text{ A}$$
$$I_2 = 1.05 \text{ A}$$
$$I'' = I_1 = 0.26 \text{ A} \quad (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 0.39 + 0.26 = 0.65 \text{ A } (\rightarrow)$$

Example 3.14 *Find the current I_x in Fig. 3.67.*

Fig. 3.67

Solution

Step I When the 30 A source is acting alone (Fig. 3.68)
From Fig. 3.68,

$$I_x' = I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_1 - I_2 = 30 \qquad \ldots\text{(ii)}$$

Fig. 3.68

Applying KVL to the outer path of the supermesh,

$$-5I_1 - 1I_2 - 4I_x' = 0$$
$$-5I_1 - I_2 - 4I_1 = 0$$
$$9I_1 + I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 3 \text{ A}$$
$$I_2 = -27 \text{ A}$$
$$I_x' = I_1 = 3 \text{ A } (\rightarrow)$$

Step II When the 20 V source is acting alone (Fig. 3.69)
Applying KVL to the mesh,

$$20 - 5I_x'' - 1I_x'' - 4I_x'' = 0$$
$$I_x'' = 2 \text{ A}(\rightarrow)$$

Step III By superposition theorem,

$$I_x = I_x' + I_x'' = 3 + 2 = 5 \text{ A}(\rightarrow)$$

Fig. 3.69

Example 3.15 *Find the current I_1. in Fig. 3.70.*

Fig. 3.70

Solution

Step I When the 5 V source is acting alone (Fig 3.71)

From the figure,

$$V_x = 5 - 10I_1'$$

Applying KVL to the mesh,

$$5 - 10I_1' - 4V_x - 2I_1' = 0$$

$$5 - 10I_1' - 4\,(5 - 10I_1') - 2I_1' = 0$$

$$5 - 10I_1' - 20 + 40I_1' - 2I_1' = 0$$

$$I_1' = \frac{15}{28} = 0.54 \text{ A}(\uparrow)$$

Fig. 3.71

Step II When the 2 A source is acting alone (Fig 3.72)

From Fig . 3.72,

$$V_x = -10I_1' \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1' = 2 \qquad \ldots\text{(ii)}$$

Fig. 3.72

Applying KVL to the outer path of the supermesh,

$$-10I_1' - 4V_x - 2I_2 = 0$$

$$-10I_1' - 4(-10I_1') - 2I_2 = 0$$

$$30I_1' - 2I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.14 \text{ A } (\uparrow)$$

$$I_2 = 2.14 \text{ A}$$

Step III By superposition theorem,

$$I_1 = I_1' + I_1'' = 0.54 + 0.14 = 0.68 \text{ A } (\uparrow)$$

Example 3.16 *Determine the current through the 10 Ω resistor in Fig. 3.73.*

Fig. 3.73

Solution

Step I When the 100 V source is acting alone (Fig. 3.74)

From the figure,

$$V_x = 5I'$$

Applying KVL to the mesh,

$$100 - 10I' + 10V_x - 5I' = 0$$

$$100 - 10I' + 10(5I') - 5I' = 0$$

$$I' = -2.86 \text{ A } (\rightarrow)$$

Fig. 3.74

Step II When the 10 A source is acting alone (Fig. 3.75)

From Fig. 3.75,

$$V_x = 5(I_1 - I_2) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$-10I_1 + 10V_x - 5(I_1 - I_2) = 0$$

$$-10I_1 + 10\{5(I_1 - I_2)\} - 5(I_1 - I_2) = 0$$

$$35I_1 - 45I_2 = 0 \qquad \ldots\text{(ii)}$$

Fig. 3.75

For Mesh 2,

$$I_2 = -10 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -12.86 \text{ A}$$

$$I_2 = -10 \text{ A}$$

$$I'' = I_1 = -12.86 \text{ A } (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = -2.86 - 12.86 = -15.72 \text{ A } (\rightarrow)$$

Example 3.17 *Find the current I in the network of Fig. 3.76.*

Fig. 3.76

Solution

Step I When the 17 V source is acting alone (Fig. 3.77)

From the figure,

$$V_x = -2I'$$

Applying KVL to the mesh,

$$-2I' - 17 - 3I' - 5V_x = 0$$

$$-2I' - 17 - 3I' - 5(-2I') = 0$$

$$I' = 3.4 \text{ A } (\rightarrow)$$

Fig. 3.77

Step II When the 1 A source is acting alone (Fig. 3.78)

From Fig. 3.78,

$$V_x = -2I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(ii)}$$

Fig. 3.78

Applying KVL to the outer path of the supermesh,

$$-2I_1 - 3I_2 - 5V_x = 0$$

$$-2I_1 - 3I_2 - 5(-2I_1) = 0$$

$$8I_1 - 3I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.6 \text{ A}$$

$$I_2 = 1.6 \text{ A}$$

$$I'' = I_2 = 1.6 \text{ A } (\rightarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.4 + 1.6 = 5 \text{ A } (\rightarrow)$$

Example 3.18

Find the voltage V_1 in Fig. 3.79.

Fig. 3.79

Solution

Step I When the 5 A source is acting alone (Fig. 3.80)

From Fig. 3.80,

$$I = \frac{V_1'}{4}$$

Applying KCL at Node 1,

$$\frac{V_1' - 4I}{1} + \frac{V_1'}{4} = 5$$

Fig. 3.80

$$V_1' - 4\left(\frac{V_1'}{4}\right) + \frac{V_1'}{4} = 5$$

$$V_1' = 20 \text{ V}$$

Step II When the 20 V source is acting alone (Fig. 3.81)
Applying KVL to the mesh,

$$4I - I - 4I - 20 = 0$$

$$I = -20 \text{ A}$$

$$V_1'' = 4I - 1(I) = 3I = 3\,(-20) = -60 \text{ V}$$

Fig. 3.81

Step III By superposition theorem,

$$V_1 = V_1' + V_1'' = 20 - 60 = -40 \text{ V}$$

Example 3.19 *Find the current in the 6 Ω resistor in Fig. 3.82.*

Fig. 3.82

Solution

Step I When the 18 V source is acting alone (Fig. 3.83)
From Fig. 3.83,

$$V_x = -I'$$

Applying KVL to the mesh,

$$18 - I' + 2V_x - 6I' = 0$$

$$18 - I' - 2I' - 6I' = 0$$

$$I' = 2 \text{ A } (\downarrow)$$

Fig. 3.83

Step II When the 3 A source is acting alone (Fig. 3.84)
From Fig. 3.84,

$$V_x = -1\,I_1 = -I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.
Writing current equation for the supermesh,

$$I_2 - I_1 = 3 \qquad \ldots\text{(ii)}$$

Fig. 3.84

Applying KVL to the outerpath of the supermesh,

$$-1I_1 + 2V_x - 6I_2 = 0$$

$$-I_1 + 2(-I_1) - 6I_2 = 0$$

$$3I_1 + 6I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -2 \text{ A}$$
$$I_2 = 1 \text{ A}$$
$$I'' = I_2 = 1 \text{ A } (\downarrow)$$

Step III By superposition theorem,

$$I_{6\,\Omega} = I' + I'' = 2 + 1 = 3 \text{ A } (\downarrow)$$

Example 3.20 *Find the current I_y in Fig. 3.85.*

Fig. 3.85

Solution

Step I When the 120 V source is acting alone (Fig. 3.86)

Applying KVL to the mesh,

$$120 - 4I_y' - 10I_y' - 8I_y' = 0$$

$$I_y' = 5.45 \text{ A } (\rightarrow)$$

Fig. 3.86

Step II When the 12 A source is acting alone (Fig. 3.87)

From Fig. 3.87,

$$I_y'' = I_1 \qquad \ldots(i)$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 12 \qquad \ldots(ii)$$

Fig. 3.87

Applying KVL to the outer path of the supermesh,

$$-4I_1 - 10I_y'' - 8I_2 = 0$$
$$-4I_1 - 10I_1 - 8I_2 = 0$$
$$14I_1 + 8I_2 = 0 \qquad \ldots(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = -4.36 \text{ A}$$
$$I_2 = 7.64 \text{ A}$$
$$Iy'' = I_1 = -4.36 \text{ A } (\rightarrow)$$

Step III When the 40 V source is acting alone (Fig. 3.88)

Applying KVL to the mesh,

$$-4\,I_y''' - 10I_y''' - 8I_y''' - 40 = 0$$

$$I_y''' = -\frac{40}{22} = -1.82 \text{ A } (\rightarrow)$$

Fig. 3.88

Step IV By superposition theorem,

$$I_y = I_y' + I_y'' + I_y''' = 5.45 - 4.36 - 1.82 = -0.73 \text{ A } (\rightarrow)$$

Example 3.21 *Find the voltage V_x in Fig. 3.89.*

Fig. 3.89

Solution

Step I When the 18 V source is acting alone (Fig. 3.90)

From Fig. 3.90,

$$V_x' = 3I$$

Applying KVL to the mesh,

$$18 - 3I - 6I - 3V_x' = 0$$
$$18 - 3I - 6I - 3\,(3I) = 0$$
$$I = 1 \text{ A}$$
$$V_x' = 3 \text{ V}$$

Fig. 3.90

Step II When the 5 A source is acting alone (Fig. 3.91)

From Fig. 3.91,

$$V_x'' = -3I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 5 \qquad \ldots\text{(ii)}$$

Fig. 3.91

Applying KVL to the outer path of the supermesh,

$$-3I_1 - 6I_2 - 3V_x'' = 0$$
$$-3I_1 - 6I_2 - 3(3I_1) = 0$$
$$12I_1 + 6I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -1.67\text{A}$$
$$I_2 = 3.33 \text{ A}$$
$$V_x'' = 3I_1 = 3(-1.67) = -5 \text{ V}$$

Step III When the 36 V source is acting alone (Fig. 3.92)

From the figure,

$$V_x''' = -3I$$

Applying KVL to the mesh,

$$36 + 3V_x''' - 6I - 3I = 0$$

Fig. 3.92

$$36 + 3V_x''' - 6\left(\frac{-V_x'''}{3}\right) - 3\left(\frac{-V_x'''}{3}\right) = 0$$

$$36 + 3V_x''' + 2V_x''' + V_x''' = 0$$

$$V_x''' = -6 \text{ V}$$

Step IV By superposition theorem,

$$V_x = V_x' + V_x'' + V_x''' = 3 - 5 - 6 = -8 \text{ V}$$

Example 3.22 *Find the voltage V in the network of Fig. 3.93.*

Fig. 3.93

Solution

Step I When the 10 V source is acting alone (Fig. 3.94)

From Fig. 3.94,

$$V' = -8I_1 \quad \text{...(i)}$$

Fig. 3.94

Applying KVL to Mesh 1,

$$-10 - 8I_1 - 15\ I_1 - 12\ (I_1 - I_2) = 0$$

$$35I_1 - 12I_2 = -10 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-12(I_2 - I_1) - 5I_2 - 8V' = 0$$

$$-12I_2 + 12I_1 - 5I_2 - 8(-8I_1) = 0$$

$$76I_1 - 17I_2 = 0 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.54 \text{ A}$$

$$I_2 = 2.4 \text{ A}$$

$$V' = -8I_1 = -8(0.54) = -4.32 \text{ V}$$

Step II When the –5 A source is acting alone (Fig. 3.95)

From Fig. 3.95,

$$V'' = -8I_1 \quad \text{...(i)}$$

Fig. 3.95

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_1 - I_2 = -5 \quad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-8I_1 - 15I_2 - 12(I_2 - I_3) = 0$$

$$-8I_1 - 27I_2 + 12I_3 = 0 \qquad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-12(I_3 - I_2) - 5I_3 - 8V'' = 0$$

$$-12I_3 + 12I_2 - 5I_3 - 8(-8I_1) = 0$$

$$64I_1 + 12I_2 - 17I_3 = 0 \qquad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 4.97 \text{ A}$$

$$I_2 = 9.97 \text{ A}$$

$$I_3 = 25.74 \text{ A}$$

$$V'' = -8I_1 = -8(-4.97) = -39.76 \text{ V}$$

Step III By superposition theorem,

$$V = V' + V'' = -4.32 - 39.76 = -44.08 \text{ V}$$

Example 3.23 *For the network shown in Fig. 3.96, find the voltage V_0.*

Fig. 3.96

Solution

Step I When the 1 A source is acting alone (Fig. 3.97)

From Fig. 3.97,

$$V_1 = 200\,I_2 \qquad \text{...(i)}$$

For Mesh 1,

$$I_1 = 1 \qquad \text{...(ii)}$$

Fig. 3.97

Applying KVL to Mesh 2,

$$0.5V_1 - 40(I_2 - I_1) - 200\,I_2 = 0$$

$$0.5(200I_2) - 40I_2 + 40I_1 - 200I_2 = 0$$

$$40I_1 - 140\,I_2 = 0 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 1 \text{ A}$$

$$I_2 = 0.29 \text{ A}$$

$$V_0' - 50\ I_1 - 200I_2 = 0$$

$$V_0' - 50(1) - 200(0.29) = 0$$

$$V_0' = 108 \text{ V}$$

Step II When the 25 V source is acting alone (Fig. 3.98)
From Fig. 3.98,

$$V_1 - 200I - 25 = 0$$

$$V_1 = 200I + 25 \quad ...(i)$$

Fig. 3.98

Applying KVL to Mesh 1,

$$0.5V_1 - 40I - 200I - 25 = 0$$

$$0.5(200I + 25) - 40I - 200I - 25 = 0$$

$$I = -0.09 \text{ A}$$

$$V_0'' = V_1 = 200I + 25 = 200(-0.09) + 25 = 7 \text{ V}$$

Step III By superposition theorem,

$$V_0 = V_0' + V_0'' = 108 + 7 = 115 \text{ V}$$

Example 3.24 *For the network shown in Fig. 3.99, find the voltage V_x.*

Fig. 3.99

Solution

Step I When the 20 V source is acting alone (Fig. 3.100)
From Fig. 3.100,

$$V_x' = 6(I_1 - I_2) \quad ...(i)$$

Fig. 3.100

Applying KVL to Mesh 1,

$$20 - 2I_1 - 6(I_1 - I_2) = 0$$

$$8I_1 - 6I_2 = 20 \quad ...(ii)$$

For Mesh 2,

$$I_2 = \frac{V_x'}{2} = \frac{6(I_1 - I_2)}{2} = 3I_1 - 3I_2$$

$$3I_1 - 4I_2 = 0 \quad ...(iii)$$

Solving Eqs (ii) and (iii),

$$I_1 = 5.71 \text{ A}$$

$$I_2 = 4.29 \text{ A}$$

$$V_x' = 6(I_1 - I_2) = 6(5.71 - 4.29) = 8.52 \text{ V}$$

Step II When the 10 A source is acting alone (Fig. 3.101)
From Fig. 3.101,

$$V_x'' = 6(I_1 - I_2) \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-2(I_1 - I_3) - 6(I_1 - I_2) = 0$$

$$8I_1 - 6I_2 - 2I_3 = 0 \quad \text{...(ii)}$$

For Mesh 2,

$$I_2 = \frac{V_x''}{2} = \frac{6(I_1 - I_2)}{2} = 3I_1 - 3I_2$$

$$3I_1 - 4I_2 = 0 \quad \text{...(iii)}$$

For Mesh 3,

$$I_3 = -10 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -5.71 \text{ A}$$

$$I_2 = -4.29 \text{ A}$$

$$I_3 = -10 \text{ A}$$

$$V_x'' = 6(I_1 - I_2) = 6(-5.71 + 4.29) = -8.52 \text{ V}$$

Fig. 3.101

Step III By superposition theorem,

$$V_x = V_x' + V_x'' = 8.52 - 8.52 = 0$$

Example 3.25

Calculate the current I in the network shown in Fig. 3.102.

Fig. 3.102

Solution

Step I When the 70 V source is acting alone (Fig. 3.103)
From Fig. 3.103,

$$I' = I_3 \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$70 - 20(I_2 - I_1) - 2(I_2 - I_3) = 0$$

$$-20I_1 + 22I_2 - 2I_3 = 70 \quad \text{...(iii)}$$

Fig. 3.103

Applying KVL to Mesh 3,

$$-2(I_3 - I_2) + 2I_1 - 10I_3 = 0$$

$$2I_1 + 2I_2 - 12I_3 = 0 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 8.94\,\text{A}$$
$$I_2 = 11.62\,\text{A}$$
$$I_3 = 3.43\,\text{A}$$
$$I' = I_3 = 3.43\,\text{A}\,(\leftarrow)$$

Step II When the 50 V source is acting alone (Fig. 3.104)
From Fig. 3.104,

$$I'' = I_3 \quad \text{...(i)}$$

Fig. 3.104

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 20(I_1 - I_2) = 0$$

$$26I_1 - 20I_2 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-20(I_2 - I_1) - 2(I_2 - I_3) = 0$$

$$-20I_1 + 22I_2 - 2I_3 = 0 \quad \text{...(iii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_2) + 2I_1 + 50 - 10I_3 = 0$$

$$2I_1 + 2I_2 - 12I_3 = -50 \quad \text{...(iv)}$$

Solving Eqs (ii), (iii), and (iv),

$$I_1 = 1.06\ \text{A}$$
$$I_2 = 1.38\ \text{A}$$
$$I_3 = 4.57\ \text{A}$$
$$I'' = I_3 = 4.57\ \text{A}\ (\leftarrow)$$

Step III By superposition theorem,

$$I = I' + I'' = 3.43 + 4.57 = 8\ \text{A}\ (\leftarrow)$$

Example 3.26 *Find the voltage V_0 in the network of Fig. 3.105.*

Fig. 3.105

Solution

Step I When the 10 V source is acting alone (Fig. 3.106)
Applying KCL at the node,

Fig. 3.106

$$\frac{V_0'-10}{5}+\frac{V_0'}{2}+\frac{V_0'-\frac{V_0'}{2}}{1}=0$$

$$\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{2}\right)V_0'=2$$

$$V_0'=1.67 \text{ V}$$

Step II When the 1A current source is acting alone (Fig. 3.107)
Applying KCL at the node,

Fig. 3.107

$$\frac{V_0''}{5}+1+\frac{V_0''}{2}+\frac{V_0''-\frac{V_0''}{2}}{1}$$

$$\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{2}\right)V_0''=-1$$

$$V_0''=-0.83 \text{ V}$$

Step III When the 4 V source is acting alone (Fig. 3.108)
Applying KCL at the node,

Fig. 3.108

$$\frac{V_0'''}{5}+\frac{V_0'''}{2}+\frac{V_0'''-4-\frac{V_0'''}{2}}{1}=0$$

$$\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{2}\right)V_0'''=4$$

$$V_0'''=3.33 \text{ V}$$

Step IV By superposition theorem,

$$V_0=V_0'+V_0''+V_0'''=1.67-0.83+3.33=4.17 \text{ V}$$

3.3 THEVENIN'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent voltage source and an equivalent series resistance. The voltage source is the voltage across the two terminals with load, if any, removed. The series resistance is the resistance of the network measured between two terminals with load removed and constant voltage source being replaced by its internal resistance (or if it is not given with zero resistance, i.e., short circuit) and constant current source replaced by infinite resistance, i.e., open circuit.*'

Fig. 3.109 *Network illustrating Thevenin's theorem*

Explanation Consider a simple network as shown in Fig. 3.110.

Fig. 3.110 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate open circuit voltage V_{Th} across points A and B as shown in Fig. 3.111.

$$V_{\text{Th}} = \frac{R_2}{R_1 + R_2} \text{ V}$$

Fig. 3.111 *Calculation of V_{Th}*

For finding series resistance R_{Th}, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 3.112.

$$R_{\text{Th}} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

Fig. 3.112 *Calculation of R_{Th}*

Thevenin's equivalent network is shown in Fig. 3.113.

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L}$$

Fig 3.113 *Thevenin's equivalent network*

If the network contains both independent and dependent sources, Thevenin's resistance R_{Th} is calculated as,

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N}$$

where I_N is the short-circuit current which would flow in a short circuit placed across the terminals A and B. Dependent sources are active at all times. They have zero values only when the control voltage or current is zero. R_{Th} may be negative in

some cases which indicates negative resistance region of the device, i.e., as voltage increases, current decreases in the region and vice-versa.

If the network contains only dependent sources then

$$V_{\text{Th}} = 0$$
$$I_N = 0$$

For finding R_{Th} in such a network, a known voltage V is applied across the terminals A and B and current is calculated through the path AB.

$$R_{\text{Th}} = \frac{V}{I}$$

or a known current source I is connected across the terminals A and B and voltage is calculated across the terminals A and B.

$$R_{\text{Th}} = \frac{V}{I}$$

Thevenin's equivalent network for such a network is shown in Fig. 3.114.

Fig. 3.114 *Thevenin's equivalent network*

Steps to be Followed in Thevenin's Theorem

1. Remove the load resistance R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Replace the network by a voltage source V_{Th} in series with resistance R_{Th}.
5. Find the current through R_L using Ohm's law.

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L}$$

Example 3.27 *Find the current through the 2 Ω resistor in Fig. 3.115.*

Fig. 3.115

Solution

Step I Calculation of V_{Th} (Fig. 3.116)

Applying KVL to the mesh,

$$40 - 5I - 20 - 10I = 0$$
$$15I = 20$$
$$I = 1.33 \text{ A}$$

Fig. 3.116

Writing the V_{Th} equation,

$$10I - V_{Th} + 10 = 0$$

$$V_{Th} = 10I + 10 = 10(1.33) + 10 = 23.33 \text{ V}$$

Fig. 3.117

Step II Calculation of R_{Th} (Fig. 3.117)

$$R_{Th} = 5 \,||\, 10 = 3.33\ \Omega$$

Step III Calculation of I_L (Fig. 3.118)

$$I_L = \frac{23.33}{3.33 + 2} = 4.38 \text{ A}$$

Fig. 3.118

Example 3.28 *Find the current through the 8 Ω resistor in Fig. 3.119.*

Fig. 3.119

Solution

Step I Calculation of V_{Th} (Fig. 3.120)

$$I = \frac{250}{5+5} = 25 \text{ A}$$

Fig. 3.120

Writing the V_{Th} equation,

$$250 - 5I - V_{Th} - 75 = 0$$

$$V_{Th} = 175 - 5I = 175 - 5(25) = 50 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.121)

$$R_{Th} = (5 \,||\, 5) + 10 = 12.5\ \Omega$$

Fig. 3.121

Step III Calculation of I_L (Fig. 3.122)

$$I_L = \frac{50}{12.5+8} = 2.44 \text{ A}$$

Fig. 3.122

Example 3.29 *Find the current through the 2 Ω resistor connected between terminals A and B in the Fig. 3.123.*

Fig. 3.123

Solution

Step I Calculation of V_{Th} (Fig. 3.124)

Applying KVL to Mesh 1,

$$2 - 2I_1 - 12(I_1 - I_2) = 0$$
$$14I_1 - 12I_2 = 2 \quad ...(i)$$

Applying KVL to Mesh 2,

$$-12(I_2 - I_1) - 1I_2 - 3I_2 - 4 = 0$$
$$-12I_1 + 16I_2 = -4 \quad ...(ii)$$

Fig. 3.124

Solving Eqs (i) and (ii),

$$I_2 = -0.4 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 3I_2 - 4 = 0$$
$$V_{Th} = 4 + 3I_2 = 4 + 3(-0.4) = 2.8 \text{ V}$$

Fig. 3.125

Step II Calculation of R_{Th} (Fig. 3.125)

$$R_{Th} = [(2 \,||\, 12) + 1] \,||\, 3 = 1.43 \,\Omega$$

Step III Calculation of I_L (Fig. 3.126)

$$I_L = \frac{2.8}{1.43+2} = 0.82 \text{ A}$$

Fig. 3.126

Example 3.30 *Find the current through the 10 Ω resistor in Fig. 3.127.*

Fig. 3.127

Solution

Step I Calculation of V_{Th} (Fig. 3.128)

Applying KVL to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$
$$7I_1 - I_2 = 10 \quad \text{.....(i)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3I_2 = 0$$
$$-I_1 + 6I_2 = 0 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = 0.24 \text{ A} \quad \text{...(ii)}$$

Writing the V_{Th} equation,

$$3I_2 - V_{Th} - 20 = 0$$
$$V_{Th} = 3I_2 - 20 = 3(0.24) - 20 = -19.28 \text{ V}$$
$$= 19.28 \text{ V (terminal } B \text{ is positive w.r.t } A)$$

Fig. 3.128

Fig. 3.129

Step II Calculation of R_{Th} (Fig. 3.129)

$$R_{Th} = [(6 \parallel 1) + 2] \parallel 3 = 1.47\ \Omega$$

Step III Calculation of I_L (Fig. 3.130)

$$I_L = \frac{19.28}{1.47 + 10} = 1.68 \text{ A}(\uparrow)$$

Fig. 3.130

Example 3.31 *Find the current through the 10 Ω resistor in Fig. 3.131.*

Fig. 3.131

Solution

Step I Calculation of V_{Th} (Fig. 3.132)
For Mesh 1,

$$I_1 = 10$$

Applying KVL to Mesh 2,

$$100 - 30I_2 - 20I_2 = 0$$

$$I_2 = 2 \text{ A}$$

Writing the V_{Th} equation,

$$5I_1 - V_{Th} - 20I_2 = 0$$

$$V_{Th} = 5I_1 - 20I_2 = 5(10) - 20(2) = 10 \text{ V}$$

Fig. 3.132

Step II Calculation of R_{Th} (Fig. 3.133)

$$R_{Th} = 5 + (20 \parallel 30) = 17\ \Omega$$

Fig. 3.133

Step III Calculation of I_L (Fig. 3.134)

$$I_L = \frac{10}{17+10} = 0.37 \text{ A}$$

Fig. 3.134

Example 3.32 *Find the current through the 40 Ω resistor in Fig. 3.135.*

Fig. 3.135

Solution

Step I Calculation of V_{Th} (Fig. 3.136)

Since the 20 Ω resistor is connected across the 25 V source, the resistor becomes redundant.

$$V_{20\,\Omega} = 25 \text{ V}$$

Fig. 3.136

Applying KVL to the mesh,

$$25 - 50I - 10I + 10 = 0$$
$$I = 0.58 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 10I + 10 = 0$$
$$V_{Th} = 10(I) - 10 = 10(0.58) - 10 = -4.2 \text{ V}$$
$$= 4.2 \text{ V(terminal } B \text{ is positive w.r.t } A)$$

Step II Calculation of R_{Th} (Fig. 3.137)

Fig. 3.137

$$R_{Th} = 50 \,||\, 10 = 8.33\ \Omega$$

Step III Calculation of I_L (Fig. 3.138)

$$I_L = \frac{4.2}{8.33 + 40} = 0.09 \text{ A}(\uparrow)$$

Fig. 3.138

Example 3.33 *Find the current through the 10 Ω resistor in Fig. 3.139.*

Fig. 3.139

Solution

Step I Calculation of V_{Th} (Fig. 3.140)

$$I_1 = \frac{50}{6+4} = 5 \text{ A}$$

$$I_2 = \frac{20}{5+15} = 1 \text{ A}$$

Fig. 3.140

Writing the V_{Th} equation,

$$4I_1 + 2 - V_{Th} - 15I_2 = 0$$

$$V_{Th} = 4I_1 + 2 - 15I_2 = 4(5) + 2 - 15(1) = 7 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.141)

$$R_{Th} = (6 \parallel 4) + (5 \parallel 15) = 6.15\ \Omega$$

Fig. 3.141

Step III Calculation of I_L (Fig. 3.142)

$$I_L = \frac{7}{6.15 + 10} = 0.43 \text{ A}$$

Fig. 3.142

Example 3.34 *Determine the current through the 24 Ω resistor in Fig. 3.143.*

Fig. 3.143

Solution

Step I Calculation of V_{Th} (Fig. 3.144)

$$I_1 = \frac{220}{30 + 50} = 2.75 \text{ A}$$

$$I_2 = \frac{220}{20 + 5} = 8.8 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} + 30I_1 - 20I_2 = 0$$

$$V_{Th} = 20I_2 - 30I_1 = 20(8.8) - 30(2.75) = 93.5 \text{ V}$$

Fig. 3.144

Step II Calculation of R_{Th} (Fig. 3.145)

Fig. 3.145

Redrawing the circuit (Fig. 3.146),

$$R_{Th} = (30 \parallel 50) + (20 \parallel 5) = 22.75\ \Omega$$

Fig. 3.146

Step III Calculation of I_L (Fig. 3.147)

$$I_L = \frac{93.5}{22.75 + 24} = 2\ \text{A}$$

Fig. 3.147

Example 3.35 *Find the current through the 3 Ω resistor in Fig. 3.148.*

Fig. 3.148

Solution

Step I Calculation of V_{Th} (Fig. 3.149)

Applying KVL to Mesh 1,

$$50 - 2I_1 - 1(I_1 - I_2) - 8(I_1 - I_2) = 0$$
$$11I_1 - 9I_2 = 50 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-4I_2 - 5I_2 - 8(I_2 - I_1) - 1(I_2 - I_1) = 0$$
$$-9I_1 + 18I_2 = 0 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 7.69 \text{ A}$$
$$I_2 = 3.85 \text{ A}$$

Fig. 3.149

Writing the V_{Th} equation,

$$V_{Th} - 5I_2 - 8(I_2 - I_1) = 0$$
$$V_{Th} = 5I_2 + 8(I_2 - I_1) = 5(3.85) + 8(3.85 - 7.69) = -11.47 \text{ V}$$
$$= 11.47 \text{ V(the terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 3.150)

Fig. 3.150

Redrawing the network (Fig. 3.151),

Fig. 3.151

Converting the upper delta into equivalent star network (Fig. 3.152),

$$R_1 = \frac{4\times 2}{4+2+5} = 0.73\ \Omega$$

$$R_2 = \frac{4\times 5}{4+2+5} = 1.82\ \Omega$$

$$R_3 = \frac{5\times 2}{4+2+5} = 0.91\ \Omega$$

Fig. 3.152

Fig. 3.153

Simplifying the network (Fig. 3.153),

$$R_{\text{Th}} = 1.82 + (1.73 \parallel 8.91) = 3.27\ \Omega$$

Fig. 3.154

Step III Calculation of I_L (Fig. 3.155)

$$I_L = \frac{11.47}{3.27+3} = 1.83\ \text{A}(\uparrow)$$

Fig. 3.155

Example 3.36 *Find the current through the 20 Ω resistor in Fig. 3.156.*

Fig. 3.156

Solution

Step I Calculation of V_{Th} (Fig. 3.157)

Applying KVL to Mesh 1,

$$45 - 120 - 15I_1 - 5(I_1 - I_2) - 10(I_1 - I_2) = 0$$

$$30I_1 - 15I_2 = -75 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$20 - 5I_2 - 10(I_2 - I_1) - 5(I_2 - I_1) = 0$$

$$-15I_1 + 20I_2 = 20 \quad \text{...(ii)}$$

Fig. 3.157

Solving Eqs (i) and (ii),

$$I_1 = -3.2 \text{ A}$$

$$I_2 = -1.4 \text{ A}$$

Writing the V_{Th} equation,

$$45 - V_{Th} - 10(I_1 - I_2) = 0$$

$$V_{Th} = 45 - 10(I_1 - I_2) = 45 - 10[-3.2 - (-1.4)] = 63 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.158)

Fig. 3.158

Converting the delta formed by resistors of 10 Ω, 5 Ω and 5 Ω into an equivalent star network (Fig. 3.159),

$$R_1 = \frac{10 \times 5}{20} = 2.5\ \Omega$$

$$R_2 = \frac{10 \times 5}{20} = 2.5\ \Omega$$

$$R_3 = \frac{5 \times 5}{20} = 1.25\ \Omega$$

Fig. 3.159

Fig. 3.160

Simplifying the network (Fig. 3.160),

$$R_{Th} = (16.25 \,||\, 2.5) + 2.5 = 4.67\ \Omega$$

Fig. 3.161

Step III Calculation of I_L

$$I_L = \frac{63}{4.67 + 20} = 2.55\ \text{A}$$

Fig. 3.162

Example 3.37 *Find the current through the 3 Ω resistor in Fig. 3.163.*

Fig. 3.163

Solution

Step I Calculation of V_{Th} (Fig. 3.164)

For Mesh 1,

$$I_1 = 6 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$42 - 12(I_2 - I_1) - 6I_2 = 0$$

$$-12I_1 + 18I_2 = 42 \qquad \ldots\text{(ii)}$$

Fig. 3.164

Solving Eqs (i) and (ii),

$$I_2 = 6.33 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} = 6I_2 = 38 \text{ V}$$

Fig. 3.165

Step II Calculation of R_{Th} (Fig. 3.165)

$$R_{Th} = 6 \,||\, 12 = 4\,\Omega$$

Step III Calculation of I_L (Fig. 3.166)

$$I_L = \frac{38}{4+3} = 5.43 \text{ A}$$

Fig. 3.166

Example 3.38

Find the current through the 30 Ω resistor in Fig. 3.167.

Fig. 3.167

Solution

Step I Calculation of V_{Th} (Fig. 3.168)

Meshes 1 and 2 form a supermesh.

Writing current equation for supermesh,

$$I_2 - I_1 = 13 \qquad \text{...(i)}$$

Writing voltage equation for supermesh,

$$150 - 15I_1 - 60I_2 - 40I_2 = 0$$

$$15I_1 + 100I_2 = 150 \qquad \text{...(ii)}$$

Fig. 3.168

Solving Eqs (i) and (ii),

$$I_1 = -10 \text{ A}$$

$$I_2 = 3 \text{ A}$$

Writing the V_{Th} equation,

$$40I_2 - V_{Th} - 50 = 0$$

$$V_{Th} = 40I_2 - 50 = 40(3) - 50 = 70 \text{ V}$$

Fig. 3.169

Step II Calculation of R_{Th} (Fig. 3.169)

$$R_{Th} = 75 \,||\, 40 = 26.09\,\Omega$$

Step III Calculation of I_L (Fig. 3.170)

$$I_L = \frac{70}{26.09 + 30} = 1.25\,\text{A}$$

Fig. 3.170

Example 3.39 *Find the current through the 20 Ω resistor in Fig. 3.171.*

Fig. 3.171

Solution

Step I Calculation of V_{Th} (Fig. 3.172)

$$V_{\text{Th}} = 100\,\text{V}$$

Fig. 3.172

Step II Calculation of R_{Th} (Fig.3.173)

Fig. 3.173

$$R_{\text{Th}} = 0$$

Step III Calculation of I_L (Fig. 3.174)

$$I_L = \frac{100}{20} = 5\,\text{A}$$

Fig. 3.174

Example 3.40 *Find the current through the 20 Ω resistor in Fig. 3.175.*

Fig. 3.175

Solution

Step I Calculation of V_{Th} (Fig. 3.176)

$$I_1 = \frac{10}{10+4} = 0.71\text{ A}$$

$$I_2 = 2\text{ A}$$

Fig. 3.176

Writing the V_{Th} equation,

$$4I_1 - V_{Th} + 8I_2 = 0$$

$$V_{Th} = 4(0.71) + 8(2) = 18.84\text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.177).

Fig. 3.177

$$R_{Th} = 10.86\ \Omega$$

Step III Calculation of I_L (Fig. 3.178)

$$I_L = \frac{18.84}{10.86+20} = 0.61\text{A}$$

Fig. 3.178

Example 3.41 *Find the current through the 5 Ω resistor in Fig. 3.179.*

Fig. 3.179

Solution

Step I Calculation of V_{Th} (Fig. 3.180)

Applying KVL to Mesh 1,

$$100-10I_1+50-2I_1-2(I_1-I_2)=0$$
$$14I_2-2I_2=150 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-2(I_2-I_1)+50-3I_2=0$$
$$-2I_1+5I_2=50 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1=12.88 \text{ A}$$
$$I_2=15.15 \text{ A}$$

Fig. 3.180

Writing the V_{Th} equation,

$$100-10I_1-V_{Th}=0$$
$$V_{Th}=100-10(12.88)=-28.8 \text{ V}$$
$$=28.8 \text{ V(terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 3.181)

$$R_{Th}=10 \parallel 3.2=2.42\ \Omega$$

Fig. 3.181

Step III Calculation of I_L (Fig. 3.182)

$$I_L = \frac{28.8}{2.42+5} = 3.88 \text{ A } (\uparrow)$$

Fig. 3.182

Example 3.42

Find the current through the 10 Ω resister in Fig. 3.183.

Fig. 3.183

Solution

Step I Calculation of V_{Th} (Fig. 3.183)

Applying KVL to Mesh 1,

$$-15 - 2I_1 - 1(I_1 - I_2) - 10 - 1I_1 = 0$$
$$4I_1 - I_2 = -25 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$10 - 1(I_2 - I_1) - 2I_2 - 1I_2 = 0$$
$$-I_1 + 4I_2 = 10 \quad \text{...(ii)}$$

Fig. 3.184

Solving Eqs (i) and (ii),

$$I_1 = -6 \text{ A}$$
$$I_2 = 1 \text{ A}$$

Writing the V_{Th} equation,

$$-V_{Th} + 2I_2 + 2I_1 = 0$$
$$V_{Th} = 2I_1 + 2I_2 = 2(-6) + 2(1) = -10 \text{ V}$$
$$= 10 \text{ V (the terminal } B \text{ is positive w.r.t. } A)$$

Step II Calculation of R_{Th} (Fig. 3.185)

Fig. 3.185

Converting the star network formed by resistors of $2\,\Omega$, $2\,\Omega$ and $1\,\Omega$ into an equivalent delta network (Fig. 3.186),

$$R_1 = 2 + 2 + \frac{2 \times 2}{1} = 8\,\Omega$$

$$R_2 = 2 + 1 + \frac{2 \times 1}{2} = 4\,\Omega$$

$$R_3 = 2 + 1 + \frac{2 \times 1}{2} = 4\,\Omega$$

Fig. 3.186

Simplifying the network (Fig. 3.187),

Fig. 3.187

$$R_{Th} = 1.33\,\Omega$$

Step III Calculation of I_L (Fig. 3.188)

$$I_L = \frac{10}{1.33+10} = 0.88 \text{ A } (\uparrow)$$

Fig. 3.188

Example 3.43 *Find the current through the 1 Ω resistor in Fig. 3.189.*

Fig. 3.189

Solution

Step I Calculation of V_{Th} (Fig. 3.190)

Fig. 3.190

Writing the current equations for Meshes 1 and 2,

$$I_1 = -3$$

$$I_2 = 1$$

Writing the V_{Th} equation,

$$4 - 2(I_1 - I_2) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 4 - 2(I_1 - I_2) = 4 - 2(-4) = 12 \text{ V}$$

Fig. 3.191

Step II Calculation of R_{Th} (Fig. 3.191)

$$R_{\text{Th}} = 2\ \Omega$$

Step III Calculation of I_L (Fig. 3.192)

$$I_L = \frac{12}{2+1} = 4 \text{ A}$$

Fig. 3.192

Example 3.44 *Find the current through the 3 Ω resistor in Fig. 3.193.*

Fig. 3.193

Solution

Step I Calculation of V_{Th} (Fig. 3.194)

Fig. 3.194

By source transformation (Fig. 3.195),

Fig. 3.195

Applying KVL to the mesh,

$$10 - 2I - 2I - 20 = 0$$
$$4I = -10$$
$$I = -2.5\,\text{A}$$

Writing the V_{Th} equation,

$$10 - 2I - V_{Th} = 0$$
$$V_{Th} = 10 - 2I = 10 - 2(-2.5) = 15\,\text{V}$$

Fig. 3.196

Step II Calculation of R_{Th} (Fig. 3.196)

$$R_{Th} = (2 \,||\, 2) + 1 = 1 + 1 = 2\,\Omega$$

Step III Calculation of I_L (Fig. 3.197)

$$I_L = \frac{15}{2+3} = 3\,\text{A}$$

Fig. 3.197

EXAMPLES WITH DEPENDENT SOURCES

Example 3.45 *Obtain the Thevenin equivalent network for the given network of Fig. 3.198 at terminals A and B.*

Fig. 3.198

Solution

Step I Calculation of V_{Th} (Fig. 3.199)

From Fig. 3.199,

$$I_1 = -2I_1$$
$$3I_1 = 0$$
$$I_1 = 0$$

Fig. 3.199

Writing the V_{Th} equation,

$$8 - 0 - V_{Th} = 0$$
$$V_{Th} = 8 \text{ V}$$

Step II Calculation of I_N (Fig. 3.200),

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 2I_1$$
$$3I_1 - I_2 = 0 \qquad \text{...(i)}$$

Fig. 3.200

Applying KVL to the outer path of the supermesh,

$$8 - 4I_1 = 0$$
$$I_1 = 2 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = 6 \text{ A}$$
$$I_N = I_2 = 6 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{8}{6} = 1.33\ \Omega$$

Fig. 3.201

Step IV Thevenin's Equivalent Network (Fig. 3.201)

Example 3.46 *Find Thevenin's equivalent network of Fig. 3.202.*

Fig. 3.202

Solution

Step I Calculation of V_{Th} (Fig. 3.203)

Fig. 3.203

$$V_x = V_{Th}$$

$$I_1 = -0.1\,V_x$$

Writing the V_{Th} equation,

$$4 - 2I_1 - V_x = 0$$

$$4 - 2(-0.1V_x) - V_x = 0$$

$$4 - 0.8V_x = 0$$

$$V_x = 5 \text{ V}$$

$$V_x = V_{Th} = 5 \text{ V}$$

Fig. 3.204

Step II Calculation of I_N (Fig. 3.204)
From Fig. 3.204,

$$V_x = 0$$

The dependent source 0.1 V_x depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e., $0.1V_x = 0$ as shown in Fig. 3.205.

Fig. 3.205

$$I_N = \frac{4}{2+3} = 0.8 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{5}{0.8} = 6.25\ \Omega$$

Fig. 3.206

Step IV Thevenin's Equivalent Network (Fig. 3.206)

Example 3.47 *Obtain the Thevenin equivalent network of Fig. 3.207 for the terminals A and B.*

Fig. 3.207

Solution

Step I Calculation of V_{Th} (Fig. 3.208)

From Fig. 3.208,

$$2 - 2I_1 - V_x = 0$$
$$V_x = 2 - 2I_1$$

For Mesh 1,

$$I_1 = -2 \text{ A}$$
$$V_x = 2 - 2(-2) = 6 \text{ V}$$

Writing the V_{Th} equation,

$$2 - 2I_1 - 0 + 4V_x - V_{Th} = 0$$
$$2 - 2(-2) - 0 + 4(6) - V_{Th} = 0$$
$$V_{Th} = 30 \text{ V}$$

Fig. 3.208

Step II Calculation of I_N (Fig. 3.209)

From Fig. 3.209,

$$V_x = 2 - 2I_1 \quad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for the supermesh

$$I_2 - I_1 = 2 \quad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$2 - 2I_1 - 1I_2 + 4V_x = 0$$
$$2 - 2I_1 - I_2 + 4(2 - 2I_1) = 0 \quad \ldots\text{(iii)}$$
$$10I_1 + I_2 = 10$$

Fig. 3.209

Solving Eqs (ii) and (iii),

$$I_1 = 0.73 \text{ A}$$
$$I_2 = 2.73 \text{ A}$$
$$I_N = I_2 = 2.73 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{2.73} = 10.98\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.210)

Fig. 3.210

Example 3.48 *Find the Thevenin equivalent network of Fig. 3.211 for the terminals A and B.*

Fig. 3.211

Solution

Step I Calculation of V_{Th} (Fig. 3.212)
Applying KVL to the mesh,

$$5-10I_1-10I_1=0$$

$$I_1=\frac{5}{20}=0.25\text{ A}$$

Fig. 3.212

Writing the V_{Th} equation,

$$5-10I_1+8I_1-0-V_{Th}=0$$

$$V_{Th}=5-2I_1=5-2(0.25)=4.5\text{ V}$$

Step II Calculation of I_N (Fig. 3.213)
Applying KVL to Mesh 1,

$$5-10I_1-10(I_1-I_2)=0$$

$$20I_1-10I_2=5 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2-I_1)+8I_1-1I_2=0$$

$$18I_1-11I_2=0 \quad \text{...(ii)}$$

Fig. 3.213

Solving Eqs (i) and (ii),

$$I_1=1.375\text{ A}$$

$$I_2=2.25\text{ A}$$

$$I_N=I_2=2.25\text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th}=\frac{V_{Th}}{I_N}=\frac{4.5}{2.25}=2\ \Omega$$

Fig. 3.214

Step IV Thevenin's Equivalent Network (Fig. 3.214)

Example 3.49 *Find V_{Th} and R_{Th} between terminals A and B of the network shown in Fig. 3.215 .*

Fig. 3.215

Solution

Step I Calculation of V_{Th} (Fig. 3.216)

$$I_x = 0$$

The dependent source $2I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $2I_x = 0$ as shown in Fig. 3.216.

Fig. 3.216

Writing the V_{Th} equation,

$$V_{\text{Th}} = 12 \times \frac{1}{1+1} = 6 \text{ V}$$

Step II Calculation of I_N (Fig. 3.217)
From Fig. 3.217,

$$I_x = \frac{V_1}{2}$$

Fig. 3.217

Applying KCL at Node 1,

$$\frac{V_1 - 12}{1} + \frac{V_1}{1} + \frac{V_1}{2} = 2I_x$$

$$V_1 + V_1 + \frac{V_1}{2} - 12 = 2\left(\frac{V_1}{2}\right)$$

$$V_1 = 8 \text{ V}$$

$$I_N = \frac{V_1}{2} = \frac{8}{2} = 4 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{6}{4} = 1.5\ \Omega$$

Example 3.50 *Obtain the Thevenin equivalent network of Fig. 3.218 for the given network at terminals a and b .*

Fig. 3.218

Solution

Step I Calculation of V_{Th} (Fig. 3.219)
Applying KCL at Node x,

$$2 = \frac{V_x}{2}$$

$$V_x = 4 \text{ V}$$

Writing the V_{Th} equation,

$$V_{Th} = V_x - 5V_x = -4V_x$$

$$= -16 \text{ V} \quad (\text{the terminal } a \text{ is negative w.r.t. } b)$$

Fig. 3.219

Step II Calculation of I_N (Fig. 3.220)
Applying KCL at Node x,

$$2 = \frac{V_x}{2} + \frac{V_x - 5V_x}{4}$$

$$2 = \frac{V_x}{2} - V_x = -\frac{V_x}{2}$$

$$V_x = -4 \text{ V}$$

$$I_N = \frac{V_x - 5V_x}{4} = -V_x = 4 \text{ A}$$

Fig. 3.220

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{-16}{4} = -4\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.221)

Fig. 3.221

Example 3.51 *Obtain the Thevenin equivalent network of Fig. 3.222 for the given network.*

Fig. 3.222

Solution

Step I Calculation of V_{Th} (Fig. 3.223)

From Fig. 3.223

$$V_x = V_{Th}$$

Applying KCL at the node,

$$\frac{V_x - 150 - \frac{1}{3}V_x}{10} + \frac{V_x}{15} + 5 = 0$$

$$V_x = 75 \text{ V}$$

$$V_{Th} = 75 \text{ V}$$

Fig. 3.223

Step II Calculation of I_N (Fig. 3.224)

Applying KCL at Node x,

$$\frac{V_x}{30} + 5 + \frac{V_x}{15} + \frac{V_x - 150 - \frac{1}{3}V_x}{10} = 0$$

$$\frac{V_x}{30} + \frac{V_x}{15} + \frac{V_x}{10} - \frac{V_x}{30} = 15 - 5$$

$$V_x = 60 \text{ V}$$

$$I_N = \frac{V_x}{30} = \frac{60}{30} = 2 \text{ A}$$

Fig. 3.224

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{75}{2} = 37.5\ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 3.225)

Fig. 3.225

Example 3.52 *Find the Thevenin's equivalent network of the network to the left of A-B in the Fig. 3.226.*

Fig. 3.226

Solution

Step I Calculation of V_{Th} (Fig. 3.227)

From Fig. 3.227,

$$I_x = I_1 - I_2 \quad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 1 \quad \ldots\text{(ii)}$$

Fig. 3.227

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 10I_x - 10I_2 = 0$$

$$-5(I_2 - I_1) - 10(I_1 - I_2) - 10I_2 = 0$$

$$5I_1 + 5I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 1 \text{ A}$$
$$I_2 = -1 \text{ A}$$
$$I_x = I_1 - I_2 = 1 - (-1) = 2 \text{ A}$$

Writing the V_{Th} equation,

$$10I_2 - 10 - V_{\text{Th}} = 0$$
$$10(-1) - 10 - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -20 \text{ V}$$

Step II Calculation of I_N (Fig. 3.228)

From Fig. 3.228,

$$I_x = I_1 - I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 1 \qquad \ldots\text{(ii)}$$

Fig. 3.228

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 10I_x - 10(I_2 - I_3) = 0$$

$$-5(I_2 - I_1) - 10(I_1 - I_2) - 10(I_2 - I_3) = 0$$

$$-5I_1 - 5I_2 + 10I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-10(I_3 - I_2) - 10 = 0$$

$$10I_2 - 10I_3 = 10 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 1 \text{ A}$$
$$I_2 = 3 \text{ A}$$
$$I_3 = 2 \text{ A}$$
$$I_N = I_3 = 2 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{-20}{2} = -10\ \Omega$$

Fig. 3.229

Step IV Thevenin's Equivalent Network (Fig. 3.229)

Example 3.53 *Find Thevenin's equivalent network at terminals A and B in the network of Fig. 3.230.*

Fig. 3.230

Solution

Since the network does not contain any independent source,

$$V_{\text{Th}} = 0$$

$$I_N = 0$$

But the R_{Th} can be calculated by applying a known voltage source of 1 V at the terminals A and B as shown in Fig. 3.231.

Fig. 3.231

$$R_{\text{Th}} = \frac{V}{I} = \frac{1}{I}$$

From Fig. 3.231,

$$V_x = 5(I_1 - I_2) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-4V_x - 2I_1 - 5(I_1 - I_2) = 0$$

$$-4\left[5(I_1 - I_2\right] - 2I_1 - 5I_1 + 5I_2 = 0$$

$$-27I_1 + 25I_2 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 4I_2 - 1 = 0$$

$$5I_1 - 9I_2 = 1 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -0.21 \text{ A}$$

$$I_2 = -0.23 \text{ A}$$

Hence, current supplied by voltage source of 1 V is 0.23 A.

$$R_{\text{Th}} = \frac{1}{0.23} = 4.35\ \Omega$$

Hence, Thevenin's equivalent network is shown in Fig. 3.232.

Fig. 3.232

Example 3.54 *Find the current in the 9 Ω resistor in Fig. 3.233.*

Fig. 3.233

Solution

Step I Calculation of V_{Th} (Fig. 3.234)

Applying KVL to the mesh,

$$20 - 4I_x + 6I_x - 6I_x = 0$$

$$I_x = 5 \text{ A}$$

Writing the V_{Th} equation,

$$6I_x - V_{Th} = 0$$

$$6(5) - V_{Th} = 0$$

$$V_{Th} = 30 \text{ V}$$

Fig. 3.234

Step II Calculation of I_N (Fig. 3.235).

From Fig. 3.235,

$$I_x = 0$$

The dependent source $6I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $6I_x = 0$ as shown in Fig. 3.236.

$$I_N = \frac{20}{4} = 5 \text{ A}$$

Fig. 3.235

(a) (b)

Fig. 3.236

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{5} = 6\ \Omega$$

Fig. 3.237

Step IV Calculation of I_L (Fig. 3.237)

$$I_L = \frac{30}{6+9} = 2 \text{ A}$$

Example 3.55 *Determine the current in the 16 Ω resistor in Fig. 3.238.*

Fig 3.238

Solution

Step I Calculation of V_{Th} (Fig. 3.239)

From Fig. 3.239,

$$I_x = 0$$

The dependent source $0.8I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, as shown in Fig. 3.240.

i.e.,

$$0.8I_x = 0$$

$$V_{Th} = 40 \text{ V}$$

Fig. 3.239

Fig. 3.240

Step II Calculation of I_N (Fig. 3.241)

From Fig. 3.241,

$$I_x = I_2 \qquad \text{...(i)}$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for the supermesh,

$$I_1 - I_2 = 0.8\, I_x = 0.8\, I_2$$

$$I_1 - 1.8\, I_2 = 0 \qquad \text{...(ii)}$$

Applying KVL to the outer path of the supermesh,

$$40 - 10\, I_1 - 6\, I_2 = 0$$

$$10\, I_1 + 6\, I_2 = 40 \qquad \text{...(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 3 \text{ A}$$

$$I_2 = \frac{5}{3} \text{ A}$$

$$I_N = I_2 = \frac{5}{3} \text{ A}$$

Fig. 3.241

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{40}{\frac{5}{3}} = 24\ \Omega$$

Step IV Calculation of I_L (Fig. 3.242)

$$I_L = \frac{40}{24+16} = 1 \text{ A}$$

Fig. 3.242

Example 3.56 *Find the current in the 6 Ω resistor in Fig. 3.243.*

Fig. 3.243

Solution

Step I Calculation of V_{Th} (Fig. 3.244)

From Fig. 3.244,

$$V_x = -1I_1 = -I_1 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = -3 \text{ A} \qquad \ldots\text{(ii)}$$

$$V_x = 3 \text{ V}$$

Fig. 3.244

Writing the V_{Th} equation,

$$18 - 1\,I_1 + 2\,V_x - V_{Th} = 0$$

$$18 + 3 + 2(3) - V_{Th} = 0$$

$$V_{Th} = 27 \text{ V}$$

Step II Calculation of I_N (Fig. 3.245)

From Fig. 3.245,

$$V_x = -I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh,

Writing current equation for supermesh,

$$I_2 - I_1 = 3 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$18 - 1I_1 + 2\,V_x = 0$$

$$18 - I_1 + 2(-I_1) = 0 \qquad \ldots\text{(iii)}$$

$$I_1 = 6 \text{ A}$$

Fig. 3.245

Solving Eqs (ii) and (iii),

$$I_2 = 9 \text{ A}$$

$$I_N = I_2 = 9 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{27}{9} = 3\ \Omega$$

Step IV Calculation of I_L (Fig. 3.246)

$$I_L = \frac{27}{3+6} = 3 \text{ A}$$

Fig. 3.246

Example 3.57 *Find the current in the 10 Ω resistor.*

Fig. 3.247

Solution

Step I Calculation of V_{Th} (Fig. 3.248)

From the figure,

$$V_x = 10 \times 5 = 50 \text{ V}$$

Writing the V_{Th} equation,

$$100 - V_{Th} + 10V_x - V_x = 0$$
$$100 - V_{Th} + 9V_x = 0$$
$$100 - V_{Th} + 9(50) = 0$$
$$V_{Th} = 550 \text{ V}$$

Fig. 3.248

Step II Calculation of I_N (Fig. 3.249)

From Fig. 3.249,

$$V_x = 5(I_N + 10)$$

Applying KVL to Mesh 1,

$$100 + 10V_x - V_x = 0$$
$$V_x = -\frac{100}{9}$$
$$-\frac{100}{9} = 5I_N + 50$$
$$I_N = -\frac{550}{45} \text{ A}$$

Fig. 3.249

Step III Calculation of R_{Th}

$$R_{Th} = \frac{550}{-\frac{550}{45}} = -45\ \Omega$$

Step IV Calculation of I_L (Fig. 3.250)

$$I_L = \frac{550}{-45 + 10} = -\frac{110}{7} \text{ A}$$

Fig. 3.250

3.4 NORTON'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent current source and an equivalent parallel resistance.*' The constant current is equal to the current which would flow in a short circuit placed across the terminals. The parallel resistance is the resistance of the network when viewed from these open-circuited terminals after all voltage and current sources have been removed and replaced by internal resistances.

Fig. 3.251 *Network illustrating Norton's theorem*

Explanation Consider a simple network as shown in Fig.3.252

Fig. 3.252 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate short circuit current I_{SC} or I_N which would flow in a short circuit placed across terminals A and B as shown in Fig. 3.253.

For finding parallel resistance R_N, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 3.254.

$$R_N = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

Norton's equivalent network is shown in Fig.3.255.

$$I_L = I_N \frac{R_N}{R_N + R_L}$$

If the network contains both independent and dependent sources, Norton's resistances R_N is calculated as

$$R_N = \frac{V_{Th}}{I_N}$$

where V_{Th} is the open-circuit voltage across terminals A and B. If the network contains only dependent sources, then

$$V_{Th} = 0$$

$$I_N = 0$$

To find R_{Th} in such network, a known voltage V or current I is applied across the terminals A and B, and the current I or the voltage V is calculated respectively.

$$R_N = \frac{V}{I}$$

Norton's equivalent network for such a network is shown in Fig. 3.256.

Fig. 3.253 *Calculation of I_N*

Fig. 3.254 *Calculation of R_N*

Fig. 3.255 *Norton's equivalent network*

Fig. 3.256 *Norton's equivalent network*

Steps to be followed in Norton's Theorem

1. Remove the load resistance R_L and put a short circuit across the terminals.
2. Find the short-circuit current I_{SC} or I_N.
3. Find the resistance R_N as seen from points A and B.
4. Replace the network by a current source I_N in parallel with resistance R_N.
5. Find current through R_L by current–division rule.

$$I_L = \frac{I_N R_N}{R_N + R_L}$$

Example 3.58 *Find the current through the 10 Ω resistor in Fig. 3.257.*

Fig. 3.257

Solution

Step I Calculation of I_N (Fig. 3.258)

Applying KVL to Mesh 1,

$$2 - 1I_1 = 0$$

$$I_1 = 2 \qquad \text{...(i)}$$

Fig. 3.258

Meshes 2 and 3 will form a supermesh.

Writing the current equation for the supermesh,

$$I_3 - I_2 = 4 \qquad \text{...(ii)}$$

Applying KVL to the supermesh,

$$-5I_2 - 15I_3 = 0 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 2\,\text{A}$$

$$I_2 = -3\,\text{A}$$

$$I_3 = 1\,\text{A}$$

$$I_N = I_1 - I_2 = 2 - (-3) = 5\,\text{A}$$

Fig. 3.259

Step II Calculation of R_N (Fig. 3.259)

$$R_N = 1 \,||\, (5+15) = 0.95\ \Omega$$

Step III Calculation of I_L (Fig.3.260)

$$I_L = 5 \times \frac{0.95}{0.95 + 10} = 0.43\,\text{A}$$

Fig. 3.260

Example 3.59 *Find the current through the 10 Ω resistor in Fig. 3.258.*

Fig. 3.261

Solution

Step I Calculation of I_N (Fig. 3.262)

Applying KVL to Mesh 1,

$$-5I_1 + 20 - 2(I_1 - I_2) = 0$$
$$7I_1 - 2I_2 = 20 \quad \text{...(i)}$$

Fig. 3.262

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 8I_2 - 12 = 0$$
$$-2I_1 + 10I_2 = -12 \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = -0.67\text{ A}$$
$$I_N = I_2 = -0.67\text{ A}$$

Fig. 3.263

Step II Calculation of R_N (Fig. 3.263)

$$R_N = (5 \parallel 2) + 8 = 9.43\ \Omega$$

Step III Calculation of I_L (Fig. 3.264)

$$I_L = 0.67 \times \frac{9.43}{9.43 + 10} = 0.33\text{ A}(\uparrow)$$

Fig. 3.264

Example 3.60 *Find the current in the 10 Ω resistor in Fig. 3.265.*

Fig. 3.265

Solution

Step I Calculation of I_N (Fig. 3.266)

The resistance of $40\,\Omega$ becomes redundant as it is connected across the 50 V source (Fig. 3.267).

Applying KVL to Mesh 1,

$$50 - 50I_1 - 20(I_1 - I_2) - 10 = 0$$
$$70I_1 - 20I_2 = 40 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$10 - 20(I_2 - I_1) = 0$$
$$-20I_1 + 20I_2 = 0 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 1\,\text{A}$$
$$I_2 = 1.5\,\text{A}$$
$$I_N = I_2 = 1.5\,\text{A}$$

Fig. 3.266

Fig. 3.267

Step II Calculation of R_N (Fig. 3.268)

$$R_N = 50 \,||\, 20 = 14.28\,\Omega$$

Fig. 3.268

Step III Calculation of I_L (Fig. 3.269)

$$I_L = 1.5 \times \frac{14.28}{14.28 + 10} = 0.88\,\text{A}$$

Fig. 3.269

Example 3.61 *Find the current through the 10 Ω resistor in Fig. 3.270.*

Fig. 3.270

Solution

Step I Calculation of I_N (Fig. 3.271)

Applying KVL to Mesh 1,

$$10 - 6I_1 - 1(I_1 - I_2) = 0$$
$$7I_1 - I_2 = 10 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) - 2I_2 - 3(I_2 - I_3) = 0$$
$$-I_1 + 6I_2 - 3I_3 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-3(I_3 - I_2) - 20 = 0$$
$$3I_2 - 3I_3 = 20 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_3 = -13.17\,\text{A}$$
$$I_N = I_3 = -13.17\,\text{A}$$

Fig. 3.271

Step II Calculation of R_N (Fig. 3.272)

$$R_N = [(6 \,||\, 1) + 2] \,||\, 3 = 1.46\,\Omega$$

Fig. 3.272

Step III Calculation of I_L (Fig. 3.273)

$$I_L = 13.17 \times \frac{1.46}{1.46 + 10} = 1.68\,\text{A}\,(\uparrow)$$

Fig. 3.273

Example 3.62 *Find the current through the 10 Ω resistor in Fig. 3.274.*

Fig. 3.274

Solution

Step I Calculation of I_N (Fig. 3.274)

Applying KVL to Mesh 1,

$$50 - 20(I_1 - I_2) - 40 = 0$$
$$20I_1 - 20I_2 = 10 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$40 - 20(I_2 - I_1) - 20I_2 - 20(I_2 - I_3) = 0$$
$$-20I_1 + 60I_2 - 20I_3 = 40 \quad \text{...(ii)}$$

Fig. 3.275

Applying KVL to Mesh 3,

$$-20(I_3 - I_2) - 30I_3 - 100 = 0$$

$$-20I_2 + 50I_3 = -100 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.81\,\text{A}$$

$$I_N = I_1 = 0.81\,\text{A}$$

Step II Calculation of R_N (Fig. 3.276)

$$R_N = [(20 \parallel 30) + 20] \parallel 20 = 12.3\,\Omega$$

Fig. 3.276

Step III Calculation of I_L (Fig. 3.277)

$$I_L = 0.81 \times \frac{12.3}{12.3 + 10} = 0.45\,\text{A}$$

Fig. 3.277

Example 3.63 *Obtain Norton's equivalent network as seen by R_L in Fig. 3.278.*

Fig. 3.278

Solution

Step I Calculation of I_N (Fig. 3.279)

Applying KVL to Mesh 1,

$$120 - 30I_1 - 60(I_1 - I_2) = 0$$

$$90I_1 - 60I_2 = 120 \qquad \text{...(i)}$$

Fig. 3.279

Applying KVL to Mesh 2,

$$-60(I_2 - I_1) + 40 - 10I_2 - 30(I_2 - I_3) = 0$$

$$-60I_1 + 100I_2 - 30I_3 = 40 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-30(I_3 - I_2) + 10 = 0$$

$$30I_2 - 30I_3 = -10 \qquad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_3 = 4.67\,\text{A}$$

$$I_N = I_3 = 4.67\,\text{A}$$

Step II Calculation of R_N (Fig. 3.280)

$$R_N = [(30 \,||\, 60) + 10] \,||\, 30 = 15\,\Omega$$

Step III Norton's equivalent network (Fig. 3.281)

Fig. 3.280

Fig. 3.281

Example 3.64 *Find the current through the 8 Ω resistor in Fig. 3.282.*

Fig. 3.282

Solution

Step I Calculation of I_N (Fig. 3.283)

Fig. 3.283

The resistor of the 4 Ω gets shorted as it is in parallel with the short circuit. Simplifying the network by source transformation (Fig. 3.284),

Fig. 3.284

Meshes 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 2 \qquad \text{...(i)}$$

Applying KVL to the supermesh,

$$60 - 12I_1 - 5 = 0$$

$$12I_1 = 55 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 4.58\,\text{A}$$
$$I_2 = 6.58\,\text{A}$$
$$I_N = I_2 = 6.58\,\text{A}$$

Fig. 3.285

Step II Calculation of R_N (Fig. 3.285)

$$R_N = 12 \parallel 4 = 3\,\Omega$$

Step III Calculation of I_L (Fig. 3.286)

$$I_L = 6.58 \times \frac{3}{3+8} = 1.79\,\text{A}$$

Fig. 3.286

Example 3.65

Find the current through the 1 Ω resistor in Fig. 3.287.

Fig. 3.287

Solution

Step I Calculation of I_N (Fig. 3.288)

Fig. 3.288

By source transformation (Fig. 3.289),

Fig. 3.289

Applying KVL to Mesh 1,

$$-3-3I_1-2(I_1-I_3)+1=0$$
$$5I_1-2I_3=-2 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-1-2(I_2-I_3)-2I_2=0$$
$$4I_2-2I_3=-1 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3-I_1)-2(I_3-I_2)=0$$
$$-2I_1-2I_2+4I_3=0 \quad \text{...(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1=-0.64\,\text{A}$$
$$I_2=-0.55\,\text{A}$$
$$I_3=-0.59\,\text{A}$$
$$I_N=I_3=-0.59\,\text{A}$$

Step II Calculation of R_N (Fig. 3.290)

$$R_N=2.2\,\Omega$$

Fig. 3.290

Step III Calculation of I_L (Fig. 3.291)

$$I_L=0.59\times\frac{2.2}{2.2+1}=0.41\,\text{A}$$

Fig. 3.291

EXAMPLES WITH DEPENDENT SOURCES

Example 3.66 *Find Norton's equivalent network across terminals A and B of Fig. 3.292.*

Fig. 3.292

Solution

Step I Calculation of V_{Th} (Fig. 3.293)

From Fig. 3.293,

$$I_2 = I_x$$
$$I_1 = -I_x$$

Applying KVL to the mesh,

$$5 - 10I_x - 5I_x - 10I_2 = 0$$
$$5 - 10I_x - 5I_x - 10I_x = 0$$
$$I_x = 0.2 \text{ A}$$
$$I_1 = -0.2 \text{ A}$$

Fig. 3.293

Writing the V_{Th} equation,

$$5 - 10I_x + 3I_1 - V_{Th} = 0$$
$$5 - 10(0.2) + 3(-0.2) - V_{Th} = 0$$
$$V_{Th} = 2.4 \text{ V}$$

Step II Calculation of I_N (Fig. 3.294)

From Fig. 3.294,

$$I_2 = I_x \quad \ldots\text{(i)}$$
$$I_1 = I_y - I_x \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 1,

$$5 - 10I_x - 5(I_x - I_y) - 10I_2 = 0$$
$$5 - 10I_x - 5I_x + 5I_y - 10I_x = 0$$
$$25I_x - 5I_y = 5 \quad \ldots\text{(iii)}$$

Fig. 3.294

Applying KVL to Mesh 2,

$$10I_2 - 5(I_y - I_x) + 3I_1 - 10I_y = 0$$
$$10I_x - 5I_y + 5I_x + 3(I_y - I_x) - 10I_y = 0$$
$$12I_x - 12I_y = 0 \quad \ldots\text{(iv)}$$

Solving Eqs (iii) and (iv),

$$I_x = 0.25 \text{ A}$$
$$I_y = 0.25 \text{ A}$$
$$I_N = I_y = 0.25 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{2.4}{0.25} = 9.6\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.295)

Fig. 3.295

Example 3.67 *For the network shown in Fig. 3.296, find Norton's equivalent network.*

Fig. 3.296

Solution

Step I Calculation of V_{Th} (Fig. 3.297)

From Fig. 3.297,

$$V_x = 2I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = -3V_x = -3(2I_2) = -6I_2 \qquad \ldots\text{(ii)}$$

For Mesh 2,

$$I_2 = 2 \qquad \ldots\text{(iii)}$$

$$I_1 = -6I_2 = -6(2) = -12 \text{ A}$$

Fig. 3.297

Writing the V_{Th} equation,

$$V_{\text{Th}} - 0 + 5I_1 + 15(I_1 - I_2) - 2I_2 = 0$$
$$V_{\text{Th}} + 5(-12) + 15(-12 - 2) - 2(2) = 0$$
$$V_{\text{Th}} = 274 \text{ V}$$

Step II Calculation of I_N (Fig. 3.298)

From Fig. 3.298,

$$V_x = 2(I_2 - I_3) \qquad \ldots\text{(i)}$$

For Mesh 2,

$$I_2 = 2 \qquad \ldots\text{(ii)}$$

Fig. 3.298

Meshes 1 and 3 will form a supermesh.
Writing the current equation for the supermesh,

$$I_3 - I_1 = 3V_x = 3\left[2(I_2 - I_3)\right] = 6I_2 - 6I_3$$

$$I_1 + 6I_2 - 7I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to the outer path of the supermesh,

$$-5I_1 - 20I_3 - 2(I_3 - I_2) - 15(I_1 - I_2) = 0$$

$$-20I_1 + 17I_2 - 22I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -0.16 \text{ A}$$

$$I_2 = 2 \text{ A}$$

$$I_3 = 1.69 \text{ A}$$

$$I_N = I_3 = 1.69 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{274}{1.69} = 162.13\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.299)

Fig. 3.299

Example 3.68 *Obtain Norton's equivalent network across A-B in the network of Fig. 3.300.*

Fig. 3.300

Solution

Step I Calculation of V_{Th} (Fig. 3.301)
From Fig. 3.301,

$$V_1 = 8(I_x - I_y) \qquad \ldots\text{(i)}$$

Fig. 3.301

Applying KVL to Mesh 1,

$$15 - 5I_x - 8(I_x - I_y) = 0$$

$$13I_x - 8I_y = 15 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-8(I_y - I_x) - 2I_y - 10I_2 = 0$$

$$8I_x - 10I_y - 10I_2 = 0 \qquad \ldots\text{(iii)}$$

For Mesh 3,

$$I_2 = 0.6V_1 = 0.6\left[8(I_x - I_y)\right]$$

$$4.8I_x - 4.8I_y - I_2 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_x = 3.28 \text{ A}$$
$$I_y = 3.45 \text{ A}$$
$$I_2 = -0.83 \text{ A}$$

Writing the V_{Th} equation,

$$15I_2 - V_{\text{Th}} = 0$$
$$15(-0.83) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -12.45 \text{ V}$$

Fig. 3.302

Step II Calculation of I_N (Fig. 3.302)
From Fig. 3.302,

$$I_2 = 0$$

The dependent source of 10 I_2 depends on the controlling variable I_2. When $I_2 = 0$, the dependent source vanishes, i.e. $10I_2 = 0$ as shown in Fig. 3.303.

Fig. 3.303

From Fig. 3.303,

$$V_1 = 8(I_x - I_y) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$15 - 5I_x - 8(I_x - I_y) = 0$$
$$13I_x - 8I_y = 15 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-8(I_y - I_x) - 2I_y = 0$$
$$-8I_x + 10I_y = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_x = 2.27 \text{ A}$$
$$I_y = 1.82 \text{ A}$$
$$V_1 = 8(I_x - I_y) = 8(2.27 - 1.82) = 3.6 \text{ V}$$

For Mesh 3,

$$I_N = 0.6V_1 = 0.6(3.6) = 2.16 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-12.45}{2.16} = -5.76\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.304)

Fig. 3.304

Example 3.69 *Find Norton's equivalent network of Fig. 3.305.*

Fig. 3.305

Fig. 3.306

Solution

Step I Calculation of V_{Th} (Fig. 3.306)

Applying KVL to the mesh,

$$2 - 2I_1 + 0.5I_1 - 1I_1 = 0$$
$$2 - 2.5I_1 = 0$$
$$I_1 = 0.8 \text{ A}$$

Writing the V_{Th} equation,

$$1I_1 - V_{Th} = 0$$
$$1(0.8) - V_{Th} = 0$$
$$V_{Th} = 0.8 \text{ V}$$

Fig. 3.307

Step II Calculation of I_N (Fig. 3.307)

When a short circuit is placed across the 1 Ω resistor, it gets shorted.

$$I_1 = 0$$

The dependent source of $0.5I_1$ depends on the controlling variable I_1. When $I_1 = 0$, the dependent source vanishes, i.e. $0.5\,I_1 = 0$ as shown in Fig. 3.308.

$$I_N = \frac{2}{2} = 1 \text{ A}$$

Fig. 3.308

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{0.8}{1} = 0.8\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.309)

Fig. 3.309

Example 3.70 *Find Norton's equivalent network at the terminals A and B of Fig. 3.310.*

Fig. 3.310

Solution

Step I Calculation of V_{Th} (Fig. 3.311)

From Fig. 3.311,

$$I_x = I_1 \qquad \ldots\text{(i)}$$

Fig. 3.311

Applying KVL to Mesh 1,

$$9 - 3(I_1 - I_2) - 6I_1 = 0$$

$$9I_1 - 3I_2 = 9 \qquad \ldots\text{(ii)}$$

For Mesh 2,

$$I_2 = 6I_x = 6I_1$$

$$6I_1 - I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = -1 \text{ A}$$

$$I_2 = -6 \text{ A}$$

Writing the V_{Th} equation,

$$9 - 3(I_1 - I_2) + 2I_2 - V_{Th} = 0$$

$$9 - 3(-1+6) + 2(-6) - V_{Th} = 0$$

$$V_{Th} = -18 \text{ V}$$

Step II Calculation of I_N (Fig. 3.312)

From Fig. 3.312,

$$I_x = I_1 - I_3 \qquad \ldots\text{(i)}$$

Fig. 3.312

Applying KVL to Mesh 1,

$$9 - 3(I_1 - I_2) - 6(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 6I_3 = 9 \qquad \ldots\text{(ii)}$$

For Mesh 2,

$$I_2 = 6I_x = 6(I_1 - I_3)$$

$$6I_1 - I_2 - 6I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) - 2(I_3 - I_2) = 0$$

$$-6I_1 - 2I_2 + 8I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 5 \text{ A}$$
$$I_2 = 3 \text{ A}$$
$$I_3 = 4.5 \text{ A}$$
$$I_N = I_3 = 4.5 \text{ A}$$

Fig. 3.313

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-18}{4.5} = -4\ \Omega$$

Step IV Norton's Equivalent Network (Fig. 3.313)

Example 3.71 *Find Norton's equivalent network to the left of terminal A-B in Fig. 3.314.*

Fig. 3.314

Solution Since the network does not contain any independent source,

$$V_{\text{Th}} = 0$$
$$I_N = 0$$

But R_N can be calculated by applying a known current source of 1 A at the terminals A and B as shown in Fig. 3.315.

Fig. 3.315

From Fig. 3.315,

$$I = \frac{V}{6}$$

Applying KCL at the node,

$$\frac{V}{6} + 0.5I + \frac{V}{4} = 1$$

$$\frac{V}{6} + 0.5\left(\frac{V}{6}\right) + \frac{V}{4} = 1$$

$$\left(\frac{1}{6} + \frac{0.5}{6} + \frac{1}{4}\right)V = 1$$

$$V = 2$$

$$R_N = \frac{V}{1} = \frac{2}{1} = 2\ \Omega$$

Hence, Norton's equivalent network is shown in Fig. 3.316.

Fig. 3.316

Example 3.72 *Find the current through the 2 Ω resistor in the network shown in Fig. 3.317.*

Fig. 3.317

Solution

Step I Calculation of V_{Th} (Fig. 3.318)

From Fig. 3.318,

$$I_x = 0$$

The dependent source of $-2\ I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e. $-2I_x = 0$ as shown in Fig. 3.319.

Fig. 3.318

$$I_1 = 2$$

Writing the V_{Th} equation,

$$-10 - V_{Th} - 4I_1 = 0$$

$$-10 - V_{Th} - 4(2) = 0$$

$$V_{Th} = -18 \text{ V}$$

Fig. 3.319

Step II Calculation of I_N (Fig. 3.320)

From Fig. 3.320,

$$I_x = I_1 \qquad \ldots\text{(i)}$$

Mesh 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 2 \qquad \ldots\text{(ii)}$$

Fig. 3.320

Applying KVL to the outer path of the supermesh,

$$-10 - 4(I_2 - I_3) = 0$$

$$-4I_2 + 4I_3 = 10 \qquad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = -(-2I_x) = 2I_x = 2I_1$$

$$2I_1 - I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 4.5 \text{ A}$$
$$I_2 = 6.5 \text{ A}$$
$$I_3 = 9 \text{ A}$$
$$I_N = I_1 = 4.5 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-18}{4.5} = -4\ \Omega$$

Step IV Calculation of I_L (Fig. 3.321)

$$I_L = 4.5 \times \frac{-4}{-4+2} = 9 \text{ A}$$

Fig. 3.321

Example 3.73 *Find the current through the 2Ω resistor in the network of Fig. 3.322.*

Fig. 3.322

Solution

Step I Calculation of V_{Th} (Fig. 3.322)

From Fig. 3.323,

$$5 + V_i + 4V_i = 0$$
$$V_i = -1 \text{ V}$$

Writing the V_{Th} equation,

$$-4V_i - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = -4V_i = -4(-1) = 4 \text{ V}$$

Fig. 3.323

Step II Calculation of I_N (Fig. 3.324)

From Fig. 3.324,

$$5 + V_i = 0$$
$$V_i = -5 \text{ V}$$

Applying KVL to the mesh,

$$-4V_i - 1I_N = 0$$
$$I_N = -4V_i = -4(-5) = 20 \text{ A}$$

Fig. 3.324

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{4}{20} = 0.2\ \Omega$$

Fig. 3.325

Step IV Calculation of I_L (Fig. 3.325)

$$I_L = 20 \times \frac{0.2}{0.2+2} = 1.82\ \text{A}$$

Example 3.74 *Find the current in the 2 Ω resistor in the network of Fig. 3.326.*

Fig. 3.326

Solution

Step I Calculation of V_{Th} (Fig. 3.327)

Meshes 1 and 2 will form a supermesh.

Writing current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(i)}$$

Applying KVL to the outer path of the supermesh,

$$10 - 1I_1 - 3I_2 = 0$$

$$I_1 + 3I_2 = 10 \qquad \ldots\text{(ii)}$$

Fig. 3.327

Solving Eqs (i) and (ii),

$$I_1 = 1.75\ \text{A}$$

$$I_2 = 2.75\ \text{A}$$

Writing the V_{Th} equation,

$$3I_2 - V_{\text{Th}} = 0$$

$$3(2.75) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 8.25\ \text{V}$$

Step II Calculation of I_N (Fig. 3.328)

From Fig. 3.328,

$$I_x = I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$10 - 1I_1 - 3(I_2 - I_3) = 0$$

Fig. 3.328

$$I_1 + 3I_2 - 3I_3 = 10 \qquad \ldots\text{(iii)}$$

For Mesh 3,

$$I_3 = 2I_x = 2I_1$$

$$2I_1 - I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = -3.5 \text{ A}$$

$$I_2 = -2.5 \text{ A}$$

$$I_3 = -7 \text{ A}$$

$$I_N = I_3 = -7 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{8.25}{-7} = -1.18\ \Omega$$

Step IV Calculation of I_L (Fig. 3.329)

$$I_L = -7 \times \frac{-1.18}{-1.18 + 2} = 10.07 \text{ A}$$

Fig. 3.329

Example 3.75 *Find the current through the 10 Ω resistor for the network of Fig. 3.330.*

Fig. 3.330

Solution

Step I Calculation of V_{Th} (Fig. 3.331)
Applying KVL to the mesh,

$$10 - 2I_x + 3I_x - 5I_x = 0$$

$$I_x = 2.5 \text{ A}$$

Writing the V_{Th} equation,

$$5I_x - V_{\text{Th}} = 0$$

$$5(2.5) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = 12.5 \text{ V}$$

Fig. 3.331

Step II Calculation of I_N (Fig. 3.332)
From Fig. 3.332,

$$I_x = 0$$

Fig. 3.332

The dependent source of 3 I_x depends on the controlling variable I_x. When $I_x = 0$, the dependent source 3 I_x vanishes, i.e. 3 $I_x = 0$ as shown in Fig. 3.333.

$$I_N = \frac{10}{2} = 5 \text{ A}$$

Fig. 3.333

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{12.5}{5} = 2.5\ \Omega$$

Step IV Calculation of I_L (Fig. 3.334)

$$I_L = 5 \times \frac{2.5}{2.5+10} = 1 \text{ A}$$

Fig. 3.334

Example 3.76 *Find the current through the 5 Ω resistor in the network of Fig. 3.335.*

Fig. 3.335

Solution

Step I Calculation of V_{Th} (Fig. 3.336)

Applying KVL to the mesh,

$$12 - 2I_x - 4I_x - 4I_x = 0$$
$$12 - 10I_x = 0$$
$$I_x = 1.2 \text{ A}$$

Fig. 3.336

Writing the V_{Th} equation,

$$12 - 2I_x - V_{\text{Th}} = 0$$
$$12 - 2(1.2) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 9.6 \text{ V}$$

Step II Calculation of I_N (Fig. 3.337)

From Fig. 3.337,

$$I_x = I_1 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$12 - 2I_1 = 0$$
$$I_1 = 6 \text{ A} \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-4I_2 - 4I_x = 0$$
$$-4I_2 - 4I_1 = 0 \quad \ldots\text{(iii)}$$

Fig. 3.337

Solving Eqs (ii) and (iii),

$$I_2 = -6 \text{ A}$$
$$I_N = I_1 - I_2 = 6 - (-6) = 12 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{9.6}{12} = 0.8\ \Omega$$

Step IV Calculation of I_L (Fig. 3.338)

$$I_L = 12 \times \frac{0.8}{0.8+5} = 1.66\ \text{A}$$

Fig. 3.338

Example 3.77 *Find the current through the 10 Ω resistor for the network of Fig. 3.339.*

Fig. 3.339

Solution

Step I Calculation of V_{Th} (Fig. 3.340)

For the mesh,

$$I = -0.5V_x = -0.5V_{\text{Th}}$$

Writing the V_{Th} equation,

$$5 - 4I - 0 - V_{\text{Th}} = 0$$

$$5 - 4(-0.5V_{\text{Th}}) - V_{\text{Th}} = 0$$

$$V_{\text{Th}} = -5\ \text{V}$$

Fig. 3.340

Step II Calculation of I_N (Fig. 3.341)

From Fig. 3.341,

$$V_x = 0$$

Fig. 3.341

The dependent source of $0.5V_x$ depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e. 0.5 $V_x = 0$ as shown in Fig. 3.342.

$$I_N = \frac{5}{4+5} = \frac{5}{9}\ \text{A}$$

Fig. 3.342

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{-5}{\frac{5}{9}} = -9\ \Omega$$

Step IV Calculation of I_L (Fig. 3.343)

$$I_L = \frac{5}{9} \times \frac{-9}{-9+10} = -5\ \text{A}$$

Fig. 3.343

Example 3.78 *Find the current through the 10 Ω resistor in the network shown in Fig. 3.344.*

Fig. 3.344

Solution

Step I Calculation of V_{Th} (Fig. 3.345)

From Fig. 3.345,

$$V_x = -25(5I_1) = -125\,I_1 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$12 - 1000I_1 - 2V_x = 0$$

$$12 - 1000I_1 - 2(-125I_1) = 0 \qquad \ldots\text{(ii)}$$

$$I_1 = 0.016 \text{ A}$$

$$V_x = -125I_1 = -125(0.016) = -2 \text{ V}$$

Fig. 3.345

Writing the V_{Th} equation,

$$V_{Th} = V_x = -2 \text{ V}$$

Step II Calculation of I_N (Fig. 3.346)

From Fig. 3.346,

$$V_x = 0$$

Fig. 3.346

The dependent source of $2V_x$ depends on the controlling variable V_x. When $V_x = 0$, the dependent source vanishes, i.e. $2\,V_x = 0$ as shown in Fig. 3.347.

$$I_1 = \frac{12}{1000} = 0.012 \text{ A}$$

$$I_N = -5I_1 = -5(0.012) = -0.06 \text{ A}$$

Fig. 3.347

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{-2}{-0.06} = 33.33\ \Omega$$

Step IV Calculation of I_L (Fig. 3.348)

$$I_L = -0.06 \times \frac{33.33}{33.33 + 10} = -0.046 \text{ A}$$

Fig. 3.348

Example 3.79 *Find the current through the 5 Ω resistor for the network of Fig. 3.349.*

Fig. 3.349

Solution

Step I Calculation of V_{Th} (Fig. 3.350)

From Fig. 3.350,

$$V_x = 2I \qquad \ldots\text{(i)}$$

For the mesh,

$$I = 2 \qquad \ldots\text{(ii)}$$

$$V_x = 2(2) = 4 \text{ V}$$

Writing the V_{Th} equation,

$$4V_x + 2I + 1I + 4 - V_{Th} = 0$$

$$4(4) + 2(2) + 2 + 4 - V_{Th} = 0$$

$$V_{Th} = 26 \text{ V}$$

Fig. 3.350

Step II Calculation of I_N (Fig. 3.351)

From Fig. 3.351,

$$V_x = 2(I_1 - I_2) \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 2 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$4V_x - 2(I_2 - I_1) - 1(I_2 - I_1) + 4 = 0$$

$$4[2(I_1 - I_2)] - 2I_2 + 2I_1 - I_2 + I_1 + 4 = 0$$

$$11I_1 - 11I_2 = -4 \qquad \ldots\text{(iii)}$$

Fig. 3.351

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$

$$I_2 = 2.36 \text{ A}$$

$$I_N = I_2 = 2.36 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{Th}}{I_N} = \frac{26}{2.36} = 11.02\ \Omega$$

Step IV Calculation of I_L (Fig. 3.352)

$$I_L = 2.36 \times \frac{11.02}{11.02 + 5} = 1.62 \text{ A}$$

Fig. 3.352

Example 3.80 *Find the current through the 1 Ω resistor in the network of Fig. 3.353.*

Fig. 3.353

Solution

Step I Calculation of V_{Th} (Fig. 3.354)

From Fig. 3.354,

$$I_x = I_1 \quad \ldots\text{(i)}$$

Fig. 3.354

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 3I_x = 3I_1$$

$$4I_1 - I_2 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$12 - 6I_1 - 3I_2 = 0$$

$$6I_1 + 3I_2 = 12 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.67 \text{ A}$$

$$I_2 = 2.67 \text{ A}$$

Writing the V_{Th} equation,

$$3I_2 - V_{Th} = 0$$

$$3(2.67) - V_{Th} = 0$$

$$V_{Th} = 8 \text{ V}$$

Fig. 3.355

Step II Calculation of I_N (Fig. 3.355)

When a short circuit is placed across a 3 Ω resistor, it gets shorted as shown in Fig. 3.356.

From Fig. 3.356,

$$I_x = I_1 \quad \ldots\text{(i)}$$

Fig. 3.356

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 3I_x = 3I_1$$

$$4I_1 - I_2 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$12 - 6I_1 = 0$$

$$I_1 = 2 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 2 \text{ A}$$
$$I_2 = 8 \text{ A}$$
$$I_N = I_2 = 8 \text{ A}$$

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{8}{8} = 1\ \Omega$$

Step IV Calculation of I_L (Fig. 3.357)

$$I_L = 8 \times \frac{1}{1+1} = 4 \text{ A}$$

Fig. 3.357

Example 3.81 *Find the current through the 1.6 Ω resistor in the network of Fig. 3.358.*

Fig. 3.358

Solution

Step I Calculation of V_{Th} (Fig. 3.359)

From Fig. 3.359,

$$I_x = I_1 - I_2 \quad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 10 \quad \ldots\text{(ii)}$$

Fig. 3.359

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + 3I_x - 6I_2 = 0$$
$$-I_2 + I_1 + 3(I_1 - I_2) - 6I_2 = 0$$
$$4I_1 - 10I_2 = 0 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 4 \text{ A}$$

Writing the V_{Th} equation,

$$6I_2 - V_{\text{Th}} = 0$$
$$6(4) - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 24 \text{ V}$$

Step II Calculation of I_N (Fig. 3.360)

When a short circuit is placed across the 3 Ω resistor, it gets shorted as shown in Fig. 3.361.

From Fig. 3.361,

$$I_x = I_1 - I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 10 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-1(I_2 - I_1) + 3I_x = 0$$

$$-I_2 + I_1 + 3(I_1 - I_2) = 0$$

$$4I_1 - 4I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$

$$I_2 = 10 \text{ A}$$

$$I_N = I_2 = 10 \text{ A}$$

Fig. 3.360

Fig. 3.361

Step III Calculation of R_N

$$R_N = \frac{V_{\text{Th}}}{I_N} = \frac{24}{10} = 2.4\ \Omega$$

Step IV Calculation of I_L (Fig. 3.362)

$$I_L = 10 \times \frac{2.4}{2.4 + 1.6} = 6 \text{ A}$$

Fig. 3.362

3.5 MAXIMUM POWER TRANSFER THEOREM

It states that '*the maximum power is delivered from a source to a load when the load resistance is equal to the source resistance.*'

Proof From Fig. 3.363,

$$I = \frac{V}{R_s + R_L}$$

Power delivered to the load $R_L = P = I^2\ R_L = \dfrac{V^2 R_L}{(Rs + R_L)^2}$

To determine the value of R_L for maximum power to be transferred to the load,

$$\frac{dP}{dR_L} = 0$$

$$\frac{dP}{dR_L} = \frac{d}{dR_L} \frac{V^2}{(R_s + R_L)^2} R_L$$

$$= \frac{V^2[(R_s + R_L)^2 - (2R_L)(R_s + R_L)]}{(R_s + R_L)^4}$$

Fig. 3.363 *Network illustrating maximum power transfer theorem*

$$(R_s + R_L)^2 - 2\,R_L(R_s + R_L) = 0$$

$$R_s^{\,2} + R_L^{\,2} + 2R_sR_L - 2R_LR_s - 2R_L^2 = 0$$

$$R_s = R_L$$

Hence, the maximum power will be transferred to the load when load resistance is equal to the source resistance.

Steps to be followed in Maximum Power Transfer Theorem

1. Remove the variable load resistor R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Find the resistance R_L for maximum power transfer.

$$R_L = R_{\text{Th}}$$

5. Find the maximum power (Fig. 3.364).

Fig. 3.364 *Thevenin's equivalent network*

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L} = \frac{V_{\text{Th}}}{2R_{\text{Th}}}$$

$$P_{\max} = I_L^2\,R_L = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}^2} \times R_{\text{Th}} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}}$$

Example 3.82 *Find the value of resistance R_L in Fig. 3.365 for maximum power transfer and calculate maximum power.*

Fig. 3.365

Solution

Step I Calculation of V_{Th} (Fig. 3.366)

Applying KVL to the mesh,

$$3 - 2I - 2I - 6 = 0$$

$$I = -0.75\,\text{A}$$

Writing the V_{Th} equation,

$$6 + 2I - V_{\text{Th}} - 10 = 0$$

$$V_{\text{Th}} = 6 + 2I - 10 = 6 + 2(-0.75) - 10 = -5.5\,\text{V}$$

$$= 5.5\ \text{V}(\text{terminal } B \text{ is positive w.r.t } A)$$

Fig. 3.366

Step II Calculation of R_{Th} (Fig. 3.367)

$$R_{Th} = (2 \parallel 2) + 2 = 3\,\Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3\,\Omega$$

Step IV Calculation of P_{max} (Fig. 3.368)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(5.5)^2}{4 \times 3} = 2.52\,\text{W}$$

Fig. 3.367

Fig. 3.368

Example 3.83 *Find the value of resistance R_L in Fig. 3.369 for maximum power transfer and calculate maximum power.*

Fig. 3.369

Solution

Step I Calculation of V_{Th} (Fig. 3.370)

$$I_2 - I_1 = 4 \qquad \text{...(i)}$$

Applying KVL to the outer path,

$$8 - 1I_1 - 5I_1 - 5I_2 - 10 = 0$$

$$-6I_1 - 5I_2 = 2 \qquad \text{...(ii)}$$

Fig. 3.370

Solving Eqs (i) and (ii),

$$I_1 = -2\,\text{A}$$

$$I_2 = 2\,\text{A} \qquad \text{...(ii)}$$

Writing the V_{Th} equation,

$$8 - 1I_1 - V_{Th} = 0$$

$$V_{Th} = 8 - I_1 = 8 - (-2) = 10\,\text{V}$$

Step II Calculation of R_{Th} (Fig. 3.371)

$$R_{Th} = 10 \parallel 1 = 0.91\,\Omega$$

Fig. 3.371

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{Th} = 0.91\,\Omega$$

Step IV Calculation of P_{max}

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(10)^2}{4\times 0.91} = 27.47 \text{ W}$$

Fig. 3.372

Example 3.84 *Find the value of the resistance R_L in Fig. 3.373 for maximum power transfer and calculate the maximum power.*

Fig. 3.373

Solution

Step I Calculation of V_{Th} (Fig. 3.374)
For Mesh 1,

$$I_1 = 50$$

Applying KVL to Mesh 2,

$$-5(I_2 - I_1) - 2I_2 - 3I_2 = 0$$
$$5I_1 - 10I_2 = 0$$
$$I_1 = 2I_2$$
$$I_2 = 25\text{ A}$$
$$V_{Th} = 3I_2 = 3(25) = 75\text{ V}$$

Fig. 3.374

Step II Calculation of R_{Th} (Fig. 3.375)

$$R_{Th} = (5+2) \,||\, 3 = 2.1\,\Omega$$

Fig. 3.375

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{Th} = 2.1\,\Omega$$

Step IV Calculation of P_{max} (Fig. 3.376)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(75)^2}{4\times 2.1} = 669.64 \text{ W}$$

Fig. 3.376

Example 3.85 *Find the value of resistance R_L in Fig. 3.377 for maximum power transfer and calculate maximum power.*

Fig. 3.377

Solution

Step I Calculation of V_{Th} (Fig. 3.378)

Writing the current equation for the supermesh,

$$I_2 - I_1 = 6 \qquad \text{...(i)}$$

Applying KVL to the supermesh,

$$10 - 5I_1 - 2I_2 = 0$$

$$5I_1 + 2I_2 = 10 \qquad \text{...(ii)}$$

Fig. 3.378

Solving Eqs (i) and (ii),

$$I_1 = -0.29\,\text{A}$$

$$I_2 = 5.71\,\text{A}$$

Writing the V_{Th} equation,

$$V_{Th} = 2I_2 = 11.42\,\text{V}$$

Step II Calculation of R_{Th} (Fig. 3.379)

$$R_{Th} = (5 \parallel 2) + 3 + 4 = 8.43\ \Omega$$

Fig. 3.379

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 8.43\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.380)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(11.42)^2}{4 \times 8.43} = 3.87\ \text{W}$$

Fig. 3.380

Example 3.86 *Find the value of resistance R_L in Fig. 3.381 for maximum power transfer and calculate the maximum power.*

Fig. 3.381

Solution

Step I Calculation of V_{Th} (Fig. 3.382)

Applying KVL to Mesh 1,

$$120 - 10I_1 - 5(I_1 - I_2) = 0$$

$$15I_1 - 5I_2 = 120 \qquad ...(i)$$

Writing current equation for Mesh 2,

$$I_2 = -6 \qquad ...(ii)$$

Solving Eqs (i) and (ii),

$$I_1 = 6\text{ A}$$

Writing the V_{Th} equation,

$$120 - 10I_1 - V_{Th} = 0$$

$$V_{Th} = 120 - 10(6) = 60\text{ V}$$

Fig. 3.382

Step II Calculation of R_{Th} (Fig. 3.383)

$$R_{Th} = 10 \parallel 5 = 3.33\ \Omega$$

Fig. 3.383

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3.33\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.384)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(60)^2}{4 \times 3.33} = 270.27\text{ W}$$

Fig. 3.384

Example 3.87 *Find the value of resistance R_L in Fig. 3.385 for maximum power transfer and calculate the maximum power.*

Fig. 3.385

Solution

Step I Calculation of V_{Th} (Fig. 3.386)

$$I_1 = 3 \qquad \ldots(i)$$

Applying KVL to Mesh 2,

$$-25(I_2 - I_1) - 10I_2 - 6I_2 = 0$$

$$-25I_1 + 41I_2 = 0 \qquad \ldots(ii)$$

Fig. 3.386

Solving Eqs (i) and (ii),

$$I_2 = 1.83 \text{ A}$$

Writing the V_{Th} equation,

$$20 + V_{Th} - 10I_2 - 6I_2 = 0$$

$$V_{Th} = -20 + 10\,(1.83) + 6\,(1.83) = 9.28 \text{ V}$$

Fig. 3.387

Step II Calculation of R_{Th} (Fig. 3.387)

$$R_{Th} = 25 \,||\, (10 + 6) = 9.76\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 9.76\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.388)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(9.28)^2}{4 \times 9.76} = 2.21 \text{ W}$$

Fig. 3.388

Example 3.88 *Find the value of resistance R_L in Fig. 3.389 for maximum power transfer and calculate maximum power.*

Fig. 3.389

Solution

Step I Calculation of V_{Th} (Fig. 3.390)

Fig. 3.390

Meshes 1 and 2 will form a supermesh.
Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(i)}$$

Writing the voltage equation for the supermesh,

$$\begin{aligned} 5 - 1I_1 - 10(I_2 - I_3) &= 0 \\ I_1 + 10I_2 - 10I_3 &= 5 \end{aligned} \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$\begin{aligned} -10(I_3 - I_2) - 2I_3 - 3I_3 &= 0 \\ -10I_2 + 15I_3 &= 0 \end{aligned} \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.38 \text{ A}$$
$$I_2 = 1.38 \text{ A}$$
$$I_3 = 0.92 \text{ A}$$

Writing the V_{Th} equation,

$$V_{\text{Th}} = 3I_3 = 2.76 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.391)

Fig. 3.391

$$R_{\text{Th}} = 6.48\ \Omega$$

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 6.48\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 3.392)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(2.76)^2}{4 \times 6.48} = 0.29 \text{ W}$$

Fig. 3.392

Example 3.89 *For the network shown in Fig. 3.393, find the value of the resistance R_L for maximum power transfer and calculate the maximum power.*

Fig. 3.393

Solution

Step I Calculation of V_{Th} (Fig. 3.394)

Fig. 3.394

$$I_2 - I_1 = 2 \qquad \ldots\text{(i)}$$

$$I_2 = -3 \text{ A} \qquad \ldots\text{(ii)}$$

Solving Eqs (i), and (ii),

$$I_1 = -5 \text{ A}$$

Writing the V_{Th} equation,

$$8 - 2I_1 - 1I_2 - V_{Th} - 6 = 0$$

$$V_{Th} = 8 - 2(-5) - (-3) - 6 = 15 \text{ V}$$

Fig. 3.395

Step II Calculation of R_{Th} (Fig. 3.395)

$$R_{Th} = 5\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 5\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.396)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(15)^2}{4 \times 5} = 11.25 \text{ W}$$

Fig. 3.396

Example 3.90 *For the value of resistance R_L in Fig. 3.397 for maximum power transfer and calculate the maximum power.*

Fig. 3.397

Solution

Step I Calculation of V_{Th} (Fig. 3.398)

Fig. 3.398

By star-delta transformation (Fig. 3.399),

$$I = \frac{100}{5+5+20+9+9} = 2.08 \text{ A}$$

Writing the V_{Th} equation,

$$100 - 5I - V_{Th} - 9I = 0$$

$$V_{Th} = 100 - 14I$$

$$= 100 - 14(2.08)$$

$$= 70.88 \text{ V}$$

Fig. 3.399

Step II Calculation of R_{Th} (Fig. 3.400)

Fig. 3.400

$$R_{Th} = 23.92\ \Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 23.92\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.401)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(70.88)^2}{4 \times 23.92} = 52.51\ \text{W}$$

Fig. 3.401

Example 3.91 *For the value of resistance R_L in Fig. 3.402 for maximum power transfer and calculate the maximum power.*

Fig. 3.402

Solution

Step I Calculation of V_{Th} (Fig. 3.403)

Applying KVL to Mesh 1,

$$80 - 5I_1 - 10(I_1 - I_2) - 20(I_1 - I_2) - 20 = 0$$

$$35I_1 - 30I_2 = 60 \quad \ldots\text{(i)}$$

Writing the current equation for Mesh 2,

$$I_2 = 2 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 3.43 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 20\,(I_1 - I_2) - 20 = 0$$

$$V_{Th} = 20(3.43 - 2) + 20 = 48.6 \text{ V}$$

Fig. 3.403

Step II Calculation of R_{Th} (Fig. 3.404)

$$R_{Th} = 15 \parallel 20 = 8.57\ \Omega$$

Fig. 3.404

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 8.57\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.405)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(48.6)^2}{4 \times 8.57} = 68.9 \text{ W}$$

Fig. 3.405

Example 3.92 *For the value of resistance R_L in Fig. 3.406 for maximum power transfer and calculate the maximum power.*

Fig. 3.406

Solution

Step I Calculation of V_{Th} (Fig. 3.407)

$$I_1 = \frac{100}{10+30} = 2.5 \text{ A}$$

$$I_2 = \frac{100}{20+40} = 1.66 \text{ A}$$

Fig. 3.407

Writing the V_{Th} equation,

$$V_{Th} + 10\, I_1 - 20 I_2 = 0$$

$$V_{Th} = 20 I_2 - 10 I_1 = 20(1.66) - 10(2.5) = 8.2 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.408)

Fig. 3.408

Redrawing the network (Fig. 3.409),

$$R_{Th} = (10 \,||\, 30) + (20 \,||\, 40) = 20.83\ \Omega$$

Fig. 3.409

Step III Value of R_L

For maximum power transfer,

$$R_L = R_{Th} = 20.83\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.410)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(8.2)^2}{4 \times 20.83} = 0.81 \text{ W}$$

Fig. 3.410

Example 3.93 *For the value of resistance R_L in Fig. 3.411 for maximum power transfer and calculate the maximum power.*

Fig. 3.411

Solution

Step I Calculation of V_{Th} (Fig. 3.412)

Applying KVL to Mesh 1,

$$72 - 6I_1 - 3(I_1 - I_2) = 0$$

$$9I_1 - 3I_2 = 72 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 4I_2 = 0$$

$$-3I_1 + 9I_2 = 0 \quad \ldots\text{(ii)}$$

Fig. 3.412

Solving Eqs (i) and (ii),

$$I_1 = 9 \text{ A}$$

$$I_2 = 3 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 6I_1 - 2I_2 = 0$$

$$V_{Th} = 6I_1 + 2I_2 = 6(9) + 2(3) = 60 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.413)

Fig. 3.413

$$R_{Th} = [(6 \parallel 3) + 2] \parallel 4 = 2\ \Omega$$

Step III Calculation of R_L
For maximum power transfer,

$$R_L = R_{Th} = 2\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.414)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(60)^2}{4 \times 2} = 450\ \text{W}$$

Fig. 3.414

Example 3.94 *For the network shown in Fig. 3.415 find the value of the resistance R_L for maximum power transfer and calculate maximum power.*

Fig. 3.415

Solution

Step I Calculation of V_{Th} (Fig. 3.416)

Fig. 3.416

By source transformation, the current source of 25 A and the 5 Ω resistor is converted into an equivalent voltage source of 125 V and a series resistor of 5 Ω. Also the voltage source of 30 V is connected across the 10 Ω resistor. Hence, the 10 Ω resistor becomes redundant (Fig. 3.417).

Fig. 3.417

Applying KCL at the node,

$$\frac{V_{Th}-125}{15}-10+\frac{V_{Th}-30}{2}=0$$

$$V_{Th}=58.81\text{ V}$$

Fig. 3.418

Step II Calculation of R_{Th} (Fig. 3.418)

$$R_{Th}=15\,||\,2=1.76\ \Omega$$

Fig. 3.419

Step III Value of R_L

For maximum power transfer

$$R_L=R_{Th}=1.76\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.420)

$$P_{max}=\frac{V_{Th}^2}{4R_{Th}}=\frac{(58.81)^2}{4\times1.76}=491.28\text{ W}$$

Fig. 3.420

Example 3.95 *For the network shown in Fig. 3.421, find the value of the resistance R_L for maximum power transfer and calculate maximum power.*

Fig. 3.421

Solution

Step I Calculation of V_{Th} (Fig. 3.422)

Fig. 3.422

Applying KVL to the outer path,

$$10 - 2I - 12 - 5I - 8 = 0$$

$$I = -\frac{10}{7} = -1.43 \text{ A}$$

Writing the V_{Th} equation,

$$8 + 5I + 6 - V_{Th} = 0$$

$$V_{Th} = 8 + 6 + 5I = 8 + 6 + 5(-1.43) = 6.85 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.423)

$$R_{Th} = (2 \parallel 5) + 2$$
$$= 3.43\ \Omega$$

Fig. 3.423

Step III Value of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3.43\ \Omega$$

Step IV Calculation of P_{max} (Fig. 3.424)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(6.85)^2}{4 \times 3.43} = 3.42 \text{ W}$$

Fig. 3.424

EXAMPLES WITH DEPENDENT SOURCES

Example 3.96 *For the network shown in Fig. 3.425, find the value of R_L for maximum power transfer. Also, calculate maximum power.*

Fig. 3.425

Solution

Step I Calculation of V_{Th} (Fig. 3.426)

Applying KVL to the mesh,

$$10I - 20I - 40I - 50 = 0$$

$$I = -1 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 40I - 50 = 0$$

$$V_{Th} - 40(-1) - 50 = 0$$

$$V_{Th} = 10 \text{ V}$$

Fig. 3.426

Step II Calculation of I_N (Fig. 3.427)

From Fig. 3.427,

$$I = I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$10I - 20I_1 = 0$$

$$10I_2 - 20I_1 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-40I_2 - 50 = 0$$

$$I_2 = -1.25 \text{ A} \qquad \ldots\text{(iii)}$$

Fig. 3.427

Solving Eqs (i), (ii) and (iii),

$$I_1 = -0.625 \text{ A}$$

$$I_N = I_1 - I_2 = -0.625 + 1.25 = 0.625 \text{ A}$$

Step III Calculation of R_N

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{10}{0.625} = 16\ \Omega$$

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 16\ \Omega$$

Step V Calculation of P_{max} (Fig. 3.428)

$$P_{max} = \frac{V_{Th}}{4\,R_{Th}} = \frac{(10)^2}{4 \times 16} = 1.56 \text{ W}$$

Fig. 3.428

Example 3.97 *For the network shown in Fig. 3.429, calculate the maximum power that may be dissipated in the load resistor R_L.*

Fig. 3.429

Solution

Step I Calculation of V_{Th} (Fig. 3.430)

From Fig. 3.430,

$$I_x = I_2 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 10 \qquad \ldots\text{(ii)}$$

Fig. 3.430

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) + 2I_x - 6I_2 = 0$$
$$-4I_2 + 4I_1 + 2I_2 - 6I_2 = 0$$
$$4I_1 - 8I_2 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 10 \text{ A}$$
$$I_2 = 5 \text{ A}$$

Writing the V_{Th} equation,

$$6I_2 - 0 - V_{Th} = 0$$
$$V_{Th} = 6I_2 = 6(5) = 30 \text{ V}$$

Step II Calculation of I_N (Fig. 3.431)

From Fig. 3.431,

$$I_x = I_2 - I_3 \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I_1 = 10 \qquad \ldots\text{(ii)}$$

Fig. 3.431

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) + 2I_x - 6(I_2 - I_3) = 0$$
$$-4I_2 + 4I_1 + 2(I_2 - I_3) - 6I_2 + 6I_3 = 0$$
$$4I_1 - 8I_2 + 4I_3 = 0 \qquad \ldots\text{(iii)}$$

Applying KVL to Mesh 3,

$$-6(I_3 - I_2) - 3I_3 = 0$$
$$6I_2 - 9I_3 = 0 \qquad \ldots\text{(iv)}$$

Solving Eqs (ii), (iii) and (iv),

$$I_1 = 10 \text{ A}$$
$$I_2 = 7.5 \text{ A}$$
$$I_3 = 5 \text{ A}$$
$$I_N = I_3 = 5 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{5} = 6\ \Omega$$

Fig. 3.432

Step IV Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 6\ \Omega$$

Step V Calculation of P_{max}

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(30)^2}{4 \times 16} = 37.5\ \text{W}$$

Example 3.98 *For the network shown in Fig. 3.433, find the value of R_L for maximum power transfer. Also, find maximum power.*

Fig. 3.433

Solution

Step I Calculation of V_{Th} (Fig. 3.434)

From Fig. 3.434,

$$V_x = -1I = -I \qquad \ldots\text{(i)}$$

For Mesh 1,

$$I = -1 \qquad \ldots\text{(ii)}$$

$$V_x = 1\ \text{V}$$

Fig. 3.434

Writing the V_{Th} equation,

$$2V_x - 1I + 2 - V_{Th} = 0$$

$$2(1) - (-1) + 2 - V_{Th} = 0$$

$$V_{Th} = 5\ \text{V}$$

Step II Calculation of I_N (Fig. 3.435)

From Fig. 3.435,

$$V_x = -1I_1 = -I_1 \qquad \ldots\text{(i)}$$

Meshes 1 and 2 will form a supermesh.

Writing the current equation for the supermesh,

$$I_2 - I_1 = 1 \qquad \ldots\text{(ii)}$$

Fig. 3.435

Applying KVL to the outer path of the supermesh,

$$2V_x - 1I_1 + 2 = 0$$

$$2(-I_1) - I_1 + 2 = 0$$

$$3I_1 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii),

$$I_1 = 0.67 \text{ A}$$
$$I_2 = 1.67 \text{ A}$$
$$I_N = I_2 = 1.67 \text{ A}$$

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{5}{1.67} = 3\ \Omega$$

Step IV Calculation of R_L
For maximum power transfer,

$$R_L = R_{\text{Th}} = 3\ \Omega$$

Step V Calculation of $P_{\max}$ (Fig. 3.436)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4\,R_{\text{Th}}} = \frac{(5)^2}{4 \times 3} = 2.08 \text{ W}$$

Fig. 3.436

Example 3.99 *What will be the value of R_L in Fig. 3.437 to get maximum power delivered to it? What is the value of this power?*

Fig. 3.437

Solution

Step I Calculation of V_{Th} (Fig. 3.438)
By source transformation,
From Fig. 3.438,

$$V_{\text{Th}} = 4I$$

Applying KVL to the mesh,

$$12 - 4I + 0.5\,V_{\text{Th}} - 4I = 0$$
$$12 - V_{\text{Th}} + 0.5\,V_{\text{Th}} - V_{\text{Th}} = 0$$
$$V_{\text{Th}} = 8 \text{ V}$$

Fig. 3.438

Step II Calculation of I_N (Fig. 3.439)
If two terminals A and B are shorted, the 4 Ω resistor gets shorted.

$$V = 0$$

Fig. 3.439

Dependent source 0.5 V depends on the controlling variable V. When $V = 0$, the dependent source vanishes, ie. 0.5 $V = 0$ as shown in Fig. 3.441.

$$I_N = \frac{12}{4} = 3 \text{ A}$$

Fig. 3.440

Fig. 3.441

Step III Calculation of R_{Th}

$$R_{\text{Th}} = \frac{V_{\text{Th}}}{I_N} = \frac{8}{3} = 2.67\ \Omega$$

Step IV *Calculation of* R_L

For maximum power transfer,

$$R_L = R_{\text{Th}} = 2.67\ \Omega$$

Step V Calculation of $P_{\max}$ (Fig. 3.442)

$$P_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{(8)^2}{4 \times 2.67} = 6 \text{ W}$$

Fig. 3.442

3.6 RECIPROCITY THEOREM

It states that '*in a linear, bilateral, active, single source network, the ratio of excitation to response remains same when the positions of excitation and response are interchanged*.'

In other words, it may be stated as 'if a single voltage source V_a in the branch 'a' produces a current I_b in the branch 'b' then if the voltage source V_a is removed and inserted in the branch 'b', it will produce a current I_b in branch 'a''.

Fig. 3.443 *Network*

Explanation Consider a network shown in Fig. 3.443.

When the voltage source V is applied at the port 1, it produces a current I at the port 2. If the positions of the excitation (source) and response are interchanged, i.e., if the voltage source is applied at the port 2 then it produces a current I at the port 1.

Fig. 3.444 *Network when excitation and response are interchanged*

The limitation of this theorem is that it is applicable only to a single-source network. This theorem is not applicable in the network which has a dependent source. This is applicable only in linear and bilateral networks. In the reciprocity theorem, position of any passive element (R, L, C) do not change. Only the excitation and response are interchanged.

Steps to be followed in Reciprocity Theorem

1. Identify the branches between which reciprocity is to be established.
2. Find the current in the branch when excitation and response are not interchanged.
3. Find the current in the branch when excitation and response are interchanged.

Example 3.100 *Calculate current I and verify the reciprocity theorem for the network shown in Fig. 3.445.*

Fig. 3.445

Solution

Case I Calculation of current I when excitation and response are not interchanged (Fig. 3.446)

Fig. 3.446

Applying KVL to Mesh 1,

$$20 - 5I_1 - 10(I_1 - I_2) = 0$$

$$15I_1 - 10I_2 = 20 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_1) - 4I_2 - 6I_2 = 0$$

$$-10I_1 + 20I_2 = 0 \qquad \ldots\text{(ii)}$$

Solving. Eqs (i) and (ii),

$$I_1 = 2 \text{ A}$$

$$I_2 = 1 \text{ A}$$

$$I = I_2 = 1 \text{ A}$$

Case II Calculation of current I when excitation and response are interchanged (Fig. 3.447).

Fig. 3.447

Applying KVL to Mesh 1,

$$-5I_1 - 10(I_1 - I_2) = 0$$

$$15I_1 - 10I_2 = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-10(I_2 - I_1) - 4I_2 - 20 - 6I_2 = 0$$

$$-10I_1 + 20I_2 = -20 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = -1 \text{ A}$$

$$I_2 = -1.5 \text{ A}$$

$$I = -I_1 = 1 \text{ A}$$

Since the current I remains the same in both the cases, reciprocity theorem is verified.

Example 3.101 *Find the current I and verify reciprocity theorem for the network shown in Fig. 3.446.*

Fig. 3.448

Solution

Case I Calculation of the current I when excitation and response are not interchanged (Fig. 3.449)

Fig. 3.449

Applying KVL to Mesh 1,

$$5 - 2I_1 - 3(I_1 - I_2) - 4(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 4I_3 = 5 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 3I_2 = 0$$

$$-3I_1 + 8I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-4(I_3 - I_1) - 4I_3 = 0$$

$$-4I_1 + 8I_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.85 \text{ A}$$

$$I_2 = 0.32 \text{ A}$$

$$I_3 = 0.43 \text{ A}$$

$$I = I_3 = 0.43 \text{ A}$$

Case II Calculation of current I when excitation and response are interchanged (Fig. 3.450).

Fig. 3.450

Applying KVL to Mesh 1,

$$-2I_1 - 3(I_1 - I_2) - 4(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 4I_3 = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 3I_2 = 0$$

$$-3I_1 + 8I_2 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-4(I_3 - I_1) + 5 - 4I_3 = 0$$

$$-4I_1 + 8I_3 = 5 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 0.43 \text{ A}$$

$$I_2 = 0.16 \text{ A}$$

$$I_3 = 0.84 \text{ A}$$

$$I = I_1 = 0.43 \text{ A}$$

Since the current I remains the same in both the cases, reciprocity theorem is verified.

Example 3.102 *Find the voltage V and verify reciprocity theorem for the network shown in Fig. 3.451.*

Fig. 3.451

Solution

Case I Calculation of the voltage V when excitation and response are not interchanged (Fig. 3.451)

For Mesh 1,

$$I_1 = 10 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-4(I_2 - I_1) - 2I_2 - 5(I_2 - I_3) = 0$$

$$-4I_1 + 11I_2 - 5I_3 = 0 \qquad \ldots\text{(ii)}$$

Fig. 3.452

Applying KVL to Mesh 3,

$$-6(I_3 - I_1) - 5(I_3 - I_2) - 8I_3 = 0$$

$$-6I_1 - 5I_2 + 19I_3 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs (i), (ii) and (iii),

$$I_1 = 10 \text{ A}$$

$$I_2 = 5.76 \text{ A}$$

$$I_3 = 4.67 \text{ A}$$

$$V = 5(I_2 - I_3) = 5(5.76 - 4.67) = 5.45 \text{ V}$$

Case II Calculation of voltage V when excitation and response are interchanged (Fig. 3.453).

Fig. 3.453

By source transformation (Fig. 3.454),

Applying KVL to Mesh 1,

$$-4I_1 - 2I_1 - 50 - 5(I_1 - I_2) = 0$$

$$11I_1 - 5I_2 = -50 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-6I_2 - 5(I_2 - I_1) + 50 - 8I_2 = 0$$

$$-5I_1 + 19I_2 = 50 \quad \ldots\text{(ii)}$$

Fig. 3.454

Solving Eqs (i) and (ii),

$$I_1 = -3.8 \text{ A}$$

$$I_2 = 1.63 \text{ A}$$

From Fig. 3.454,

$$V + 4I_1 + 6I_2 = 0$$

$$V + 4(-3.8) + 6(1.63) = 0$$

$$V = 5.42 \text{ V}$$

Since the voltage V is same in both the cases, the reciprocity theorem is verified.

3.7 MILLMAN'S THEOREM

It states that '*if there are n voltage sources* $V_1, V_2, \ldots, V_n$ *with internal resistances* $R_1, R_2, \ldots, R_n$ *respectively connected in parallel then these voltage sources can be replaced by a single voltage source* V_m *and a single series resistance* R_m, '(Fig. 3.455).

Fig. 3.455 *Millman's network*

where
$$V_m = \frac{V_1G_1 + V_2G_2 + \ldots + V_nG_n}{G_1 + G_2 + \ldots + G_n}$$

and
$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

Explanation By source transformation, each voltage source in series with a resistance can be converted to a current source in parallel with a resistance as shown in Fig. 3.456.

Fig. 3.456 *Equivalent network*

Let I_m be the resultant current of the parallel current sources and R_m be the equivalent resistance as shown in Fig. 3.457.

Fig. 3.457 *Equivalent network*

$$I_m = I_1 + I_2 + \ldots + I_n = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \ldots + \frac{V_n}{R_n} = V_1 G_1 + V_2 G_2 + \ldots V_n G_n$$

$$\frac{1}{R_m} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

$$G_m = G_1 + G_2 + \ldots + G_n$$

By source transformation, the parallel circuit can be converted into a series circuit as shown in Fig. 3.458.

Fig. 3.458 *Millman's equivalent network*

$$V_m = I_m R_m = \frac{I_m}{G_m} = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

Dual of Millman's Theorem

It states that '*if there are n current sources $I_1, I_2, \ldots, I_n$ with internal resistances $R_1, R_2, \ldots, R_n$ respectively, connected in series then these current sources can be replaced by a single current source I_m and a single parallel resistance R_m*' (Fig. 3.459).

Fig. 3.459 *Millman's network*

where
$$I_m = \frac{I_1 R_1 + I_2 R_2 + \ldots + I_n R_n}{R_1 + R_2 + \ldots + R_n}$$

$$R_m = R_1 + R_2 + \ldots + R_n$$

Steps to be followed in Millman's Theorem

1. Remove the load resistance R_L.
2. Find Millman's voltage across points A and B.

$$V_m = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

3. Find the resistance R_m between points A and B.

$$R_m = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

4. Replace the network by a voltage source V_m in series with the resistance R_m.
5. Find the current through R_L using ohm's law.

$$I_L = \frac{V_m}{R_m + R_L}$$

Example 3.103 *Find Millman's equivalent for the left of the terminals A-B in Fig. 3.460.*

Fig. 3.460

Solution By source transformation, the network is redrawn as shown in Fig. 3.461.

Fig. 3.461

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2} = \frac{6\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right)}{\frac{1}{4} + \frac{1}{2}} = 3.33 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2} = \frac{1}{\frac{1}{4} + \frac{1}{2}} = 1.33\ \Omega$$

Step III Millman's Equivalent Network (Fig. 3.462)

Fig. 3.462

Example 3.104 *Find the current through the 10 Ω resistor in the network of Fig. 3.463.*

Fig. 3.463

Solution

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{5\left(\frac{1}{2}\right) - 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{6}\right)}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} = 2.73 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} = 1.09\ \Omega$$

Step III Calculation of I_L (Fig. 3.464)

$$I_L = \frac{2.73}{1.09 + 10} = 0.25 \text{ A}$$

Fig. 3.464

Example 3.105 *Find the current through the 10 Ω resistor in the network of Fig. 3.465.*

Fig. 3.465

Solution

Since the 40 Ω resistor is connected in parallel with the 50 V source, it becomes redundant. The network can be redrawn as shown in Fig. 3.466.

Fig. 3.466

Step I Calculation of V_m

$$V_m = \frac{V_1 G_1 + V_2 G_2}{G_1 + G_2} = \frac{50\left(\frac{1}{50}\right) - 100\left(\frac{1}{20}\right)}{\frac{1}{50} + \frac{1}{20}} = -57.15 \text{ V}$$

Step II Calculation of R_m

$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2} = \frac{1}{\frac{1}{50} + \frac{1}{20}} = 14.29\ \Omega$$

Step III Calculation of I_L (Fig. 3.467)

$$I_L = \frac{57.15}{14.29 + 10} = 2.35\ \text{A}$$

Fig. 3.467

Example 3.106 *Draw Millman's equivalent network across terminals AB in the network of Fig. 3.468.*

Fig. 3.468

Step I By source transformation, the network is redrawn as shown in Fig. 3.469.

Fig. 3.469

Step II Applying Millman's theorem at terminals *CD,*

$$V_{m_1} = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{-8\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right) + 6(1)}{\frac{1}{2} + \frac{1}{2} + 1} = 3\ \text{V}$$

$$R_{m_1} = \frac{1}{G_{m_1}} = \frac{1}{G_1 + G_2 + G_3} = \frac{1}{\frac{1}{2} + \frac{1}{2} + 1} = 0.5\ \Omega$$

Step III Applying Millman's theorem at terminals *CA*,

$$V_{m_2} = \frac{V_4\, G_4 + V_5\, G_5}{G_4 + G_5} = \frac{6\left(\frac{1}{2}\right) + 3(1)}{\frac{1}{2} + 1} = 4\ \text{V}$$

$$R_{m_2} = \frac{1}{G_{m_2}} = \frac{1}{G_4 + G_4} = \frac{1}{\frac{1}{2} + 1} = 0.67\ \Omega$$

Step IV Millman's Equivalent Network (Fig. 3.470)
Simplifying Fig. 3.470 further, the Millman's equivalent network is shown in fig. 3.471.

Fig. 3.470

Fig. 3.471

3.8 TELLEGEN'S THEOREM

It states that '*the algebraic sum of the powers in all branches of the network at any instant is zero*'.

$$\sum_{k=1}^{n} V_k\ I_k = 0$$

This condition is valid for the network which obeys kirchhoff's voltage and current laws.

Explanation Consider a network shown in Fig. 3.472.

Fig. 3.472 *Network illustrating Tellegen's theorem*

According to Tellegen's theorem,

$$\sum_{k=1}^{6} V_k\ I_k = V_1\, I_1 + V_2\, I_2 + V_3\, I_3 + V_4\, I_4 + V_5\, I_5 + V_6\, I_6$$

$$= V_1\, I_1 + (V_1 - V_4) I_2 + (V_1 - V_6) I_3 + V_4\, I_4 + (V_4 - V_6) I_5 + V_6\ I_6$$

$$= V_1 (I_1 + I_2 + I_3) + V_4 (-I_2 + I_4 + I_5) + V_6 (-I_3 - I_5 + I_6)$$

Applying KCL at Node 1,

$$I_1 + I_2 + I_3 = 0$$

Applying KCL at Node 4,

$$I_2 = I_4 + I_5$$

Applying KCL at Node 6,

$$I_5 + I_3 = I_6$$

$$\sum_{k=1}^{6} V_k\ I_k = V_1(0) + V_4(0) + V_6(0) = 0$$

Hence, Tellegen's theorem is verified.

Example 3.107 *Determine the power supplied by the source in the network of Fig. 3.473.*

$V_1 = 5$ V, $I_1 = 2$ A
$V_2 = 10$ V, $I_2 = 0.8$ A
$V_3 = 3$ V, $I_3 = 6$ A
$V_4 = 7$ V, $I_4 = 3$ A
$V_5 = 8$ V, $I_5 = 4$ A

Fig. 3.473

Solution By Tellegen's theorem,

Power supplied by the source $= V_1 I_1 + V_2 I_2 + V_3 I_3 + V_4 I_4 + V_5 I_5$

$$= 5 \times 2 + 10 \times 0.8 + 3 \times 6 + 7 \times 3 + 8 \times 4 = 89 \text{ W}$$

Example 3.108 *Prove the Tellegen's theorem for the network shown in Fig. 3.474.*

Fig. 3.474

$V_1 = 8$ V, $V_2 = 4$ V, $V_4 = 2$ V
$I_1 = 4$ A, $I_2 = 2$ A, $I_3 = 1$ A

Solution Applying KVL to Mesh 1,

$$V_1 - V_2 - V_3 = 0$$

$$V_3 = V_1 - V_2 = 8 - 4 = 4 \text{ V}$$

Applying KVL to Mesh 2,

$$V_3 - V_4 - V_5 = 0$$

$$V_5 = V_3 - V_4 = 4 - 2 = 2 \text{ V}$$

Applying KVL to Mesh 3,

$$-V_6 + V_4 + V_2 = 0$$
$$V_6 = V_2 + V_4 = 4 + 2 = 6 \text{ V}$$

Applying KCL at Node 1,

$$I_1 + I_2 + I_6 = 0$$
$$I_6 = -I_1 - I_2 = -4 - 2 = -6 \text{ A}$$

Applying KCL at Node 2,

$$I_2 = I_3 + I_4$$
$$I_4 = I_2 - I_3 = 2 - 1 = 1 \text{ A}$$

Applying KCL at Node 3,

$$I_5 = I_4 + I_6 = 1 + (-6) = -5 \text{ A}$$

$$\sum_{b=1}^{6} V_b\, I_b = V_1 I_1 + V_2 I_2 + V_3 I_3 + V_4 I_4 + V_5 I_5 + V_6 I_6 = 8\times 4 + 4\times 2 + 4\times 1 + 2\times 1 + 2\times(-5) + 6\times(-6) = 0$$

Hence, Tellegen's theorem is verified.

Example 3.109 *Find the value of source V_2 in Fig. 3.475, using Tellegen' theorem if the power absorbed by V_2 is 20 W.*

Fig. 3.475

Solution Finding Thevenin's equivalent network across voltage source V_2 (Fig. 3.476).

Fig. 3.476

Fig. 3.477

$$I_1 = \frac{100}{10+10} = 5 \text{ A}$$
$$V_{Th} = 10 I_1 = 10(5) = 50 \text{ V}$$

$$R_{Th} = (10 \,||\, 10) + 5 + 5 = 15\ \Omega$$

Thevenin's equivalent network is shown in Fig. 3.478.

Fig. 3.478

$$V_2 I = 20$$

Applying Tellegen's theorem,

$$-50I + 15I^2 + V_2 I = 0$$

$$15I^2 - 50I + 20 = 0$$

$$I = 2.87 \text{ or } 0.46 \text{ A}$$

$$V_2 = \frac{20}{I} = \frac{20}{2.87} = 6.97 \text{ V}$$

or

$$V_2 = \frac{20}{0.46} = 43.48 \text{ V}$$

3.9 SUBSTITUTION THEOREM

It states that '*any branch in a network can be substituted by a different branch without disturbing the voltages and currents in the entire network provided the new branch has the same set of terminal voltage and current as the original network.*'

Explanation Any branch can be replaced by any combination of elements that will have the same voltage across it and same current through it as the original branch (Fig. 3.479).

Fig. 3.479 *Substitution theorem*

Steps to be followed in Substitution Theorem

1. Find the branch voltage and branch current in the network.
2. Replace the branch by an independent voltage source and find the voltage and current in the network.
3. Replace the branch by an independent current source and find the voltage and current in the network.

Example 3.110 *In the network of Fig. 3.480 verify substitution theorem by replacing the 6 Ω resistor by (a) a voltage source, and (b) a current source.*

Fig. 3.480

Solution

Step I Calculation of current through 6 Ω resistor (Fig. 3.481)

Applying KVL to Mesh 1,

$$100 - 2I_1 - 8(I_1 - I_2) = 0$$

$$10I_1 - 8I_2 = 100 \qquad \ldots\text{(i)}$$

Fig. 3.481

Applying KVL to Mesh 2,

$$-8(I_2 - I_1) - 4I_2 - 6I_2 = 0$$

$$-8I_1 + 18I_2 = 0 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 15.52 \text{ A}$$

$$I_2 = 6.9 \text{ A}$$

$$I_{6\,\Omega} = I_2 = 6.9 \text{ A}$$

$$V_{6\,\Omega} = 6I_2 = 6(6.9) = 41.4 \text{ V}$$

Step II When the 6 Ω resistor is replaced by the voltage source of 41.4 V (Fig. 3.482)

Applying KVL to Mesh 1,

$$100 - 2I_1 - 8(I_1 - I_2) = 0$$

$$10I_1 - 8I_2 = 100 \qquad \ldots\text{(i)}$$

Fig. 3.482

Applying KVL to Mesh 2,

$$-8(I_2 - I_1) - 4I_2 - 41.4 = 0$$

$$-8I_1 + 12I_2 = -41.4 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 51.51 \text{ A}$$

$$I_2 = 6.9 \text{ A}$$

By replacing the 6 Ω resistor by a voltage source has not affected the currents I_1 and I_2. Hence, the substitution theorem is verified.

Step III When the 6 Ω resistor is replaced by current source of 6.9 A (Fig. 3.483)

Applying KVL to Mesh 1,

$$100 - 2I_1 - 8(I_1 - I_2) = 0$$

$$10I_1 - 8I_2 = 100 \qquad \ldots\text{(i)}$$

Fig. 3.483

For Mesh 2,

$$I_2 = 6.9 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$I_1 = 15.52 \text{ A}$$

$$I_2 = 6.9 \text{ A}$$

By replacing the 6 Ω resistor by a current source has not affected the currents I_1 and I_2. Hence, the substitution theorem is verified.

3.10 COMPENSATION THEOREM

It states that '*in any linear bilateral active network, if any branch carrying a current I has its resistance R changed by an amount δR, the resulting changes that occur in the other branches are the same as those which would have been produced by an opposing voltage source of value V_c (IδR) introduced into the modified branch.*'

Explanation Consider a network shown in Fig. 3.484 (a), having load resistance R.

Fig. 3.484 *Network illustrating compensation theorem*

$$I = \frac{V_{Th}}{R_{Th} + R}$$

If the load resistance R be changed to $R + \delta R$ as shown in Fig. 3.484(b) then the current flowing in the circuit is

$$I' = \frac{V_{Th}}{R_{Th} + R + \delta R}$$

The change in the current is

$$\delta I = I' - I = \frac{V_{Th}}{R_{Th} + R + \delta R} - \frac{V_{Th}}{R_{Th} + R}$$

$$= -\frac{V_{Th}\ \delta R}{(R_{Th} + R + \delta R)\,(R_{Th} + R)}$$

$$= -\frac{V_{Th}}{R_{Th} + R}\,\frac{\delta R}{R_{Th} + R + \delta R}$$

$$= -\frac{I\ \delta R}{R_{Th} + R + \delta R}$$

$$= -\frac{V_c}{R_{Th} + R + \delta R}$$

where $V_c = I\ \delta R$ and is called the *compensation voltage*.

δI has the same direction as I. This shows that the change in current δI due to a change in any branch in a linear network can be calculated by determining the current in that branch in a network obtained from the original network by removing all the independent sources and placing a voltage source called compensation source in series with the branch whose value is $V_c = I\ \delta R$, whose I is the current through the branch before its resistance is changed and δR is the change in resistance. (Fig. 3.485).

Fig. 3.485 *Compensation network*

Example 3.111 *In the network of Fig. 3.486, the resistance of 4 Ω is changed to 2 Ω. Verify the compensation theorem.*

Fig. 3.486

Solution

Step I Calculation of change in current

Finding Thevenin's equivalent network across the 4 Ω resistor,

$$V_{Th} = 1 \times \frac{8}{8+1} = 0.89 \text{ V}$$

$$R_{Th} = \frac{1 \times 8}{1+8} = 0.89\ \Omega$$

Thevenin's equivalent network is shown in Fig. 3.487.

Fig. 3.487

$$I_L = \frac{0.89}{0.89+4} = 0.18 \text{ A}$$

When resistance of 4 Ω is changed to 2 Ω,

$$I_L' = \frac{0.89}{0.89+2} = 0.31 \text{ A}$$

$$\delta I_L = I_L' - I_L = 0.31 - 0.18 = 0.13 \text{ A}$$

Fig. 3.488

Step II Calculation of change of current by compensation theorem

$$\delta R = 2 - 4 = -2\ \Omega$$

$$R + \delta R = 4 - 2 = 2\ \Omega$$

$$V_c = I_L \delta R = 0.18 \times (-2) = -0.36 \text{ V}$$

The compensating network is shown in Fig. 3.489.

Fig. 3.489

$$\delta I_L = \frac{0.36}{0.89+2} = 0.12 \text{ A}$$

Since change in current is same in both the steps, the compensation theorem is verified.

Exercises

Superposition Theorem

3.1 Find the current through the 10 Ω resistor in Fig. 3.490.

Fig. 3.490

[0.37 A]

3.2 Find the current through the 8 Ω resistor in Fig. 3.491.

Fig. 3.491

[16.2 A]

3.3 Find the potential across the 3 Ω resistor in Fig. 3.492.

Fig. 3.492

[3.3 V]

3.4 Calculate the current through the 10 Ω resistor in Fig. 3.493.

Fig. 3.493

[1.62 A]

3.5 Find the current through the 1Ω resistor in Fig. 3.494.

Fig. 3.494

[0.41 A]

3.6 Find the current through the 4 Ω resistor in Fig. 3.495.

Fig. 3.495

[1.33 A]

3.7 Find the current I_x in Fig. 3.496.

Fig. 3.496

[−1.143 A]

3.8 Find the voltage V_x in Fig. 3.497.

Fig. 3.497

[−38.5 V]

3.9 Determine the voltages V_1 and V_2 in Fig. 3.498.

Fig. 3.498

[6 V,12 V]

3.10 Find the voltage V_x in Fig. 3.499.

Fig. 3.499

[7.5 V]

Thevenin's Theorem

3.11 Find the current through the 5 Ω resistor in Fig. 3.500.

Fig. 3.500

[3.87 A]

3.12 Find the current through the 6 Ω resistor in Fig. 3.501.

Fig. 3.501

[1.26 A]

3.13 Find the current through the 6 Ω resistor in Fig. 3.502.

Fig. 3.502

[2.04 A]

3.14 Find the current through the 2 Ω resistor connected between terminals *A* and *B* in Fig. 3.503.

Fig. 3.503

[1.26 A]

3.15 Find the current through the 5 Ω resistor in Fig. 3.504.

Fig. 3.504

[4.67 A]

3.16 Find the current through the 20 Ω resistor in Fig. 3.505.

Fig. 3.505

[1.54 A]

3.17 Calculate the current through the 10 Ω resistor in Fig. 3.506.

Fig. 3.506

[1.62 A]

3.18 Determine Thevenin's equivalent network for figures 3.507 to 3.510 shown below.

(i)

Fig. 3.507

[−58 V, 12 Ω]

(ii)

Fig. 3.508

[9.09 V, 9.09 Ω]

(iii)

Fig. 3.509

[8 V, 2.66 Ω]

(iv)

Fig. 3.510

[150 V, 20 Ω]

3.19 Find the current I_x in Fig. 3.511.

Fig. 3.511

[4 A]

3.20 Find the current in the 24 Ω resistor in Fig. 3.512.

Fig. 3.512

[0.225 A]

Norton's Theorem

3.21 Find the current through the 10 Ω resistor in Fig. 3.513.

Fig. 3.513

[0.68 A]

3.22 Find the current through the 20 Ω resistor in Fig. 3.514.

Fig. 3.514

[0.61 A]

3.23 Find the current through the 2 Ω resistor in Fig. 3.515.

Fig. 3.515

[5 A]

3.24 Find the current through the 5 Ω resistor in Fig. 3.516.

Fig. 3.516

[4.13 A]

3.25 Find Norton's equivalent circuit for the portion of network shown in Fig. 3.517 to the left of *ab*. Hence obtain the current in the 10 Ω resistor.

Fig. 3.517

[0.053 A]

3.26 Find Norton's equivalent network and hence find the current in the 10 Ω resistor in Fig. 3.518.

Fig. 3.518

[0.25 A]

3.27 Find Norton's equivalent network In Fig. 3.519.

Fig. 3.519

[0.533 A, 31 Ω]

Maximum Power Transfer Theorem

3.28 Find the value of the resistance R_L in Fig. 3.520 for maximum power transfer and calculate the maximum power.

Fig. 3.520

[1.75 Ω,1.29 W]

3.29 Find the value of the resistance R_L in Fig. 3.521 for maximum power transfer and calculate the maximum power.

Fig. 3.521

[2.36 Ω, 940 W]

3.30 Find the value of the resistance R_L in Fig. 3.522 for maximum power transfer and calculate the maximum power.

Fig. 3.522

[2.18 Ω, 29.35 W]

3.31 Find the value of the resistance R_L in Fig. 3.523 for maximum power transfer and calculate the maximum power.

Fig. 3.523

[3 Ω, 2.52 W]

3.32 Find the value of the resistance R_L in Fig. 3.524 for maximum power transfer and calculate the maximum power.

Fig. 3.524

[1.76 Ω, 490.187 W]

Objective-Type Questions

3.1 The value of the resistance R connected across the terminals A and B in Fig. 3.525, which will absorb the maximum power is

Fig. 3.525

(a) 4 kΩ (b) 4.11 kΩ
(c) 8 kΩ (d) 9 k Ω

3.2 Superposition theorem is not applicable to networks containing

(a) nonlinear elements
(b) dependent voltage source
(c) dependent current source
(d) transformers

3.3 The value of R required for maximum power transfer in the network shown in Fig. 3.526 is

Fig. 3.526

(a) 2 Ω (b) 4 Ω (c) 8 Ω (d) 16 Ω

3.4 In the network of Fig. 3.527, the maximum power is delivered to R_L if its value is

Fig. 3.527

(a) 16 Ω (b) $\frac{40}{3}$ Ω
(c) 60 Ω (d) 20 Ω

3.5 The maximum power that can be transferred to the load R_L from the voltage source in Fig. 3.528 is

Fig. 3.528

(a) 1 W (b) 10 W
(c) 0.25 W (d) 0.5 W

3.6 For the circuit shown in Fig. 3.529, Thevenin's voltage and Thevenin's equivalent resistance at terminals a-b is

Fig. 3.529

(a) 5 V and 2 Ω (b) 7.5 V and 2.5 Ω
(c) 4 V and 2 Ω (d) 3 V and 2.5 Ω

3.7 The value of R_L in Fig. 3.530 for maximum power transfer is

Fig. 3.530

(a) 3 Ω (b) 1.125 Ω
(c) 4.1785 Ω (d) none of these

Answers to Objective-Type Questions

3.1 (a)	3.2 (a)	3.3 (c)	3.4 (a)	3.5 (c)	3.6 (b)	3.7 (a)

4

Single-Phase ac Circuits

4.1 INTRODUCTION

An alternating waveform changes its magnitude and direction periodically. Many times alternating voltages and currents are represented by a sinusoidal waveform. A sinusoidal waveform is more useful than other waveforms, like square, triangular, sawtooth etc. Any waveform can be written in terms of sinusoidal function with the help of Fourier series. The derivative and integration of sinusoidal waveform result in sinusoidal waveform only. The sinusoidal waveform can be easily analyzed and generated.

4.2 GENERATION OF ALTERNATING VOLTAGES

An alternating voltage can be generated either by rotating a coil in a stationary magnetic field or by a rotating a magnetic field within a stationary coil. In both the cases, the magnetic field is cut by the conductors or coils and an emf is induced in the coil according to Faraday's laws of electromagnetic induction. The magnitude of the induced emf depends upon the number of turns of the coil, the strength of the magnetic field and the speed at which the coil or magnetic field rotates.

Consider a rectangular coil of N turns of area A m^2 and rotating in anti-clockwise direction with angular velocity of ω radians per second in a uniform magnetic field as shown in Fig. 4.1.

Let ϕ_m be the maximum flux cutting the coil when its axis coincides with XX' axis (reference position of the coil). Thus when the coil is along XX', the flux linking with it is maximum, i.e., ϕ_m. When the coil is along YY', i.e., parallel to the lines of flux, the flux linking with it is zero.

The coil rotates through an angle $\theta = \omega t$ at any instant t. At this instant, the flux linking with the coil is

$$\phi = \phi_m \cos \omega t$$

(a)

(b)

Fig. 4.1 *Generation of alternating voltage*

According to Faraday's laws of electromagnetic induction,

$$e = -N\frac{d\phi}{dt} = -N\frac{d}{dt}(\phi_m \cos\omega t) = N\,\phi_m\,\omega \sin\omega t = E_m \sin\omega t$$

where $E_m = N\phi_m\omega$

$= \text{maximum value of induced emf}$

When $\omega t = 0, \quad \sin\omega t = 0, \quad e = 0$

When $\omega t = \frac{\pi}{2}, \quad \sin\frac{\pi}{2} = 1 \quad e = E_m$

If the induced emf is plotted against time, a sinusoidal waveform is obtained as shown in Fig. 4.2.

Fig. 4.2 *Sinusoidal wave form*

4.3 TERMS RELATED TO ALTERNATING QUANTITIES

1. ***Waveform*** A waveform is a graph in which the instantaneous value of any quantity is plotted against time. Figure 4.3 shows a few waveforms.
2. ***Cycle*** One complete set of positive and negative values of an alternating quantity is termed a cycle.
3. ***Frequency*** The number of cycles per second of an alternating quantity is known as its frequency. It is denoted by f and is measured in hertz (Hz) or cycles per second (c/s).
4. ***Time Period*** The time taken by an alternating quantity to complete one cycle is called its time period. It is denoted by T and is measured in seconds.

$$T = \frac{1}{f}$$

Fig. 4.3 *Alternating waveforms*

5. ***Amplitude*** The maximum positive or negative value of an alternating quantity is called the amplitude.
6. ***Phase*** The phase of an alternating quantity is the time that has elapsed since the quantity has last passed through zero point of reference.
7. ***Phase Difference*** This term is used to compare the phases of two alternating quantities. Two alternating quantities are said to be in phase when they reach their maximum and zero values at the same time. Their maximum value may be different in magnitude.

A leading alternating quantity is one which reaches its maximum or zero value earlier compared to the other quantity.

A lagging alternating quantity is one which attains its maximum or zero value later than the other quantity.

A plus (+) sign, when used in connection with the phase difference, denotes 'lead' whereas a minus (–) sign denotes 'lag'.

$$v_A = V_m \sin \omega t$$
$$v_B = V_m \sin(\omega t + \phi)$$

Here, the quantity B leads A by a phase angle ϕ as shown in Fig. 4.4.

Fig. 4.4 *Phase difference*

4.4 ROOT MEAN SQUARE (rms) OR EFFECTIVE VALUE

Normally, the current is measured by the amount of work it will do or the amount of heat it will produce. Hence, rms or effective value of alternating current is defined as that value of steady current(direct current) which will do the same amount of work in the same time or would produce the same heating effect as when the alternating current is applied for the same time.

Fig. 4.5 *Mid-ordinate method*

Figure 4.5 shows the positive half cycle of a non-sinusoidal alternating current waveform. The waveform is divided in m equal intervals of time each of duration $\frac{t}{m}$. Let the average values of instantaneous currents during these intervals be respectively i_1, i_2, …, i_m. This waveform is applied to a circuit consisting of a resistance of R ohms. The work done in different intervals will be $\left(i_1^2 R \times \frac{t}{m}\right), \left(i_2^2 R \times \frac{t}{m}\right), \ldots, \left(i_m^2 R \times \frac{t}{m}\right)$ joules.

Thus, the total work done in t seconds on applying an alternating current waveform to a resistance

$$R = \frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m} \times Rt \text{ joules}$$

Let I be the value of the direct current that while flowing through the same resistance does the same amount of work in the same time t.

$$I^2 Rt = \frac{i_1^2 + i_2^2 + \ldots i_m^2}{m} \times Rt$$
$$I^2 = \frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m}$$

Hence, rms value of the alternating current is given by

$$I_{\text{rms}} = \sqrt{\frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m}} = \sqrt{\text{Mean value of}(i)^2}$$

The rms value of any current $i(t)$ over the specified interval t_1 to t_2 is expressed mathematically as

$$I_{\text{rms}} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} i^2(t)\,dt}$$

The rms value of an alternating current is of considerable importance in practice because the ammeters and voltmeters record the rms value of alternating current and voltage respectively.

The rms Value of a Sinusoidal Waveform

Figure 4.6 shows a sinusoidal waveform.

Fig. 4.6 *Sinusoidal waveform*

$$v = V_m \sin\theta \qquad 0 < \theta < 2\pi$$

$$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\, d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} V_m^2 \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{2\pi} \sin^2\theta\, d\theta} = \sqrt{\frac{V_m^2}{2\pi}\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta} = \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{2\pi}{2} - 0 - 0 + 0\right]} = \frac{V_m}{\sqrt{2}} = 0.707\, V_m$$

Crest or Peak or Amplitude Factor It is defined as the ratio of maximum value to rms value of the given quantity.

$$\text{Peak factor } (k_p) = \frac{\text{Maximum value}}{\text{rms value}}$$

4.5 AVERAGE VALUE

The average value of an alternating quantity is defined as the arithmetic mean of all the values over one complete cycle.

In case of a symmetrical alternating waveform (whether sinusoidal or non-sinusoidal), the average value over a complete cycle is zero. Hence, in such a case, the average value is obtained over half the cycle only. Referring to Fig. 4.5, the average value of the current is given by

$$I_{\text{avg}} = \frac{i_1 + i_2 + \ldots + i_m}{m}$$

The average value of any current $i(t)$ over the specified interval t_1 to t_2 is expressed mathematically as

$$I_{\text{avg}} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} i(t)\, dt$$

Fig. 4.7 *Sinusoidal waveform*

Average Value of a Sinusoidal Waveform

Figure 4.7 shows a sinusoidal waveform.

$$v = V_m \sin\theta \qquad 0 < \theta < 2\pi$$

Since this is a symmetrical waveform, the average value is calculated over half the cycle.

$$V_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} V_m \sin\theta\, d\theta = \frac{V_m}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{V_m}{\pi}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{V_m}{\pi}\left[1+1\right] = \frac{2V_m}{\pi} = 0.637\ V_m$$

Form Factor It is defined as the ratio of rms value to the average value of the given quantity.

$$\text{Form factor}\,(k_f) = \frac{\text{rms value}}{\text{Average value}}$$

Example 4.1 *An alternating current takes 3.375 ms to reach 15 A for the first time after becoming instantaneously zero. The frequency of the current is 40 Hz. Find the maximum value of the alternating current.*

Solution

$$i = 15\text{ A}, \quad t = 3.375\text{ ms}, \quad f = 40\text{ Hz}$$

$$i = I_m \sin 2\pi ft$$

$$15 = I_m \sin\,(2 \times 180 \times 40 \times 3.375 \times 10^{-3}) \qquad \text{(angle in degrees)}$$

$$I_m = 20\text{ A}$$

Example 4.2 *An alternating current of 50 c/s frequency has a maximum value of 100 A. (a) Calculate its value* $\frac{1}{600}$ *second after the instant the current is zero. (b) In how many seconds after the zero value will the current attain the value of 86.6 A?*

Solution

$$f = 50\text{ c/s}, \quad I_m = 100\text{ A}$$

(a) Value of current $\frac{1}{600}$ second after the instant the current is zero

$$i = I_m \sin 2\pi ft$$

$$= 100 \sin\left(2 \times 180 \times 50 \times \frac{1}{600}\right) \qquad \text{(angle in degrees)}$$

$$= 50\text{ A}$$

(b) Time at which current will attain the value of 86.6 A after the zero value

$$i = I_m \sin 2\pi ft$$

$$86.6 = 100 \sin\,(2 \times 180 \times 50 \times t) \qquad \text{(angle in degrees)}$$

$$t = \frac{1}{300}\text{ second}$$

Example 4.3 *An alternating current varying sinusoidally with a frequency of 50 c/s has an rms value of 20 A. Write down the equation for the instantaneous value and find this value at (a) 0.0025 s, and (b) 0.0125 s after passing through zero and increasing positively. (c) At what time, measured from zero, will the value of the instantaneous current be 14.14 A?*

Solution

$$f = 50\text{ c/s}, \quad I_{\text{rms}} = 20\text{ A}$$

$$I_m = I_{\text{rms}} \times \sqrt{2} = 20\sqrt{2} = 28.28\text{ A}$$

Equation of current, $i = I_m \sin 2\pi ft$

$$= 28.28 \sin (100\pi \times t)$$

$$= 28.28 \sin (100 \times 180 \times t) \quad \text{(angle in degrees)}$$

(a) Value of current at $t = 0.0025$ second

$$i = 28.28 \sin (100 \times 180 \times 0.0025) \quad \text{(angle in degrees)}$$

$$= 20 \text{ A}$$

(b) Value of current at $t = 0.0125$ second

$$i = 28.28 \sin (100 \times 180 \times 0.0125) \quad \text{(angle in degrees)}$$

$$= -20 \text{ A}$$

(c) Time at which value of instantaneous current will be 14.14 A

$$i = 28.28 \sin (100 \times 180 \times t)$$

$$14.14 = 28.28 \sin 18000\, t \quad \text{(angle in degrees)}$$

$$t = 1.66 \text{ ms}$$

Example 4.4 *An alternating current of 60 Hz frequency has a maximum value of 110 A. Calculate (a) time required to reach 90 A after the instant current is zero and increasing positively, and (b) its value* $\frac{1}{600}$ *second after the instant current is zero and its value decreasing thereafter.*

Solution $f = 60$ Hz, $I_m = 110$ A

(a) Time required to reach 90 A after the instant current is zero and increasing positively

$$i = I_m \sin 2\pi ft$$

$$90 = 110 \sin (2 \times 180 \times 60 \times t) \quad \text{(angle in degrees)}$$

$$t = 2.54 \text{ ms}$$

Fig. 4.8

(b) Value of current $\frac{1}{600}$ second after the instant current is zero and decreasing thereafter

From Fig. 4.8,

$$t = \frac{T}{2} + \frac{1}{600} = \frac{1}{2f} + \frac{1}{600}$$

$$= \frac{1}{2 \times 60} + \frac{1}{600} = 0.01 \text{ s}$$

$$i = I_m \sin 2\pi ft$$

$$= 110 \sin (2 \times 180 \times 60 \times 0.01) \quad \text{(angle in degrees)}$$

$$= -64.66 \text{ A}$$

Example 4.5 *A sinusoidal wave of 50 Hz frequency has its maximum value of 9.2 A. What will be its value at (a) 0.002 s after the wave passes through zero in the positive direction, and (b) 0.0045 s after the wave passes through the positive maximum.*

Solution $f = 50$ Hz, $I_m = 9.2$ A

(a) Value of current at 0.002 s after the wave passes through zero in the positive direction

$$i = I_m \sin 2\pi ft$$
$$= 9.2 \sin (2 \times 180 \times 50 \times 0.002) \quad \text{(angle in degrees)}$$
$$= 5.41 \text{ A}$$

(b) Value of current at 0.0045 s after the wave passes through the positive maximum
From Fig. 4.9,

$$t = \frac{T}{4} + 0.0045 = \frac{1}{4f} + 0.0045$$
$$= \frac{1}{4 \times 50} + 0.0045 = 9.5 \text{ ms}$$

Fig. 4.9

$$i = I_m \sin 2\pi ft$$
$$= 9.2 \sin (2 \times 180 \times 50 \times 9.5 \times 10^{-3}) \quad \text{(angle in degrees)}$$
$$= 1.44 \text{ A}$$

Example 4.6 *An alternating current varying sinusoidally with a frequency of 50 Hz has an rms value of current of 20 A. At what time measured from negative maximum value will the instantaneous current be $10\sqrt{2}$ A?*

Solution $f = 50$ Hz, $I_{rms} = 20$ A

(a) Time at which instantaneous current will be $10\sqrt{2}$ A

$$i = 10\sqrt{2} \text{ A} = 14.14 \text{ A}$$
$$I_m = I_{rms} \times \sqrt{2} = 20\sqrt{2} = 28.28 \text{ A}$$
$$i = I_m \sin 2\pi ft$$
$$14.14 = 28.28 \sin (2 \times 180 \times 50 \times t) \quad \text{(angle in degrees)}$$
$$t = 1.67 \text{ ms}$$

(b) Time, measured from negative maximum value, at which instantaneous current will be $10\sqrt{2}$ A
From Fig. 4.10,

$$t = \frac{T}{4} + 1.67 \times 10^{-3} = \frac{1}{4f} + 1.67 \times 10^{-3}$$
$$= \frac{1}{4 \times 50} + 1.67 \times 10^{-3} = 6.67 \text{ ms}$$

Fig. 4.10

Example 4.7 *An alternating current is given by i = 14.14 sin 377 t. Find (a) rms value of the current, (b) frequency, (c) instantaneous value of the current when t = 3 ms, and (d) time taken by the current to reach 10 A for first time after passing through zero.*

Solution $i = 14.14 \sin 377\, t$

(a) The rms value of the current

$$I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{14.14}{\sqrt{2}} = 10 \text{ A}$$

(b) Frequency

$$2\pi f = 377$$

$$f = \frac{377}{2\pi} = 60 \text{ Hz}$$

(c) Instantaneous value of the current when $t = 3$ ms

$$i = 14.14 \sin (377 \times 3 \times 10^{-3}) \quad \text{(angle in radians)}$$
$$= 12.79 \text{ A}$$

(d) Time taken by the current to reach 10 A for the first time after passing through zero

$$i = 14.14 \sin 377\, t \quad \text{(angle in radians)}$$
$$10 = 14.14 \sin 377\, t$$
$$t = 2.084 \text{ ms}$$

Example 4.8 *An alternating current varying sinusoidally at 50 Hz has its rms value of 10 A. Write down an equation for the instantaneous value of the current. Find the value of the current at (a) 0.0025 second after passing through the positive maximum value, and (b) 0.0075 second after passing through zero value and increasing negatively.*

Solution $f = 50$ Hz, $I_{\text{rms}} = 10$ A

(a) Equation for instantaneous value of the current

$$I_m = I_{\text{rms}} \times \sqrt{2} = 10\sqrt{2} = 14.14 \text{ A}$$

$$i = I_m \sin 2\pi f t$$
$$= 14.14 \sin (2 \times 180 \times 50 \times t) \quad \text{(angle in degrees)}$$
$$= 14.14 \sin (18000\, t)$$

(b) Value of the current at 0.0025 s after passing through the positive maximum value

From Fig. 4.11,

$$t = \frac{T}{4} + 0.0025 = \frac{1}{4f} + 0.0025$$

$$= \frac{1}{4 \times 50} + 0.0025 = 7.5 \text{ ms}$$

$$i = 14.14 \sin(18000 \times 7.5 \times 10^{-3}) \quad \text{(angle in degrees)}$$
$$= 10 \text{ A}$$

Fig. 4.11

(c) Value of the current 0.0075 s after passing through zero value and increasing negatively.

From Fig. 4.12,

$$t = \frac{T}{2} + 0.0075 = \frac{1}{2f} + 0.0075$$

$$= \frac{1}{2 \times 50} + 0.0075 = 17.5 \text{ ms}$$

$$i = 14.14 \sin (18000 \times 17.5 \times 10^{-3}) \quad \text{(angle in degrees)}$$
$$= -10 \text{ A}$$

Fig. 4.12

Example 4.9 *Draw a neat sketch in each case of the waveform and write expressions of instantaneous value for the following:*

(a) Sinusoidal current of amplitude 10 A, 50 Hz passing through its zero value at $\omega t = \frac{\pi}{3}$ *and increasing positively*

(b) Sinusoidal current of amplitude 8 A, 50 Hz passing through its zero value at $\omega t = -\frac{\pi}{6}$ *and increasing positively.*

Solution

(a) The current waveform is lagging in nature.
From Fig. 4.13,

$$i = I_m \sin(2\pi ft - \phi) = 10 \sin\left(2\pi \times 50 \times t - \frac{\pi}{3}\right)$$

$$= 10 \sin\left(100\pi t - \frac{\pi}{3}\right)$$

Fig. 4.13

(b) The current waveform is leading in nature.
From Fig. 4.14,

$$i = I_m \sin(2\pi ft + \phi) = 8 \sin\left(2\pi \times 50 \times t + \frac{\pi}{6}\right)$$

$$= 8 \sin\left(100\pi t + \frac{\pi}{6}\right)$$

Fig. 4.14

Example 4.10 *The instantaneous current is given by* $l = 7.071 \sin\left(157.08t - \frac{\pi}{4}\right)$ *. Find its effective value, periodic time and the instant at which it reaches its positive maximum value. Sketch the waveform from* $t = 0$ *over one complete cycle.*

Solution

$$i = 7.7071 \sin\left(157.08t - \frac{\pi}{4}\right)$$

(a) Effective value

$$I_{\text{eff}} = I_{\text{rms}} = \frac{I_m}{\sqrt{2}} = \frac{7.071}{\sqrt{2}} = 5\text{ A}$$

(b) Periodic time

$$2\pi f = 157.08$$

$$f = 25\text{ Hz}$$

$$T = \frac{1}{f} = \frac{1}{25} = 0.04\text{ s}$$

(c) Instant at which the current reaches its positive maximum value i.e., $i = 7.071$ A

From Fig. 4.14,

$$7.071 = 7.071\sin\left(157.08t - \frac{\pi}{4}\right)$$

$$1 = \sin(157.08t - 0.785)$$

$$t = 0.015 \text{ s}$$

(d) The waveform is shown in Fig. 4.15.

Fig. 4.15

Example 4.11 *A 60 Hz sinusoidal current has an instantaneous value of 7.07 A at t = 0 and rms value of $10\sqrt{2}$ A. Assuming the current wave to enter positive half at t = 0, determine (a) expression for instantaneous current, (b) magnitude of the current at t = 0.0125 second, and (c) magnitude of the current at t = 0.025 second after t = 0.*

Solution

$$f = 60 \text{ Hz}, \quad i(0) = 7.07 \text{ A}, \quad I_{\text{rms}} = 10\sqrt{2} \text{ A}$$

(a) Expression for instantaneous current

$$I_m = I_{\text{rms}} \times \sqrt{2} = 10\sqrt{2} \times \sqrt{2} = 20 \text{ A}$$

Since $i = 7.07$ A at $t = 0$, the current is leading in nature.
At $t = 0$

$$i = I_m \sin(2\pi ft + \phi)$$

$$7.07 = 20\sin(2\pi \times 60 \times 0 + \phi)$$

$$\phi = 20.7°$$

$$i = 20\sin(120\pi t + 20.7°)$$

Fig. 4.16

(b) Magnitude of current at $t = 0.0125$ second

$$i = 20 \sin(120 \times 180 \times 0.0125 + 20.7°) \quad \text{(angle in degrees)}$$

$$= -18.7 \text{ A}$$

(c) Magnitude of current at $t = 0.025$ second after $t = 0$
From Fig. 4.16,

Time corresponding to a phase shift of 20.7° $= \dfrac{20.7°}{360°} \times T = \dfrac{20.7°}{360°} \times \dfrac{1}{f} = \dfrac{20.7°}{360°} \times \dfrac{1}{60} = 0.958 \text{ ms}$

Time 0.025 second after $t = 0$

$$t = 0.958 \times 10^{-3} + 0.025 = 25.958 \text{ ms}$$

$$i = 20 \sin(120 \times 180 \times 25.958 \times 10^{-3} + 20.7°) \quad \text{(angle in degrees)}$$

$$= -13.22 \text{ A}$$

Example 4.12 *A 50 Hz sinusoidal voltage applied to a single-phase circuit has an rms value of 200 V. Its value at t = 0 is $\left(\sqrt{2} \times 200\right)$ V positive. The current drawn by the circuit is 5 A (rms) and lags behind the voltage by one-sixth of a cycle. Write the expressions for the instantaneous values of voltage and current. Sketch their waveforms and find their values at t = 0.0125 second.*

Solution

$$f = 50 \text{ Hz}, \quad V_{\text{rms}} = 200 \text{ V}, \quad I_{\text{rms}} = 5 \text{ A}$$
$$v(0) = \sqrt{2} \times 200 = 282.84 \text{ V}$$

(a) Instantaneous value of voltage

$$V_m = V_{\text{rms}} \times \sqrt{2} = 200\sqrt{2} = 282.84 \text{ V}$$
$$v = V_m \sin(2\pi ft + \phi)$$

At $t = 0$,

$$282.84 = 282.84 \sin(0 + \phi)$$
$$\sin \phi = 1$$
$$\phi = 90°$$
$$v = 282.84 \sin(2\pi \times 50 \times t + 90°) = 282.84 \sin(100\pi t + 90°)$$

(b) Instantaneous value of current
The current lags behind the voltage by one-sixth of a cycle.

$$\phi = \frac{1}{6} \times 360° = 60°$$
$$I_m = I_{\text{rms}} \times \sqrt{2} = 5\sqrt{2} = 7.07 \text{ A}$$
$$i = I_m \sin(2\pi ft + 90° - 60°) = 7.07 \sin(2\pi \times 50 \times t + 30\ ) = 7.07 \sin(100\pi t + 30\ )$$

(c) Voltage and current waveforms are shown in Fig. 4.17.

(d) Value of voltage at $t = 0.0125$ s
$v = 282.84 \sin(100 \times 180 \times 0.0125 + 90°)$
(angle in degrees)
$= 200$ V

(e) Value of current at $t = 0.0125$ s
$i = 7.07 \sin(100 \times 180 \times 0.0125 + 30°)$
(angle in degrees)
$= -6.83$ A

Fig. 4.17

Example 4.13 *At the instant t = 0, the instantaneous value of a 50 Hz sinusoidal current is 5 A and increases in magnitude further. Its rms value is 10 A. (a) Write the expression for its instantaneous value. (b) Find the current at t = 0.01 s and t = 0.015 s. (c) Sketch the waveform indicating these values.*

Solution

$$f = 50 \text{ Hz}, \quad i(0) = 5 \text{ A}, \quad I_{\text{rms}} = 10 \text{ A}$$

(a) Expression for instantaneous value of current

$$I_m = I_{\text{rms}} \times \sqrt{2} = 10\sqrt{2} = 14.14 \text{ A}$$

Since $i = 5$ A at $t = 0$, the current is leading in nature.

$$i = I_m \sin(2\pi ft + \phi)$$

At $t = 0$,

$$5 = 14.14 \sin(2 \times 180 \times 50 \times 0 + \phi) \quad \text{(angle in degrees)}$$
$$5 = 14.14 \sin\phi$$
$$\phi = 20.7°$$
$$i = 14.14 \sin(100\pi t + 20.7°)$$

(b) Current at $t = 0.01$ s

$$i = 14.14 \sin(100 \times 180 \times 0.01 + 20.7°) \quad \text{(angle in degrees)}$$
$$= -5 \text{ A}$$

(c) Current at $t = 0.015$ s

$$i = 14.14 \sin(100 \times 180 \times 0.015 + 20.7°) \quad \text{(angle in degrees)}$$
$$= -13.23 \text{ A}$$

(d) Waveform

The waveform is shown in Fig. 4.18.

Fig. 4.18

Example 4.14 *In a certain circuit supplied from 50 Hz mains, the potential difference has a maximum value of 500 V and the current has a maximum value of 10 A. At the instant t = 0, the instantaneous values of potential difference and current are 400 V and 4 A respectively, both increasing in the positive direction. State expressions for instantaneous values of potential difference and current at time t. Calculate the instantaneous values at time t = 0.015 second. Find the phase angle between potential difference and current.*

Solution $f = 50$ Hz, $V_m = 500$ V, $I_m = 10$ A $v(0) = 400$ V, $i(0) = 4$ A

(a) Expression for instantaneous value of potential difference and current
Since $v = 400$ V and $i = 4$ A at $t = 0$, the voltage and current waveforms are leading in nature.

(i) $$v = V_m \sin(2\pi ft + \phi_1)$$

At $t = 0$

$$400 = 500 \sin(2\pi \times 50 \times 0 + \phi_1)$$
$$\phi_1 = 53.13°$$
$$v = 500 \sin(100\pi t + 53.13°)$$

(ii) $$i = I_m \sin(2\pi ft + \phi_2)$$

At $t = 0$

$$4 = 10 \sin(2\pi \times 50 \times 0 + \phi_2)$$
$$\phi_2 = 23.58°$$
$$i = 10 \sin(100\pi t + 23.58°)$$

(b) Instantaneous values of potential difference and current at t = 0.015 second

(i)
$$v = 500 \sin(100 \times 180 \times 0.015 + 53.13°)$$
$$= -300 \text{ V}$$
(angle in degrees)

(ii)
$$i = 10 \sin(100 \times 180 \times 0.015 + 23.58°)$$
$$= -9.17 \text{ A}$$
(angle in degrees)

(c) Phase angle between potential difference and current
$$\phi = \phi_1 - \phi_2 = 53.13° - 23.58° = 29.55°$$

Example 4.15 *Find the following parameters of a voltage v = 200 sin 314 t: (a) frequency (b) form factor, and (c) crest factor.*

Solution
$$v = 200 \sin 314\, t$$

(a) Frequency
$$v = V_m \sin 2\pi ft$$
$$2\pi f = 314$$
$$f = \frac{314}{2\pi} = 50 \text{ Hz}$$

For a sinusoidal waveform,
$$V_{\text{avg}} = \frac{2V_m}{\pi}$$
$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

(b) Form factor
$$k_f = \frac{V_{\text{rms}}}{V_{\text{avg}}} = \frac{\frac{V_m}{\sqrt{2}}}{\frac{2V_m}{\pi}} = 1.11$$

(c) Crest factor
$$k_p = \frac{V_m}{V_{\text{rms}}} = \frac{V_m}{\frac{V_m}{\sqrt{2}}} = 1.414$$

Example 4.16 *A non-sinusoidal voltage has a form factor of 1.2 and peak factor of 1.5. If the average value of the voltage is 10 V, calculate (a) rms value, and (b) maximum value.*

Solution
$$k_f = 1.2, \quad k_p = 1.5, \quad V_{\text{avg}} = 10 \text{ V}$$

(a) rms value

$$k_f = \frac{V_{\text{rms}}}{V_{\text{avg}}}$$

$$1.2 = \frac{V_{\text{rms}}}{10}$$

$$V_{\text{rms}} = 12\text{ V}$$

(b) Maximum value

$$k_p = \frac{V_m}{V_{\text{rms}}}$$

$$1.5 = \frac{V_m}{12}$$

$$V_m = 18\text{ V}$$

Example 4.17 *The waveform of a voltage has a form factor of 1.15 and a peak factor of 1.5. If the maximum value of the voltage is 4500 V, calculate the average value and rms value of the voltage.*

Solution $k_f = 1.15, \quad k_p = 1.5, \quad V_m = 4500\text{ V}$

(a) rms value of the voltage

$$k_p = \frac{V_m}{V_{\text{rms}}}$$

$$1.5 = \frac{4500}{V_{\text{rms}}}$$

$$V_{\text{rms}} = 3000\text{ V}$$

(b) Average value of the voltage

$$k_f = \frac{V_{\text{rms}}}{V_{\text{avg}}}$$

$$1.15 = \frac{3000}{V_{\text{avg}}}$$

$$V_{\text{avg}} = 2608.7\text{ V}$$

Example 4.18 *A 50 Hz sinusoidal current has a peak factor of 1.4 and a form factor of 1.1. Its average value is 20 A. The instantaneous value of the current is 15 A at t = 0. Write the equation of the current and draw its waveform.*

Solution $f = 50\text{ Hz}, \quad k_p = 1.4, \quad k_f = 1.1, \quad I_{\text{avg}} = 20\text{ A}, \quad i(0) = 15\text{ A}$

(a) Equation of current

$$k_f = \frac{I_{\text{rms}}}{I_{\text{avg}}}$$

$$1.1 = \frac{I_{\text{rms}}}{20}$$

$$I_{\text{rms}} = 22 \text{ A}$$

$$k_p = \frac{I_m}{I_{\text{rms}}}$$

$$1.4 = \frac{I_m}{22}$$

$$I_m = 30.8 \text{ A}$$

Since $i = 15$ A at $t = 0$, the current is leading in nature as shown in Fig. 4.19.

$$i = I_m \sin(2\pi ft + \phi)$$

At $t = 0$

$$15 = 30.8 \sin\ (2\pi \times 50 \times 0 + \phi)$$

$$\phi = 29.14°$$

$$i = 30.8 \sin\ (100\pi t + 29.14°)$$

(b) The waveform is shown in Fig. 4.19.

Fig. 4.19

Example 4.19 *Find the average value and rms value of the waveform shown in Fig. 4.20.*

Fig. 4.20

Solution $v = V_m \sin\theta \qquad 0 < \theta < \pi$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} V_m \sin\theta\, d\theta = \frac{V_m}{\pi}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{V_m}{\pi}[1+1] = 0.637\ V_m$$

(b) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2(\theta)\, d\theta} = \sqrt{\frac{1}{\pi}\int_0^{\pi} V_m^2 \sin^2\theta\, d\theta} = \sqrt{\frac{V_m^2}{\pi}\int_0^{\pi} \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta} = \sqrt{\frac{V_m^2}{\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi}} = \sqrt{\frac{V_m^2}{\pi}\left[\frac{\pi}{2}-\frac{\sin 2\pi}{4}-0+\frac{\sin 0}{4}\right]} = 0.707\ V_m$$

Example 4.20 *Find the average and rms values of the waveform shown in Fig. 4.21.*

Fig. 4.21

Solution

$$v = V_m \sin\theta \qquad 0 < \theta < \pi$$
$$= 0 \qquad \pi < \theta < 2\pi$$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta = \frac{1}{2\pi}\left[\int_0^{\pi} V_m \sin\theta\,d\theta + \int_{\pi}^{2\pi} 0\,d\theta\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} V_m \sin\theta\,d\theta = \frac{V_m}{2\pi}\left[-\cos\theta\right]_0^{\pi} = \frac{V_m}{2\pi}[1+1] = 0.318\,V_m$$

(b) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\,d\theta} = \sqrt{\frac{1}{2\pi}\left[\int_0^{\pi} V_m^2 \sin^2\theta\,d\theta + \int_{\pi}^{2\pi} 0\,d\theta\right]}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta\,d\theta} = \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi}\sin^2\theta\,d\theta} = \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi}} = \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\pi}{2}-\frac{\sin 2\pi}{4}-0+\frac{\sin 0}{4}\right]} = 0.5\ V_m$$

Example 4.21 *Find the average value and rms value of the waveform shown in Fig. 4.22.*

Fig. 4.22

Solution

$$v = 0 \qquad 0 < \theta < \frac{\pi}{4}$$

$$= V_m \sin\theta \qquad \frac{\pi}{4} < \theta < \pi$$

$$= 0 \qquad \pi < \theta < 2\pi$$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\, d\theta = \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\pi} V_m \sin\theta\, d\theta$$

$$= \frac{V_m}{2\pi}\left[-\cos\theta\right]_{\frac{\pi}{4}}^{\pi} = \frac{V_m}{2\pi}[1+0.707] = 0.272 \text{ Vm}$$

(b) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\, d\theta} = \sqrt{\frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\pi} V_m^2 \sin^2\theta\, d\theta} = \sqrt{\frac{V_m^2}{2\pi}\int_{\frac{\pi}{4}}^{\pi} \sin^2\theta\, d\theta} = \sqrt{\frac{V_m^2}{2\pi}\int_{\frac{\pi}{4}}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{4}}^{\pi}} = \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} - \frac{\pi}{8} + \frac{\sin\frac{\pi}{2}}{4}\right]} = 0.476\, V_m$$

Example 4.22 *A full-wave rectified wave is clipped at 70.7% of its maximum value as shown in Fig. 4.23. Find its average and rms values.*

Fig. 4.23

Solution

$$v = V_m \sin\theta \qquad 0 < \theta < \frac{\pi}{4}$$

$$= 0.707\, V_m \qquad \frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

$$= V_m \sin\theta \qquad \frac{3\pi}{4} < \theta < \pi$$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\, d\theta = \frac{1}{\pi}\left[\int_0^{\frac{\pi}{4}} V_m \sin\theta\, d\theta + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 0.707\, V_m\, d\theta + \int_{\frac{3\pi}{4}}^{\pi} V_m \sin\theta\, d\theta\right]$$

$$= \frac{V_m}{\pi}\left\{\left[-\cos\theta\right]_0^{\frac{\pi}{4}} + 0.707\left[\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \left[-\cos\theta\right]_{\frac{3\pi}{4}}^{\pi}\right\} = \frac{V_m}{\pi}(0.293 + 1.11 + 0.293) = 0.54\ V_m$$

(b) rms value of the waveform

$$V_{\text{avg}} = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2(\theta)\, d\theta} = \sqrt{\frac{1}{\pi}\left[\int_0^{\frac{\pi}{4}} V_m^2 \sin^2\theta\, d\theta + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (0.707V_m)^2\, d\theta + \int_{\frac{3\pi}{4}}^{\pi} V_m^2 \sin^2\theta\, d\theta\right]}$$

$$= \sqrt{\frac{V_m^2}{\pi}\left\{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}} + 0.499\left[\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\frac{3\pi}{4}}^{\pi}\right\}} = 0.584\ V_m$$

Example 4.23 *Find the rms value of the waveform shown in Fig. 4.24.*

Fig. 4.24

Solution The equation of the waveform is given by $v = V_m \sin(\theta + \phi)$ where ϕ is the phase difference. When $\theta = 0$, $v = 0.866\ V_m$.

$$0.866\ V_m = V_m \sin(\theta + \phi)$$

$$0.866 = \sin(0 + \phi)$$

$$\phi = \sin^{-1}(0.866) = \frac{\pi}{3}$$

$$v = V_m \sin\left(\theta + \frac{\pi}{3}\right)$$

The time period of a complete sine wave is always 2π. Since some part of the waveform is chopped from both the sides,

$$\text{Time period} = 2\pi - \frac{\pi}{3} - \frac{\pi}{3} = \frac{4\pi}{3}$$

$$V_{\text{rms}} = \sqrt{\frac{1}{\frac{4\pi}{3}}\int_0^{\frac{4\pi}{3}} V_m^2 \sin^2\left(\theta + \frac{\pi}{3}\right) d\theta} = \sqrt{\frac{3}{4\pi}\int_0^{\frac{4\pi}{3}} V_m^2 \sin^2\left(\theta + \frac{\pi}{3}\right) d\theta}$$

$$= \sqrt{\frac{3V_m^2}{4\pi}\int_0^{\frac{4\pi}{3}}\left[\frac{1-\cos 2\left(\theta+\frac{\pi}{3}\right)}{2}\right]d\theta} = \sqrt{\frac{3V_m^2}{4\pi}\left[\frac{\theta}{2}-\frac{\sin 2\left(\theta+\frac{\pi}{3}\right)}{4}\right]_0^{\frac{4\pi}{3}}} = 0.776\ V_m$$

Example 4.24 *Find the average and rms values of the waveform shown in Fig. 4.25.*

Fig. 4.25

Solution

$$v = V_m \qquad 0 < t < \frac{T}{2}$$

$$= 0 \qquad \frac{T}{2} < t < T$$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\left[\int_0^{\frac{T}{2}} V_m\,dt + \int_{\frac{T}{2}}^{T} 0\,dt\right] = \frac{1}{T}\int_0^{\frac{T}{2}} V_m\,dt$$

$$= \frac{V_m}{T}[t]_0^{\frac{T}{2}} = \frac{V_m}{T}\cdot\frac{T}{2} = 0.5\ V_m$$

(b) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2\,(t)\,dt} = \sqrt{\frac{1}{T}\int_0^{\frac{T}{2}} V_m^2\,dt} = \sqrt{\frac{V_m^2}{T}[t]_0^{\frac{T}{2}}}$$

$$= \sqrt{\frac{V_m^2}{T}\cdot\frac{T}{2}} = 0.707\ V_m$$

Example 4.25 *Find the average and rms values of the waveform shown in Fig. 4.26.*

Fig. 4.26

Solution $\quad v = \frac{V_m}{T}t \qquad 0 < t < T$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\int_0^T \frac{V_m}{T}\,t\,dt = \frac{V_m}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{V_m}{T^2}\cdot\frac{T^2}{2} = 0.5\,V_m$$

(b) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} = \sqrt{\frac{1}{T}\int_0^T \frac{V_m^2}{T^2}\cdot t^2\,dt}$$

$$= \sqrt{\frac{V_m^2}{T^3}\left[\frac{t^3}{3}\right]_0^T} = \sqrt{\frac{V_m^2}{T^3}\left[\frac{T^3}{3}\right]} = 0.577\,V_m$$

Example 4.26 *Find the average and rms values of the waveform shown in Fig. 4.27.*

Fig. 4.27

Solution

$$v = t \qquad 0 < t < 1$$
$$= 1 \qquad 1 < t < 2$$

(a) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 1\,dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{t^2}{2}\right]_0^1 + [t]_1^2\right\} = \frac{1}{2}\left[\frac{1}{2} - 0 + 2 - 1\right] = 0.75\text{ V}$$

(b) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} = \sqrt{\frac{1}{2}\left[\int_0^1 t^2\,dt + \int_1^2 (1)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{2}\left\{\left[\frac{t^3}{3}\right]_0^1 + [t]_1^2\right\}} = \sqrt{\frac{1}{2}}\left[\frac{1}{3} - 0 + 2 - 1\right] = 0.816\text{ V}$$

Example 4.27 *Find the average and rms values of the waveform shown in Fig. 4.28.*

Fig. 4.28

Solution

(a) Average value of the waveform

$$V_{avg} = \frac{0+40+60+80+100+80+60+40}{8} = 57.5 \text{ V}$$

(b) rms value of the waveform

$$V_{rms} = \sqrt{\frac{0^2+(40)^2+(60)^2+(80)^2+(100)^2+(80)^2+(60)^2+(40)^2}{8}} = 64.42 \text{ V}$$

Example 4.28 *Find the average and rms values of the waveform shown in Fig. 4.29.*

Fig. 4.29

Solution

(a) Average value of the waveform

$$V_{avg} = \frac{0+10+20}{3} = 10 \text{ V}$$

(b) rms value of the waveform

$$V_{rms} = \sqrt{\frac{0^2+(10)^2+(20)^2}{3}} = 12.9 \text{ V}$$

Example 4.29 *Find the effective value of the resultant current which carries simultaneously a direct current of 10 A and a sinusoidally alternating current with a peak value of 10 A.*

Solution

Figure 4.30 shows the current waveforms.

Fig. 4.30

$$i = i_1 + i_2 = 10 + 10 \sin \theta$$

$$I_{\text{eff}} = I_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2(\theta)\, d\theta} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} (10 + 10 \sin\theta)^2\, d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} (100 + 200 \sin\theta + 100 \sin^2 \theta)\, d\theta} = \sqrt{\frac{100}{2\pi}\int_0^{2\pi} (1 + 2\sin\theta + \sin^2 \theta)\, d\theta}$$

$$= \sqrt{\frac{100}{2\pi}\int_0^{2\pi}\left[1 + 2 \sin \theta + \left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta} = \sqrt{\frac{100}{2\pi}\left[\theta - 2 \cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}}$$

$$= \sqrt{\frac{100}{2\pi}\left[2\pi - 2\cos 2\pi + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} - 0 + 2\cos 0 - 0 + \frac{\sin 0}{4}\right]} = \sqrt{\frac{100}{2\pi}\left[2\pi - 2 + \frac{2\pi}{2} + 2\right]}$$

$$= 12.25 \text{ A}$$

Example 4.30 *Find the effective value of a resultant current in a wire which carries simultaneously a direct current of $i_1 = 10$ A and alternating current given by $i_2 = 12 \sin \omega t + 6 \sin\left(3\omega t - \frac{\pi}{6}\right) + 4 \sin\left(5\omega t + \frac{\pi}{3}\right)$.*

Solution

$$i_1 = 10 \text{ A}$$

$$i_2 = 12 \sin \omega t + 6 \sin\left(3\omega t - \frac{\pi}{6}\right) + 4 \sin\left(5\omega t + \frac{\pi}{3}\right)$$

$$i = i_1 + i_2 = 10 + 12 \sin \omega t + 6 \sin\left(3\omega t - \frac{\pi}{6}\right) + 4 \sin\left(5\omega t + \frac{\pi}{3}\right)$$

$$= 10 + 12 \sin\theta + 6 \sin (3\theta - 30°) + 4 \sin (5\theta + 60°)$$

$$I_{\text{eff}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2(\theta)\, d\theta} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \left[10 + 12 \sin\theta + 6\sin(3\theta - 30°) + 4\sin(5\theta + 60°)\right]^2 d\theta} = 14.07 \text{ A}$$

Example 4.31 *Find the relative heating effects of two current waves of equal peak value, one sinusoidal and the other, rectangular in shape.*

Solution Figure 4.31 shows the two current waves.

Fig. 4.31

$$\text{rms value of the rectangular wave} = I_m$$

$$\text{rms value of the sinusoidal current wave} = \frac{I_m}{\sqrt{2}}$$

$$\text{Heating effect due to the rectangular current wave} = (I_m)^2 RT$$

$$\text{Heating effect due to the sinusoidal current wave} = \left(\frac{I_m}{\sqrt{2}}\right)^2 RT = \frac{(I_m)^2}{2} RT$$

$$\text{Relative heating effects} = \frac{(I_m)^2}{2} RT : (I_m)^2 RT = \frac{1}{2} : 1 = 1 : 2$$

4.6 PHASOR REPRESENTATIONS OF ALTERNATING QUANTITIES

The alternating quantities are represented by phasors. A phasor is a line of definite length rotating in an anticlockwise direction at a constant angular velocity ω. The length of a phasor is equal to the maximum value of the alternating quantity, and the angular velocity is equal to the angular velocity of alternating quantity.

As shown in Fig. 4.32, consider a phasor $OP = I_m$, where I_m is the maximum value of the alternating current. Let this phasor rotate in an anticlockwise direction at a uniform angular velocity of ω radians/second. The projection of the phasor OP on the Y-axis at any instant gives the instantaneous value of that alternating current.

$$OM = OP \sin \omega t = I_m \sin \omega t = i$$

Fig. 4.32 *Representation of alternating quantities in terms of phasors*

Thus, if we plot the projections of the phasor on the Y-axis versus its angular position point by point, a sinusoidal alternating current waveform is obtained.

Phasor Diagram Using rms Values Sinusoidal alternating currents and voltages can be represented by phasors. Electrical measuring instruments like ammeters and voltmeters are calibrated to read the rms values of ac quantities. Hence, instead of using maximum values, it is more convenient to draw phasor diagrams using rms values of alternating quantities. However, such a phasor diagram will not generate a sine wave of proper amplitude unless the length of the phasor is multiplied by $\sqrt{2}$.

Example 4.32 *Two currents i_1 and i_2 are given by the expressions* $i_1 = 10\sin\left(\omega t + \frac{\pi}{4}\right)$ *and* $i_2 = 8\sin\left(\omega t - \frac{\pi}{3}\right)$. *Find (a) $i_1 + i_2$, and (b) $i_1 - i_2$. Express the answers in the form $i = I_m \sin(\omega t \pm \phi)$.*

Solution

$$i_1 = 10\sin\left(\omega t + \frac{\pi}{4}\right)$$

$$i_2 = 8\sin\left(\omega t - \frac{\pi}{3}\right)$$

(a) Let phasors $\mathbf{I}_1$ and $\mathbf{I}_2$ represent the alternating currents i_1 and i_2 respectively in terms of their maximum values.

(i) *Analytical method*:
From Fig. 4.33,

Fig. 4.33

Resolving $\mathbf{I}_1$ and $\mathbf{I}_2$ into x- and y- components,

$$\Sigma x = 10\cos(45°) + 8\cos(-60°) = 11.07$$

$$\Sigma y = 10\sin(45°) + 8\sin(-60°) = 0.14$$

$$\text{Magnitude of } (\mathbf{I}_1 + \mathbf{I}_2) = \sqrt{(\Sigma x)^2 + (\Sigma y)^2}$$

$$= \sqrt{(11.07)^2 + (0.14)^2}$$

$$= 11.07$$

$$\text{Phase angle } \phi = \tan^{-1}\left(\frac{\Sigma y}{\Sigma x}\right) = \tan^{-1}\left(\frac{0.14}{11.07}\right) = 0.72°$$

$$i = i_1 + i_2 = 11.07\,(\omega t + 0.72°)$$

(ii) *Graphical method*: The phasor sum $\mathbf{I}_1 + \mathbf{I}_2$ is obtained by adding phasors $\mathbf{I}_1$ and $\mathbf{I}_2$ by the parallelogram law as shown in Fig. 4.34.

Fig. 4.34

(b) (i) *Analytical method*:
From Fig. 4.35,
Resolving $\mathbf{I}_1$ and $-\mathbf{I}_2$ into x- and y-components,

$$\sum x = 10\cos(45°) - 8\cos(-60°) = 3.07$$
$$\sum y = 10\sin(45°) - 8\sin(-60°) = 14$$

Fig. 4.35

$$\text{Magnitude of } (\mathbf{I}_1 - \mathbf{I}_2) = \sqrt{(\sum x)^2 + (\sum y)^2}$$
$$= \sqrt{(3.07)^2 + (14)^2}$$
$$= 14.33 \text{ A}$$

$$\text{Phase angle } \phi = \tan^{-1}\left(\frac{\sum y}{\sum x}\right)$$
$$= \tan^{-1}\left(\frac{14}{3.07}\right) = 77.63$$
$$i = i_1 - i_2 = 14.33 \sin(\omega t + 77.63)$$

(ii) *Graphical method*: The phasor sum $\mathbf{I}_1 - \mathbf{I}_2$ is obtained by adding phasors $\mathbf{I}_1$ and $-\mathbf{I}_2$ by the parallelogram law as shown in Fig. 4.36.

Fig. 4.36

Example 4.33 *Three voltages are represented by* $v_1 = 10 \sin \omega t$, $v_2 = 20 \sin\left(\omega t - \frac{\pi}{6}\right)$ *and* $v_3 = 30 \sin\left(\omega t + \frac{\pi}{4}\right)$. *Find the magnitude and phase angle of the resultant voltage.*

Solution

$$v_1 = 10 \sin \omega t$$
$$v_2 = 20 \sin\left(\omega t - \frac{\pi}{6}\right)$$
$$v_3 = 30 \sin\left(\omega t + \frac{\pi}{4}\right)$$

Let phasors $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ represent the alternating voltages v_1, v_2 and v_3 respectively in terms of their maximum values.

(a) *Analytical method*:
From Fig. 4.37,

Resolving $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ into x- and y-components,

$$\sum x = 10 + 20\cos(-30°) + 30\cos(45°) = 48.53$$
$$\sum y = 20\sin(-30°) + 30\sin(45°) = 11.21$$

Magnitude of $(\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3) = \sqrt{(\sum x)^2 + (\sum y)^2}$

$$= \sqrt{(48.53)^2 + (11.21)^2} = 49.81 \text{ V}$$

Phase angle $\phi = \tan^{-1}\left(\dfrac{\sum y}{\sum x}\right) = \tan^{-1}\left(\dfrac{11.21}{48.53}\right) = 13°$

$$v = 49.81 \sin(\omega t + 13°)$$

Fig. 4.37

(b) *Graphical method*: The phasor sum $\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3$ is obtained by first adding phasors $\mathbf{V}_1$ and $\mathbf{V}_2$ by the parallelogram law and then adding $\mathbf{V}_3$ to the resultant of $\mathbf{V}_1$ and $\mathbf{V}_2$ by the parallelogram law as shown in Fig. 4.38.

Fig. 4.38

Example 4.34 *The instantaneous voltages across each of the four coils connected in series are given by* $v_1 = 100 \sin\omega t$, $v_2 = 250 \cos \omega t$, $v_3 = 150 \sin\left(\omega t + \dfrac{\pi}{6}\right)$, $v_4 = 200 \sin\left(\omega t - \dfrac{\pi}{4}\right)$. *Determine the resultant voltage by analytical method.*

Solution

$$v_1 = 100 \sin \omega t, \quad v_2 = 250 \cos \omega t = 250 \sin(\omega t + 90°)$$
$$v_3 = 150 \sin\left(\omega t + \frac{\pi}{6}\right), \quad v_4 = 200 \sin\left(\omega t - \frac{\pi}{4}\right)$$

Let phasors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_4$ represent the instantaneous voltages v_1, v_2, v_3, and v_4 respectively in terms of their maximum values as shown in Fig. 4.39.

Fig. 4.39

Resolving $\mathbf{V}_1$, $\mathbf{V}_2$ $\mathbf{V}_3$ and $\mathbf{V}_4$ into x- and y-components,

$$\sum x = 100 + 250\cos(90°) + 150\cos(30°) + 200\cos(-45°) = 371.33$$
$$\sum y = 250\sin(90°) + 150\sin(30°) + 200\sin(-45°) = 183.58$$

$$\text{Magnitude of } (\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \mathbf{V}_4) = \sqrt{(\sum x)^2 + (\sum y)^2} = \sqrt{(371.33)^2 + (183.58)^2} = 414.23 \text{ V}$$

$$\text{Phase angle } \phi = \tan^{-1}\left(\frac{\sum y}{\sum x}\right) = \tan^{-1}\left(\frac{183.58}{371.33}\right) = 26.31$$

$$v = v_1 + v_2 + v_3 + v_4 = 414.23 \sin(\omega t + 26.31°)$$

4.7 MATHEMATICAL REPRESENTATIONS OF PHASORS

A phasor can be represented in four forms.

1. Rectangular Form

$$\mathbf{V} = X \pm jY$$

$$\text{Magnitude of phasor, } V = \sqrt{X^2 + Y^2}$$

$$\text{Phase angle } \phi = \tan^{-1}\left(\frac{Y}{X}\right)$$

2. Trigonometric Form

$$\mathbf{V} = V(\cos\phi \pm j\sin\phi)$$

3. Exponential Form

$$\mathbf{V} = Ve^{\pm j\phi}$$

4. Polar Form

$$\mathbf{V} = V\angle \pm \phi$$

Significance of Operator j The operator j is used in rectangular form. It is used to indicate anticlockwise rotation of a phasor through 90°. Mathematically,

$$j = \sqrt{-1}$$

Whenever a phasor is multiplied by j, the phasor is rotated once in the anticlockwise direction through 90°. The power of j represents the number of times the phasor should be rotated through 90° in the anticlockwise directon.

Example 4.35 *Two sinusoidal currents are given as* $i_1 = 10\sqrt{2}\sin\omega t$, $i_2 = 20\sqrt{2}\sin(\omega t + 60°)$. *Find the expression for the sum of these currents.*

Solution

$$i_1 = 10\sqrt{2}\sin\omega t, \quad i_2 = 20\sqrt{2}\sin(\omega t + 60°)$$

Writing currents i_1 and i_2 in the phasor form,

$$\mathbf{I}_1 = \frac{10\sqrt{2}}{\sqrt{2}}\angle 0° = 10\angle 0°$$

$$\mathbf{I}_2 = \frac{20\sqrt{2}}{\sqrt{2}}\angle 60° = 20\angle 60°$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10\angle 0° + 20\angle 60° = 26.46\angle 40.89$$

$$i = 26.46\sqrt{2}\sin(\omega t + 40.89°) = 37.42\sin(\omega t + 40.89)$$

Example 4.36 *The following three sinusoidal currents flow into the junction* $i_1 = 3\sqrt{2}\sin\omega t$, $i_2 = 5\sqrt{2}\sin(\omega t + 30°)$ *and* $i_3 = 6\sqrt{2}\sin(\omega t - 120°)$. *Find the expression for the resultant current which leaves the junction.*

Solution

$$i_1 = 3\sqrt{2}\sin\omega t, \quad i_2 = 5\sqrt{2}\sin(\omega t + 30°), \quad i_3 = 6\sqrt{2}\sin(\omega t - 120°)$$

Writing currents i_1, i_2 and i_3 in the phasor form,

$$\mathbf{I}_1 = \frac{3\sqrt{2}}{\sqrt{2}}\angle 0° = 3\angle 0°$$

$$\mathbf{I}_2 = \frac{5\sqrt{2}}{\sqrt{2}}\angle 30° = 5\angle 30°$$

$$\mathbf{I}_3 = \frac{6\sqrt{2}}{\sqrt{2}}\angle -120° = 6\angle -120°$$

The resultant current which leaves the junction is given by

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 3\angle 0° + 5\angle 30° + 6\angle -120° = 5.1\angle -31.9°$$

$$i = 5.1\sqrt{2}\sin(\omega t - 31.9°) = 7.21\sin(\omega t - 31.9°)$$

Example 4.37 *In a circuit, four currents as indicated below, are meeting at a point. Find the resultant current.*

$i_1 = 5\sin\omega t$, $\quad i_2 = 10\sin(\omega t - 30°)$

$i_3 = 5\cos(\omega t - 30°)$ $\quad i_4 = -10\sin(\omega t + 45°)$

Solution

$$i_1 = 5\sin\omega t, \quad i_2 = 10\sin(\omega t - 30°),$$
$$i_3 = 5\cos(\omega t - 30°) = 5\sin(\omega t + 60°)$$
$$i_4 = -10\sin(\omega t + 45°) = 10\sin(\omega t + 225°)$$

Writing currents i_1, i_2, i_3, and i_4 in the phasor form,

$$\mathbf{I}_1 = \frac{5}{\sqrt{2}}\angle 0° = 3.54\angle 0°$$

$$\mathbf{I}_2 = \frac{10}{\sqrt{2}}\angle -30° = 7.07\angle -30°$$

$$\mathbf{I}_3 = \frac{5}{\sqrt{2}}\angle 60° = 3.54\angle 60°$$

$$\mathbf{I}_4 = \frac{10}{\sqrt{2}}\angle 225° = 7.07\angle 225°$$

Resultant current $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 = 3.54\angle 0° + 7.07\angle -30° + 3.54\angle 60° + 7.07\angle 225°$

$$= 8.44\angle -40.36°$$

$$i = 8.44\sqrt{2}\sin(\omega t - 40.36°) = 11.94\sin(\omega t - 40.36°)$$

Example 4.38 *Find the resultant voltage and its equation for the given voltages.*

$$e_1 = 20\sin\omega t, \qquad e_2 = 30\sin\left(\omega t - \frac{\pi}{4}\right), \qquad e_3 = 40\cos\left(\omega t + \frac{\pi}{6}\right)$$

Solution

$$e_1 = 20\sin\omega t, \quad e_2 = 30\sin\left(\omega t - \frac{\pi}{4}\right) = 30\sin(\omega t - 45°), \quad e_3 = 40\cos\left(\omega t + \frac{\pi}{6}\right) = 40\sin(\omega t + 120°)$$

Writing voltages e_1, e_2, and e_3 in the phasor form,

$$\mathbf{E}_1 = \frac{20}{\sqrt{2}}\angle 0° = 14.14\angle 0°, \ \mathbf{E}_2 = \frac{30}{\sqrt{2}}\angle -45° = 21.21\angle -45°, \ \mathbf{E}_3 = \frac{40}{\sqrt{2}}\angle 120° = 28.28\angle 120°$$

Resultant voltage $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = 14.14\angle 0° + 21.21\angle -45° + 28.28\angle 120° = 17.75\angle 32.33°$

$$e = 17.75\sqrt{2}\sin(\omega t + 32.33°) = 25.1\sin(\omega t + 32.33°)$$

Example 4.39 *Obtain the sum of the three voltages.*

$$v_1 = 147.3\cos(\omega t + 98.1°), \quad v_2 = 294.6\cos(\omega t - 45°), \quad v_3 = 88.4\sin(\omega t + 135°)$$

Solution

$$v_1 = 147.3\cos(\omega t + 98.1°) = 147.3\sin(\omega t + 188.1°)$$

$$v_2 = 294.6\cos(\omega t - 45°) = 294.6\sin(\omega t + 45°), \quad v_3 = 88.4\sin(\omega t + 135°)$$

Writing voltages v_1, v_2 and v_3 in the phasor form,

$$\mathbf{V}_1 = \frac{147.3}{\sqrt{2}}\angle 188.1° = 104.16\angle 188.1°$$

$$\mathbf{V}_2 = \frac{294.6}{\sqrt{2}}\angle 45° = 208.31\angle 45°$$

$$\mathbf{V}_3 = \frac{88.4}{\sqrt{2}}\angle 135° = 62.51\angle 135°$$

Resultant voltage $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 104.16\angle 188.1° + 208.31\angle 45° + 62.51\angle 135° = 176.82\angle 90°$

$$v = 176.82\sqrt{2}\sin(\omega t + 90°) = 250.06\sin(\omega t + 90°)$$

Example 4.40 *Find vectorially the resultant of the following four voltages.*

$$e_1 = 25 \sin \omega t, \qquad e_2 = 30 \sin\left(\omega t + \frac{\pi}{6}\right),$$

$$e_3 = 30 \cos \omega t, \qquad e_4 = 20 \sin\left(\omega t - \frac{\pi}{6}\right)$$

Solution

$$e_1 = 25 \sin \omega t, \quad e_2 = 30 \sin\left(\omega t + \frac{\pi}{6}\right) = 30 \sin(\omega t + 30°)$$

$$e_3 = 30 \cos \omega t = 30 \sin(\omega t + 90°), \quad e_4 = 20 \sin\left(\omega t - \frac{\pi}{6}\right) = 20 \sin(\omega t - 30°)$$

Writing voltages e_1, e_2, e_3 and e_4 in the phasor form,

$$\mathbf{E}_1 = \frac{25}{\sqrt{2}} \angle 0° = 17.68 \angle 0°$$

$$\mathbf{E}_2 = \frac{30}{\sqrt{2}} \angle 30° = 21.21 \angle 30°$$

$$\mathbf{E}_3 = \frac{30}{\sqrt{2}} \angle 90° = 21.21 \angle 90°$$

$$\mathbf{E}_4 = \frac{20}{\sqrt{2}} \angle -30° = 14.14 \angle -30°$$

Resultant voltage $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 = 17.68 \angle 0° + 21.21 \angle 30° + 21.21 \angle 90° + 14.14 \angle -30°$

$$= 54.26 \angle 27.13°$$

$$e = 54.26 \sqrt{2} \sin(\omega t + 27.13°) = 76.74 \sin(\omega t + 27.13°)$$

Example 4.41 *Two currents are represented by* $i_1 = 15 \sin\left(\omega t + \frac{\pi}{3}\right)$ *and* $i_2 = 25 \sin\left(\omega t + \frac{\pi}{4}\right)$. *These currents are fed into a common conductor. Find the total current in the form* $i = I_m \sin(\omega t + \phi)$. *If the conductor has a resistance of 10 Ω, what will be the energy loss in 24 hours?*

Solution

$$i_1 = 15 \sin\left(\omega t + \frac{\pi}{3}\right), \quad i_2 = 25 \sin\left(\omega t + \frac{\pi}{4}\right)$$

$$R = 10\ \Omega, \quad t = 24 \text{ hours} = 86400 \text{ seconds}$$

Writing currents i_1 and i_2 in the phasor form,

$$\mathbf{I}_1 = \frac{15}{\sqrt{2}} \angle 60° = 10.61 \angle 60°$$

$$\mathbf{I}_2 = \frac{25}{\sqrt{2}} \angle 45° = 17.68 \angle 45°$$

Total current $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10.61 \angle 60° + 17.68 \angle 45° = 28.06 \angle 50.62°$

$$i = 28.06 \sqrt{2} \sin(\omega t + 50.62°) = 39.68 \sin(\omega t + 50.62°)$$

Energy loss in 24 hours, $E = I^2 Rt$ where I is the rms value of the current

$$E = (28.06)^2 \times 10 \times 86400 = 6.8 \times 10^8 \text{ J}$$

Example 4.42 *The voltage drops across four series-connected impedances are given:*

$$v_1 = 60\ sin\left(\omega t + \frac{\pi}{6}\right), \qquad v_2 = 75\ sin\left(\omega t - \frac{5\pi}{6}\right),$$

$$v_3 = 100\ cos\left(\omega t + \frac{\pi}{4}\right), \qquad v_4 = V_{4m}\ sin\ (\omega t + \theta_4)$$

Solution

Calculate the values of V_{4m} and ϕ_4 if the voltage applied across the series circuit is $140\ sin\left(\omega t + \frac{3\pi}{5}\right)$.

$$v_1 = 60\sin\left(\omega t + \frac{\pi}{6}\right) = 60\sin(\omega t + 30°), \quad v_2 = 75\sin\left(\omega t - \frac{5\pi}{6}\right) = 75\sin(\omega t - 150°)$$

$$v_3 = 100\cos\left(\omega t + \frac{\pi}{4}\right) = 100\sin(\omega t + 135°), \quad v = 140\sin\left(\omega t + \frac{3\pi}{5}\right) = 140\sin(\omega t + 108°)$$

Writing voltages v_1, v_2, v_3 and v in the phasor form,

$$\mathbf{V}_1 = \frac{60}{\sqrt{2}}\angle 30° = 42.43\angle 30°$$

$$\mathbf{V}_2 = \frac{75}{\sqrt{2}}\angle -150° = 53.03\angle -150°$$

$$\mathbf{V}_3 = \frac{100}{\sqrt{2}}\angle 135° = 70.71\angle 135°$$

$$\mathbf{V} = \frac{140}{\sqrt{2}}\angle 108° = 98.99\angle 108°$$

For series–connected impedances,

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \mathbf{V}_4$$

$$\mathbf{V}_4 = \mathbf{V} - \mathbf{V}_1 - \mathbf{V}_2 - \mathbf{V}_3 = 98.99\angle 108° - 42.43\angle 30° - 53.03\angle -150° - 70.71\angle 135° = 57.13\angle 59.96°$$

$$v_4 = 57.13\sqrt{2}\ \sin(\omega t + 59.96°) = 80.79\ \sin(\omega t + 59.96°)$$

$$V_{4m} = 80.79\ \text{V}$$

$$\phi_4 = 59.96°$$

Example 4.43 *A circuit consists of three parallel branches. The branch currents are given as $i_1 = 10\ sin\omega t$, $i_2 = 20\ sin(\omega t + 60°)$ and $i_3 = 7.5\ sin(\omega t - 30°)$. Find the resultant current and express it in the form $i = I_m\ sin(\omega t + \phi)$. If the supply frequency is 50 Hz, calculate the resultant current when (a) $t = 0$ (b) $t = 0.001$ s.*

Solution $\quad i_1 = 10\ \sin\ \omega t, \quad i_2 = 20\ \sin\ (\omega t + 60°), \quad i_3 = 7.5\ \sin\ (\omega t - 30°)$

Writing currents i_1, i_2, i_3 in the phasor form,

$$\mathbf{I}_1 = \frac{10}{\sqrt{2}}\angle 0° = 7.07\angle 0°$$

$$\mathbf{I}_2 = \frac{20}{\sqrt{2}}\angle 60° = 14.14\angle 60°$$

Resultant current $\quad \mathbf{I}_3 = \frac{7.5}{\sqrt{2}}\angle -30° = 5.3\angle -30°$

$$\mathbf{I}_1 = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3$$
$$= 7.07\angle 0° + 14.14\angle 60° + 5.3\angle -30° = 21.04\angle 27.13°$$
$$i = 21.04\sqrt{2}\sin(\omega t + 27.13°) = 29.76\sin(\omega t + 27.13°)$$

(a) Resultant current at $t = 0$

$$i = 29.76\sin(0 + 27.13°) = 13.57 \text{ A}$$

(b) Resultant current at $t = 0.001$ s

$$i = 29.76\sin(2\pi ft + 27.13°)$$
$$= 29.76\sin(2\times 180\times 50\times 0.001 + 27.13°)$$
$$= 21.09 \text{ A}$$

(angle in degrees)

Example 4.44 *Two currents, $\mathbf{I}_1 = 10\angle 50°$ A and $\mathbf{I}_2 = 5\angle -100°$ A, flow in a single-phase ac circuit. Estimate (a) $\mathbf{I}_1 + \mathbf{I}_2$ (b) $\mathbf{I}_1 \cdot \mathbf{I}_2$ (c) $\dfrac{\mathbf{I}_1}{\mathbf{I}_2}$.*

Solution $\mathbf{I}_1 = 10\angle 50°\text{A}, \quad \mathbf{I}_2 = 5\angle -100°\text{A}$

(a) $\mathbf{I}_1 + \mathbf{I}_2 = 10\angle 50° + 5\angle -100° = 6.2\angle 26.21°$ A

(b) $\mathbf{I}_1 \cdot \mathbf{I}_2 = (10\angle 50°)(5\angle -100°) = 50\angle -50°$ A

(c) $$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{10\angle 50°}{5\angle -100°} = 2\angle 150° \text{ A}$$

Example 4.45 *Two voltages having rms values of 50 V and 75 V have a phase difference of 60°. Find the resultant sum of these two voltages.*

Solution $V_1 = 50$ V, $V_2 = 75$ V, $\phi = 60°$

Let $\mathbf{V}_1 = 50\angle 0°$ V

$\mathbf{V}_2 = 75\angle -60°$ V

Resultant voltage $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = 50\angle 0° + 75\angle -60° = 108.97\angle -36.58°$ V

Example 4.46 *Two single-phase alternators supply 300 A and 400 A respectively at a phase difference of 20° to a common load. Find the resultant current and its phase relation to its component.*

Solution $I_1 = 300$ A, $I_2 = 400$ A, $\phi = 20°$

Let $\mathbf{I}_1 = 300\angle 0°$ A

$\mathbf{I}_2 = 400\angle -20°$ A

Resultant current $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 300\angle 0° + 400\angle -20° = 689.59\angle -11.44°$ A

Example 4.47 *Two voltage sources have equal emfs and a phase difference α. When they are connected in series, the voltage is 200 V. When one source is reversed, the voltage is 15 V. Find their emfs and phase angle α.*

Solution $\mathbf{E}_1 = E\angle 0°, \quad \mathbf{E}_2 = E\angle \alpha°, \quad E_1 = E_2 = E$

Fig. 4.40

When two sources are connected in series [Fig. 4.40 (a)],

$$\sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\alpha} = 200$$

$$\sqrt{E^2 + E^2 + 2E^2\cos\alpha} = 200$$

$$2E^2 + 2E^2\cos\alpha = 40000 \qquad \ldots\text{(i)}$$

When one source is reversed [Fig. 4.40 (b)],

$$\sqrt{E_1^2 + E_2^2 - 2E_1E_2\cos\alpha} = 15$$

$$\sqrt{E^2 + E^2 - 2E^2\cos\alpha} = 15$$

$$2E^2 - 2E^2\cos\alpha = 225 \qquad \ldots\text{(ii)}$$

Adding Eqs (i) and (ii),

$$4E^2 = 40225$$

$$E^2 = 10056.25$$

$$E = 100.28 \text{ V}$$

From Eq. (i),

$$2E^2 + 2E^2\cos\alpha = 40000$$

$$20112.5 + 20112.5\cos\alpha = 40000$$

$$\cos\alpha = 0.988$$

$$\alpha = 8.58°$$

Example 4.48 *Two sinusoidal sources of emf have rms values of E_1 and E_2 and a phase difference of α. When connected in series, the resultant voltage is 41.1 V. When one of the sources is reversed, the resultant emf is 17.52 V. When phase displacement is made zero, the resultant emf is 42.5 V. Calculate E_1, E_2 and α.*

Solution $\mathbf{E}_1 = E_1\angle 0°, \quad \mathbf{E}_2 = E_2\angle \alpha°$

Fig. 4.41

When two sources are connected in series [Fig. 4.41 (a)],

$$\sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos\alpha} = 41.1$$

$$E_1^2 + E_2^2 + 2E_1E_2 \cos\alpha = 1689.21 \qquad \ldots\text{(i)}$$

When one of the sources is reversed [Fig. 4.41 (b)],

$$\sqrt{E_1^2 + E_2^2 - 2E_1E_2 \cos\alpha} = 17.52$$

$$E_1^2 + E_2^2 - 2E_1E_2 \cos\alpha = 306.95 \qquad \ldots\text{(ii)}$$

When phase displacement is made zero [Fig. 4.41 (c)],

$$\sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos 0°} = 42.5$$

$$E_1 + E_2 = 42.5$$

$$E_1 = 42.5 - E_2 \qquad \ldots\text{(iii)}$$

Adding Eqs (i) and (ii),

$$2(E_1^2 + E_2^2) = 1996.16$$

$$E_1^2 + E_2^2 = 998.08$$

From Eq. (iii),

$$(42.5 - E_2)^2 + E_2^2 = 998.08$$

$$1806.25 - 85E_2 + E_2^2 + E_2^2 = 998.08$$

$$E_2^2 - 42.5E_2 + 404.09 = 0 \qquad \ldots\text{(iv)}$$

Solving Eq. (iv),

$$E_2 = 28.14 \text{ V or } E_2 = 14.36 \text{ V}$$

$$E_1 = 14.36 \text{ V or } E_1 = 28.14 \text{ V}$$

Subtracting Eq. (ii) from Eq. (i),

$$4E_1E_2 \cos\alpha = 1382.26$$

$$4 \times 14.36 \times 28.14 \cos\alpha = 1382.26$$

$$\cos\alpha = 0.855$$

$$\alpha = 31.24°$$

4.8 BEHAVIOUR OF A PURE RESISTOR IN AN ac CIRCUIT

Consider a pure resistor R connected across an alternating voltage source v as shown in Fig. 4.42. Let the alternating voltage be $v = V_m \sin \omega t$.

Fig. 4.42 *Purely resistive circuit*

1. Current The alternating current i is given by

$$i = \frac{v}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t \qquad \ldots\left(I_m = \frac{V_m}{R}\right)$$

where I_m is the maximum value of the alternating current. From the voltage and current equation, it is clear that the current is in phase with the voltage in a purely resistive circuit.

2. Waveforms Figure 4.43 shows the voltage and current waveforms.

Fig. 4.43 *Waveforms*

3. Phasor Diagram Figure 4.44 shows the phasor diagram.

Fig. 4.44 *Phasor diagram*

4. Impedance It is the resistance offered to the flow of current in an ac circuit. In a purely resistive circuit,

$$Z = \frac{V}{I} = \frac{V_m}{I_m} = \frac{V_m}{\frac{V_m}{R}} = R$$

5. Phase Difference Since the voltage and current are in phase with each other, the phase difference is zero.

$$\phi = 0°$$

6. Power Factor It is defined as the cosine of the angle between the voltage and current phasors.

$$\text{Power factor} = \cos \phi = \cos (0°) = 1$$

7. Power Instantaneous power p is given by

$$\begin{aligned} p &= vi \\ &= V_m \sin \omega t \; I_m \sin \omega t \\ &= V_m I_m \sin^2 \omega t \\ &= \frac{V_m I_m}{2}(1 - \cos 2\omega t) \\ &= \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t \end{aligned}$$

The power waveform is shown in Fig. 4.45. The power consists of a constant part $\frac{V_m I_m}{2}$ and a fluctuating part $\frac{V_m I_m}{2} \cos 2\omega t$. The frequency of the fluctuating power is twice the applied voltage frequency and its average value over one complete cycle is zero.

$$\text{Average power } P = \frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} = VI$$

Fig. 4.45 *Power waveform*

Thus, power in a purely resistive circuit is equal to the product of rms values of voltage and current.

4.9 BEHAVIOUR OF A PURE INDUCTOR IN AN ac CIRCUIT

Consider a pure inductor L connected across an alternating voltage v as shown in Fig. 4.46. Let the alternating voltage be $v = V_m \sin \omega t$.

Fig. 4.46 *Purely inductive circuit*

1. Current The alternating current i is given by

$$\begin{aligned} i &= \frac{1}{L}\int v\, dt \\ &= \frac{1}{L}\int V_m \sin \omega t\, dt \\ &= \frac{V_m}{\omega L}(-\cos \omega t) \\ &= \frac{V_m}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) \\ &= I_m \sin\left(\omega t - \frac{\pi}{2}\right) \qquad \ldots\left(I_m = \frac{V_m}{\omega L}\right) \end{aligned}$$

where I_m is the maximum value of the alternating current. From the voltage and current equation, it is clear that the current lags behind the voltage by 90° in a purely inductive circuit.

2. Waveforms Figure 4.47 shows the voltage and current waveforms.

Fig. 4.47 *Waveforms*

3. Phasor Diagram Figure 4.48 shows the phasor diagram.

Fig. 4.48 *Phasor diagram*

4. Impedance In a purely inductive circuit,

$$Z = \frac{V}{I} = \frac{V_m}{I_m} = \frac{V_m}{\dfrac{V_m}{\omega L}} = \omega L$$

The quantity ωL is called inductive reactance, is denoted by X_L and is measured in ohms.
For a dc supply, $f = 0 \quad \therefore \quad X_L = 0$
Thus, an inductor acts as a short circuit for a dc supply.

5. Phase Difference It is the angle between the voltage and current phasors.

$$\phi = 90°$$

6. Power Factor It is defined as the cosine of the angle between the voltage and current phasors.

$$\text{pf} = \cos\phi = \cos(90°) = 0$$

7. Power Instantaneous powers p is given by

$$\begin{aligned} p &= vi \\ &= V_m \sin \omega t \; I_m \sin\left(\omega t - \frac{\pi}{2}\right) \\ &= -V_m I_m \sin \omega t \cos \omega t \\ &= -\frac{V_m I_m}{2} \sin 2\,\omega t \end{aligned}$$

Fig. 4.49 *Power waveform*

The power waveform is shown in Fig. 4.49
The average power for one complete cycle, $P = 0$.
Hence, power consumed by a purely inductive circuit is zero.

4.10 BEHAVIOUR OF A PURE CAPACITOR IN AN ac CIRCUIT

Consider a pure capacitor C connected across an alternating voltage v as shown in Fig. 4.50. Let the alternating voltage be $v = V_m \sin \omega t$.

Fig. 4.50 *Purely capacitive circuit*

1. *Current* The alternating current i is given by

$$\begin{aligned} i &= C\frac{dv}{dt} \\ &= C\frac{d}{dt}(V_m \sin \omega t) \\ &= \omega C\, V_m \cos \omega t \\ &= \omega C\, V_m \sin(\omega t + 90°) \\ &= I_m \sin(\omega t + 90°) \qquad \ldots(I_m = \omega C\, V_m) \end{aligned}$$

where I_m is the maximum value of the alternating current. From the voltage and current equation, it is clear that the current leads the voltage by 90° in a purely capacitive circuit.

2. *Waveforms* Figure 4.51 shows the voltage and current waveforms.

Fig. 4.51 *Waveforms*

3. *Phasor Diagram* Figure 4.52 shows the phasor diagram.

Fig. 4.52 *Phasor diagram*

4. *Impedance* In a purely capacitive circuit,

$$Z = \frac{V}{I} = \frac{V_m}{I_m} = \frac{V_m}{\omega C\, V_m} = \frac{1}{\omega C}$$

The quantity $\frac{1}{\omega C}$ is called capacitive reactance, is denoted by X_C and is measured in ohms.
For a dc supply, $f = 0 \therefore X_C = \infty$
Thus, the capacitor acts as an open circuit for a dc supply.

5. *Phase Difference* It is the angle between the voltage and current phasors.

$$\phi = 90°$$

6. *Power Factor* It is defined as the cosine of the angle between the voltage and current phasors.

$$\text{pf} = \cos\phi = \cos(90°) = 0$$

7. *Power* Instantaneous power p is given by

$$\begin{aligned} p &= vi \\ &= V_m \sin \omega t\, I_m \sin(\omega t + 90°) \\ &= V_m I_m \sin \omega t \cos \omega t \\ &= \frac{V_m I_m}{2} \sin 2\omega t \end{aligned}$$

The power waveform is shown in Fig. 4.53.
The average power for one complete cycle, $P = 0$.
Hence, power consumed by a purely capacitive circuit is zero.

Fig. 4.53 *Power waveform*

Example 4.49 *An ac circuit consists of a pure resistance of 10 ohms and is connected across an ac supply of 230 V, 50 Hz. Calculate (a) current, (b) power consumed, (c) power factor, and (d) write down the equations for voltage and current.*

Solution $R = 10\ \Omega, \quad V = 230\text{ V}, \quad f = 50\text{ Hz}$

(a) Current

$$I = \frac{V}{R} = \frac{230}{10} = 23\text{ A}$$

(b) Power consumed

$$P = VI = 230 \times 23 = 5290\text{ W}$$

(c) Power factor
Since the voltage and current are in phase with each other, $\phi = 0°$

$$\text{pf} = \cos\phi = \cos(0°) = 1$$

(d) Voltage and current equations

$$V_m = \sqrt{2}\, V = \sqrt{2} \times 230 = 325.27\text{ V}$$

$$I_m = \sqrt{2} I = \sqrt{2} \times 23 = 32.53\text{ A}$$

$$\omega = 2\pi f = 2\pi \times 50 = 314.16\text{ rad/s}$$

$$v = V_m \sin\omega t = 325.27 \sin 314.16\, t$$

$$i = I_m \sin\omega t = 32.53 \sin 314.16\, t$$

Example 4.50 *An inductive coil having negligible resistance and 0.1 henry inductance is connected across a 200 V, 50 Hz supply. Find (a) inductive reactance, (b) rms value of current, (c) power, (d) power factor, and (e) equations for voltage and current.*

Solution $L = 0.1$ H, $V = 200$ V, $f = 50$ Hz

(a) Inductive reactance

$$X_L = 2\pi f L = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

(b) rms value of current

$$I = \frac{V}{X_L} = \frac{200}{31.42} = 6.37 \text{ A}$$

(c) Power

Since the current lags behind the voltage by 90° in purely inductive circuit, $\phi = 90°$

$$P = VI \cos \phi = 200 \times 6.37 \times \cos (90°) = 0$$

(d) Power factor

$$\text{pf} = \cos \phi = \cos (90°) = 0$$

(e) Equations for voltage and current

$$V_m = \sqrt{2}\ V = \sqrt{2} \times 200 = 282.84 \text{ V}$$

$$I_m = \sqrt{2}\ I = \sqrt{2} \times 6.37 = 9 \text{ A}$$

$$\omega = 2\pi f = 2\pi \times 50 = 314.16 \text{ rad/s}$$

$$v = V_m \sin \omega t = 282.84 \sin 314.16\, t$$

$$i = I_m \sin\left(\omega t - \frac{\pi}{2}\right) = 9 \sin\left(314.16\, t - \frac{\pi}{2}\right)$$

Example 4.51 *The voltage and current through circuit elements are*

$$v = 100 \sin (314\, t + 45°) \text{ volts}$$
$$i = 10 \sin (314\, t + 315°) \text{ amperes}$$

(a) Identify the circuit elements. (b) Find the value of the elements. (c) Obtain an expression for power.

Solution

$$v = 100 \sin (314\, t + 45°)$$
$$i = 10 \sin (314\, t + 315°)$$
$$= 10 \sin (314\, t + 315° - 360°)$$
$$= 10 \sin (314\, t - 45°)$$

(a) Identification of elements

From voltage and current equations, it is clear that the current i lags behind the voltage by 90°. Hence, the circuit element is an inductor.

(b) Value of elements

$$X_L = \frac{V}{I} = \frac{V_m}{I_m} = \frac{100}{10} = 10\ \Omega$$

$$X_L = \omega L$$

$$10 = 314\, L$$

$$L = 31.8 \text{ mH}$$

(c) Expression for power

$$p = -\frac{V_m I_m}{2}\sin 2\omega t = -\frac{100\times 10}{2}\sin(2\times\ 314\,t) = -500\sin 628\,t$$

Example 4.52 *A capacitor has a capacitance of 30 microfarads which is connected across a 230 V, 50 Hz supply. Find (a) capacitive reactance, (b) rms value of current, (c) power, (d) power factor, and (v) equations for voltage and current.*

Solution $C = 30\ \mu\text{F},\quad V = 230\ \text{V},\quad f = 50\ \text{Hz}$

(a) Capacitive reactance

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi\times 50\times 30\times 10^{-6}} = 106.1\ \Omega$$

(b) rms value of current

$$I = \frac{V}{X_C} = \frac{230}{106.1} = 2.17\ \text{A}$$

(c) Power

Since the current leads the voltage by 90° in purely capacitive circuit, $\phi = 90°$

$$P = VI\cos\phi = 230\times 2.17\times\cos(90°) = 0$$

(d) Power factor

$$\text{pf} = \cos\phi = \cos(90°) = 0$$

(e) Equations for voltage and current

$$V_m = \sqrt{2}\,V = \sqrt{2}\times 230 = 325.27\ \text{V}$$

$$I_m = \sqrt{2}\,I = \sqrt{2}\times 2.17 = 3.07\ \text{A}$$

$$\omega = 2\pi f = 2\pi\times 50 = 314.16\ \text{rad/s}$$

$$v = V_m\sin\omega t = 325.27\sin 314.16\,t$$

$$i = I_m\sin\left(\omega t + \frac{\pi}{2}\right) = 3.07\sin\left(314.16\,t + \frac{\pi}{2}\right)$$

4.11 SERIES *RL* CIRCUIT

Figure 4.54 shows a pure resistor R connected in series with a pure inductor L across an alternating voltage $v = V_m\sin\omega t$

Let V and I be the rms values of applied voltage and current.

Potential difference across the resistor $= V_R = R\,I$

Potential difference across the inductor $= V_L = X_L I$

The voltage $\mathbf{V}_R$ is in phase with the current $\mathbf{I}$ whereas the voltage $\mathbf{V}_L$ leads the current $\mathbf{I}$ by 90°.

Fig. 4.54 *RL circuit*

1. Phase Diagram

Figure 4.55 shows the phasor diagram of series *RL* circuit.

Fig. 4.55 *(a) Phasor Diagram, (b) Voltage triangle*

2. Impedance

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L = R\mathbf{I} + jX_L\mathbf{I} = (R + jX_L)\,\mathbf{I}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R + jX_L = \mathbf{Z}$$

$$\mathbf{Z} = Z\angle\phi$$

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$$

$$\phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

The quantity Z is called the *complex impedance* of the series RL circuit.

3. Impedance Triangle

Figure 4.56 shows the impedance triangle of series RL circuit.

Fig. 4.56 *Impedance triangle*

4. Current From the phasor diagram, it is clear that the current I lags behind the voltage V by an angle ϕ. If the applied voltage is given by $v = V_m \sin \omega t$ then the current equation will be

$$i = I_m \sin(\omega t - \phi)$$

where $$I_m = \frac{V_m}{Z}$$

and $$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

5. Waveforms Figure 4.57 shows the voltage and current waveforms.

Fig. 4.57 *Waveforms*

6. Power Instantaneous power p is given by

$$\begin{aligned} p &= vi \\ &= V_m \sin \omega t \, I_m \sin(\omega t - \phi) \\ &= V_m I_m \sin \omega t \sin(\omega t - \phi) \\ &= V_m I_m \left[\frac{\cos\phi - \cos(2\omega t - \phi)}{20} \right] \\ &= \frac{V_m I_m}{2} \cos\phi - \frac{V_m I_m}{2} \cos(2\omega t - \phi) \end{aligned}$$

Thus, power consists of a constant part $\frac{V_m I_m}{2} \cos\phi$ and a fluctuating part $\frac{V_m I_m}{2} \cos(2\omega t - \phi)$. The frequency of the fluctuating part is twice the applied voltage frequency and its average value over one complete cyce is zero.

$$\text{Average power } P = \frac{V_m I_m}{2} \cos\phi = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos\phi = VI \cos\phi$$

Thus, power is dependent upon the in-phase component of the current. The average power is also called *active power* and is measured in watts.

We know that a pure inductor and capacitor consume no power because all the power received from the source in a half cycle is returned to the source in the next half cycle. This circulating power is called *reactive power*. It is a product of the voltage and reactive component of the current, i.e., $I \sin\phi$ and is measured in VAR (volt-ampere-reactive).

Reactive power $Q = VI \sin\phi$.

The product of voltage and current is known as *apparent power* (S) and is measured in volt-ampere (VA).

$$S = VI = \sqrt{P^2 + Q^2}$$

7. Power Triangle In terms of circuit components,

$$\cos\phi = \frac{R}{Z}$$

and

$$V = ZI$$

$$P = VI \cos\phi = ZII \frac{R}{Z} = I^2 R$$

$$Q = VI \sin\phi = ZII \frac{X_L}{Z} = I^2 X_L$$

$$S = VI = Z\,I\,I = I^2 Z$$

Fig. 4.58 *Power triangle*

Figure 4.58 shows the power triangle of series RL circuit.

8. Power Factor It is defined as the cosine of the angle between the voltage and current phasors.

$$\text{pf} = \cos\phi$$

From voltage triangle, $\text{pf} = \frac{V_R}{V}$

From impedance triangle, $\text{pf} = \frac{R}{Z}$

From power triangle, $\text{pf} = \dfrac{P}{S}$

In case of series *RL* circuit, the power factor is lagging in nature.

Example 4.53 *An alternating voltage of 80 + j60 V is applied to a circuit and the current flowing is 4 – j2 A. Find the (a) impedance, (b) phase angle, (c) power factor, and (d) power consumed.*

Solution $\mathbf{V} = 80 + j\,60$ V, $\mathbf{I} = 4 - j\,2$ A

(a) Impedance

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{80 + j60}{4 - j2} = \frac{100\,\angle 36.87°}{4.47\angle -26.56°} = 22.37\,\angle 63.43°\ \Omega$$

$$Z = 22.37\ \Omega$$

(b) Phase angle

$$\phi = 63.43°$$

(c) Power factor

$$\text{pf} = \cos\phi = \cos(63.43°) = 0.447 \text{ (lagging)}$$

(d) Power consumed

$$P = VI\cos\phi = 100 \times 4.47 \times 0.447 = 199.81 \text{ W}$$

Example 4.54 *The voltage and current in a circuit are given by* $\mathbf{V} = 150\,\angle 30°$ *V and* $\mathbf{I} = 2\,\angle -15°$ *A. If the circuit works on a 50 Hz supply, determine impedance, resistance, reactance, power factor and power loss considering the circuit as a simple series circuit.*

Solution $\mathbf{V} = 150\,\angle 30°$ V, $\mathbf{I} = 2\,\angle -15°$ A, $f = 50$ Hz

(a) Impedance

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{150\angle 30°}{2\angle -15°} = 75\,\angle 45°\ \Omega = 53.03 + j53.03\ \Omega$$

$$Z = 75\ \Omega$$

(b) Resistance

$$R = 53.03\ \Omega$$

(c) Reactance

$$X = 53.03\ \Omega$$

(d) Power factor

$$\phi = 45°$$

$$\text{pf} = \cos\phi = \cos(45°) = 0.707 \text{ (lagging)}$$

(e) Power loss

$$P = VI\cos\phi = 150 \times 2 \times 0.707 = 212.1 \text{ W}$$

Example 4.55 *An rms voltage of 100* $\angle 0°$ *V is applied to a series combination of* Z_1 *and* Z_2 *when* $\mathbf{Z}_1 = 20\,\angle 30°\ \Omega$. *The effective voltage drop across* $\mathbf{Z}_1$ *is known to be* $40\,\angle -30°$ *V. Find the reactive component of* $\mathbf{Z}_2$.

Solution

$$\mathbf{V} = 100\angle 0°\text{ V}, \quad \mathbf{Z}_1 = 20\angle 30°\Omega, \quad \mathbf{V}_1 = 40\angle -30°\text{ V}$$

$$\mathbf{I} = \frac{\mathbf{V}_1}{\mathbf{Z}_1} = \frac{40\angle -30°}{20\angle 30°} = 2\angle -60°\text{A}$$

$$\mathbf{Z} = \frac{\mathbf{V}_1}{\mathbf{I}} = \frac{100\angle 0°}{2\angle -60°} = 50\angle 60° = 25 + j43.3\ \Omega$$

$$\mathbf{Z}_1 = 20\angle 30° = 17.32 + j10\ \Omega$$

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$$

$$\mathbf{Z}_2 = \mathbf{Z} - \mathbf{Z}_1 = 25 + j43.3 - 17.32 - j10 = 7.68 + j33.3\ \Omega$$

Reactive component of $\mathbf{Z}_2 = 33.3\ \Omega$

Example 4.56 *A voltage v(t) = 177 sin (314 t + 10°) is applied to a circuit. It causes a steady-state current to flow, which is described by i(t) = 14.14 sin (314 t – 20°). Determine the power factor and average power delivered to the circuit.*

Solution $v(t) = 177 \sin(314\,t + 10°)$, $i(t) = 14.14 \sin(314\,t - 20°)$

(a) Power factor

Current $i(t)$ lags behind voltage $v(t)$ by 30°.

$$\phi = 30°$$

$$\text{pf} = \cos\phi = \cos(30°) = 0.866 \text{ (lagging)}$$

(b) Average power

$$P = VI\cos\phi = \frac{177}{\sqrt{2}} \times \frac{14.14}{\sqrt{2}} \times 0.866 = 1083.7\text{ W}$$

Example 4.57 *When a sinusoidal voltage of 120 V(rms) is applied to a series RL circuit, it is found that there occurs a power dissipation of 1200 W and a current flow given by i(t) = 28.3 sin (314t – ϕ). Find the circuit resistance and inductance.*

Solution $V = 120$ V, $P = 1200$ W, $i(t) = 28.3 \sin(314t - \phi)$

(a) Resistance

$$I = \frac{28.3}{\sqrt{2}} = 20.01\text{ A}$$

$$P = V\,I\cos\phi$$

$$1200 = 120 \times 20.01 \times \cos\phi$$

$$\cos\phi = 0.499$$

$$\phi = 60.02°$$

$$Z = \frac{V}{I} = \frac{120}{20.01} = 6\ \Omega$$

$$\mathbf{Z} = Z\angle\phi = 6\angle 60.02° = 3 + j5.2\ \Omega$$

$$R = 3\ \Omega$$

(b) Inductance

$$X_L = 5.2\ \Omega$$

$$X_L = \omega L$$

$$5.2 = 314 \times L$$

$$L = 0.0165\text{ H}$$

Example 4.58 *In a series circuit containing resistance and inductance, the current and voltage are expressed as* $i(t) = 5\sin\left(314t + \frac{2\pi}{3}\right)$ *and* $v(t) = 20\sin\left(314t + \frac{5\pi}{6}\right)$ *(a) What is the impedance of the circuit? (b) What are the values of resistance, inductance, and power factor? (c) What is the average power drawn by the circuit?*

Solution

$$i(t) = 5\sin\left(314t + \frac{2\pi}{3}\right), \quad v(t) = 20\sin\left(314t + \frac{5\pi}{6}\right)$$

(a) Impedance

$$Z = \frac{V}{I} = \frac{V_m}{I_m} = \frac{20}{5} = 4\ \Omega$$

(b) Power factor, resistance and inductance

Current $i(t)$ lags behind voltage $v(t)$ by an angle $\phi = 150° - 120° = 30°$

$$\text{pf} = \cos\phi = \cos(30°) = 0.866 \text{ (lagging)}$$
$$\mathbf{Z} = 4\angle 30° = 3.464 + j2\ \Omega$$
$$R = 3.464\ \Omega$$
$$X_L = 2\ \Omega$$
$$X_L = \omega L$$
$$2 = 314 \times L$$
$$L = 6.37 \text{ mH}$$

(c) Average power

$$P = VI\cos\phi = \frac{20}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \times 0.866 = 43.3 \text{ W}$$

Example 4.59 *A series circuit consists of a non-inductive resistance of 6 Ω and an inductive reactance of 10 Ω. When connected to a single-phase ac supply, it draws a current i(t) = 27.89 sin (628t−45°). Calculate (a) the voltage applied to the series circuit in the form* $V_m \sin(\omega t \pm \phi)$*, (b) inductance, and (c) power drawn by the circuit.*

Solution

$$R = 6\ \Omega, \quad X_L = 10\ \Omega, \quad i(t) = 27.89\sin(628t - 45°)$$

(a) Voltage applied to the series circuit

$$\mathbf{Z} = R + jX_L = 6 + j10 = 11.66\angle 59.04°\ \Omega$$
$$\mathbf{I} = \frac{27.89}{\sqrt{2}}\angle -45° = 19.72\angle -45° \text{ A}$$
$$\mathbf{V} = \mathbf{Z}\,\mathbf{I} = (11.66\angle 59.04°)\ (19.72\angle -45°) = 229.95\angle 14.04° \text{ V}$$
$$v = 229.95\sqrt{2}\sin(\omega t + 14.04°) = 325.2\sin(\omega t + 14.04°)$$

(b) Inductance

$$X_L = 10\ \Omega$$
$$X_L = \omega L$$
$$10 = 628 \times L$$
$$L = 15.9 \text{ mH}$$

(c) Power drawn by the circuit

$$P = VI\cos\phi = 229.95 \times 19.72 \times \cos(59.04°) = 2332.78 \text{ W}$$

Example 4.60 *When an inductive coil is connected to a dc supply at 240 V, the current in it is 16 A. When the same coil is connected to an ac supply at 240 V, 50 Hz, the current is 12.27 A. Calculate (a) resistance, (b) impedance, (c) reactance, and (d) inductance of the coil.*

Solution

$$\text{For dc}: V = 240 \text{ V}, \qquad I = 16 \text{ A}$$
$$\text{For ac}: V = 240 \text{ V}, \qquad I = 12.27 \text{ A}$$

(a) Resistance

When an inductive coil is connected to a dc supply,

$$f = 0$$
$$X_L = 2\pi fL = 0$$

The coil behaves like a pure resistor.

$$R = \frac{V}{I} = \frac{240}{16} = 15\ \Omega$$

(b) Impedance

When the coil is connected to an ac supply,

$$Z = \frac{V}{I} = \frac{240}{12.27} = 19.56\ \Omega$$

(c) Reactance

$$X_L = \sqrt{Z^2 - R^2} = \sqrt{(19.56)^2 - (15)^2} = 12.55\ \Omega$$

(d) Inductance

$$X_L = 2\pi fL$$
$$12.55 = 2\pi \times 50 \times L$$
$$L = 0.04 \text{ H}$$

Example 4.61 *An inductive coil draws 10 A current and consumes 1 kW power from a 200 V, 50 Hz ac supply. Determine (a) impedance in Cartesian and polar forms, (b) power factor, and (c) reactive and apparent power.*

Solution

$$I = 10 \text{ A}, \quad P = 1 \text{ kW}, \quad V = 200 \text{ V}, \quad f = 50 \text{ Hz}$$

(a) Impedance in Cartesian and polar forms

$$Z = \frac{V}{I} = \frac{200}{10} = 20\ \Omega$$
$$P = VI\cos\phi$$
$$1000 = 200 \times 10 \times \cos\phi$$
$$\cos\phi = 0.5$$
$$\phi = 60°$$

Expressing $\mathbf{Z}$ in polar form,

$$\mathbf{Z} = Z\angle\,\phi^\circ = 20\,\angle 60^\circ\ \Omega$$

Expressing $\mathbf{Z}$ in Cartesian form,

$$\mathbf{Z} = 10 + j17.32\ \Omega$$

(b) Power factor

$$\text{pf} = \cos\phi = \cos(60^\circ) = 0.5\ \text{(lagging)}$$

(c) Reactive and apparent power

$$Q = VI\sin\phi = 200\times 10\times\sin\,(60^\circ) = 1.73\ \text{kVAR}$$

$$\text{S} = VI = 200\times 10 = 2\ \text{kVA}$$

Example 4.62 *In the circuit of Fig. 4.59, a coil connected across a 250 V, 50 Hz supply takes a current of 10 A at 0.8 lagging power factor. What will be the power taken by the choke coil when connected across a 200 V, 25 Hz supply? Also calculate resistance and inductance of the coil.*

Fig. 4.59

Solution

$$V_1 = 250\ \text{V},\quad f_1 = 50\ \text{Hz},\quad I_1 = 10\ \text{A},\quad \text{pf} = 0.8\ \text{(lagging)}$$

$$V_2 = 200\text{V},\quad f_2 = 25\ \text{Hz}$$

(a) Resistance and inductance of the coil

$$Z_1 = \frac{V_1}{I_1} = \frac{250}{10} = 25\ \Omega$$

$$\phi_1 = \cos^{-1}(0.8) = 36.87^\circ$$

$$\mathbf{Z}_1 = Z_1\angle\phi_1 = 25\angle 36.87^\circ = 20 + j15\ \Omega$$

$$R = 20\ \Omega$$

$$X_{L_1} = 15\ \Omega$$

$$X_{L_1} = 2\pi f_1\ L$$

$$15 = 2\pi\times 50\times L$$

$$L = 0.0477\ \text{H}$$

(b) Power taken by the choke coil when connected across a 200 V, 50 Hz supply.

$$X_{L_2} = 2\pi f_2 L = 2\pi\times 25\times 0.0477 = 7.49\ \Omega$$

$$\mathbf{Z}_2 = R + jX_{L_2} = 20 + j7.49 = 21.36\ \angle 20.53^\circ\ \Omega$$

$$Z_2 = 21.36\ \Omega$$

$$\phi_2 = 20.53^\circ$$

$$I_2 = \frac{V_2}{Z_2} = \frac{200}{21.36} = 9.36\ \text{A}$$

$$P = V_2\ I_2\ \cos\phi_2 = 200\times 9.36\times\cos\ (20.53^\circ) = 1.753\ \text{kW}$$

Example 4.63 *A load of 22 kW operates at 0.8 lagging power factor when connected to a 420 V, single-phase, 50 Hz source. Find (a) current in the load, (b) power factor angle, (c) impedance, (d) resistance of load, (e) reactance of load, (f) voltage and current equations.*

Solution $P = 22$ kW, pf = 0.8 (lagging), $V = 420$ V, $f = 50$ Hz

(a) Current in the load

$$P = VI\cos\phi$$
$$22\times10^3 = 420\times I\times0.8$$
$$I = 65.48 \text{ A}$$

(b) Power-factor angle

$$\phi = \cos^{-1}(0.8) = 36.87°$$

(c) Impedance

$$Z = \frac{V}{I} = \frac{420}{65.48} = 6.41\ \Omega$$

(d) Resistance of load

$$R = Z\cos\phi = 6.41\times0.8 = 5.13\ \Omega$$

(e) Reactance of load

$$X_L = Z\sin\phi = 6.41\times\sin(36.87°) = 3.85\ \Omega$$

(f) Voltage and current equations

$$v = V_m\sin 2\pi ft = 420\sqrt{2}\sin(2\pi\times50)\,t = 593.97\sin 100\,\pi t$$

The current lags behind the voltage by 36.87°.

$$i = I_m\sin(2\pi ft-\phi) = 65.48\sqrt{2}\sin(2\pi\times50t-36.87°)$$
$$= 92.6\sin(100\,\pi t-36.87°)$$

Example 4.64 *A choke coil is connected in series with fixed resistor as shown in Fig. 4.60. A 240 V, 50 Hz supply is applied and a current of 2.5 A flows. If the voltage drops across the coil and fixed resistor are 140 V and 160 V respectively, calculate the value of the fixed resistance, the resistance and inductance of the coil, and power drawn by the coil.*

Fig. 4.60

Solution $V = 240$ V, $f = 50$ Hz, $I = 2.5$ A, $V_{\text{coil}} = 140$ V, $V_R = 160$ V

(a) Resistance of fixed resistor

$$R = \frac{V_R}{I} = \frac{160}{2.5} = 64\ \Omega$$

(b) Resistance and inductance of the coil

$$Z_{\text{coil}} = \frac{V_{\text{coil}}}{I} = \frac{140}{2.5} = 56\ \Omega$$

$$Z_{\text{coil}} = \sqrt{r^2 + X_L^2} = 56$$

$$r^2 + X_L^2 = 3136 \qquad \ldots\text{(i)}$$

$$Z = \frac{V}{I} = \frac{240}{2.5} = 96\ \Omega$$

$$\mathbf{Z} = (R + r) + jX_L$$

$$Z = \sqrt{(R + r)^2 + X_L^2}$$

$$96 = \sqrt{(64 + r)^2 + X_L^2}$$

$$(64 + r)^2 + X_L^2 = 9216 \qquad \ldots\text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii),

$$(64 + r)^2 - r^2 = 6080$$

$$4096 + 128r + r^2 - r^2 = 6080$$

$$128r = 1984$$

$$r = 15.5\ \Omega$$

Substituting the value of r in Eq. (i),

$$(15.5)^2 + X_L^2 = 3136$$

$$X_L^2 = 2895.75$$

$$X_L = 53.81\ \Omega$$

$$X_L = 2\pi fL$$

$$53.81 = 2\pi \times 50 \times L$$

$$L = 0.17\ \text{H}$$

(c) Power drawn by the coil

$$P_{\text{coil}} = I^2 r = (2.5)^2 \times 15.5 = 96.875\ \text{W}$$

Example 4.65 *A 100 Ω resistor is connected in series with a choke coil as shown in Fig. 4.61. When a 400 V, 50 Hz supply is applied to this combination, the voltages across the resistance and the choke coil are 200 V and 300 V respectively. Find the power consumed by the choke coil. Also, calculate the power factor of the choke coil and the power factor of the circuit.*

Fig. 4.61

Solution $R = 100\ \Omega$, $V = 400$ V, $f = 50$ Hz, $V_R = 200$ V, $V_{\text{coil}} = 300$ V

(a) Power consumed by choke coil

$$I = \frac{V_R}{R} = \frac{200}{100} = 2 \text{ A}$$

$$Z_{\text{coil}} = \frac{V_{\text{coil}}}{I} = \frac{300}{2} = 150\ \Omega$$

$$\sqrt{r^2 + X_L^2} = 150$$

$$r^2 + X_L^2 = 22500 \qquad \ldots\text{(i)}$$

$$Z = \frac{V}{I} = \frac{400}{2} = 200\ \Omega$$

$$\mathbf{Z} = (R + r) + jX_L$$

$$Z = \sqrt{(R + r)^2 + X_L^2} = 200$$

$$(100 + r)^2 + X_L^2 = 40000 \qquad \ldots\text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii),

$$(100 + r)^2 - r^2 = 17500$$

$$10000 + 200r + r^2 - r^2 = 17500$$

$$200r = 7500$$

$$r = 37.5\ \Omega$$

Substituting the value of r in Eq. (i),

$$(37.5)^2 + X_L^2 = 22500$$

$$X_L^2 = 21093.75$$

$$X_L = 145.24\ \Omega$$

$$P_{\text{coil}} = I^2 r = (2)^2 \times 37.5 = 50 \text{ W}$$

(b) Power factor of the choke coil

$$\text{pf}_{\text{coil}} = \frac{r}{Z_{\text{coil}}} = \frac{37.5}{150} = 0.25 \text{ (lagging)}$$

(c) Power factor of the circuit

$$\text{pf}_{\text{circuit}} = \frac{R + r}{Z} = \frac{100 + 37.5}{200} = 0.6875 \text{ (lagging)}$$

Example 4.66 *A resistor of 25 Ω is connected in series with a choke coil as shown in Fig. 4.62. The series combination when connected across a 250 V, 50 Hz supply, draws a current of 4 A which lags behind the voltage by 65°. Calculate (a) resistance and inductance of the coil, (b) total power, (c) power consumed by resistance, and (d) power consumed by choke coil.*

Fig. 4.62

Solution $R = 25\ \Omega$, $V = 250$ V, $f = 50$ Hz, $I = 4$ A, $\phi = 65°$

(a) Resistance and inductance of the coil

$$Z = \frac{V}{I} = \frac{250}{4} = 62.5\ \Omega$$

$$\mathbf{Z} = Z\angle\phi = 62.5\angle 65° = 26.41 + j56.64\ \Omega$$

But $$\mathbf{Z} = (R + r) + jX_L$$

$$X_L = 56.64\ \Omega$$

$$R + r = 26.41$$

$$r = 26.41 - 25 = 1.41\ \Omega$$

$$X_L = 2\pi fL$$

$$56.64 = 2\pi \times 50 \times L$$

$$L = 0.18\ \text{H}$$

(b) Total power

$$P = I^2(R + r) = (4)^2 \times 26.41 = 422.56\ \text{W}$$

(c) Power consumed by resistance

$$P_R = I^2 R = (4)^2 \times 25 = 400\ \text{W}$$

(d) Power consumed by choke coil

$$P_{\text{coil}} = I^2 r = (4)^2 \times 1.41 = 22.56\ \text{W}$$

Example 4.67 *When a resistor and a coil in series are connected to a 240 V supply as shown in Fig. 4.63, a current of 3 A flows, lagging 37° behind the supply voltage. The voltage across the coil is 171 volts. Find the resistance and reactance of the coil, and the resistance of the resistor.*

Fig. 4.63

Solution $V = 240$ V, $I = 3$ A, $\phi = 37°$, $V_{coil} = 171$ V

(a) Resistance and reactance of the coil

$$Z_{coil} = \frac{V_{coil}}{I} = \frac{171}{3} = 57\ \Omega$$

$$\sqrt{r^2 + X_L^2} = 57$$

$$r^2 + X_L^2 = 3249$$

$$Z = \frac{V}{I} = \frac{240}{3} = 80\ \ \Omega$$

$$\mathbf{Z} = Z\angle\phi = 80\angle 37° = 63.89 + j48.15\ \ \Omega$$

But

$$\mathbf{Z} = (R + r) + jX_L$$

$$X_L = 48.15\ \Omega$$

$$r^2 + X_L^2 = 3249$$

$$r^2 + (48.15)^2 = 3249$$

$$r^2 = 931.04$$

$$r = 30.51\ \Omega$$

(b) Resistance of the resistor

$$R + r = 63.89$$

$$R + 30.51 = 63.89$$

$$R = 33.38\ \Omega$$

Example 4.68 *A choke coil and a resistor are connected in series across a 230 V, 50 Hz ac supply as shown in Fig. 4.64. The circuit draws a current of 2 A at 0.866 lagging pf. The voltage drop across the resistor is 100 V. Calculate the power factor of the choke coil.*

Fig. 4.64

Solution $V = 230$ V, $f = 50$ Hz, $I = 2$ A, pf = 0.866 (lagging), $V_R = 100$ V

$$R = \frac{V_R}{I} = \frac{100}{2} = 50\ \Omega$$

$$Z = \frac{V}{I} = \frac{230}{2} = 115\ \Omega$$

$$\text{pf} = 0.866 \text{ (lagging)}$$

$$\phi = \cos^{-1}(0.866) = 30°$$

$$\mathbf{Z} = Z\angle\phi = 115\angle 30° = 99.59 + j57.5\ \Omega$$

$$R + r = 99.59$$

$$50 + r = 99.59$$

$$r = 49.59\ \Omega$$

$$X_L = 57.5\ \Omega$$

$$Z_{\text{coil}} = \sqrt{r^2 + X_L^2} = \sqrt{(49.59)^2 + (57.5)^2} = 75.93\ \Omega$$

$$\text{pf}_{\text{coil}} = \frac{r}{Z_{\text{coil}}} = \frac{49.59}{75.93} = 0.653 \text{ (lagging)}$$

Example 4.69 *A circuit consists of a pure resistor and coil in series as shown in Fig. 4.65. Power dissipated in the resistor and in the coil are 1000 W and 250 W respectively. The voltage drops across the resistor and the coil are 200 V and 300 V respectively. Determine (a) value of pure resistance, (b) resistance and reactance of the coil, (c) combined resistance of the circuit, (d) combined impedance, and (e) supply voltage.*

Fig. 4.65

Solution $P_R = 1000$ W, $P_{\text{coil}} = 250$ W, $V_R = 200$ V, $V_{\text{coil}} = 300$ V

(a) Value of pure resistance

$$P_R = \frac{V_R^2}{R}$$

$$1000 = \frac{(200)^2}{R}$$

$$R = 40\ \Omega$$

(b) Resistance and reactance of the coil

$$V_R = RI$$
$$200 = 40I$$
$$I = 5\text{ A}$$
$$P_{\text{coil}} = I^2 r$$
$$250 = (5)^2 \times r$$
$$r = 10\ \Omega$$
$$Z_{\text{coil}} = \frac{V_{\text{coil}}}{I} = \frac{300}{5} = 60\ \Omega$$
$$X_L = \sqrt{Z_{\text{coil}}^2 - r^2} = \sqrt{(60)^2 - (10)^2} = 59.2\ \Omega$$

(c) Combined resistance of the circuit

$$R_T = R + r = 40 + 10 = 50\ \Omega$$

(d) Combined impedance

$$Z = \sqrt{(R+r)^2 + X_L^2} = \sqrt{(50)^2 + (59.2)^2} = 77.5\ \Omega$$

(e) Supply voltage

$$V = ZI = 77.5 \times 5 = 387.5\text{ V}$$

Example 4.70 *In the circuits of Fig. 4.66, a coil A takes 2 A at a power factor of 0.8 lagging with an applied p.d. of 10 V. A second coil B takes 2 A with a power of 0.7 lagging with an applied voltage of 5 V. What voltage will be required to produce a total current of 2 A with coils A and B in series? Find the power factor in this case.*

Fig. 4.66

Solution

Coil A: $I_A = 2$ A, $\text{pf}_A = 0.8$ (lagging), $V_A = 10$ V
Coil B: $I_B = 2$ A, $\text{pf}_B = 0.7$ (lagging), $V_B = 5$ V

For Coil A,

$$\phi_A = \cos^{-1}(0.8) = 36.87°$$
$$Z_A = \frac{V_A}{I_A} = \frac{10}{2} = 5\ \Omega$$
$$\mathbf{Z}_A = Z_A \angle\phi_A = 5\angle 36.87° = 4 + j3\ \Omega$$
$$r_A = 4\ \Omega$$
$$X_A = 3\ \Omega$$

For Coil B,

$$\phi_B = \cos^{-1}(0.7) = 45.57°$$

$$Z_B = \frac{V_B}{I_B} = \frac{5}{2} = 2.5\ \Omega$$

$$\mathbf{Z}_B = Z_B\angle\phi_B = 2.5\angle 45.57° = 1.75 + j1.78\ \Omega$$

$$r_B = 1.75\ \Omega$$

$$X_B = 1.78\ \Omega$$

When coils A and B are connected in series as shown in Fig. 4.67,

Fig. 4.67

$$\mathbf{Z} = r_A + jX_A + r_B + jX_B = 4 + j3 + 1.75 + j1.78 = 5.75 + j4.78 = 7.48\angle 39.74°\ \Omega$$

$$Z = 7.48\ \Omega$$

$$\phi = 39.74°$$

$$V = ZI = 7.48 \times 2 = 14.96\ \text{V}$$

$$\text{pf} = \cos\phi = \cos(39.74°) = 0.77\ \text{(lagging)}$$

Example 4.71 *In the circuits of Fig. 4.68, a voltage of 100 V is applied to a coil A, the current taken is 8 A and the power is 120 W. When applied to a coil B, the current is 10 A and the power is 500 W. What current and power will be taken when 100 V is applied to the two coils connected in series?*

Fig. 4.68

Solution

Coil A: $V_A = 100$ V, $I_A = 8$ A, $P_A = 120$ W

Coil B: $V_B = 100$ V, $I_B = 10$ A, $P_B = 500$ W

For Coil A,

$$Z_A = \frac{V_A}{I_A} = \frac{100}{8} = 12.5\ \Omega$$

$$P_A = I_A^2 r_A$$
$$120 = (8)^2 \times r_A$$
$$r_A = 1.875\ \Omega$$
$$X_A = \sqrt{Z_A^2 - r_A^2} = \sqrt{(12.5)^2 - (1.875)^2} = 12.36\ \Omega$$

For Coil B,
$$Z_B = \frac{V_B}{I_B} = \frac{100}{10} = 10\ \Omega$$
$$P_B = I_B^2 r_B$$
$$500 = (10)^2 \times r_B$$
$$r_B = 5\ \Omega$$
$$X_B = \sqrt{Z_B^2 - r_B^2} = \sqrt{(10)^2 - (5)^2} = 8.66\ \Omega$$

When coils A and B are connected in series as shown in Fig. 4.69,

Fig. 4.69

$$\mathbf{Z} = r_A + jX_A + r_B + jX_B = 1.875 + j12.36 + 5 + j8.66 = 6.875 + j21.02 = 22.11\angle 71.89°\ \Omega$$
$$Z = 22.11\ \Omega$$
$$\phi = 71.89°$$
$$I = \frac{V}{Z} = \frac{100}{22.11} = 4.52\ \text{A}$$
$$P = I^2(r_A + r_B) = (4.52)^2 \times (6.875) = 140.64\ \text{W}$$

Example 4.72 *In a particular circuit, a voltage of 10 V at 25 Hz produces 100 mA, while the same voltage at 75 Hz produces 60 mA. Find the values of components of the circuit.*

Solution

$V_1 = 10$ V, $f_1 = 25$ Hz, $I_1 = 100$ mA

$V_2 = 10$ V, $f_2 = 75$ Hz, $I_2 = 60$ mA

Case (a) $$Z_1 = \frac{V_1}{I_1} = \frac{100}{100 \times 10^{-3}} = 100\ \Omega$$

Case (b) $$Z_2 = \frac{V_2}{I_2} = \frac{10}{60 \times 10^{-3}} = 166.67\ \Omega$$

As frequency increases, impedance of the circuit increases. In a series RL circuit, inductive reactance X_L increases with frequency. Hence, impedance increases. The circuit consists of a resistance R and an inductance L.

$$Z_1 = \sqrt{R^2 + X_{L_1}^2} = \sqrt{R^2 + (2\pi \times 25 \times L)^2} = 100\ \Omega$$

$$R^2 + (50\pi L)^2 = 10000 \qquad \ldots\text{(i)}$$

$$Z_2 = \sqrt{R^2 + X_{L_2}^2} = \sqrt{R^2 + (2\pi \times 75 \times L)^2} = 166.67\ \Omega$$

$$R^2 + (150\pi L)^2 = 27778.89 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$R = 88.1\ \Omega$$
$$L = 0.3\ \text{H}$$

Example 4.73 *When 1 A is passed through three coils A, B and C in series, the voltages across them are 6 V, 3 V and 8 V respectively on a dc supply and 7 V, 5 V and 10 V respectively on an ac supply. Find the power factor and the power dissipated in each coil and the power factor of the whole circuit.*

Solution $I = 1$ A

On dc supply, $V_A = 6$ V, $V_B = 3$ V, $V_C = 8$ V

On dc supply, $V_A = 7$ V, $V_B = 5$ V, $V_C = 10$ V

For dc supply, $f = 0$

$$X_L = 2\pi fL = 0$$

The coils behave like pure resistors.

$$R_A = \frac{V_A}{I} = \frac{6}{1} = 6\ \Omega$$

$$R_B = \frac{V_B}{I} = \frac{3}{1} = 3\ \Omega$$

$$R_C = \frac{V_C}{I} = \frac{8}{1} = 8\ \Omega$$

For ac supply,

$$Z_A = \frac{V_A}{I} = \frac{7}{1} = 7\ \Omega$$

$$Z_B = \frac{V_B}{I} = \frac{5}{1} = 5\ \Omega$$

$$Z_C = \frac{V_C}{I} = \frac{10}{1} = 10\ \Omega$$

$$X_A = \sqrt{Z_A^2 - R_A^2} = \sqrt{(7)^2 - (6)^2} = 3.6\ \Omega$$

$$X_B = \sqrt{Z_B^2 - R_B^2} = \sqrt{(5)^2 - (3)^2} = 4\ \Omega$$

$$X_C = \sqrt{Z_C^2 - R_C^2} = \sqrt{(10)^2 - (8)^2} = 6\ \Omega$$

(a) Power factor of Coil A

$$\text{pf}_A = \frac{R_A}{Z_A} = \frac{6}{7} = 0.857\ \text{(lagging)}$$

(b) Power factor of Coil B

$$\text{pf}_B = \frac{R_B}{Z_B} = \frac{3}{5} = 0.6 \text{ (lagging)}$$

(c) Power factor of Coil C

$$\text{pf}_C = \frac{R_C}{Z_C} = \frac{8}{10} = 0.8 \text{ (lagging)}$$

(d) Power dissipated in Coil A

$$P_A = I^2 R_A = (1)^2 \times 6 = 6\,\text{W}$$

(e) Power dissipated in Coil B

$$P_B = I^2 R_B = (1)^2 \times 3 = 3\,\text{W}$$

(f) Power dissipated in Coil C

$$P_C = I^2 R_C = (1)^2 \times 8 = 8\,\text{W}$$

(g) Power factor of the whole circuit

$$\mathbf{Z} = R_A + jX_A + R_B + jX_B + R_C + jX_C = 6 + j3.6 + 3 + j4 + 8 + j6 = 17 + j13.6 = 21.77\angle 38.68°\ \Omega$$

$$\text{pf}_T = \cos(38.68°) = 0.78 \text{ (lagging)}$$

Example 4.74 *An air-cored coil takes 5 A of current and consumes 600 W of power when connected across a 200 V, 50 Hz ac supply. Calculate the value of the current drawn by the coil if the supply frequency increases to 60 Hz.*

Solution

$$I = 5\text{ A}, \quad P = 600\text{ W}, \quad V = 200\text{ V}$$

For $f = 50$ Hz,

$$Z = \frac{V}{I} = \frac{200}{5} = 40\ \Omega$$

$$P = I^2 r$$

$$600 = (5)^2 \times r$$

$$r = 24\ \Omega$$

$$X_L = \sqrt{Z^2 - r^2} = \sqrt{(40)^2 - (24)^2} = 32\ \Omega$$

$$X_L = 2\pi f L$$

$$32 = 2\pi \times 50 \times L$$

$$L = 0.1019\text{ H}$$

For $f = 60$ Hz,

$$X_L = 2\pi \times 60 \times 0.1019 = 38.4\ \Omega$$

$$r = 24\ \Omega$$

$$Z = \sqrt{r^2 + X_L^2} = \sqrt{(24)^2 + (38.4)^2} = 45.28\ \Omega$$

$$I = \frac{V}{Z} = \frac{200}{45.28} = 4.417\text{ A}$$

Example 4.75 *When an iron-cored choking coil is connected to a 12 V dc supply, it draws a current of 2.5 A and when it is connected to a 230 V, 50 Hz supply, it draws a 2 A current and consumes 50 W of power. Determine for this value of current (a) power loss in the iron core, (b) inductance of the coil, (c) power factor, and (d) value of the series resistance which is equivalent to the effect of iron loss.*

Solution

For dc	$V = 12$ V,	$I = 2.5$ A	
For ac	$V = 230$ V,	$I = 2$ A,	$P = 50$ W

In an iron-cored coil, there are two types of losses.

(a) Losses in core known as core or iron loss
(b) Losses in winding known as copper loss

$$P = I^2R + P_i$$

$$\frac{P}{I^2} = R + \frac{P_i}{I^2}$$

$$R_T = R + \frac{P_i}{I^2}$$

where R is the resistance of the coil and $\frac{P_i}{I^2}$ is the resistance which is equivalent to the effect of iron loss.

For dc supply,

$$f = 0$$

$$X_L = 0$$

$$R = \frac{12}{2.5} = 4.8\ \Omega$$

For ac supply,

$$Z = \frac{230}{2} = 115\ \Omega$$

(a) Iron loss

$$P_i = P - I^2R = 50 - (2)^2 \times 4.8 = 30.8 \text{ W}$$

$$R_T = \frac{P}{I^2} = \frac{50}{(2)^2} = 12.5\ \Omega$$

$$X_L = \sqrt{Z^2 - R_T^2} = \sqrt{(115)^2 - (12.5)^2} = 114.3\ \Omega$$

(b) Inductance

$$X_L = 2\pi fL$$

$$114.3 = 2\pi \times 50 \times L$$

$$L = 0.363 \text{ H}$$

(c) Power factor

$$\text{pf} = \frac{R_T}{Z} = \frac{12.5}{115} = 0.108 \text{ (lagging)}$$

(d) The series resistance equivalent to the effect of iron loss

$$R_i = \frac{P_i}{I^2} = \frac{30.8}{(2)^2} = 7.7\ \Omega$$

Example 4.76 *An iron-cored coil takes 4 A at a power factor of 0.5 when connected to a 200 V, 50 Hz supply. When the iron core is removed and the voltage is reduced to 40 V, the current rises to 5 A at a pf of 0.8. Find the iron loss in the core and inductance in each case.*

Solution

With iron core	$I = 4$ A,	pf = 0.5,	$V = 200$ V
Without iron core	$I = 5$ A,	pf = 0.8,	$V = 40$ V

(a) Inductance of the coil

(i) When the iron core is removed,

$$Z = \frac{V}{I} = \frac{40}{5} = 8\ \Omega$$

$$\text{pf} = \frac{R}{Z}$$

$$0.8 = \frac{R}{8}$$

$$R = 6.4\ \Omega$$

$$X_L = \sqrt{Z^2 - R^2} = \sqrt{(8)^2 - (6.4)^2} = 4.8\ \Omega$$

$$X_L = 2\pi fL$$

$$4.8 = 2\pi \times 50 \times L$$

$$L = 0.0153\ \text{H}$$

(ii) With iron core,

$$Z = \frac{V}{I} = \frac{200}{4} = 50\ \Omega$$

$$\text{pf} = \frac{R_T}{Z}$$

$$0.5 = \frac{R_T}{50}$$

$$R_T = 25\ \Omega$$

$$X_L = \sqrt{Z^2 - R_T^2} = \sqrt{(50)^2 - (25)^2} = 43.3\ \Omega$$

$$X_L = 2\pi fL$$

$$43.3 = 2\pi \times 50 \times L$$

$$L = 0.1378\ \text{H}$$

(b) Iron loss

$$P_i = P - I^2R = VI\cos\phi - I^2R = 200 \times 4 \times 0.5 - (4)^2 \times 6.4 = 297.6\ \text{W}$$

4.12 SERIES *RC* CIRCUIT

Figure 4.70 shows a pure resistor R connected in series with a pure capacitor C across an alternating voltage $v = V_m \sin \omega t$.

Let V and I be the rms values of applied voltage and current.

Potential difference across the resistor = $V_R = R\,I$

Potential difference across the capacitor = $V_C = X_C I$

The voltage $\mathbf{V}_R$ is in phase with the current $\mathbf{I}$ whereas voltage $\mathbf{V}_c$ lags behind the current $\mathbf{I}$ by 90°.

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_C$$

Fig. 4.70 *Series RC circuit*

1. *Phasor Diagram* Since the same current flows through R and C, the current I is taken as a reference phasor. Figure 4.71 shows the phasor diagram of series RC circuit.

Fig. 4.71

2. *Impedance*

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_C = R\mathbf{I} - jX_C\mathbf{I} = (R - jX_C)\mathbf{I}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R - jX_C = \mathbf{Z}$$

$$\mathbf{Z} = Z\angle -\phi$$

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$$

$$\phi = \tan^{-1}\left(\frac{X_C}{R}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

The quantity **Z** is called the *complex impedance* of the series RC circuit.

3. *Impedance Triangle*

Figure 4.72 shows the impedance triangle of series RC circuit.

Fig. 4.72 *Impedance triangle*

4. *Current* From the phasor diagram, it is clear that the current I leads the voltage V by an angle ϕ. If the applied voltage is given by $v = V_m \sin \omega t$ then the current equation will be

$$i = I_m \sin(\omega t + \phi)$$

where $$I_m = \frac{V_m}{Z}$$

and $$\phi = \tan^{-1}\left(\frac{X_C}{R}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

5. *Waveforms* Figure 4.73 shows the voltage and current waveforms.

Fig. 4.73 *Waveforms*

6. *Power*

Active power $P = VI\cos\phi = I^2R$

Reactive power $Q = VI\sin\phi = I^2X_C$

Apparent power $S = VI = I^2Z$

7. *Power Triangle* Figure 4.74 shows the power triangle of series *RC* circuit.

Fig. 4.74 *Power triangle*

8. *Power Factor* It is defined as the cosine of the angle between voltage and current phasors.

$$\text{pf} = \cos\phi$$

From voltage triangle, $\text{pf} = \dfrac{V_R}{V}$

From impedance triangle $\text{pf} = \dfrac{R}{Z}$

From power triangle, $\text{pf} = \dfrac{P}{S}$

In case of series *RC* circuit, the power factor is leading in nature.

Example 4.77 *The voltage applied to a circuit is e = 100 sin (ωt + 30°) and the current flowing in the circuit is i = 15 sin (ωt + 60°). Determine impedance, resistance, reactance, power factor and power.*

Solution

$$e = 100\sin(\omega t + 30°), \quad i = 15\sin(\omega t + 60°)$$

(a) Impedance

$$\mathbf{E} = \frac{100}{\sqrt{2}}\angle 30° \text{ V}$$

$$\mathbf{I} = \frac{15}{\sqrt{2}}\angle 60° \text{ A}$$

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{I}} = \frac{\frac{100}{\sqrt{2}}\angle 30^\circ}{\frac{15}{\sqrt{2}}\angle 60^\circ} = 6.67\angle -30^\circ = 5.77 - j3.33 = R - jX_C$$

$$Z = 6.67\ \Omega$$

(b) Resistance

$$R = 5.77\ \Omega$$

(c) Reactance

$$X_C = 3.33\ \Omega$$

(d) Power factor

$$\text{pf} = \cos\phi = \cos(30)^\circ = 0.866 \text{ (leading)}$$

(e) Power

$$P = EI\cos\phi = \frac{100}{\sqrt{2}} \times \frac{15}{\sqrt{2}} \times 0.866 = 649.5\text{ W}$$

Example 4.78 *A series circuit consumes 2000 W at 0.5 leading power factor, when connected to 230 V, 50 Hz ac supply. Calculate (a) current, (b) kVA, and (c) kVAR.*

Solution $P = 2000$ W, pf = 0.5 (leading), $V = 230$ V

(a) Current

$$P = VI\cos\phi$$

$$2000 = 230 \times I \times 0.5$$

$$I = 17.39\text{ A}$$

(b) Apparent power

$$S = VI = \frac{P}{\cos\phi} = \frac{2000}{0.5} = 4\text{kVA}$$

(c) Reactive power

$$\phi = \cos^{-1}(0.5) = 60^\circ$$

$$Q = VI\sin\phi = 230 \times 17.39 \times \sin(60^\circ) = 3.464\text{ kVAR}$$

Example 4.79 *A resistor R in series with a capacitor C is connected to a 240 V, 50 Hz ac supply. Find the value of C so that R absorbs 300 W at 100 V. Find also the maximum charge and maximum stored energy in C.*

Solution $V = 240$ V, $f = 50$ Hz, $P = 300$ W, $V_R = 100$ V

(a) Value of C

$$P = \frac{V_R^2}{R}$$

$$300 = \frac{(100)^2}{R}$$

$$R = 33.33\ \Omega$$

$$P = I^2 R$$

$$300 = I^2 \times 33.33$$

$$I = 3\text{ A}$$

$$Z = \frac{V}{I} = \frac{240}{3} = 80\ \Omega$$

$$X_C = \sqrt{Z^2 - R^2} = \sqrt{(80)^2 - (33.33)^2} = 72.72\ \Omega$$

$$X_C = \frac{1}{2\pi fC}$$

$$72.72 = \frac{1}{2\pi \times 50 \times C}$$

$$C = 43.77\mu\text{F}$$

(b) Maximum charge

$$V_C = \sqrt{V^2 - V_R^2} = \sqrt{(240)^2 - (100)^2} = 218.17\ \text{V}$$

$$V_{C\max} = 218.17 \times \sqrt{2} = 308.54\ \text{V}$$

$$Q_{\max} = CV_{C\max} = 43.77 \times 10^{-6} \times 308.54 = 0.0135\ \text{C}$$

(c) Maximum stored energy

$$E_{\max} = \frac{1}{2} C\ (V_{C\max})^2 = \frac{1}{2} \times 43.77 \times 10^{-6} \times (308.54)^2 = 2.08\ \text{J}$$

Example 4.80 *A capacitor of 35 μF is connected in series with a variable resistor. The circuit is connected across 50 Hz mains. Find the value of the resistor for a condition when the voltage across the capacitor is half the supply voltage.*

Solution

$$\text{C} = 35\ \mu\text{F}, \quad f = 50\ \text{Hz}, \quad V_C = \frac{1}{2}\text{V}$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 35 \times 10^{-6}} = 90.946\ \Omega$$

$$V_C = \frac{1}{2}\text{V}$$

$$X_C I = \frac{1}{2} ZI$$

$$X_C = \frac{1}{2} Z$$

$$Z = 2X_C$$

$$Z = \sqrt{R^2 + X_C^2}$$

$$(2X_C)^2 = R^2 + X_C^2$$

$$R^2 = 3X_C^2 = 3 \times (90.946)^2 = 24813.35$$

$$R = 157.5\ \Omega$$

Example 4.81 *A voltage of 125 V at 50 Hz is applied across a non-inductive resistor connected in series with a capacitor. The current is 2.2 A. The power loss in the resistor is 96.8 W. Calculate the resistance and capacitance.*

Solution $V = 125$ V, $f = 50$ Hz, $I = 2.2$ A, $P = 96.8$ W

(a) Resistance

$$Z = \frac{V}{I} = \frac{125}{2.2} = 56.82 \text{ A}$$

$$P = I^2 R$$

$$96.8 = (2.2)^2 \times R$$

$$R = 20\ \Omega$$

(b) Capacitance

$$X_C = \sqrt{Z^2 - R^2} = \sqrt{(56.82)^2 - (20)^2} = 53.18\ \Omega$$

$$X_C = \frac{1}{2\pi fC}$$

$$53.18 = \frac{1}{2\pi \times 50 \times C}$$

$$C = 59.85\ \mu\text{F}$$

Example 4.82 *A resistor and a capacitor are connected across a 250 V supply. When the supply frequency is 50 Hz, the current drawn is 5 A. When the frequency is increased to 60 Hz, it draws 5.8 A. Find the values of R and C and the power drawn in the second case.*

Solution $V = 250$ V, $f_1 = 50$ Hz, $I_1 = 5$ A, $f_2 = 60$ Hz, $I_2 = 5.8$ A

(a) Values of R and C

For $f_1 = 50$ Hz,

$$Z_1 = \frac{V}{I_1} = \frac{250}{5} = 50\ \Omega$$

$$Z_1 = \sqrt{R^2 + \left(\frac{1}{2\pi f_1 C}\right)^2} = \sqrt{R^2 + \left(\frac{1}{100\pi C}\right)^2}$$

$$R^2 + \left(\frac{1}{100\pi C}\right)^2 = 2500 \qquad \text{...(i)}$$

For $f_2 = 60$ Hz,

$$Z_2 = \frac{V}{I_2} = \frac{250}{5.8} = 43.1\ \Omega$$

$$Z_2 = \sqrt{R^2 + \left(\frac{1}{2\pi f_2 C}\right)^2} = \sqrt{R^2 + \left(\frac{1}{120\pi C}\right)^2}$$

$$R^2 + \left(\frac{1}{120\pi C}\right)^2 = 1857.9\ \Omega \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$R = 19.96\ \Omega$$

$$C = 69.4\ \mu\text{F}$$

(b) Power drawn in the second case

$$P_2 = I_2^2 R = (5.8)^2 \times 19.96 = 671.45 \text{ W}$$

Table showing various relationships for *R, L,* and *C*

	R	*L*	*C*
Voltage	$V_m \sin \omega t$	$V_m \sin \omega t$	$V_m \sin \omega t$
Current	$I_m \sin \omega t$	$I_m \sin(\omega t - 90°)$	$I_m \sin(\omega t + 90°)$
Waveform			
Phasor Diagram			
Impedance	R	$j\omega L$	$\frac{1}{j\omega C} = -j\frac{1}{\omega C}$
Phase Difference	0°	90°	90°
Power Factor	1	0	0
Power	VI	0	0

Table showing various relationships for series *RL* and series *RC circuits*

	Series RL Circuit	*Series RC Circuit*
Voltage	$V_m \sin \omega t$	$V_m \sin \omega t$
Current	$I_m \sin(\omega t - \phi)$	$I_m \sin(\omega t + \phi)$
Waveform		
Phasor Diagram		
Impedance	$R + jX_L, \quad Z\angle$	$R - jX_C, Z\angle -\phi$

Phase Difference	$0° < \phi < 90°$	$0° < \phi < 90°$
Power Factor	$\frac{V_R}{V} = \frac{R}{Z} = \frac{P}{S}$	$\frac{V_R}{V} = \frac{R}{Z} = \frac{P}{S}$
Power	$P = VI\cos\phi = I^2R$ $Q = VI\sin\phi = I^2X_L$ $S = VI = I^2Z$	$P = VI\cos\phi = I^2R$ $Q = VI\sin\phi = I^2X_C$ $S = VI = I^2Z$

4.13 SERIES *RLC* CIRCUIT

Figure 4.75 shows a pure resistor R, pure inductor L and pure capacitor C connected in series across an alternating voltage $v = V_m \sin \omega t$.

Fig. 4.75 *Series RLC circuit*

Let V and I be the rms values of the applied voltage and current.
Potential difference across the resistor = $V_R = R\,I$
Potential difference across the inductor = $V_L = X_L I$
Potential difference across the capacitor = $V_C = X_C I$

The voltage $\mathbf{V}_R$ is in phase with the current $\mathbf{I}$, the voltage $\mathbf{V}_L$ leads the current $\mathbf{I}$ by 90° and the voltage $\mathbf{V}_C$ lags behind the current $\mathbf{I}$ by 90°.

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

1. *Phasor Diagram* Since the same current flows through R, L and C, the current I is taken as a reference phasor.

Case (i) $X_L > X_C$ (Fig. 4.76)

Fig. 4.76 *Phasor diagram*

The reactance X will be inductive in nature and the circuit will behave like a series RL circuit.

Case (ii) $X_C > X_L$ (Fig. 4.77)

Fig. 4.77 *Phasor diagram*

The reactance X will be capacitive in nature and the circuit will behave like a series RC circuit.

2. Impedance

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C = R\mathbf{I} + jX_L I - jX_C\mathbf{I} = [R + j(X_L - X_C)]\mathbf{I}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R + j(X_L - X_C) = \mathbf{Z}$$

$$\mathbf{Z} = Z\angle\phi$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

3. Impedance Triangles

Figure 4.78 shows the impedance triangles of series RLC circuit for two cases.

Case (i) $X_L > X_C$

Case (ii) $X_C > X_L$

Fig. 4.78 *Impedance triangles*

4. Current Equation

If the applied voltage is given by $v = V_m \sin \omega t$ then current equation will be

$$i = I_m \sin(\omega t \pm \phi)$$

'−' sign is used when $X_L > X_C$.
'+' sign is used when $X_C > X_L$.

5. Waveforms

Figure 4.79 shows the voltage and current waveforms.

Case (i) $X_L > X_C$

Case (ii) $X_C > X_L$

Fig. 4.79 *Waveforms*

6. Power

Average power $P = VI \cos \phi = I^2R$

Reactive power $Q = VI \sin \phi = I^2X$

Apparent power $S = VI = I^2Z$

7. Power Triangles Figure 4.80 shows the power triangles of series *RLC* circuit.

Case (i) $X_L > X_C$ ***Case (ii)*** $X_C > X_L$

Fig. 4.80 *Power triangles*

8. Power Factor It is defined as the cosine of the angle between voltage and current phasors.

$$\text{pf} = \cos \phi$$

$$\text{pf} = \frac{V_R}{V} = \frac{R}{Z} = \frac{P}{S}$$

Example 4.83 *A resistor of 20 Ω, inductor of 0.05 H and a capacitor of 50 μF are connected in series as shown in Fig. 4.81. A supply voltage 230 V, 50 Hz is connected across the series combination. Calculate the following: (a) impedance, (b) current drawn by the circuit, (c) phase difference and power factor, and (d) active and reactive power consumed by the circuit.*

Fig. 4.81

Solution $R = 20\ \Omega$, $L = 0.05$ H, $C = 50\ \mu$F, $V = 230$ V, $f = 50$ Hz

(a) Impedance

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.05 = 15.71\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.66\ \Omega$$

$$\mathbf{Z} = R + jX_L - jX_C = 20 + j15.71 - j63.66 = 51.95\ \angle -67.36°\ \Omega$$

$$Z = 51.95\ \Omega$$

(b) Phase difference

$$\phi = 67.36°$$

(c) Current

$$I = \frac{V}{Z} = \frac{230}{51.95} = 4.43\ \text{A}$$

(d) Power factor

$$\text{pf} = \cos\phi = \cos(67.36°) = 0.385\ \text{(leading)}$$

(e) Active power

$$P = VI\cos\phi = 230 \times 4.43 \times 0.385 = 392.28\ \text{W}$$

(f) Reactive power

$$Q = VI\sin\phi = 230 \times 4.43 \times \sin(67.36°) = 940.39\ \text{VAR}$$

Example 4.84 *A circuit consists of a pure resistor, a pure inductor, and a capacitor connected in series as shown in Fig. 4.82. When the circuit is supplied with 100 V, 50 Hz supply, the voltages across inductor and resistor are 240 V and 90 V respectively. If the circuit takes a 10 A leading current, calculate (a) value of inductance, resistance and capacitance, (b) power factor of the circuit, and (c) voltage across the capacitor.*

Fig. 4.82

Solution $V = 100$ V, $f = 50$ Hz, $V_L = 240$ V, $V_R = 90$ V, $I = 10$ A

(a) Value of inductance, resistance and capacitance

$$R = \frac{V_R}{I} = \frac{90}{10} = 9\ \Omega$$

$$X_L = \frac{V_L}{I} = \frac{240}{10} = 24\ \Omega$$

$$Z = \frac{V}{I} = \frac{100}{10} = 10\ \Omega$$

$$\mathbf{Z} = R + jX_L - jX_C = R - j(X_C - X_L)$$

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$

$$10 = \sqrt{(9)^2 + (X_C - 24)^2}$$

$$X_C = 28.36\ \Omega$$

$$X_L = 2\pi fL$$

$$24 = 2\pi \times 50 \times L$$
$$L = 0.076\ H$$
$$X_C = \frac{1}{2\pi fC}$$
$$28.36 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 112.24\ \mu\text{F}$$

(b) Power factor of the circuit

$$\text{pf} = \frac{R}{Z} = \frac{9}{10} = 0.9 \text{ (leading)}$$

(c) Voltage across the capacitor

$$V_C = X_C I = 28.36 \times 10 = 283.6 \text{ V}$$

Example 4.85 *Two impedances $\mathbf{Z}_1 = 40\angle 30°\ \Omega$ and $\mathbf{Z}_2 = 30\angle 60°\ \Omega$ are connected in series across a single-phase 230 V, 50 Hz supply as shown in Fig. 4.83. Calculate the (a) current drawn, (b) pf, and (c) power consumed by the circuit.*

Fig. 4.83

Solution

$$\mathbf{Z}_1 = 40\angle 30°\ \Omega, \quad \mathbf{Z}_2 = 30\angle 60°\ \Omega, \quad V = 230 \text{ V}$$

(a) Current drawn

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 = 40\ \angle 30° + 30\ \angle 60° = 67.66\ \angle 42.81°\ \Omega$$
$$I = \frac{V}{Z} = \frac{230}{67.66} = 3.4 \text{ A}$$

(b) Power factor

$$\text{pf} = \cos\phi = \cos(42.81°) = 0.734 \text{ (lagging)}$$

(c) Power consumed

$$P = VI\cos\phi = 230 \times 3.4 \times 0.734 = 573.99 \text{ W}$$

Example 4.86 *A circuit takes a current of 3 A at a power factor of 0.6 lagging when connected to 115 V, 50 Hz supply. Another circuit takes a current of 5 A at a power factor of 0.707 leading when connected to the same supply. If the two circuits are connected in series across a 230 V, 50 Hz supply, calculate (a) current, (b) power consumed, and (c) power factor.*

Solution

Circuit 1: $I_1 = 3$ A, $\text{pf}_1 = 0.6$ (lagging), $V_1 = 115$ V
Circuit 2: $I_2 = 5$ A, $\text{pf}_2 = 0.707$ (leading), $V_2 = 115$ V

$$\text{V} = 230 \text{ V}$$

For Circuit 1,

$$\phi_1 = \cos^{-1}(0.6) = 53.13°$$

$$Z_1 = \frac{V_1}{I_1} = \frac{115}{3} = 38.33\ \Omega$$

$$\mathbf{Z}_1 = Z_1\angle\phi_1 = 38.33\angle 53.13°\ \Omega$$

For Circuit 2,

$$\phi_2 = \cos^{-1}(0.707) = 45°$$

$$Z_2 = \frac{V_2}{I_2} = \frac{115}{5} = 23\ \Omega$$

$$\mathbf{Z}_2 = Z_2\angle\phi_2 = 23\angle -45°\ \Omega$$

When the two circuits are connected in series,

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 = 38.33\angle 53.13° + 23\angle -45° = 41.82\angle 20.14°\ \Omega$$

(a) Current

$$I = \frac{V}{Z} = \frac{230}{41.82} = 5.5\ \text{A}$$

(b) Power consumed

$$P = VI\cos\phi = 230 \times 5.5 \times \cos(20.14°) = 1.187\ \text{kW}$$

(c) Power factor

$$\text{pf} = \cos\phi = \cos(20.14°) = 0.939\ \text{(lagging)}$$

Example 4.87 *Two impedances Z_1 and Z_2, having the same numerical value, are connected in series. If Z_1 has a pf of 0.866 lagging and Z_2 has a pf of 0.8 leading, calculate the pf of the series combination.*

Solution

$$\text{pf}_1 = 0.866\ \text{(lagging)}, \quad \text{pf}_2 = 0.8\ \text{(leading)}, \quad Z_1 = Z_2 = Z$$

$$\phi_1 = \cos^{-1}(0.866) = 30°$$

$$\phi_2 = \cos^{-1}(0.8) = 36.87°$$

$$\mathbf{Z}_1 = Z\angle\phi_1 = Z\angle 30° = 0.866\,Z + j0.5\,Z\ \Omega$$

$$\mathbf{Z}_2 = Z\angle -\phi_2 = Z\angle -36.87° = 0.8\,Z - j0.6\,Z\ \Omega$$

For a series combination,

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 = 0.866\,Z + j0.5\,Z + 0.8\,Z - j0.6\,Z = 1.66\,Z - j0.1\,Z = Z(1.666 - j0.1) = 1.668\,Z\angle -3.43°\ \Omega$$

$$\text{pf} = \cos(3.43°) = 0.9982$$

Example 4.88 *A coil of 3 Ω resistance and an inductance of 0.22 H is connected in series with an imperfect capacitor as shown in Fig. 4.84. When such a series circuit is connected across a 200 V, 50 Hz supply, it has been observed that their combined impedance is $(3.8 + j6.4)\ \Omega$. Calculate the resistance and capacitance of the imperfect capacitor.*

Fig. 4.84

Solution $r = 3\ \Omega$, $L = 0.22$ H, $V = 200$ V, $f = 50$ Hz, $Z = 3.8 + j6.4\ \Omega$

(a) Resistance of the imperfect capacitor

$$\mathbf{Z} = 3.8 + j6.4\ \Omega$$
$$X_L = 2\pi fL = 2\pi \times 50 \times 0.22 = 69.12\ \Omega$$
$$\mathbf{Z} = 3 + j69.12 + R - jX_C = (3 + R) + j(69.12 - X_C)$$
$$3 + R = 3.8$$
$$R = 0.8\ \Omega$$

(b) Capacitance of the imperfect capacitor

$$69.12 - X_C = 6.4$$
$$X_C = 62.72\ \Omega$$
$$X_C = \frac{1}{2\pi fC}$$
$$62.72 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 50.75\ \mu\text{F}$$

Example 4.89 *An RLC series circuit has a current which lags the applied voltage by 45°. The voltage across the inductance has a maximum value equal to twice the maximum value of voltage across the capacitor. Voltage across the inductance is 300 sin (1000t) and R = 20 Ω. Find the value of inductance and capacitance.*

Solution $\phi = 45°$, $v_L = 300 \sin(1000t)$, $R = 20\ \Omega$

(a) Value of inductance

$$V_{L(\max)} = 2V_{C(\max)}$$
$$V_L = 2V_C$$
$$IX_L = 2I\,X_C$$
$$X_L = 2X_C$$
$$\cos\phi = \frac{R}{Z}$$
$$\cos(45°) = \frac{20}{Z}$$
$$Z = 28.28\ \Omega$$

For a series *RLC* circuit,

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$28.28 = \sqrt{(20)^2 + (2X_C - X_C)^2} = \sqrt{400 + X_C{}^2}$$

$$X_C = 20\ \Omega$$

$$X_L = 2X_C = 40\ \Omega$$

$$X_L = \omega L$$

$$40 = 1000 \times L$$

$$L = 0.04\ \text{H}$$

(b) Value of capacitance

$$X_C = \frac{1}{\omega C}$$

$$20 = \frac{1}{1000 \times C}$$

$$C = 50\ \mu\text{F}$$

Example 4.90 *A coil having a power factor of 0.5 is in series with a 79.57 μF capacitor and when connected across a 50 Hz supply, the p.d. across the coil is equal to the p.d. across the capacitor. Find the resistance and inductance of the coil.*

Solution

$$\text{pf}_{\text{coil}} = 0.5, \quad C = 79.57\ \mu\text{F}, \quad f = 50\ \text{Hz}, \quad V_{\text{coil}} = V_C$$

(a) Resistance of the coil

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 79.57 \times 10^{-6}} = 40\ \Omega$$

$$V_{\text{coil}} = V_C$$

$$IZ_{\text{coil}} = IX_C$$

$$Z_{\text{coil}} = X_C = 40\ \Omega$$

$$\text{pf}_{\text{coil}} = \cos\phi = \frac{R}{Z_{\text{coil}}}$$

$$0.5 = \frac{R}{40}$$

$$R = 20\ \Omega$$

(b) Inductance of the coil

$$X_L = \sqrt{Z_{\text{coil}}^2 - R^2} = \sqrt{(40)^2 - (20)^2} = 34.64\ \Omega$$

$$X_L = 2\pi fL$$

$$34.64 = 2\pi \times 50 \times L$$

$$L = 0.11\ \text{H}$$

Example 4.91 *A 250 V, 50 Hz voltage is applied to a coil having a resistance of 5 Ω and an inductance of 9.55 H in series with a capacitor C. If the voltage across the coil is 300 V, find the value of C.*

Solution

$$V = 250 \text{ V}, \quad f = 50 \text{ Hz}, \quad R = 5\ \Omega, \quad L = 9.55 \text{ H}, \quad V_{\text{coil}} = 300 \text{ V}$$

$$X_L = 2\pi fL = 2\pi \times 50 \times 9.55 = 3000\ \Omega$$

$$Z_{\text{coil}} = \sqrt{R^2 + X_L^2} = \sqrt{(5)^2 + (3000)^2} = 3000\ \Omega$$

$$I = \frac{V_{\text{coil}}}{Z_{\text{coil}}} = \frac{300}{3000} = 0.1 \text{ A}$$

$$Z = \frac{V}{I} = \frac{250}{0.1} = 2500\ \Omega$$

When $X_L > X_C$,

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$2500 = \sqrt{(5)^2 + (3000 - X_C)^2}$$

$$X_C = 500\ \Omega$$

$$X_C = \frac{1}{2\pi fC}$$

$$500 = \frac{2}{2\pi \times 50 \times C}$$

$$C = 6.37\ \mu\text{F}$$

When $X_C > X_L$,

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$

$$2500 = \sqrt{(5)^2 + (X_C - 300)^2}$$

$$X_C = 5500\ \Omega$$

$$X_C = \frac{1}{2\pi fC}$$

$$5500 = \frac{1}{2\pi \times 50 \times C}$$

$$C = 0.58\ \mu\text{F}$$

Example 4.92 *Draw the phasor diagram for the series circuit shown in Fig. 4.85 when the current in the circuit is 2 A. Find the values of V_1 and V_2 and show these voltages on the phasor diagram.*

Fig. 4.85

Solution

$$\mathbf{Z}_1 = j3 + 6 - j8 = 6 - j5 = 7.81 \angle -39.8^\circ\ \Omega$$

$$\mathbf{Z}_2 = 5 + j3 + 6 - j8 + 4 = 15 - j5 = 15.81 \angle -18.43^\circ\ \Omega$$

$$I = 2 \text{ A}$$

(a) Values of V_1 and V_2

Let
$$\mathbf{I} = 2\angle 0° \text{ A}$$
$$\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I} = (7.81\angle -39.8°)(2\angle 0°) = 15.62\angle -39.8° \text{ V}$$
$$\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I} = (15.81\angle -18.43°)(2\angle 0°) = 31.62\angle -18.43° \text{ V}$$

(b) Phasor diagram

The Phasor diagram is shown in Fig. 4.86.

Fig. 4.86

Example 4.93 *Draw a vector diagram for the circuit shown in Fig. 4.87 indicating terminal voltages V_1 and V_2 and the current. Find the value of (a) current, (b) V_1 and V_2, and (c) power factor.*

Fig. 4.87

Solution

(a) Current

$$X_{L_1} = 2\pi fL_1 = 2\pi \times 50 \times 0.05 = 15.71\ \Omega$$
$$X_{L_2} = 2\pi fL_2 = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$
$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.66\ \Omega$$
$$\mathbf{Z} = 10 + j15.71 + 20 + j31.42 - j63.66 = 30 - j16.53 = 34.25\angle -28.85°\ \Omega$$
$$I = \frac{V}{Z} = \frac{200}{34.25} = 5.84 \text{ A}$$

(b) V_1 and V_2

Let
$$\mathbf{I} = 5.84\angle 0° \text{ A}$$
$$\mathbf{Z} = 10 + j15.71 = 18.62\angle 57.52\ \Omega$$
$$\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I} = (18.62\angle 57.52°)(5.84\angle 0°) = 108.74\angle 57.52° \text{ V}$$

$$\mathbf{Z}_2 = 20 + j31.42 - j63.66 = 20 - j32.24 = 37.94\angle -58.19° \ \Omega$$
$$\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I} = (37.94\angle -58.19°)(5.84\ \angle 0°) = 221.57\angle -58.19° \text{ V}$$

(c) Power factor

$$\text{pf} = \cos\phi = \cos(28.85°) = 0.875 \text{ (leading)}$$

(d) Vector diagram

The vector diagram is shown in Fig. 4.88.

Fig. 4.88

Example 4.94 *Find the values of R and C so that $V_x = 3V_y$, V_x and V_y are in quadrature in the circuit of Fig. 4.89.*

Fig. 4.89

Solution

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.0255 = 8\ \Omega$$
$$\mathbf{Z}_x = 6 + j8 = 10\ \angle 53.13°\ \Omega$$
$$V_x = 3V_y$$
$$IZ_x = 3IZ_y$$
$$Z_x = 3Z_y$$

V_x and V_y are in quadrature, i.e., phase angle between V_x and V_y is 90°. Hence, the angle between Z_x and Z_y will be 90°. The impedance Z_y is capcitive in nature.

$$\mathbf{Z}_y = Z_y\angle -\phi$$
$$\mathbf{Z}_y = \frac{10}{3}\angle(53.13 - 90)° = 3.33\angle -36.87° = 2.66 - j2\ \Omega$$
$$R = 2.66\ \Omega$$
$$X_C = 2\ \Omega$$
$$X_C = \frac{1}{2\pi fC}$$
$$2 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 1.59 \text{ mF}$$

Example 4.95 *In the circuit of Fig. 4.90, a pure resistor R, a choke coil and a pure capacitor of 15.91 μF are connected in series across a supply of V volts and carries a current of 0.25 A. The voltage across the choke coil is 40 V, the voltage across the capacitor is 50 V and the voltage across the resistor is 20 V. The voltage across the combination of R and the choke coil is 45 V. Calculate (a) supply voltage, (b) frequency, and (c) power loss in the choke coil.*

Fig. 4.90

Solution $C = 15.91\ \mu\text{F}$, $I = 0.25$ A, $V_{\text{coil}} = 40$ V, $V_C = 50$ V, $V_R = 20$ V, $V_{R\text{-coil}} = 45$ V

(a) Supply voltage

$$R = \frac{V_R}{I} = \frac{20}{0.25} = 80\ \Omega$$

$$Z_{\text{coil}} = \frac{V_{\text{coil}}}{I} = \frac{40}{0.25} = 160\ \Omega$$

$$X_C = \frac{V_C}{I} = \frac{50}{0.25} = 200\ \Omega$$

$$Z_{R\text{-coil}} = \frac{V_{R\text{-coil}}}{I} = \frac{45}{0.25} = 180\ \Omega$$

$$Z_{\text{coil}} = \sqrt{r^2 + X_L^2} = 160$$

$$r^2 + X_L^2 = 25600 \qquad \ldots\text{(i)}$$

$$\mathbf{Z}_{R\text{-coil}} = (R + r) + j\,X_L$$

$$Z_{R\text{-coil}} = \sqrt{(R+r)^2 + X_L^2}$$

$$180 = \sqrt{(80+r)^2 + X_L^2}$$

$$(80+r)^2 + X_L^2 = 32400 \qquad \ldots\text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii),

$$(80 + r)^2 - r^2 = 6800$$

$$6400 + 160\,r + r^2 - r^2 = 6800$$

$$160\,r = 400$$

$$r = 2.5\ \Omega$$

Substituting the value of r in Eq. (i),

$$(2.5)^2 + X_L^2 = 25600$$

$$X_L^2 = 25593.75$$

$$X_L = 159.98\ \Omega$$

$$Z = R + r + jX_L - jX_C = 80 + 2.5 + j\,159.98 - j200 = 82.5 - j40.02 = 91.69\angle -25.88°\ \Omega$$
$$V = ZI = 91.69 \times 0.25 = 22.92\ \text{V}$$

(b) Frequency

$$X_C = \frac{1}{2\pi fC}$$

$$200 = \frac{1}{2\pi f \times 15.91 \times 10^{-6}}$$

$$f = 50.02\ Hz$$

(c) Power loss in the choke coil

$$P_{\text{coil}} = I^2 r = (0.25)^2 \times 2.5 = 0.156\ \text{W}$$

Example 4.96 *Two impedances, one inductive and the other capacitive, are connected in series across the voltage of 120∠30° V and a frequency of 50 Hz as shown in Fig. 4.91. The current flowing in the circuit is 3∠−15°A. If one of the impedances is (10 + j48.3) Ω, find the other. Also calculate the values of L and C in the impedances.*

Fig. 4.91

Solution $\mathbf{V} = 120\angle 30°$ V, $f = 50$ Hz, $\mathbf{I} = 3\angle -15°$ A, $\mathbf{Z}_1 = 10 + j48.3\ \Omega$

(a) Impedance Z_2

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle 30°}{3\angle -15°} = 40\angle 45° = 28.28 + j28.28\ \Omega$$

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$$

$$\mathbf{Z}_2 = \mathbf{Z} - \mathbf{Z}_1 = 28.28 + j28.28 - 10 - j48.3 = 18.28 - j20.02\ \Omega$$

(b) Value of L

$$\mathbf{Z}_1 = 10 + j48.3 = R_1 + jX_L$$

$$X_L = 48.3\ \Omega$$

$$X_L = 2\pi fL$$

$$48.3 = 2\pi \times 50 \times L$$

$$L = 0.1537\ \text{H}$$

(c) Value of C

$$\mathbf{Z}_2 = 18.28 - j20.02 = R_2 - jX_C$$
$$X_C = 20.02\ \Omega$$
$$X_C = \frac{1}{2\pi fC}$$
$$20.02 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 159\ \mu\text{F}$$

4.14 PARALLEL ac CIRCUITS

In parallel circuits, resistor, inductor and capacitor or any combination of these elements are connected across same supply. Hence, the voltage is same across each branch of the parallel ac circuit. The total current supplied to the circuit is equal to the phasor sum of the branch currents.

Fig. 4.92 *Parallel ac circuit*

For the parallel ac circuit shown in Fig. 4.92,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$
$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$

where Y represents the admittance of the circuit and is defined as the reciprocal of impedance. The real part of admittance is called conductance (G) and the imaginary part is called susceptance (B) and these are measured in mhos ($\mho$) or siemens (S).

If $\mathbf{Z}_1 = R + jX_L$, and $\mathbf{Z}_2 = -jX_C$,

$$\frac{1}{\mathbf{Z}} = \frac{1}{R + jX_L} + \frac{1}{-jX_C}$$
$$= \frac{R - jX_L}{R^2 + X_L^2} + j\frac{1}{X_C}$$
$$= \frac{R}{R^2 + X_L^2} + j\left(\frac{1}{X_C} - \frac{X_L}{R^2 + X_L^2}\right)$$
$$= G + jB$$

where, $$G = \frac{R}{R^2 + X_L^2}$$

$$B = \frac{1}{X_C} - \frac{X_L}{R^2 + X_L^2}$$

The current in the parallel ac circuit can be found as the phasor sum of the branch currents,

i.e., $$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$$

Note: 1. For a series *RL* circuit,

$$\mathbf{Z} = R + jX_L$$

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{R + jX_L} = \frac{R - jX_L}{R^2 + X_L^2} = \frac{R}{R^2 + X_L^2} - j\frac{X_L}{R^2 + X_L^2} = G - jB_L$$

where $$G = \frac{R}{R^2 + X^2} \text{ and } B_L = \frac{X_L}{R^2 + X_L^2}$$

The admittance triangle for series *RL* circuit is shown in Fig. 4.93.

Fig. 4.93 *Admittance triangle*

2. For a series *RC* circuit,

$$\mathbf{Z} = R - jX_C$$

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{R - jX_C} = \frac{R + jX_C}{R^2 + X_C^2} = \frac{R}{R^2 + X_C^2} + j\frac{X_C}{R^2 + X_C^2} = G + jB_C$$

where $G = \dfrac{R}{R^2 + X_C^2}$ and $B_C = \dfrac{X_C}{R^2 + X_C^2}$

The admittance triangle for series *RC* circuit is shown in Fig. 4.94.

Fig. 4.94 *Admittance triangle*

Example 4.97 *A coil having a resistance of 50 Ω and an inductance of 0.02 H is connected in parallel with a capacitor of 25 μF across a single-phase 200 V, 50 Hz supply as shown in Fig. 4.95. Calculate the current in coil and capacitance. Calculate also the total current drawn, total pf and total power consumed by the circuit.*

Fig. 4.95

Solution $R = 50\ \Omega$, $L = 0.02$ H, $C = 25\ \mu$F, $V = 200$ V, $f = 50$ Hz

(a) Current in coil and capacitance

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 25 \times 10^{-6}} = 127.32\ \Omega$$

$$\mathbf{Z}_2 = R + jX_L = 50 + j6.28 = 50.39\angle 7.16°\ \Omega$$

$$\mathbf{Z}_2 = -jX_C = -j127.32 = 127.32\angle -90°\ \Omega$$
$$\mathbf{V} = 200\angle 0° = 200\ \text{V}$$
$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{200}{50.39\angle 7.16°} = 3.97\angle -7.16°\text{A}$$
$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{200}{127.32\angle -90°} = 1.57\angle 90°\text{A}$$

(b) Total current

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 3.97\angle -7.16° + 1.57\angle 90° = 4.08\angle 15.27°\text{A}$$

(c) Total pf

$$\text{pf} = \cos\phi = \cos(15.27°) = 0.965\ \text{(lagging)}$$

(d) Total power consumed

$$P = VI\cos\phi = 200 \times 4.08 \times 0.965 = 787.44\ \text{W}$$

Example 4.98 *Two impedances $\mathbf{Z}_1 = 30\angle 45°\ \Omega$ and $\mathbf{Z}_2 = 45\angle 30°\ \Omega$ are connected in parallel across a single-phase 230 V, 50 Hz supply. Calculate (a) current drawn by each branch, (b) total current, and (c) overall power factor. Also draw the phasor diagram indicating the current drawn by each branch and the total current, taking the supply voltage as reference.*

Solution $\mathbf{Z}_1 = 30\angle 45°\ \Omega,\quad \mathbf{Z}_2 = 45\angle 30°\ \Omega,\quad V = 230\ \text{V}$

(a) Current drawn by each branch

Let $\mathbf{V} = 230\angle 0° = 230\ \text{V}$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{230}{30\angle 45°} = 7.67\angle -45°\ \text{A}$$
$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{230}{45\angle 30°} = 5.11\angle -30°\ \text{A}$$

(b) Total current

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 7.67\angle -45° + 5.11\angle -30° = 12.67\angle -39.01°\ \text{A}$$

(c) Overall power factor

$$\text{pf} = \cos\phi = \cos(39.01°) = 0.777\ \text{(lagging)}$$

(d) Phasor diagram

The phasor diagram is shown in Fig. 4.96.

Fig. 4.96

Example 4.99 *Two circuits, the impedances of which are given by $\mathbf{Z}_1 = (10 + j15)$ ohms and $\mathbf{Z}_2 = (6 - j8)$ ohms, are connected in parallel across an ac supply as shown in Fig. 4.97. If the total current supplied is 15 A, what is the power taken by each branch?*

Fig. 4.97

Solution

$$\mathbf{Z}_1 = 10 + j15\ \Omega, \quad \mathbf{Z}_2 = 6 - j8\ \Omega, \quad I = 15\ \text{A}$$

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(10 + j\,15)(6 - j8)}{10 + j\,15 + 6 - j8} = 10.32\angle -20.45°\ \Omega$$

$$V = ZI = 10.32 \times 15 = 154.8\ \text{V}$$

Let

$$\mathbf{V} = 154.8\angle 0° = 154.8\ \text{V}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{154.8}{10 + j\,15} = 8.59\angle -56.31°\ \text{A}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{154.8}{6 - j8} = 15.48\angle 53.13°\text{A}$$

$$P_1 = I_1^2 R_1 = (8.59)^2 \times 10 = 737.88\ \text{W}$$

$$P_2 = I_2^2 R_2 = (15.48)^2 \times 6 = 1437.78\ \text{W}$$

Example 4.100 *A circuit consists of a 25 Ω resistor, 64 mH inductor and 80 μF capacitor connected in parallel across a 110 V, 50 Hz single-phase supply as shown in Fig. 4.98. Calculate the individual currents drawn by each element, the total current drawn from the supply and the overall power factor of the circuit. Draw the phasor diagram.*

Fig. 4.98

Solution

$$R = 25\ \Omega, \quad L = 64\ \text{mH}, \quad C = 80\ \mu\text{F}, \quad V = 110\ \text{V}, \quad f = 50\ \text{Hz}$$

(a) Individual currents

$$X_L = 2\pi fL = 2\pi \times 50 \times 64 \times 10^{-3} = 20.11\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 80 \times 10^{-6}} = 39.79\ \Omega$$

$$\mathbf{Z}_1 = R = 25\ \Omega$$

$$\mathbf{Z}_2 = jX_L = j\,20.11\ \Omega$$

$$\mathbf{Z}_3 = -jX_C = -j\,39.79\ \Omega$$

Let $\mathbf{V} = 110\angle 0° = 110$ V

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{110}{25} = 4.4\angle 0° \text{A}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{110}{j20.11} = 5.47\angle -90° \text{ A}$$

$$\mathbf{I}_3 = \frac{\mathbf{V}}{\mathbf{Z}_3} = \frac{110}{-j39.79} = 2.76\angle 90° \text{ A}$$

(b) Total current

$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3$

$= 4.4\angle 0° + 5.47\angle -90° + 2.76\angle 90° = 5.17\angle -31.63°$ A

(c) Overall power factor

pf = cos ϕ = cos (31.63°) = 0.851 (lagging)

(d) Phasor diagram

The phasor diagram is shown in Fig. 4.99.

Fig. 4.99

Example 4.101 *An ac circuit connected across a 200 V, 50 Hz, supply has two branches A and B. Branch A draws a current of 4 A at 0.8 lagging power factor, while the total current drawn by the parallel combination is 5 A at unity power factor as shown in Fig. 4.100, Find (a) current and power factor of branch B, and (b) admittances of branch A and B, and their parallel combination both in polar and rectangular forms.*

Fig. 4.100

Solution V = 200 V, f = 50 Hz, $I_A = 4\ A$, $\text{pf}_A = 0.8$ (lagging), $I = 5$ A, pf = 1

(a) Current and power factor of Branch B

$$\phi_A = \cos^{-1}(0.8) = 36.87°$$

$$\phi = \cos^{-1}(1) = 0°$$

$$\mathbf{I}_A = I_A \angle -\phi_A = 4 \angle -36.87° \text{ A}$$

$$\mathbf{I} = I \angle \phi = 5 \angle 0° \text{ A}$$

$$\mathbf{I} = \mathbf{I}_A + \mathbf{I}_B$$

$$\mathbf{I}_B = \mathbf{I} - \mathbf{I}_A = 5 \angle 0° - 4 \angle -36.87° = 3 \angle 53.13° \text{ A}$$

$$\text{pf}_B = \cos(53.13°) = 0.6 \quad \text{(leading)}$$

(b) Admittances of branches A and B and their parallel combination

Let $\mathbf{V} = 200 \angle 0° = 200$ V

$$\mathbf{Y}_A = \frac{\mathbf{I}_A}{\mathbf{V}} = \frac{4 \angle -36.87°}{200 \angle 0°} = 0.02 \angle -36.87° \ \mho$$

$$\mathbf{Y}_B = \frac{\mathbf{I}_B}{\mathbf{V}} = \frac{3 \angle 53.13°}{200 \angle 0°} = 0.015 \angle 53.13° \ \mho$$

$$\mathbf{Y} = \mathbf{Y}_A + \mathbf{Y}_B = 0.02 \angle -36.87° + 0.015 \angle 53.13° \ \mho = 0.025 \angle 0° \ \mho$$

Example 4.102 *Two circuits A and B are connected in parallel to a 115 V, 50 Hz supply as shown in Fig. 4.101. The total current taken by the combination is 10 A at unity power factor. Circuit A consists of a 10 Ω resistor and 200 μF capacitor connected in series. Circuit B consists of a resistor and an inductor in series. Determine (a) current, (b) power factor, (c) impedance, (d) resistance, and (e) reactance of the circuit B.*

Fig. 4.101

Solution V = 115 V, f = 50 Hz, I = 10 A, pf = 1, $R_A = 10\ \Omega$, C = 200 μF

(a) Current I_B

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 200 \times 10^{-6}} = 15.92\ \Omega$$

$$\mathbf{Z}_A = 10 - j\,15.92 = 18.8 \angle -57.87°\ \Omega$$

$$\mathbf{I}_A = \frac{\mathbf{V}}{\mathbf{Z}_A} = \frac{115}{18.8 \angle -57.87°} = 6.12 \angle 57.87°\ \text{A}$$

$$\mathbf{I} = 10 \angle 0°\ \text{A}$$

$$\mathbf{I}_B = \mathbf{I} - \mathbf{I}_A = 10 \angle 0° - 6.12 \ \angle 57.87° = 8.5 \angle -37.54°\ \text{A}$$

(b) Power factor of Circuit B

$$\text{pf}_B = \cos(37.54°) = 0.79 \text{ (lagging)}$$

(c) Impedance of Circuit B

$$Z_B = \frac{\mathbf{V}}{\mathbf{I_B}} = \frac{115}{8.5 \angle -37.54°} = 13.53 \angle 37.54°\ \Omega = 10.73 + j\,8.24\ \Omega$$

(d) Resistance of Circuit B

$$R_B = 10.73\ \Omega$$

(e) Reactance of Circuit B

$$X_B = 8.24\ \Omega$$

Example 4.103 *Two circuits, the impedances of which are given by $\mathbf{Z}_1 = (6 + j8)\ \Omega$ and $\mathbf{Z}_2 = (8 - j6)\ \Omega$, and are connected in parallel as shown in Fig. 4.102. If the applied voltage to the combination is 100 V, find (a) current and pf of each branch, (b) overall current and pf of the combination, and (c) power consumed by each impedance.*

Fig. 4.102

Solution $\mathbf{Z}_1 = 6 + j8\ \Omega, \quad \mathbf{Z}_2 = 8 - j6\ \Omega, \quad V = 100\ \text{V}$

(a) Current and pf of each branch

$$\mathbf{V} = 100\ \angle 0° = 100\ \text{V}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{100}{6 + j8} = 10\ \angle -53.13°\text{A}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{100}{8 - j6} = 10\ \angle 36.9°\text{A}$$

$$\cos\phi_1 = \cos(53.13°) = 0.6\ \text{(lagging)}$$
$$\cos\phi_2 = \cos(36.9°) = 0.8\ \text{(leading)}$$

(b) Overall current and pf of the combination

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10\ \angle -53.13° + 10\ \angle 36.9° = 14.14\ \angle -8.13°\ \text{A}$$

$$\text{pf} = \cos\phi = \cos(8.13°) = 0.989\ \text{(lagging)}$$

(c) Power consumed by each impedance

$$P_1 = I_1^2 R_1 = (10)^2 \times (6) = 600\ \text{W}$$

$$P_2 = I^2 R_2 = (10)^2 \times (8) = 800\ \text{W}$$

Example 4.104 *Two impedances $R_1 - jX_{C_1}$ and $R_2 + jX_{L_2}$ are connected in parallel across a supply voltage $v = 100\sqrt{2}\sin 314\,t$. The current flowing through two impedances are $i_1 = 10\sqrt{2}\sin\left(314\,t + \frac{\pi}{4}\right)$ and $i_2 = 10\sqrt{2}\sin\left(314\,t - \frac{\pi}{4}\right)$ respectively. Find the equation for instantaneous value of total current drawn from the supply. Also find values of R_1, R_2, X_{C_1} and X_{L_2}.*

Solution $\mathbf{Z}_1 = R_1 - jX_{C_1}, \quad \mathbf{Z}_2 = R_2 + jX_{L_2}, \quad v = 100\sqrt{2}\sin 314\,t$

$$i_1 = 10\sqrt{2}\sin\left(314t + \frac{\pi}{4}\right), \quad i_2 = 10\sqrt{2}\sin\left(314t - \frac{\pi}{4}\right)$$

(a) Instantaneous value of total current
Writing v, i_1 and i_2 in polar form,

$$\mathbf{V} = 100\ \angle 0°\ \text{V}$$
$$\mathbf{I}_1 = 10\ \angle 45° \text{A}$$
$$\mathbf{I}_2 = 10\ \angle -45° \text{A}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10\ \angle 45° + 10\ \angle -45° = 14.14\ \angle 0°\ \text{A}$$
$$i = I_m \sin 2\pi ft = 14.14\sqrt{2}\ \sin 314\ t = 20 \sin 314\ t$$

(b) Values of R_1, R_2, X_{C_1} and X_{L_2}.

$$\mathbf{Z}_1 = \frac{\mathbf{V}}{\mathbf{I}_1} = \frac{100\ \angle 0°}{10\ \angle 45°} = 10\ \angle -45° = 7.07 - j7.07\ \Omega$$
$$R_1 = 7.07\ \Omega$$
$$X_{C_1} = 7.07\ \Omega$$
$$\mathbf{Z}_2 = \frac{\mathbf{V}}{\mathbf{I}_2} = \frac{100\ \angle 0°}{10\ \angle -45°} = 10\ \angle 45° = 7.07 + j7.07\ \Omega$$
$$R_2 = 7.07\ \Omega$$
$$X_{L_2} = 7.07\ \Omega$$

Example 4.105 *An impedance of (7 + j5) Ω is connected in parallel with another impedance of (10 – j8) Ω across a 230 V, 50 Hz supply as shown in Fig. 4.103. Calculate (a) admittance, conductance and susceptance of the combined circuit, and (b) total current and power factor.*

Fig. 4.103

Solution $\mathbf{Z}_1 = 7 + j5\ \Omega, \quad \mathbf{Z}_2 = 10 - j8\ \Omega, \quad V = 230\ \text{V}$

(a) Admittance, conductance and susceptance of the combined circuit

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{7 + j5} = 0.12\ \angle -35.54°\ \mho$$
$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} = \frac{1}{10 - j8} = 0.08\ \angle 38.66°\ \mho$$
$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 0.12\ \angle -35.54° + 0.08\ \angle 38.66°$$
$$= 0.16\ \angle -7.04°\ \mho = 0.16 - j0.02\ \mho$$

$$Y = 0.16 \text{ ℧}$$
$$G = 0.16 \text{ ℧}$$
$$B = 0.02 \text{ ℧}$$

(b) Total current and power factor

$$\mathbf{I} = \mathbf{V}\,\mathbf{Y} = (230\angle 0°)\,(0.16\angle -7.04°) = 36.8\angle -7.04° \text{ A}$$
$$I = 36.8 \text{ A}$$

$$\text{pf} = \cos\phi = \cos(7.04°) = 0.99 \text{ (lagging)}$$

Example 4.106 *Two impedances Z_1 and Z_2 are connected in parallel across a 200 V, 50 Hz ac supply. The current drawn by the impedance Z_1 is 4 A at 0.8 lagging pf. The total current drawn from the supply is 5 A at unity pf. Calculate the impedance Z_2.*

Solution

$$V = 200 \text{ V}, \quad I_1 = 4 \text{ A at 0.8 lagging pf}, \quad I = 5 \text{ A at unity pf}$$

$$\mathbf{I}_1 = 4\angle -\cos^{-1}(0.8) = 4\angle -36.87° \text{ A}$$
$$\mathbf{I} = 5\angle \cos^{-1}(1) = 5\angle 0° \text{ A}$$
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$$
$$\mathbf{I}_2 = \mathbf{I} - \mathbf{I}_1 = 5\angle 0° - 4\angle -36.87° = 3\angle 53.13° \text{ A}$$
$$\mathbf{Z}_2 = \frac{\mathbf{V}}{\mathbf{I}_2} = \frac{200\angle 0°}{3\angle 53.13°} = 66.67\angle -53.13° \ \Omega$$

Example 4.107 *Compute Z_{eq} and Y_{eq} for the circuit as shown in Fig. 4.104.*

Fig. 4.104

Solution

$$\mathbf{Z}_1 = j5\ \Omega, \quad \mathbf{Z}_2 = 5 + j8.66\ \Omega, \quad \mathbf{Z}_3 = 15\ \Omega, \quad \mathbf{Z}_4 = -j10\ \Omega$$

$$\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4 = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4}$$

$$= \frac{1}{j5} + \frac{1}{5 + j8.66} + \frac{1}{15} + \frac{1}{-j10} = 0.22\ \angle -57.99° \text{ ℧}$$

$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{0.22\ \angle -57.99°} = 4.54\ \angle 57.99° \ \Omega$$

Example 4.108 *Find currents* I_1 and I_2 *in Fig. 4.105.*

Fig. 4.105

Solution $\mathbf{Z}_1 = 3 - j4\ \Omega, \quad \mathbf{Z}_2 = 10\ \Omega, \quad \mathbf{I} = 25 \angle 90° \text{A}$

By current-division rule,

$$\mathbf{I}_1 = \mathbf{I}\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = (25 \angle 90°)\frac{10}{3 - j4 + 10} = 18.38 \angle 107.1° \text{ A}$$

$$\mathbf{I}_2 = \mathbf{I} - \mathbf{I}_1 = 25\angle 90° - 18.38 \angle 107.1° = 9.19 \angle 54° \text{ A}$$

Example 4.109 *Three impedances of* $25 \angle 53.1° \Omega$, $5 \angle -53.1° \Omega$ *and* $10 \angle 36.9° \Omega$ *are connected in parallel. The combination is in series with another impedance of* $14.14 \angle 45° \Omega$. *Calculate the equivalent impedance of the circuit.*

Solution

$$\mathbf{Z}_1 = 25 \angle 53.1° \ \Omega, \quad \mathbf{Z}_2 = 5 \angle -53.1° \ \Omega, \quad \mathbf{Z}_3 = 10 \angle 36.9° \ \Omega, \quad \mathbf{Z}_4 = 14.14 \angle 45° \ \Omega$$

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} = \frac{1}{25 \angle 53.1°} + \frac{1}{5 \angle -53.1°} + \frac{1}{10 \angle 36.9°} = 0.23 \angle 16.86 \ °\mho$$

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = 4.27 \angle -16.86 \ ° \ \Omega$$

$$\mathbf{Z}_{eq} = \mathbf{Z} + \mathbf{Z}_4 = 4.27 \angle -16.86° + 14.14 \angle 45° = 16.58 \angle 31.87° \ \Omega$$

Example 4.110 *A voltage of* $200 \angle 25°$ *V is applied to a circuit composed of two parallel branches. If the branch currents are* $10 \angle 40°$ *A* and $20 \angle -30°$ *A, determine the kVA, kVAR and kW in each branch. Also, calculate the pf of the combined load.*

Solution $\mathbf{V} = 200 \angle 25° \text{ V}, \quad \mathbf{I}_1 = 10 \angle 40° \text{ A}, \quad \mathbf{I}_2 = 20 \angle -30° \text{A}$

Phase difference between V and I_1

$$\phi_1 = 40° - 25° = 15°$$

Phase difference between V and I_2

$$\phi_2 = 25° - (-30°) = 55°$$
$$\cos \phi_1 = \cos (15°) = 0.97 \text{ (leading)}$$
$$\cos \phi_2 = \cos (55°) = 0.57 \text{ (lagging)}$$

(a) kVA, kVAR and kW for the branch current of $10 \angle 40°$ A

$$P_1 = VI_1 \cos \phi_1 = 200 \times 10 \times 0.97 = 1.94 \text{ kW}$$
$$Q_1 = VI_1 \sin \phi_1 = 200 \times 10 \times \sin(15°) = 0.52 \text{ kVAR}$$
$$S_1 = VI_1 = 200 \times 10 = 2 \text{ kVA}$$

(b) kVA, kVAR and kW for the branch current of $20 \angle -30°$ A

$$P_2 = VI_2 \cos \phi_2 = 200 \times 20 \times 0.57 = 2.28 \text{ kW}$$
$$Q_2 = VI_2 \sin \phi_2 = 200 \times 20 \times \sin(55°) = 3.28 \text{ kVAR}$$
$$S_2 = VI_2 = 200 \times 20 = 4 \text{ kVA}$$

(c) Power factor of the combined load

$$I_1 = 10 \angle 40° \text{ A}$$
$$I_2 = 20 \angle -30° \text{ A}$$
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10 \angle 40° + 20 \angle -30° = 25.24 \angle -8.14° \text{A}$$
$$\phi = 25° - (-8.14°) = 33.14°$$
$$\text{pf} = \cos(33.14) = 0.84 \text{ (lagging)}$$

Example 4.111 *The load taken from a supply consists of a (a) lamp load of 10 kW at unity power factor, (b) motor load of 80 kVA at 0.8 power factor lagging, and (c) motor load of 40 kVA at 0.7 power factor lagging. Calculate the total load taken from the supply in kW and in kVA and the power factor of the combined load.*

Solution

Lamp load: $P_1 = 10$ kW, $\text{pf}_1 = 1$
Motor load: $S_2 = 80$ kVA, $\text{pf}_2 = 0.8$ (lagging)
Motor load: $S_3 = 40$ kVA, $\text{pf}_3 = 0.7$ (lagging)

(a) Total load in kW
For motor loads,

$$P_2 = S_2 \times \text{pf}_2 = 80 \times 0.8 = 64 \text{ kW}$$
$$P_3 = S_3 \times \text{pf}_3 = 40 \times 0.7 = 28 \text{ kW}$$
$$P = P_1 + P_2 + P_3 = 10 + 64 + 28 = 102 \text{ kW}$$

(b) Total load in kVA
For lamp load,

$$S_1 = \frac{P_1}{\text{pf}_1} = \frac{10}{1} = 10 \text{ kVA}$$
$$S = S_1 + S_2 + S_3 = 10 + 80 + 40 = 130 \text{ kVA}$$

(c) Power factor of the combined load

$$\text{pf} = \frac{P}{S} = \frac{102}{130} = 0.785 \text{ (lagging)}$$

Example 4.112 *Two circuits have the same numerical value of impedances. The pf of one is 0.8 lagging and that of the other is 0.6 lagging. What is the pf of combination if they are connected in parallel?*

Solution

$$\text{pf}_1 = 0.8 \text{ (lagging)}, \quad \text{pf}_2 = 0.6 \text{ (lagging)}$$

$$\mathbf{Z}_1 = Z \angle \cos^{-1}(0.8) = Z \angle 36.87° \ \Omega$$

$$\mathbf{Z}_2 = Z \angle \cos^{-1}(0.6) = Z \angle 53.13° \ \Omega$$

For parallel combination,

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(Z \angle 36.87°)(Z \angle 53.13°)}{Z \angle 36.87° + Z \angle 53.13°}$$

$$= \frac{Z^2 \angle 90°}{Z(1.4 + j1.4)} = \frac{Z^2 \angle 90°}{1.98Z \angle 45°} = 0.505\, Z \angle 45° \ \Omega$$

$$\text{pf} = \cos(45°) = 0.707 \text{ (lagging)}$$

Example 4.113 *When a 240 V, 50 Hz supply is fed to a 15 Ω resistor in parallel with an inductor, the total current is 22.1 A as shown in Fig. 4.106. What value must the frequency have for the total current to be 34 A?*

Fig. 4.106

Solution $V = 240$ V, $R = 15\ \Omega$, $I = 22.1$ A

Let

$$\mathbf{V} = 240 \angle 0° \text{ V}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{R} = \frac{240 \angle 0°}{15 \angle 0°} = 16 \angle 0° = 16 \text{ A}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}}{jX_L} = \frac{240 \angle 0°}{X_L \angle 90°} = \frac{240}{X_L} \angle -90° = -j\frac{240}{X_L} \text{ A}$$

$$\mathbf{I} = 16 - j\frac{240}{X_L}$$

$$\sqrt{(16)^2 + \left(\frac{240}{X_L}\right)^2} = 22.1$$

$$256 + \frac{57600}{X_L^2} = 488.41$$

$$X_L = 15.74\ \Omega$$

$$X_L = 2\pi fL$$

$$15.74 = 2\pi \times 50 \times L$$
$$L = 0.05 \text{ H}$$

Let the new frequency be f for the total current of 34 A.

$$\sqrt{(16)^2 + \left(\frac{240}{2\pi f \times 0.05}\right)^2} = 34$$
$$256 + \frac{57600}{0.0987 f^2} = 1156$$
$$f = 25.47 \text{ Hz}$$

Example 4.114 *Determine the current in the circuit of Fig. 4.107. Also, find the power consumed as well as pf.*

Fig. 4.107

Solution

$$X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 0.01 = 3.14\ \Omega$$
$$X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$
$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 50 \times 200 \times 10^{-6}} = 15.92\ \Omega$$
$$\mathbf{Z}_1 = 6 + j3.14\ \Omega$$
$$\mathbf{Z}_2 = 4 + j6.28\ \Omega$$
$$\mathbf{Z}_3 = 2 - j15.92\ \Omega$$

$$\mathbf{Z} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} = (6 + j3.14) + \frac{(4 + j6.28)(2 - j15.92)}{(4 + j6.28) + (2 - j15.92)} = 17.27\ \angle 30.75°\ \Omega$$
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100\ \angle 0°}{17.27 \angle 30.75°} = 5.79\ \angle -30.75° \text{ A}$$
$$P = VI\cos\phi = 100 \times 5.79 \times \cos(30.75°) = 497.94 \text{ W}$$
$$\text{pf} = \cos\phi = \cos(30.75°) = 0.86 \text{ (lagging)}$$

Example 4.115 *Two impedances $\mathbf{Z}_A = (4 + j3)\ \Omega$ and $\mathbf{Z}_B = (10 - j7)\ \Omega$ are connected in parallel and impedance $\mathbf{Z}_C = (6 + j5)\ \Omega$ is connected in series with parallel combination of $\mathbf{Z}_A$ and $\mathbf{Z}_B$ as shown*

in Fig. 4.108. If the voltage applied across the circuit is 200 V at 59 Hz, calculate (a) currents flowing in Z_A, Z_B and Z_C, and (b) total power factor of the circuit.

Fig. 4.108

Solution $\boldsymbol{Z}_A = 4 + j3\ \Omega,\quad \boldsymbol{Z}_B = 10 - j7\ \Omega,\quad \boldsymbol{Z}_C = 6 + j5\ \Omega,\quad V = 200\text{ V},\quad f = 59\text{ Hz}$

(a) Currents flowing in Z_A, Z_B and Z_C

$$\boldsymbol{Z}_A = 4 + j3 = 5\angle 36.87°\ \Omega$$

$$\boldsymbol{Z}_B = 10 - j7 = 12.21\angle -35°\ \Omega$$

$$\boldsymbol{Z}_C = 6 + j5\ \Omega = 7.81\angle 39.81°\ \Omega$$

$$\boldsymbol{Z} = \frac{\boldsymbol{Z}_A\boldsymbol{Z}_B}{\boldsymbol{Z}_A + \boldsymbol{Z}_B} + \boldsymbol{Z}_C = \frac{(5\angle 36.87°)(12.21\angle -35°)}{5\angle 36.87° + 12.21\angle -35°} + 7.81\angle 39.81° = 11.8\angle 32.17°\ \Omega$$

$$\boldsymbol{I}_C = \frac{\boldsymbol{V}}{\boldsymbol{Z}} = \frac{200\angle 0°}{11.8\angle 32.17°} = 16.95\ \angle -32.17°\text{A}$$

$$\boldsymbol{I}_A = \boldsymbol{I}_C\frac{\boldsymbol{Z}_B}{\boldsymbol{Z}_A + \boldsymbol{Z}_B} = (16.95\angle -32.17°)\frac{(12.21\angle -35°)}{5\angle 36.87° + 12.21\angle -35°} = 14.21\angle -51.21°\text{A}$$

$$\mathbf{I}_B = \mathbf{I}_C - \mathbf{I}_A = 16.95\angle -32.17° - 14.21\angle -51.21° = 5.82\angle 20.64°\ \text{A}$$

(b) Total power factor of the circuit

$$\text{pf} = \cos(32.17°) = 0.85\ \text{(lagging)}$$

Example 4.116 *In a series–parallel circuit, the parallel branches A and B are in series with Branch C as shown in Fig. 4.109. The impedances are* $\boldsymbol{Z}_A = (4 + j3)\ \Omega$, $\boldsymbol{Z}_B = (4 - j\frac{16}{3})\ \Omega$ *and* $\boldsymbol{Z}_C = (2 + j8)\ \Omega$. *If the current* $I_c = (25 + j0)$ *A, determine the branch currents, voltages and the total voltage. Hence, calculate active and reactive powers for each branch and the whole circuit.*

Fig. 4.109

Solution $\mathbf{Z}_A = 4 + j3\ \Omega,\quad \mathbf{Z}_B = 4 - j\dfrac{16}{3}\ \Omega,\quad \mathbf{Z}_C = 2 + j8\ \Omega,\quad \mathbf{I}_C = 25 + j0 = 25\angle 0°\ \text{A}$

(a) Branch currents
By current-division rule,

$$\mathbf{I}_A = \mathbf{I}_C \frac{\mathbf{Z}_B}{\mathbf{Z}_A + \mathbf{Z}_B} = (25 + j0)\frac{\left(4 - j\dfrac{16}{3}\right)}{4 + j3 + 4 - j\dfrac{16}{3}} = 20\angle -36.87°\ \text{A}$$

$$\mathbf{I}_B = \mathbf{I}_C - \mathbf{I}_A = 25\angle 0° - 20\angle -36.87° = 15\angle 53.13°\ \text{A}$$

(b) Branch voltages and total voltage

$$\mathbf{V}_A = \mathbf{V}_B = \mathbf{Z}_A I_A = (4 + j3)\,(20\angle -36.87°) = 100\angle -1.02°\ \text{V}$$

$$\mathbf{V}_C = \mathbf{Z}_C \mathbf{I}_C = (2 + j8)\,(25\angle 0°) = 206.16\angle 75.96°\ \text{V}$$

$$\mathbf{V} = \mathbf{V}_A + \mathbf{V}_C = 100\angle -1.02° + 206.16\angle 75.16° = 249.7\angle 52.27°\ \text{V}$$

(c) Active and reactive powers for each branch and the whole circuit

$$P_A = I_A^2\ R_A = (20)^2 \times 4 = 1600\ \text{W}$$

$$P_B = I_B^2\ R_B = (15)^2 \times 4 = 900\ \text{W}$$

$$P_C = I_C^2\ R_C = (25)^2 \times 2 = 1250\ \text{W}$$

$$P = P_A + P_B + P_C = 1600 + 900 + 1250 = 3750\ \text{W}$$

$$Q_A = I_A^2\ X_A = (20)^2 \times 3 = 1200\ \text{VAR}$$

$$Q_B = I_B^2\ X_B = (15)^2 \times \frac{16}{3} = 1200\ \text{VAR}$$

$$Q_C = I_C^2 X_C = (25)^2 \times 8 = 5000\ \text{VAR}$$

$$Q = Q_A + Q_B + Q_C = 1200 + 1200 + 5000 = 7400\ \text{VAR}$$

Example 4.117 *Find the applied voltage V_{AB} so that a 10 A current may flow through the capacitor in Fig. 4.110.*

Fig. 4.110

Solution

$$X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 0.0191 = 6\ \Omega$$

$$X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 0.0318 = 10\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 398 \times 10^{-6}} = 8\ \Omega$$

$$\mathbf{Z}_1 = 2 + j6 = 6.32\ \angle 71.56°\ \Omega$$

$$\mathbf{Z}_2 = 7 - j8 = 10.63\ \angle -48.8°\ \Omega$$

$$\mathbf{Z}_3 = 8 + j10 = 12.8\ \angle 51.34°\ \Omega$$

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} + \mathbf{Z}_3 = \frac{(2 + j6)(7 - j8)}{(2 + j6 + 7 - j8)} + (8 + j10) = 19.91\ \angle 45.53°\ \Omega$$

Let $\mathbf{I}_2 = 10\ \angle\ 0°\text{A}$

$$\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 = (10.63\ \angle -48.8°)\ (10\ \angle\ 0°) = 106.3\ \angle -48.8°\ \text{V}$$

$$\mathbf{V}_1 = \mathbf{V}_2 = 106.3\ \angle -48.8°\ \text{V}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}_1} = \frac{106.3\ \angle -48.8°}{6.32\ \angle\ 71.56°} = 16.82\ \angle -120.36°\ \text{A}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 16.82\ \angle -120.36° + 10\ \angle\ 0° = 14.58\ \angle -84.09°\ \text{A}$$

$$\mathbf{V}_{AB} = \mathbf{Z}\ \mathbf{I} = (19.91\ \angle\ 45.53°)\ (14.58\ \angle -84.09°) = 290.28\ \angle -38.56°\ \text{V}$$

Example 4.118 *If a voltage of 150 V applied between terminals A and B produces a current of 32 A for the circuit shown in Fig. 4.111, calculate the value of the resistance R and pf of the circuit.*

Fig. 4.111

Solution

$$V = 150\ \text{V}, \quad I = 32\ \text{A}$$

$$Z = \frac{150}{32} = 4.687\ \Omega$$

$$\mathbf{Z} = \frac{(5)(j4)}{5 + j4} + R = \frac{(5)(4\ \angle\ 90°)}{64\ \angle\ 38.66°} + R = 3.125\ \angle\ 51.34° + R = 1.95 + j2.44 + R$$

$$\sqrt{(1.95 + R)^2 + (2.44)^2} = 4.687$$

$$(1.95 + R)^2 + (2.44)^2 = (4.687)^2$$

$$(1.95 + R^2) = (4.687)^2 - (244)^2$$

$$1.95 + R = 4$$

$$R = 2.05\ \Omega$$

$$\text{pf} = \frac{\text{Total resistance}}{\text{Total impedance}} = \frac{1.95 + 2.05}{4.687} = 0.853 \text{ (lagging)}$$

Example 4.119 *An impedance of R + jX ohms is connected in parallel with another impedance of –j5 ohms as shown in Fig. 4.112. The combination is then connected in series with a pure resistance of 2 Ω. When connected across a 100 V, 50 Hz ac supply, the total current drawn by the circuit is 20 A and the total power consumed by the circuit is 2 kW. Calculate (a) currents through parallel branches, and (b) R and L.*

Fig. 4.112

Solution

$$P = 2 \text{ kW}, \quad V = 100 \text{ V}, \quad I = 20 \text{ A}$$

(a) Currents through parallel branches

$$P = VI \cos \phi$$

$$2000 = 100 \times 20 \times \cos \phi$$

$$\cos \phi = 1$$

$$\phi = 0°$$

$$\mathbf{V} = 100 \angle 0° \text{ V}$$

$$\mathbf{I} = 20 \angle 0° \text{ A}$$

$$\mathbf{V}_R = 2 \times 20° \angle 0° = 40 \angle 0° \text{ V}$$

$$\mathbf{V}_P = \mathbf{V} - \mathbf{V}_R = 100 \angle 0° - 40 \angle 0° = 60 \angle 0° \text{ V}$$

$$\mathbf{I}_C = \frac{\mathbf{V}_P}{\mathbf{Z}_C} = \frac{60 \angle 0°}{5 \angle -90°} = 12 \angle 90° \text{ A}$$

$$\mathbf{I}_X = \mathbf{I} - \mathbf{I}_C = 20 \angle 0° - 12 \angle 90° = 23.32 \angle -30.96° \text{ A}$$

(b) Resistance and inductance

$$\mathbf{Z}_X = \frac{60 \angle 0°}{23.32 \angle -30.96°} = 2.57 \angle 30.96° = 2.2 + j1.32\ \Omega$$

$$R = 2.2\ \Omega$$

$$X_L = 1.32\ \Omega$$

$$X_L = 2\pi fL$$
$$1.32 = 2\pi \times 50 \times L$$
$$L = 4.2 \text{ mH}$$

Example 4.120 *The circuit of Fig. 4.113 takes 12 A at a lagging power factor and dissipates 1800 W. The reading of the voltmeter is 200 V. Find R_1, X_1 and X_2.*

Fig. 4.113

Solution $I = 12$ A, $P = 1800$ W

Let $\mathbf{I} = 12 \angle 0°$ A

$$Z_2 = \frac{200}{12} = 16.67\ \Omega$$
$$Z_2 = \sqrt{R_2^2 + X_2^2}$$

$$16.67 = \sqrt{(10)^2 + X_2^2}$$
$$277.88 = 100 + X_2^2$$
$$X_2 = 13.33\ \Omega$$
$$\mathbf{V}_2 = (12 \angle 0°)\,(10 + j13.33) = (12 \angle 0°)\,(16.67 \angle 53.13°) = 200 \angle 53.13°\ \text{V}$$
$$P = VI \cos \phi$$
$$1800 = 200 \times 12 \times \cos \phi$$
$$\cos \phi = 0.75$$
$$\phi = 41.41°$$

Applied voltage $\mathbf{V}_{\text{req}} = 200 \angle 41.41°$ V

Voltage across parallel branches $= 200 \angle 41.41° - 200 \angle 53.13° = 40.84 \angle -42.73°$ V

Current through capacitor $= \dfrac{40.84 \angle -42.73°}{20 \angle -90°} = 2.04 \angle 47.27°$A

Current through R_1 and $X_1 = 12 \angle 0° - 2.04 \angle 47.27° = 10.72 \angle -8.03°$ A

$$\mathbf{Z}_1 = \frac{40.84 \angle -42.73°}{10.72 \angle -8.03°} = 3.81 \angle -34.7° = 3.13 - j2.17$$
$$R_1 = 3.13\ \Omega$$
$$X_1 = 2.17\ \Omega$$

Example 4.121 *For the circuit shown in Fig. 4.114, calculate (a) total admittance, total conductance and total susceptance, (b) total current and total pf, and (c) value of pure capacitance to be connected in parallel with the above combination to make the total pf unity.*

Fig. 4.114

Solution

(a) Total admittance, total conductance and total susceptance

$$\mathbf{Z}_1 = 6 + j8 = 10 \angle 53.13° \ \Omega$$

$$\mathbf{Z}_2 = 8 - j6 = 10 \angle -36.87° \ \Omega$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{10 \angle 53.13°} = 0.1 \angle -53.13° \ ℧$$

$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} = \frac{1}{10 \angle -36.87°} = 0.1 \angle 36.87° \ ℧$$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 0.1 \angle -53.13° + 0.1 \angle 36.87° = 0.14 - j0.02 \ ℧ = 0.14 \angle -8.13° \ ℧$$

$$Y = 0.14 \ ℧$$

$$G = 0.14 \ ℧$$

$$B = 0.02 \ ℧$$

(b) Total current and total pf

Let $\mathbf{V} = 200 \angle 0° \text{ V}$

$$\mathbf{I} = \mathbf{V}\,\mathbf{Y} = (200 \angle 0°)\,(0.14 \angle -8.13°) = 28 \angle -8.13° \text{ A}$$

$$\text{pf} = \cos(8.13°) = 0.989 \text{ (lagging)}$$

(c) Value of pure capacitance to make total pf unity

Since the current lags behind voltage, the circuit is inductive in nature. In order to make the total pf unity, a pure capacitor is connected in parallel so that pf becomes unity and imaginary part of $\mathbf{Y}_{\text{req}}$ becomes zero.

$$\mathbf{Y}_{\text{req}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3$$

$$\mathbf{Y}_{\text{req}} = 0.14 - j0.02 + j0.02 = 0.14$$

$$Y_3 = Y_C = \frac{1}{X_C} = 0.02 \ ℧$$

$$X_C = 50 \ \Omega$$

$$C = \frac{1}{2\pi \times 50 \times 50} = 63.66 \ \mu\text{F}$$

4.15 LOCUS DIAGRAM

Locus diagrams are useful in analysing the behaviour of an *RLC* circuit when one of its parameters, i.e, *R*, *L* or *C* is varied keeping frequency and voltage constant. The magnitude and phase of a current phasor in the circuit depends upon the values of *R*, *L* and *C* and frequency of the supply.

The path traced by the terminus of the current phasor when any one parameter *R*, *L* or *C* is varied keeping *f* and *V* constant is called *current locus*.

Locus diagrams can also be drawn for impedance and admittance when frequency is varied.

4.15.1 Locus Diagram of a Series *RL* Circuit

Figure 4.115 shows a series *RL* circuit with variable resistance and constant reactance.

Fig. 4.115

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_L^2}}$$

In a series *RL* circuit, current *I* lags behind voltage *V* by an angle ϕ. Figure 4.116 shows the phasor diagram and impedance triangle for a series *RL* circuit.

Fig. 4.116 *Phasor diagram and impedance triangle of a series RL circuit*

$$\cos\phi = \frac{R}{Z}$$

$$\sin\phi = \frac{X_L}{Z} \qquad \ldots(4.1)$$

The current *I* has two components, I_x and I_y,

$$I_x = I\cos\phi = \frac{V}{Z}\frac{R}{Z} = \frac{VR}{Z^2} = \frac{VR}{R^2 + X_L{}^2} \qquad \ldots(4.2)$$

$$I_y = -I\sin\phi = -\frac{V}{Z}.\frac{X_L}{Z} = -\frac{VX_L}{Z^2} = -\frac{VX_L}{R^2 + X_L{}^2} \qquad \ldots(4.3)$$

Squaring and adding Eqs (4.2) and (4.3),

$$I_x^{\,2} + I_y^{\,2} = \frac{V^2R^2}{(R^2 + X_L^{\,2})^2} + \frac{V^2X_L^{\,2}}{(R^2 + X_L)^2}, \; = \frac{V^2(R^2 + X_L^2)}{(R^2 + X_L^{\,2})^2}, \; = \frac{V^2}{R^2 + X_L^2}$$

From Eq. (4.3),

$$R^2 + X_L^2 = -\frac{VX_L}{I_y}$$

$$I_x^{\,2} + I_y^{\,2} = -\frac{VI_y}{X_L}$$

$$I_x^{\,2} + I_y^{\,2} + \frac{VI_y}{X_L} = 0$$

Adding $\left(\frac{V}{2X_L}\right)^2$ to both the sides,

$$I_x^{\,2} + (I_y + \frac{V}{2X_L})^2 = \left(\frac{V}{2X_L}\right)^2 \qquad \ldots(4.4)$$

Equation (4.4) represent a circle with radius $\frac{V}{2X_L}$ and centre $\left(0, -\frac{V}{2X_L}\right)$.

Thus, the locus of the extremity of a current phasor is a circle. But the maximum possible value for ϕ is 90°. When $R = 0$, the locus will be a semicircle Hence, the locus diagram for a series *RL* circuit with variable resistance is a semicircle of radius $\frac{V}{2X_L}$ and centre $\left(0, -\frac{V}{2X_L}\right)$. The locus diagram is shown in Fig. 4.117.

Fig. 4.117 *Locus diagram of a series RL circuit for variable R*

Similarly, the locus diagram of a series *RL* circuit with variable reactance and constant resistance can be drawn. From Eq. (4.3),

$$R^2 + X_L^2 = \frac{VR}{I_x}$$

$$I_x^{\,2} + I_y^{\,2} = \frac{V}{R} I_x$$

$$I_x^{\,2} + I_y^{\,2} - \frac{V}{R} I_x = 0$$

Adding $\left(\frac{V}{2R}\right)^2$ on both the sides,

$$\left(I_x - \frac{V}{2R}\right)^2 + I_y^{\,2} = \left(\frac{V}{2R}\right)^2 \qquad \ldots(4.5)$$

Equation (4.5) represents a circle with radius $\frac{V}{2R}$ and centre $\left(\frac{V}{2R}, 0\right)$. Thus, the locus of the extremity of a current phasor is a semicircle of radius $\frac{V}{2R}$ and centre $\left(\frac{V}{2R}, 0\right)$. The locus diagram is shown in Fig. 4.118

Fig. 4.118 *Locus diagram of a series RL circuit for variable X_L*

4.15.2 Locus Diagram of a Series *RC* Circuit

Figure 4.119 shows a series *RC* circuit with variable resistance and constant reactance.

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_C^2}}$$

Fig. 4.119 *Series RC circuit*

In a series *RC* circuit, current *I* leads voltage *V* by an angle ϕ. Figure 4.120 shows the phasor diagram and impedance triangle for series *RC* circuit.

$$\cos\phi = \frac{R}{Z}$$

$$\sin\phi = \frac{X_C}{Z}$$

Fig. 4.120 *Phasor diagram and impedance triangle of a series RC circuit*

It can be similarly proved that the locus of the extremity of a current phasor is a semicircle with radius $\frac{V}{2X_C}$ and centre $\left(0, \frac{V}{2X_C}\right)$. The locus diagram is shown in Fig. 4.121.

For a series *RC* circuit with variable reactance and constant resistance, the locus of the extremity of a current phasor is a semicircle with radius $\frac{V}{2R}$ and centre $\left(\frac{V}{2R}, 0\right)$. The locus diagram is shown in Figure 4.122.

Fig. 4.121 *Locus diagram of a series RC circuit with variable R*

Fig. 4.122 *Locus diagram of a series RC circuit with variable X_C.*

Exercises

4.1 An alternating current varying sinusoidally with a frequency of 50 Hz has an rms value of 20 A. Write down the equation for the instantaneous value and find this value (i) 0.0025 second, and (ii) 0.0125 second after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A?

$$\left[i = 28.28 \sin 100\,\pi t,\ 20\text{ A}, -20\text{ A}, \frac{1}{300}\text{ s}\right]$$

4.2 A certain waveform has a form factor of 1.2 and a peak factor of 1.5. If the maximum value is 100, find the rms value and average value.

[66.67 V, 55.67 V]

4.3 Find the root mean square value, the average value and the form factor of the resultant current in a wire that carries simultaneously a direct current of 5 A and a sinusoidal alternating current with an amplitude of 5 A.

[6.12 A, 5 A, 1.224]

4.4 Find the relative heating effects of two in-phase current waveform of equal peak values and time periods, but one sinusoidal and the other triangular.

[3:2]

4.5 Prove that if a dc current of I amperes is superimposed in a conductor by an ac current of maximum value I amperes, the root mean square value of its resultant is $\sqrt{\frac{3}{2}}I$.

4.6 Find the average values and rms values of the waveforms shown in Fig. 4.123 to Fig. 4.126.

Fig. 4.123

[0.543 V_m, 0.674 V_m]

Fig. 4.124

[0.622 V_m, 0.687 V_m]

Fig. 4.125

[7.5 V, 8.666 V]

Fig. 4.126

[0.67 V_m, 0.745 V_m]

4.7 Find the rms value of the periodic waveform with time period T given in Fig. 4.127.

Fig. 4.127

[7.45 V]

4.8 A voltage wave has the variation shown in Fig. 4.128.

(i) Find the average and effective values of the voltage.

(ii) If this voltage is applied to a 10 Ω resistance, find the dissipated power.

Fig. 4. 128

[10 V, 12.91 V, 16.67 W]

4.9 Find the resultants of the following voltages:

(i) $v_1 = 60 \cos\theta$,

$v_2 = 40 \sin\left(\theta - \frac{\pi}{3}\right)$,

$v_3 = 15 \sin\theta$

[43.22 sin (θ + 35.92°)]

(ii) $e_1 = 50 \sin \omega t$,

$e_2 = 100 \sin (\omega t + 120°)$

$e_3 = 120 \sin (\omega t - 30°)$

[107.27 sin (ωt + 14.35°)]

(iii) $v_1 = 4\sqrt{2} \sin (\omega t + 135°)$,

$v_2 = -4\sqrt{3} \sin (\omega t + 60°)$,

$v_3 = 4 \cos (\omega t - 150°)$

[7.73 sin (ωt – 135°)]

(iv) $e_1 = 25 \sin \omega t$,

$e_2 = 10 \sin\left(\omega t + \frac{\pi}{6}\right)$,

$e_3 = 30 \cos \omega t$,

$e_4 = 20 \sin\left(\omega t - \frac{\pi}{4}\right)$

[52.14 sin (ωt + 23.57°)]

(v) $v_1 = 10 \sin \omega t$, $v_2 = -10 \cos \omega$

[14.14 sin (ωt – 45°)]

4.10 Three alternating currents $i_1 = 141 \sin\left(\omega t + \frac{\pi}{4}\right)$, $i_2 = 30 \sin\left(\omega t + \frac{\pi}{2}\right)$ and $i_3 = 20 \sin\left(\omega t + \frac{\pi}{6}\right)$ are fed into a

common conductor. Find graphically or otherwise the equation of the resultant current and its rms value.

[167.5 sin ($\omega t + 45.6°$), 118.54]

4.11 Four wires, p, q, r, s are connected to a common point. The currents in lines p, q and r are $6 \sin\left(\omega t + \frac{\pi}{3}\right)$, $5 \cos\left(\omega t + \frac{\pi}{3}\right)$ and $3 \cos\left(\omega t + \frac{\pi}{3}\right)$ respectively. Find the current in the wire s.

[9.99 sin ($\omega t - 66.84°$)]

4.12 A sinusoidal voltage $V_m \sin \omega t$ is applied to three parallel branches yielding branch currents as follows:

$$i_1 = 14.14 \sin(\omega t - 45°)$$

$$i_2 = 28.3 \cos(\omega t - 60°)$$

$$i_3 = 7.07 \sin(\omega t + 60°)$$

Find the complete expression for the source current. Draw the phasor diagram in terms of effective values. Use the voltage as reference.

[39.4 sin ($\omega t + 15.1°$)]

4.13 A sinusoidal voltage $V_m \sin \omega t$ is applied to three parallel branches. Two of the branch currents are given by

$$i_1 = 14.14 \sin(\omega t - 37°)$$

$$i_2 = 28.28 \cos(\omega t - 143°)$$

The source current is found to be $i = 63.8 \sin(\omega t + 12.8°)$.

(i) Find the effective value of the current in the third branch.

(ii) Write the complete time expression for the instantaneous value of the current in part (i).

(iii) Draw the phasor diagram showing the source current and the three branch currents. Use voltage as the reference phasor.

[39.95 A, 56.51 sin ($\omega t + 53.12°$)]

4.14 A voltage wave $e(t) = 170 \sin 120\, t$ produces a net current of $14.14 \sin 120\, t + 7.07 \cos(120\, t + 30°)$.

(i) Express the effective value of the current as a single phasor quantity.

(ii) Draw the phasor diagram. Show the components of the current as well as the resultant.

(iii) Determine the power delivered by the source.

[8.66 $\angle 30°$ A, 901.7 W]

4.15 The instantaneous values of two alternating voltages are represented by $v_1 = 60 \sin\theta$ and $v_2 = 40 \sin(\theta - \pi/3)$. Derive an expression for the instantaneous value of

(i) the sum, and

(ii) the difference of these voltages.

[87.17 sin ($\theta - 23.41°$), 52.91 sin ($\theta + 40.89°$)]

4.16 A 250 V, 50 Hz voltage is applied across a circuit consisting of a pure resistance of 20 Ω. Determine

(i) the current flowing through the circuit, and (ii) power absorbed by the circuit. Give the expressions for the voltage and current.

[(i) 12.5 A, (ii) 3.125 kW, (iii) $v = 353.6 \sin 314\, t$, $i = 17.68 \sin 314\, t$]

4.17 A purely inductive circuit allows a current 20 A to flow through a 230 V, 50 Hz supply. Find (i) inductive reactance, (ii) inductance of the coil, (iii) power absorbed, and (iv) equations for voltage and current.

[(i) 11.5 Ω,
(ii) 36.62 mH
(iii) 0
(iv) $v = 325.27 \sin 314\, t$,
$i = 28.28 \sin\left(314t - \frac{\pi}{2}\right)$]

4.18 A capacitor connected to a 230 V, 50 Hz supply draws a 15 A current. What current will it draw when the capacitance and frequency are both reduced to half?

[3.75 A]

4.19 Calculate the impedance of the circuit shown in Fig. 4.129.

Fig. 4.129

[4 Ω]

4.20 An ac circuit has the following voltage and current: v = 325 sin 314 t, i = 65 sin (314 t – 1.57). Find (i) frequency, (ii) rms value of voltage and current, (iii) impedance, and (iv) power factor.

[50 Hz, 229.81 V, 45.96 A, 5 Ω 0.99 (lagging)]

4.21 Two sources of an electromotive force represented respectively by 200 sin ωt and 200 $\sin(\omega t + \pi/6)$ are in series. Express the resultant in vector notation with reference to 200 $\sin \omega t$ and calculate the rms current and power supplied to a circuit of 8 + j6 Ω impedance.

[273.2 ∠15° V, 27.32 ∠–21.87° A, 5971 W]

4.22 The voltage applied to a series circuit consisting of two pure elements is given by v = 180 $\sin \omega t$ and the resulting current is given by i = 2.5 $\sin(\omega t - 45°)$. Find the average power taken by the circuit and the values of the elements. [159 W, 50.91 Ω, 50.91 Ω]

4.23 Find an expression for the current and calculate the power when a voltage represented by v = 283 $\sin 100\pi t$ is applied to a coil having R = 50 Ω and L = 0.159 H.

[$4 \sin(100\pi t - \pi/4)$, 400 W]

4.24 A voltage $V = (150 + j180)$ V is applied across an impedance and the current is found to be $I = (5 - j4)$ A. Determine (i) scalar impedance, (ii) reactance, and (iii) power consumed.

[36.6 Ω, 36.6 Ω, 30 W]

4.25 The current in a series circuit of R = 5 Ω and L = 30 mH lags the applied voltage by 72°. Determine the source frequency and the impedance Z. [81.63 Hz, 16.18 Ω]

4.26 Current flowing through an inductive circuit is 15 sin ($\omega t + \pi/4$). When the voltage applied across it is 30 cos ωt, find the pf of the circuit.

[0.707 (lagging)]

4.27 Voltage and current in an ac circuit are given by

$v = 200 \sin 377t$ $\qquad$ $i = 8 \sin(377t - \pi/6)$

Determine true power, reactive power and apparent power drawn by the circuit.

[692.82 W, 400 VAR, 800 VA]

4.28 A current of 5 A flows through a non-inductive resistor in series with a choke coil when supplied at 250 V, 50 Hz. If the voltage drops across the coil and non-inductive resistor are 200 V and 125 V respectively, calculate the resistance and inductance of the impedance coil, value of non-inductive resistor and power drawn by the coil. Draw the vector diagram.

[5.5 Ω, 0.126 H, 25 Ω, 137.5 W]

4.29 Two coils A and B are connected in series across a 200 V, 50 Hz ac supply. The power input to the circuit is 2 kW and 1.15 kVAR. If the resistance and reactance of the coil A are 5 Ω and 8 Ω respectively, calculate resistance and reactance of the coil B. Also, calculate the active power consumed by coils A and B.

[10.3 Ω, 0.642 Ω, 665.3 W, 1334.7 W]

4.30 A choking coil and a pure resistor are connected in series across a supply of 230 V, 50 Hz. The voltage drop across the resistor is 100 V and that across the chocking coil is 150 V. Find the voltage drop across the inductance and resistance of the choking coil. Hence, find their values if the current is 1 A.

[109.98 V, 102 V, 0.366 H, 102 Ω]

4.31 For Fig. 4.130, find R and L.

Fig. 4.130

[4.928 Ω, 0.0266 H]

4.32 Voltage and current for a circuit with two pure elements in series are expressed as follows:

$$v(t) = 170\sin\left(6280t + \frac{\pi}{3}\right) \text{ volts}$$

$$i(t) = 8.5\sin\left(6280t + \frac{\pi}{2}\right) \text{ amperes}$$

Sketch the two waveforms. Determine (i) frequency, (ii) power stating its nature, and (iii) values of the elements.

[(ii) 0.866 (leading) (iii) 17.32 Ω, 16 μF]

4.33 A load consisting of a capacitor in series with a resistor has an impedance of 50 Ω and a pf of 0.707 leading. The load is connected in series with a 40 Ω resistor across an ac supply and the resulting current is of 3 A. Determine the supply voltage and overall phase angle.

[249.69 V, 25.135°]

4.34 A capacitive load takes 10 kVA and 5 kVAR, when connected to a 200 V, 50 Hz ac supply. Calculate (i) resistance, (ii) capacitance, (iii) active power, and (iv) pf.

[3.464 Ω, 1.59×10^{-3} F, 8.66 kW, 0.866 (leading)]

4.35 A resistor of 100 Ω is connected in series with a 50 μF capacitor to a 50 Hz, 200 V supply. Find:

(i) impedance (ii) current
(iii) power factor (iv) phase angle
(v) voltage across the resistor and across the capacitor.

[118.6 Ω, 1.69 A,0.845 (leading), 32.48°, 168.712 V, 107.42 V]

4.36 In an *RLC* series circuit, the voltage across the resistor, inductor and capacitor are 10 V, 15 V and 10 V respectively. What is the supply voltage?

[11.18 V]

4.37 A resistor *R*, inductor $L = 0.01$ H and capacitor *C* are connected in series. When a voltage of 400 sin (3000 t – 10°) is applied to a series combination, the current flowing is $10\sqrt{2}$ cos (3000 t – 55°). Find *R* and *C*.

[20 Ω, 6.66 μF]

4.38 A resistor of 20 Ω, an inductor of 0.2 H and a capacitor of 100 μF are connected in series across a 220 V, 50 Hz supply. Determine

(i) impedance
(ii) current flowing through the circuit
(iii) voltage drop across each element
(iv) power consumed

[36.9 Ω, 5.96 A, 119.2 V, 374.52 V, 189.64 V, 710.43 W]

4.39 A series *RLC* circuit having a resistance of 8 Ω, an inductance of 80 mH and a capacitance of 100 μF is connected across a 150 V, 50 Hz supply. Calculate (i) current, (ii) power factor, and (iii) voltage across the inductance and capacitance.

[(i) 14.35 A (ii) 0.766 (leading) (iii) 360.47 V, 457.04 V]

4.40 An emf of 100 V, 50 Hz is applied to an impedance $\mathbf{Z}_1 = 8 + j6$ Ω. An impedance $\mathbf{Z}_2$ is added in series with $\mathbf{Z}_1$. The current becomes half of the original and leads it by 20°. Determine $\mathbf{Z}_2$. [(11.14 – j0.2) Ω]

4.41 In a series *RLC* circuit, the inductor is imperfect and the capacitor has no leakage but has a lead resistance. When the circuit is connected to a 50 Hz supply, the voltage drop across the resistor and the inductor is 200 V. The voltage drop across the resistor and the capacitor is 250 V. The voltage drop across the resistor alone is 150 V. The voltage drop across the capacitor and inductor are 125 V and 100 V respectively. If the current flowing through the circuit is 1 A, find graphically

(i) capacitance
(ii) lead resistance of the capacitor
(iii) inductance
(iv) resistance of the inductor
(v) applied voltage

[33.5 μF, 81.25 Ω, 0.3 H, 25 Ω, 256.26 V]

4.42 Two impedances of 10 ∠ 30° Ω and 20 ∠–45° Ω are connected in series. Calculate the power factor of the series combination.

[0.9281 (lagging)]

4.43 Two impedances Z_1 and Z_2 are connected in series across a 230 V, 50 Hz ac supply. The total current drawn by the series combination in 2.3 A. The pf of Z_1 is 0.8 lagging. The voltage drop across Z_1 is twice the voltage drop across Z_2 and it is 90° out of phase with it. Determine the value of Z_2. [44.719 Ω]

4.44 In the arrangement shown in Fig. 4.131, $C = 20$ microfarads and the current flowing through the circuit is 0.345 A. If the voltages are as indicated, find the applied voltage, frequency and loss in the iron-cored inductor. Draw the phasor diagram.

Fig. 4.131

[34.2 V, 50 Hz, 1.89 W]

4.45 Two impedances of $14 + j5\ \Omega$ and $18 + j10\ \Omega$ are connected in parallel across a 200 V, 50 Hz supply, Determine (i) admittance of each branch and the entire circuit, (ii) current in each branch and total current, (iii) power and power factor of each branch, (iv) total power factor, and (v) draw phasor diagram.

[$0.067\angle -19.6°$ ℧, $0.048\angle -29.05°$ ℧, $0.115\angle -23.75°$ ℧, $13.45\angle -19.65°$A, $9.6\angle -29.05°$A, $22.91\angle -23.75°$ A, 2532.64 W, 1658.8 W, 0.94 (lagging), 0.874 (lagging), 0.916 (lagging)]

4.46 In a parallel *RC* circuit shown in Fig. 4.132, $i_R = 15 \cos (5000\, t - 30°)$ amperes. Obtain the current in capacitance.

Fig. 4.132

[$74.95 \sin (5000\, t + 150°)$]

4.47 A voltage $v = 100 \sin (5000\, t + \pi/4)$ is applied across a three-branch circuit as shown in Fig. 4.133. Find the rms value of the current in the inductor.

Fig. 4.133

[7.07 A]

4.48 A circuit comprises of a conductance G in parallel with a susceptance B. Calculate the admittance $G + jB$ if the impedance is $10 + j5\ \Omega$.

[0.08 S, 0.04 S]

4.49 If an admittance of a circuit is $(8 + j6)$ ℧ and the circuit current is $2\angle 30°$A, find the pf of the circuit. Also, calculate the apparent power in the circuit.

[0.8 (leading), 0.2 VA]

4.50 A resistor of 50 Ω, and inductor of 0.15 H and a capacitor of 100 μF are connected in parallel across a 100 V, 50 Hz ac supply. Calculate (i) current in each circuit, (ii) resultant current. Draw individual phasor diagrams and the overall phasor diagram.

[$2\angle 0°$ A, $2.12\angle -90°$A, $3.14\angle 90°$ A, $2.25\angle 27.02°$ A]

4.51 The power dissipated in the coil A is 300 W and in the coil B is 400 W. Each coil takes a current of 5 A when connected to a 110 V, 50 Hz supply. Find the current drawn when the coils are connected in parallel.

[$9.93\angle -50.11°$A]

4.52 Find the total impedance, supply current and pf of the entire circuit.

Fig. 4.134

[$10.06\angle 36.68°\ \Omega$, $19.88\angle -36.68°$ A, 0.8 (lagging)]

4.53 Determine kVA, kVAR and kW consumed by the two impedances $\mathbf{Z}_1 = (20 + j37.7)\ \Omega$ and $\mathbf{Z}_2 = (50 + j0)\ \Omega$, when connected in parallel across a 230 *V*, 50 *Hz* supply.

[1.971 kVA, 1.095 kVAR, 1.64 kW]

4.54 In the parallel circuit shown in Fig. 4.135, the power in the 3 Ω resistor is 666 W and the total volt-amperes taken by the circuit is 3370 VA. The power factor of the whole circuit is 0.937 leading. Find *Z*.

Fig. 4.135

[$(2 - j2)\ \Omega$]

4.55 A coil is connected across a non-inductive resistor of 120 Ω. When a 240 V, 50 Hz supply is applied to this circuit, the coil draws a current of 5 A and the total current is 6 A. Determine the power and the power factor of (i) the coil, and (ii) the whole circuit.

[420 W, 0.35 (lagging), 900 W, 0.625 (lagging)]

4.56 A coil takes a current of 10 A and dissipates 1410 W when connected to 220 V, 50 Hz supply. If another coil is connected in parallel with it, the total current taken from the supply is 20 A at a power factor of 0.868. Determine the current and the overall power factor when the coils are connected in series across the same supply.

[5.55 A, 0.8499 (lagging)]

4.57 Draw an impedance triangle between terminals *AB* in Fig. 4.136, labelling its sides with appropriate values and units.

Fig. 4.136

[$R = 3.84\ \Omega$, $X = 1.12\ \Omega$ (inductive), $Z = 4\ \Omega$]

4.58 Draw the impedance triangle between terminals *AB* in Fig. 4.137, for the following conditions:

(i) $X_C = 8\ \Omega$, $R = 6\ \Omega$, $X_L = 4\ \Omega$

(ii) $X_C = 4\ \Omega$, $R = 6\ \Omega$, $X_L = 6\ \Omega$

Label its side with appropriate values and units.

Fig. 4.137

[(i) $R = 1.8461\ \Omega$, $X = 5.2308\ \Omega$, (capacitive), $Z = 5.547\ \Omega$

(ii) $R = 3\ \Omega$, $X = 1\ \Omega$, (capacitive), $Z = 3.1623\ \Omega$]

4.59 Draw an admittance triangle between terminals *AB* in Fig. 4.138, labelling its sides with appropriate values and units in case of

(i) $X_L = 4\ \Omega$, and $X_C = 8\ \Omega$

(ii) $X_L = 10\ \Omega$, and $X_C = 5\ \Omega$

Fig. 4.138

[(i) $G = 0.015$ ℧, $B_L = 0.123$ ℧, $Y = 0.124$ ℧, (ii) $G = 0.0098$ ℧, $B_C = 0.099$ ℧, $Y = 0.1$ ℧]

4.60 An inductive impedance *BC* is connected in series with a parallel combination *AB* consisting of a capacitor and a non-inductive resistor. The circuit constants are so adjusted that the current in the parallel branches *AB* are equal, and that the voltage across *AB* is equal to and in quadrature with the voltage across *BC*. When a voltage of 200 V is applied to *AC*, the total power absorbed is 1200 W. Calculate the circuit constants and draw a vector diagram.

[$AB = R = 33.32\ \Omega$ in parallel with $X_C = 33.32\ \Omega$, $BC = (16.66 + j16.66)\ \Omega$]

4.61 A 100 Ω resistor, shunted by a 0.4 H inductor is in series with a capacitor *C*. A voltage of 250 V at

50 Hz is applied to the circuit. Find

(i) the value of C to give unity power factor

(ii) the total current

(iii) current in the inductive branch

[65.4 μF, 4.08 A, 2.55 A]

4.62 A non-inductive 10 Ω resistor is in series with a coil of 1.3 Ω resistance and 0.018 H inductance. If a voltage of maximum value of 100 V at a frequency of 100 Hz is applied to this circuit, what will be the voltage across the resistor?

[62.54 V]

4.63 Two impedances $\mathbf{Z}_1 = 10 - j15\ \Omega$ and $\mathbf{Z}_2 = 4 + j8\ \Omega$ are connected in parallel. The supply voltage is 100 V, 25 Hz. Calculate (i) the admittance, conductance and susceptance of the combined circuit, and (ii) total current drawn and pf.

[0.097 ℧, 0.081 ℧, 0.054 ℧, 9.7 A, 0.83 (lagging)]

4.64 A voltage of 200 ∠ 53.13° V is applied across two impedances in parallel. The values of the impedances are $(12 + j16)\ \Omega$ and $(10 - j20)\ \Omega$. Determine kVA, kVAR and kW in each branch and the pf of the whole circuit.

[2 kVA, 1.2 kW, 1.6 kVAR, 1.788 kVA, 0.8 kW, 1.6 kVAR, unity pf]

4.65 For the circuit shown in Fig. 4.139, evaluate the current through and voltage across each element.

Fig. 4.139

$$\left[\begin{array}{l}\mathbf{I}_1 = 5\angle -30^\circ\ \text{A},\ \mathbf{I}_2 = 8.66\angle 60^\circ\ \text{A},\\ \mathbf{V}_1 = 100\angle -30^\circ\ \text{V},\ \mathbf{V}_2 = 100\angle -30^\circ\ \text{V},\end{array}\right.$$

$$\left[\mathbf{V}_{10\Omega} = 100\angle -30^\circ\ \text{V}\right]$$

4.66 Two impedances Z_1 and Z_2 are connected in parallel. The first branch takes a leading current of 16 A and has a resistance of 5 Ω, while the second branch takes a lagging current at a pf of 0.8. The total power supplied is 5 kW, the applied voltage being $(100 + j200)$ V. Determine branch currents and total current.

[16∠132.46° A, 20.8 ∠26.57° A, 22.49 A]

4.67 For the parallel branch shown in Fig. 4.140, find the value of R_2 when the overall power factor is 0.92 lag.

Fig. 4.140

[1.35 Ω]

4.68 In a series–parallel circuit, two parallel branches A and B are in series with C. The impedances are $\mathbf{Z}_A = (10 + j8)\ \Omega$, and $\mathbf{Z}_B = (9 - j6)\ \Omega$ and $\mathbf{Z}_C$ $(3 + j2)\ \Omega$. If the voltage across Z_C is 100 ∠0° V, determine the values I_A and I_B.

[15.7∠–73.39° A, 18.59 ∠–1.04° A]

4.69 A capacitor is placed in parallel with two inductive loads. The current through the first inductor is 20 A at 30° lag and the current through the second is 40 A at 60° lag. What must be the current in the capacitor so that the current in the external circuit is of unity power factor?

[44.64 ∠90° A]

4.70 A circuit of 15 Ω resistance and 12 Ω inductive reactance is connected in parallel with another circuit consisting of a resistor of 25 Ω in series with a capacitive reactance of 17 Ω. This combination is energized from a 200 V, 40 Hz mains. Find the branch currents, total current and power factor of the circuit. It is desired to raise the power factor of this circuit to unity by connecting a capacitor in parallel. Determine the value of the capacitance of the capacitor.

[10.42 ∠–38.56° A, 6.61 ∠34.21° A, 13.95 ∠–11.56° A, 0.98 (lagging), 54.9 μF]

4.71 A resistor of 30 Ω and a capacitor of unknown value are connected in parallel across a 110 V, 50 Hz supply. The combination draws a current of 5 A from the supply. Find the value of the unknown capacitance of the capacitor. This combination is again connected across a 110 V supply of unknown frequency. It is observed that the total current drawn from the mains falls to 4 A. Determine the frequency of the supply.

[98.58 μF, 23.68 Hz]

4.72 Two reactive circuits have an impedance of 20 Ω each. One of them has a lagging power factor of 0.8 and the other has a leading power factor of 0.6. Find (i) voltage necessary to send a current of 10 A through the two in series, and (ii) current drawn from 200 V supply if the two are connected in parallel. Draw a phasor diagram in each case.

[282.8 V, 14.14 A]

4.73 Inductor loads of 0.8 kW and 1.2 kW at lagging power factors of 0.8 and 0.6 respectively are connected across a 200 V, 50 Hz supply. Find the total current, power factor and the value of the capacitor to be put in parallel to both to raise the overall power factor to 0.9 lagging.

[14.87 A, 0.673 (lagging), 98 μF]

Objective-Type Questions

Choose the correct alternative in the following questions:

4.1 A current is said to be alternating when it changes in
(a) magnitude only
(b) direction only
(c) both magnitude and direction
(d) neither magnitude nor direction

4.2 An alternating current of 50 Hz frequency and 100 A maximum value is given as
(a) $i = 200 \sin 628\, t$
(b) $i = 100 \sin 314\, t$
(c) $i = 100\sqrt{2} \sin 314\, t$
(d) $i = 100\sqrt{2} \sin 157\, t$

4.3 An alternating current of 50 Hz frequency has a maximum value of 100 A. Its value $\frac{1}{600}$ second after the instant the current is zero will be
(a) 25 A (b) 12.5 A
(c) 50 A (d) 75 A

4.4 A sinusoidal voltage varies from zero to a maximum of 250 V. The voltage at the instant of 60° of the cycle will be
(a) 150 V (b) 216.5 V
(c) 125 V (d) 108.25 V

4.5 An alternating current is given by the expression $i = 200 \sin\left(314t + \frac{\pi}{3}\right)$ amperes.
The maximum value and frequency of the current are
(a) 200 A, 50 Hz (b) $100\sqrt{2}, 50$ Hz
(c) 200 A,100 Hz (d) 200 A, 25 Hz

4.6 The average value of the current $i = 200 \sin t$ from $t = 0$ to $t = \frac{\pi}{2}$ is
(a) 400 π (b) $\frac{400}{\pi}$
(c) $\frac{1}{400}$ (d) $\frac{\pi}{400}$

4.7 The rms value of half-wave rectified current is 50 A. Its rms value for full wave rectification would be
(a) 100 A (b) 70.7 A
(c) $\frac{50}{\pi}$ A (d) $\frac{100}{\pi}$ A

4.8 When the two quantities are in quadrature, the phase angle between them will be
(a) 45° (b) 90°
(c) 135° (d) 60°

4.9 The phase difference between the two waveforms can be compared when they
(a) have the same frequency
(b) have the same peak value
(c) have the same effective value
(d) are sinusoidal.

4.10 If two sinusoids of the same frequency but of different amplitude and phase difference are added, the resultant is a
(a) sinusoid of the same frequency
(b) sinusoid of double the original frequency
(c) sinusoid of half the original frequency
(d) non-sinusoid

4.11 Two alternating currents represented as $i_1 = 4 \sin \omega t$ and $i_2 = 10 \sin\left(\omega t + \frac{\pi}{3}\right)$ are fed into a common conductor. The rms value of the resultant current is
(a) 9.62 A (b) 8.83 A
(c) 12.48 A (d) 13.6 A

4.12 A constant current of 2.8 A exists in a resistor. The rms value of the current is
(a) 2.8 A (b) 2 A
(c) 1.4 A (d) undefined.

4.13 The current through a resistor has a waveform as shown in Fig. 4.141. The reading shown by a moving-coil ammeter will be

Fig. 4.141

(a) $\frac{5}{\sqrt{2}}$ A (b) $\frac{2.5}{\sqrt{2}}$ A (c) $\frac{5}{\pi}$ A (d) 0

4.14 Which of the following statements about the two alternating currents shown in Fig. 4.142 is correct?

Fig. 4.142

(a) The peak values of i_1 and i_2 are different.
(b) The rms values of i_1 and i_2 are different.
(c) The time period of current i_1 is more than that of the current i_2.
(d) The frequency of current i_1 is more than that of the current i_2.

4.15 The alternating voltage $e = 200 \sin 314\, t$ is applied to a device which offers an ohmic resistance of 20 Ω to the flow of current in one direction while entirely preventing the flow in the opposite direction. The overage value of the current will be
(a) 5 A (b) 3.18 A
(c) 1.57 A (d) 1.10 A

4.16 A 10 mH inductor carries a sinusoidal current of 1 A rms at a frequency of 50 Hz. The average power dissipated by the inductor is
(a) 0 (b) 0.25 W
(c) 0.5 W (d) 1 W

4.17 The power factor of an ordinary bulb is
(a) zero (b) unity
(c) slightly more than unity
(d) slightly less than unity

4.18 When a sinusoidal voltage is applied across *RL* series circuit having $R = X_L$, the phase angle will be
(a) 90° (b) 45°lag
(c) 45° lead (d) 90° lead

4.19 An ac source having voltage $e = 110 \sin\left(\omega t + \frac{\pi}{3}\right)$ is connected in an ac circuit. If the current drawn from the circuit varies as $i = 5 \sin\left(\omega t - \frac{\pi}{3}\right)$, the impedance of the circuit will be
(a) 22 Ω (b) 16 Ω
(c) 30.8 Ω (d) none of these

4.20 An ac source of 200 V rms supplies active power of 600 W and reactive power of 800 VAR. The rms current drawn from the source is
(a) 10 A (b) 5 A
(c) 3.75 A (d) 2.5 A

4.21 The voltage phasor of a circuit is $10 \angle 15°$ V and the current phasor is $2 \angle -45°$ A. The active and reactive powers in the circuit are

(a) 10 W and 17.32 VAR
(b) 5 W and 8.66 VAR
(c) 20 W and 60 VAR
(d) $20\sqrt{2}$ and $10\sqrt{2}$ VAR

4.22 In a two element series circuit, the applied voltage and resultant current are respectively $v(t) = 50 + 50 \sin (5 \times 10^3 t)$ and $i(t) = 11.2 \sin (5 \times 10^3 t + 63.4°)$. The nature of the elements would be
(a) *RL* (b) *RC*
(c) *LC* (d) neither *R*, nor *L*, nor *C*

4.23 A boiler at home is switched on to the ac mains supplying power at 230 V, 50 Hz. The frequency of instantaneous power consumed is
(a) 0 (b) 50 Hz
(c) 100 Hz (d) 150 Hz

4.24 In a series *RLC* circuit, $V_R = 3$ V, $V_L = 14$ V, $V_C = 10$ V. The input voltage to the circuit is
(a) 10 V (b) 5 V
(c) 27 V (d) 24 V

4.25 A_1, A_2 and A_3 are ideal ammeters, shown in Fig. 4.143. If A_1 reads 5 A, A_2 reads 12 A then A_3 should read

Fig. 4. 143

(a) 7 A (b) 12 A
(c) 13 A (d) 17 A

4.26 A series *RLC* circuit consisting of $R = 10\ \Omega$, $X_L = 20\ \Omega$ and $X_C = 20\ \Omega$ is connected across an ac supply of 200 V rms. The rms voltage across the capacitor is
(a) $200 \angle -90°$ V
(b) $200 \angle 90°$ V
(c) $400 \angle -90°$ V
(d) $400 \angle 90°$ V

4.27 In Fig. 4.144, A_1, A_2, and A_3, are ideal ammeters. If A_1 and A_3 read 5 A and 13 A respectively, the reading of A_2 will be

Fig. 4.144

(a) 8 A (b) 12 A
(c) 18 A (d) 10 A

4.28 In Fig. 4.145, $i(t)$ under steady state is

Fig. 4.145

(a) zero
(b) 5
(c) 7.07 sin *t*
(d) 7.07 sin(*t* – 45°)

4.29 The circuit shown with $R = \frac{1}{3}\Omega$, $L = \frac{1}{4}$H, $C = 3$ F in Fig. 4.146 has input voltage $v(t) = \sin 2t$. The resulting current $i(t)$ is

Fig. 4.146

(a) 5 sin (2*t* + 53.1°)
(b) 5 sin (2*t* – 53.1°)
(c) 25 sin (2*t* + 53.1°)
(d) 25 sin (2*t* – 53.1°)

Answers to Objective-Type Questions

4.1 (c)	4.2 (b)	4.3 (c)	4.4 (b)	4.5 (a)	4.6 (b)	4.7 (b)	4.8 (b)
4.9 (a)	4.10 (a)	4.11 (c)	4.12 (a)	4.13 (c)	4.14 (c)	4.15 (b)	4.16 (a)
4.17 (d)	4.18 (b)	4.19 (a)	4.20 (b)	4.21 (a)	4.22 (b)	4.23 (c)	4.24 (b)
4.25 (c)	4.26 (c)	4.27 (b)	4.28 (d)	4.29 (a)			

5

Resonance

5.1 INTRODUCTION

Resonance is a very important phenomenon in many electrical applications. The study of resonance is very useful in the telecommunication field. A circuit containing reactance is said to be in resonance if the voltage across the circuit is in phase with the current through it. At resonance, the circuit thus behaves as a pure resistor and the net reactance is zero.

5.2 SERIES RESONANCE

Consider the series *RLC* circuit as shown in Fig. 5.1. The impedance of the circuit is

$$\mathbf{Z} = R + jX_L - jX_C = R + j(X_L - X_C)$$

At resonance, the circuit is purely resistive.

$$X_L - X_C = 0$$

$$X_L = X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

where f_0 is called the resonant frequency of the circuit.

Fig. 5.1 *Series circuit*

1. Power Factor

$$\text{Power factor} = \cos\phi = \frac{R}{Z}$$

At resonance $Z = R$

$$\text{Power factor} = \frac{R}{R} = 1$$

2. Current Since impedance is minimum, the current is maximum at resonance. Thus, the circuit accepts more current and as such, an *RLC* circuit under resonance is called an *accepator circuit*.

$$I_0 = \frac{V}{Z} = \frac{V}{R}$$

3. Voltage At resonance,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 L I_0 = \frac{1}{\omega_0 C} I_0$$

$$V_{L_0} = V_{C_0}$$

Thus, potential difference across inductor equal to potential difference across capacitor being equal and opposite cancel each other. Also, since I_0 is maximum, V_{L_0} and V_{C_0} will also be maximum. Thus, voltage magnification takes place during resonance. Hence, it is also referred to as voltage magnification circuit.

4. Phasor Diagram Figure 5.2 shows the phasor diagram of a series resonant circuit.

Fig. 5.2 *Phasor diagram*

5. Behaviour of R, L and C with Change in Frequency Resistance remains constant with the change in frequencies. Inductive reactance X_L is directly proportional to frequency *f*. It can be drawn as a straight line passing through the origin. Capacitive reactance X_C is inversely proportional to the frequency *f*. It can be drawn as a rectangular hyperbola in the fourth quadrant. Figure 5.3 shows the behaviour of *R*, *L* and *C* with change in frequency.

Fig. 5.3 *Behaviour of R, L and C with change in frequency*

Total impedance $\mathbf{Z} = R + j(X_L - X_C)$

(a) When $f < f_0$, impedance is capacitive and decreases up to f_0. The power factor is leading in nature.
(b) At $f = f_0$, impedance is resistive. The power factor is unity.
(c) When $f > f_0$, impedance is inductive and goes on increasing beyond f_0. The power factor is lagging in nature. Figure 5.4 shows the impedance curve of series resonant circuit.

Fig. 5.4 *Impedance curve*

6. *Bandwidth* For the series *RLC* circuit, bandwidth is defined as the range of frequencies for which the power delivered to *R* is greater than or equal to $\frac{P_0}{2}$ where P_0 is the power delivered to *R* at resonance.

From the shape of the resonance curve shown in Fig. 5.5, it is clear that there are two frequencies for which the power delivered to *R* is half the power at resonance. For this reason, these frequencies are referred as those corresponding to the half-power points. The magnitude of the current at each half-power point is the same.

Fig. 5.5 *Resonance curve*

Hence,
$$I_1^2 R = \frac{1}{2} I_0^2 R = I_2^2 R$$

where the subscript 1 denotes the lower half point and the subscript 2, the higher half point. It follows then that

$$I_1 = I_2 = \frac{I_0}{\sqrt{2}} = 0.707 I_0$$

Accordingly, the bandwidth may be identified on the resonance curve as the range of frequencies over which the magnitude of the current is equal to or greater than 0.707 of the current at resonance. In Fig. 5.5, the bandwidth is $\omega_2 - \omega_1$.

Expression for Bandwidth Generally, at any frequency ω,

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad \ldots\text{(i)}$$

At half-power points,

$$I = \frac{I_0}{\sqrt{2}}$$

But

$$I_0 = \frac{V}{R}$$

$$I = \frac{V}{\sqrt{2}R} \qquad \ldots\text{(ii)}$$

From Eqs (i) and (ii),

$$\frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V}{\sqrt{2}R}$$

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}R$$

Squaring both the sides,

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\omega L - \frac{1}{\omega C} \pm R = 0$$

$$\omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0$$

$$\boldsymbol{\omega} = \pm\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

For low values of R, the term $\left(\frac{R^2}{4L^2}\right)$ can be neglected in comparison with the term $\frac{1}{LC}$.

Then ω is given by,

$$\omega = \pm\frac{R}{2L} \pm \sqrt{\frac{1}{LC}} = \pm\frac{R}{2L} \pm \frac{1}{\sqrt{LC}}$$

The resonant frequency for this circuit is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega = \pm\frac{R}{2L} + \omega_0 \qquad \text{(considering only posititve sign of } \omega_0\text{)}$$

$$\omega_1 = \omega_0 - \frac{R}{2L}$$

and $$\omega_2 = \omega_0 + \frac{R}{2L}$$

$$f_1 = f_0 - \frac{R}{4\pi L}$$

and $$f_2 = f_0 + \frac{R}{4\pi L}$$

$$\text{Bandwidth} = \omega_2 - \omega_1 = \frac{R}{L}$$

or $$\text{Bandwidth} = f_2 - f_1 = \frac{R}{2\pi L}$$

7. *Quality Factor* It is measure of voltage magnification in the series resonant circuit. It is also a measure of selectivity or sharpness of the series resonant circuit.

$$Q_0 = \frac{\text{Voltage across inductor or capacitor}}{\text{Voltage at resonance}} = \frac{V_{L_0}}{V} = \frac{V_{C_0}}{V}$$

Substituting values of V_{L_0} and V,

$$Q_0 = \frac{I_0 X_{L_0}}{I_0 R} = \frac{X_{L_0}}{R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Substituting values of ω_0,

$$Q_0 = \frac{\left(\frac{1}{\sqrt{LC}}\right)L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Example 5.1 *A series RLC circuit has the following parameter values: R = 10 Ω, L = 0.01 H, C = 100 μF. Compute the resonant frequency, bandwidth, and lower and upper frequencies of the bandwidth.*

Solution $R = 10\ \Omega, \quad L = 0.01\ \text{H}, \quad C = 100\ \mu\text{F}$

(a) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01 \times 100 \times 10^{-6}}} = 159.15\ \text{Hz}$$

(b) Bandwidth

$$BW = \frac{R}{2\pi L} = \frac{10}{2\pi \times 0.01} = 159.15\ \text{Hz}$$

(c) Lower frequency of bandwidth

$$f_1 = f_0 - \frac{BW}{2} = 159.15 - \frac{159.15}{2} = 79.58\ \text{Hz}$$

(d) Upper frequency of bandwidth

$$f_2 = f_0 + \frac{BW}{2} = 159.15 + \frac{159.15}{2} = 238.73\ \text{Hz}$$

Example 5.2 *An RLC series circuit with a resistance of 10 Ω, inductance of 0.2 H and a capacitance of 40 μF is supplied with a 100 V supply at variable frequency. Find the following w.r.t. the series resonant circuit:*

(a) frequency of which resonance takes place
(b) current
(c) power
(d) power factor
(e) voltage across RLC at that time
(f) quality factor
(g) half-power points
(h) resonance and phasor diagrams

Solution $R = 10\ \Omega, \quad L = 0.2\text{ H}, \quad C = 40\ \mu\text{F}, \quad V = 100\text{ V}$

(a) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 40 \times 10^{-6}}} = 56.3\text{ Hz}$$

(b) Current

$$I_0 = \frac{V}{R} = \frac{100}{10} = 10\text{ A}$$

(c) Power

$$P_0 = I_0^2 R = (10)^2 \times 10 = 1000\text{ W}$$

(d) Power factor

$$\text{pf} = 1$$

(e) Voltage across R, L, C

$$V_{R_0} = R\,I_0 = 10 \times 10 = 100\text{ V}$$

$$V_{L_0} = X_{L_0}\,I_0 = 2\pi f_0 L I_0 = 2\pi \times 56.3 \times 0.2 \times 10 = 707.5\text{ V}$$

$$V_{C_0} = X_{C_0} I_0 = \frac{1}{2\pi f_0 C} I_0 = \frac{1}{2\pi \times 56.3 \times 40 \times 10^{-6}} \times 10 = 707.5\text{ V}$$

(f) Quality factor

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.2}{40 \times 10^{-6}}} = 7.07$$

(g) Half-power points

$$f_1 = f_0 - \frac{R}{4\pi L} = 56.3 - \frac{10}{4\pi \times 0.2} = 52.32\text{ Hz}$$

$$f_2 = f_0 + \frac{R}{4\pi L} = 56.3 + \frac{10}{4\pi \times 0.2} = 60.3\text{ Hz}$$

(h) Resonance and phasor diagram are shown in Fig. 5.6.

Fig. 5.6

Example 5.3 *A series RLC circuit is connected to a 200 V ac supply. The current drawn by the circuit at resonance is 20 A. The voltage drop across the capacitor is 5000 V at series resonance. Calculate resistance and inductance if capacitance is 4 μF. Also, calculate the resonant frequency.*

Solution $V = 200\text{ V}, \quad I_0 = 20\text{ A}, \quad V_{C_0} = 5000\text{ V}, \quad C = 4\ \mu\text{F}$

(a) Resistance

$$R = \frac{V}{I_0} = \frac{200}{20} = 10\ \Omega$$

(b) Resonant frequency

$$X_{C_0} = \frac{V_{C_0}}{I_0} = \frac{5000}{20} = 250\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$250 = \frac{1}{2\pi \times f_0 \times 4 \times 10^{-6}}$$

$$f_0 = 159.15\text{ Hz}$$

(c) Inductance
At resonance

$$X_{C_0} = X_{L_0} = 250\ \Omega$$

$$X_{L_0} = 2\pi f_0 L$$

$$250 = 2\pi \times 159.15 \times L$$

$$L = 0.25\text{ H}$$

Example 5.4 *A resistor and a capacitor are connected in series with a variable inductor. When the circuit is connected to a 230 V, 50 Hz supply, the maximum current obtained by varying the inductance is 2 A. The voltage across the capacitor is 500 V. Calculate the resistance, inductance and capacitance of the circuit.*

Solution $V = 230\text{ V}, \quad f_0 = 50\text{ Hz}, \quad I_0 = 2\text{ A}, \quad V_{C_0} = 500\text{ V}$

$$I_0 = 2\text{ A}$$
$$V_{C_0} = 500\text{ V}$$

(a) Resistance

$$R = \frac{V}{I_0} = \frac{230}{2} = 115\ \Omega$$

(b) Capacitance

$$X_{C_0} = \frac{V_{C_0}}{I_0} = \frac{500}{2} = 250\ \Omega$$
$$X_{C_0} = \frac{1}{2\pi f_0 C}$$
$$250 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 12.73\ \mu\text{F}$$

(c) Inductance
At resonance

$$X_{C_0} = X_{L_0} = 250\ \Omega$$
$$X_{L_0} = 2\pi f_0 L$$

$$250 = 2\pi \times 50 \times L$$
$$L = 0.795\text{ H}$$

Example 5.5 *A coil of 2 Ω resistance and 0.01 H inductance is connected in series with a capacitor across 200 V mains. What must be the capacitance in order that maximum current occurs at a frequency of 50 Hz? Find also the current and the voltage across the capacitor.*

Solution $R = 2\ \Omega, \quad L = 0.01\text{ H}, \quad V = 200\text{ V}, \quad f_0 = 50\text{ Hz}$

(a) Capacitance

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$50 = \frac{1}{2\pi\sqrt{0.01 \times C}}$$
$$C = 1013.2\ \mu\text{F}$$

(b) Current

$$I_0 = \frac{V}{R} = \frac{200}{2} = 100\text{ A}$$

(c) Voltage across capacitor

$$V_{C_0} = I_0 X_{C_0} = I_0 \frac{1}{2\pi f_0 C} = 100 \times \frac{1}{2\pi \times 50 \times 1013.2 \times 10^{-6}} = 314.16 \text{ V}$$

Example 5.6 *A series RLC circuit has a quality factor of 5 at 50 rad/s. The current flowing through the circuit at resonance is 10 A and the supply voltage is 100 V. Find the circuit constants.*

Solution

$$Q_0 = 5, \quad \omega_0 = 50 \text{ rad/s}, \quad I_0 = 10 \text{ A}, \quad V = 100 \text{ V}$$

$$I_0 = \frac{V}{R}$$

$$10 = \frac{100}{R}$$

$$R = 10\ \Omega$$

$$Q_0 = \frac{\omega_0 L}{R}$$

$$5 = \frac{50 \times L}{10}$$

$$L = 1 \text{ H}$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$5 = \frac{1}{10}\sqrt{\frac{1}{C}}$$

$$C = 400\ \mu\text{F}$$

Example 5.7 *A series RLC circuit has R = 2.5 Ω, C = 100 μF and a variable inductance. The applied voltage is 50 V at 800 rad/s. The inductance is varied till the voltage across the resistance is maximum. Find (a) value of inductance under this condition, (b) Q, (c) current, and (d) voltage across resistance, capacitance and inductance.*

Solution

$$R = 2.5\ \Omega, \quad C = 100\ \mu\text{F}, \quad V = 50 \text{ V}, \quad \omega = 800 \text{ rad/s}$$

(a) Value of inductance

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$800 = \frac{1}{\sqrt{L \times 100 \times 10^{-6}}}$$

$$L = 0.0156 \text{ H}$$

(b) Quality factor

$$Q_0 = \frac{\omega_0 L}{R} = \frac{800 \times 0.0156}{2.5} = 5$$

(c) Current

$$I_0 = \frac{V}{R} = \frac{50}{2.5} = 20 \text{ A}$$

(d) Voltage across resistance, capacitance and inductance

$$V_{R_0} = I_0 \; R = 20 \times 2.5 = 50 \text{ V}$$
$$V_{L_0} = Q_0 \; V = 5 \times 50 = 250 \text{ V}$$
$$V_{C_0} = Q_0 \; V = 5 \times 50 = 250 \text{ V}$$

Example 5.8 *A series RLC circuit which resonates at 500 kHz has R = 25 Ω, L = 100 μH and C = 1000 pF. Determine the quality factor, new value of C required to resonate at 500 kHz when the value of L is doubled, and the new quality factor.*

Solution $f_0 = 500 \text{ kHz}, \quad R = 25\ \Omega, \quad L = 100\ \mu\text{H}, \quad C = 1000 \text{ pF}$

(a) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{25}\sqrt{\frac{100 \times 10^{-6}}{1000 \times 10^{-12}}} = 12.65$$

(b) New value of C when $L = 200\ \mu\text{H}$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$500 \times 10^3 = \frac{1}{2\pi\sqrt{200 \times 10^{-6} \times C}}$$
$$C = 506.61 \text{ pF}$$

(c) New Q factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{25}\sqrt{\frac{200 \times 10^{-6}}{506.61 \times 10^{-12}}} = 25.13$$

Example 5.9 *Find the values of R,L and C in a series RLC circuit that resonates at 1.5 kHz and consumes 50 W from a 50 V ac source operating at the resonant frequency. The bandwidth is 0.75 kHz.*

Solution $f_0 = 1.5 \text{ kHz}, \quad P = 50 \text{ W}, \quad V = 50 \text{ V}, \quad BW = 0.75 \text{ kHz}$

$$P = \frac{V^2}{R}$$
$$50 = \frac{(50)^2}{R}$$
$$R = 50\ \Omega$$

$$BW = \frac{R}{2\pi L}$$

$$0.75\times10^3 = \frac{50}{2\pi\times L}$$

$$L = 0.011\text{ H}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$1.5\times10^3 = \frac{1}{2\pi\sqrt{0.011\times C}}$$

$$C = 1.023\ \mu\text{F}$$

Example 5.10 *A 50 Ω resistor is connected in series with an inductor having internal resistance, a capacitor and 100 V variable frequency supply as shown in Fig. 5.7. At a frequency of 200 Hz, a maximum current of 0.7 A flows through the circuit and voltage across the capacitor is 200 V. Determine the circuit constants.*

Fig. 5.7

Solution

$$V = 100\text{ V},\quad R = 50\ \Omega,\quad f_0 = 200\text{ Hz},\quad I_0 = 0.7\text{ A}$$

$$V_{C_0} = 200\text{ V}$$

$$V_{C_0} = I_0 X_{C_0}$$

$$200 = 0.7 X_{C_0}$$

$$X_{C_0} = 285.71\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$285.71 = \frac{1}{2\pi\times200\times C}$$

$$C = 2.79\ \mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$200 = \frac{1}{2\pi\sqrt{L\times2.79\times10^{-6}}}$$

$$L = 0.23\text{ H}$$

At resonance, total resistance is

$$R_T = \frac{V}{I} = \frac{100}{0.7} = 142.86\ \Omega$$
$$R_T = R + r$$
$$142.86 = 50 + r$$
$$r = 92.86\ \ \Omega$$

Example 5.11 *In the circuit shown in Fig. 5.8, a maximum current of 0.1 A flows through the circuit when the capacitor is at 5 μF with a fixed frequency and a voltage of 5 V. Determine the frequency at which the circuit resonates, the bandwidth, quality factor Q and the value of resistance at resonant frequency.*

Fig. 5.8

Solution

$$I_0 = 0.1\text{ A},\quad C = 5\ \mu\text{F},\quad V = 5\text{ V},\quad L = 0.1\text{ H}$$

$$R = \frac{V}{I_0} = \frac{5}{0.1} = 50\ \Omega$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times5\times10^{-6}}} = 225.08\text{ Hz}$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{50}\sqrt{\frac{0.1}{5\times10^{-6}}} = 2.83$$

$$BW = \frac{R}{2\pi L} = \frac{50}{2\pi\times0.1} = 79.58\text{ Hz}$$

Example 5.12 *A series RLC circuit has R = 10 Ω and L = 60 mH. At a frequency of 25 Hz, the pf of the circuit is 45° lead. At what frequency will the circuit be resonant?*

Solution

$$R = 10\ \Omega,\quad L = 60\text{ mH}$$
$$X_L = 2\pi fL = 2\pi\times25\times60\times10^{-3} = 9.42\ \Omega$$

For a series *RLC* circuit,

$$\tan\phi = \frac{X_C - X_L}{R}$$
$$\tan 45° = \frac{X_C - 9.42}{10}$$

$$1 = \frac{X_C - 9.42}{10}$$

$$X_C = 19.42\ \Omega$$

$$X_C = \frac{1}{2\pi fC}$$

$$19.42 = \frac{1}{2\pi \times 25 \times C}$$

$$C = 327.82\ \mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{60\times10^{-3}\times327.82\times10^{-6}}} = 35.89\ \text{Hz}$$

Example 5.13 *A voltage $v(t) = 10 \sin \omega t$ is applied to a series RLC circuit. At the resonant frequency of the circuit, the voltage across the capacitor is found to be 500 V. The bandwidth of the circuit is known to be 400 rad/s and the impedance of the circuit at resonance is 100 Ω. Determine inductance and capacitance, resonant frequency, upper and lower cut-off frequencies.*

Solution $v(t) = 10 \sin \omega t, \quad V_{C_0} = 500\ \text{V}, \quad BW = 400\ \text{rad/s}, \quad R = 100\ \Omega$

(a) Inductance and capacitance

$$V = \frac{V_m}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.07\ \text{V}$$

$$I_0 = \frac{V}{R} = \frac{7.07}{100} = 0.0707\ \text{A}$$

$$BW = \frac{R}{L}$$

$$400 = \frac{100}{L}$$

$$L = 0.25\ \text{H}$$

$$Q_0 = \frac{V_{C_0}}{V} = \frac{500}{7.07} = 70.72$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$70.72 = \frac{1}{100}\sqrt{\frac{0.25}{C}}$$

$$C = 4.99\ \text{nF}$$

(b) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\times\sqrt{0.25\times4.99\times10^{-9}}} = 4506.09\ \text{Hz}$$

(c) Lower cut-off frequency

$$f_1 = f_0 - \frac{R}{4\pi L} = 4506.09 - \frac{100}{4\pi \times 0.25} = 4474.26 \text{ Hz}$$

(d) Upper cut-off frequency

$$f_2 = f_0 + \frac{R}{4\pi L} = 4506.09 + \frac{100}{4\pi \times 0.25} = 4537.92 \text{ Hz}$$

Example 5.14 *A series resonant circuit has an impedance of 500 Ω at resonant frequency. Cut-off frequencies are 10 kHz and 100 Hz. Determine (a) resonant frequency, (b) value of L, C, and (c) quality factor at resonant frequency.*

Solution $R = 500\ \Omega, \quad f_1 = 100 \text{ Hz}, \quad f_2 = 10 \text{ kHz}$

(a) Resonant frequency

$$BW = f_2 - f_1 = 10000 - 100 = 9900 \text{ Hz}$$

$$f_1 = f_0 - \frac{R}{4\pi L} \qquad \text{...(i)}$$

$$f_2 = f_0 + \frac{R}{4\pi L} \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii),

$$f_1 + f_2 = 2f_0$$

$$f_0 = \frac{f_1 + f_2}{2} = \frac{100 + 10000}{2} = 5050 \text{ Hz}$$

(b) Values of L and C

$$BW = \frac{R}{2\pi L}$$

$$9900 = \frac{500}{2\pi L}$$

$$L = 8.038 \text{ mH}$$

$$X_{L_0} = 2\pi f_0 L = 2\pi \times 5050 \times 8.038 \times 10^{-3} = 255.05\ \Omega$$

At resonance $X_{L_0} = X_{C_0} = 255.05\ \Omega$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$255.05 = \frac{1}{2\pi \times 5050 \times C}$$

$$C = 0.12\ \mu\text{F}$$

(c) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{500}\sqrt{\frac{8.038 \times 10^{-3}}{0.12 \times 10^{-6}}} = 0.5176$$

Example 5.15 *Impedance of a circuit is observed to be capacitive and decreasing from 1 Hz to 100 Hz. Beyond 100 Hz, the impedance starts increasing. Find the values of circuit elements if the power drawn by this circuit is 100 W at 100 Hz, when the current is 1 A. The power factor of the circuit at 70 Hz is 0.707.*

Solution $f_0 = 100\text{ Hz}, \quad P_0 = 100\text{ W}, \quad I_0 = 1\text{A}, \quad (\text{pf})_{70\text{ Hz}} = 0.707$

The impedance of the circuit is capacitive and decreasing from 1 Hz to 100 Hz. Beyond 100 Hz, the impedance starts increasing.

$$f_0 = 100\text{ Hz}$$
$$P_0 = I_0^2 R$$
$$100 = (1)^2 \times R$$
$$R = 100\ \Omega$$
$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$100 = \frac{1}{2\pi\sqrt{LC}}$$
$$LC = 2.53 \times 10^{-6} \qquad \ldots\text{(i)}$$

Power factor at 70 Hz is 0.707.

$$\frac{R}{Z} = 0.707$$
$$\frac{100}{Z} = 0.707$$
$$Z = 141.44\ \Omega$$

$$\text{Impedance at 70 Hz} = Z_{70} = \sqrt{R^2 + (X_C - X_L)^2}$$
$$141.44 = \sqrt{(100)^2 + \left(\frac{1}{2\pi \times 70 \times C} - 2\pi \times 70 \times L\right)^2}$$
$$\frac{2.27 \times 10^{-3}}{C} - 439.82\ L = 100.02 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii),

$$L = 0.2187\text{ H}$$
$$C = 11.58\ \mu\text{F}$$

Example 5.16 *A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor. When the capacitor is set to 500 pF, the current has its maximum value while it is reduced to one-half when the capacitance is 600 pF. Find resistance, inductance and Q-factor of inductor.*

Solution $f_0 = 1\text{ MHz}, \quad C_1 = 500\text{ pF}, \quad C_2 = 600\text{ pF}$

(a) Resistance and inductance of inductor

At resonance

$$C = 500\text{ pF}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$10^6 = \frac{1}{2\pi\sqrt{L \times 500 \times 10^{-12}}}$$

$$L = 0.05\text{ mH}$$

$$X_L = 2\pi f_0 L = 2\pi \times 10^6 \times 0.05 \times 10^{-3} = 314.16\ \Omega$$

When capacitance is 600 pF, the current reduces to one-half of the current at resonance.

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 10^6 \times 600 \times 10^{-12}} = 265.26\ \Omega$$

$$I = \frac{1}{2} I_0$$

$$\frac{V}{Z} = \frac{1}{2}\frac{V}{R}$$

$$Z = 2R$$

$$\sqrt{R^2 + (X_L - X_C)^2} = 2R$$

$$R^2 + (314.16 - 265.26)^2 = 4R^2$$

$$3R^2 = 2391.21$$

$$R = 28.23\ \Omega$$

(b) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{28.23}\sqrt{\frac{0.05 \times 10^{-3}}{500 \times 10^{-12}}} = 11.2$$

Example 5.17 *In the circuit shown in Fig. 5.9, determine the circuit constants when the circuit draws a maximum current at 10 μF with a 10 V, 100 Hz supply. When the capacitance is changed to 12 μF, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/s. Also find the maximum current that flows through the circuit.*

Fig. 5.9

Solution

$$C = 10\ \mu\text{F}, \quad V = 10\ \text{V}, \quad f_0 = 100\ \text{Hz}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$100 = \frac{1}{2\pi\sqrt{L \times 10 \times 10^{-6}}}$$

$$L = 0.25\ \text{H}$$

When $C = 12\ \mu\text{F}$, $\omega = 900$ rad/s

$$I = 0.707\ I_0$$

$$\frac{V}{Z} = 0.707\ \frac{V}{R}$$

$$Z = \frac{R}{0.707}$$

$$\sqrt{R^2 + (X_L - X_C)^2} = \frac{R}{0.707}$$

$$\sqrt{R^2 + \left(900 \times 0.25 - \frac{1}{900 \times 12 \times 10^{-6}}\right)^2} = \frac{R}{0.707}$$

$$R = 132.37\ \Omega$$

$$Q_0 = \frac{\omega L}{R} = \frac{900 \times 0.25}{132.37} = 1.7$$

$$I_0 = \frac{V}{R} = \frac{10}{132.37} = 0.076\ \text{A}$$

Example 5.18 *A series RLC circuit is excited from a constant voltage variable frequency source. The current in the circuit becomes maximum at a frequency of* $\frac{600}{2\pi}$ *Hz and falls to half the maximum value at* $\frac{400}{2\pi}$ *Hz. If the resistance in the circuit is 3Ω, find L and C.*

Solution

$$f_0 = \frac{600}{2\pi}\ \text{Hz}, \quad R = 3\ \Omega$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$\frac{600}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$LC = 2.78 \times 10^{-6} \qquad \ldots\text{(i)}$$

At $f = \frac{400}{2\pi}$ Hz,

$$I = \frac{I_0}{2}$$

$$\frac{V}{Z} = \frac{V}{2R}$$

$$Z = 2R$$

$$\sqrt{R^2 + (X_L - X_C)^2} = 2R$$

$$R^2 + (X_L - X_C)^2 = 4R^2$$

$$(X_L - X_C)^2 = 3R^2 = 3(3)^2 = 27$$

$$\left(2\pi \times \frac{400}{2\pi} \times L - \frac{1}{2\pi \times \frac{400}{2\pi} \times C}\right)^2 = 27$$

$$\left(400\ L - \frac{1}{400C}\right)^2 = 27 \qquad \ldots\text{(ii)}$$

$$\left(400 \times \frac{2.78 \times 10^{-6}}{C} - \frac{1}{400\ C}\right)^2 = 27$$

$$C = 267.12\ \mu\text{F}$$

$$L = \frac{2.78 \times 10^{-6}}{C} = \frac{2.78 \times 10^{-6}}{267.12 \times 10^{-6}} = 10.41\ \text{mH}$$

5.3 PARALLEL RESONANCE

Consider a parallel circuit consisting of a coil and a capacitor as shown in Fig. 5.10. The impedances of two branches are

Fig. 5.10 *Parallel circuit*

$$\mathbf{Z}_1 = R + jX_L$$

$$\mathbf{Z}_2 = -jX_C$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{R + jX_L} = \frac{R - jX_L}{R^2 + X_L^2}$$

$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} = \frac{1}{-jX_C} = \frac{j}{X_C}$$

Admittance of the circuit $\quad \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \dfrac{R - jX_L}{R^2 + X_L^2} + \dfrac{j}{X_C} = \dfrac{R}{R^2 + X_L^2} - j\left(\dfrac{X_L}{R^2 + X_L^2} - \dfrac{1}{X_C}\right)$

At resonance, the circuit is purely resistive. Hence, the condition for resonance is

$$\frac{X_L}{R^2 + X_L^2} - \frac{1}{X_C} = 0$$

$$\frac{X_L}{R^2 + X_L^2} = \frac{1}{X_C}$$

$$X_L X_C = R^2 + X_L^2$$

$$\omega_0 L \frac{1}{\omega_0 c} = R^2 + \omega_0^2 L^2$$

$$\frac{L}{C} = R^2 + \omega_0^2 L^2$$

$$\omega_0^2 L^2 = \frac{L}{C} - R^2$$

$$\omega_0{}^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

where f_0 is called the resonant frequency of the circuit.
If R is very small as compared to L then

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

1. *Dynamic Impedance of a Parallel Circuit* At resonance, the circuit is purely resistive. The real part of admittance is $\frac{R}{R^2 + X_L^2}$. Hence, the dynamic impedance at resonance is given by

$$Z_D = \frac{R^2 + X_L^2}{R}$$

At resonance,

$$R^2 + X_L^2 = X_L X_C = \frac{L}{C}$$

$$Z_D = \frac{L}{CR}$$

2. *Current* Since impedance is maximum at resonance, the current is minimum at resonance.

$$I_0 = \frac{V}{Z_D} = \frac{V}{\frac{L}{CR}} = \frac{VCR}{L}$$

3. *Phasor Diagram* At resonance, power factor of the circuit is unity and the total current drawn by the circuit is in phase with the voltage. This will happen only when the current I_C is equal to the reactive component of the current in the inductive branch, i.e., $I_C = I_L \sin\phi$

Fig. 5.11 *Phasor diagram*

Hence, at resonance

$$I_C = I_L \sin\phi$$

and

$$I = I_L \cos\phi$$

The phasor diagram of parallel resonant circuit is shown in Fig. 5.11.

4. *Behaviour of Conductance G, Inductive Susceptance B_L and Capacitive Susceptance with Change in Frequency* Conductance remains constant with the change in frequencies. Inductive susceptance B_L is

$$B_L = \frac{1}{jX_L} = -j\frac{1}{X_L} = -j\frac{1}{2\pi fL}$$

It is inversely proportional to the frequency. Thus, it decreases with the increase in the frequency. Hence, it can be drawn as a rectangular hyperbola in the fourth quadrant.

Capacitive susceptance B_C is

$$B_C = \frac{1}{-jX_C} = -j\frac{1}{X_C} = j2\pi fC$$

It is directly proportional to the frequency. It can be drawn as a straight line passing through the origin. Figure 5.12 shows the behaviour of G, B_L and B_C with change in frequency.

Fig. 5.12 *Behaviour of G, B_L and B_c with change in frequency*

(a) When $f < f_0$, inductive susceptance predominates. Hence, the current lags behind the voltage and the power factor is lagging in nature.

(b) When $f = f_0$, net susceptance is zero. Hence, the admittance is minimum and impedance is maximum. At f_0, the current is in phase with the voltage and the power factor is unity.

(c) When $f > f_0$, capacitive susceptance predominates. Hence, the current leads the voltage and power factor is leading in nature.

Fig. 5.13 *Impedance curve*

Figure 5.13 shows the impedance curve of a parallel resonant circuit.

5. *Bandwidth* The bandwidth of a parallel resonant circuit is defined in the same way as that for a series resonant circuit. Figure 5.14 shows the resonance curve of a parallel resonant circuit.

6. *Quality Factor* It is a measure of current magnification in a parallel resonant circuit.

$$Q_0 = \frac{\text{Current through inductor or capacitor}}{\text{Current at resonance}} = \frac{I_{C_0}}{I_0}$$

Fig. 5.14 *Resonance curve*

Substituting values of I_{C_0} and I_0,

$$Q_0 = \frac{\dfrac{V}{X_{C_0}}}{\dfrac{VCR}{L}} = \frac{\dfrac{1}{X_{C_0}}}{\dfrac{CR}{L}} = \frac{\omega_0 C}{\dfrac{CR}{L}} = \frac{\omega_0 L}{R}$$

Neglecting the resistance R, the resonant frequency ω_0 is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q_0 = \frac{\left(\dfrac{1}{\sqrt{LC}}\right)L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

5.4 COMPARISON OF SERIES AND PARALLEL RESONANT CIRCUITS

Parameter	Series Circuit	Parallel Circuit
Current at resonance	$I = \dfrac{V}{R}$ and is maximum	$I = \dfrac{VCR}{L}$ and is minimum
Impedance at resonance	$Z = R$ and is minimum	$Z = \dfrac{L}{CR}$ and is maximum
Power factor at resonance	Unity	Unity
Resonant frequency	$f_0 = \dfrac{1}{2\pi\sqrt{LC}}$	$f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$
Q-factor	$Q = \dfrac{2\pi f_0 L}{R}$	$Q = \dfrac{2\pi f_0 L}{R}$
It magnifies	Voltage across L and C	Current through L and C

Example 5.19 *A coil having an inductance of L henries and a resistance of 12 Ω is connected in parallel with a variable capacitor. At $\omega = 2.3 \times 10^6$ rad/s, resonance is achieved and at this instant, capacitance $C = 0.021\ \mu F$. Find the inductance of the coil.*

Solution

$$R = 12\ \Omega, \quad \omega_0 = 2.3 \times 10^6 \text{ rad/s}, \quad C = 0.021\ \mu\text{F}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$2.3\times10^6 = \sqrt{\frac{1}{L\times0.021\times10^{-6}} - \frac{(12)^2}{L^2}}$$

$$L = 89.7\ \mu\text{H}$$

Example 5.20 *A coil of 20 Ω resistance has an inductance of 0.2 H and is connected in parallel with a condenser of 100 μF capacitance. Calculate the frequency at which this circuit will be have as a non-inductive resistance. Find also the value of dynamic resistance.*

Solution $R = 20\ \Omega,\quad L = 0.2\ \text{H},\quad C = 100\ \mu\text{F}$

(a) Resonant frequency

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{0.2\times100\times10^{-6}} - \left(\frac{20}{0.2}\right)^2} = 31.83\ \text{Hz}$$

(b) Dynamic resistance

$$Z_D = \frac{L}{CR} = \frac{0.2}{100\times10^{-6}\times20} = 100\ \Omega$$

Example 5.21 *In the circuit shown in Fig. 5.15, an inductance at 0.1 H having a Q_0 of 5 is in parallel with a capacitor. Determine the value of capacitance and coil resistance at a resonant frequency of 500 rad/s.*

Fig. 5.15

Solution $L = 0.1\ \text{H},\quad Q_0 = 5,\quad \omega_0 = 500\ \text{rad/s}$

$$Q_0 = \frac{\omega_0 L}{R}$$

$$5 = \frac{500\times0.1}{R}$$

$$R = 10\ \Omega$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$500 = \sqrt{\frac{1}{0.1\times C} - \frac{(10)^2}{(0.1)^2}}$$

$$C = 38.46\ \mu\text{F}$$

Example 5.22 *A coil of 10 Ω resistance and 0.2 H inductance is connected in parallel with a variable capacitor across a 220 V, 50 Hz supply. Calculate (a) the capacitance of the capacitor for resonance, (b) the dynamic impedance of the circuit, and (c) supply current.*

Solution $R = 10\ \Omega, \quad L = 0.2\ \text{H}, \quad V = 220\ \text{V}, \quad f_0 = 50\ \text{Hz}$

(a) Capacitance of the capacitor

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$50 = \frac{1}{2\pi}\sqrt{\frac{1}{0.2 \times C} - \frac{(10)^2}{(0.2)^2}}$$

$$C = 49.41\ \mu\text{F}$$

(b) Dynamic impedance of the circuit

$$Z_D = \frac{L}{CR} = \frac{0.2}{49.41 \times 10^{-6} \times 10} = 404.78\ \Omega$$

(c) Supply current

$$I = \frac{V}{Z_D} = \frac{220}{404.78} = 0.543\ \text{A}$$

Example 5.23 *Find the value of the dynamic impedance of the circuit shown in Fig. 5.16 at a frequency of 500 kHz and for bandwidth of operation equal to 20 kHz. The resistance of the coil is 5 Ω.*

Fig. 5.16

Solution $f_0 = 500\ \text{kHz}, \quad BW = 20\ \text{kHz}, \quad R = 5\ \Omega$

$$BW = \frac{R}{2\pi L}$$

$$20 \times 10^3 = \frac{5}{2\pi L}$$

$$L = 39.79\ \mu\text{H}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$500 \times 10^3 = \frac{1}{2\pi}\sqrt{\frac{1}{39.79 \times 10^{-6} \times C} - \frac{(5)^2}{(39.79 \times 10^{-6})^2}}$$

$$C = 2.54 \text{ nF}$$

$$Z_D = \frac{L}{CR} = \frac{39.79 \times 10^{-6}}{2.54 \times 10^{-9} \times 5} = 3.13 \text{ k } \Omega$$

Example 5.24 *A coil having a resistance of 20 Ω and an inductance of 200 μH is connected in parallel with a variable capacitor. This parallel combination is connected in series with a resistance of 8000 Ω. A voltage of 230 V at a frequency of 10^6 Hz is applied across the circuit as shown in Fig. 5.17. Calculate (a) the value of capacitance at resonance, (b) Q factor of the circuit, (c) dynamic impedance of the circuit, and (d) total circuit current.*

Fig. 5.17

Solution $R = 20\ \Omega$, $L = 200\ \mu\text{H}$, $f_0 = 10^6$ Hz, $V = 230$ V, $R_S = 8000\ \Omega$

(a) Value of capacitance

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$10^6 = \frac{1}{2\pi}\sqrt{\frac{1}{200 \times 10^{-6} \times C} - \frac{(20)^2}{(200 \times 10^{-6})^2}}$$

$$C = 126.65 \text{ pF}$$

(b) Quality factor

$$Q_0 = \frac{2\pi f_0 L}{R} = \frac{2\pi \times 10^6 \times 200 \times 10^{-6}}{20} = 62.83$$

(c) Dynamic impedance

$$Z_D = \frac{L}{CR} = \frac{200 \times 10^{-6}}{126.65 \times 10^{-12} \times 20} = 78958\ \Omega$$

(d) Current

Total impedance of the circuit at resonance $= Z_D + R_S = 78958 + 8000 = 86958\ \Omega$

$$I = \frac{230}{86958} = 2.65 \text{ mA}$$

Example 5.25 *Derive the expression for resonant frequency for the parallel circuit shown in Fig. 5.18. Also calculate the impedance and current at resonance.*

Fig. 5.18

Solution $\mathbf{Z}_1 = R, \quad \mathbf{Z}_2 = jX_L, \quad \mathbf{Z}_3 = -jX_C$

(a) Resonant frequency of the circuit
For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}$$

$$\mathbf{Y} = \frac{1}{R} + \frac{1}{jX_L} + \frac{1}{-jX_C} = \frac{1}{R} - j\frac{1}{X_L} + j\frac{1}{X_C} = \frac{1}{R} - j\left(\frac{1}{X_L} - \frac{1}{X_C}\right)$$

At resonance, the circuit is purely resistive. Hence, the condition for resonance is

$$\frac{1}{X_L} - \frac{1}{X_C} = 0$$

$$X_L = X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

where f_0 is called the resonant frequency of the circuit.

(b) Impedance at resonance.
At resonance, the circuit is purely resistive. Hence, the imaginary part of $\mathbf{Y}$ is zero.

$$Y_D = \frac{1}{R}$$

$$Z_D = R$$

(c) Current at resonance

$$I_0 = \frac{V}{Z_D} = \frac{V}{R}$$

Example 5.26 *Derive the expression for the resonant frequency of the parallel circuit as shown in Fig. 5.19.*

Fig. 5.19

Solution

$$\mathbf{Z}_1 = R_1 + jX_L$$
$$\mathbf{Z}_2 = R_2 - jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{R_1 + jX_L} + \frac{1}{R_2 - jX_C} = \frac{R_1 - jX_L}{R_1^2 + X_L^2} + \frac{R_2 + jX_C}{R_2^2 + X_C^2} = \frac{R_1}{R_1^2 + X_L^2} + \frac{R_2}{R_2^2 + X_C^2} - j\left(\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2}\right)$$

At resonance, the circuit is purely resistive. Hence, the condition for resonance is

$$\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2} = 0$$

$$\frac{X_L}{R_1^2 + X_L^2} = \frac{X_C}{R_2^2 + X_C^2}$$

$$\frac{\omega_0 L}{R_1^2 + \omega_0^2 L^2} = \frac{\dfrac{1}{\omega_0 C}}{R_2^2 + \dfrac{1}{\omega_0^2 C^2}}$$

$$\frac{\omega_0^2 LC}{R_1^2 + \omega_0^2 L^2} = \frac{\omega_0^2 C^2}{R_2^2 \omega_0^2 C^2 + 1}$$

$$LC(R_2^2 \omega_0^2 C^2 + 1) = C^2 (R_1^2 + \omega_0^2 L^2)$$

$$\omega_0^2 R_2^2 LC^3 + LC = C^2 R_1^2 + \omega_0^2 L^2 C^2$$

$$\omega_0^2 R_2^2 LC^3 - \omega_0^2 L^2 C^2 = C^2 R_1^2 - LC$$

$$\omega_0^2 LC^2(CR_2^2 - L) = C(CR_1^2 - L)$$

$$\omega_0^2 LC(CR_2^2 - L) = CR_1^2 - L$$

$$\omega_0^2 = \frac{CR_1^2 - L}{LC(CR_2^2 - L)}$$

$$\omega_0 = \sqrt{\frac{CR_1^2 - L}{LC(CR_2^2 - L)}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{CR_1^2 - L}{CR_2^2 - L}}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{CR_1^2 - L}{CR_2^2 - L}}$$

where f_0 is called the resonant frequency of the circuit.

Example 5.27 *Derive the expression for resonant frequency for the parallel circuit shown in Fig. 5.20.*

Fig. 5.20

Solution $\mathbf{Z}_1 = r + jX_L, \quad \mathbf{Z}_2 = R, \quad \mathbf{Z}_3 = -jX_C$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}$$

$$\mathbf{Y} = \frac{1}{r + jX_L} + \frac{1}{R} + \frac{1}{-jX_C} = \frac{r - jX_L}{r^2 + X_L{}^2} + \frac{1}{R} + j\frac{1}{X_C} = \left(\frac{r}{r^2 + X_L{}^2} + \frac{1}{R}\right) + j\left(\frac{1}{X_C} - \frac{X_L}{r^2 + X_L{}^2}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{1}{X_C} - \frac{X_L}{r^2 + X_L{}^2} = 0$$

$$\frac{1}{X_C} = \frac{X_L}{r^2 + X_L{}^2}$$

$$X_L\, X_C = r^2 + X_L{}^2$$

$$\omega_0 L \frac{1}{\omega_0 C} = r^2 + \omega_0{}^2 L^2$$

$$\frac{L}{C} = r^2 + \omega_0{}^2 L^2$$

$$\omega_0{}^2 L^2 = \frac{L}{C} - r^2$$

$$\omega_0{}^2 = \frac{1}{LC} - \frac{r^2}{L^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{r^2}{L^2}}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{r^2}{L^2}}$$

where f_0 is called the resonant frequency of the circuit.

Example 5.28 *Find the value of R in the circuit shown in Fig. 5.21 to achieve resonance.*

Fig. 5.21

Solution

$$\mathbf{Z}_1 = 10 + j10\ \Omega, \quad \mathbf{Z}_2 = R - j2\ \Omega$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{10 + j10} + \frac{1}{R - j2} = \frac{10 - j10}{100 + 100} + \frac{R + j2}{R^2 + 4} = \left(\frac{10}{200} + \frac{R}{R^2 + 4}\right) - j\left(\frac{10}{200} - \frac{2}{R^2 + 4}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{10}{200} - \frac{2}{R^2 + 4} = 0$$

$$\frac{2}{R^2 + 4} = \frac{1}{20}$$

$$R^2 + 4 = 40$$

$$R^2 = 36$$

$$R = 6\ \Omega$$

Example 5.29 *Find the value of R_1 such that the circuit given in Fig. 5.22 is in resonance.*

Fig. 5.22

Solution

$$\mathbf{Z}_1 = R_1 + j6\ \Omega, \quad \mathbf{Z}_2 = 10 - j4\ \Omega$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{R_1 + j6} + \frac{1}{10 - j4} = \frac{R_1 - j6}{R_1^2 + 36} + \frac{10 + j4}{100 + 16} = \left(\frac{R_1}{R_1^2 + 36} + \frac{10}{116}\right) - j\left(\frac{6}{R_1^2 + 36} - \frac{4}{116}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{6}{R_1^2 + 36} - \frac{4}{116} = 0$$

$$\frac{6}{R_1^2 + 36} = \frac{4}{116}$$

$$R_1^2 + 36 = \frac{116 \times 6}{4} = 174$$

$$R_1^2 = 138$$

$$R_1 = 11.75\ \Omega$$

Example 5.30 *Find the value of C in the circuit shown in Fig. 5.23 to give resonance.*

Fig. 5.23

Solution

$$\mathbf{Z}_1 = 20 + j37.7\ \Omega, \quad \mathbf{Z}_2 = 10 - jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{20 + j37.7} + \frac{1}{10 - jX_C} = \frac{20 - j37.7}{400 + 1421.29} + \frac{10 + jX_C}{100 + X_C^2}$$

$$= \left(\frac{20}{1821.29} + \frac{10}{100 + X_C^2}\right) - j\left(\frac{37.7}{1821.29} - \frac{X_C}{100 + X_C^2}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{37.7}{1821.29} - \frac{X_C}{100 + X_C^2} = 0$$

$$\frac{X_C}{100 + X_C^2} = 0.021$$

$$0.021\, X_C^2 - X_C + 2.1 = 0$$

Solving the quadratic equation,

$$X_C = 45.41\ \Omega \quad \text{or} \quad X_C = 2.2\ \Omega$$

$$\frac{1}{2\pi \times 50 \times C} = 45.41 \quad \text{or} \quad \frac{1}{2\pi \times 50 \times C} = 2.2$$

$$C = 70.1\ \mu\text{F} \quad \text{or} \quad C = 1.44\ \text{mF}$$

Example 5.31 *Find the value of L at which the circuit resonates at a frequency of 1000 rad/s in the circuit shown in Fig. 5.24.*

Fig. 5.24

Solution $\mathbf{Z}_1 = 5 + jX_L\ \Omega, \quad \mathbf{Z}_2 = 10 - j12\ \Omega$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{5 + jX_L} + \frac{1}{10 - j12} = \frac{5 - jX_L}{25 + X_L^2} + \frac{10 + j12}{100 + 144} = \left(\frac{5}{25 + X_L^2} + \frac{10}{244}\right) - j\left(\frac{X_L}{25 + X_L^2} - \frac{12}{244}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{X_L}{25+X_L^2}-\frac{12}{244}=0$$

$$\frac{X_L}{25+X_L^2}=\frac{12}{244}$$

$$12X_L^2-244X_L+300=0$$

Solving the quadratic equation,

$$X_L=19.01 \quad \text{or} \quad X_L=1.31\,\Omega$$

$$2\pi\times 50\times L=19.01 \quad \text{or} \quad 2\pi\times 50\times L=1.31$$

$$L=19.01\text{ mH} \quad \text{or} \quad L=1.31\text{ mH}$$

Example 5.32 *Two impedances $Z_1 = (20 + j10)\ \Omega$ and $Z_2 = (10 - j30)\ \Omega$ are connected in parallel and this combination is connected in series with $Z_3 = (30 + jX)\ \Omega$. Find the value of X which will produce resonance.*

Solution $\mathbf{Z}_1 = 20 + j10\ \Omega,\quad \mathbf{Z}_2 = 10 - j30\ \Omega,\quad \mathbf{Z}_3 = 30 + jX\ \Omega$

$$\mathbf{Z}=\frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1+\mathbf{Z}_2}+\mathbf{Z}_3=\frac{(20+j10)\,(10-j30)}{20+j10+10-j30}+(30+jX)=19.23-j3.85+30+jX=49.23+j(X-3.85)$$

At resonance, the circuit is purely resistive, i.e., imaginary part of $\mathbf{Z}$ is equal to zero.

$$X-3.85=0$$

$$X=3.85\,\Omega$$

Example 5.33 *Impedances Z_2 and Z_3 in parallel are in series with an impedance Z_1 across a 100 V, 50 Hz ac supply. $Z_1 = (6.25 + j1.25)\ \Omega$, $Z_2 = 5\ \Omega$, $Z_3 = (5 - jX_C)\ \Omega$. Determine the value of capacitance of X_C such that the total current of the circuit will be in phase with the total voltage.*

Solution $\mathbf{Z}_1 = 6.25 + j1.25\ \Omega,\quad \mathbf{Z}_2 = 5\ \Omega,\quad \mathbf{Z}_3 = 5 - jX_C\ \Omega$

$$\mathbf{Z}=\frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2+\mathbf{Z}_3}+\mathbf{Z}_1=\frac{5(5-jX_C)}{5+5-jX_C}+(6.25+j1.25)$$

$$=\frac{25-j5X_C}{10-jX_C}+6.25+j1.25=\frac{25-j5X_C}{10-jX_C}\times\frac{10+jX_C}{10+jX_C}+6.25+j1.25$$

$$=\frac{250+5X_C^{\,2}}{100+X_C^{\,2}}-j\frac{25X_C}{100+X_C^{\,2}}+6.25+j1.25=\left(6.25+\frac{250+5X_C^2}{100+X_C^2}\right)-j\left(\frac{25X_C}{100+X_C^2}-1.25\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Z}$ is equal to zero.

$$\frac{25X_C}{100+X_C^{\,2}}-1.25=0$$

$$\frac{25X_C}{100+X_C^{\,2}}=1.25$$

$$1.25\,X_C^{\,2}-25X_C+125=0$$

$$X_C^2 - 20X_C + 100 = 0$$
$$X_c = 10$$
$$\frac{1}{2\pi \times 50 \times C} = 10$$
$$C = 318.31\ \mu\text{F}$$

Example 5.34 *Find the frequency at which the circuit shown in Fig. 5.25 will be at resonance. If the capacitor and inductors are interchanged, what would be the value of resonant frequency?*

Fig. 5.25

Solution $\mathbf{Z}_1 = j\omega\ \Omega, \quad \mathbf{Z}_2 = -\frac{j}{2\omega}\Omega, \quad \mathbf{Z}_3 = 1\,\Omega$

$$\mathbf{Z} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} = j\omega + \frac{\left(-\frac{j}{2\omega}\right)(1)}{-\frac{j}{2\omega}+1} = j\omega - \frac{j}{2\omega - j} = j\omega - \frac{j(2\omega + j)}{4\omega^2+1} = \frac{1}{4\omega^2+1} + j\left(\omega - \frac{2\omega}{4\omega^2+1}\right)$$

At resonance, the circuit is purely resistive, i.e., imaginary part of $\mathbf{Z}$ is equal to zero.

$$\omega - \frac{2\omega}{4\omega^2+1} = 0$$
$$\omega = \frac{2\omega}{4\omega^2+1}$$
$$\frac{1}{2} = \frac{1}{4\omega^2+1}$$
$$4\omega^2 + 1 = 2$$
$$4\omega^2 - 1 = 0$$
$$\omega = \pm 0.5 \text{ rad/s}$$

Considering positive values of ω,

$$\omega = 0.5 \text{ rad/s}$$

If the capacitor and inductor are interchanged,

$$\mathbf{Z} = -\frac{j}{2\omega} + \frac{(1)(j\omega)}{1+j\omega} = -\frac{j}{2\omega} + \frac{j\omega(1-j\omega)}{1+\omega^2} = -\frac{j}{2\omega} + \frac{(\omega^2 + j\omega)}{1+\omega^2} = \frac{\omega^2}{1+\omega^2} - j\left(\frac{1}{2\omega} - \frac{\omega}{1+\omega^2}\right)$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Z}$ is equal to zero.

$$\frac{1}{2\omega} - \frac{\omega}{1+\omega^2} = 0$$

$$\frac{1}{2\omega} = \frac{\omega}{1+\omega^2}$$

$$2\omega^2 - \omega^2 - 1 = 0$$

$$\omega^2 - 1 = 0$$

$$\omega = \pm 1 \text{ rad/s}$$

Considering positive values of ω,

$$\omega = 1 \text{ rad/s}$$

Example 5.35 *A 400 V, 50 Hz supply feeds energy to a parallel circuit consisting of $10\angle 30°\ \Omega$ branch and $10\angle -60°\ \Omega$ branch. Determine the impedance of a circuit element such that if connected in series with the source, the system comes at resonance.*

Solution

$$\mathbf{Z}_p = \frac{(10\angle 30°)\,(10\angle -60°)}{10\angle 30° + 10\angle -60°} = 7.07\ \angle -15° = 6.83 - j1.83\ \Omega$$

$$\mathbf{Z}_{\text{eq}} = \mathbf{Z} + \mathbf{Z}_p = \mathbf{Z} + 6.83 - j1.83$$

At resonance, the circuit is purely resistive, i.e., the imaginary part of $\mathbf{Z}$ is equal to zero. Hence $\mathbf{Z}$ must be equal to $j1.83\ \Omega$.

$$\mathbf{Z} = j1.83\ \Omega = jX_L$$

$$X_L = 1.83$$

$$2\pi \times 50 \times L = 1.83$$

$$L = 5.82 \text{ mH}$$

Example 5.36 *A coil of 400 Ω resistance and 318 μH inductance is connected in parallel with a capacitor and the circuit resonates at 1 MHz. If a second capacitor of 23.5 pF capacitance is connected in parallel with the first capacitor, find the frequency at which the circuit resonates.*

Solution

$$R = 400\ \Omega, \quad L = 318\ \mu\text{H}, \quad f_0 = 1\ \text{MHz}, \quad C_2 = 23.5\ \text{pF}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC_1} - \frac{R^2}{L^2}}$$

$$10^6 = \frac{1}{2\pi}\sqrt{\frac{1}{318\times 10^{-6}\times C_1} - \frac{(400)^2}{(318\times 10^{-6})^2}}$$

$$C_1 = 76.59 \text{ pF}$$

When capacitor of 23.5 pF is connected in parallel with $C_1 = 76.59$ pF,

$$C_T = C_1 + C_2 = 76.59 + 23.5 = 100.09 \text{ pF}$$

The new resonant frequency $f_0' = \frac{1}{2\pi}\sqrt{\frac{1}{LC_T} - \frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{318\times10^{-6}\times100.09\times10^{-12}} - \frac{(400)^2}{(318\times10^{-6})^2}}$

$= 869.34$ kHz

Example 5.37 *A series circuit consisting of a 12 Ω resistor, 0.3 H inductor and a variable capacitor is connected across a 100 V, 50 Hz ac supply as shown in Fig. 5.26. The capacitance value is adjusted to obtain maximum current. Find the capacitance value. Now, the supply frequency is raised to 60 Hz, the voltage remaining same at 100 V. Find the value of the capacitance C_2 to be connected across the above series circuit so that current drawn from the supply is minimum.*

Fig. 5.26

Solution $R = 12\ \Omega,\quad L = 0.3\text{ H},\quad V = 100\text{ V},\quad f = 50\text{ Hz}$

(a) Value of capacitance C_1

The resonance occurs at $f = 50$ Hz

$$f_0 = \frac{1}{2\pi\sqrt{LC_1}}$$

$$50 = \frac{1}{2\pi\sqrt{0.3\times C_1}}$$

$$C_1 = 33.77\ \mu\text{F}$$

(b) Value of capacitance C_2

The resonance occurs at 60 Hz for the circuit shown in Fig. 5.27.

Fig. 5.27

$$X_L = 2\pi f_0 L = 2\pi\times60\times0.3 = 113.1\ \Omega$$

$$X_{C_1} = \frac{1}{2\pi f_0 C_1} = \frac{1}{2\pi\times60\times33.77\times10^{-6}} = 78.55\ \Omega$$

$$\mathbf{Z}_1 = 12 + j113.1 - j78.55 = 36.57\ \angle70.85°\ \Omega$$

$$\mathbf{Y}_1 = \frac{1}{Z_1} = \frac{1}{36.57\ \angle70.85°} = 0.027\angle70.85° = 8.86\times10^{-3} - j0.0255\ \mho$$

$$\mathbf{Y}_{\text{req}} = \mathbf{Y}_1 + \mathbf{Y}_2 = 8.86\times10^{-3} - j0.0255 + \mathbf{Y}_2$$

At resonance, the imaginary part of $\mathbf{Y}_{\text{req}}$ becomes zero.

$$\mathbf{Y}_2 = j0.0255\ \mho$$

$$\mathbf{Y}_2 = \frac{1}{X_{C_2}} = 0.0255\ \mho$$

$$X_{C_2} = 39.22\ \Omega$$

$$X_{C_2} = \frac{1}{2\pi f_0 C_2}$$

$$39.22 = \frac{1}{2\pi \times 60 \times C_2}$$

$$C_2 = 67.63\ \mu\text{F}$$

Example 5.38 *Find active and reactive components of the current taken by a series circuit consisting of a coil of inductance 0.1 H and a resistance of 8 Ω and a capacitor of 120 μF connected to a 240 V, 50 Hz supply. Find the value of the capacitor that has to be connected in parallel with the above series circuit so that the pf of the entire circuit is unity.*

Solution

$$L = 0.1\ \text{H},\quad R = 8\ \Omega,\quad C = 120\ \mu\text{F},\quad V = 240\ \text{V},\quad f = 50\ \text{Hz}$$

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 120 \times 10^{-6}} = 26.53\ \Omega$$

$$\mathbf{Z}_1 = R + jX_L - jX_C = 8 + j31.42 - j26.53 = 9.38\ \angle 31.44°\ \Omega$$

$$I = \frac{V}{Z_1} = \frac{240}{9.43} = 25.45\ \text{A}$$

Active component of current $= I\cos\phi = 25.45\cos(31.44°) = 21.71$ A

Reactive component of current $= I\sin\phi = 25.45\sin(31.44°) = 13.27$ A

When a capacitor is connected in parallel with the above series circuit, to make pf of the entire circuit unity,

$$\mathbf{Z}_1 = R + jX_L - jX_C = 8 + j31.42 - j26.53 = 8 + j4.89\ \Omega$$

$$\mathbf{Z}_2 = -jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{8 + j4.89} + \frac{1}{-jX_C} = \frac{8 - j4.89}{64 + 23.91} + j\frac{1}{X_C} = \frac{8}{87.91} + j\left(\frac{1}{X_C} - \frac{4.89}{87.91}\right)$$

At resonance, pf is unity and the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{1}{X_C} - \frac{4.89}{87.91} = 0$$

$$\frac{1}{X_C} = 0.056$$

$$X_C = 17.86$$

$$\frac{1}{2\pi \times 50 \times C} = 17.86$$

$$C = 178.23\ \mu\text{F}$$

Example 5.39 *A coil of 10 Ω resistance and 0.5 H inductance is connected in series with a capacitor. On applying a sinusoidal voltage the current is maximum when the frequency is 50 Hz. A second capacitor is connected in parallel with the circuit. What capacitance should it have so that the combination acts as a non-inductive resistor at 100 Hz?*

Solution

$$R = 10\ \Omega, \quad L = 0.5\ \text{H}, \quad f_0 = 50\ \text{Hz}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$50 = \frac{1}{2\pi\sqrt{0.5 \times C}}$$

$$C = 20.26\ \mu\text{F}$$

When a second capacitor is connected in parallel with the circuit so that the combination acts as a non-inductive resistor at 100 Hz,

$$X_L = 2\pi \times 100 \times 0.5 = 314.16\ \Omega$$

$$X_C = \frac{1}{2\pi \times 100 \times 20.26 \times 10^{-6}} = 78.56\ \Omega$$

$$\mathbf{Z}_1 = 10 + j314.16 - j78.56 = 10 + j235.6\ \Omega$$

$$\mathbf{Z}_2 = -jX_C$$

For a parallel circuit,

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

$$\mathbf{Y} = \frac{1}{10 + j235.6} + \frac{1}{-jX_C} = \frac{10 - j235.6}{100 + 55507.36} + j\frac{1}{X_C} = \frac{10}{55607.36} + j\left(\frac{1}{X_C} - \frac{235.6}{55607.36}\right)$$

At resonance, the circuit acts as a non-inductive resistor, i.e., the imaginary part of $\mathbf{Y}$ is equal to zero.

$$\frac{1}{X_C} - \frac{235.6}{55607.36} = 0$$

$$X_C = 236.02$$

$$\frac{1}{2\pi \times 100 \times C} = 236.02$$

$$C = 6.7\ \mu\text{F}$$

Example 5.40 *A coil having a resistance and inductance of 15 Ω and 8 mH respectively is connected in parallel with another coil having a resistance and inductance of 4 Ω and 18 mH as shown in Fig. 5.28. If this parallel combination is to be replaced by a single coil, calculate the value of resistance and inductance*

of that coil. What value of capacitance should be connected in parallel with this coil in order to get unity power factor? Assume operating frequency to be 50 Hz.

Fig. 5.28

Solution $R_1 = 15\ \Omega,\quad L_1 = 8\text{ mH},\quad R_2 = 4\ \Omega,\quad L_2 = 18\text{ mH},\quad f = 50\text{ Hz}$

(a) Value of resistance and inductance of the coil

$$X_{L_1} = 2\pi f L_1 = 2\pi \times 50 \times 8 \times 10^{-3} = 2.51\ \Omega$$

$$X_{L_2} = 2\pi f L_2 = 2\pi \times 50 \times 18 \times 10^{-3} = 5.65\ \Omega$$

$$\mathbf{Z}_1 = R_1 + jX_{L_1} = 15 + j2.51 = 15.21\angle 9.5°\ \Omega$$

$$\mathbf{Z}_2 = R_2 + jX_{L_2} = 4 + j5.65 = 6.92\angle 54.7°\ \Omega$$

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(15.21\angle 9.5°)(6.92\angle 54.7°)}{15.21\angle 9.5° + 6.92\angle 54.7°} = 5.09\angle 40.96°\ \Omega = 3.84 + j3.34\ \Omega$$

$$R = 3.84\ \Omega$$

$$X_L = 3.34\ \Omega$$

$$X_L = 2\pi f L$$

$$3.34 = 2\pi \times 50 \times L$$

$$L = 10.63\text{ mH}$$

(b) Value of capacitance

When a capacitance C is connected in parallel with this coil as shown in Fig. 5.29, power factor becomes unity. Hence, the circuit behaves like a parallel resonant circuit at 50 Hz.

Fig. 5.29

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$50 = \frac{1}{2\pi}\sqrt{\frac{1}{10.63 \times 10^{-3} \times C} - \frac{(3.84)^2}{(10.63 \times 10^{-3})^2}}$$

$$C = 410.46\ \mu\text{F}$$

Exercises

5.1 A series *RLC* circuit has the following parameter values:
$R = 10\ \Omega$, $L = 0.014$ H, $C = 100\ \mu$F. Compute the following.

(i) resonant frequency in rad/s
(ii) quality factor of the circuit
(iii) bandwidth
(iv) lower and upper frequency points of the bandwidth

[845.2 rad/s, 1.185, 714.3 rad/s
487.9 rad/s, 1202.1 rad/s]

5.2 A series *RLC* circuit as $R = 500\ \Omega$, $L = 60$ mH and $C = 0.053\ \mu$F. Estimate

(i) circuit impedance at 50 Hz
(ii) resonance frequency in rad/s
(iii) quality factor of the circuit
(iv) bandwidth
(v) current at resonance frequency with an applied voltage of 30 V

[0.0041 Ω, 17733 rad/s 2.128,
8333.33 rad/s 60 mA]

5.3 A choking coil of 10 Ω resistance and 0.1 H inductance is connected in series with a capacitor of 200 μF capacitance across a supply voltage of 230 V, 50 Hz ac. Find the following w.r.t. the resonant circuit.

(i) Neat circuit diagram indicating all the relevant parameters.
(ii) At what frequency will the circuit resonate.
(iii) Value of reactances at resonance condition
(iv) Impedance of the coil at resonance condition
(v) Voltage across the coil and voltage across the capacitance
(vi) Half-power frequencies f_1 and f_2
(vii) Q factor of the circuit
(vii) Power and power factor

[35.59 Hz, 22.36 Ω, 10 Ω,
744.28 V, 514.28 V, 27.63 Hz,
43.55 Hz, 2.24, 5.29 kW, 1]

5.4 A coil of 40 Ω resistance and 0.75 H inductance forms part of a series circuit for which the resonant frequency is 55 Hz. If the supply is 250 V, 50 Hz, find

(i) the line current
(ii) the power factor
(iii) the voltage across the coil

[3.85 A, 0.616 (leading), 920.07 V]

5.5 An *RLC* series circuit of 10 Ω resistance should be designed to have a bandwidth of 100 Hz. Determine the values of L and C so that the circuit resonates at 250 Hz.

[0.0159 H, 25 μF]

5.6 A resistor and capacitor are connected in series with a variable inductor. When the circuit is connected to a 220 V, 50 Hz supply, the maximum current obtained by varying the inductance is 0.314 A. The voltage across the capacitor is 800 V. Calculate the resistance, inductance and capacitance of the circuit.

[700.63 Ω, 1.25 μF, 8.10 H]

5.7 A coil of 2 Ω resistance and 0.01 H inductance is connected in series with a capacitor across 230 V mains. What must be the capacitance, in order that maximum current occurs at a frequency of 50 Hz? Find also the current and the voltage across the capacitor.

[1 mF, 115 A, 361.28 V]

5.8 A choking coil of 10-ohm resistance and 0.1 H inductance is connected in series with a capacitor of 200 μF capacitance. Calculate the current, the coil voltage and the capacitor voltage. The supply voltage is 230 V at 50 Hz. At what frequency will the circuit resonate? Calculate the voltages at resonant frequency across the coil and capacitor. For this, assume that supply voltage is 230 V of variable frequency.

[12.47 A, 411.17, 198.47 V, 35.6 Hz,
563.3 V, 514.2 V]

5.9 An inductive coil having an inductance of 0.04 H and a resistance of 25 Ω has been connected in series with another inductive coil of 0.2 H

inductance and 15 Ω resistance. The whole circuit has been energized from 230 V, 50 Hz mains. Calculate power dissipation in each coil and power factor of the whole circuit. Draw the phasor diagram. Suggest a suitable capacitor for the above circuit to resonate at 50 Hz.

[181.57 W, 108.95 W, 0.468 (lagging), 42.22 μF]

5.10 A voltage $v(t) = 10\sqrt{2}\sin \omega t$ is applied to a series *RLC* circuit. At resonant frequency, voltage across the capacitor is found to be 500 V. The bandwidth of the circuit is known to be 400 rad/s and impedance at resonance, is 10 Ω. Determine resonant frequency, upper and lower cut-off frequencies, *L* and *C*.

[3.183 kHz, 3.151 kHz, 3.214 kHz, 0.025 H, 0.1 μF]

5.11 An inductive coil having a resistance of 20 Ω and an inductance of 0.2 H is connected in parallel with a 20 μF capacitor with variable frequency, 230 V supply. Find the resonant frequency at which the total current taken from the supply is in phase with the supply voltage. Also, find the value of this current.

[19.49 Hz, 4.6 A]

Objective-Type Questions

5.1 For a series *RLC* resonant circuit, the total reactance at the lower half power frequency is

(a) $\sqrt{2}R\angle 45°$ (b) $\sqrt{2}R\angle -45°$
(c) R (d) $-R$

5.2 In a series *RLC* high *Q* circuit, the current peaks at a frequency

(a) equal to the resonant frequency
(b) greater than the resonant frequency
(c) less than the resonant frequency
(d) none of the above.

5.3 In a series *RLC* circuit, $R = 2\text{k}\,\Omega$, $L = 1$ H, $C = \frac{1}{400}$ μF. The resonant frequency is

(a) 2×10^4 Hz (b) $\frac{1}{\pi}\times 10^4$ Hz
(c) 10^4 Hz (d) $2\pi\times 10^4$ Hz

5.4 For a series resonant circuit at low frequency, circuit impedance is _____ and at high frequency circuit impedance is _____.

(a) capacitive, inductive
(b) inductive, capacitive
(c) resistive, inductive
(d) capacitive, resistive

5.5 A circuit with a resistor, inductor and capacitor in series in resonant of f_0 Hz. If all the component values are now doubled, the new resonant frequency is

(a) $2f_0$ (b) f_0
(c) $\frac{f_0}{4}$ (d) $\frac{f_0}{2}$

5.6 In a series *RLC* circuit at resonance, the magnitude of the voltage developed across the capacitor

(a) is always zero
(b) can never be greater than the input voltage
(c) can be greater than the input voltage, however it is 90° out of phase with the input voltage
(d) can be greater than the input voltage and is in phase with the input voltage

5.7 The power in a series *RLC* circuit will be half of that at resonance when the magnitude of the current is equal to

(a) $\frac{V}{2R}$ (b) $\frac{V}{\sqrt{3}R}$
(c) $\frac{V}{\sqrt{2}R}$ (d) $\frac{\sqrt{2}V}{R}$

5.8 A series *RLC* circuit has a resonance frequency of 1 kHz and a quality factor *Q* of 100. If each of *R*, *L* and *C* is doubled from its original value, the new *Q* of the circuit is

(a) 25 (b) 50
(c) 100 (d) 200

5.9 A series *RLC* circuit has a Q of 100 and an impedance of $(100 + j0)\,\Omega$ at its resonant angular frequency of 10^7 rad/s. The values of R and L are

(a) $100\,\Omega, 10^3$ H (b) $100\,\Omega, 10^{-3}$ H
(c) $10\,\Omega, 10$ H (d) $10\,\Omega, 0.1$ H

5.10 The following circuit in Fig. 5.30 resonates at

Fig. 5.30

(a) all frequencies (b) 0.5 rad/s
(c) 5 rad/s (d) 1 rad/s

5.11 At resonance, the parallel circuit shown in Fig. 5.31 behaves like

Fig. 5.31

(a) an open circuit
(b) a short circuit
(c) a pure resistor of value R
(d) a pure resistor of value much higher than R

Answers to Objective-Type Questions

5.1 (d)	5.2 (a)	5.3 (b)	5.4 (a)	5.5 (d)	5.6 (c)
5.7 (c)	5.8 (b)	5.9 (b)	5.10 (b)	5.11 (d)	

6

Network Theorems (Application to ac Circuits)

6.1 INTRODUCTION

We have discussed the network theorems with reference to resistive load and dc sources. Now, all the theorems will be discussed when a network consists of ac sources, resistors, inductors and capacitors. All the theorems are also valid for ac sources.

6.2 MESH ANALYSIS

Mesh analysis is useful if a network has a large number of voltage sources. In this method, currents are assigned in each mesh. We can write mesh equations by Kirchhoff's voltage law in terms of unknown mesh currents,

Example 6.1 *Find mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the network of Fig. 6.1.*

Fig. 6.1

Solution Applying KVL to Mesh 1,

$$100\angle 45^\circ - (3 + j4)\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3 + j14)\mathbf{I}_1 - j10\mathbf{I}_2 = 100\angle 45^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) + j10(\mathbf{I}_2) = 0$$

$$j10\mathbf{I}_1 = 0 \qquad \ldots\text{(ii)}$$

$$\mathbf{I}_1 = 0$$

Substituting $\mathbf{I}_1$ in Eq. (i),

$$-j10\mathbf{I}_2 = 100\angle 45^\circ$$

$$\mathbf{I}_2 = \frac{100\angle 45^\circ}{-j10} = 10\angle 135^\circ \text{ A}$$

Example 6.2 *Find mesh current $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ in the network of Fig. 6.2.*

Fig. 6.2

Solution Applying KVL to Mesh 1,

$$10\angle 30° - (5 - j2)\,\mathbf{I}_1 - 3\,(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(8 - j2)\,\mathbf{I}_1 - 3\mathbf{I}_2 = 10\angle 30^\circ \qquad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-3\,(\mathbf{I}_2 - \mathbf{I}_1) - j5\mathbf{I}_2 - 5\,(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-3\mathbf{I}_1 + (8 + j5)\,\mathbf{I}_2 - 5\mathbf{I}_3 = 0 \qquad \ldots \text{(ii)}$$

Applying KVL to Mesh 3,

$$-5\,(\mathbf{I}_3 - \mathbf{I}_2) - (2 - j2)\,\mathbf{I}_3 = 0$$

$$-5\mathbf{I}_2 + (7 - j2)\,\mathbf{I}_3 = 0 \qquad \ldots \text{(iii)}$$

Writing Eqs (i), (ii) and (iii),

$$\begin{bmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 30° \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 30° & -3 & 0 \\ 0 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}} = 1.43\angle 38.7° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 8 - j2 & 10\angle 30° & 0 \\ -3 & 0 & -5 \\ 0 & 0 & 7 - j2 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}} = 0.693\angle -2.2° \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 8 - j2 & -3 & 10\angle 30° \\ -3 & 8 + j5 & 0 \\ 0 & -5 & 0 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}} = 0.476\angle 13.8° \text{ A}$$

Example 6.3 *In the network of Fig. 6.3, find the value of $\mathbf{V}_2$ so that the current through (2 + j3) ohm impedance is zero.*

Fig. 6.3

Solution Applying KVL to Mesh 1,

$$30\angle 0° - 5\mathbf{I}_1 - j5(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(5 + j5)\mathbf{I}_1 - j5\mathbf{I}_2 = 30\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j5(\mathbf{I}_2 - \mathbf{I}_1) - (2 + j3)\mathbf{I}_2 - 6(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-j5\mathbf{I}_1 + (8 + j8)\mathbf{I}_2 - 6\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-6(\mathbf{I}_3 - \mathbf{I}_2) - 4\mathbf{I}_3 - \mathbf{V}_2 = 0$$

$$-6\mathbf{I}_2 + 10\mathbf{I}_3 = -\mathbf{V}_2 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 30\angle 0° \\ 0 \\ -\mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5+j5 & 30\angle 0° & 0 \\ -j5 & 0 & -6 \\ 0 & -\mathbf{V}_2 & 10 \end{vmatrix}}{\begin{vmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{vmatrix}} = 0$$

$$(5 + j5)(-6\mathbf{V}_2) - (30)(-j50) = 0$$

$$\mathbf{V}_2 = \frac{j1500}{30 + j30} = 35.36\angle 45° \text{ V}$$

Example 6.4 *Find the value of the current $\mathbf{I}_3$ in the network shown in Fig. 6.4.*

Fig. 6.4

Solution Applying KVL to Mesh 1,

$$20\angle 0° - (4 - j4)\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) - 10(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

$$(14 + j6)\mathbf{I}_1 - j10\mathbf{I}_2 - 10\mathbf{I}_3 = 20\angle 0^\circ \qquad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) - 10\angle 30^\circ - 20\mathbf{I}_2 - (4 - j4)(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-j10\mathbf{I}_1 + (24 + j6)\mathbf{I}_2 - (4 - j4)\mathbf{I}_3 = -10\angle 30^\circ \qquad \ldots \text{(ii)}$$

Applying KVL to Mesh 3,

$$-10(\mathbf{I}_3 - \mathbf{I}_1) - (4 - j4)(\mathbf{I}_3 - \mathbf{I}_2) - 20\mathbf{I}_3 = 0$$

$$-10\mathbf{I}_1 - (4 - j4)\mathbf{I}_2 + (34 - j4)\mathbf{I}_3 = 0 \qquad \ldots \text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 14 + j6 & -j10 & -10 \\ -j10 & 24 + j6 & -(4 - j4) \\ -10 & -(4 - j4) & 34 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 20\angle 0° \\ -10\angle 30° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 14 + j6 & -j10 & 20\angle 0° \\ -j10 & 24 + j6 & -10\angle 30° \\ -10 & -(4 - j4) & 0 \end{vmatrix}}{\begin{vmatrix} 14 + j6 & -10 & -10 \\ -j10 & 24 + j6 & -(4 - j4) \\ -10 & -(4 - j4) & 34 - j4 \end{vmatrix}} = 0.44\angle -14° \text{ A}$$

Example 6.5 *Find the voltage $\mathbf{V}_{AB}$ in the network of Fig. 6.5.*

Fig. 6.5

Solution Applying KVL to Mesh 1,

$$-96\,\mathbf{I}_1 - (100 + 4 + j200)(\mathbf{I}_1 - \mathbf{I}_2) + 10\angle 0° = 0$$

$$(200 + j200)\,\mathbf{I}_1 - (104 + j200)\,\mathbf{I}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-(1 - j50 - 100)\,\mathbf{I}_2 - (100 + 4 + j200)(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

$$-(104 + j200)\,\mathbf{I}_1 + (205 + j150)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 200 + j200 & -(104 + j200) \\ -(104 + j200) & 205 + j150 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -(104+j200) \\ 0 & 205+j150 \end{vmatrix}}{\begin{vmatrix} 200+j200 & -(104+j200) \\ -(104+j200) & 205+j150 \end{vmatrix}} = 0.051\angle 2.72\times 10^{-3}\,^\circ\text{A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 200+j200 & 10\angle 0^\circ \\ -(104+j200) & 0 \end{vmatrix}}{\begin{vmatrix} 200+j200 & -(104+j200) \\ -(104+j200) & 205+j150 \end{vmatrix}} = 0.045\angle 26.34^\circ \text{ A}$$

$$\begin{aligned}\mathbf{V}_{AB} &= 100\mathbf{I}_2 - (4+j200)(\mathbf{I}_1 - \mathbf{I}_2) \\ &= 100(0.045\angle 26.34^\circ) - (4+j200)(0.051\angle 2.72\times 10^{-3}\,^\circ - 0.045\angle 26.34^\circ) \\ &= 0.058\angle -92.65^\circ \text{ V}\end{aligned}$$

Example 6.6 *For the network shown in Fig. 6.6, find the voltage across the capacitor.*

Fig. 6.6

Solution Applying KVL to Mesh 1,

$$5\angle 0^\circ - (1+j2)\mathbf{I}_1 - 2(\mathbf{I}_1 - \mathbf{I}_2) - (1+j3)(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

$$(4+j5)\mathbf{I}_1 - 2\mathbf{I}_2 - (1+j3)\mathbf{I}_3 = 5\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-2(\mathbf{I}_2 - \mathbf{I}_1) - 3\mathbf{I}_2 + j2(\mathbf{I}_2 - \mathrm{I}_3) = 0$$

$$-2\mathbf{I}_1 + (5-j2)\mathbf{I}_2 + j2\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-(1+j3)(\mathbf{I}_3 - \mathbf{I}_1) + j2(\mathbf{I}_3 - \mathrm{I}_2) - (1+j1)\mathbf{I}_3 = 0$$

$$-(1+j3)\mathbf{I}_1 + j2\mathbf{I}_2 + (2+j2)\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 5\angle 0^\circ \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4+j5 & 5\angle 0^\circ & -(1+j3) \\ -2 & 0 & j2 \\ -(1+j3) & 0 & 2+j2 \end{vmatrix}}{\begin{vmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{vmatrix}} = 0.65\angle 130.51^\circ \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 4+j5 & -2 & 5\angle 0^\circ \\ -2 & 5-j2 & 0 \\ -(1+j3) & j2 & 0 \end{vmatrix}}{\begin{vmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{vmatrix}} = 0.91\angle -21.51^\circ \text{ A}$$

$$\mathbf{V}_c = (-j2)(\mathbf{I}_3 - \mathbf{I}_2) = (-j2)(0.91\angle -21.51^\circ - 0.65\angle 130.51^\circ) = 3.03\angle -123.12^\circ \text{ V}$$

Example 6.7 *Find the voltage across the 2 Ω resistor in the network of Fig. 6.7.*

Fig. 6.7

Solution For Mesh 1,

$$\mathbf{I}_1 = 2\angle 30^\circ \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-(2-j2)(\mathbf{I}_2 - \mathbf{I}_1) - j1\mathbf{I}_2 - 8\angle 45^\circ = 0$$

$$(2-j2)\mathbf{I}_1 - (2-j1)\mathbf{I}_2 = 8\angle 45^\circ \qquad \text{...(ii)}$$

Substituting $\mathbf{I}_1$ in Eq. (i),

$$(2-j2)(2\angle 30^\circ) - (2-j1)\mathbf{I}_2 = 8\angle 45^\circ$$

$$\mathbf{I}_2 = \frac{-(8\angle 45^\circ) + (2-j2)(2\angle 30^\circ)}{2-j1} = 3.19\angle -65^\circ \text{ A}$$

$$\mathbf{V}_{2\Omega} = 2(\mathbf{I}_1 - \mathbf{I}_2) = 2(2\angle 30^\circ - 3.19\angle -65^\circ) = 7.82\angle 84.37^\circ \text{ V}$$

Example 6.8 *Find the current through 3 Ω resistor in the network of Fig. 6.8.*

Fig. 6.8

Solution Applying KVL to Mesh 1,

$$10\angle 0° - j2\mathbf{I}_1 - 3\mathbf{I}_1 - 1(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(4 + j2)\mathbf{I}_1 - \mathbf{I}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$\mathbf{I}_3 - \mathbf{I}_2 = 1\angle 0° \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-1(\mathbf{I}_2 - \mathbf{I}_1) - 5\mathbf{I}_3 - j1\mathbf{I}_3 = 0$$

$$\mathbf{I}_1 - \mathbf{I}_2 - (5 + j1)\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4+j2 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -(5+j1) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 1\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0° & -1 & 0 \\ 1\angle 0° & -1 & 1 \\ 0 & -1 & -(5+j1) \end{vmatrix}}{\begin{vmatrix} 4+j2 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -(5+j1) \end{vmatrix}} = 2.11\angle -28.01° \text{ A}$$

$$\mathbf{I}_{3\Omega} = \mathbf{I}_1 = 2.11\angle -28.01° \text{ A}$$

Example 6.9 *Find the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the network of Fig. 6.9.*

Fig. 6.9

Solution From Fig. 6.9,

$$\mathbf{V}_x = -j3(\mathbf{I}_1 - \mathbf{I}_2) \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$9\angle 0° - 6\,\mathbf{I}_1 + j3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(6 - j3)\,\mathbf{I}_1 + j3\mathbf{I}_2 = 9\angle 0° \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$j3(\mathbf{I}_2 - \mathbf{I}_1) + 2\mathbf{V}_x - 3\,\mathbf{I}_2 = 0$$

$$j3\mathbf{I}_2 - j3\mathbf{I}_1 + 2[-j3(\mathbf{I}_1 - \mathbf{I}_2)] - 3\,\mathbf{I}_2 = 0$$

$$j9\mathbf{I}_1 + (3 - j9)\,\mathbf{I}_2 = 0 \qquad \text{...(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 6 - j3 & j3 \\ j9 & 3 - j9 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 9\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 9\angle 0° & j3 \\ 0 & 3 - j9 \end{vmatrix}}{\begin{vmatrix} 6 - j3 & j3 \\ j9 & 3 - j9 \end{vmatrix}} = 1.3\angle 2.49° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 6 - j3 & 9\angle 0° \\ j9 & 0 \end{vmatrix}}{\begin{vmatrix} 6 - j3 & j3 \\ j9 & 3 - j9 \end{vmatrix}} = 1.24\angle -15.95° \text{ A}$$

Example 6.10 *Find the voltage across the 4 Ω resistor in the network of Fig. 6.10.*

Fig. 6.10

Solution From Fig. 6.10,

$$\mathbf{I}_x = \mathbf{I}_1 \qquad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-2\mathbf{I}_1 + 6\angle 30° + j1(\mathbf{I}_1 - \mathbf{I}_2) - 2\mathbf{I}_x = 0$$

$$-2\mathbf{I}_1 + 6\angle 30° + j1\mathbf{I}_1 - j1\mathbf{I}_2 - 2\mathbf{I}_1 = 0$$

$$(4 - j1)\,\mathbf{I}_1 + j1\mathbf{I}_2 = 6\angle 30° \qquad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$2\mathbf{I}_x + j1(\mathbf{I}_2 - \mathbf{I}_1) - j2\mathbf{I}_2 - 4\mathbf{I}_2 = 0$$

$$2\mathbf{I}_1 + j1\mathbf{I}_2 - j1\,\mathbf{I}_1 - j2\mathbf{I}_2 - 4\mathbf{I}_2 = 0$$

$$(2 - j1)\,\mathbf{I}_1 - (4 + j1)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4 - j1 & j1 \\ 2 - j1 & -(4 + j1) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 6\angle 30^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4 - j1 & 6\angle 30^\circ \\ 2 - j1 & 0 \end{vmatrix}}{\begin{vmatrix} 4 - j1 & j1 \\ 2 - j1 & -(4 + j1) \end{vmatrix}} = 0.74\angle -2.91^\circ \text{ A}$$

$$V_{4\Omega} = 4\mathbf{I}_2 = 4\,(0.74\angle -2.91^\circ) = 2.96\angle -2.91^\circ \text{ V}$$

6.3 NODE ANALYSIS

Node analysis uses Kirchhoff's current law for finding currents and voltages in a network. For ac networks, Kirchhoff's current law states that the phasor sum of currents meeting at a point is equal to zero.

Example 6.11 *In the network shown in Fig. 6.11, determine $\mathbf{V}_a$ and $\mathbf{V}_b$.*

Fig. 6.11

Solution Applying KCL at Node a,

$$\frac{\mathbf{V}_a - 10\angle 0^\circ}{j6} + \frac{\mathbf{V}_a}{-j6} + \frac{\mathbf{V}_a - \mathbf{V}_b}{3} = 0$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)\mathbf{V}_a - \frac{1}{3}\mathbf{V}_b = \frac{10\angle 0^\circ}{j6}$$

$$0.33\mathbf{V}_a - 0.33\mathbf{V}_b = 1.67\angle -90^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node b,

$$\frac{\mathbf{V}_b - \mathbf{V}_a}{3} + \frac{\mathbf{V}_b}{j4} + \frac{\mathbf{V}_b}{j1} = 0$$

$$-\frac{1}{3}\mathbf{V}_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right)\mathbf{V}_b = 0$$

$$-0.33\mathbf{V}_a + (0.33 - j1.25)\mathbf{V}_b = 0 \qquad \ldots\text{(ii)}$$

Adding Eqs (i) and (ii),

$$-j1.25\mathbf{V}_b = 1.67\angle -90^\circ$$

$$\mathbf{V}_b = \frac{1.67\angle -90^\circ}{-j1.25} = 1.34\angle 0^\circ \text{ V}$$

Substituting $\mathbf{V}_b$ in Eq. (i),

$$0.33\mathbf{V}_a - 0.33(1.34\angle 0^\circ) = 1.67\angle -90^\circ$$

$$\mathbf{V}_a = \frac{1.73\angle 75.17^\circ}{0.33} = 5.24\angle -75.17^\circ \text{ V}$$

Example 6.12 *For the network shown in Fig. 6.12, find the voltages $\mathbf{V}_1$ and $\mathbf{V}_2$.*

Fig. 6.12

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1 - 50\angle 0^\circ}{5} + \frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{4} = 0$$

$$\left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right)\mathbf{V}_1 - \frac{1}{4}\mathbf{V}_2 = 10\angle 0^\circ$$

$$(0.45 - j0.5)\mathbf{V}_1 - 0.25\mathbf{V}_2 = 10\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{4} + \frac{\mathbf{V}_2}{-j2} + \frac{\mathbf{V}_2 - 50\angle 90^\circ}{2} = 0$$

$$-\frac{1}{4}\mathbf{V}_1 + \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right)\mathbf{V}_2 = 25\angle 90^\circ$$

$$-0.25\mathbf{V}_1 + (0.75 + j0.5)\mathbf{V}_2 = 25\angle 90^\circ \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 25\angle 90^\circ \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -0.25 \\ j25 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = 24.7\angle 72.25^\circ \text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.45 - j0.5 & 10\angle 0^\circ \\ -0.25 & 25\angle 90^\circ \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = 34.34\angle 52.82^\circ \text{ V}$$

Example 6.13 *Find the voltage $\mathbf{V}_{AB}$ in the network of Fig. 6.13.*

Fig. 6.13

Solution Applying KCL at Node 1,

$$10\angle 0° = \frac{\mathbf{V}_1 - \mathbf{V}_2}{2} + \frac{\mathbf{V}_1}{3 + j4}$$

$$\left(\frac{1}{2} + \frac{1}{3 + j4}\right)\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2 = 10\angle 0°$$

$$(0.62 - j0.16)\mathbf{V}_1 - 0.5\mathbf{V}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{2} + \frac{\mathbf{V}_2}{j5} + \frac{\mathbf{V}_2}{j10} = 0$$

$$-\frac{1}{2}\mathbf{V}_1 + \left(\frac{1}{2} + \frac{1}{j5} + \frac{1}{j10}\right)\mathbf{V}_2 = 0$$

$$-0.5\mathbf{V}_1 + (0.5 - j0.3)\mathbf{V}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0° & -0.5 \\ 0 & 0.5 - j0.3 \end{vmatrix}}{\begin{vmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{vmatrix}} = 21.8\angle 56.42° \text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.62 - j0.16 & 10\angle 0° \\ -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{vmatrix}} = 18.7\angle 87.42° \text{ V}$$

$$\mathbf{V}_A = \mathbf{V}_2 = 18.7\angle 87.42° \text{ V}$$

$$\mathbf{V}_B = \frac{\mathbf{V}_1}{3 + j4}(j4) = \frac{21.8\angle 56.42°}{3 + j4}(j4) = 17.45\angle 93.32° \text{ V}$$

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B = (18.7\angle 87.42°) - (17.45\angle 93.32°) = 2.23\angle 34.1° \text{ V}$$

Example 6.14 *Find the node voltages V_1 and V_2 in the network of Fig. 6.14.*

Fig. 6.14

Solution Applying KCL at Node 1,

$$\frac{\mathbf{V}_1}{2}+\frac{\mathbf{V}_1-\mathbf{V}_2}{-j1}=2\mathbf{V}_2$$

$$\left(\frac{1}{2}+\frac{1}{-j1}\right)\mathbf{V}_1-\left(2-\frac{1}{j1}\right)\mathbf{V}_2=0$$

$$(0.5+j1)\,\mathbf{V}_1-(2+j1)\mathbf{V}_2=0 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2-\mathbf{V}_1}{-j1}+\frac{\mathbf{V}_2}{j2}=2\angle 30^\circ$$

$$\frac{1}{j1}\mathbf{V}_1+\left(\frac{1}{-j1}+\frac{1}{j2}\right)\mathbf{V}_2=2\angle 30^\circ$$

$$-j1\,\mathbf{V}_1+j\,0.5\,\mathbf{V}_2=2\angle 30^\circ \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.5+j1 & -(2+j1) \\ -j1 & j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 2\angle 30^\circ \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1=\frac{\begin{vmatrix} 0 & -(2+j1) \\ 2\angle 30^\circ & j0.5 \end{vmatrix}}{\begin{vmatrix} 0.5+j1 & -(2+j1) \\ -j1 & j0.5 \end{vmatrix}}=2.46\angle 130.62^\circ \text{ V}$$

$$\mathbf{V}_2=\frac{\begin{vmatrix} 0.5+j1 & 0 \\ -j1 & 2\angle 30^\circ \end{vmatrix}}{\begin{vmatrix} 0.5+j1 & -(2+j1) \\ -j1 & j0.5 \end{vmatrix}}=1.23\angle 167.49^\circ \text{ V}$$

Example 6.15 *In the network of Fig 6.15, find the voltage V_2 which results in zero current through 4 Ω resistor.*

Fig. 6.15

Solution Applying KCL at Node 1,

$$\frac{\mathbf{V}_1 - 50\angle 0^\circ}{5} + \frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_3}{4} = 0$$

$$\left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right)\mathbf{V}_1 - \frac{1}{4}\mathbf{V}_3 = 10\angle 0^\circ$$

$$(0.45 - j0.5)\mathbf{V}_1 - 0.25\mathbf{V}_3 = 10\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node 3,

$$\frac{\mathbf{V}_3 - \mathbf{V}_1}{4} + \frac{\mathbf{V}_3}{-j2} + \frac{\mathbf{V}_3 - \mathbf{V}_2}{2} = 0$$

$$-\frac{1}{4}\mathbf{V}_1 + \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right)\mathbf{V}_3 = 0.5\mathbf{V}_2$$

$$-0.25\mathbf{V}_1 + (0.75 + j0.5)\mathbf{V}_3 = 0.5\mathbf{V}_2 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0.5\mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -0.25 \\ 0.5\mathbf{V}_2 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{10(0.75 + j0.5) + 0.125\,\mathbf{V}_2}{0.55\angle -15.95^\circ}$$

$$\mathbf{V}_3 = \frac{\begin{vmatrix} 0.45 - j\,0.5 & 10\angle 0^\circ \\ -0.25 & 0.5\mathbf{V}_2 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{0.5\,\mathbf{V}_2(0.45 - j0.5) + 2.5}{0.55\angle -15.95^\circ}$$

$$\mathbf{I}_{4\,\Omega} = \frac{\mathbf{V}_1 - \mathbf{V}_3}{4} = 0$$

$$\mathbf{V}_1 = \mathbf{V}_3$$

$$\frac{10(0.75 + j0.5) + 0.125\mathbf{V}_2}{0.55\angle -15.95^\circ} = \frac{0.5\mathbf{V}_2(0.45 - j0.5) + 2.5}{0.55\angle -15.95^\circ}$$

$$7.5 + 0.125\mathbf{V}_2 - j\,5 = 2.5 + 0.225\mathbf{V}_2 - j0.25\mathbf{V}_2$$

$$5 + j\,5 = \mathbf{V}_2(0.1 - j0.25)$$

$$\mathbf{V}_2 = \frac{5 + j5}{0.1 - j0.25} = 26.26\angle 113.2^\circ \text{ V}$$

Example 6.16 *Find the voltage across the capacitor in the network of Fig. 6.16.*

Fig. 6.16

Solution Nodes 1 and 2 will form a supernode.
Writing the voltage equation for the supernode,

$$\mathbf{V}_1 - \mathbf{V}_2 = 12\angle 30° \qquad \ldots(i)$$

Applying KCL to the supernode,

$$\frac{\mathbf{V}_1}{j1} + \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2}{-j2} = 2\angle 60°$$

$$(-j1)\,\mathbf{V}_1 + (0.5 + j0.5)\,\mathbf{V}_2 = 2\angle 60° \qquad \ldots(ii)$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 1 & -1 \\ -j1 & 0.5 + j0.5 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 12\angle 30° \\ 2\angle 60° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 1 & 12\angle 30° \\ -j1 & 2\angle 60° \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -j1 & 0.5 + j0.5 \end{vmatrix}} = 18.55\angle 157.42° \text{ V}$$

$$\mathbf{V}_c = \mathbf{V}_2 = 18.55\angle 157.42° \text{ V}$$

6.4 SUPERPOSITION THEOREM

The superposition theorem can be used to analyse an ac network containing more than one source. The superposition theorem states that *in a network containing more than one voltage source or current source, the total current or voltage in any branch of the network is the phasor sum of currents or voltages produced in that branch by each source acting separately*. As each source is considered, all of the other sources are replaced by their internal impedances. This theorem is valid only for linear systems.

Example 6.17 *Find the current through the 3 + j4 ohm impedance.*

Fig. 6.17

Solution

Step I When the 50 ∠90° V source is acting alone (Fig. 6.18)

$$\mathbf{Z}_T = 5 + \frac{(3 + j4)(j5)}{3 + j9} = 6.35\angle 23.2° \ \Omega$$

$$\mathbf{I}_T = \frac{50\angle 90°}{6.35\angle 23.2°} = 7.87\angle 66.8° \text{ A}$$

Fig. 6.18

By current division rule,

$$\mathbf{I}' = (7.87\angle 66.8°)\left(\frac{j5}{3+j9}\right) = 4.15\angle 85.3°\ \text{A}(\downarrow)$$

Step II When the 50∠0° V source is acting alone (Fig. 6.19)

$$\mathbf{Z}_T = j5 + \frac{5(3+j4)}{8+j4} = 6.74\angle 68.2°\ \Omega$$

$$\mathbf{I}_T = \frac{50\angle 0°}{6.74\angle 68.2°} = 7.42\angle -68.2°\ \text{A}$$

Fig. 6.19

By current division rule,

$$\mathbf{I}'' = (7.42\angle -68.2°)\left(\frac{5}{8+j4}\right) = 4.15\angle -94.77°\ \text{A}(\uparrow) = 4.15\angle 85.3°\ \text{A}(\downarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' = 4.15\ \angle 85.3° + 4.15\ \angle 85.3° = 8.31\ \angle 85.3°\text{A}\ (\downarrow)$$

Example 6.18 *Determine the voltage across the (2 + j5) ohm impedance for the network shown in Fig.6.20.*

Fig. 6.20

Solution

Step I When the 50∠0° V source is acting alone (Fig. 6.21)

$$\mathbf{I} = \frac{50\angle 0°}{2+j4+j5} = 5.42\angle -77.47°\ \text{A}$$

Voltage cross $(2+j5)\ \Omega$ impedance

$$\mathbf{V}' = (2+j5)\ (5.42\ \angle -77.47°) = 29.16\ \angle -9.28°\ \text{V}$$

Fig. 6.21

Step II When the 20∠30° A source is acting alone (Fig. 6.22)
By current division rule,

$$\mathbf{I} = (20\angle 30°)\left(\frac{j4}{2+j9}\right) = 8.68\angle 42.53°\ \text{A}$$

Voltage across $(2+j5)\ \Omega$ impedance

$$\mathbf{V}'' = (2+j5)\ (8.68\ \angle 42.53°) = 46.69\ \angle 110.72°\ \text{V}$$

Fig. 6.22

Step III By superposition theorem,

$$\mathbf{V} = \mathbf{V}' + \mathbf{V}'' = 29.16\ \angle{-9.28°} + 46.69\ \angle 110.72° = 40.85\ \angle 72.53°\ \text{V}$$

Example 6.19 *Determine the voltage $\mathbf{V}_{AB}$ for the network shown in Fig. 6.23.*

Fig. 6.23

Solution

Step I When the 50∠0° V source is acting alone (Fig. 6.24)

Fig. 6.24

$$\mathbf{V}'_{AB} = 50\angle 0°\ \text{V}$$

Step II When the 4∠0° A source is acting alone (Fig. 6.25)

Fig. 6.25

$$\mathbf{V}''_{AB} = 0$$

Step III By superposition theorem,

$$\mathbf{V}_{AB} = \mathbf{V}'_{AB} + \mathbf{V}''_{AB} = 50\angle 0° + 0 = 50\angle 0°\ \text{V}$$

Example 6.20 *Find the current **I** in the network shown in Fig. 6.26.*

Fig. 6.26

Solution

Step I When the 13∠25° V source is acting alone (Fig. 6.27)

Fig. 6.27

$$\mathbf{I}' = \frac{13\angle 25^\circ}{6 - j2} = 2.057\angle 43.43^\circ \text{ A} \quad (\rightarrow)$$

Step II When the 20∠−30° V source is acting alone (Fig. 6.28)

Fig. 6.28

$$\mathbf{I}'' = \frac{20\angle -30^\circ \text{ V}}{6 - j2} = 3.16\angle -11.57^\circ \text{ A}(\leftarrow) = 3.16\angle 168.43^\circ \text{ A}(\rightarrow)$$

Step III When the 3∠50° A source is acting alone (Fig. 6.29)

Fig. 6.29

By current division rule,

$$\mathbf{I}''' = 3\angle 50° \times \frac{2-j5}{6-j2} = 2.56\angle 0.23°\ \text{A}(\leftarrow) = 2.56\angle -179.77°\ \text{A}(\rightarrow)$$

Step IV By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' + \mathbf{I}''' = 2.057\ \angle 43.13° + 3.16\ \angle 168.43° + 2.56\ \angle -179.77°\ \text{A} = 4.62\ \angle 153.99°\ \text{A}\ (\rightarrow)$$

Example 6.21 *Find the current through the j3 Ω reactance in the network of Fig 6.30.*

Fig. 6.30

Solution

Step I When the 5∠30° V source is acting alone (Fig. 6.31)

Fig. 6.31

When a short circuit is placed across $j15\ \Omega$ reactance, it gets shorted as shown in Fig 6.32.

Fig. 6.32

$$\mathbf{I}' = \frac{5\angle 30°}{-j5 + j3} = 2.5\angle 120°\ \text{A}(\leftarrow)$$

Step II When the 10∠60° V source is acting alone (Fig. 6.33)

Fig. 6.33

When a short circuit is placed across the –*j*2 Ω reactance, it gets shorted as shown in Fig. 6.34

Fig. 6.34

$$\mathbf{I}'' = \frac{10\angle 60°}{-j5 + j3} = 5\angle 150° \text{ A } (\rightarrow) = 5\angle -30° \text{ A } (\leftarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' = 2.5\angle 120° + 5\angle -30° = 3.1\angle -6.21° \text{ A} \quad (\leftarrow)$$

Example 6.22 *Find the current* $\mathbf{I}_0$ *in the network of Fig. 6.35.*

Fig. 6.35

Solution

Step I When the 10∠30° V source is acting alone (Fig. 6.36)

$$\mathbf{Z}_T = 6 + \frac{j4(8 - j2)}{j4 + 8 - j2} = 8.64\angle 24.12° \ \Omega$$

$$\mathbf{I}_T = \frac{10\angle 30°}{8.64\angle 24.12°} = 1.16\angle 5.88° \text{ A}$$

Fig. 6.36

By current division rule,

$$\mathbf{I}_0' = 1.16\angle 5.88° \times \frac{j4}{8 - j2 + j4} = 0.56\angle 81.84° \text{ A} \quad (\downarrow)$$

Step II When the $2\angle 0°$ A source is acting alone (Fig. 6.37)

Fig. 6.37

The network can be redrawn as shown in Fig. 6.38.

Fig. 6.38

By current division rule,

$$\mathbf{I}_0'' = 2\angle 0° \times \frac{1.85 + j2.77}{1.85 + j2.77 + 8 - j2} = 0.67\angle 51.83° \text{ A } (\downarrow)$$

Step III By superposition theorem,

$$\mathbf{I}_0 = \mathbf{I}_0' + \mathbf{I}_0'' = 0.56\angle 81.84° + 0.67\angle 51.83° = 1.19\angle 65.46° \text{ A } (\downarrow)$$

Example 6.23 *Find the current through the j5 Ω branch for the network shown in Fig. 6.39.*

Fig. 6.39

Solution

Step I When the 10∠0° V source is acting alone (Fig. 6.40)

Fig. 6.40

$$\mathbf{Z}_T = j5 + \frac{3(-j4)}{3-j4} = 4.04\angle 61.66°\ \Omega$$

$$\mathbf{I}' = \frac{10\angle 0°}{4.04\angle 61.66°} = 2.48\angle -61.66° \text{A} \quad (\rightarrow)$$

Step II When the 15∠90° V source is acting alone (Fig. 6.41)

$$\mathbf{Z}_T = 3 + \frac{(j5)(-j4)}{j5-j4} = 20.22\angle -81.47°\ \Omega$$

$$\mathbf{I}_T = \frac{15\angle 90°}{20.22\angle -81.47°} = 0.74\angle 171.47°\ \text{A}$$

Fig. 6.41

By current division rule,

$$\mathbf{I}'' = 0.74\angle 171.47° \times \frac{-j4}{-j4+j5} = 2.96\angle -8.53°\ \text{A}\ (\leftarrow) = 2.96\angle 171.47°\ \text{A} \quad (\rightarrow)$$

Step III When the 20 ∠0° V source is acting along (Fig. 6.42)

Fig. 6.42

$$\mathbf{Z}_T = -j4 + \frac{3(j5)}{3+j5} = 3.47\angle -50.51°\ \Omega$$

$$\mathbf{I}_T = \frac{20\angle 0°}{3.47\angle -50.51°} = 5.76\angle 50.51°\ \text{A}$$

By current division rule,

$$\mathbf{I}''' = 5.76\angle 50.51° \times \frac{3}{3+j5} = 2.96\angle -8.53°\ \text{A}\ (\leftarrow) = 2.96\angle 171.47°\ \text{A}\ (\rightarrow)$$

Step IV By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' + \mathbf{I}''' = 2.48\angle -61.66° + 2.96\angle 171.47° + 2.96\angle 171.47° = 4.86\angle -164.41° \text{ A}$$

Example 6.24 *Find the voltage drop across the capacitor for the network shown in Fig. 6.43.*

Fig. 6.43

Solution

Step I When the 10∠0° V source is acting alone (Fig. 6.44)

$$\mathbf{Z}_T = 4 + \frac{(2+j5)(2-j2)}{2+j5+2-j2}$$

$$= 7\angle -5.91° \ \Omega$$

$$\mathbf{I}_T = \frac{10\angle 0°}{7\angle -5.91°} = 1.43\angle 5.91° \text{ A}$$

Fig. 6.44

By current division rule,

$$\mathbf{I}' = (1.43\angle 5.91°)\left(\frac{2+j5}{2+j5+2-j2}\right) = 1.54\angle 37.24° \text{ A} \quad (\rightarrow)$$

Step II When the 20∠45° V source is acting alone (Fig. 6.45)

Fig. 6.45

$$\mathbf{Z}_T = (2-j2) + \frac{4(2+j5)}{4+2+j5} = 4.48\angle -8.84° \ \Omega$$

$$\mathbf{I}'' = \frac{20\angle 45°}{4.48\angle -8.84°} = 4.46\angle 53.84° \text{ A} \quad (\leftarrow) = -4.46\angle 53.84° \text{ A} \quad (\rightarrow)$$

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' = 1.54\angle 37.24 - 4.46\angle 53.84° = 3.01\angle -117.78° \text{ A}$$

$$\mathbf{V}_c = (-j2)\mathbf{I} = (-j2)\ (3.01\angle -117.78°) = 6.02\angle 152.22° \text{ V}$$

Example 6.25 *Find the node voltage $\mathbf{V}_2$ in the network of Fig. 6.46.*

Fig. 6.46

Solution

Step I When the $10\angle 0°$ A source is acting alone (Fig. 6.47)

Fig. 6.47

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1'}{5} + \frac{\mathbf{V}_1' - \mathbf{V}_2'}{5\angle 30°} = 10\angle 0°$$

$$\left(\frac{1}{5} + \frac{1}{5\angle 30°}\right)\mathbf{V}_1' - \frac{1}{5\angle 30°}\mathbf{V}_2' = 10\angle 0°$$

$$(0.37 - j0.1)\mathbf{V}_1' - (0.17 - j0.1)\mathbf{V}_2' = 10\angle 0° \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2' - \mathbf{V}_1'}{5\angle 30°} + \frac{\mathbf{V}_2'}{2} + \frac{\mathbf{V}_2'}{j10} = 0$$

$$-\frac{1}{5\angle 30°}\mathbf{V}_1' + \left(\frac{1}{5\angle 30°} + \frac{1}{2} + \frac{1}{j10}\right)\mathbf{V}_2' = 0$$

$$-(0.17 - j0.1)\mathbf{V}_1' + (0.67 - j0.2)\mathbf{V}_2' = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.37 - j0.1 & -(0.17 - j0.1) \\ -(0.17 - j0.1) & 0.67 - j0.2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_2' = \frac{\begin{vmatrix} 0.37 - j0.1 & 10\angle 0° \\ -(0.17 - j0.1) & 0 \end{vmatrix}}{\begin{vmatrix} 0.37 - j0.1 & -(0.17 - j0.1) \\ -(0.17 - j0.1) & 0.67 - j0.2 \end{vmatrix}} = 8.57\angle -3.36° \text{ V}$$

Step II When the $5\angle 0°$ A source is acting alone (Fig. 6.48)

Fig. 6.48

$$\frac{\mathbf{V}_2''}{5\angle 30° + 5} + \frac{\mathbf{V}_2''}{2} + \frac{\mathbf{V}_2''}{j10} = 5\angle 0°$$

$$(0.61\angle -11.93°)\mathbf{V}_2'' = 5\angle 0°$$

$$\mathbf{V}_2'' = 8.2\angle 11.93° \text{ V}$$

Step III By superposition theorem,

$$\mathbf{V}_2 = \mathbf{V}_2' + \mathbf{V}_2'' = 8.57\angle -3.36° + 8.2\angle 11.93° = 16.62\angle 4.12° \text{ V}$$

Example 6.26

Find current through inductor in the network of Fig. 6.49.

Fig. 6.49

Solution

Step I When the $8\angle 135°$ V source is acting alone (Fig. 6.50)
Applying KVL to the mesh,

$$8\angle 135° - (-j1)\mathbf{I}' - j2\mathbf{I}' = 0$$

$$\mathbf{I}' = \frac{8\angle 135°}{j1} = 8\angle 45° \text{ A } (\leftarrow) = 8\angle -135° \text{ A } (\rightarrow)$$

Fig. 6.50

Step II When the $2\angle 0°$ A source is acting alone (Fig. 6.51)

Fig. 6.51

The network can be redrawn as shown in Fig. 6.52.
By current division rule,

$$\mathbf{I}'' = 2\angle 0°\left(\frac{-j1}{-j1+j2}\right) = 2\angle 0°\left(\frac{-j1}{j1}\right) = 2\angle 180°\ \text{A}(\rightarrow)$$

Fig. 6.52

Step III When the $2\angle 90°$ A source is acting alone (Fig. 6.53)

Fig. 6.53

The network can be redrawn as shown in Fig. 6.54.
By current division rule,

$$\mathbf{I}''' = 2\angle 90°\left(\frac{-j1}{-j1+j2}\right) = 2\angle -90°\ \text{A}\ (\leftarrow) = 2\angle 90°\ \text{A}\ (\rightarrow)$$

Fig. 6.54

Step III By superposition theorem,

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}'' + \mathbf{I}''' = 8\angle -135° + 2\angle 180° + 2\angle 90° = 8.49\angle -154.47°\text{A}$$

Example 6.27 *Determine the source voltage $\mathbf{V}_S$ so that the current through 2 Ω resistor is zero in the network of Fig. 6.55.*

Fig. 6.55

Solution

Step I When the voltage source $\mathbf{V}_s$ is acting alone (Fig. 6.56)

Fig. 6.56

Appling KVL to Mesh 1,

$$\mathbf{V}_s - 3\mathbf{I}_1' - j3(\mathbf{I}_1' - \mathbf{I}_2') = 0$$

$$(3 + j3)\mathbf{I}_1' - j3\mathbf{I}_2' = \mathbf{V}_s \qquad \ldots\text{(i)}$$

Appling KVL to Mesh 2,

$$-j3(\mathbf{I}_2' - \mathbf{I}_1') - 2\mathbf{I}_2' + j3(\mathbf{I}_2' - \mathbf{I}_3') = 0$$

$$-j3\mathbf{I}_1' + 2\mathbf{I}_2' + j3\mathbf{I}_3' = 0 \qquad \ldots\text{(ii)}$$

Appling KVL to Mesh 3,

$$-j3(\mathbf{I}_3' - \mathbf{I}_2') - 4\mathbf{I}_3' = 0$$

$$j3\mathbf{I}_2' + (4 - j3)\mathbf{I}_3' = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1' \\ \mathbf{I}_2' \\ \mathbf{I}_3' \end{bmatrix} = \begin{bmatrix} \mathbf{V}_s \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2' = \frac{\begin{vmatrix} 3+j3 & \mathbf{V}_s & 0 \\ -j3 & 0 & j3 \\ 0 & 0 & 4-j3 \end{vmatrix}}{\begin{vmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{vmatrix}} = \frac{(9 + j12)\,\mathbf{V}_s}{\Delta}$$

Step II When the 20 ∠90° V source is acting alone (Fig. 6.57)

Fig. 6.57

Applying KVL to Mesh 1,

$$-3\mathbf{I}_1'' - j3(\mathbf{I}_1'' - \mathbf{I}_2'') = 0$$

$$(3 + j3)\,\mathbf{I}_1'' - j3\mathbf{I}_2'' = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j3(\mathbf{I}_2'' - \mathbf{I}_1'') - 2\mathbf{I}_2'' + j3(\mathbf{I}_2'' - \mathbf{I}_3'') = 0$$

$$-j3\mathbf{I}_1'' + 2\mathbf{I}_2'' + j3\mathbf{I}_3'' = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$j3(\mathbf{I}_3'' - \mathbf{I}_2'') - 4\mathbf{I}_3'' - 20\angle 90° = 0$$

$$j3\mathbf{I}_2'' + (4 - j3)\mathbf{I}_3'' = -20\angle 90° \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1'' \\ \mathbf{I}_2'' \\ \mathbf{I}_3'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -20\angle 90° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2'' = \frac{\begin{vmatrix} 3+j3 & 0 & 0 \\ -j3 & 0 & j3 \\ 0 & -20\angle 90° & 4-j3 \end{vmatrix}}{\begin{vmatrix} 3+j3 & -j3 & 0 \\ -j3 & 2 & j3 \\ 0 & j3 & 4-j3 \end{vmatrix}} = \frac{-180 - j180}{\Delta}$$

Step III By superposition theorem,

$$\mathbf{I}_2 = \mathbf{I}_2' + \mathbf{I}_2'' = \frac{(9+j12)\mathbf{V}_s + (-180 - j180)}{\Delta} = 0$$

$$(9+j12)\mathbf{V}_s + (-180 - j180) = 0$$

$$(9+j12)\mathbf{V}_s = 180 + j180$$

$$\mathbf{V}_s = 16.97\angle -8.13° \text{ V}$$

6.5 THEVENIN'S THEOREM

Thevenin's theorem gives us a method for simplifying a network. In Thevenin's theorem, *any linear network can be replaced by a voltage source* $\mathbf{V}_{\text{Th}}$ *in series with an impedance* $\mathbf{Z}_{\text{Th}}$.

Example 6.28 *Obtain Thevenin's equivalent network for the terminals A and B in Fig. 6.58.*

Fig. 6.58

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.59)

Fig. 6.59

Applying KVL to the mesh,

$$50\angle 0° - (3 - j4)\,\mathbf{I} - (4 + j6)\,\mathbf{I} = 0$$

$$\mathbf{I} = \frac{50\angle 0°}{(3 - j4) + (4 + j6)} = 6.87\angle -15.95° \text{ A}$$

$$\mathbf{V}_{Th} = (4 + j6)\,\mathbf{I} = (4 + j6)\,(6.87\angle -15.95°) = 49.5\angle 40.35° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.60)

$$\mathbf{Z}_{Th} = (j5 - j4) + \frac{(3 - j4)(4 + j6)}{(3 - j4) + (4 + j6)} = 4.83\angle -1.13° \ \Omega$$

Fig. 6.60

Step III Thevenin's Equivalent Network (Fig. 6.61)

Fig. 6.61

Example 6.29 *Find Thevenin's equivalent network for Fig. 6.62.*

Fig. 6.62

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.63)

Fig. 6.63

Applying KVL to Mesh 1,

$$10\angle 30° - (5 - j2)\,\mathbf{I}_1 - 3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(8 - j2)\,\mathbf{I}_1 - 3\mathbf{I}_2 = 10\angle 30° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-3(\mathbf{I}_2 - \mathbf{I}_1) - j5\,\mathbf{I}_2 - 5\,\mathbf{I}_2 = 0$$

$$-3\mathbf{I}_1 + (8 + j5)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form;

$$\begin{bmatrix} 8 - j2 & -3 \\ -3 & 8 + j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 30° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 8 - j2 & 10\angle 30° \\ -3 & 0 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 \\ -3 & 8 + j5 \end{vmatrix}} = 0.433\angle 9.7°\ \text{A}$$

$$\mathbf{V}_{Th} = 5\mathbf{I}_2 = 5(0.433\angle 9.7°) = 2.16\angle 9.7°\ \text{V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.64)

$$\mathbf{Z}_{Th} = \left[\left\{\frac{(5 - j2)3}{5 - j2 + 3}\right\} + j5\right] \| 5$$

$$= [1.94 - j0.265 + j5] \| 5 = (1.94 + j4.735) \| 5$$

$$= \frac{(1.94 + j4.735)5}{6.94 + j4.735} = 3.04\angle 33.4°\ \Omega$$

Fig. 6.64

Step III Thevenin's equivalent Network (Fig. 6.65)

Fig. 6.65

Example 6.30 *Obtain Thevenin's equivalent network for Fig. 6.66.*

Fig. 6.66

Solution

Step I Calculation of $\mathbf{V}_{Th}$

Fig. 6.67

Applying KVL to the mesh,

$$(2 + j6 - j4)\mathbf{I} - 5\angle 90° = 0$$

$$\mathbf{I} = \frac{5\angle 90°}{2 + j2} = 1.77\angle 45° \text{ A}$$

$$\mathbf{V}_{Th} = (-j4)\,\mathbf{I} + 5\angle 90° - 10\angle 0° = (4\angle -90°)\,(1.77\angle 45°) + 5\angle 90° - 10\angle 0° = 18\angle 146.31° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.67)

Fig. 6.68

$$\mathbf{Z}_{Th} = 4 + \frac{(2 + j6)(-j4)}{2 + j2} = 11.3\angle -44.93° \ \Omega$$

Step III Thevenin's Equivalent Network

Fig. 6.69

Example 6.31 *Obtain Thevenin's equivalent network for Fig. 6.70.*

Fig. 6.70

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.71)
By current division rule,

$$\mathbf{I} = \frac{(10\angle 0°)(j15)}{5 - j5 + j15} = 13.42\angle 26.57° \text{ A}$$

$$\mathbf{V}_{Th} = (-j5)\,\mathbf{I}$$
$$= (5\angle -90°)\,(13.42\angle 26.57°) = 67.08\angle -63.43° \text{ V}$$

Fig. 6.71

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.72)

$$\mathbf{Z}_{Th} = \frac{(-j5)(5 + j15)}{-j5 + 5 + j15} = 7.07\angle -81.86° \ \Omega$$

Fig. 6.72

Step III Thevenin's Equivalent Network

Fig. 6.73

Example 6.32 *Obtain Thevenin's equivalent network for Fig. 6.74.*

Fig. 6.74

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig 6.75)

Fig. 6.75

$$\mathbf{I}_1 = \frac{20\angle 0°}{21+12+j24} = 0.49\angle -36.02° \text{ A}$$

$$\mathbf{I}_2 = \frac{20\angle 0°}{80+j60} = 0.2\angle -36.86° \text{ A}$$

$$\begin{aligned}\mathbf{V}_{Th} &= (12+j24)\,\mathbf{I}_1 - (30+j60)\,\mathbf{I}_2 \\ &= (26.83\ \angle 63.43°)\,(0.49\ \angle -36.02°) - (67.08\ \angle 63.43°)\,(0.2\ \angle -36.86°) \\ &= 0.33\ \angle 171.12° \text{ V}\end{aligned}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.76)

Fig. 6.76

$$\mathbf{Z}_{Th} = \frac{21(12+j24)}{33+j24} + \frac{50(30+j60)}{80+j60} = 47.4\angle 26.8° \ \Omega$$

Step III Thevenin's Equivalent Network

Fig. 6.77

Example 6.33 *Find Thevenin's equivalent network across terminals A and B for Fig. 6.78.*

Fig. 6.78

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.79)

Fig. 6.79

Applying KCL at the node,

$$\frac{\mathbf{V}_{Th}}{1+j2}+\frac{\mathbf{V}_{Th}-10\angle 90^\circ}{5}=2\angle 45^\circ$$

$$\left(\frac{1}{1+j2}+\frac{1}{5}\right)\mathbf{V}_{Th}=2\angle 45^\circ+2\angle 90^\circ$$

$$(0.57\angle -45^\circ)\mathbf{V}_{Th}=3.7\angle 67.5^\circ$$

$$\mathbf{V}_{Th}=6.49\angle 112.5^\circ \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.80)

Fig. 6.80

$$\mathbf{Z}_{Th}=\frac{5(1+j2)}{5+1+j2}=1.77\angle 45^\circ\ \Omega$$

Step III Thevenin's Equivalent Network (Fig. 6.81)

Fig. 6.81

Example 6.34 *Find the current through the* $(5 + j2)\ \Omega$ *impedance in the network of Fig. 6.82.*

Fig. 6.82

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.83)

Fig. 6.83

Applying KCL at the node,

$$\frac{\mathbf{V}_1 - 20\angle 0^\circ}{5} + \frac{\mathbf{V}_1}{2 - j2} = 20\angle 0^\circ$$

$$\left(\frac{1}{5} + \frac{1}{2 - j2}\right)\mathbf{V}_1 = 20\angle 0^\circ + 4\angle 0^\circ$$

$$0.51\angle 29.05^\circ\ \mathbf{V}_1 = 24\angle 0^\circ$$

$$\mathbf{V}_1 = 47.06\angle -29.05^\circ \text{ V}$$

$$\mathbf{V}_{Th} = \mathbf{V}_1 = 47.06\angle -29.05^\circ \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.84)

Fig. 6.84

$$\mathbf{Z}_{Th} = 3 + \frac{5(2-j2)}{5+2-j2} = 4.79\angle -11.35^\circ\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 6.85)

Fig. 6.85

$$\mathbf{I}_L = \frac{47.06\angle -29.05^\circ}{4.79\angle -11.35^\circ + 5 + j2} = 4.73\angle -39.96^\circ\ \text{A}$$

Example 6.35 *Find the current through the 5 Ω resistor in the network of Fig. 6.86.*

Fig. 6.86

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.87)

Fig. 6.87

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1}{4}+\frac{\mathbf{V}_1-\mathbf{V}_2}{j5}+6\angle 0^\circ=0$$

$$\left(\frac{1}{4}+\frac{1}{j5}\right)\mathbf{V}_1-\frac{1}{j5}\mathbf{V}_2=-6\angle 0^\circ$$

$$(0.25-j0.2)\mathbf{V}_1+j0.2\mathbf{V}_2=-6\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2-\mathbf{V}_1}{j5}+\frac{\mathbf{V}_2}{-j2}=4\angle 0^\circ$$

$$\left(-\frac{1}{j5}\right)\mathbf{V}_1+\left(\frac{1}{j5}-\frac{1}{j2}\right)\mathbf{V}_2=4\angle 0^\circ$$

$$j0.2\,\mathbf{V}_1+j0.3\,\mathbf{V}_2=4\angle 0^\circ \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix}0.25-j0.2 & j0.2\\ j0.2 & j0.3\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}=\begin{bmatrix}-6\angle 0^\circ\\ 4\angle 0^\circ\end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1=\frac{\begin{vmatrix}-6\angle 0^\circ & j0.2\\ 4\angle 0^\circ & j0.3\end{vmatrix}}{\begin{vmatrix}0.25-j0.2 & j0.2\\ j0.2 & j0.3\end{vmatrix}}=20.8\angle -126.87^\circ \text{ V}$$

$$\mathbf{V}_{\text{Th}}=\mathbf{V}_1=20.8\angle -126.87^\circ \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 6.88)

Fig. 6.88

$$\mathbf{Z}_{\text{Th}}=\frac{4(-j2+j5)}{4-j2+j5)}=2.4\angle 53.13^\circ\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 6.89)

$$\mathbf{I}_L=\frac{20.8\angle -126.87^\circ}{2.4\angle 53.13^\circ+5}=3.1\angle -143.47^\circ \text{ A}$$

Fig. 6.89

Example 6.36 *In the network of Fig. 6.90, find the current through the 10 Ω resistor.*

Fig. 6.90

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.91)

Applying KVL to the mesh,

$$j2\mathbf{I} - 1\mathbf{I} - 10\angle 0° - 5\mathbf{I} = 0$$

$$(j2-6)\mathbf{I} = 10\angle 0°$$

$$\mathbf{I} = 1.58\angle -161.57° \text{ A}$$

Writing $\mathbf{V}_{Th}$ equation,

$$5\mathbf{I} + 10\angle 0° - 5\angle 30° - 0 - \mathbf{V}_{Th} = 0$$

$$5(1.58\angle -161.57°) - 10\angle 0° - 5\angle 30° - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 5.32\angle -110.06° \text{ V}$$

Fig. 6.91

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.92)

$$\mathbf{Z}_{Th} = 2 + \frac{5(1-j2)}{5+1-j2} = 3.48\angle -21.04° \ \Omega$$

Fig. 6.92

Step III Calculation of $\mathbf{I}_L$ (Fig. 6.93)

Fig. 6.93

$$\mathbf{I}_L = \frac{5.32\angle -110.06°}{3.48\angle -21.04° + 10} = 0.4\angle -104.67° \text{ A}$$

Example 6.37 *Find the current through $(4+j6)\ \Omega$ impedance in the network of Fig. 6.94.*

Fig. 6.94

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.95)

Fig. 6.95

Applying KVL to the mesh,

$$100\angle 0° - 2\mathbf{I} - j5\mathbf{I} - 3\mathbf{I} + j5\mathbf{I} - 50\angle 90° = 0$$

$$\mathbf{I} = 22.36\angle -26.57° \text{ A}$$

Writing $\mathbf{V}_{Th}$ equation,

$$\mathbf{V}_{Th} - 3\mathbf{I} + j5\mathbf{I} - 50\angle 90° = 0$$

$$\mathbf{V}_{Th} - (3 - j5)(22.36\angle -26.57°) - 50\angle 90° = 0$$

$$\mathbf{V}_{Th} = 80.61\angle -82.88° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.96)

Fig. 6.96

$$\mathbf{Z}_{Th} = \frac{(2+j5)(3-j5)}{2+j5+3-j5} = 6.28\angle 9.16° \ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 6.97)

Fig. 6.97

$$\mathbf{I}_L = \frac{80.61\angle -82.88°}{6.28\angle 9.16° + 4 + j6} = 6.52\angle -117.34° \text{ A}$$

Example 6.38 *Obtain Thevenin's equivalent network across terminals A and B in Fig. 6.98.*

Fig. 6.98

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.99)

Applying KVL to the mesh,

$$10\angle 0° - 4\mathbf{I} + j1\mathbf{I} - 2\mathbf{I} = 0$$

$$\mathbf{I} = 1.64\angle 9.46° \text{ A}$$

Fig. 6.99

Writing $\mathbf{V}_{Th}$ equation,

$$10\angle 0° - 4\mathbf{I} - 0 - \mathbf{V}_{Th} = 0$$

$$10\angle 0° - 4(1.64\angle 9.46°) - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 3.69\angle -17° \text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 6.100)

From Fig. 6.100,

$$\mathbf{I} = \mathbf{I}_1$$

Applying KVL to Mesh 1,

$$10\angle 0° - 4\mathbf{I}_1 + j1(\mathbf{I}_1 - \mathbf{I}_2) - 2\mathbf{I} = 0$$

$$10\angle 0° - 4\mathbf{I}_1 + j1\mathbf{I}_1 - j1\mathbf{I}_2 - 2\mathbf{I}_1 = 0$$

$$(6 - j1)\mathbf{I}_1 + j1\mathbf{I}_2 = 10\angle 0° \quad \ldots(i)$$

Fig. 6.100

Applying KVL to Mesh 2,

$$2\mathbf{I} + j1(\mathbf{I}_2 - \mathbf{I}_1) - j2\mathbf{I}_2 = 0$$

$$2\mathbf{I}_1 + j1\mathbf{I}_2 - j1\mathbf{I}_1 - j_2\mathbf{I}_2 = 0$$

$$(2 - j1)\mathbf{I}_1 - j1\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 6 - j1 & j1 \\ 2 - j1 & -j1 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 6 - j1 & 10\angle 0° \\ 2 - j1 & 0 \end{vmatrix}}{\begin{vmatrix} 6 - j1 & j1 \\ 2 - j1 & -j1 \end{vmatrix}} = 2.71\angle -102.53° \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 2.71\angle -102.53° \text{ A}$$

Step III Calculation of $\mathbf{Z}_{\text{Th}}$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{3.69\angle -17°}{2.71\angle -102.53°} = 1.36\angle 85.53° \ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 6.101)

Fig. 6.101

Example 6.39 *Find Thevenin's equivalent network across terminals A and B for Fig. 6.102.*

Fig. 6.102

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 6.103)

From Fig. 6.103,

$$\mathbf{I} = -0.2\,\mathbf{V}_x \qquad \ldots \text{(i)}$$

Writing $\mathbf{V}_{\text{Th}}$ equation,

$$-\mathbf{I} + 5\angle 0° - 0 - \mathbf{V}_x = 0$$

Fig. 6.103

$$0.2\mathbf{V}_x + 5\angle 0° - \mathbf{V}_x = 0$$
$$\mathbf{V}_x = 6.25\angle 0° \text{ V}$$
$$\mathbf{V}_{\text{Th}} = \mathbf{V}_x = 6.25\angle 0° \text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 6.104)

Fig. 6.104

From Fig. 6.104,

$$\mathbf{V}_x = 0$$

The dependent source depends on the controlling variable $\mathbf{V}_x$. When $\mathbf{V}_x = 0$, the dependent source vanishes, i.e. $0.2\mathbf{V}_x = 0$ as shown in Fig. 6.105.

$$\mathbf{I}_N = \frac{5\angle 0°}{1+2+j4} = 1\angle -53.13° \text{ A}$$

Fig. 6.105

Step III Calculation of $\mathbf{Z}_{\text{Th}}$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{6.25\angle 0°}{1\angle -53.13°} = 6.25\angle 53.13° \ \Omega$$

Step IV Thevenin's Equivalent Network (Fig. 6.106)

Fig. 6.106

6.6 NORTON'S THEOREM

Norton's theorem states that *any linear network can be replaced by a current source* $\mathbf{I}_N$ *parallel with an impedance* $\mathbf{Z}_N$ *where* $\mathbf{I}_N$ *is the current flowing through the short-circuited path placed across the terminals.*

Example 6.40 *Obtain Norton's equivalent network between terminals A and B as shown in Fig. 6.107.*

Fig. 6.107

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 6.108)

When a short circuit is placed across $(4 - j4)\ \Omega$ impedance, it gets shorted as shown in Fig. 6.109.

Fig. 6.108

Fig. 6.109

$$\mathbf{I}_N = \frac{25\angle 0^\circ}{3 + j4} = 5\angle -53.13^\circ \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 6.110)

$$\mathbf{Z}_N = \frac{(3 + j4)(4 - j5)}{3 + j4 + 4 - j5} = 4.53\angle 9.92^\circ\ \Omega$$

Fig. 6.110

Step III Norton's Equivalent Network

Fig. 6.111

Example 6.41 *Obtain Norton's equivalent network at the terminals A and B in Fig. 6.112.*

Fig. 6.112

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 6.113)

Fig. 6.113

By series-parallel reduction technique (Fig. 6.114)

Fig. 6.114

$$\mathbf{I}_N = (10\angle 30°)\left(\frac{1.62\angle 58.24°}{1.62\angle 58.24° + 5}\right) = 2.69\angle 75° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 6.115)

Fig. 6.115

$$\mathbf{Z}_N = 5 + \frac{(1+j2)(4+j4)}{1+j2+4+j4} = 6.01\angle 13.24° \ \Omega$$

Step III Norton's Equivalent Network (Fig. 6.116)

Fig. 6.116

Example 6.42 *Find Norton's equivalent network across terminals A and B in Fig. 6.117.*

Fig. 6.117

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 6.118)

Fig. 6.118

When a short circuit is placed across the $(3+j4)\ \Omega$ impedance, it gets shorted as shown in Fig. 6.119.

Fig. 6.119

By source transformation, the network is redrawn as shown in Fig. 6.120.

Fig. 6.120

$$I_N = 4\angle 45° + 2.5\angle 90° = 6.03\angle 62.04° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 6.121)

Fig. 6.121

$$\mathbf{Z}_N = \frac{10(3+j4)}{10+3+j4} = 3.68\angle 36.03° \ \Omega$$

Step III Norton's Equivalent Network (Fig. 6.122)

Fig. 6.122

Example 6.43 *Obtain the Norton's equivalent network for Fig. 6.123.*

Fig. 6.123

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 6.124)

Fig. 6.124

By source transformation, the network can be redrawn as shown in Fig. 6.125.
Writing KVL equations in matrix form,

$$\begin{bmatrix} 5 & j5 \\ j5 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5 & 50\angle 0° \\ j5 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & j5 \\ j5 & 0 \end{vmatrix}} = 10\angle -90° \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 10\angle -90° \text{ A}$$

Fig. 6.125

Step II Calculation of $\mathbf{Z}_N$ (Fig. 6.126)

Fig. 6.126

$$\mathbf{Z}_N = j5 + \frac{(5+j5)(-j5)}{5+j5-j5} = 5\ \Omega$$

Step III Norton's Equivalent Network (Fig. 6.127)

Fig. 6.127

Example 6.44 *Obtain the Norton's equivalent network for Fig. 6.128.*

Fig. 6.128

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 6.129)

Writing KVL equations in matrix form,

$$\begin{bmatrix} 15-j2 & -10+j2 & -5 \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 15+j2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 45° \\ 0 \\ 0 \end{bmatrix}$$

Fig. 6.129

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 15-j2 & 10\angle 45° & -5 \\ -10+j2 & 0 & 0 \\ -5 & 0 & 15+j2 \end{vmatrix}}{\begin{vmatrix} 15-j2 & -10+j2 & -5 \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 15+j2 \end{vmatrix}} = 1\angle 41.28° \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 15-j2 & -10+j2 & 10\angle 45° \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 15-j2 & -10+j2 & -5 \\ -10+j2 & 15-j2 & 0 \\ -5 & 0 & 15+j2 \end{vmatrix}} = 0.49\angle 37.41° \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_3 - \mathbf{I}_2 = 0.49\angle 37.41 - 1\angle 41.28° = 0.51\angle -135° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 6.130)

Fig. 6.130

$$\mathbf{Z}_N = \frac{5(10-j2)}{5+10-j2} + \frac{5(10+j2)}{5+10+j2} = 6.72\ \Omega$$

Step III Norton's Equivalent Network (Fig. 6.131)

Fig. 6.131

Example 6.45 *Find the current through the 8 Ω resistor in the Network of Fig. 6.132.*

Fig. 6.132

Solution

Step I Calculation of $\mathbf{I}_N$ (Fig. 6.133)

Fig. 6.133

When a short circuit is placed across the $(10+j4)\ \Omega$ impedance, it gets shorted as shown in fig. 6.134.

Fig. 6.134

By source transformation, the network is redrawn as shown in Fig. 6.135.

Fig. 6.135

$$\mathbf{I}_N = 4\angle 0° + 5\angle 0° = 9\angle 0° \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$ (Fig. 6.136)

Fig. 6.136

$$\mathbf{Z}_N = \frac{5(10 + j4)}{5 + 10 + j4} = 3.47\angle 6.87° \ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 6.137)

$$\mathbf{I}_L = \frac{9\angle 0°}{3.47\angle 6.87° + 8} = 0.79\angle -2.08° \text{ A}$$

Fig. 6.137

Example 6.46 *Obtain Norton's equivalent network across the terminals A and B in Fig. 6.138.*

Fig. 6.138

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.139)

Fig. 6.139

$$\mathbf{I} = \frac{10\angle 0°}{100 + j10} = 0.1\angle -5.71° \text{ A}$$

Writing $\mathbf{V}_{Th}$ equation,

$$10\angle 0° - 100\,\mathbf{I} - (-j5)(5\mathbf{I}) - \mathbf{V}_{Th} = 0$$

$$10\angle 0° - 100(0.1\angle -5.71°) + (j5)(5)(0.1\angle -5.71°) - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 3.5\angle 85.1° \text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 6.140)

Fig. 6.140

By source transformation, the network is redrawn as shown in Fig. 6.141.

Fig. 6.141

From Fig. 6.141,

$$\mathbf{I} = \mathbf{I}_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$10\angle 0° - 100\,\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(100 + j10)\mathbf{I}_1 - j10\mathbf{I}_2 = 10\angle 0° \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) + j5\mathbf{I}_2 + j25\mathbf{I} = 0$$

$$-j10\mathbf{I}_2 + j10\mathbf{I}_1 + j5\mathbf{I}_2 + j25\mathbf{I}_1 = 0$$

$$j35\mathbf{I}_1 - j5\mathbf{I}_2 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 100 + j10 & -j10 \\ j35 & -j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 100 + j10 & 10\angle 0° \\ j35 & 0 \end{vmatrix}}{\begin{vmatrix} 100 + j10 & -j10 \\ j35 & -j5 \end{vmatrix}} = 0.6\angle 30.96° \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 0.6\angle 30.96° \text{ A}$$

Step III Calculation of $\mathbf{Z}_N$

$$\mathbf{Z}_N = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{3.5\angle 85.1°}{0.6\angle 30.96°} = 5.83\angle 54.14° \ \Omega$$

Step IV Norton's Equivalent Network (Fig. 6.142)

Fig. 6.142

6.7 MAXIMUM POWER TRANSFER THEOREM

This theorem is used to determine the value of load impedance for which the source will transfer maximum power.

Consider a simple network as shown in Fig. 6.143. There are three possible cases for load impedance $\mathbf{Z}_L$.

Fig. 6.143

Case (i) When the load impedance is variable resistance (Fig. 6.144)

$$\mathbf{I}_L = \frac{\mathbf{V}_s}{\mathbf{Z}_s + \mathbf{Z}_L} = \frac{\mathbf{V}_s}{R_s + jX_s + R_L}$$

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_s|}{\sqrt{(R_s + R_L)^2 + X_s^2}}$$

Fig. 6.144 *Purely resistive load*

The power delivered to the load is

$$P_L = |\mathbf{I}_L|^2 R_L = \frac{|\mathbf{V}_s|^2 R_L}{\sqrt{(R_s + R_L)^2 + X_s^2}}$$

For power to be maximum,

$$\frac{dP_L}{dR_L} = 0$$

$$|\mathbf{V}_s|^2 \left[\frac{\{(R_s + R_L)^2 + X_s^2\} - 2R_L(R_s + R_L)}{[(R_s + R_L)^2 + X_s^2]^2} \right] = 0$$

$$(R_s + R_L)^2 + X_s^2 - 2R_L(R_s + R_L) = 0$$

$$R_s^2 + 2R_sR_L + R_L^2 + X_s^2 - 2R_LR_s - 2R_L^2 = 0$$

$$R_s^2 + X_s^2 - R_L^2 = 0$$

$$R_L^2 = R_s^2 + X_s^2$$

$$R_L = \sqrt{R_s^2 + X_s^2} = |Z_s|$$

Hence, load resistance R_L should be equal to the magnitude of the source impedance for maximum power transfer.

Case (ii) When the load impedance is a complex impedance with variable resistance and variable reactance (Fig. 6.145)

$$\mathbf{I}_L = \frac{\mathbf{V}_s}{\mathbf{Z}_s + \mathbf{Z}_L}$$

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_s|}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

Fig. 6.145 *Complex impedance load*

The power delivered to the load is

$$P_L = |\mathbf{I}_L|^2 R_L = \frac{|\mathbf{V}_S|^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

For maximum value of P_L, denominator of the equation should be small, ie. $X_L = -X_s$.

$$P_L = \frac{|\mathbf{V}_S|^2 R_L}{(R_s + R_L)^2}$$

Differentiating the above equation w.r.t. R_L and equating to zero,

$$\frac{dP_L}{dR_L} = |\mathbf{V}_s|^2 \left[\frac{(R_s + R_L)^2 - 2\,R_L\,(R_s + R_L)}{(R_s + R_L)^2} \right] = 0$$

$$(R_s + R_L)^2 - 2\,R_L\,(R_s + R_L) = 0$$

$$R_s^2 + 2\,R_s R_L + R_L^2 - 2\,R_L R_s - 2R_L^2 = 0$$

$$R_s^2 - R_L^2 = 0$$

$$R_L^2 = R_s^2$$

$$R_L = R_s$$

Hence, load resistance R_L should be equal to source resistance R_L and load reactance X_L should be equal to negative value of source reactance for maximum power transfer.

$$\mathbf{Z}_L = \mathbf{Z}_s^* = R_s - jX_s$$

i.e. load impedance should be a complex conjugate of the source impedance.

Case (iii) When the load impedance is a complex impedance with variable resistance and fixed reactance (Fig. 6.146)

$$\mathbf{I}_L = \frac{\mathbf{V}_s}{\mathbf{Z}_s + \mathbf{Z}_L}$$

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_s|}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

Fig. 6.146 *Complex impedance load*

The power delivered to the load is

$$P_L = |\mathbf{I}_L|^2\, R_L = \frac{|\mathbf{V}_s|^2\, R_L}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

For maximum power,

$$\frac{dP_L}{dR_L} = 0$$

$$|\mathbf{V}_s|^2 \left[\frac{(R_s + R_L)^2 + (X_s + X_L)^2 - 2\,R_L(R_s + R_L)}{\{(R_s + R_L)^2 + (X_s + X_L)^2\}^2} \right] = 0$$

$$(R_s + R_L)^2 + (X_s + X_L)^2 - 2\,R_L(R_s + R_L) = 0$$

$$R_s^2 + 2\,R_s\,R_L + R_L^2 + (X_s + X_L)^2 - 2\,R_L R_s - 2\,R_L^2 = 0$$

$$R_s^2 + (X_s + X_L)^2 - R_L^2 = 0$$

$$R_L^2 = R_s^2 + (X_s + X_L)^2$$

$$R_L = \sqrt{R_s^2 + (X_s + X_L)^2}$$

$$= |R_s + j(X_s + X_L)|$$

$$= |R_s + jX_s + jX_L|$$

$$= |Z_s + jX_L|$$

Hence, load resistance R_L should be equal to the magnitude of the impedance $\mathbf{Z}_s + jX_L$, i.e. $|\mathbf{Z}_s + jX_L|$ for maximum power transfer.

Example 6.47 *For maximum power transfer, find the value of $\mathbf{Z}_L$ in the network of Fig. 6.147 if (i) $\mathbf{Z}_L$ is an impedance, and (ii) $\mathbf{Z}_L$ is pure resistance.*

Fig. 6.147

Solution $\mathbf{Z}_s = (6 - j8)\ \Omega$

(i) If $\mathbf{Z}_L$ is an impedance

For maximum power transfer, $\mathbf{Z}_L = \mathbf{Z}_s^* = (6 + j8)\ \Omega$

(ii) If $\mathbf{Z}_L$ is a resistance

For maximum power transfer, $\mathbf{Z}_L = |\mathbf{Z}_S| = |6 + j8| = 10\ \Omega$

Example 6.48 *For the maximum power transfer, find the value of $\mathbf{Z}_L$ in the network of Fig. 6.148 for the following cases:*

(i) $\mathbf{Z}_L$ is variable resistance, (ii) $\mathbf{Z}_L$ is complex impedance, with variable resistance and variable reactance, and (iii) $\mathbf{Z}_L$ is complex impedance with variable resistance and fixed reactance of j5 Ω.

Fig. 6.148

Solution Thevenin's impedance can be calculated by replacing voltage source by a short circuit and current source by an open circuit.

Fig. 6.149

$$\mathbf{Z}_{\text{Th}} = \frac{3(2+j5)}{3+2+j5} = (2.1+j0.9)\ \Omega$$

For maximum power transfer, value of $\mathbf{Z}_L$ will be,

(i) $\mathbf{Z}_L$ is variable resistance

$$\mathbf{Z}_L = |\mathbf{Z}_{\text{Th}}| = |2.1+j0.9| = 2.28\ \Omega$$

(ii) $\mathbf{Z}_L$ is complex impedance with variable resistance and variable reactance

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = (2.1-j0.9)\ \Omega$$

(iii) $\mathbf{Z}_L$ is complex impedance with variable resistance and fixed reactance of $j5\ \Omega$

$$\mathbf{Z}_L = |\mathbf{Z}_{\text{Th}} + j5| = |2.1+j0.9+j5| = 6.26\ \Omega$$

Example 6.49 *Find the impedance $\mathbf{Z}_L$ so that maximum power can be transferred to it in the network of Fig. 6.150. Find maximum power.*

Fig. 6.150

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 6.151)

Fig. 6.151

$$\mathbf{Z}_T = 3 + \frac{j3(3-j3)}{3+j3-j3} = 6.71\angle 26.57°\ \Omega$$

$$\mathbf{I}_T = \frac{5\angle 0°}{6.71\angle 26.57°} = 0.75\angle -26.57°\ \text{A}$$

By current division rule,

$$\mathbf{I} = 0.75\angle -26.57° \times \frac{j3}{3+j3-j3} = 0.75\angle 63.43°\ \text{A}$$

$$\mathbf{V}_{\text{Th}} = (-j3)(0.75\angle 63.43°) = 2.24\angle -26.57°\ \text{V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.152)

$$\mathbf{Z}_{Th} = [(3 \parallel j3) + 3] \parallel (-j3)$$
$$= 3 \angle -53.12° \ \Omega$$
$$= (1.8 - j2.4) \ \Omega$$

Fig. 6.152

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be a complex conjugate of the source impedance.

$$\mathbf{Z}_L = (1.8 + j2.4) \ \Omega$$

Step IV Calculation of P_{max} (Fig. 6.153)

Fig. 6.153

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{4R_L} = \frac{|2.24|^2}{4 \times 1.8} = 0.7 \text{ W}$$

Example 6.50 *Find the value of $\mathbf{Z}_L$ for maximum power transfer in the network shown and find maximum power.*

Fig. 6.154

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.155)

$$\mathbf{I}_1 = \frac{100\angle 0°}{5 + j10} = 8.94\angle -63.43° \text{ A}$$

$$\mathbf{I}_2 = \frac{100\angle 0°}{7 - j20} = 4.72\angle 70.7° \text{ A}$$

Fig. 6.155

$$\mathbf{V}_{\text{Th}} = \mathbf{V}_A - \mathbf{V}_B = (8.94\angle -63.43°)(j10) - (4.72\angle 70.7°)(-j20) = 71.76\angle 97.3° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 6.156)

Fig. 6.156

$$\mathbf{Z}_{\text{Th}} = \frac{5(j10)}{5+j10} + \frac{7(-j20)}{7-j20} = \frac{50\angle 90°}{11.18\angle 63.43°} + \frac{140\angle -90°}{21.19\angle -70.7°} = (10.23 - j0.18)\ \Omega$$

Step III For maximum power transfer, the load impedance should be complex conjugate of the source impedance.

$$\mathbf{Z}_L = (10.23 + j0.18)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 6.157)

Fig. 6.157

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{|71.76|^2}{4\times 10.23} = 125.84 \text{ W}$$

Example 6.51 *Find the value of load impedance $\mathbf{Z}_L$ so that maximum power can be transferred to it in the network of Fig. 6.158. Find maximum power.*

Fig. 6.158

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 6.159)

Fig. 6.159

$$\mathbf{I} = \frac{50\angle 45°}{3+2+j10} = 4.47\angle -18.43° \text{ A}$$

$$\mathbf{V}_{\text{Th}} = (2+j10)\,\mathbf{I} = (2+j10)(4.47\angle -18.43°) = 45.6\angle 60.26° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 6.160)

Fig. 6.160

$$\mathbf{Z}_{\text{Th}} = \frac{3(2+j10)}{3+2+j10} = (2.64+j0.72)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be complex conjugate of the source impedance.

$$\mathbf{Z}_L = (2.64 - j0.72)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 6.161)

Fig. 6.161

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{|45.6|^2}{4\times 2.64} = 196.91 \text{ W}$$

Example 6.52 *Determine the load $\mathbf{Z}_L$ required to be connected in the network of Fig. 6.162 for maximum power transfer. Determine the maximum power drawn.*

Fig. 6.162

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$ (Fig. 6.163)

Fig. 6.163

$$\mathbf{I}_2 = 4\angle 0° \times \frac{2}{6+j1} = 1.315\angle -9.46° \text{ A}$$

$$\mathbf{V}_{\text{Th}} = 4\mathbf{I}_2 = 4(1.315\angle -9.46°) = 5.26\angle -9.46° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 6.164)

$$\mathbf{Z}_{\text{Th}} = \frac{4(2+j1)}{4+2+j1} = 1.47\angle 17.1° = (1.41 + j0.43)\ \Omega$$

Fig. 6.164

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be the complex conjugate of the source impedance.

$$\mathbf{Z}_L = (1.41 - j0.43)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 6.165)

Fig. 6.165

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{|5.26|^2}{4 \times 1.41} = 4.91 \text{ W}$$

Example 6.53 *In the network shown in Fig. 6.166, find the value of $\mathbf{Z}_L$ for which the power transferred will be maximum. Also find maximum power.*

Fig. 6.166

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.167)

Fig. 6.167

Applying KVL to the mesh,

$$10\angle 0° - (5\angle 60°)\,\mathbf{I} - (10\angle -30°)\,\mathbf{I} - 5\angle 90° = 0$$

$$11.18\angle -26.57° - (11.18\angle -3.43°)\,\mathbf{I} = 0$$

$$\mathbf{I} = 1\angle -23.14° \text{ A}$$

Writing $\mathbf{V}_{Th}$ equation,

$$10\angle 0° - (5\angle 60°)\mathbf{I} - \mathbf{V}_{Th} = 0$$

$$10\angle 0° - (5\angle 60°)(1\angle -23.14°) - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 6.71\angle -26.56° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$ (Fig. 6.168)

Fig. 6.168

$$\mathbf{Z}_{Th} = \frac{(5\angle 60°)(10\angle -30°)}{5\angle 60° + 10\angle -30°} = 4.47\angle 33.43° \ \Omega = (3.73 + j2.46)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be the complex conjugate of the source impedance.

$$Z_L = Z_{Th}^* = (3.73 - j2.46)\ \Omega$$

Step IV Calculation of P_{max} (Fig. 6.169)

Fig. 6.169

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{4R_L} = \frac{(6.71)^2}{4 \times 3.73} = 3.02\ \text{W}$$

Example 6.54 *In the network shown in Fig. 6.170, find the value of $\mathbf{Z}_L$ so that power transfer from the source is maximum. Also find maximum power.*

Fig. 6.170

Solution

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.171)

Applying Star-delta transformation (Fig. 6.172)

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \frac{(j9)(j9)}{j9 + j9 + j9} = j3\ \Omega$$

$\mathbf{V}_{Th}$ = Voltage drop across $(8 + j3)\Omega$ impedence

$$= (8 + j3)\frac{10\angle 0°}{8 + j3 + j3} = 8.54\angle -16.31°\ \text{V}$$

Fig. 6.171

Fig. 6.172

Step II Calculation of $\mathbf{Z}_{\text{Th}}$ (Fig. 6.173)

Fig. 6.173

$$\mathbf{Z}_{\text{Th}} = j3 + \frac{j3(8+j3)}{j3+8+j3)} = 5.51\angle 82.49^\circ\ \Omega = (0.72 + j5.46)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be the complex conjugate of the source impedance.

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = (0.72 - j5.46)\ \Omega$$

Step IV Calculation of $P_{\max}$ (Fig. 6.174)

Fig. 6.174

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{(8.54)^2}{4 \times 0.72} = 25.32\ \text{W}$$

Example 6.55 *For the network shown in Fig. 6.175, find the value of $\mathbf{Z}_L$ that will transfer maximum power from the source. Also find maximum power.*

Fig. 6.175

Solution

Fig. 6.176

Step I Calculation of $\mathbf{V}_{Th}$ (Fig. 6.176)

From Fig 6.176,

$$\mathbf{V}_x = 4\mathbf{I}$$

Applying KVL to the mesh,

$$100\angle 0° - 4\mathbf{I} - j\,10\,\mathbf{I} - 5\mathbf{V}_x = 0$$

$$100\angle 0° - (4 + j10)\,\mathbf{I} - 5(4\mathbf{I}) = 0$$

$$\mathbf{I} = \frac{100\,\angle 0°}{24 + j10} = 3.85\,\angle -22.62°\text{ A}$$

Writing $\mathbf{V}_{Th}$ equation,

$$100\,\angle\,0° - 4\mathbf{I} - \mathbf{V}_{Th} = 0$$

$$100\angle 0° - 4(3.85\,\angle -22.62°) - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = 86\,\angle 3.95°\text{ V}$$

Step II Calculation of $\mathbf{I}_N$ (Fig. 6.177)

From Fig. 6.177,

$$\mathbf{V}_x = 4\mathbf{I}_1$$

Applying KVL to Mesh 1,

$$100\,\angle 0° - 4\mathbf{I}_1 = 0$$

$$\mathbf{I}_1 = 25\text{ A}$$

Fig. 6.177

Applying KVL to Mesh 2,

$$-j10\mathbf{I}_2 - 5\mathbf{V}_x = 0$$

$$-j10\mathbf{I}_2 - 5(4\mathbf{I}_1) = 0$$

$$-j10\mathbf{I}_2 - 5(100) = 0$$

$$\mathbf{I}_2 = 50\angle 90°\text{A}$$

$$\mathbf{I}_N = \mathbf{I}_1 - \mathbf{I}_2 = 25 - 50\angle 90° = 55.9\angle -63.43°\text{A}$$

Step III Calculation of $\mathbf{Z}_{Th}$

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_{Th}}{\mathbf{I}_N} = \frac{86\angle 3.95°}{55.9\angle -63.43°} = 1.54\angle 67.38°\ \Omega = (0.59 + j1.42)\ \Omega$$

Step IV Calculation of $\mathbf{Z}_L$

For maximum power transfer, $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = (0.59 - j1.42)\ \Omega$

Step V Calculation of $P_{\max}$ (Fig. 6.178)

Fig. 6.178

$$P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{4R_L} = \frac{(86)^2}{4 \times 0.59} = 3133.9 \text{ W}$$

6.8 RECIPROCITY THEOREM

The Reciprocity theorem states that '*In a linear, bilateral, active, single-source network, the ratio of excitation to response remains same when the positions of excitation and response are interchanged.*'

Example 6.56 *Find the current through the 6 Ω resistor and verify the reciprocity theorem.*

Fig. 6.179

Solution

Case I Calculation of current **I** when excitation and response are not interchanged (Fig. 6.180)

Fig. 6.180

Applying KVL to Mesh 1,

$$5\angle 0° - 1\mathbf{I}_1 - j1(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(1 + j1)\mathbf{I}_1 - j1\mathbf{I}_2 = 5\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j1(\mathbf{I}_2 - \mathbf{I}_1) + j1\mathbf{I}_2 - 2\mathbf{I}_2 = 0$$

$$-j1\mathbf{I}_1 + 2\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 1+j1 & -j1 \\ -j1 & 2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 5\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 1+j1 & 5\angle 0° \\ -j1 & 0 \end{vmatrix}}{\begin{vmatrix} 1+j1 & -j1 \\ -j1 & 2 \end{vmatrix}} = 1.39\angle 56.31° \text{ A}$$

$$\mathbf{I} = \mathbf{I}_2 = 1.39\angle 56.31° \text{ A}$$

Case II Calculation of current **I** when excitation and response are interchanged (Fig. 6.181)

Fig. 6.181

Applying KVL to Mesh 1,

$$-1\mathbf{I}_1 - j1(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(1+j1)\mathbf{I}_1 - j1\mathbf{I}_2 = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j1(\mathbf{I}_2 - \mathbf{I}_1) + j1\mathbf{I}_2 - 2\mathbf{I}_2 - 5\angle 0° = 0$$

$$-j1\mathbf{I}_1 + 2\mathbf{I}_2 = -5\angle 0° \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 1+j1 & -j1 \\ -j1 & 2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -5\angle 0° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 0 & -j1 \\ -5\angle 0° & 2 \end{vmatrix}}{\begin{vmatrix} 1+j1 & -j1 \\ -j1 & 2 \end{vmatrix}} = 1.38\angle -123.69° \text{ A}$$

$$\mathbf{I} = -\mathbf{I}_1 = 1.39\angle 56.31° \text{ A}$$

Since the current **I** is same in both the cases, the reciprocity theorem is verified.

Example 6.57 *In the network of Fig. 6.182, find the voltage $\mathbf{V}_x$ and verify the reciprocity theorem.*

Fig. 6.182

Solution

Case I Calculation of voltage $\mathbf{V}_x$ when excitation and response are interchanged. (Fig. 6.183)

Fig. 6.183

By current division rule,

$$\mathbf{I}_x = (20\angle 90°)\frac{(10+j5)}{(10+j5)+(j5-j2)} = 17.46\angle 77.91° \text{ A}$$

$$\mathbf{V}_x = (-j2)\mathbf{I}_x = (-j2)(17.46\angle 77.91°) = 34.92 \angle -12.09° \text{ V}$$

Case II Calculation of voltage $\mathbf{V}_x$ when excitation and response are interchanged (Fig. 6.184)

Fig. 6.184

$$\mathbf{I}_x = (20\angle 90°)\frac{(-j2)}{(-j2)+(10+j5+j5)} = 3.12\angle -38.66°\text{A}$$

$$\mathbf{V}_x = (10+j5)\mathbf{I}_x = (10+j5)(3.12\angle -38.66°) = 34.88\angle -12.09°\text{V}$$

Since the voltage $\mathbf{V}_x$ is same in both the cases, the reciprocity theorem is verified.

Example 6.58 *Find **I** and verify the reciprocity theorem for the network shown in Fig. 6.185.*

Fig. 6.185

Solution

Case I Calculation of **I** when excitation and response are not interchanged (Fig. 6.186)

Fig. 6.186

Applying KVL to Mesh 1,

$$10\angle 45^\circ - (3+j4)\mathbf{I}_1 - 4(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(7+j4)\mathbf{I}_1 - 4\mathbf{I}_2 = 10\angle 45^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-4(\mathbf{I}_2 - \mathbf{I}_1) - (1-j2)\,\mathbf{I}_2 - j3(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-4\mathbf{I}_1 + (5+j1)\mathbf{I}_2 - j3\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-j3(\mathbf{I}_3 - \mathbf{I}_2) - 2\mathbf{I}_3 - j2\mathbf{I}_3 = 0$$

$$-j3\mathbf{I}_2 + (2+j5)\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 45^\circ \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 7+j4 & -4 & 10\angle 45^\circ \\ -4 & 5+j1 & 0 \\ 0 & -j3 & 0 \end{vmatrix}}{\begin{vmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{vmatrix}} = 0.704\angle 30.72^\circ \text{ A}$$

$$\mathbf{I} = \mathbf{I}_3 = 0.704\angle 30.72^\circ \text{A}$$

Case II Calculation of **I** when excitation and response are interchanged (Fig. 6.187)

Fig. 6.187

Applying KVL to Mesh 1,

$$-(3+j4)\mathbf{I}_1 - 4(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(7+j4)\mathbf{I}_1 - 4\mathbf{I}_2 = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-4(\mathbf{I}_2 - \mathbf{I}_1) - (1-j2)\,\mathbf{I}_2 - j3(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-4\mathbf{I}_1 + (5+j1)\mathbf{I}_2 - j3\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-j3(\mathbf{I}_3 - \mathbf{I}_2) - 2\mathbf{I}_3 - j2\mathbf{I}_3 + 10\angle 45° = 0$$

$$-j3\mathbf{I}_2 + (2+j5)\mathbf{I}_3 = 10\angle 45° \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10\angle 45° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 0 & -4 & 0 \\ 0 & 5+j1 & -j3 \\ 10\angle 45° & -j3 & 2+j5 \end{vmatrix}}{\begin{vmatrix} 7+j4 & -4 & 0 \\ -4 & 5+j1 & -j3 \\ 0 & -j3 & 2+j5 \end{vmatrix}} = 0.704\angle 30.72°\text{A}$$

$$\mathbf{I} = \mathbf{I}_1 = 0.704\angle 30.72°\text{A}$$

Since the current **I** is same in both the cases, the reciprocity theorem, is verified.

6.9 MILLMAN'S THEOREM

Millman's theorem states that '*If there are n voltage sources* $\boldsymbol{V}_1$, $\boldsymbol{V}_2$. … $\boldsymbol{V}_n$ *with internal impedances* $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$, … $\boldsymbol{Z}_n$ *respectively connected in parallel then these voltage sources can be replaced by a single voltage source* $\boldsymbol{V}_m$ *and a single series impedance* $\boldsymbol{Z}_m$.

$$\mathbf{V}_m = \frac{\mathbf{V}_1\,\mathbf{Y}_1 + \mathbf{V}_2\,\mathbf{Y}_2 + \ldots + \mathbf{V}_n\,\mathbf{Y}_n}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

Example 6.59 *Find the current through the 40 Ω resistor for the network shown in Fig. 6.188.*

Fig. 6.188

Solution

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{(10\angle 0°)\left(\dfrac{1}{30 - j20}\right) + (20\angle 0°)\left(\dfrac{1}{10 + j20}\right)}{\dfrac{1}{30 - j20} + \dfrac{1}{10 + j20}} = 18.2\angle -15.95°\text{V}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\dfrac{1}{30 - j20} + \dfrac{1}{10 + j20}} = 20.15\angle 29.74°\ \Omega$$

Step III Calculation of $\mathbf{I}_L$ (Fig. 6.189)

Fig. 6.189

$$\mathrm{I}_L = \frac{18.2\angle -15.95°}{20.15\angle 29.74° + 40} = 0.31\angle -25.81°\ \text{A}$$

Example 6.60 *Find the current **I** in the network shown in Fig. 6.190.*

Fig. 6.190

Solution The network is redrawn as shown in Fig. 6.191.

Fig. 6.191

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{0\left(\frac{1}{40}\right) + (20\angle 0°)\left(\frac{1}{10 + j20}\right)}{\frac{1}{40} + \frac{1}{10 + j20}} = 14.86\angle -21.8° \text{ V}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\frac{1}{40} + \frac{1}{10 + j20}} = 16.61\angle 41.63° \ \Omega$$

Step III Calculation of **I** (Fig. 6.192)

Fig. 6.192

$$\mathbf{I} = \frac{10 - 14.86\angle -21.8°}{30 - j20 + 16.61\angle 41.63°} = 0.15\angle 136.47° \text{ A}$$

Example 6.61 *Apply the dual Millman theorem and find the power loss in the (2 + j2) Ω impedance.*

Fig. 6.193

Solution

Step I Calculation of $\mathbf{I}_m$

$$\mathbf{I}_m = \frac{\mathbf{I}_1\mathbf{Z}_1 + \mathbf{I}_2\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(10)(4)+(20)(5)}{4+5} = 15.56\text{ A}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \mathbf{Z}_1 + \mathbf{Z}_2 = 4+5 = 9\ \Omega$$

Step III Calculation of P (Fig. 6.194)

Fig. 6.194

By current division rule,

$$\mathbf{I}_L = 15.56 \times \frac{9}{9+2+j2} = 12.53\angle -10.3°\text{ A}$$

$$P = \mathbf{I}_L^2 R_L = (12.53)^2 \times 2 = 314\text{ W}$$

Example 6.62 *In the network shown in Fig. 6.195, what load Z_L will receive the maximum power. Also find maximum power.*

Fig. 6.195

Solution

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{(50\angle 0°)\left(\dfrac{1}{5+j5}\right) + (25\angle 90°)\left(\dfrac{1}{3-j4}\right)}{\dfrac{1}{5+j5} + \dfrac{1}{3-j4}} = 9.81\angle -78.69°\text{V}$$

Step II Calculation of $\mathbf{Z}_m$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\dfrac{1}{5+j5} + \dfrac{1}{3-j4}} = 4.39\angle -15.26°\ \Omega = (4.23 - j1.15)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = (4.23 + j1.15)\ \Omega$

Step IV Calculation of $P_{\max}$ (Fig. 6.196)

Fig. 6.196

$$P_{\max} = \frac{|\mathrm{V}_m|^2}{4R_L} = \frac{(9.81)^2}{4 \times 4.23} = 5.69 \text{ W}$$

6.10 TELLEGEN'S THEOREM

Tellegen's theorem states that '*The algebraic sum of the powers in all branches of the network at any instant is zero*'.

$$\sum_{k=1}^{n} \mathbf{V}_k \mathbf{I}_k = 0$$

This condition is valid for the network which obeys Kirchhoff's voltage and current laws.

Example 6.63 *Verify Tellegen's theorem for the network shown in Fig. 6.197.*

Fig. 6.197

$V_0 = 20\ V$, $V_1 = 16\ V$

$V_2 = 2\ V$, $V_3 = 4\ V$

$V_4 = 14\ V$, $V_5 = 6\ V$

$V_0 = -16\ A$ $I_1 = 12\ A$

$I_2 = 2\ A$, $I_3 = 14\ A$

$I_4 = 4\ A$, $I_5 = 2\ A$

Solution

$$\sum_{b=0}^{5} \mathbf{V}_b \mathbf{I}_b = \mathbf{V}_0\mathbf{I}_0 + \mathbf{V}_1\mathbf{I}_1 + \mathbf{V}_2\mathbf{I}_2 + \mathbf{V}_3\mathbf{I}_3 + \mathbf{V}_4\mathbf{I}_4 + \mathbf{V}_5\mathbf{I}_5$$

$$= 20(-16) + 16 \times 12 + 2 \times 2 + 4 \times 14 + 14 \times 4 + 6 \times 2 = 0$$

Hence, Tellegen's theorem is verified.

Example 6.64 *In the network shown in Fig. 6.198, check the validity of Tellegen's theorem.*

$\mathbf{V}_1 = 10\,V, \quad \mathbf{V}_2 = 4\,V, \quad \mathbf{V}_4 = 6\,V$
$\mathbf{I}_1 = 2\,A, \quad \mathbf{I}_2 = 2\,A, \quad \mathbf{I}_3 = 4\,A$

Fig. 6.198

Solution Applying KVL to Mesh 1,

$$\mathbf{V}_1 - \mathbf{V}_2 - \mathbf{V}_3 = 0$$
$$\mathbf{V}_3 = \mathbf{V}_1 - \mathbf{V}_2 = 10 - 4 = 6 \text{ V}$$

Applying KVL to Mesh 2,

$$\mathbf{V}_3 - \mathbf{V}_4 - \mathbf{V}_5 = 0$$
$$\mathbf{V}_5 = \mathbf{V}_3 - \mathbf{V}_4 = 6 - 6 = 0$$

Applying KVL to Mesh 3,

$$-\mathbf{V}_6 + \mathbf{V}_4 + \mathbf{V}_2 = 0$$
$$\mathbf{V}_6 = \mathbf{V}_2 + \mathbf{V}_4 = 4 + 6 = 10 \text{ V}$$

Applying KCL at Node 1,

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_6 = 0$$
$$\mathbf{I}_6 = -\mathbf{I}_1 - \mathbf{I}_2 = -2 - 2 = -4 \text{ A}$$

Applying KCL at Node 2,

$$\mathbf{I}_2 = \mathbf{I}_3 + \mathbf{I}_4$$
$$\mathbf{I}_4 = \mathbf{I}_2 - \mathbf{I}_3 = 2 - 4 = -2 \text{ A}$$

Applying KCL at Node 3,

$$\mathbf{I}_5 = \mathbf{I}_4 + \mathbf{I}_6 = -2 + (-4) = -6 \text{ A}$$

$$\sum_{b=1}^{6} \mathbf{V}_b \mathbf{I}_b = \mathbf{V}_1\mathbf{I}_1 + \mathbf{V}_2\mathbf{I}_2 + \mathbf{V}_3\mathbf{I}_3 + \mathbf{V}_4\mathbf{I}_4 + \mathbf{V}_5\mathbf{I}_5 + \mathbf{V}_6\mathbf{I}_6$$
$$= 10 \times 2 + 4 \times 2 + 6 \times 4 + 6 \times (-2) + 0 \times (-6) + 10 \times (-4)$$
$$= 0$$

Hence, Tellegen's theorem is verified.

6.11 SUBSTITUTION THEOREM

The substitution theorem states that '*Any branch in a network can be substituted by a different branch without disturbing the voltages and currents in the entire network, provided the new branch has the same set of terminal voltage and current as the original network.*'

Example 6.65 *In the network of Fig. 6.199, verify the substitution theorem by replacing the 6 Ω resistor by (a) a voltage source (b) a current source.*

Fig. 6.199

Solution:

Step I Calculation of current through and voltage across the 6 Ω resistor (Fig. 6.200)

Fig. 6.200

Applying KVL to Mesh 1,

$$100\angle 0° - 20\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(20 + j10)\,\mathbf{I}_1 - j10\mathbf{I}_2 = 100\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) - (6 + j8)\,\mathbf{I}_2 = 0$$

$$-j10\mathbf{I}_1 + (6 + j18)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 20 + j10 & -j10 \\ -j10 & 6 + j18 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 100\angle 0° & -j10 \\ 0 & 6 + j18 \end{vmatrix}}{\begin{vmatrix} 20 + j10 & -j10 \\ -j10 & 6 + j18 \end{vmatrix}} = 4.5\angle -13° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 20 + j10 & 100\angle 0° \\ -j10 & 0 \end{vmatrix}}{\begin{vmatrix} 20 + j10 & -j10 \\ -j10 & 6 + j18 \end{vmatrix}} = 2.37\angle 5.44° \text{ A}$$

$$\mathbf{I}_{6\Omega} = \mathbf{I}_2 = 2.37\angle 5.44° \text{ A}$$
$$\mathbf{V}_{6\Omega} = 6\mathbf{I}_2 = 6(2.37\angle 5.44°) = 14.22\angle 5.44° \text{ V}$$

Step II When 6 Ω resistor is replaced by voltage source of 14.22∠5.44° V (Fig. 6.201)

Fig. 6.201

Applying KVL to Mesh 1,

$$100\angle 0° - 20\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$
$$(20 + j10)\mathbf{I}_1 - j10\mathbf{I}_2 = 100\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) - 14.22\angle 5.44° - j8\mathbf{I}_2 = 0$$
$$-j10\mathbf{I}_1 + j18\mathbf{I}_2 = -14.22\angle 5.44° \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 20 + j10 & -j10 \\ -j10 & j18 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0° \\ -14.22\angle 5.44° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 100\angle 0° & -j10 \\ -14.22\angle 5.44° & j18 \end{vmatrix}}{\begin{vmatrix} 20 + j10 & -j10 \\ -j10 & j18 \end{vmatrix}} = 4.5\angle -13° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 20 + j10 & 100\angle 0° \\ -j10 & -14.22\angle 5.44° \end{vmatrix}}{\begin{vmatrix} 20 + j10 & -j10 \\ -j10 & j18 \end{vmatrix}} = 2.37\angle 5.44° \text{ A}$$

Since currents $\mathbf{I}_1$ and $\mathbf{I}_2$ remain the same, substitution theorem is verified.

Step III When the 6 Ω resistor is replaced by a current source of 2.37∠5.44° A (Fig. 6.202)

Fig. 6.202

Applying KVL to Mesh 1,

$$100\angle 0° - 20\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(20 + j10)\mathbf{I}_1 - j10\mathbf{I}_2 = 100\angle 0° \quad \text{...(i)}$$

For Mesh 2,

$$\mathbf{I}_2 = 2.37\angle 5.44° \text{ A} \quad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$\mathbf{I}_1 = 4.5\angle -13° \text{ A}$$

Since currents $\mathbf{I}_1$ and $\mathbf{I}_2$ remain the same, substitution theorem is verified.

6.12 COMPENSATION THEOREM

The compensation theorem states that '*In any linear bilateral active network, if any branch carrying a current* **I** *has its impedance* **Z** *changed by an amount* δ**Z**, *the resulting changes that occur in the other branches are the same as those which would have been provided by an opposing voltage source of value* $\mathbf{V}_C$ (**I** δ**Z**) *introduced into the modified branch*.'

Example 6.66 *In the network of Fig. 6.203, the impedance (3+j2) Ω is changed to (5+j3) Ω. Determine the change in current drawn from the source by direct calculation and then verify the result by compensation theorem.*

Fig. 6.203

Solution

Step I Calculation of change in current

$$\mathbf{I} = \frac{10\angle 0°}{5+3+j2} = 1.21\angle -14.04° \text{ A}$$

When impedance (3 + *j*2) Ω is changed to (5+*j*3) Ω,

$$\mathbf{I}' = \frac{10\angle 0°}{5+5+j3} = 0.96\angle -16.7° \text{ A}$$

$$\delta\mathbf{I} = \mathbf{I}' - \mathbf{I} = 0.96\angle -16.7° - 1.21\angle -14.04° = 0.25\angle 176.02° \text{ A}$$

Step II Calculation of change in current by compensation theorem

$$\delta\mathbf{Z} = (5 + j3) - (3 + j2) = (2 + j1)\ \Omega$$

$$\mathbf{V}_c = \delta Z\mathbf{I} = (2 + j1)(1.21\angle -14.04°) = 2.71\angle 12.53° \text{ V}$$

The compensating network is as shown in Fig. 6.204.

$$\delta \mathbf{I} = -\frac{2.71\angle 12.53°}{5+3+j2+2+j1}$$
$$= 0.26\angle 175.83° \text{ A}$$

Fig. 6.204

Example 6.67 *In the network of Fig. 6.205, the impedance (5 + j2) Ω is changed to (1 + j1) Ω. Find the change in current drawn from the supply by direct calculation and then verify by the compensation theorem.*

Fig. 6.205

Solution

Step I Calculation of change in current

Finding Thevenin's equivalent circuit across impedance of (5 + *j*2) Ω (Fig.6.206)

Fig. 6.206

$$\mathbf{Z}_{\text{Th}} = \frac{5(6-j3)}{5+6-j3} = 2.94\angle -11.31° \ \Omega$$

$$\mathbf{I} = \frac{20\angle 0°}{5+6-j3} = 1.75\angle 15.26° \text{ A}$$

$$\mathbf{V}_{\text{Th}} = (6-j3)\mathbf{I} = (6-j3)(1.75\angle 15.26°) = 11.74\angle -11.31° \text{ V}$$

Thevenin's equivalent network is shown in Fig. 6. 207.

Fig. 6.207

$$\mathbf{I}_L = \frac{11.74\angle -11.31°}{2.94\angle -11.31° + 5 + j2} = 1.47\angle -21.55° \text{ A}$$

When impedance (5 + *j*2) Ω is changed to (1 + *j*1) Ω,

$$\mathbf{I}'_L = \frac{11.74\angle -11.31°}{2.94\angle -11.31° + 1 + j1} = 3\angle -17.53° \text{ A}$$

$$\delta \mathbf{I}_L = \mathbf{I}'_L - \mathbf{I}_L = 3\angle -17.53° - 1.44\angle -25° = 1.53\angle -13.69° \text{ A}$$

Step II Calculation of change in current by compensation theorem

$$\delta \mathbf{Z} = (1 + j1) - (5 + j2) = (-4 - j1)\ \Omega$$

$$\mathbf{V}_c = \mathbf{I}_L \delta \mathbf{Z} = (1.44\angle -25°)(-4 - j1) = 5.94\angle 169.04°\ \text{V}$$

The compensating network is shown in Fig. 6.208.

$$\delta \mathbf{I}_L = -\frac{5.94\angle 169.04°}{2.94\angle -11.31° + 5 + j2 - 4 - j1}$$

$$= 1.52\angle -17.18°\ \text{A}$$

Fig. 6.208

Exercises

MESH ANALYSIS

6.1 Find the current through the 3 + *j*4 Ω impedance in Fig. 6.209.

Fig. 6.209

[0]

6.2 In the network of Fig. 6.210, find $\mathbf{V}_0$.

Fig. 6.210

[1.56 ∠128.7° V]

6.3 Find the current $\mathbf{I}_3$ in the network of Fig. 6.211.

Fig. 6.211

[11.6 ∠113.2° A]

6.4 In the network of Fig.6.212, find $\mathbf{V}_2$ which results in zero current through the 4 Ω resistor.

Fig. 6.212

[26.3 ∠113.2° V]

NODE ANALYSIS

6.5 For the network shown in Fig. 6.213, find the voltage $\mathbf{V}_{AB}$.

Fig. 6.213

[75.4 ∠55.2° V]

6.6 Find the voltages at nodes 1 and 2 in the network of Fig. 6.214.

Fig. 6.214

[15.95 ∠49.94° V, 12.9 ∠55.5° V]

6.7 In the network of Fig. 6.215, find the current in the 10 ∠30° V source.

Fig. 6.215

[1.44 ∠38.8° A]

SUPERPOSITION THEOREM

6.8 For the network shown in Fig. 6.216, find the current in the 10 Ω resistor.

Fig. 6.216

[73.4 ∠−21.84° A]

6.9 In the network of Fig. 6.217, find the current through capacitance.

Fig. 6.217

[4.86 ∠80.8° A]

THEVENIN'S THEOREM

6.10 Obtain Thevenin's equivalent network for the network shown in Fig. 6.218.

Fig. 6.218

[0.192 ∠−43.4° V, 88.7 ∠11.55° Ω]

6.11 Obtain Thevenin's equivalent network for Fig. 6.219.

Fig. 6.219

[11.17 ∠−63.4° V, 10.6 ∠45° Ω]

NORTON'S THEOREM

6.12 Find Norton's equivalent network for Fig. 6.220.

Fig. 6.220

[2.77 ∠–33.7° A, 2.5 + *j*12.5 Ω]

6.13 Find the current through the (3 + *j*4) Ω impedance in the network of Fig. 6.221.

Fig. 6.221

[8.3 ∠85.2° A]

MAXIMUM POWER TRANSFER THEOREM

6.14 Determine the maximum power delivered to the load in the network shown in Fig. 6.222.

Fig. 6.222

[1032.35 W]

6.15 For the network shown in Fig. 6.223, find the value of $\mathbf{Z}_L$ that will receive the maximum power. Determine also this power.

Fig. 6.223

[3.82 – *j*1.03 Ω, 54.5 W]

Objective-Type Questions

6.1 In Fig. 6.224, the equivalent impedance seen across terminals *a*, *b*, is

Fig. 6.224

(a) $\frac{16}{3}$ Ω (b) $\frac{8}{3}$ Ω

(C) $\left(\frac{8}{3}+j12\right)$ Ω (d) none of the above

6.2 The Thevenin equivalent voltage $\mathbf{V}_{Th}$ appearing between the terminals *A* and *B* of the network shown in Fig. 6.225 is given by

Fig. 6.225

(a) *j*16(3 – *j*4) (b) *j*16(3 + *j*4)

(c) 16(3 + *j*4) (d) 16 (3 – *j*4)

6.3 A source of angular frequency of 1 rad/s has a source impedance consisting of a 1 Ω resistance

in series with a 1 H inductance. The load that will obtain the maximum power transfer is

(a) 1 Ω resistance
(b) 1 Ω resistance in parallel with 1 H inductance
(c) 1 Ω resistance in series with 1 F capacitance
(d) 1 Ω resistance in parallel with 1 F capacitance

6.4 For the network shown in Fig. 6.226, the instantaneous current $\mathbf{I}_1$ is

Fig. 6.226

(a) $\frac{10\sqrt{3}}{2}\angle 90°$ A (b) $\frac{10\sqrt{3}}{2}\angle -90°$ A

(c) 5 ∠60° A (d) 5∠−60° A

6.5 In the network shown in Fig. 6.227, the current supplied by the sinusoidal current source **I** is

Fig. 6.227

(a) 28 A
(b) 4 A
(c) 20 A
(d) cannot be determined

6.6 In the network of Fig. 5.228, the magnitudes of $\mathbf{V}_L$ and $\mathbf{V}_C$ are twice that of $\mathbf{V}_R$. The inductance of the coil is

Fig. 6.228

(a) 2.14 mH (b) 5.3 H
(c) 31.8 mH (d) 1.32 H

6.7 Phase angle of the current **I** with respect to the voltage $\mathbf{V}_1$ in the circuit shown in Fig. 6.229.

Fig. 6.229

(a) 0° (b) 45°
(c) −45° (d) −90°

Answers to Objective-Type Questions

6.1 (b) 6.2 (d) 6.3 (c) 6.4 (a) 6.5 (c) 6.6 (c) 6.7 (d)

7

Coupled Circuits

7.1 INTRODUCTION

Two circuits are said to be coupled circuits when energy transfer takes place from one circuit to the other without having any electrical connection between them. Such coupled circuits are frequently used in network analysis and synthesis. Common examples of coupled circuits are transformer, gyrator, etc. In this chapter, we will discuss self and mutual inductance, magnetically coupled circuits, dot conventions and tuned circuits.

7.2 SELF-INDUCTANCE

Consider a coil of N turns carrying a current i as shown in Fig. 7.1.

When current flows through the coil, a flux ϕ is produced in the coil. The flux produced by the coil links with the coil itself. If the current flowing through the coil changes, the flux linking the coil also changes. Hence, an emf is induced in the coil. This is known as self-induced emf. The direction of this emf is given by *Lenz's law*.

Fig. 7.1 *Coil carrying current*

We know that

$$\phi \propto i$$

$$\frac{\phi}{i} = k, \text{ a constant}$$

$$\phi = k\, i$$

Hence, rate of change of flux $= k \times$ rate of change of current

$$\frac{d\phi}{dt} = k\frac{di}{dt}$$

According to Faraday's laws of electromagnetic induction, a self-induced emf can be expressed as

$$v = -N\frac{d\phi}{dt} = -Nk\frac{di}{dt} = -N\frac{\phi}{i}\frac{di}{dt} = -L\frac{di}{dt}$$

where $L = \dfrac{N\phi}{i}$ and is called coefficient of self-inductance.

The property of a coil that opposes any change in the current flowing through it is called self-inductance or inductance of the coil. If the current in the coil is increasing, the self-induced emf is set up in such a direction so

as to oppose the rise in current, i.e., the direction of self-induced emf is opposite to that of the applied voltage. Similarly, if the current in the coil is decreasing, the self-induced emf will be in the same direction as the applied voltage. Self-inductance does not prevent the current from changing, it serves only to delay the change.

7.3 MUTUAL INDUCTANCE

If the flux produced by one coil links with the other coil, placed closed to the first coil, an emf is induced in the second coil due to change in the flux produced by the first coil. This is known as mutually induced emf.

Consider two coils 1 and 2 placed adjacent to each other as shown in Fig. 7.2. Let Coil 1 has N_1 turns while Coil 2 has N_2 turns.

Fig. 7.2 *Two adjacent coils*

If a current i_1 flows in Coil 1, flux is produced and a part of this flux links Coil 2. The emf induced in Coil 2 is called mutually induced emf.

We know that

$$\phi_2 \propto i_1$$

$$\frac{\phi_2}{i_1} = k, \text{ a constant}$$

$$\phi_2 = k\, i_1$$

Hence, rate of change of flux = $k \times$ rate of change of current i_1

$$\frac{d\phi_2}{dt} = k\frac{di_1}{dt}$$

According to Faraday's law of electromagnetic induction, the induced emf is expressed as

$$v_2 = -N_2\frac{d\phi_2}{dt} = -N_2\, k\frac{di_1}{dt} = -N_2\frac{\phi_2}{i_1}\frac{di_1}{dt} = -M\frac{di_1}{dt}$$

where $M = \dfrac{N_2\,\phi_2}{i_1}$ and is called *coefficient of mutual inductance*.

7.4 COEFFICIENT OF COUPLING (k)

The coefficient of coupling (k) between coils is defined as fraction of magnetic flux produced by the current in one coil that links the other.

Consider two coils having number of turns N_1 and N_2 respectively. When a current i_1 is flowing in Coil 1 and is changing, an emf is induced in Coil 2.

$$M = \frac{N_2\,\phi_2}{i_1}$$

Let
$$k_1 = \frac{\phi_2}{\phi_1}$$
$$\phi_2 = k_1\ \phi_1$$
$$M = \frac{N_2\ k_1\ \phi_1}{i_1} \qquad \ldots(7.1)$$

If the current i_2 is flowing in Coil 2 and is changing, an emf is induced in Coil 1,

$$M = \frac{N_1\ \phi_1}{i_2}$$

Let
$$k_2 = \frac{\phi_1}{\phi_2}$$
$$\phi_1 = k_2\ \phi_2$$
$$M = \frac{N_1\ k_2\ \phi_2}{i_2} \qquad \ldots(7.2)$$

Multiplying Eqs (7.1) and (7.2),

$$M^2 = k_1 k_2 \times \frac{N_1\phi_1}{i_1} \times \frac{N_2\phi_2}{i_2} = k^2 L_1 L_2$$
$$M = k\sqrt{L_1 L_2}$$

where
$$k = \sqrt{k_1 k_2}$$

7.5 INDUCTANCES IN SERIES

1. *Cumulative Coupling* Figure 7.3 shows two coils 1 and 2 connected in series, so that currents through the two coils are in the same direction in order to produce flux in the same direction. Such a connection of two coils is known as *cumulative coupling*.

Fig. 7.3 *Cumulative coupling*

Let L_1 = coefficient of self-inductance of Coil 1

L_2 = coefficient of self-inductance of Coil 2

M = coefficient of mutual inductance

If the current in the coil increases by di amperes in dt seconds then

Self-induced emf in Coil 1 $= -L_1 \dfrac{di}{dt}$

Self-induced emf in Coil 2 $= -L_2 \dfrac{di}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= -M \dfrac{di}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= -M \dfrac{di}{dt}$

Total induced emf
$$v = -(L_1 + L_2 + 2M)\frac{di}{dt} \qquad \ldots(7.3)$$

If L is the equivalent inductance then total induced emf in that single coil would have been

$$v = -L\frac{di}{dt} \qquad \ldots(7.4)$$

Equating Eqs (7.3) and (7.4),

$$L = L_1 + L_2 + 2M$$

2. *Differential Coupling* Figure 7.4 shows the coils connected in series but the direction of current in Coil 2 is now opposite to that in 1. Such a connection of two coils is known as *differential coupling*.

Hence, total induced emf in coils 1 and 2.

Fig. 7.4 *Differential coupling*

$$v = -L_1\frac{di}{dt} - L_2\frac{di}{dt} + 2M\frac{di}{dt} = -(L_1 + L_2 - 2M)\frac{di}{dt}$$

Coils 1 and 2 connected in series can be considered as a single coil with equivalent inductance L. The induced emf in the equivalent single coil with same rate of change of current is given by,

$$v = -L\frac{di}{dt}$$

$$-L\frac{di}{dt} = -(L_1 + L_2 - 2M)\frac{di}{dt}$$

$$L = L_1 + L_2 - 2M$$

7.6 INDUCTANCES IN PARALLEL

1. *Cumulative Coupling* Figure 7.5 shows two coils 1 and 2 connected in parallel such that fluxes produced by the coils act in the same direction. Such a connection of two coils is known as cumulative coupling.

Let L_1 = coefficient of self-inductance of Coil 1

L_2 = coefficient of self-inductance of Coil 2

M = coefficient of mutual inductance

If the current in the coils changes by di amperes in dt seconds then

Fig. 7.5 *Cumulative coupling*

Self-induced emf in Coil 1 $= -L_1\dfrac{di_1}{dt}$

Self-induced emf in Coil 2 $= -L_2\dfrac{di_2}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= -M\dfrac{di_2}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= -M\dfrac{di_1}{dt}$

Total induced emf in Coil 1 $= -L_1\dfrac{di_1}{dt} - M\dfrac{di_2}{dt}$

Total induced emf in Coil 2 $= -L_2 \dfrac{di_2}{dt} - M\dfrac{di_1}{dt}$

As both the coils are connected in parallel, the emf induced in both the coils must be equal.

$$-L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} = -L_2 \frac{di_2}{dt} - M\frac{di_1}{dt}$$

$$L_1 \frac{di_1}{dt} - M\frac{di_1}{dt} = L_2 \frac{di_2}{dt} - M\frac{di_2}{dt}$$

$$(L_1 - M)\frac{di_1}{dt} = (L_2 - M)\frac{di_2}{dt}$$

$$\frac{di_1}{dt} = \left(\frac{L_2 - M}{L_1 - M}\right)\frac{di_2}{dt} \qquad ...(7.5)$$

Now,

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 - M}{L_1 - M}\right)\frac{di_2}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 - M}{L_1 - M} + 1\right)\frac{di_2}{dt}$$

$$= \left(\frac{L_1 + L_2 - 2M}{L_1 - M}\right)\frac{di_2}{dt} \qquad ...(7.6)$$

If L is the equivalent inductance of the parallel combination then the induced emf is given by

$$v = -L\frac{di}{dt}$$

Since induced emf in parallel combination is same as induced emf in any one coil,

$$L\frac{di}{dt} = L_1 \frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}\left(L_1 \frac{di_1}{dt} + M\frac{di_2}{dt}\right)$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2 - M}{L_1 - M}\right)\frac{di_2}{dt} + M\frac{di_2}{dt}\right]$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2 - M}{L_1 - M}\right) + M\right]\frac{di_2}{dt} \qquad ...(7.7)$$

Substituting Eq. (7.6) in Eq. (7.7),

$$\left(\frac{L_1 + L_2 - 2M}{L_1 - M}\right)\frac{di_2}{dt} = \frac{1}{L}\left[L_1\left(\frac{L_2 - M}{L_1 - M}\right) + M\right]\frac{di_2}{dt}$$

$$L = \frac{L_1\left(\dfrac{L_2 - M}{L_1 - M}\right) + M}{\dfrac{L_1 + L_2 - 2M}{L_1 - M}}$$

$$= \frac{L_1L_2 - L_1M + L_1M - M^2}{L_1 + L_2 - 2M}$$

$$= \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

2. *Differential Coupling* Figure 7.6 shows two coils 1 and 2 connected in parallel such that fluxes produced by the coils act in the opposite direction. Such a connection of two coils is known as differential coupling.

Fig. 7.6 *Differential coupling*

Self-induced emf in Coil 1 $= -L_1 \dfrac{di_1}{dt}$

Self-induced emf in Coil 2 $= -L_2 \dfrac{di_2}{dt}$

Mutually induced emf in Coil 1 due to change of current in Coil 2 $= M \dfrac{di_2}{dt}$

Mutually induced emf in Coil 2 due to change of current in Coil 1 $= M \dfrac{di_1}{dt}$

Total induced emf in Coil 1 $= -L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt}$

Total induced emf in Coil 2 $= -L_2 \dfrac{di_2}{dt} + M \dfrac{di_1}{dt}$

As both the coils are connected in parallel, the emf induced in the coils must be equal.

$$-L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$L_1 \frac{di_1}{dt} + M \frac{di_1}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_2}{dt}$$

$$(L_1 + M)\frac{di_1}{dt} = (L_2 + M)\frac{di_2}{dt}$$

$$\frac{di_1}{dt} = \left(\frac{L_2 + M}{L_1 + M}\right)\frac{di_2}{dt} \qquad \text{...(7.8)}$$

Now,

$$i = i_1 + i_2$$

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 + M}{L_1 + M}\right)\frac{di_2}{dt} + \frac{di_2}{dt}$$

$$= \left(\frac{L_2 + M}{L_1 + M} + 1\right)\frac{di_2}{dt}$$

$$= \left(\frac{L_1 + L_2 + 2M}{L_1 + M}\right)\frac{di_2}{dt} \qquad \text{...(7.9)}$$

If L is the equivalent inductance of the parallel combination then the induced emf is given by

$$v = -L\frac{di}{dt}$$

Since induced emf in parallel combination is same as induced emf in any one coil,

$$L\frac{di}{dt} = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}\left(L_1\frac{di_1}{dt} - M\frac{di_2}{dt}\right)$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right)\frac{di_2}{dt} - M\frac{di_2}{dt}\right]$$

$$= \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right) - M\right]\frac{di_2}{dt} \qquad ...(7.10)$$

Substituting Eq. (7.9) in Eq. (7.10),

$$\left(\frac{L_1+L_2+2M}{L_1+M}\right)\frac{di_2}{dt} = \frac{1}{L}\left[L_1\left(\frac{L_2+M}{L_1+M}\right) - M\right]\frac{di_2}{dt}$$

$$L = \frac{L_1\left(\frac{L_2+M}{L_1+M}\right) - M}{\frac{L_1+L_2+2M}{L_1+M}}$$

$$= \frac{L_1L_2 + L_1M - L_1M - M^2}{L_1+L_2+2M}$$

$$= \frac{L_1L_2 - M^2}{L_1+L_2+2M}$$

Example 7.1 *The combined inductance of two coils connected in series is 0.6 H or 0.1 H depending on relative directions of currents in the two coils. If one of the coils has a self-inductance of 0.2 H, find (a) mutual inductance, and (b) coefficient of coupling.*

Solution $L_1 = 0.2\text{ H}, \quad L_{\text{diff}} = 0.1\text{ H}, \quad L_{\text{cum}} = 0.6\text{ H}$

(a) Mutual inductance

$$L_{\text{cum}} = L_1 + L_2 + 2M = 0.6 \qquad ...(i)$$

$$L_{\text{diff}} = L_1 + L_2 - 2M = 0.1 \qquad ...(ii)$$

Adding Eqs (i) and (ii),

$$2(L_1 + L_2) = 0.7$$

$$L_1 + L_2 = 0.35$$

$$L_2 = 0.35 - 0.2 = 0.15\text{ H}$$

Subtracting Eqs (ii) from Eqs (i),

$$4M = 0.5$$
$$M = 0.125 \text{ H}$$

(b) Coefficient of coupling

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.125}{\sqrt{0.2 \times 0.15}} = 0.72$$

Example 7.2 *Two coils with a coefficient of coupling of 0.6 between them are connected in series so as to magnetise in (a) same direction, and (b) opposite direction. The total inductance in the same direction is 1.5 H and in the opposite direction is 0.5 H. Find the self-inductance of the coils.*

Solution

$$k = 0.6, \quad L_{\text{diff}} = 0.5 \text{ H}, \quad L_{\text{cum}} = 1.5 \text{ H}$$

$$L_{\text{diff}} = L_1 + L_2 - 2M = 0.5 \qquad \text{...(i)}$$
$$L_{\text{cum}} = L_1 + L_2 + 2M = 1.5 \qquad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii),

$$4M = 1$$
$$M = 0.25 \text{ H}$$

Adding Eq. (i) and (ii),

$$2(L_1 + L_2) = 2$$
$$L_1 + L_2 = 1 \qquad \text{...(iii)}$$

$$k = \frac{M}{\sqrt{L_1 L_2}}$$
$$0.6 = \frac{0.25}{\sqrt{L_1 L_2}}$$
$$L_1 L_2 = 0.1736 \qquad \text{...(iv)}$$

Solving Eqs (iii) and (iv),

$$L_1 = 0.22 \text{ H}$$
$$L_2 = 0.78 \text{ H}$$

Example 7.3 *Two coils having self-inductances of 4 mH and 7 mH respectively are connected in parallel. If the mutual inductance between them is 5 mH, find the equivalent inductance.*

Solution

$$L_1 = 4 \text{ mH}, \quad L_2 = 7 \text{ mH}, \quad M = 5 \text{ mH}$$

For cumulative coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} = \frac{4 \times 7 - (5)^2}{4 + 7 - 2(5)} = 3 \text{ mH}$$

For differential coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} = \frac{4 \times 7 - (5)^2}{4 + 7 + 2(5)} = 0.143 \text{ mH}$$

Example 7.4 *Two inductors are connected in parallel. Their equivalent inductance when the mutual inductance aids the self-inductance is 6 mH and it is 2 mH when the mutual inductance opposes the self-inductance. If the ratio of the self- inductances is 1:3 and the mutual inductance between the coils is 4 mH, find the self-inductances.*

Solution

$$L_{\text{cum}} = 6 \text{ mH}, \quad L_{\text{diff}} = 2 \text{ mH}, \quad \frac{L_1}{L_2} = 1.3, \quad M = 4 \text{ mH}$$

For cumulative coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$6 = \frac{L_1 L_2 - (4)^2}{L_1 + L_2 - 2(4)}$$

$$6 = \frac{L_1 L_2 - 16}{L_1 + L_2 - 8} \qquad \text{...(i)}$$

For differential coupling,

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

$$2 = \frac{L_1 L_2 - (4)^2}{L_1 + L_2 + 8}$$

$$2 = \frac{L_1 L_2 - 16}{L_1 + L_2 + 8} \qquad \text{...(ii)}$$

From Eqs (i) and (ii),

$$2(L_1 + L_2 + 8) = 6(L_1 + L_2 - 8)$$

$$L_1 + L_2 + 8 = 3L_1 + 3L_2 - 24$$

$$L_1 + L_2 = 16$$

But

$$\frac{L_1}{L_2} = 1.3$$

$$1.3\, L_2 + L_2 = 16$$

$$2.3\, L_2 = 16$$

$$L_2 = 6.95 \text{ mH}$$

$$L_1 = 1.3\, L_2 = 9.035 \text{ mH}$$

7.7 DOT CONVENTION

Consider two coils of inductances L_1 and L_2 respectively connected in series as shown in Fig. 7.7. Each coil will contribute the same mutual flux (since it is in a series connection, the same current flows through L_1 and L_2) and hence, same mutual inductance (M). If the mutual fluxes of the two coils aid each other as

shown in Fig 7.7 (a), the inductances of each coil will be increased by M, i.e., the inductance of coils will become $(L_1 + M)$ and $(L_2 + M)$. If the mutual fluxes oppose each other as shown in Fig. 7.7 (b), inductance of the coils will become $(L_1 - M)$ and $(L_2 - M)$. Whether the two mutual fluxes aid to each other or oppose will depend upon the manner in which coils are wound. The method described above is very inconvenient because we have to include the pictures of the coils in the circuit. There is another simple method of defining the directions of currents in the coils. This is known as dot convention.

Fig. 7.7 *Dot convention*

Figure 7.7 shows the schematic connection of the two coils. It is not possible to state from Fig. 7.7(a) and Fig. 7.7(b) whether the mutual fluxes are additive or in opposition. However dot convention removes this confusion.

If the current enters from both the dotted ends of Coil 1 and Coil 2, the mutual fluxes of the two coils aid each other as shown in Fig. 7.7(c). If the current enters from the dotted end of Coil 1 and leaves from the dotted end of Coil 2, the mutual fluxes of the two coils oppose each other as shown in Fig. 7.7(d).

When two mutual fluxes aid each other, the mutual inductance is positive and polarity of the mutually induced emf is same as that of the self-induced emf. When two mutual fluxes oppose each other, the mutual inductance is negative and polarity of the mutually induced emf is opposite to that of the self-induced emf.

Example 7.5 *Obtain the dotted equivalent circuit for Fig. 7.8 shown below.*

Fig. 7.8

Solution The current in the two coils is shown in Fig. 7.9. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

Fig. 7.9

From Fig. 7.9, it is seen that, the flux ϕ_1 is in upward direction in Coil 1, and flux ϕ_2 is in downward direction in Coil 2. Hence, fluxes are opposing each other. The mutual inductances are negative and mutually induced emfs have opposite polarities as that of self-induced emf. The dots are placed in two coils to illustrate these conditions. Hence, current $i(t)$ enters from the dotted end in Coil 1 and leaves from the dotted end in Coil 2.

The dotted equivalent circuit is shown in Fig. 7.10.

Fig. 7.10

Example 7.6 *Obtain the dotted equivalent circuit for the circuit of Fig. 7.11.*

Fig. 7.11

Solution The current in the three coils is shown in Fig. 7.12. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

Fig. 7.12

From Fig. 7.12, it is seen that the flux is towards the left in Coil 1, towards the right in Coil 2 and towards the left in Coil 3. Hence, fluxes ϕ_1 and ϕ_2 oppose each other in coils 1 and 2, fluxes ϕ_2 and ϕ_3 oppose each other in coils 2 and 3, and fluxes ϕ_1 and ϕ_3 aid each other in coils 1 and 3. The dots are placed in three coils to illustrate these conditions. Hence, current enters from the dotted end in Coil 1, leaves from the dotted end in Coil 2 and enters from the dotted end in Coil 3.

The dotted equivalent circuit is shown in Fig. 7.13.

Fig. 7.13

Example 7.7 *Obtain the dotted equivalent circuit for the circuit shown in Fig. 7.14.*

Fig. 7.14

Solution The current in the three coils is shown in Fig. 7.15. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

From Fig. 7.15, it is seen that all the three fluxes ϕ_1, ϕ_2, ϕ_3 aid each other. Hence, all the mutual reactances are positive and mutually induced emfs have same polarities as that of self-induced emfs. The dots are placed in three coils to illustrate these conditions. Hence, currents enter from the dotted end in each of the three coils. The dotted equivalent circuit is shown in Fig. 7.16.

Fig. 7.15

Fig. 7.16

Example 7.8 *Obtain the dotted equivalent circuit for the coupled circuit of Fig. 7.17.*

Fig. 7.17

Solution The current in the two coils is shown in Fig. 7.18 . The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

Fig. 7.18

From Fig. 7.18, it is seen that the flux ϕ_1 is in clockwise direction in Coil 1 and in anti-clockwise direction in Coil 2. Hence, fluxes are opposing each other. The dots are placed in two coils to illustrate these conditions. Hence, current enters from the dotted end in Coil 1 and leaves from the dotted end in Coil 2. The dotted equivalent circuit is shown in Fig. 7.19.

Fig. 7.19

Example 7.9 *Find the equivalent inductance of the network shown in Fig. 7.20.*

Fig. 7.20

Solution

$$L = (L_1 + M_{12} + M_{13}) + (L_2 + M_{23} + M_{21}) + (L_3 + M_{31} + M_{32})$$
$$= (1 + 0.5 + 1) + (2 + 1 + 0.5) + (5 + 1 + 1)$$
$$= 13 \text{ H}$$

Example 7.10 *Find the equivalent inductance of the network shown in Fig. 7.21.*

Fig. 7.21

Solution

$$L = (L_1 + M_{12} - M_{13}) + (L_2 - M_{23} + M_{21}) + (L_3 - M_{31} - M_{23})$$
$$= (10 + 2 - 1) + (5 - 1 + 2) + (6 - 1 - 1) = 21 \text{ H}$$

Example 7.11 *Find the equivalent inductance of the network shown in Fig. 7.22.*

Fig. 7.22

Solution

$$M_{12} = M_{21} = k_1\sqrt{L_1L_2} = 0.33\sqrt{(12)(14)} = 4.28 \text{ H}$$

$$M_{23} = M_{32} = k_2\sqrt{L_2L_3} = 0.37\sqrt{(14)(14)} = 5.18 \text{ H}$$

$$M_{31} = M_{13} = k_3\sqrt{L_3L_1} = 0.65\sqrt{(12)(14)} = 8.42 \text{ H}$$

$$L = (L_1 - M_{12} + M_{13}) + (L_2 - M_{23} - M_{21}) + (L_3 + M_{31} - M_{32})$$

$$= (12 - 4.28 + 8.42) + (14 - 5.18 - 4.28) + (14 + 8.42 - 5.18)$$

$$= 37.92 \text{ H}$$

Example 7.12 *Find the equivalent inductance of the network shown in Fig. 7.23.*

Fig. 7.23

Solution For Coil A,

$$L_A = L_1 - M_{12} = 15 - 8 = 7 \text{ H}$$

For Coil B,

$$L_B = L_2 - M_{12} = 16 - 8 = 8 \text{ H}$$

$$\frac{1}{L} = \frac{1}{L_A} + \frac{1}{L_B} = \frac{1}{7} + \frac{1}{8} = \frac{15}{56}$$

$$L = \frac{56}{15} = 3.73 \text{ H}$$

Example 7.13 *Find the equivalent inductance of the network shown in Fig. 7.24.*

Fig. 7.24

Solution For Coil A,

$$L_A = L_1 + M_{12} - M_{13} = 25 + 10 - 10 = 25 \text{ H}$$

For Coil B,

$$L_B = L_2 - M_{23} + M_{21} = 35 - 15 + 10 = 25 \text{ H}$$

For Coil C,

$$L_C = L_3 - M_{32} - M_{31} = 35 - 15 - 10 = 10 \text{ H}$$

$$\frac{1}{L} = \frac{1}{L_A} + \frac{1}{L_B} + \frac{1}{L_C} = \frac{1}{25} + \frac{1}{25} + \frac{1}{10} = \frac{9}{50}$$

$$L = \frac{50}{9} = 5.55 \text{ H}$$

Example 7.14 *Find the equivalent impedance across the terminals A and B in Fig. 7.25.*

Fig. 7.25

Solution

$$\mathbf{Z}_1 = 5\Omega, \quad \mathbf{Z}_2 = (2 + j4)\ \Omega, \quad \mathbf{Z}_3 = (3 + j3)\ \Omega, \quad \mathbf{Z}_M = j2\ \Omega$$

$$\mathbf{Z} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3 - \mathbf{Z}_M^2}{\mathbf{Z}_2 + \mathbf{Z}_3 - 2\mathbf{Z}_M} = 5 + \frac{(2 + j4)(3 + j3) - (j2)^2}{2 + j4 + 3 + j3 - 2(j2)} = 6.9 \angle 24.16° \ \Omega$$

7.8 COUPLED CIRCUITS

Consider two coils located physically close to one another as shown in Fig. 7.26.

When current i_1 flows in the first coil and $i_2 = 0$ in the second coil, flux ϕ_1 is produced in the coil. A fraction of this flux also links the second coil and induces a voltage in this coil. The voltage v_1 induced in the first coil is

$$v_1 = L_1 \left.\frac{di_1}{dt}\right|_{i_2 = 0}$$

The voltage v_2 induced in the second coil is

$$v_2 = M \left.\frac{di_1}{dt}\right|_{i_2 = 0}$$

Fig. 7.26 *Coupled circuit*

The polarity of the voltage induced in the second coil depends on the way the coils are wound and it is usually indicated by dots. The dots signify that the induced voltages in the two coils (due to single current) have the same polarities at the dotted ends of the coils. Thus, due to i_1, the induced voltage v_1 must be positive at the dotted end of Coil 1. The voltage v_2 is also positive at the dotted end in Coil 2.

The same reasoning applies if a current i_2 flows in Coil 2 and $i_1 = 0$ in Coil 1. The induced voltages v_2 and v_1 are

$$v_2 = L_2 \left.\frac{di_2}{dt}\right|_{i_1 = 0}$$

and

$$v_1 = M \left.\frac{di_2}{dt}\right|_{i_1 = 0}$$

The polarities of v_1 and v_2 follow the dot convention. The voltage polarity is positive at the doted end of inductor L_2 when the current direction for i_2 is as shown in Fig. 7.26. Therefore, the voltage induced in Coil 1 must be positive at the dotted end also.

Now if both currents i_1 and i_2 are present, by using superposition principle, we can write

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

This can be represented in terms of dependent sources, as shown in Fig. 7.27.

Fig. 7.27 *Equivalent circuit*

Now consider the case when the dots are placed at the opposite ends in the two coils, as shown in Fig. 7.28.

Fig. 7.28 *Coupled circuit*

Due to i_1, with $i_2 = 0$, the dotted end in Coil 1 is positive, so the induced voltage in Coil 2 is positive at the dot, which is the reverse of the designated polarity for v_2. Similarly, due to i_2, with $i_1 = 0$, the dotted ends have negative polarities for the induced voltages. The mutually induced voltages in both cases have polarities that are the reverse of terminal voltages and the equations are

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

This can be repressed in terms of dependent sources as shown in Fig. 7.29.

Fig. 7.29 *Equivalent circuit*

The various cases are summarized in the table shown in Fig. 7.30.

Fig. 7.30 *Coupled circuits for various cases*

Example 7.15 *Write mesh equations for the network shown in Fig. 7.31.*

Fig. 7.31

Solution Coil 1 is magnetically coupled to Coil 2. Similarly, Coil 2 is magnetically coupled with Coil 1 and Coil 3. By applying dot convention, the equivalent circuit is drawn with the dependent sources.

The equivalent circuit in terms of dependent sources is shown in Fig. 7.32.

Fig. 7.32

(a) In Coil 1, there is a mutually induced emf due to current $(i_1 - i_2)$ in Coil 2. The polarity of the mutually induced emf is same as that of self-induced emf because currents i_1 and $(i_1 - i_2)$ enter in respective coils from the dotted ends.

(b) In Coil 2, there are two mutually induced emfs, one due to current i_1 in Coil 1 and the other due to current i_2 in Coil 3. The polarity of the mutually induced emf in Coil 2 due to the current i_1 is same as that of the self-induced emf because currents i_1 and $(i_1 - i_2)$ enter in respective coils from dotted ends. The polarity of the mutually induced emf in Coil 2 due to the current i_2 is opposite to that of the self-induced emf because current $(i_1 - i_2)$ leaves from the dotted end in Coil 2 and the current i_2 enters from the dotted end in Coil 3.

(c) In Coil 3, there is a mutually induced emf due to the current $(i_1 - i_2)$ in Coil 2. The polarity of the mutually induced emf is opposite to that of self-induced emf because the current $(i_1 - i_2)$ leaves from the dotted end in Coil 2 and the current i_2 enters from the dotted end in Coil 3.

Applying KVL to Mesh 1,

$$v_1(t) - R_1\, i_1 - L_1 \frac{di_1}{dt} - M_{12}\frac{d}{dt}(i_1 - i_2) - R_2(i_1 - i_2) - L_2\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{di_2}{dt} - M_{12}\frac{di_1}{dt} = 0$$

$$(R_1 + R_2)\, i_1 + (L_1 + L_2 + 2M_{12})\frac{di_1}{dt} - R_2 i_2 - (L_2 + M_{12} + M_{23})\frac{di_2}{dt} = v_1(t) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$M_{12}\frac{di_1}{dt} - M_{23}\frac{di_2}{dt} - L_2\frac{d}{dt}(i_2 - i_1) - R_2(i_2 - i_1) - L_3\frac{di_2}{dt} + M_{23}\frac{d}{dt}(i_1 - i_2) - R_3 i_2 = 0$$

$$-R_2\, i_1 - (L_2 + M_{12} + M_{23})\frac{di_1}{dt} + (R_2 + R_3)\, i_2 + (L_2 + L_3 + 2M_{23})\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

Example 7.16 *Write KVL equations for the circuit shown in Fig. 7.33.*

Fig. 7.33

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.34.

Fig. 7.34

Applying KVL to Mesh 1,

$$v_1(t) - R_1\, i_1 - L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0$$

$$R_1\, i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = v_1(t) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$M\frac{di_1}{dt} - L_2\frac{di_2}{dt} - R_3(i_2 - i_3) - \frac{1}{C}\int_0^t (i_2 - i_3)\, dt - v_2(t) = 0$$

$$M\frac{di_1}{dt} - L_2\frac{di_2}{dt} - R_3(i_2 - i_3) - \frac{1}{C}\int_0^t (i_2 - i_3)\,dt = v_2(t) \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-R_2 i_3 - \frac{1}{C}\int_0^t (i_3 - i_2)dt - R_3(i_3 - i_2) = 0 \qquad \ldots\text{(iii)}$$

Example 7.17 *Write down the mesh equations for the network shown in Fig. 7.35.*

Fig. 7.35

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.36.

Fig. 7.36

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - \mathbf{Z}_1\mathbf{I}_1 - j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(\mathbf{Z}_1 + j\omega L_1 + \mathbf{Z}_2)\,\mathbf{I}_1 - (\mathbf{Z}_2 - j\omega M)\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-\mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_1) + j\omega M\mathbf{I}_1 - j\omega L_2\mathbf{I}_2 - \mathbf{Z}_L\mathbf{I}_2 = 0$$

$$-(\mathbf{Z}_2 - j\omega M)\,\mathbf{I}_1 + (\mathbf{Z}_2 + j\omega L_2 + \mathbf{Z}_L)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Example 7.18 *Write mesh equations for the network shown in Fig. 7.37.*

Fig. 7.37

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.38.

Fig. 7.38

Applying KVL to Mesh 1,

$$v(t) - Ri_1 - L_1\frac{d}{dt}(i_1 - i_2) - M_{12}\frac{d}{dt}(i_2 - i_3) + M_{13}\frac{di_3}{dt} = 0$$

$$Ri_1 + L_1\frac{d}{dt}(i_1 - i_2) + M_{12}\frac{d}{dt}(i_2 - i_3) - M_{13}\frac{di_3}{dt} = v(t) \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-M_{13}\frac{di_3}{dt} + M_{12}\frac{d}{dt}(i_2 - i_3) - L_1\frac{d}{dt}(i_2 - i_1) - L_2\frac{d}{dt}(i_2 - i_3) - M_{21}\frac{d}{dt}(i_1 - i_2) - M_{23}\frac{di_3}{dt} = 0$$

$$M_{13}\frac{di_3}{dt} - M_{12}\frac{d}{dt}(i_2 - i_3) + L_1\frac{d}{dt}(i_2 - i_1) + L_2\frac{d}{dt}(i_2 - i_3) + M_{21}\frac{d}{dt}(i_1 - i_2) + M_{23}\frac{di_3}{dt} = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$M_{23}\frac{di_3}{dt} + M_{21}\frac{d}{dt}(i_1 - i_2) - L_2\frac{d}{dt}(i_3 - i_2) - L_3\frac{di_3}{dt} + M_{31}\frac{d}{dt}(i_1 - i_2) - M_{32}\frac{d}{dt}(i_2 - i_3) - \frac{1}{C}\int i_3\, dt = 0$$

$$-M_{23}\frac{di_3}{dt} - M_{21}\frac{d}{dt}(i_1 - i_2) + L_2\frac{d}{dt}(i_3 - i_2) + L_3\frac{di_3}{dt} - M_{31}\frac{d}{dt}(i_1 - i_2) + M_{32}\frac{d}{dt}(i_2 - i_3) + \frac{1}{C}\int i_3\, dt = 0 \qquad \ldots\text{(iii)}$$

Example 7.19 *Write KVL equations for the network shown in Fig. 7.39.*

Fig. 7.39

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.40.

Fig. 7.40

Applying KVL to Loop 1,

$$v(t) - R(i_1 + i_2 + i_3) - L_1 \frac{di_1}{dt} - M_{12} \frac{di_2}{dt} + M_{13} \frac{di_3}{dt} = 0$$

$$R(i_1 + i_2 + i_3) + L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} - M_{13} \frac{di_3}{dt} = v(t) \qquad \ldots\text{(i)}$$

Applying KVL to Loop 2,

$$v(t) - R(i_1 + i_2 + i_3) - L_2 \frac{di_2}{dt} - M_{21} \frac{di_1}{dt} - M_{23} \frac{di_3}{dt} = 0$$

$$R(i_1 + i_2 + i_3) + L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt} + M_{23} \frac{di_3}{dt} = v(t) \qquad \ldots\text{(ii)}$$

Applying KVL to Loop 3,

$$v(t) - R(i_1 + i_2 + i_3) - L_3 \frac{di_3}{dt} + M_{31} \frac{di_1}{dt} - M_{32} \frac{di_2}{dt} - \frac{1}{C}\int i_3 \, dt = 0$$

$$R(i_1 + i_2 + i_3) + L_3 \frac{di_3}{dt} - M_{31} \frac{di_1}{dt} + M_{32} \frac{di_2}{dt} + \frac{1}{C}\int i_3 \, dt = v(t) \qquad \ldots\text{(iii)}$$

Example 7.20 *In the network shown in Fig. 7.41, find the voltages V_1 and V_2.*

Fig. 7.41

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.42.

Fig. 7.42

From Fig. 7.42,

$$i_3 = i_1 + i_2 = 5\,e^{-t} + 10\,e^{-t} = 15\,e^{-t}\text{ A}$$

$$v_1 = 3\frac{di_1}{dt} + 2\frac{di_3}{dt} = 3\frac{d}{dt}(5\,e^{-t}) + 2\frac{d}{dt}(15\,e^{-t}) = -15\,e^{-t} - 30\,e^{-t} = -45\,e^{-t}\text{V}$$

$$v_2 = 5\frac{di_3}{dt} + 2\frac{di_1}{dt} = 5\frac{d}{dt}(15\,e^{-t}) + 2\frac{d}{dt}(5\,e^{-t}) = -75\,e^{-t} - 10\,e^{-t} = -85\,e^{-t}\text{V}$$

Example 7.21 *In the network shown in Fig. 7.43, find the voltages V_1 and V_2.*

Fig. 7.43

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.44.

Fig. 7.44

From Fig. 7.44,

$$i_3 = i_1 + i_2 = 10\,e^{-t} + 10\,e^{-t} = 20\,e^{-t}\text{A}$$

$$v_1 = 2\frac{di_1}{dt} - 2\frac{di_3}{dt} = 2\frac{d}{dt}(10\,e^{-t}) - 2\frac{d}{dt}(20\,e^{-t}) = -20\,e^{-t} + 40\,e^{-t} = 20\,e^{-t}\text{A}$$

$$v_2 = 4\frac{di_3}{dt} - 2\frac{di_1}{dt} = 4\frac{d}{dt}(20\,e^{-t}) - 2\frac{d}{dt}(10\,e^{-t}) = -80\,e^{-t} + 20\,e^{-t} = -60\,e^{-t}\text{A}$$

Example 7.22 *Calculate the current $i_2(t)$ in the coupled circuit of Fig. 7.45.*

Fig. 7.45

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.46.

Fig. 7.46

Applying KVL to Mesh 1,

$$30 \sin t - 0.2\frac{di_1}{dt} + 0.1\frac{di_2}{dt} = 0 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-0.2\frac{di_2}{dt}+0.1\frac{di_1}{dt}=0$$

$$\frac{di_1}{dt}=2\frac{di_2}{dt} \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$30\sin t-0.2\left(2\frac{di_2}{dt}\right)+0.1\frac{di_2}{dt}=0$$

$$0.3\frac{di_2}{dt}=30\sin t$$

$$\frac{di_2}{dt}=100\sin t$$

$$di_2=100\sin t\,dt$$

Integrating both the sides,

$$i_2(t)=100\int_0^t \sin t\,dt$$

$$=100\left[-\cos t\right]_0^t$$

$$=100\,(1-\cos t)$$

Example 7.23 *Find the voltage V_2 in the circuit shown in Fig. 7.47 such that the current in the left-hand loop (Loop 1) is zero.*

Fig. 7.47

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.48.

Fig. 7.48

Applying KVL to Loop 1,

$$5\angle 0° - 2\,\mathbf{I}_1 - j4\mathbf{I}_1 + j2\mathbf{I}_2 = 0$$

$$(2 + j4)\,\mathbf{I}_1 - j2\mathbf{I}_2 = 5\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Loop 2,

$$-j2\mathbf{I}_1 - j3\mathbf{I}_2 - 1\mathbf{I}_2 - \mathbf{V}_2 = 0$$

$$-j2\mathbf{I}_1 - (1 + j3)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 2+j4 & -j2 \\ -j2 & -(1+j3) \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 5\angle 0° \\ \mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 5\angle 0° & -j2 \\ \mathbf{V}_2 & -(1+j3) \end{vmatrix}}{\begin{vmatrix} 2+j4 & -j2 \\ -j2 & -(1+j3) \end{vmatrix}}$$

But $\mathbf{I}_1 = 0$.

$$\therefore -(5\angle 0°)(1 + j3) + j2\,\mathbf{V}_2 = 0$$

$$\mathbf{V}_2 = \frac{(5\angle 0°)(1 + j3)}{j2} = 7.91\angle -18.43°\ \text{V}$$

Example 7.24 *Determine the ratio* $\dfrac{\mathbf{V}_2}{\mathbf{V}_1}$ *in the circuit of Fig. 7.49, if* $\mathbf{I}_1 = 0$.

Fig. 7.49

Solution The equivalent circuit in terms of dependent sources is as shown in Fig. 7.50.

Fig. 7.50

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 8\mathbf{I}_1 - j8\mathbf{I}_1 - j2\mathbf{I}_2 = 0$$

$$(8 + j8)\,\mathbf{I}_1 + j2\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(i)}$$

Putting $\mathbf{I}_1 = 0$ in Eq (i),

$$j2\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - 2\mathbf{I}_2 - j2\mathbf{I}_2 - j2\mathbf{I}_1 = 0$$
$$j2\mathbf{I}_1 + (2 + j2)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots\text{(iii)}$$

Putting $\mathbf{I}_1 = 0$ in Eq (iii),

$$(2 + j2)\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots\text{(iv)}$$

From Eqs (ii) and (iv),

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{(2 + j2)\mathbf{I}_2}{j2\mathbf{I}_2} = \frac{2 + j2}{j2} = 1.41 \angle -45^\circ \text{ V}$$

Example 7.25 *For the coupled circuit shown in Fig. 7.51, find input impedance at terminals A and B.*

Fig. 7.51

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.52.

Fig. 7.52

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 3\mathbf{I}_1 - j4\mathbf{I}_1 - j3\,(\mathbf{I}_1 - \mathbf{I}_2) - j5\,(\mathbf{I}_1 - \mathbf{I}_2) - j3\mathbf{I}_1 = 0$$
$$(3 + j15)\,\mathbf{I}_1 - j8\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$j3\,\mathbf{I}_1 - j5\,(\mathbf{I}_2 - \mathbf{I}_1) + j8\,\mathbf{I}_2 = 0$$
$$j8\,\mathbf{I}_1 + j3\,\mathbf{I}_2 = 0$$
$$\mathbf{I}_2 = -\frac{j8}{j3}\mathbf{I}_1 = -2.67\,\mathbf{I}_1 \qquad \ldots\text{(ii)}$$

Substituting Eq (ii) in Eq (i),

$$(3+j15)\mathbf{I}_1 - j8\,(-2.67\,\mathbf{I}_1) = \mathbf{V}_1$$

$$(3+j\,36.36)\,\mathbf{I}_1 = \mathbf{V}_1$$

$$\mathbf{Z}_i = \frac{\mathbf{V}_1}{\mathbf{I}_1} = (3+j\,36.36)\,\Omega = 36.48\,\angle\,85.28^\circ\,\Omega$$

Example 7.26 *Find equivalent impedance of the network shown in Fig. 7.53.*

Fig. 7.53

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.54.

Fig. 7.54

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - 2\mathbf{I}_1 - j2\mathbf{I}_1 + j4\,(\mathbf{I}_1 - \mathbf{I}_2) - j3\,(\mathbf{I}_1 - \mathbf{I}_2) + j4\,\mathbf{I}_1 - 5(\mathbf{I}_1 - \mathbf{I}_2) + j6\mathbf{I}_2 = 0$$

$$(7 - j3)\,\mathbf{I}_1 - (5 + j5)\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j6\mathbf{I}_2 - 5(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_1 - j3(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_2 + j6(\mathbf{I}_1 - \mathbf{I}_2) + j5\mathbf{I}_2 = 0$$

$$(5 + j5)\mathbf{I}_1 = (5 + j4)\,\mathbf{I}_2$$

$$\mathbf{I}_2 = \left(\frac{5 + j5}{5 + j4}\right)\mathbf{I}_1 \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$(7 - j3)\mathbf{I}_1 - (5 + j5)\left(\frac{5 + j5}{5 + j4}\right)\mathbf{I}_1 = \mathbf{V}_1$$

$$\mathbf{Z}_i = \frac{\mathbf{V}_1}{\mathbf{I}_1} = 7 - j3 - \frac{(5 + j5)\,(5 + j5)}{5 + j4} = 5.63 \angle -47.15^\circ\ \Omega$$

Example 7.27 *Find the voltage across the 5 Ω resistor in Fig. 7.55 using mesh analysis.*

Fig. 7.55

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.56.

Fig. 7.56

Applying KVL to Mesh 1,

$$50 \angle 0^\circ - j5\,\mathbf{I}_1 - j\,5.66\,\mathbf{I}_2 - (3 - j4)\,(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3 + j\,1)\,\mathbf{I}_1 - (3 - j\,9.66)\,\mathbf{I}_2 = 50 \angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-(3 - j4)\,(\mathbf{I}_2 - \mathbf{I}_1) - j\,10\mathbf{I}_2 - j\,5.66\,\mathbf{I}_1 - 5\mathbf{I}_2 = 0$$

$$-(3 - j\,9.66)\,\mathbf{I}_1 + (8 + j6)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j1 & -(3-j9.66) \\ -(3-j9.66) & 8+j6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 3+j1 & 50\angle 0° \\ -(3-j9.66) & 0 \end{vmatrix}}{\begin{vmatrix} 3+j1 & -(3-j9.66) \\ -(3-j9.66) & 8+j6 \end{vmatrix}} = 3.82\angle -112.14° \text{ A}$$

$$\mathbf{V}_{5\,\Omega} = 5\,\mathbf{I}_2 = 5\,(3.82\angle -112.14°) = 19.1\angle -112.14° \text{ V}$$

Example 7.28 *Find the voltage across the 5 Ω resistor in Fig. 7.57 using mesh analysis.*

Fig. 7.57

Solution For a magnetically coupled circuit,

$$\begin{aligned} X_M &= k\sqrt{X_{L_1} X_{L_2}} \\ &= 0.8\sqrt{(5)\,(10)} \\ &= 5.66\ \Omega \end{aligned}$$

The equivalent circuit in terms of dependent sources is shown in Fig. 7.58.

Fig. 7.58

Applying KVL to Mesh 1,

$$50\angle 0° - j5\,\mathbf{I}_1 + j\,5.66\,\mathbf{I}_2 - (3-j4)\,(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3+j\,1)\,\mathbf{I}_1 - (3+j\,1.66)\,\mathbf{I}_2 = 50\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-(3-j4)(\mathbf{I}_2-\mathbf{I}_1)-j\,10\mathbf{I}_2+j\,5.66\,\mathbf{I}_1-5\mathbf{I}_2=0$$

$$-(3+j\,1.66)\,\mathbf{I}_1+(8+j6)\,\mathbf{I}_2=0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j\,1 & -(3+j\,1.66) \\ -(3+j\,1.66) & 8+j\,6 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}=\begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2=\frac{\begin{vmatrix} 3+j\,1 & 50\angle 0^\circ \\ -(3+j\,1.66) & 0 \end{vmatrix}}{\begin{vmatrix} 3+j\,1 & -(3+j\,1.66) \\ -(3+j\,1.66) & 8+j6 \end{vmatrix}}=8.62\angle -24.79^\circ \text{ A}$$

$$\mathbf{V}_{5\,\Omega}=5\,\mathbf{I}_2=5\,(8.62\angle -24.79^\circ)=43.1\angle -24.79^\circ \text{ A}$$

Example 7.29 *Find the current through the capacitor in Fig. 7.59 using mesh analysis.*

Fig. 7.59

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.60.

Fig. 7.60

Applying KVL to Mesh 1,

$$50\angle 45^\circ-(3+j4)\,\mathbf{I}_1-j3(\mathbf{I}_1-\mathbf{I}_2)-j5\,(\mathbf{I}_1-\mathbf{I}_2)-j3\,\mathbf{I}_1=0$$

$$(3+j\,15)\,\mathbf{I}_1-j\,8\,\mathbf{I}_2=50\angle 45^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$j3\,\mathbf{I}_1 - j5\,(\mathbf{I}_2 - \mathbf{I}_1) + j8\,\mathbf{I}_2 = 0$$

$$-j8\,\mathbf{I}_1 - j3\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 50\angle 45° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 3+j15 & 50\angle 45° \\ -j8 & 0 \end{vmatrix}}{\begin{vmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{vmatrix}} = 3.66\angle -310.33°\ \text{A}$$

$$\mathbf{I}_C = \mathbf{I}_2 = 3.66\angle -310.33°\text{A}$$

Example 7.30 *Find the voltage across the 15 Ω resistor in Fig. 7.61 using mesh analysis.*

Fig. 7.61

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.62.

Fig. 7.62

Applying KVL to Mesh 1,

$$120\angle 0° - 20\,\mathbf{I}_1 - j20(\mathbf{I}_1 - \mathbf{I}_2) - j5\,\mathbf{I}_2 = 0$$

$$(20 + j\,20)\,\mathbf{I}_1 - j\,15\,\mathbf{I}_2 = 120\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$j5\,\mathbf{I}_2 - j\,20\,(\mathbf{I}_2 - \mathbf{I}_1) - j10\,\mathbf{I}_2 - j5\,(\mathbf{I}_1 - \mathbf{I}_2) - 15\,\mathbf{I}_2 = 0$$
$$-j15\,\mathbf{I}_1 + (15 + j20)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 20 + j\,20 & -j15 \\ -j15 & 15 + j\,20 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 20 + j\,20 & 120\angle 0^\circ \\ -j15 & 0 \end{vmatrix}}{\begin{vmatrix} 20 + j\,20 & -j15 \\ -j15 & 15 + j\,20 \end{vmatrix}} = 2.53\angle 10.12^\circ \text{A}$$

$$\mathbf{V}_{15\,\Omega} = 15\,\mathbf{I}_2 = 15\,(2.53\angle 10.12^\circ) = 37.95\angle 10.12^\circ \text{V}$$

Example 7.31 *Find the current through the 6 Ω resistor in Fig. 7.63 using mesh analysis.*

Fig. 7.63

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.64.

Fig. 7.64

Applying KVL to Mesh 1,

$$120\angle 0^\circ - 4\,\mathbf{I}_1 - j3(\mathbf{I}_1 - \mathbf{I}_2) + j2\,\mathbf{I}_2 = 0$$
$$(4 + j\,3)\,\mathbf{I}_1 - j\,5\,\mathbf{I}_2 = 120\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j\,2\,\mathbf{I}_2 - j\,3\,(\mathbf{I}_2 - \mathbf{I}_1) - j8\,\mathbf{I}_2 + j2\,(\mathbf{I}_1 - \mathbf{I}_2) - 6\,\mathbf{I}_2 = 0$$

$$-j5\,\mathbf{I}_1 + (6 + j15)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 4 + j\,3 & -j5 \\ -j5 & 6 + j\,15 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4 + j\,3 & 120\angle 0^\circ \\ -j5 & 0 \end{vmatrix}}{\begin{vmatrix} 4 + j\,3 & -j5 \\ -j5 & 6 + j\,15 \end{vmatrix}} = 7.68\angle 2.94^\circ\text{A}$$

Example 7.32 *Determine the mesh current $\mathbf{I}_3$ in the network of Fig. 7.65.*

Fig. 7.65

Solution The equivalent circuit in terms of dependent sources is shown in Fig. 7.66.

Fig. 7.66

Applying KVL to Mesh 1,

$$200\angle 30° - 4(\mathbf{I}_1 - \mathbf{I}_3) - j4(\mathbf{I}_1 - \mathbf{I}_2) + j5\,\mathbf{I}_3 = 0$$

$$(4 + j\,4)\,\mathbf{I}_1 - j\,4\,\mathbf{I}_2 - (4 + j5)\,\mathbf{I}_3 = 200\angle 30° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j\,5\,\mathbf{I}_3 - j\,4\,(\mathbf{I}_2 - \mathbf{I}_1) - (7 - j8)\,(\mathbf{I}_2 - \mathbf{I}_3) - (6 - j\,4)\,\mathbf{I}_2 = 0$$

$$-j4\,\mathbf{I}_1 + (13 - j8)\,\mathbf{I}_2 - (7 - j13)\,\mathbf{I}_3 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-j\,16\,\mathbf{I}_3 + j\,5\,(\mathbf{I}_1 - \mathbf{I}_2) - 12\,\mathbf{I}_3 - (7 - j\,8)\,(\mathbf{I}_3 - \mathbf{I}_2) - 4(\mathbf{I}_3 - \mathbf{I}_1) = 0$$

$$-(4 + j5)\,\mathbf{I}_1 - (7 - j13)\,\mathbf{I}_2 + (23 + j8)\,\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs. (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4 + j4 & -j4 & -(4 + j5) \\ -j4 & 13 - j8 & -(7 - j13) \\ -(4 + j5) & -(7 - j13) & 23 + j8 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 200\angle 30° \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 4 + j4 & -j4 & 200\angle 30° \\ -j4 & 13 - j8 & 0 \\ -(4 + j5) & -(7 - j13) & 0 \end{vmatrix}}{\begin{vmatrix} 4 + j4 & -j4 & -(4 + j5) \\ -j4 & 13 - j8 & -(7 - j13) \\ -(4 + j5) & -(7 - j13) & 23 + j8 \end{vmatrix}} = 16.28\angle 16.87°\text{A}$$

Example 7.33 *Obtain the dotted equivalent circuit for the coupled circuit shown in Fig. 7.67 and find mesh currents. Also find the voltage across the capacitor.*

Fig. 7.67

Solution The currents in the coils are as shown in Fig. 7.68. The corresponding flux due to current in each coil is also drawn with the help of right-hand thumb rule.

Fig. 7.68

From Fig. 7.68, it is seen that two fluxes ϕ_1 and ϕ_2 aid each other. Hence, dots are placed at the two coils as shown in Fig. 7.69.

Fig. 7.69

The equivalent circuit in terms of dependent sources is shown in Fig. 7.70.

Fig. 7.70

Applying KVL to Mesh 1,

$$10 \angle 0° - (5 + j5)\,\mathbf{I}_1 - j2\,\mathbf{I}_2 + j10\,(\mathbf{I}_1 + \mathbf{I}_2) = 0$$

$$(5 - j\,5)\,\mathbf{I}_1 - j\,8\,\mathbf{I}_2 = 10 \angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-j10\,(\mathbf{I}_2 + \mathbf{I}_1) + j5\,\mathbf{I}_2 - j2\,\mathbf{I}_1 + 5\,\mathbf{I}_2 - 10 \angle 90° = 0$$

$$-j\,8\,\mathbf{I}_1 + (5 - j5)\,\mathbf{I}_2 = 10 \angle 90° \qquad \ldots\text{(ii)}$$

Writing Eqs. (i) and (ii) in matrix form,

$$\begin{bmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 10\angle 90° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0° & -j8 \\ 10\angle 90° & 5-j5 \end{vmatrix}}{\begin{vmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{vmatrix}} = 0.72\angle -82.97°\text{A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5-j5 & 10\angle 0° \\ -j8 & 10\angle 90° \end{vmatrix}}{\begin{vmatrix} 5-j5 & -j8 \\ -j8 & 5-j5 \end{vmatrix}} = 1.71\angle 106.96°\text{A}$$

$$\mathbf{V}_C = -j10\,(\mathbf{I}_1 + \mathbf{I}_2) = (-j10)\,(0.72\angle -82.97° + 1.71\angle 106.96°\text{ A})$$
$$= 10.08\angle 24.03°\text{V}$$

Example 7.34 *Find Thevenin's and Norton's equivalent network at terminals A-B for Fig. 7.71.*

Fig. 7.71

Solution

Step I Calculation of $\mathbf{V}_{Th}$

The equivalent circuit in terms of dependent sources is shown in Fig. 7.72.

Fig. 7.72

Applying KVL to Mesh,

$$10\angle 0° - 4\mathbf{I} - j\,10\,\mathbf{I} + j6\,\mathbf{I} - j5\,\mathbf{I} + j6\,\mathbf{I} - 4\mathbf{I} = 0$$

$$10\angle 0° - (8 + j3)\,\mathbf{I} = 0$$

$$\mathbf{I} = \frac{10\angle 0°}{8 + j3} = 1.17\angle -20.56°\text{A}$$

Writing the $\mathbf{V}_{Th}$ equation,

$$4\mathbf{I} - j6\,\mathbf{I} + j5\,\mathbf{I} - 0 - \mathbf{V}_{Th} = 0$$

$$(4 - j1)\,\mathbf{I} - \mathbf{V}_{Th} = 0$$

$$\mathbf{V}_{Th} = (4 - j1)\,(1.17\angle -20.56°)$$

$$= 4.82\angle -34.6°\text{V}$$

Step II Calculation of $\mathbf{I}_N$

The equivalent circuit in terms of dependent sources is shown in Fig. 7.73.

Fig. 7.73

Applying KVL to Mesh 1,

$$10\angle 0° - 4\mathbf{I}_1 - j\,10\,\mathbf{I}_1 + j6\,(\mathbf{I}_1 - \mathbf{I}_2) - j\,5(\mathbf{I}_1 - \mathbf{I}_2) + j6\,\mathbf{I}_1 - 4(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(8 + j3)\,\mathbf{I}_1 - (4 - j1)\,\mathbf{I}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$-4(\mathbf{I}_2 - \mathbf{I}_1) - j6\,\mathbf{I}_1 - j\,5(\mathbf{I}_2 - \mathbf{I}_1) - 3\mathbf{I}_2 = 0$$

$$-(4 - j1)\,\mathbf{I}_1 + (7 + j5)\,\mathbf{I}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs. (i) and (ii) in matrix form,

$$\begin{bmatrix} 8 + j\,3 & -(4 - j1) \\ -(4 - j1) & 7 + j\,5 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 8 + j\,3 & 10\angle 0° \\ -(4 - j1) & 0 \end{vmatrix}}{\begin{vmatrix} 8 + j\,3 & -(4 - j1) \\ -(4 - j1) & 7 + j5 \end{vmatrix}} = 0.56\angle -83.39°\text{A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 0.56 \angle -83.39° \text{A}$$

Step III Calculation of $\mathbf{Z}_{\text{Th}}$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{4.82 \angle -34.6°}{0.56 \angle -83.39°} = 8.61 \angle 48.79° \ \Omega$$

Step IV Thevenin's Equivalent Network

Fig. 7.74

Step V Norton's Equivalent Network

Fig. 7.75

Example 7.35 *Find the value of Z_L connected across terminals A and B in Fig. 7. 76 to receive the maximum power when (a) Z_L is complex impedance, and (b) Z_L is pure resistance.*

Fig. 7.76

Solution

Step I Calculation of $\mathbf{V}_{\text{Th}}$

The equivalent circuit in terms of dependent sources is shown in Fig. 7.77.

Fig. 7.77

Applying KVL to the mesh,

$$50 \angle 0° - 5\mathbf{I} - j\,8\,\mathbf{I} - j\,4\,\mathbf{I} - j\,10\,\mathbf{I} - j\,4\,\mathbf{I} = 0$$

$$50 \angle 0° - (5 + j\,26)\,\mathbf{I} = 0$$

$$\mathbf{I} = \frac{50 \angle 0°}{5 + j\,26} = 1.89 \angle -79.11° \text{ A}$$

Writing the $\mathbf{V}_{\text{Th}}$ equation,

$$j\,4\,\mathbf{I} + j\,10\,\mathbf{I} - \mathbf{V}_{\text{Th}} = 0$$

$$j\,14\,\mathbf{I} - \mathbf{V}_{\text{Th}} = 0$$

$$\mathbf{V}_{\text{Th}} = (j\,14)\,(1.89 \angle -79.11°) = 26.46 \angle 10.89° \text{ V}$$

Step II Calculation of $\mathbf{I}_N$

The equivalent circuit in terms of dependent sources is shown in Fig. 7.78.

Fig. 7.78

Applying KVL to Mesh 1,

$$50 \angle 0° - 5\mathbf{I}_1 - j\,8\,\mathbf{I}_1 - j\,4\,(\mathbf{I}_1 - \mathbf{I}_2) - j\,10\,(\mathbf{I}_1 - \mathbf{I}_2) - j\,4\,\mathbf{I}_1 = 0$$

$$(5 + j26)\,\mathbf{I}_1 - j\,14\,\mathbf{I}_2 = 50 \angle 0° \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$j\,4\,\mathbf{I}_1 - j\,10\,(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

$$j\,14\,\mathbf{I}_1 - j\,10\,\mathbf{I}_2 = 0$$

$$\mathbf{I}_1 = 0.71\,\mathbf{I}_2 \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$(5 + j26)\,(0.71\,\mathbf{I}_2) - j\,14\,\mathbf{I}_2 = 50 \angle 0°$$

$$(5.7 \angle 51.48°)\,\mathbf{I}_2 = 50 \angle 0°$$

$$\mathbf{I}_2 = 8.77 \angle -51.48° \text{ A}$$

$$\mathbf{I}_N = \mathbf{I}_2 = 8.77 \angle -51.48° \text{ A}$$

Step III Calculation of $\mathbf{Z}_{\text{Th}}$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{I}_N} = \frac{26.46 \angle 10.89}{5.7 \angle -51.48} = 3.02 \angle 62.37° \ \Omega = (1.4 + j\,2.67)\ \Omega$$

Step IV Calculation of $\mathbf{Z}_L$

(i) When $\mathbf{Z}_L$ is complex impedance

For maximum power transfer, $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = (1.4 - j\,2.67)\ \Omega$

(ii) When $\mathbf{Z}_L$ is pure resistance

For maximum power transfer, $\mathbf{Z}_L = |\mathbf{Z}_{\text{Th}}| = |1.4 + j\,2.67| = 3.02\ \Omega$

7.9 CONDUCTIVELY COUPLED EQUIVALENT CIRCUITS

For simplifying circuit analysis, it is desirable to replace a magnetically coupled circuit with an equivalent circuit called conductively coupled circuit. In this circuit, no magnetic coupling is involved. The dot convention is also not needed in the conductively coupled circuit.

Consider a coupled circuit as shown in Fig. 7.79.

Fig. 7.79 *Coupled circuit*

The equivalent circuit in terms of dependent sources is shown in Fig. 7.80.

Fig. 7.80 *Equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - j\omega L_1\,\mathbf{I}_1 - j\omega M\,\mathbf{I}_2 = 0$$

$$j\omega L_1\,\mathbf{I}_1 + j\omega M\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(7.11)$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - j\omega L_2\,\mathbf{I}_2 - j\omega M\,\mathbf{I}_1 = 0$$

$$j\omega M\,\mathbf{I}_1 + j\omega L_2\,\mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots(7.12)$$

Writing Eqs (7.11) and (7.12) in matrix form,

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \qquad \ldots(7.13)$$

Consider a T-network as shown in Fig. 7.81.

Fig. 7.81 *T-network*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - \mathbf{Z}_1\ \mathbf{I}_1 - \mathbf{Z}_3(\mathbf{I}_1 + \mathbf{I}_2) = 0$$
$$(\mathbf{Z}_1 + \mathbf{Z}_3)\ \mathbf{I}_1 + \mathbf{Z}_3\ \mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots (7.14)$$

Applying KVL to Mesh 2,

$$\mathbf{V}_2 - \mathbf{Z}_2\ \mathbf{I}_2 - \mathbf{Z}_3(\mathbf{I}_2 + \mathbf{I}_1) = 0$$
$$\mathbf{Z}_3\ \mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z}_3)\ \mathbf{I}_2 = \mathbf{V}_2 \qquad \ldots (7.15)$$

Writing Eqs (7.14) and (7.15) in matrix form,

$$\begin{bmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & \mathbf{Z}_3 \\ \mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Comparing matrix equations,

$$\mathbf{Z}_1 + \mathbf{Z}_3 = j\ \omega\ L_1$$
$$\mathbf{Z}_3 = j\ \omega\ M$$
$$\mathbf{Z}_2 + \mathbf{Z}_3 = j\ \omega\ L_2$$

Solving these equations,

$$\mathbf{Z}_1 = j\ \omega\ L_1 - j\ \omega\ M = j\ \omega(L_1 - M)$$
$$\mathbf{Z}_2 = j\ \omega\ L_2 - j\ \omega\ M = j\ \omega(L_2 - M)$$
$$\mathbf{Z}_3 = j\ \omega\ M$$

Hence, the conductively coupled circuit of a magnetically coupled circuit is shown in Fig. 7.82.

Fig. 7.82 *Conductively coupled equivalent circuit*

Example 7.36 *Find the conductively coupled equivalent circuit for the network shown in Fig. 7.83.*

Fig. 7.83

Solution The current $\mathbf{I}_1$ leaves from the dotted end and $\mathbf{I}_2$ enters from the dotted end. Hence, mutual inductance M is negative.

In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 - M) = j\omega L_1 - j\omega M = j3 - j2 = j1\ \Omega$$

$$\mathbf{Z}_2 = j\omega(L_2 - M) = j\omega L_2 - j\omega M = j5 - j2 = j3\ \Omega$$

$$\mathbf{Z}_3 = j\omega M = j2\ \Omega$$

The conductively coupled equivalent circuit is shown in Fig. 7.84.

Fig. 7.84

Example 7.37 *Draw the conductively coupled equivalent circuit of Fig. 7.85.*

Fig. 7.85

Solution The current $\mathbf{I}_1$ enters from the dotted end and $\mathbf{I}_2$ leaves from the dotted end. Hence, the mutual inductance M is negative.

In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 - M) = j\omega L_1 - j\omega M = j5 - j6 = -j1\ \Omega$$
$$\mathbf{Z}_2 = j\omega(L_2 - M) = j\omega L_2 - j\omega M = j10 - j6 = j4\ \Omega$$
$$\mathbf{Z}_3 = j\omega M = j6\ \Omega$$

The conductively coupled equivalent circuit is shown in Fig. 7.86.

Fig. 7.86

Example 7.38 *Find the conductively coupled equivalent circuit of the network in Fig. 7.87.*

Fig. 7.87

Solution The currents $\mathbf{I}_1$ and $\mathbf{I}_2$ leave from the dotted terminals. Hence, mutual inductance is positive. In the conductively coupled equivalent circuit,

$$\mathbf{Z}_1 = j\omega(L_1 + M) = j\omega L_1 + j\omega M = j4 + j2 = j6\ \Omega$$
$$\mathbf{Z}_2 = j\omega(L_2 + M) = j\omega L_2 + j\omega M = j2 + j2 = j4\ \Omega$$
$$\mathbf{Z}_3 = -j\omega M = -j2\ \Omega$$

The conductively coupled equivalent circuit is shown in Fig. 7.88.

Fig. 7.88

7.10 TUNED CIRCUITS

At radio frequencies, the amplification of signals is done by tuned circuits. There are two types of tuned circuits—single-tuned circuit and double-tuned circuit. In the single-tuned circuit, only the secondary is tuned. In the double-tuned circuits, both primary and secondary of the coupled circuits are tuned. The tuned circuits are used in TV receivers to couple the antenna to the RF stage. In RF circuit design, the tuned circuits are generally employed for obtaining maximum power transfer to the load connected to secondary or for obtaining maximum possible value of the secondary voltage.

7.10.1 Single-Tuned Circuit

Consider a single-tuned circuit as shown in Fig. 7.89.

Fig. 7.89 *Single-tuned circuit*

The conductively coupled equivalent circuit is shown in Fig. 7.90.

Fig. 7.90 *Conductively coupled equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - R_1\mathbf{I}_1 - j\omega(L_1 - M)\mathbf{I}_1 - j\omega M(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\,\mathbf{I}_2 = \mathbf{V}_1 \qquad \ldots(7.16)$$

Applying KVL to Mesh 2,

$$-j\omega M(\mathbf{I}_2 - \mathbf{I}_1) - j\omega(L_2 - M)\mathbf{I}_2 - R_2\mathbf{I}_2 + j\frac{1}{\omega C_2}\mathbf{I}_2 = 0$$

$$-j\omega M\,\mathbf{I}_1 + \left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right]\mathbf{I}_2 = 0 \qquad \ldots(7.17)$$

From Eq. (7.17),

$$\mathbf{I}_1 = \frac{R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}{j\omega M}\mathbf{I}_2 \qquad \ldots(7.18)$$

Substituting Eq. (7.18) in Eq. (7.16),

$$\left(R_1 + j\omega L_1\right)\left[\frac{R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}{j\omega M}\right]\mathbf{I}_2 - j\omega M\mathbf{I}_2 = \mathbf{V}_1$$

$$\mathbf{I}_2\left\{\frac{\left(R_1 + j\omega L_1\right)\left[R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right] + \omega^2 M^2}{j\omega M}\right\} = \mathbf{V}_1$$

$$\mathbf{I}_2 = \frac{j\omega M\ \mathbf{V}_1}{(R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}$$

1. Output Voltage

$$\mathbf{V}_2 = -j\left(\frac{1}{\omega C_2}\right)\mathbf{I}_2 = \frac{1}{j\omega C_2}\mathbf{I}_2$$

$$= \frac{MV_1}{C_2\left\{(R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

2. Voltage Amplification

$$A = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{M}{C_2\left\{(R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

If $R_1 >> j\omega L_1$ and secondary circuit resonates at

$$\omega_0 = \frac{1}{\sqrt{L_2 C_2}}$$

then, at the resonant frequency $\omega_{0,}$

$$I_2 = \frac{\omega_0 M V_1}{R_1 R_2 + \omega_0^2 M^2}$$

$$V_2 = \frac{MV_1}{C_2(R_1 R_2 + \omega_0^2 M^2)}$$

$$A = \frac{M}{C_2(R_1 R_2 + \omega_0^2 M^2)}$$

Output current, output voltage and voltage amplification depends on the mutual inductance M at resonant frequency. The maximum amplification or maximum output voltage can be obtained by

$$\frac{dA}{dM} = 0$$

$$\frac{C_2(R_1 R_2 + \omega_0^2 M^2) - 2C_2\omega_0^2 M^2}{\left[C_2(R_1 R_2 + \omega_0^2 M^2)\right]^2} = 0$$

$$R_1 R_2 + \omega_0^2 M^2 - 2\omega_0^2 M^2 = 0$$

$$\omega_0^2 M^2 = R_1 R_2$$

$$M = \frac{\sqrt{R_1 R_2}}{\omega_0}$$

where $M = \frac{\sqrt{R_1 R_2}}{\omega_0}$ is the critical value of mutual inductance.

Hence, maximum value of I_2, V_2 and A is

$$I_{2_{\max}} = \frac{V_1}{2\sqrt{R_1 R_2}}$$

$$V_{2_{\max}} = \frac{V_1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

$$A_{\max} = \frac{1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

The variation of output voltage V_0 with angular frequency ω for different values of coefficient of coupling k is shown in Fig. 7.91.

Fig. 7.91 *Variation of V_0 with ω for different values of k*

7.10.2 Double-Tuned Circuit

Consider a double-tuned circuit with tuning capacitors placed both in primary and secondary circuits as shown in Fig. 7.92.

Fig. 7.92 *Double-tuned circuit*

The conductively coupled equivalent circuit is shown in Fig. 7.93.

Fig. 7.93 *Conductively coupled equivalent circuit*

Applying KVL to Mesh 1,

$$\mathbf{V}_1 - R_1\mathbf{I}_1 + j\frac{1}{\omega C_2}\mathbf{I}_1 - j\omega(L_1 - M)\mathbf{I}_1 - j\omega M(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$\left(R_1 + j\omega L_1 - j\frac{1}{\omega C_1}\right)\mathbf{I}_1 - j\omega M\mathbf{I}_2 = \mathbf{V}_1 \qquad ...(7.19)$$

Applying KVL to Mesh 2,

$$-j\omega M(\mathbf{I}_2 - \mathbf{I}_1) - j\omega(L_2 - M)\mathbf{I}_2 - R_2\mathbf{I}_2 + j\frac{1}{\omega C_2}\mathbf{I}_2 = 0$$

$$-j\omega M\mathbf{I}_1 + \left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right]\mathbf{I}_2 = 0 \qquad ...(7.20)$$

From Eq. (7.20),

$$\mathbf{I}_1 = \frac{R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)}{j\omega M}\mathbf{I}_2 \qquad ...(7.21)$$

Substituting Eq. (7.21) in Eq. (7.19),

$$\left(R_1 + j\omega L_1 - j\frac{1}{\omega C_2}\right)\left[\frac{R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)}{j\omega M}\right]\mathbf{I}_2 - j\omega M\mathbf{I}_2 = \mathbf{V}_1$$

$$\mathbf{I}_2\left\{\frac{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}{j\omega M}\right\} = \mathbf{V}_1$$

$$\mathbf{I}_2 = \frac{j\omega M\mathbf{V}_1}{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right) + \omega^2 M^2\right]}$$

1. Output Voltage

$$\mathbf{V}_2 = -j\left(\frac{1}{\omega C_2}\right)\mathbf{I}_2 = \frac{1}{j\omega C_2}\mathbf{I}_2$$

$$= \frac{M\mathbf{V}_1}{C_2\left\{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

2. Voltage Amplification

$$A = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{M}{C_2\left\{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_2}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2\right\}}$$

If the primary and secondary circuits resonate at the same frequency ω_0,

$$\omega_0 = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}$$

Then, at the resonant frequency $\omega_{0,}$

$$I_2 = \frac{\omega_0 M V_1}{R_1 R_2 + \omega_0^2 M^2}$$

$$V_2 = \frac{M V_1}{C_2 (R_1 R_2 + \omega_0^2 M^2)}$$

$$A = \frac{M}{C_2 (R_1 R_2 + \omega_0^2 M^2)}$$

The maximum amplification or maximum output voltage can be obtained by,

$$\frac{dA}{dM} = 0$$

$$\frac{C_2 (R_1 R_2 + \omega_0^2 M^2) - 2C_2 \omega_0^2 M^2}{\left[C_2 (R_1 R_2 + \omega_0^2 M^2)\right]^2} = 0$$

$$\left(R_1 R_2 + \omega_0^2 M^2\right) - 2\omega_0^2 M^2 = 0$$

$$\omega_0^2 M^2 = R_1 R_2$$

$$M = \frac{\sqrt{R_1 R_2}}{\omega_0}$$

where $M = \dfrac{\sqrt{R_1 R_2}}{\omega_0}$ is the critical value of mutual inductance.

Hence, maximum value of $I_{2,}$ V_2 and A is

$$I_{2_{\max}} = \frac{V_1}{2\sqrt{R_1 R_2}}$$

$$V_{2_{\max}} = \frac{V_1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

$$A_{\max} = \frac{1}{2\omega_0 C_2 \sqrt{R_1 R_2}}$$

Fig. 7.94 *Variation of V_0 with ω for different values of k*

The variation of the output voltage V_0 with the angular frequency ω for different values of coefficient of coupling k is shown in Fig. 7.94.

Example 7.39 *For the single-tuned circuit shown in Fig. 7.95, determine (a) the resonant frequency, (b) Output voltage at resonance, and (c) maximum output voltage. Assume $R_1 >> \omega_0 L_1$ and $k = 0.8$.*

Fig. 7.95

Solution

$$M = k\sqrt{L_1 L_2} = 0.8\sqrt{(1\times10^{-6})(60\times10^{-6})} = 6.2\ \mu\text{H}$$

(a) Resonant frequency

$$\omega_0 = \frac{1}{\sqrt{L_2 C_2}} = \frac{1}{\sqrt{(60\times10^{-6})(0.1\times10^{-6})}} = 0.41\times10^6\ \text{rad/s}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{0.41\times10^6}{2\pi} = 0.0653\ \text{MHz} = 65.3\ \text{kHz}$$

(b) Output voltage at resonance

$$V_0 = \frac{MV_1}{C_2\left(R_1R_2 + \omega_0^2 M^2\right)} = \frac{6.2\times10^{-6}\times50}{0.1\times10^{-6}\left[(10\times5)+\left(0.42\times10^6\right)^2\left(6.2\times10^{-6}\right)^2\right]} = 54.6\ \text{V}$$

(c) Maximum output voltage

$$V_{2_{\max}} = \frac{V_1}{2\omega_0 C_2\sqrt{R_1R_2}} = \frac{50}{2(0.42\times10^6)(0.1\times10^{-6})\sqrt{10\times5}} = 84.18\ \text{V}$$

Example 7.40 *The resonant frequency of the tuned circuit shown in Fig. 7.96 is 1000 rad/s. Calculate the self-inductances of two coils and critical value of mutual inductance.*

Fig. 7.96

Solution

$$\omega_0 = \frac{1}{\sqrt{L_1C_1}} = \frac{1}{\sqrt{L_2C_2}}$$

$$L_1 = \frac{1}{\omega_0^2 C_1} = \frac{1}{(1000)^2\left(1\times10^{-6}\right)} = 1\ \text{H}$$

$$L_2 = \frac{1}{\omega_0^2 C_2} = \frac{1}{(1000)^2\left(2\times10^{-6}\right)} = 0.5\ \text{H}$$

$$M_c = \frac{\sqrt{R_1R_2}}{\omega_0} = \frac{\sqrt{5\times3}}{1000} = 3.87\ \text{mH}$$

Example 7.41 *The tuned frequency of a double-tuned circuit shown in Fig. 7.97 is 10^4 rad/s. If the source voltage is 2 V and has a resistance of 0.1 Ω, calculate the maximum output voltage.*

Fig. 7.97

Solution

$$M_C = \frac{\sqrt{(R_1 + R_S)R_2}}{\omega_0} = \frac{\sqrt{(0.01+0.1)(0.1)}}{10^4} = 10.49\ \mu\text{H}$$

$$\omega_0 = \frac{1}{\sqrt{L_2 C_2}}$$

$$10^4 = \frac{1}{\sqrt{(25\times10^{-6})C_2}}$$

$$C_2 = 0.4\ \text{mF}$$

$$V_{0_{max}} = \frac{V_1}{2\omega_0 C_2\sqrt{(R_S + R_1)R_2}} = \frac{2}{2\times10^4\times0.4\times10^{-3}\times\sqrt{(0.1+0.01)(0.1)}} = 2.38\ \text{V}$$

Exercises

7.1 Two coupled coils have inductances of 0.8 H and 0.2 H. The coefficient of coupling is 0.90. Find the mutual inductance and the turns ratio $\frac{N_1}{N_2}$.

[0.36 H, 2]

7.2 Two coils with coefficient of coupling 0.5 are connected in such a way that they magnetise (i) in the same direction, and (ii) in opposite directions. The corresponding equivalent inductances are 1.9 H and 0.7 H. Find self-inductances of the two coils and the mutual inductance between them.

[0.4 H, 0.9 H, 0.3 H]

7.3 Two coils having 3000 and 2000 turns are wound on a magnetic ring. 60% of the flux produced in the first coil links with the second coil. A current of 3 A produce a flux of 0.5 mwb in the first coil and 0.3 mwb in the second coil. Determine the mutual inductance and coefficient of coupling.

[0.2 H, 0.63]

7.4 Find the equivalent inductance of the network shown in Fig. 7.98.

Fig. 7.98

[10 H]

7.5 Find the effective inductance of the network shown in Fig. 7.99.

Fig. 7.99

[4.8 H]

7.6 Write mesh equations of the network shown in Fig. 7.100.

Fig. 7.100

$$\left[\begin{array}{l} v = i_1R_1 + L_1\dfrac{d}{dt}(i_1 - i_2) + M_{12}\dfrac{di_2}{dt} + M_{13}\dfrac{di_3}{dt} \\ R_2(i_3 - i_2) + R_3i_3 + L_3\dfrac{di_3}{dt} + M_{13}\dfrac{d}{dt}(i_1 - i_2) \\ -M_{23}\dfrac{di_2}{dt} = 0 \end{array}\right]$$

7.7 Find the input impedance at terminals *AB* of the coupled circuits shown in Fig. 7.101 to 7.103.

(i)

Fig. 7.101

(ii)

Fig. 7.102

(iii)

Fig. 7.103

$$\left[\begin{array}{ll} \text{(a)}\,(3 + j36.3)\,\Omega & \text{(b)}\,(1 + j1.5)\,\Omega \\ \text{(c)}\,(6.22 + j4.65)\,\Omega & \end{array}\right]$$

7.8 In the coupled circuit shown in Fig. 7.104, find $\mathbf{V}_2$ for which $\mathbf{I}_1 = 0$. What voltage appears at the 8 Ω inductive reactance under this condition?

Fig. 7.104

[141.5∠ – 45°V, 100∠0°V]

7.9 For the coupled circuit shown in Fig. 7.105, find the components of the current $\mathbf{I}_2$ resulting from each source $\mathbf{V}_1$ and $\mathbf{V}_2$.

Fig. 7.105

[0.77∠112.6°A, 1.72∠86.05°A]

7.10 Find the voltage across the 5 Ω resistor in the network shown in Fig. 7.106.

Fig. 7.106

[19.2∠ – 33.02°V]

7.11 Find the power dissipated in the 5 Ω resistor in the network of Fig. 7.107.

Fig. 7.107

[668.16 W]

7.12 Obtain Thevenin's and Norton's equivalent network at terminals *AB* of the coupled circuit shown in Fig. 7.108.

Fig. 7.108

[7.07∠45° V, 1.04∠ – 27.9°A, 6.8∠72.9° Ω]

7.13 For the circuit shown in Fig. 7.109, what load connected to terminals *A* and *B* absorbs maximum power and what is this power?

Fig. 7.109

[54.1∠ – 56.3° Ω, 83.3 W]

7.14 Find the current **I** in the circuit of Fig. 7.110.

Fig. 7.110

[7.38∠39.4° A]

7.15 Obtain a conductively coupled circuit for the circuit shown in Fig. 7.111.

Fig. 7.111

Fig. 7.112

Objective-Type Questions

7.1 Two coils are wound on a common magnetic core. The sign of mutual inductance M for finding out effective inductance of each coil is positive if the

(a) two coils are wound in the same sense.
(b) fluxes produced by the two coils are equal
(c) fluxes produced by the coils act in the same direction
(d) fluxes produced by the two coils act in opposition

7.2 When two coils having self-inductances of L_1 and L_2 are coupled through a mutual inductance M, the coefficient of coupling k is given by

(a) $k = \dfrac{M}{\sqrt{2L_1L_2}}$ (b) $k = \dfrac{M}{\sqrt{L_1L_2}}$

(c) $k = \dfrac{2M}{\sqrt{L_1L_2}}$ (d) $k = \dfrac{L_1L_2}{M}$

7.3 The overall inductance of two coils connected in series, with mutual inductance aiding self-inductance is L_1; with mutual inductance opposing self-inductance, the overall inductance is L_2. The mutual inductance M is given by

(a) $L_1 + L_2$ (b) $L_1 - L_2$

(c) $\frac{1}{4}(L_1 - L_2)$ (d) $\frac{1}{2}(L_1 + L_2)$

7.4 Consider the following statements:
The coefficient of coupling between two oils depends upon
1. Orientation of the coils
2. Core material
3. Number of turns on the two coils
4. Self-inductance of the two coils
of these statements,
(a) 1, 2 and 3 are correct
(b) 1 and 2 are correct
(c) 3 and 4 are correct
(d) 1, 2 and 4 are correct

7.5 Two coupled coils connected in series have an equivalent inductance of 16 mH or 8 mH depending on the inter connection.
Then the mutual inductance M between the coils is

(a) 12 mH (b) $8\sqrt{2}$ mH
(c) 4 mH (d) 2 mH

7.6 Two coupled coils with $L_1 = L_2 = 0.6$ H have a coupling coefficient of $k = 0.8$. The turns ratio $\dfrac{N_1}{N_2}$ is

(a) 4 (b) 2
(c) 1 (d) 0.5

7.7 The coupling between two magnetically coupled coils is said to be ideal if the coefficient of coupling is

(a) zero (b) 0.5
(c) 1 (d) 2

7.8 The mutual inductance between two coupled coils is 10 mH . If the turns in one coil are doubled and that in the other are halved then the mutual inductance will be

(a) 5 mH (b) 10 mH
(c) 14 mH (d) 20 mH

7.9 Two perfectly coupled coils each of 1 H self-inductance are connected in parallel so as to aid each other. The overall inductance in henrys is

(a) 2 (b) 1
(c) $\frac{1}{2}$ (d) Zero

7.10 The impedance Z as shown in Fig.7.113 is

Fig. 7.113

(a) $j\,29\ \Omega$ (b) $j9\ \Omega$
(c) $j19\ \Omega$ (d) $j39\ \Omega$

Answers to Objective-Type Questions

1. (c)	2. (b)	3. (c)	4. (d)	5. (d)	6. (c)
7. (c)	8. (b)	9. (b)	10. (b)		

8

Three-Phase Circuits

8.1 INTRODUCTION

A system which generates a single alternating voltage and current is termed a *single-phase system*. It utilises only one winding. A *polyphase system* utilises more than one winding. It will produce as many induced voltages as the number of windings.

A three-phase system consists of three separate but identical windings displaced by 120 electrical degrees from each other as shown in Fig. 8.1. When these three windings are rotated in an anticlockwise direction with constant angular velocity in a uniform magnetic field, the emfs induced in each winding have the same magnitude and frequency but are displaced 120° from one another.

Fig. 8.1 *Three-phase system*

The instantaneous values of generated voltage in windings RR_1, YY_1 and BB_1 are given by

$$e_R = E_m \sin\theta$$
$$e_Y = E_m \sin(\theta - 120°)$$
$$e_B = E_m \sin(\theta - 240°)$$

where E_m is the maximum value of the induced voltage in each winding. The waveforms of these three voltages are shown in Fig. 8.2.

Fig. 8.2 *Voltage waveforms*

Figure 8.3 shows the phasor diagram of these three induced voltages.

Fig. 8.3 *Phasor diagram*

8.2 ADVANTAGES OF A THREE-PHASE SYSTEM

1. In a single-phase system, the instantaneous power is fluctuating in nature. However, in a three-phase system, it is constant at all times.
2. The output of a three-phase system is greater than that of a single-phase system.
3. Transmission and distribution of a three-phase system is cheaper than that of a single-phase system.
4. Three-phase motors are more efficient and have higher power factors than single-phase motors of the same frequency.
5. Three-phase motors are self-starting whereas single-phase motors are not self-starting.

8.3 SOME DEFINITIONS

1. ***Phase Sequence*** The sequence in which the voltages in the three phases reach the maximum positive value is called the *phase sequence* or *phase order*. From the phasor diagram of a three-phase system, it is clear that the voltage in the coil *R* attains maximum positive value first, next in the coil *Y* and then in the coil *B*. Hence, the phase sequence is *R-Y-B*.

2. ***Phase Voltage*** The voltage induced in each winding is called the *phase voltage*.

3. ***Phase Current*** The current flowing through each winding is called the *phase current*.

4. ***Line Voltage*** The voltage available between any pair of terminals or lines is called the *line voltage*.

5. ***Line Current*** The current flowing through each line is called the *line current*.

6. ***Symmetrical or Balanced System*** A three-phase system is said to be balanced if the
   (a) voltages in the three phases are equal in magnitude and differ in phase from one another by 120°, and
   (b) currents in the three phases are equal in magnitude and differ in phase from one another by 120°.

7. ***Balanced Load*** The load is said to be balanced if loads connected across the three phases are identical, i.e., all the loads have the same magnitude and power factor.

8.4 INTERCONNECTION OF THREE PHASES

In a three-phase system, there are three windings. Each winding has two terminals, viz., 'start' and 'finish'. If a separate load is connected across each winding as shown in Fig. 8.4, six conductors are required to transmit and distribute power. This will make the system complicated and expensive.

Fig. 8.4 *Non-interlinked three-phase system*

In order to reduce the number of conductors, the three windings are connected in the following two ways:

1. Star, or Wye, connection
2. Delta, or Mesh, connection

8.5 STAR, OR WYE, CONNECTION

In this method, similar terminals (start or finish) of the three windings are joined together as shown in Fig. 8.5. The common point is called star or *neutral point*.

Figure 8.6 shows a three-phase system in star connection.

Fig. 8.5 *Three-phase star connection*

Fig. 8.6 *Three-phase, four-wire system*

This system is called a three-phase, four-wire system. If three identical loads are connected to each phase, the current flowing through the neutral wire is the sum of the three currents I_R, I_Y and I_B. Since the impedances are identical, the three currents are equal in magnitude but differ in phase from one another by 120°.

$$i_R = I_m \sin\theta$$

$$i_Y = I_m \sin(\theta - 120°)$$

$$i_B = I_m \sin(\theta - 240°)$$

$$i_R + i_Y + i_B = I_m \sin\theta + I_m \sin(\theta - 120°) + I_m \sin(\theta - 240°) = 0$$

Hence, the neutral wire can be removed without any way affecting the voltages or currents in the circuit as shown in Fig. 8.7. This constitutes a three-phase, three-wire system. If the load is not balanced, the neutral wire carries some current.

Fig. 8.7 *Three-phase, three-wire system*

8.6 DELTA, OR MESH, CONNECTION

In this method, dissimilar terminals of the three windings are joined together, i.e., the 'finish' terminal of one winding is connected to the 'start' terminal of the other winding, and so on, as shown in Fig. 8.8. This system is also called a three-phase, three-wire system.

For a balanced system, the sum of the three-phase voltages round the closed mesh is zero. The three emfs are equal in magnitude but differ in phase from one another by 120°.

Fig. 8.8 *Three-phase delta connection*

$$e_R = E_m \sin\theta$$
$$e_Y = E_m \sin(\theta - 120°)$$
$$e_B = E_m \sin(\theta - 240°)$$
$$e_R + e_Y + e_B = E_m \sin\theta + E_m \sin(\theta - 120°) + E_m \sin(\theta - 240°) = 0$$

8.7 VOLTAGE, CURRENT AND POWER RELATIONS IN A BALANCED STAR-CONNECTED LOAD

8.7.1 Relation between Line Voltage and Phase Voltage

Figure 8.9 shows a balanced star-connected load.

Since the system is balanced, the three-phase voltages V_{RN}, V_{YN}, and V_{BN} are equal in magnitude and differ in phase from one another by 120°.

Let $$V_{RN} = V_{YN} = V_{BN} = V_{ph}$$

where V_{ph} indicates the rms value of phase voltage.

$$\mathbf{V}_{RN} = V_{ph}\angle 0°$$
$$\mathbf{V}_{YN} = V_{ph}\angle -120°$$
$$\mathbf{V}_{BN} = V_{ph}\angle -240°$$

Let $$V_{RY} = V_{YB} = V_{BR} = V_L$$

Fig. 8.9 *Star connection*

where V_L indicates the rms value of line voltage.
Applying Kirchhoff's voltage law,

$$\begin{aligned}\mathbf{V}_{RY} &= \mathbf{V}_{RN} + \mathbf{V}_{NY}\\ &= \mathbf{V}_{RN} - \mathbf{V}_{YN}\\ &= V_{ph}\angle 0° - V_{ph}\angle -120°\\ &= (V_{ph} + j0) - (-0.5\,V_{ph} - j0.866\,V_{ph})\\ &= 1.5\,V_{ph} + j0.866\,V_{ph}\\ &= \sqrt{3}\,V_{ph}\angle 30°\end{aligned}$$

Similarly,

$$\mathbf{V}_{YB} = \mathbf{V}_{YN} + \mathbf{V}_{NB} = \sqrt{3}\,V_{ph}\angle 30°$$

$$\mathbf{V}_{BR} = \mathbf{V}_{BN} + \mathbf{V}_{NR} = \sqrt{3}\,V_{ph}\angle 30°$$

Thus, in a star-connected, three-phase system, $V_L = \sqrt{3}\,V_{ph}$ and line voltages lead respective phase voltages by 30°.

8.7.2 Relation between Line Current and Phase Current

From Fig. 8.9, it is clear that line current is equal to the phase current.

$$I_L = I_{ph}$$

8.7.3 Phasor Diagram (Lagging Power Factor)

Figure 8.10 shows the phasor diagram of a balanced star-connected inductive load.

Fig. 8.10 *Phasor diagram*

8.7.4 Power

The total power in a three-phase system is the sum of powers in the three phases. For a balanced load, the power consumed in each load phase is the same.

Total active power $P = 3 \times$ power in each phase $= 3\ V_{ph}\ I_{ph} \cos \phi$
In a star-connected, three-phase system,

$$V_{ph} = \frac{V_L}{\sqrt{3}}$$

$$I_{ph} = I_L$$

$$P = 3 \times \frac{V_L}{\sqrt{3}} \times I_L \times \cos \phi = \sqrt{3}\ V_L\ I_L \cos \phi$$

where ϕ is the phase difference between phase voltage and corresponding phase current.
Similarly, total reactive power $Q = 3V_{ph}\ I_{ph} \sin \phi = \sqrt{3}\ V_L I_L \sin \phi$
Total apparent power $S = 3V_{ph} I_{ph} = \sqrt{3} V_L I_L$

8.8 VOLTAGE, CURRENT AND POWER RELATIONS IN A BALANCED DELTA-CONNECTED LOAD

8.8.1 Relation between Line Voltage and Phase Voltage

Figure 8.11 shows a balanced delta-connected load.

Fig. 8.11 *Delta connection*

From Fig. 8.11, it is clear that line voltage is equal to phase voltage.

$$V_L = V_{ph}$$

8.8.2 Relation between Line Current and Phase Current

Since the system is balanced, the three-phase currents I_{RY}, I_{YB}, and I_{BR} are equal in magnitude but differ in phase from one another by 120°.

Let $$I_{RY} = I_{YB} = I_{BR} = I_{ph}$$

where I_{ph} indicates rms value of the phase current.

$$\mathbf{I}_{RY} = I_{ph}\angle 0°$$

$$\mathbf{I}_{YB} = I_{ph}\angle -120°$$

$$\mathbf{I}_{BR} = I_{ph}\angle -240°$$

Let $$I_R = I_Y = I_B = I_L$$

where I_L indicates rms value of the line current.
Applying Kirchhoff's current law,

$$\mathbf{I}_R + \mathbf{I}_{BR} = \mathbf{I}_{RY}$$

$$\begin{aligned}\mathbf{I}_R &= \mathbf{I}_{RY} - \mathbf{I}_{BR}\\ &= I_{ph}\angle 0° - I_{ph}\angle -240°\\ &= (I_{ph} + j0) - (-0.5\,I_{ph} + j0.866\,I_{ph})\\ &= 1.5\,I_{ph} - j0.866\,I_{ph}\\ &= \sqrt{3}\,I_{ph}\angle -30°\end{aligned}$$

Similarly,

$$\mathbf{I}_Y = \mathbf{I}_{YB} - \mathbf{I}_{RY} = \sqrt{3}\,I_{ph}\angle -30°$$

$$\mathbf{I}_B = \mathbf{I}_{BR} - \mathbf{I}_{YB} = \sqrt{3}\,I_{ph}\angle -30°$$

Thus, in a delta-connected, three-phase system, $I_L = \sqrt{3}\,I_{ph}$ and line currents lag behind the respective phase currents by 30°.

8.8.3 Phasor Diagram (Lagging Power Factor)

Figure 8.12 shows the phasor diagram of a balanced delta connected inductive load.

Fig. 8.12 *Phasor diagram*

8.8.4 Power

$$P = 3V_{ph}I_{ph}\cos\phi$$

In a delta-connected, three-phase system,

$$V_{ph} = V_L$$

$$I_{ph} = \frac{I_L}{\sqrt{3}}$$

$$P = 3\times V_L \times \frac{I_L}{\sqrt{3}} \times \cos\phi = \sqrt{3}\,V_L I_L \cos\phi$$

Total reactive power $Q = 3\,V_{ph} I_{ph} \sin\phi = \sqrt{3}\,V_L I_L \sin\phi$

Total apparent power $S = 3\,V_{ph} I_{ph} = \sqrt{3}\,V_L I_L$

8.9 BALANCED Y/Δ AND Δ/Y CONVERSIONS

Any balanced star-connected system can be converted into the equivalent delta-connected system and vice versa. For a balanced star-connected load,

$$\text{Line voltage} = V_L$$
$$\text{Line current} = I_L$$
$$\text{Impedance/phase} = Z_Y$$

$$V_{ph} = \frac{V_L}{\sqrt{3}}$$
$$I_{ph} = I_L$$
$$Z_Y = \frac{V_{ph}}{I_{ph}} = \frac{V_L}{\sqrt{3}\,I_L}$$

For an equivalent delta-connected system, the line voltages and currents must have the same values as in the star-connected system, i.e.,

$$\text{Line voltage} = V_L$$
$$\text{Line current} = I_L$$
$$\text{Impedance/phase} = Z_\Delta$$

$$V_{ph} = V_L$$
$$I_{ph} = \frac{I_L}{\sqrt{3}}$$
$$Z_\Delta = \frac{V_{ph}}{I_{ph}} = \frac{V_L}{\frac{I_L}{\sqrt{3}}} = \sqrt{3}\frac{V_L}{I_L} = 3Z_Y$$
$$Z_Y = \frac{1}{3} Z_\Delta$$

Thus, when three equal phase impedances are connected in delta, the equivalent star impedance is one-third of the delta impedance.

8.10 RELATION BETWEEN POWER IN DELTA AND STAR SYSTEMS

Let a balanced load be connected in star having impedance per phase as Z_{ph}.
For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{V_L}{\sqrt{3}\,Z_{ph}}$$
$$I_L = I_{ph} = \frac{V_L}{\sqrt{3}\,Z_{ph}}$$

Now
$$P_Y = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times V_L \times \frac{V_L}{\sqrt{3} Z_{ph}} \times \cos\phi = \frac{V_L^2}{Z_{ph}} \cos\phi$$

For a delta-connected load,
$$V_{ph} = V_L$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{V_L}{Z_{ph}}$$
$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \frac{V_L}{Z_{ph}}$$

Now
$$P_\Delta = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times V_L \times \sqrt{3} \frac{V_L}{Z_{ph}} \times \cos\phi = 3 \frac{V_L^2}{Z_{ph}} \cos\phi = 3\, P_Y$$
$$P_Y = \frac{1}{3} P_\Delta$$

Thus, power consumed by a balanced star-connected load is one-third of that in the case of a delta-connected load.

8.11 COMPARISON BETWEEN STAR AND DELTA CONNECTIONS

Star Connection	Delta Connection
1. $V_L = \sqrt{3}\, V_{ph}$	1. $V_L = V_{ph}$
2. $I_L = I_{ph}$	2. $I_L = \sqrt{3}\, I_{ph}$
3. Line voltage leads the respective phase voltage by 30°.	3. Line current lags behind the respective phase current by 30°
4. Power in star connection is one-third of power in delta connection.	4. Power in delta connection is 3 times of the power in star connection.
5. Three-phase, three-wire and three-phase, four-wire systems are possible.	5. Only three-phase, three-wire system is possible.
6. The phasor sum of all the phase currents is zero.	6. The phasor sum of all the phase voltages is zero.

Example 8.1 *Three equal impedances, each of (8 + j10) ohms, are connected in star. This is further connected to a 440 V, 50 Hz, three-phase supply. Calculate (a) phase voltage, (b) phase angle, (c) phase current, (d) line current, (e) active power, and (f) reactive power.*

Solution
$$\mathbf{Z}_{ph} = (8 + j10)\ \Omega, \quad V_L = 440\ \text{V}, \quad f = 50\ \text{Hz}$$

For a star-connected load,

(a) Phase voltage
$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03\ \text{V}$$

(b) Phase angle

$$\mathbf{Z}_{ph} = 8 + j10 = 12.81 \angle 51.34° \ \Omega$$
$$Z_{ph} = 12.81 \ \Omega$$
$$\phi = 51.34°$$

(c) Phase current

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{254.03}{12.81} = 19.83 \text{ A}$$

(d) Line current

$$I_L = I_{ph} = 19.83 \text{ A}$$

(e) Active power

$$P = \sqrt{3} V_L I_L \cos\phi = \sqrt{3} \times 440 \times 19.83 \times \cos(51.34°) = 9.44 \text{ kW}$$

(f) Reactive power

$$Q = \sqrt{3} V_L I_L \sin\phi = \sqrt{3} \times 440 \times 19.83 \times \sin(51.34°) = 11.81 \text{ kVAR}$$

Example 8.2 *A balanced delta-connected load of impedance (8 − j6) ohms per phase is connected to a three-phase, 230 V, 50 Hz supply. Calculate (a) power factor, (b) line current, and (c) reactive power.*

Solution

$$\mathbf{Z}_{ph} = (8 - j6) \ \Omega, \quad V_L = 230 \text{ V}, \quad f = 50 \text{ Hz}$$

For a delta-connected load,

(a) Power factor

$$\mathbf{Z}_{ph} = 8 - j6 = 10 \angle -36.87° \ \Omega$$
$$Z_{ph} = 10 \ \Omega$$
$$\phi = 36.87°$$
$$\text{pf} = \cos\phi = \cos(36.87°) = 0.8 \text{ (leading)}$$

(b) Line current

$$V_{ph} = V_L = 230 \text{ V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230}{10} = 23 \text{ A}$$
$$I_L = \sqrt{3} I_{ph} = \sqrt{3} \times 23 = 39.84 \text{ A}$$

(c) Reactive power

$$Q = \sqrt{3} V_L I_L \sin\phi = \sqrt{3} \times 230 \times 39.84 \times \sin(36.87°) = 9.52 \text{ kVAR}$$

Example 8.3 *Three coils, each having a resistance and an inductance of 8 Ω and 0.02 H respectively, are connected in star across a three-phase, 230 V, 50 Hz supply. Find the (a) power factor, (b) line current, (c) power, (d) reactive volt-amperes, and (e) total volt-amperes.*

Solution

$$R = 8 \ \Omega, \quad L = 0.02 \text{ H}, \quad V_L = 230 \text{ V}, \quad f = 50 \text{ Hz}$$

For a star-connected load,

(a) Power factor

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$
$$\mathbf{Z}_{ph} = R + jX_L = 8 + j6.28 = 10.17\ \angle 38.13°\ \Omega$$
$$Z_{ph} = 10.17\ \Omega$$
$$\phi = 38.13°$$
$$\text{pf} = \cos\phi = \cos(38.13°) = 0.786 \text{ (lagging)}$$

(b) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{230}{\sqrt{3}} = 132.79 \text{ V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{132.79}{10.17} = 13.05 \text{ A}$$
$$I_L = I_{ph} = 13.05 \text{ A}$$

(c) Power

$$P = \sqrt{3}\, V_L\, I_L \cos\phi = \sqrt{3} \times 230 \times 13.05 \times 0.786 = 4.088 \text{ kW}$$

(d) Reactive volt-amperes

$$Q = \sqrt{3}\, V_L\, I_L \sin\phi = \sqrt{3} \times 230 \times 13.05 \times \sin(38.13°) = 3.21 \text{ kVAR}$$

(e) Total volt-amperes

$$S = \sqrt{3}\, V_L\, I_L = \sqrt{3} \times 230 \times 13.05 = 5.198 \text{ kVA}$$

Example 8.4 *Three coils, each having a resistance of 8 Ω and an inductance of 0.02 H, are connected in delta to a three-phase, 400 V, 50 Hz supply. Calculate the (a) line current, and (b) power absorbed.*

Solution $R = 8\ \Omega,\quad L = 0.02 \text{ H},\quad V_L = 400 \text{ V},\quad f = 50 \text{ Hz}$

For a delta-connected load,

(a) Line current

$$V_L = V_{ph} = 400 \text{ V}$$
$$X_L = 2\pi fL = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$
$$\mathbf{Z}_{ph} = R + jX_L = 8 + j6.28 = 10.17\ \angle 38.13°\ \Omega$$
$$Z_{ph} = 10.17\ \Omega$$
$$\phi = 38.13°$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{10.17} = 39.33 \text{ A}$$
$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 39.33 = 68.12 \text{ A}$$

(b) Power absorbed

$$P = \sqrt{3}\, V_L\, I_L \cos\phi = \sqrt{3} \times 400 \times 68.12 \times \cos(38.13°) = 37.12 \text{ kW}$$

Example 8.5 *The three equal impedances of each of 10 ∠ 60°Ω, are connected in star across a three-phase, 400 V, 50 Hz supply. Calculate the (a) line voltage and phase voltage, (b) power factor and active power consumed. (c) If the same three impedances are connected in delta to the same source of supply, what is the active power consumed?*

Solution $\mathbf{Z}_{ph} = 10\angle 60°\ \Omega, \quad V_L = 400\ \text{V}, \quad f = 50\ \text{Hz}$

For a star-connected load,

(a) Line voltage and phase voltage

$$V_L = 400\ \text{V}$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94\ \text{V}$$

(b) Power factor and active power consumed

$$\phi = 60°$$

$$\text{pf} = \cos\phi = \cos(60°) = 0.5\ \text{(lagging)}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{10} = 23.094\ \text{A}$$

$$I_L = I_{ph} = 23.094\ \text{A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 23.094 \times 0.5 = 8\ \text{kW}$$

(c) Active power consumed for delta-connected load

$$V_L = 400\ \text{V}$$

$$Z_{ph} = 10\ \Omega$$

$$V_{ph} = V_L = 400\ \text{V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{10} = 40\ \text{A}$$

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 40 = 69.28\ \text{A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 69.28 \times \cos(60°) = 24\ \text{kW}$$

Example 8.6 *Three similar coils A,B, and C are available. Each coil has a 9 Ω resistance and a 12 Ω reactance. They are connected in delta to a three-phase, 440 V, 50 Hz supply. Calculate for this load, the (a) phase current, (b) line current, (c) power factor, (d) total kVA, (e) active power, and (f) reactive power. If these coils are connected in star across the same supply, calculate all the above quantities.*

Solution $R = 9\ \Omega, \quad X_L = 12\ \Omega, \quad V_L = 440\ \text{V}, \quad f = 50\ \text{Hz}$

For a delta-connected load,

(a) Phase current

$$V_L = V_{ph} = 440\ \text{V}$$

$$\mathbf{Z}_{ph} = R + jX_L = 9 + j12 = 15\ \angle 53.13°\ \Omega$$

$$Z_{ph} = 15\ \Omega$$

$$\phi = 53.13°$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{440}{15} = 29.33 \text{ A}$$

(b) Line current

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 29.33 = 50.8 \text{ A}$$

(c) Power factor

$$\text{pf} = \cos\phi = \cos(53.13°) = 0.6 \text{ (lagging)}$$

(d) Total kVA

$$S = \sqrt{3}\, V_L I_L = \sqrt{3} \times 440 \times 50.8 = 38.71 \text{ kVA}$$

(e) Active power

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 440 \times 50.8 \times 0.6 = 23.23 \text{ kW}$$

(f) Reactive power

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 440 \times 50.8 \times \sin(53.13°) = 30.97 \text{ kVAR}$$

If these coils are connected in star across the same supply,

(a) Phase current

$$V_L = 440 \text{ V}$$

$$Z_{ph} = 15\ \Omega$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{254.03}{15} = 16.94 \text{ A}$$

(b) Line current

$$I_L = I_{ph} = 16.94 \text{ A}$$

(c) Power factor

$$\text{pf} = 0.6 \text{ (lagging)}$$

(d) Total kVA

$$S = \sqrt{3}\, V_L I_L = \sqrt{3} \times 440 \times 16.94 = 12.91 \text{ kVA}$$

(e) Active power

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 440 \times 16.94 \times 0.6 = 7.74 \text{ kW}$$

(f) Reactive power

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 440 \times 16.94 \times \sin(53.13°) = 12.33 \text{ kVAR}$$

Example 8.7 *A 415 V, 50 Hz, three-phase voltage is applied to three star-connected identical impedances. Each impedance consists of a resistance of 15 Ω, a capacitance of 177 μF and an inductance of 0.1 henry in series. Find the (a) power factor, (b) phase current, (c) line current, (d) active power, (e) reactive power, and (f) total VA. Draw a neat phasor diagram. If the same impedances are connected in delta, find the (a) line current, and (b) power consumed.*

Solution $V_L = 415$ V, $f = 50$ Hz, $R = 15\ \Omega$, $C = 177\ \mu$F, $L = 0.1$ H

For a star-connected load,

(a) Power factor

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 177 \times 10^{-6}} = 17.98\ \ \Omega$$

$$\mathbf{Z}_{ph} = R + jX_L - jX_C = 15 + j31.42 - j17.98 = 15 + j13.44 = 20.14\ \angle 41.86°\ \Omega$$

$$Z_{ph} = 20.14\ \Omega$$

$$\phi = 41.86°$$

$$\text{pf} = \cos\phi = \cos(41.86°) = 0.744 \text{ (lagging)}$$

(b) Phase current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{239.6}{20.14} = 11.9 \text{ A}$$

(c) Line current

$$I_L = I_{ph} = 11.9 \text{ A}$$

(d) Active power

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 415 \times 11.9 \times 0.744 = 6.36 \text{ kW}$$

(e) Reactive power

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 415 \times 11.9 \times \sin(41.86°) = 5.71 \text{ kVAR}$$

(f) Total VA

$$S = \sqrt{3}\, V_L I_L = \sqrt{3} \times 415 \times 11.9 = 8.55 \text{ kVA}$$

The phasor diagram is shown in Fig. 8.13.

Fig. 8.13

If the same impedances are connected in delta,

(a) Line current

$$V_L = V_{ph} = 415 \text{ V}$$
$$Z_{ph} = 20.14\ \Omega$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{415}{20.14} = 20.61 \text{ A}$$
$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 20.61 = 35.69 \text{ A}$$

(b) Power consumed

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 415 \times 35.69 \times 0.774 = 19.09 \text{ kW}$$

Example 8.8 *Each phase of a delta-connected load consists of a 50 mH inductor in series with a parallel combination of a 50 Ω resistor and a 50 μF capacitor. The load is connected to a three-phase, 550 V, 800 rad/s ac supply. Find the (a) power factor, (b) phase current, (c) the line current, (d) power drawn, (e) reactive power, and (f) kVA rating of the load.*

Solution $L = 50$ mH, $R = 50\ \Omega$, $C = 50\ \mu$F, $V_L = 550$ V, $\omega = 800$ rad/s

For a delta-connected load,

(a) Power factor

$$X_L = \omega L = 800 \times 50 \times 10^{-3} = 40\ \Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{800 \times 50 \times 10^{-6}} = 25\ \Omega$$
$$\mathbf{Z}_{ph} = jX_L + \frac{R(-jX_C)}{R - jX_C} = j40 + \frac{50(-j25)}{50 - j25} = 10 + j20 = 22.36\ \angle 63.43^\circ\ \Omega$$
$$\mathbf{Z}_{ph} = 22.36\ \Omega$$
$$\phi = 63.43^\circ$$
$$\text{pf} = \cos\phi = \cos(63.43^\circ) = 0.447 \text{ (lagging)}$$

(b) Phase current

$$V_L = V_{ph} = 550 \text{ V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{550}{22.36} = 24.6 \text{ A}$$

(c) Line current

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 24.6 = 42.61 \text{ A}$$

(d) Power drawn

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 550 \times 42.61 \times 0.447 = 18.14 \text{ kW}$$

(e) Reactive power

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 550 \times 42.61 \times \sin(63.43^\circ) = 36.3 \text{ kVAR}$$

(f) kVA rating of the load

$$S = \sqrt{3}\, V_L I_L = \sqrt{3} \times 550 \times 42.61 = 40.59 \text{ kVA}$$

Example 8.9 *A balanced star-connected load is supplied from a symmetrical three-phase 400 volts, 50 Hz system. The current in each phase is 30 A and lags 30° behind the phase voltage. Find the (a) phase voltage, (b) resistance and reactance per phase, (c) load inductance per phase, and (d) total power consumed.*

Solution

$$V_L = 400 \text{ V}, \quad f = 50 \text{ Hz}, \quad I_{ph} = 30 \text{ A}, \quad \phi = 30°$$

For a star-connected load,

(a) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

(b) Resistance and reactance per phase

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{30} = 7.7\ \Omega$$

$$Z_{ph} = Z_{ph}\angle\phi = 7.7\angle 30° = (6.67 + j3.85)\ \Omega$$

$$R_{ph} = 6.67\ \Omega$$

$$X_{ph} = 3.85\ \Omega$$

(c) Load inductance per phase

$$X_{ph} = 2\pi f L_{ph}$$

$$3.85 = 2\pi \times 50 \times L_{ph}$$

$$L_{ph} = 0.01225 \text{ H}$$

(d) Total power consumed

$$P = 3V_{ph}I_{ph}\cos\phi = 3 \times 230.94 \times 30 \times \cos(30°) = 18 \text{ kW}$$

Example 8.10 *A symmetrical three-phase 400 V system supplies a basic load of 0.8 lagging power factor and is connected in star. If the line current is 34.64 A, find the (a) impedance, (b) resistance and reactance per phase, (c) total power, and (d) total reactive voltamperes.*

Solution

$$V_L = 400 \text{ V}, \quad \text{pf} = 0.8 \text{ (lagging)}, \quad I_L = 34.64 \text{ A}$$

For a star-connected load,

(a) Impedance

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = I_L = 34.64 \text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{34.64} = 6.67\ \Omega$$

(b) Resistance and reactance per phase

$$\text{pf} = \cos\phi = 0.8 \text{ (lagging)}$$

$$\phi = \cos^{-1}(0.8) = 36.87°$$

$$Z_{ph} = Z_{ph}\angle\phi = 6.67\angle 36.87° = (5.33 + j4)\ \Omega$$

$$R_{ph} = 5.33\ \Omega$$
$$X_{ph} = 4\ \Omega$$

(c) Total power

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 34.64 \times 0.8 = 19.19 \text{ kW}$$

(d) Total reactive volt-amperes

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 400 \times 34.64 \times \sin(36.87°) = 14.4 \text{ kVAR}$$

Example 8.11 *A balanced star-connected load is supplied by a 415 V, 50 Hz three-phase system. Current in each phase is 20 A and lags 30° behind its phase voltage. Find the (a) phase voltage, (b) power, and (c) circuit parameters. Also, find power consumed when the same load is connected in delta across the same supply.*

Solution

$$V_L = 415 \text{ V}, \quad f = 50 \text{ Hz}, \quad I_{ph} = 20 \text{ A}, \quad \phi = 30°$$

For a star-connected load,

(a) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6 \text{ V}$$

(b) Power

$$I_L = I_{ph} = 20 \text{ A}$$
$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 415 \times 20 \times \cos(30°) = 12.45 \text{ kW}$$

(c) Circuit parameters

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{239.6}{20} = 11.98\ \Omega$$
$$\mathbf{Z}_{ph} = Z_{ph}\angle\phi = 11.98\angle 30° = (10.37 + j6)\ \Omega$$
$$R_{ph} = 10.37\ \Omega$$
$$X_{ph} = 6\ \Omega$$

Since

$$X_{ph} = 2\pi f L_{ph}$$
$$6 = 2\pi \times 50 \times L_{ph}$$
$$L_{ph} = 19.1 \text{ mH}$$

(d) Power consumed by same delta load across the same supply

$$P_\Delta = 3P_Y = 3 \times 12.45 \times 10^3 = 37.35 \text{ kW}$$

Example 8.12 *Three identical coils connected in delta to a 440 V, three-phase supply take a total power of 50 kW and a line current of 90 A. Find the (a) phase current, (b) power factor, and (c) apparent power taken by the coils.*

Solution

$$V_L = 440 \text{ V}, \quad P = 50 \text{ kW}, \quad I_L = 90 \text{ A}$$

For a delta-connected load,

(a) Phase current

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{90}{\sqrt{3}} = 51.96 \text{ A}$$

(b) Power factor

$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$50 \times 10^3 = \sqrt{3} \times 440 \times 90 \times \cos\phi$$
$$\text{pf} = \cos\phi = 0.73 \text{ (lagging)}$$

(c) Apparent power

$$S = \sqrt{3}\, V_L I_L = \sqrt{3} \times 440 \times 90 = 68.59 \text{ kVA}$$

Example 8.13 *Three similar choke coils are connected in star to a three-phase supply. If the line current is 15 A, the total power consumed is 11 kW and the volt-ampere input is 15 kVA, find the line and phase voltages, the VAR input and the reactance and resistance of each coil. If these coils are now connected in delta to the same supply, calculate phase and line currents, active and reactive power.*

Solution $I_L = 15 \text{ A}, \quad P = 11 \text{ kW}, \quad S = 15 \text{ kVA}$

For a star-connected load,

(a) Line voltage

$$S = \sqrt{3}\, V_L I_L$$
$$15 \times 10^3 = \sqrt{3} \times V_L \times 15$$
$$V_L = 577.35 \text{ V}$$

(b) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{577.35}{\sqrt{3}} = 333.33 \text{ V}$$

(c) VAR input

$$\cos\phi = \frac{P}{S} = \frac{11 \times 10^3}{15 \times 10^3} = 0.733$$
$$\phi = 42.86°$$
$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 577.35 \times 15 \times \sin(42.86°) = 10.2 \text{ kVAR}$$

(d) Reactance and resistance of coil

$$I_{ph} = I_L = 15 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{333.33}{15} = 22.22\ \Omega$$
$$R = Z_{ph} \cos\phi = 22.22 \times 0.733 = 16.29\ \Omega$$
$$X_L = Z_{ph} \sin\phi = 22.22 \times \sin(42.86°) = 15.11\ \Omega$$

If these coils are now connected in delta,

(a) Phase current

$$V_{ph} = V_L = 577.35 \text{ V}$$
$$Z_{ph} = 22.22\ \Omega$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{577.35}{22.22} = 25.98 \text{ A}$$

(b) Line current

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 25.98 = 45 \text{ A}$$

(c) Active power

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 577.35 \times 45 \times 0.733 = 32.98 \text{ kW}$$

(d) Reactive power

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 577.35 \times 45 \times \sin(42.86°) = 30.61 \text{ kVAR}$$

Example 8.14 *A three-phase, star-connected source feeds 1500 kW at 0.85 power factor lag to a balanced mesh-connected load. Calculate the current, its active and reactive components in each phase of the source and the load. The line voltage is 2.2 kV.*

Solution $P = 1500 \text{ kW}, \quad \text{pf} = 0.85 \text{ (lagging)}, \quad V_L = 2.2 \text{ kV}$

For a mesh or delta-connected load,

(a) Line current

$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$1500 \times 10^3 = \sqrt{3} \times 2.2 \times 10^3 \times I_L \times 0.85$$
$$I_L = 463.12 \text{ A}$$

(b) Active component of current in each phase of the load

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{463.12}{\sqrt{3}} = 267.38 \text{ A}$$
$$I_{ph} \cos\phi = 267.38 \times 0.85 = 227.27 \text{ A}$$

(c) Reactive component of current in each phase of the load

$$I_{ph} \sin\phi = 267.38 \times \sin(\cos^{-1} 0.85) = 267.38 \times 0.526 = 140.85 \text{ A}$$

For a star-connected source, the phase current in the source will be the same as the line current drawn by the load.

(d) Active component of this current in each phase of the source

$$I_L \cos\phi = 463.12 \times 0.85 = 393.65 \text{ A}$$

(e) Reactive component of this current in each phase of the source

$$I_L \sin\phi = 463.12 \times 0.526 = 243.6 \text{ A}$$

Example 8.15 *A three-phase, 208-volt generator supplies a total of 1800 W at a line current of 10 A when three identical impedances are arranged in a wye connection across the line terminals of the generator. Compute the resistive and reactive components of each phase impedance.*

Solution $V_L = 208 \text{ V}, \quad P = 1800 \text{ W}, \quad I_L = 10 \text{ A}$

For a wye-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{208}{\sqrt{3}} = 120.09 \text{ V}$$
$$I_{ph} = I_L = 10 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{120.09}{10} = 12\ \Omega$$
$$P = \sqrt{3} V_L I_L \cos\phi$$
$$1800 = \sqrt{3} \times 208 \times 10 \times \cos\phi$$
$$\cos\phi = 0.5$$
$$\phi = 60°$$
$$R_{ph} = Z_{ph} \cos\phi = 12 \times 0.5 = 6\ \Omega$$
$$X_{ph} = Z_{ph} \sin\phi = 12 \times \sin(60°) = 10.39\ \Omega$$

Example 8.16 *A balanced, three-phase, star-connected load of 100 kW takes a leading current of 80 A, when connected across a three-phase, 1100 V, 50 Hz supply. Find the circuit constants of the load per phase.*

Solution $P = 100$ kW, $I_L = 80$ A, $V_L = 1100$ V, $f = 50$ Hz

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{1100}{\sqrt{3}} = 635.08 \text{ V}$$
$$I_{ph} = I_L = 80 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{635.08}{80} = 7.94\ \Omega$$
$$P = \sqrt{3} V_L I_L \cos\phi$$
$$100 \times 10^3 = \sqrt{3} \times 1100 \times 80 \times \cos\phi$$
$$\cos\phi = 0.656 \text{ (leading)}$$
$$\phi = 49°$$
$$R_{ph} = Z_{ph} \cos\phi = 7.94 \times 0.656 = 5.21\ \Omega$$
$$X_{ph} = Z_{ph} \sin\phi = 7.94 \times \sin(49°) = 6\ \Omega$$

This reactance will be capacitive in nature as the current is leading.

$$X_C = \frac{1}{2\pi f C}$$
$$6 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 530.52\ \mu\text{F}$$

Example 8.17 *Three identical impedances are connected in delta to a three-phase supply of 400 V. The line current is 34.65 A, and the total power taken from the supply is 14.4 kW. Calculate the resistance and reactance values of each impedance.*

Solution $V_L = 400\text{ V}, \quad I_L = 34.65\text{ A}, \quad P = 14.4\text{ kW}$

For a delta-connected load,

$$V_L = V_{ph} = 400\text{ V}$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{34.65}{\sqrt{3}} = 20\text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{400}{20} = 20\ \Omega$$

$$P = \sqrt{3}\, V_L I_L \cos\phi$$

$$14.4 \times 10^3 = \sqrt{3} \times 400 \times 34.65 \times \cos\phi$$

$$\cos\phi = 0.6$$

$$\phi = 53.13°$$

$$R_{ph} = Z_{ph}\cos\phi = 20 \times 0.6 = 12\ \Omega$$

$$X_{ph} = Z_{ph}\sin\phi = 20 \times \sin(53.13°) = 16\ \Omega$$

Example 8.18 *A balanced, three-phase load connected in delta draws a power of 10.44 kW at 200 V at a power factor of 0.5 lead. Find the values of the circuit elements and the reactive volt-amperes drawn.*

Solution $P = 10.44\text{ kW}, \quad V_L = 200\text{ V}, \quad \text{pf} = 0.5\text{ (leading)}$

For a delta-connected load,

(a) Values of circuit elements

$$V_L = V_{ph} = 200\text{ V}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi$$

$$10.44 \times 10^3 = \sqrt{3} \times 200 \times I_L \times 0.5$$

$$I_L = 60.28\text{ A}$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{60.28}{\sqrt{3}} = 34.8\text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{200}{34.8} = 5.75\ \Omega$$

$$R_{ph} = Z_{ph}\cos\phi = 5.75 \times 0.5 = 2.875\ \Omega$$

$$X_{ph} = Z_{ph}\sin\phi = 5.75 \times \sin(\cos^{-1} 0.5) = 5.747 \times 0.866 = 4.98\ \Omega$$

(b) Reactive volt-amperes drawn

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 200 \times 60.28 \times 0.866 = 18.08\text{ kVAR}$$

Example 8.19 *Each leg of a balanced, delta-connected load consists of a 7 Ω resistance in series with a 4 Ω inductive reactance. The line-to-line voltages are*

$$\mathbf{E}_{ab} = 2360\angle 0^\circ\ V, \quad \mathbf{E}_{bc} = 2360\angle -120^\circ\ V, \quad \mathbf{E}_{ca} = 2360\angle 120^\circ\ V$$

Determine (a) *phase current I_{ab}, I_{bc}, and I_{ca} both magnitude and phase)*
(b) *each line current and its associated phase angle*
(c) *the load power factor*

Solution $R = 7\ \Omega, \quad X_L = 4\ \Omega, \quad V_L = 2360\ \text{V}$

For a delta-connected load,
(a) Phase current

$$V_{ph} = V_L = 2360\ \text{V}$$
$$\mathbf{Z}_{ph} = 7 + j4 = 8.06\angle 29.74^\circ\ \Omega$$
$$\mathbf{I}_{ab} = \frac{\mathbf{E}_{ab}}{\mathbf{Z}_{ph}} = \frac{2360\angle 0^\circ}{8.06\angle 29.74^\circ} = 292.8\angle -29.74^\circ\ \text{A}$$
$$\mathbf{I}_{bc} = \frac{\mathbf{E}_{bc}}{\mathbf{Z}_{ph}} = \frac{2360\angle -120^\circ}{8.06\angle 29.74^\circ} = 292.8\angle -149.71^\circ\ \text{A}$$
$$\mathbf{I}_{ca} = \frac{\mathbf{E}_{ca}}{\mathbf{Z}_{ph}} = \frac{2360\angle 120^\circ}{8.06\angle 29.74^\circ} = 292.8\angle 90.26^\circ\ \text{A}$$

(b) Line current
In a delta-connected, three-phase system, line currents lag behind respective phase currents by 30°.

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 292.8 = 507.14\ \text{A}$$
$$\mathbf{I}_{La} = 507.14\angle -59.71^\circ \text{A}$$
$$\mathbf{I}_{Lb} = 507.14\angle -179.71^\circ \text{A}$$
$$\mathbf{I}_{Lc} = 507.14\angle 60.26^\circ \text{A}$$

(c) Load power factor

$$\text{pf} = \cos(29.74^\circ) = 0.868\ \text{(lagging)}$$

Example 8.20 *A three-phase, 200 kW, 50 Hz, delta-connected induction motor is supplied from a three-phase, 440 V, 50 Hz supply system. The efficiency and power factor of the three-phase induction motor are 91% and 0.86 respectively. Calculate (a) line currents, (b) currents in each phase of the motor, (c) active, and (d) reactive components of phase current.*

Solution $P_o = 200\ \text{kW}, \quad V_L = 440\ \text{V}, \quad f = 50\ \text{Hz}, \quad \eta = 91\%, \quad \text{pf} = 0.86$

For a delta-connected load (induction motor),
(a) Line current

$$\eta = \frac{\text{Output power}}{\text{Input power}} = \frac{P_o}{P_i}$$
$$0.91 = \frac{200 \times 10^3}{P_i}$$
$$P_i = 219.78\ \text{kW}$$

$$P_i = \sqrt{3}\, V_L I_L \cos\phi$$
$$219.78 \times 10^3 = \sqrt{3} \times 440 \times I_L \times 0.86$$
$$I_L = 335.3 \text{ A}$$

(b) Currents in each phase of motor

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{335.3}{\sqrt{3}} = 193.6 \text{ A}$$

(c) Active component of phase current

$$I_{ph} \cos\phi = 193.6 \times 0.86 = 166.5 \text{ A}$$

(d) Reactive component of phase current

$$I_{ph} \sin\phi = 193.6 \times \sin(\cos^{-1} 0.86) = 193.6 \times 0.51 = 98.7 \text{ A}$$

Example 8.21 *A three-phase, 400 V, star-connected alternator supplies a three-phase, 112 kW, mesh-connected induction motor of efficiency and power factor 0.88 and 0.86 respectively. Find the (a) current in each motor phase, (b) current in each alternator phase, and (c) active and reactive components of current in each motor phase, (d) active and reactive components of current in each alternator phase.*

Solution $V_L = 400$ V, $P_o = 112$ kW, $\eta = 0.88$, pf $= 0.86$

For a mesh-connected load (induction motor),

(a) Current in each motor phase

$$V_{ph} = V_L = 400 \text{ V}$$
$$\eta = \frac{\text{Output power}}{\text{Input power}} = \frac{P_o}{P_i}$$
$$0.88 = \frac{112 \times 10^3}{P_i}$$
$$P_i = 127.27 \text{ kW}$$
$$P_i = \sqrt{3}\, V_L I_L \cos\phi$$
$$127.27 \times 10^3 = \sqrt{3} \times 400 \times I_L \times 0.86$$
$$I_L = 213.6 \text{ A}$$
$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{213.6}{\sqrt{3}} = 123.32 \text{ A}$$

Current in a star-connected alternator phase will be same as the line current drawn by the motor.

(b) Current in each alternator phase

$$I_L = 213.6 \text{ A}$$

(c) Active component of current in each phase of a motor

$$I_{ph} \cos\phi = 123.32 \times 0.86 = 105.06 \text{ A}$$

Reactive component of current in each phase of the motor

$$I_{ph} \sin\phi = 123.32 \times \sin(\cos^{-1} 0.86) = 123.32 \times 0.51 = 62.89 \text{ A}$$

(d) Active component of current in each alternator phase

$$I_L \cos\phi = 213.6 \times 0.86 = 183.7 \text{ A}$$

Reactive component of current in each alternator phase

$$I_L \sin\phi = 213.6 \times \sin(\cos^{-1} 0.86) = 213.6 \times 0.51 = 108.94 \text{ A}$$

Example 8.22 *Three similar resistors are connected in star across 400 V, three-phase lines. The line current is 5 A. Calculate the value of each resistor. To what value should the line voltage be changed to obtain the same line current with the resistors connected in delta?*

Solution $V_L = 400$ V, $I_L = 5$ A

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = I_L = 5 \text{ A}$$

$$Z_{ph} = R_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{5} = 46.19\ \Omega$$

For a delta-connected load,

$$I_L = 5 \text{ A}$$

$$R_{ph} = 46.19\ \Omega$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{5}{\sqrt{3}} \text{A}$$

$$V_{ph} = I_{ph}\ R_{ph} = \frac{5}{\sqrt{3}} \times 46.19 = 133.33 \text{ V}$$

$$V_L = 133.33 \text{ V}$$

Voltage needed is one-third of the star value.

Example 8.23 *Three 100 Ω, non-inductive resistors are connected in (a) star, and (b) delta across a 400 V, 50 Hz, three-phase supply. Calculate the power taken from the supply in each case. If one of the resistors is open-circuited, what would be the value of total power taken from the mains in each of the two cases?*

Solution $V_L = 400$ V, $Z_{ph} = 100\ \Omega$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{100} = 2.31 \text{ A}$$

$$I_L = I_{ph} = 2.31 \text{ A}$$

$$\cos\phi = 1 \qquad \text{(For pure resistor, pf = 1)}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 2.31 \times 1 = 1600.41 \text{ W}$$

For a delta-connected load,

$$V_{ph} = V_L = 400 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{100} = 4 \text{ A}$$

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 4 = 6.93 \text{ A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 6.93 \times 1 = 4801.24 \text{ W}$$

When One of the Resistors is Open-circuited

Fig. 8.14 *Delta connection*

1. ***Star Connection*** The circuit consists of two 100 Ω resistors in series across a 400 V supply as shown in Fig. 8.14.

$$\text{Currents in lines } A \text{ and } C = \frac{400}{200} = 2 \text{ A}$$

$$\text{Power taken from the mains} = 400 \times 2 = 800 \text{ W}$$

Hence, when one of the resistors is open circuited, the power consumption is reduced by half.

Fig. 8.15 *Delta connection*

2. ***Delta Connection*** In this case, currents in A and C remains as usual 120° out of phase with each other as shown in Fig. 8.15.

$$\text{Currents in each phase} = \frac{400}{100} = 4 \text{ A}$$

$$\text{Power taken from the mains} = 2 \times 4 \times 400 = 3200 \text{ W}$$

Hence, when one of the resistors is open-circuited, the power consumption is reduced by one-third.

Example 8.24 *Three identical impedances of 10∠30° Ω each are connected in star and another set of three identical impedances of 18∠60° Ω are connected in delta. If both the sets of impedances are connected across a balanced, three-phase 400 V, supply, find the line current, total volt-amperes, active power and reactive power.*

Solution

$$\mathbf{Z}_Y = 10\angle 30° \ \Omega, \quad \mathbf{Z}_\Delta = 18\angle 60° \ \Omega, \quad V_L = 400 \text{ V}$$

Three identical delta impedances can be converted into equivalent star impedances.

$$\mathbf{Z}'_Y = \frac{\mathbf{Z}_\Delta}{3} = \frac{18\angle 60°}{3} = 6\angle 60° \ \Omega$$

Now two star-connected impedances of $10\angle 30° \ \Omega$ and $6\angle 60° \ \Omega$ are connected in parallel across a three-phase supply.

$$\mathbf{Z}_{eq} = \frac{(10\angle 30°)(6\angle 60°)}{10\angle 30° + 6\angle 60°} = 3.87\angle 48.83° \ \Omega$$

For a star-connected load,

(a) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{V_{ph}}{Z_{eq}} = \frac{230.94}{3.87} = 59.67 \text{ A}$$

$$I_L = I_{ph} = 59.67 \text{ A}$$

(b) Total volt-amperes

$$S = \sqrt{3} V_L I_L = \sqrt{3} \times 400 \times 59.67 = 41.34 \text{ kVA}$$

(c) Active power

$$P = \sqrt{3} V_L I_L \cos\phi = \sqrt{3} \times 400 \times 59.67 \times \cos(48.83°) = 27.21 \text{ kW}$$

(d) Reactive power

$$Q = \sqrt{3} V_L I_L \sin\phi = \sqrt{3} \times 400 \times 59.67 \times \sin(48.83°) = 31.12 \text{ kVAR}$$

Example 8.25 *Three star-connected impedances $\mathbf{Z}_Y = (20 + j37.7)\ \Omega$ per phase are connected in parallel with three delta-connected impedances $\mathbf{Z}_\Delta = (30 - j159.3)\ \Omega$ per phase. The line voltage is 398 V. Find the line current, pf, active and reactive power taken by the combination.*

Solution $\mathbf{Z}_Y = 20 + j37.7 = 42.68\angle 62.05°\ \Omega$, $\mathbf{Z}_\Delta = 30 - j159.3 = 162.1\angle -79.3°\ \Omega$, $V_L = 398$ V

Three identical delta-connected impedances can be converted into equivalent star impedances.

$$\mathbf{Z}'_Y = \frac{162.1\angle -79.3°}{3} = 54.03\angle -79.3°\ \Omega$$

Now two star-connected impedances of $42.68\angle 62.05°\ \Omega$ and $54.03\angle -79.3°\ \Omega$ are connected in parallel across the three-phase supply.

$$\mathbf{Z}_{eq} = \frac{(42.68\angle 62.05°)(54.03\angle -79.3°)}{42.68\angle 62.05° + 54.03\angle -79.3°} = 68.33\ \angle 9.88°\ \Omega$$

For a star-connected load,

(a) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{398}{\sqrt{3}} = 229.79 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{V_{ph}}{Z_{eq}} = \frac{229.79}{68.33} = 3.36 \text{ A}$$

$$I_L = I_{ph} = 3.36 \text{ A}$$

(b) Power factor

$$\text{pf} = \cos\phi = \cos(9.88°) = 0.99 \text{ (lagging)}$$

(c) Active power

$$P = \sqrt{3} V_L I_L \cos\phi = \sqrt{3} \times 398 \times 3.36 \times 0.99 = 2.29 \text{ kW}$$

(d) Reactive power

$$Q = \sqrt{3} V_L I_L \sin\phi = \sqrt{3} \times 398 \times 3.36 \times \sin(9.88°) = 397.43 \text{ VAR}$$

Example 8.26 *Three coils, each having a resistance of 20 Ω and a reactance of 15 Ω, are connected in star to a 400 V, three-phase, 50 Hz supply. Calculate(a) line current, (b) power supplied, and (c) power factor. If three capacitors, each of same capacitance, are connected in delta to the same supply so as to form a parallel circuit with the above coils, calculate the capacitance of each capacitor to obtain a resultant power factor of 0.95 lagging.*

Solution $R_{ph} = 20\ \Omega, \quad X_{ph} = 15\ \Omega, \quad V_L = 400\ \text{V}$

For a star-connected load,

(a) Line current

$$\mathbf{Z}_{ph} = R_{ph} + jX_{ph} = 20 + j15 = 25\angle 36.87°\ \Omega$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94\ \text{V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{25} = 9.24\ \text{A}$$

$$I_L = I_{ph} = 9.24\ \text{A}$$

(b) Power supplied

$$P_1 = \sqrt{3}\,V_L I_L \cos\phi_1 = \sqrt{3}\times 400\times 9.24\times\cos(36.87°) = 5.12\ \text{kW}$$

(c) Power factor

$$\text{pf} = \cos\phi_1 = \cos(36.87°) = 0.8\ \text{(lagging)}$$

(d) Value of capacitance of each capacitor

$$Q_1 = \sqrt{3}\,V_L I_L \sin\phi_1 = \sqrt{3}\times 400\times 9.24\times\sin(36.87°) = 3.84\ \text{kVAR}$$

When capacitors are connected in delta to the same supply,

$$\text{pf} = 0.95$$

$$\phi_2 = \cos^{-1}(0.95) = 18.19°$$

$$\tan\phi_2 = \tan(18.19°) = 0.33$$

Since capacitors do not absorb any power, power remains the same even when capacitors are connected. But reactive power changes.

$$P_2 = 5.12\ \text{kW}$$

$$Q_2 = P_2 \tan\phi_2 = 5.12\times 0.33 = 1.69\ \text{kVAR}$$

Difference in reactive power is supplied by three capacitors.
From Fig. 8.16,

$$Q = Q_1 - Q_2 = 3.84 - 1.69 = 2.15\ \text{kVAR}$$

$$Q = \sqrt{3}\,V_L I_L \sin\phi$$

$$2.15\times 10^3 = \sqrt{3}\times 400\times I_L\times\sin(90°)$$

$$I_L = 3.1\ \text{A}$$

Fig. 8.16

$$I_{ph} = \frac{I_L}{\sqrt{3}} = 1.79 \text{ A}$$

$$I_{ph} = \frac{V_{ph}}{X_C} = V_{ph} \times 2\pi fC$$

$$C = \frac{I_{ph}}{V_{ph} \times 2\pi f} = \frac{1.79}{400 \times 2\pi \times 50} = 14.24 \ \mu\text{F}$$

8.12 THREE-PHASE UNBALANCED CIRCUITS

If the loads connected across the three phases are not identical to each other, i.e., the loads have different magnitude and power factors, the loads are said to be unbalanced. The phase currents in delta and phase or line current in star connection differ in unbalanced loading giving rise to flow of current, in a neutral wire.

$$I_N = I_R + I_Y + I_B$$

There may be three cases of unbalanced loads:

(i) Unbalanced delta-connected load
(ii) Unbalanced three-wire star-connected load
(iii) Unbalanced four-wire star-connected load

8.12.1 Unbalanced Delta-connected Load

Figure 8.17 shows an unbalanced delta-connected load connected to a balanced three-phase supply.
For a delta-connected load,

Let
$$V_L = V_{ph}$$
$$\mathbf{V}_{RY} = V_{ph}\angle 0°$$
$$\mathbf{V}_{YB} = V_{ph}\angle -120°$$
$$\mathbf{V}_{BR} = V_{ph}\angle -240°$$

Phase currents are given by

$$\mathbf{I}_{RY} = \frac{\mathbf{V}_{RY}}{\mathbf{Z}_{RY}}$$

$$\mathbf{I}_{YB} = \frac{\mathbf{V}_{YB}}{\mathbf{Z}_{YB}}$$

$$\mathbf{I}_{BR} = \frac{\mathbf{V}_{BR}}{\mathbf{Z}_{BR}}$$

Fig. 8.17 *Unbalanced delta-connected load*

Line currents are given by

$$\mathbf{I}_R = \mathbf{I}_{RY} - \mathbf{I}_{BR}$$
$$\mathbf{I}_Y = \mathbf{I}_{YB} - \mathbf{I}_{RY}$$
$$\mathbf{I}_B = \mathbf{I}_{BR} - \mathbf{I}_{YB}$$

8.12.2 Unbalanced Four-wire Star-connected Load

Figure 8.18 shows an unbalanced star-connected load connected to a balanced 3-phase, 4-wire supply. The neutral point N of the load is connected to the neutral point O of the supply. Hence, voltage across the three load impedances is equal to the phase voltage of the supply.

Fig. 8.18 *Unbalanced four-wire star-connected load*

For a star-connected load,

$$V_L = \sqrt{3}\, V_{ph}$$

Let

$$\mathbf{V}_{RN} = V_{ph}\angle 0°$$
$$\mathbf{V}_{YN} = V_{ph}\angle -120°$$
$$\mathbf{V}_{BN} = V_{ph}\angle -240°$$

Phase currents are given by

$$\mathbf{I}_R = \frac{\mathbf{V}_{RN}}{\mathbf{Z}_R}$$
$$\mathbf{I}_Y = \frac{\mathbf{V}_{YN}}{\mathbf{Z}_Y}$$
$$\mathbf{I}_B = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B}$$

For a star-connected load, phase currents are equal to the line currents.
The current in a neutral wire is given by

$$\mathbf{I}_N = \mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B$$

8.12.3 Unbalanced Three-wire Star-connected Load

Figure 8.19 shows an unbalanced three-wire star-connected load.

Fig. 8.19 *Unbalanced three-wire star-connected load*

If the neutral point N of the load is not connected to the neutral point O of the supply, there exists a voltage between the supply neutral point and the load neutral point. The load phase voltage is not the same as the

supply phase voltage. There are many methods to solve such unbalanced star-connected loads. Two most commonly used methods are

(i) Star-delta transformation
(ii) Millman's theorem,

In star-delta transformation, the star-connected loads, are replaced by an equivalent delta-connected load.

$$\mathbf{Z}_{RY} = \mathbf{Z}_R + \mathbf{Z}_Y + \frac{\mathbf{Z}_R\mathbf{Z}_Y}{\mathbf{Z}_B}$$

$$\mathbf{Z}_{YB} = \mathbf{Z}_Y + \mathbf{Z}_B + \frac{\mathbf{Z}_Y\mathbf{Z}_B}{\mathbf{Z}_R}$$

$$\mathbf{Z}_{BR} = \mathbf{Z}_B + \mathbf{Z}_R + \frac{\mathbf{Z}_B\mathbf{Z}_R}{\mathbf{Z}_Y}$$

The problem is then solved as an unbalanced delta-connected load. The line currents so calculated are equal to the line currents of the original star-connected load.

An unbalanced three-wire star-connected load can also be solved by Millman's theorem. Let V_{RO}, V_{YO} and V_{BO} be the phase voltages of the supply which are equal in magnitude but differ in phase from one another by 120°. Let V_{RN}, V_{YN} and V_{BN} be the load phase voltages which are unequal in magnitude as well as differ in phase by unequal angles. The voltage between the neutral points, N and O, i.e, V_{NO}, is calculated using Millman's theorem.

$$\mathbf{V}_{NO} = \frac{\mathbf{V}_{RO}\,\mathbf{Y}_R + \mathbf{V}_{YO}\,\mathbf{Y}_Y + \mathbf{V}_{BO}\,\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B}$$

where Y_R, Y_Y and Y_B are the admittances of the branches of the unbalanced star-connected load. The voltage across phase impedance, i.e., load phase voltage can be calculated as,

$$\mathbf{V}_{RN} = \mathbf{V}_{RO} - \mathbf{V}_{NO}$$
$$\mathbf{V}_{YN} = \mathbf{V}_{YO} - \mathbf{V}_{NO}$$
$$\mathbf{V}_{BN} = \mathbf{V}_{BO} - \mathbf{V}_{NO}$$

The phase currents in the load are given by

$$\mathbf{I}_R = \frac{\mathbf{V}_{RN}}{\mathbf{Z}_R}$$

$$\mathbf{I}_Y = \frac{\mathbf{V}_{YN}}{\mathbf{Z}_Y}$$

$$\mathbf{I}_B = \frac{\mathbf{V}_{BR}}{\mathbf{Z}_B}$$

The line currents are equal to the phase currents for a star-connected load.

Example 8.27 *A three-phase supply with a line voltage of 250 V has an unbalanced delta-connected load as shown in Fig. 8.20.*

Fig. 8.20

Determine (a) phase currents, (b) line currents, (c) total active power and (d) total reactive power, if phase sequence is ABC.

Solution Let $\mathbf{V}_{AB} = 250\angle 0°$ V, $\mathbf{V}_{BC} = 250\angle -120°$ V, $\mathbf{V}_{CA} = 250\angle -240°$ V,

$$\mathbf{Z}_{AB} = 20\angle 0° \ \Omega, \quad \mathbf{Z}_{BC} = 15\angle 20° \ \Omega, \quad \mathbf{Z}_{CA} = 25\angle 90° \ \Omega$$

(a) Phase currents

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_{AB}} = \frac{250\angle 0°}{20\angle 0°} = 12.5\angle 0° \text{A}$$

$$\mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_{BC}} = \frac{250\angle -120°}{15\angle 20°} = 16.67\angle -140° \text{A}$$

$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_{CA}} = \frac{250\angle -240°}{25\angle 90°} = 10\angle 30° \text{A}$$

(b) Line currents

$$\mathbf{I}_A = \mathbf{I}_{AB} - \mathbf{I}_{CA} = 12.5\angle 0° - 10\angle 30° = 6.3\angle -52.48° \text{A}$$

$$\mathbf{I}_B = \mathbf{I}_{BC} - \mathbf{I}_{AB} = 16.67\angle -140° - 12.5\angle 0° = 27.45\angle -157.02° \text{A}$$

$$\mathbf{I}_C = \mathbf{I}_{CA} - \mathbf{I}_{BC} = 10\angle 30° - 16.67\angle -140° = 26.57\angle 36.25° \text{A}$$

(c) Total active power

$$P_{AB} = V_{AB} \ I_{AB} \cos \phi_{AB} = 250 \times 12.5 \times \cos(0°) = 3.13 \text{ kW}$$

$$P_{BC} = V_{BC} \ I_{BC} \cos \phi_{BC} = 250 \times 16.67 \times \cos(20°) = 3.92 \text{ kW}$$

$$P_{CA} = V_{CA} \ I_{CA} \cos \phi_{CA} = 250 \times 25 \times \cos(90°) = 0$$

$$P = P_{AB} + P_{BC} + P_{CA} = 3.13 + 3.92 + 0 = 7.05 \text{ kW}$$

(d) Total reactive power

$$Q_{AB} = V_{AB} \ I_{AB} \sin \phi_{AB} = 250 \times 12.5 \times \sin(0°) = 0$$

$$Q_{BC} = V_{BC} \ I_{BC} \sin \phi_{BC} = 250 \times 16.67 \times \sin(20°) = 1.43 \text{ kVAR}$$

$$Q_{CA} = V_{CA} \ I_{CA} \sin \phi_{CA} = 250 \times 25 \times \sin(90°) = 6.25 \text{ kVAR}$$

$$Q = Q_{AB} + Q_{BC} + Q_{CA} = 0 + 1.43 + 6.25 = 7.68 \text{ kVAR}$$

Example 8.28 *In the circuit of Fig. 8.21, a 400 V, 50 Hz, 3-phase supply of phase sequence ABC is supplied to a delta-connected load consisting of a 100 Ω resistor between lines A and B, a 378 mH inductor between lines B and C, and a 37.8 μF capacitor between lines C and A. Determine phase and line currents.*

Fig. 8.21

Solution $V_L = 400$ V, $f = 50$ Hz, $R = 100\ \Omega$, $L = 318$ mH, $C = 31.8\ \mu$F

Let

$$\mathbf{V}_{AB} = 400\angle 0°\ \text{V}$$
$$\mathbf{V}_{BC} = 400\angle -120°\ \text{V}$$
$$\mathbf{V}_{CA} = 400\angle -240°\ \text{V}$$

$$X_L = 2\pi fL = 2\pi \times 50 \times 318 \times 10^{-3} = 99.9\ \Omega$$
$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 31.8 \times 10^{-6}} = 103.35\ \Omega$$

$$\mathbf{Z}_{AB} = R = 100\angle 0°\ \Omega$$
$$\mathbf{Z}_{BC} = jX_L = j99.9 = 99.9\ \angle 90°\ \Omega$$
$$\mathbf{Z}_{CA} = -jX_C = -j103.35 = 103.35\ \angle -90°\ \Omega$$

(a) Phase currents

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_{AB}} = \frac{400\angle 0°}{100\angle 0°} = 4\angle 0°\text{A}$$
$$\mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_{BC}} = \frac{400\angle -120°}{99.9\angle 90°} = 4\angle 150°\text{A}$$
$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_{CA}} = \frac{400\angle -240°}{103.35\angle -90°} = 3.87\angle -150°\text{A}$$

(b) Line currents

$$\mathbf{I}_A = \mathbf{I}_{AB} - \mathbf{I}_{CA} = 4\angle 0° - 3.87\angle -150° = 7.6\angle 14.75°\text{A}$$
$$\mathbf{I}_B = \mathbf{I}_{BC} - \mathbf{I}_{AB} = 4\angle 150° - 4\angle 0° = 7.73\angle 165°\text{A}$$
$$\mathbf{I}_C = \mathbf{I}_{CA} - \mathbf{I}_{BC} = 3.87\angle -150° - 4\angle 150° = 3.94\angle -88.36°\text{A}$$

Example 8.29 *A three-phase, 400 V, 4-wire system of Fig. 8.22 has a star-connected load with* $\mathbf{Z}_A = (10 + j0)\ \Omega$, $\mathbf{Z}_B = (15 + j10)\ \Omega$, $\mathbf{Z}_C = (0 + j5)\ \Omega$. *Find the line currents and current through neutral conductor.*

Fig. 8.22

Solution

$$V_L = 400 \text{ V}$$
$$\mathbf{Z}_A = (10 + j10)\ \Omega = 10\angle 0°\ \Omega$$
$$\mathbf{Z}_B = (15 + j10)\ \Omega = 18.03\angle 33.69\ \Omega$$
$$\mathbf{Z}_C = (0 + j5)\ \Omega = 5\angle 90°\ \Omega$$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

The phase sequence is assumed as *A-B-C*.

$$\mathbf{V}_{AN} = 230.94\angle 0° \text{ V}$$
$$\mathbf{V}_{BN} = 230.94\angle -120° \text{ V}$$
$$\mathbf{V}_{CN} = 230.94\angle -240° \text{ V}$$

Phase currents

$$\mathbf{I}_A = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_A} = \frac{230.94\angle 0°}{10\angle 0°} = 23.09\angle 0° \text{A}$$
$$\mathbf{I}_B = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B} = \frac{230.94\angle -120°}{18.03\angle 33.69°} = 12.81\angle -153.69° \text{A}$$
$$\mathbf{I}_C = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_C} = \frac{230.94\angle -240°}{5\angle 90°} = 46.19\angle 30° \text{A}$$

Line currents are equal to the phase currents in a star-connected load.

$$\mathbf{I}_{L_1} = \mathbf{I}_A = 23.09\angle 0° \text{A}$$
$$\mathbf{I}_{L_2} = \mathbf{I}_B = 12.81\angle -153.69° \text{A}$$
$$\mathbf{I}_{L_3} = \mathbf{I}_C = 46.19\angle 30° \text{A}$$

Current through neutral conductor

$$\mathbf{I}_N = \mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C = 23.09\angle 0° + 12.81\angle -153.69° + 46.19\angle 30° = 54.47\angle 18.65° \text{A}$$

Example 8.30 *A 3-phase, 4 wire, 208 V, CBA system as shown in Fig. 8.23 has a star-connected load with* $\mathbf{Z}_A = 5\angle 0°\ \Omega$, $\mathbf{Z}_B = 5\angle 30°\ \Omega$, $\mathbf{Z}_C = 10\angle -60°\ \Omega$. *Obtain the phase currents, line currents and current through neutral wire.*

Fig. 8.23

Solution $V_L = 208$ V, $\mathbf{Z}_A = 5\angle 0°\ \Omega$, $\mathbf{Z}_B = 5\angle 30°\ \Omega$, $\mathbf{Z}_C = 10\angle -60°\ \Omega$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{208}{\sqrt{3}} = 120.09 \text{ V}$$

$$\mathbf{V}_{CN} = 120.09\angle 0° \text{V}$$

$$\mathbf{V}_{BN} = 120.09\angle -120° \text{ V}$$

$$\mathbf{V}_{CN} = 120.09\angle -240° \text{ V}$$

(a) Phase currents

$$\mathbf{I}_C = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_C} = \frac{120.09\angle 0°}{10\angle -60°} = 12\angle 60° \text{ A}$$

$$\mathbf{I}_B = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B} = \frac{120.09\angle -120°}{5\angle 30°} = 24.02\angle -150° \text{ A}$$

$$\mathbf{I}_A = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_A} = \frac{120.09\angle -240°}{5\angle 0°} = 24.02\angle 120° \text{ A}$$

(b) Line currents are equal to the phase currents in a star-connected load.

$$\mathbf{I}_{L_1} = \mathbf{I}_C = 12\angle 60° \text{A}$$

$$\mathbf{I}_{L_2} = \mathbf{I}_B = 24.02\angle -150° \text{A}$$

$$\mathbf{I}_{L_3} = \mathbf{I}_A = 24.02\angle 120° \text{A}$$

(c) Current through neutral wire

$$\mathbf{I}_N = \mathbf{I}_C + \mathbf{I}_B + \mathbf{I}_A = 12\angle 60° + 24.02\angle -150° + 24.02\angle 120° = 32.97\angle 144.42° \text{ A}$$

Example 8.31 *A symmetrical 440 V, 3-phase system supplied a star-connected load with the branch impedances* $\mathbf{Z}_R = 10\ \Omega$, $\mathbf{Z}_Y = j5\ \Omega$, $\mathbf{Z}_B = -j5\ \Omega$ *as shown in Fig. 8.24. Calculate line currents and voltage across each phase impedance by Millman's theorem. The phase sequence is RYB.*

Fig. 8.24

Solution $V_L = 440$ V, $\mathbf{Z}_R = 10\ \Omega$, $\mathbf{Z}_Y = j5\ \Omega$, $\mathbf{Z}_B = -j5\ \Omega$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03 \text{ V}$$

Let O be the neutral point of the supply system.

Let
$$V_{RO} = 254.03\angle 0° \text{ V}$$
$$V_{YO} = 254.03\angle -120° \text{ V}$$
$$V_{BO} = 254.03\angle -240° \text{ V}$$

$$\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{10} = \frac{1}{10\angle 0°} = 0.1\angle 0° \ \mho$$

$$\mathbf{Y}_Y = \frac{1}{\mathbf{Z}_Y} = \frac{1}{j5} = \frac{1}{5\angle 90°} = 0.2\angle -90° \ \mho$$

$$\mathbf{Y}_B = \frac{1}{\mathbf{Z}_B} = \frac{1}{-j5} = \frac{1}{5\angle -90°} = 0.2\angle 90° \ \mho$$

By Millman's theorem,

$$\mathbf{V}_{NO} = \frac{\mathbf{V}_{RO}\,\mathbf{Y}_R + \mathbf{V}_{YO}\,\mathbf{Y}_Y + \mathbf{Y}_{BO}\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B}$$

$$= \frac{(254.03\angle 0°)(0.1\angle 0°) + (254.03\angle -120°)(0.2\angle -90°) + (254.03\angle -240°)(0.2\angle 90°)}{0.1\angle 0° + 0.2\angle -90° + 0.2\angle 90°}$$

$$= 625.96\angle 180° \text{ V}$$

Voltage across phase impedance

$$\mathbf{V}_{RN} = \mathbf{V}_{RO} - \mathbf{V}_{NO} = 254.03\angle 0° - 625.96\angle 180° = 880\angle 0° \text{ V}$$
$$\mathbf{V}_{YN} = \mathbf{V}_{YO} - \mathbf{V}_{NO} = 254.03\angle -120° - 625.96\angle 180° = 545.29\angle -23.79° \text{ V}$$
$$\mathbf{V}_{BN} = \mathbf{V}_{BO} - \mathbf{V}_{NO} = 254.03\angle -240° - 625.96\angle 180° = 545.29\angle 23.79° \text{ V}$$

Phase currents/ line currents

$$\mathbf{I}_R = \frac{\mathbf{V}_{RN}}{\mathbf{Z}_R} = \frac{880\angle 0°}{10\angle 0°} = 88\angle 0° \text{ A}$$
$$\mathbf{I}_Y = \frac{\mathbf{V}_{YN}}{\mathbf{Z}_Y} = \frac{545.29\angle -23.79°}{5\angle 90°} = 109.06\angle -113.79° \text{ A}$$
$$\mathbf{I}_B = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B} = \frac{545.29\angle 23.79°}{5\angle -90°} = 109.06\angle 113.79° \text{ A}$$

Example 8.32 *In the circuit of Fig. 8.25, a symmetrical 3-phase 100 V, three-wire supply feeds an unbalanced star-connected load, with impedances of the load as* $\mathbf{Z}_R = 5\angle 0°\ \Omega$, $\mathbf{Z}_Y = 2\angle 90°\ \Omega$ *and* $\mathbf{Z}_B = 4\angle -90°\ \Omega$. *Find the line currents and voltage across the impedance using star-delta transformation method.*

Fig. 8.25

Solution $V_L = 100$ V, $\mathbf{Z}_R = 5\angle 0°\ \Omega$, $\mathbf{Z}_Y = 2\angle 90°\ \Omega$, $\mathbf{Z}_B = 4\angle -90°\ \Omega$

The unbalanced star-connected load can be converted into equivalent delta-connected load by star-delta transformation technique.

$$\mathbf{Z}_{RY} = \mathbf{Z}_R + \mathbf{Z}_Y + \frac{\mathbf{Z}_R\mathbf{Z}_Y}{\mathbf{Z}_B} = 5\angle 0° + 2\angle 90° + \frac{(5\angle 0°)(2\angle 90°)}{4\angle -90°} = 3.2\angle 38.66°\ \Omega$$
$$\mathbf{Z}_{YB} = \mathbf{Z}_Y + \mathbf{Z}_B + \frac{\mathbf{Z}_Y\mathbf{Z}_B}{\mathbf{Z}_R} = 2\angle 90° + 4\angle -90° + \frac{(2\angle 90°)(4\angle -90°)}{5\angle 0°} = 2.56\angle -51.34°\ \Omega$$
$$\mathbf{Z}_{BR} = \mathbf{Z}_B + \mathbf{Z}_R + \frac{\mathbf{Z}_B\mathbf{Z}_R}{\mathbf{Z}_Y} = 4\angle -90° + 5\angle 0° + \frac{(4\angle -90°)(5\angle 0°)}{2\angle 90°} = 6.4\angle -141.34°\ \Omega$$

The equivalent delta-connected load is shown in Fig. 8.26.
For a delta-connected load,

$$V_{ph} = V_L = 100 \text{ V}$$

Let $\mathbf{V}_{RY} = 100\angle 0°$ V

$$\mathbf{V}_{YB} = 100\angle -120° \text{ V}$$
$$\mathbf{V}_{BR} = 100\angle -240° \text{ V}$$

Phase currents

$$\mathbf{I}_{RY} = \frac{\mathbf{V}_{RY}}{\mathbf{Z}_{RY}} = \frac{100\angle 0°}{3.2\angle 38.66°} = 31.25\angle -38.66° \text{ A}$$
$$\mathbf{I}_{YB} = \frac{\mathbf{V}_{YB}}{\mathbf{Z}_{YB}} = \frac{100\angle -120°}{2.56\angle -51.34°} = 39.06\angle -68.66° \text{ A}$$
$$\mathbf{I}_{BR} = \frac{\mathbf{V}_{BR}}{\mathbf{Z}_{BR}} = \frac{100\angle -240°}{6.4\angle -141.34°} = 15.63\angle -98.66° \text{ A}$$

Fig. 8.26

Line currents

$$\mathbf{I}_R = \mathbf{I}_{RY} - \mathbf{I}_{BR} = 31.25\angle -38.66° - 15.63\angle -98.66° = 27.06\angle -8.65° \text{ A}$$
$$\mathbf{I}_Y = \mathbf{I}_{YB} - \mathbf{I}_{RY} = 39.06\angle -68.66° - 31.25\angle -38.66° = 19.7\angle -121.14° \text{ A}$$
$$\mathbf{I}_B = \mathbf{I}_{BR} - \mathbf{I}_{YB} = 15.63\angle -98.66° - 39.06\angle -68.66° = 26.69\angle 128.36° \text{ A}$$

These line currents are equal to the line currents of the original star-connected load.
For a star-connected load,

$$I_{ph} = I_L$$
$$\mathbf{I}_{RN} = 27.06\angle -8.65° \text{ A}$$
$$\mathbf{I}_{YN} = 19.7\angle -121.14° \text{ A}$$
$$\mathbf{I}_{BN} = 26.69\angle 128.36° \text{ A}$$
$$\mathbf{V}_{RN} = \mathbf{Z}_R\ \mathbf{I}_{RN} = (5\angle 0°)(27.06\angle -8.65°) = 135.3\angle -8.65° \text{ V}$$
$$\mathbf{V}_{YN} = \mathbf{Z}_Y\ \mathbf{I}_{YN} = (2\angle 90°)(19.7\angle -121.14°) = 39.4\angle -31.14° \text{ V}$$
$$\mathbf{V}_{BN} = \mathbf{Z}_B\ \mathbf{I}_{BN} = (4\angle -90°)(26.69\angle 128.36°) = 106.76\angle 38.36° \text{ V}$$

Example 8.33 *A three-phase three-wire unbalanced load is star-connected as shown in Fig. 8.27. The phase voltage of two of the arms are $\mathbf{V}_{RN} = 100\angle -10°$ V and $\mathbf{V}_{YN} = 150\angle 100°$ V. Calculate voltage between star point of the load and the supply neutral.*

Fig. 8.27

Solution $\mathbf{V}_{RN} = 100\angle -10°$ V, $\mathbf{V}_{YN} = 150\angle 100°$ V

Let $\mathbf{V}_{RO} = V\angle 0°$

$\mathbf{V}_{YO} = V\angle -120°$

From Fig. 8.27,

$$\mathbf{V}_{RO} = \mathbf{V}_{RN} + \mathbf{V}_{NO}$$

$$\mathbf{V}_{NO} = \mathbf{V}_{RO} - \mathbf{V}_{RN} = V\angle 0° - 100°\angle -10° \quad \ldots\text{(i)}$$

Similarly,

$$\mathbf{V}_{NO} = \mathbf{V}_{YO} - \mathbf{V}_{YN} = V\angle -120° - 150\angle 100° \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$V\angle 0° - 100\angle -10° = V\angle -120° - 150\angle 100°$$

$$V\angle 0° - V\angle -120° = 100\angle -10° - 150\angle 100°$$

$$\sqrt{3}\, V\angle 30° = 206.79\angle -52.97°$$

$$V = 119.39\angle -82.97$$

$$\mathbf{V}_{NO} = 119.39\angle -82.97 - 100\angle -10 = 131.88\angle -129.67° \text{ V}$$

8.13 MEASUREMENT OF THREE-PHASE POWER

In a three-phase system, total power is the sum of powers in three phases. The power is measured by wattmeter. It consists of two coils: (i) Current coil, and (ii) Voltage coil. Current coil is connected in series with the load and it senses current. Voltage coil is connected across supply terminals and it senses voltages.

There are three methods to measure three-phase power:

1. Three-wattmeter method
2. Two-wattmeter method
3. One-wattmeter method

8.13.1 Three-Wattmeter Method

This method is used for balanced as well as unbalanced loads. Three wattmeters are inserted in each of the three phases of the load whether star connected or delta connected as shown in Fig. 8.28. Each wattmeter will measure the power consumed in each phase.

For balanced load, $W_1 = W_2 = W_3$

For unbalanced load, $W_1 \neq W_2 \neq W_3$

Total power $P = W_1 + W_2 + W_3$

Fig. 8.28 *Three-wattmeter method*

8.13.2 Two-Wattmeter Method

This method is used for balanced as well as unbalanced loads. The current coils of the two wattmeters are inserted in any two lines and the voltage coil of each wattmeter is joined to a third line. The load may be star or delta connected as shown in Fig. 8.29. The sum of the two wattmeter readings gives three-phase power.

Fig. 8.29 *Two-wattmeter method*

Total power $P = W_1 + W_2$

8.13.3 One-Wattmeter Method

This method is used for balanced loads only. When the load is balanced, total power is given by

$$P = 3\,V_{ph} I_{ph} \cos\phi$$

Hence, one wattmeter is used to measure power in one phase. The wattmeter reading is then multiplied by three to obtain three-phase power. The load may be star or delta connected as shown in Fig. 8.30.

Fig. 8.30 *One-wattmeter method*

8.13.4 Measurement of Power by Two-wattmeter Method

Figure 8.31 shows a balanced star-connected load and this load may be assumed to be inductive. Let V_{RN}, V_{YN} and V_{BN} be the three phase voltages and I_R, I_Y and I_B be the phase currents. The phase currents will lag behind their respective phase voltages by angle ϕ.

Current through current coil of $W_1 = I_R$

Voltage across voltage coil of $W_1 = V_{RB} = V_{RN} + V_{NB} = V_{RN} - V_{BN}$

Figure 8.31 shows the phasor diagram of a balanced star-connected inductive load.

Fig. 8.31 *Phasor diagram*

From the phasor diagram, it is clear that the phase angle between V_{RB} and I_R is $(30° - \phi)$.

$$W_1 = V_{RB} I_R \cos(30° - \phi)$$

Current through current coil of $W_2 = I_Y$

Voltage across voltage coil of $W_2 = V_{YB} = V_{YN} + V_{NB} = V_{YN} - V_{BN}$

From the phasor diagram, it is clear that phase angle between V_{YB} and I_Y is $(30° + \phi)$.

$$W_2 = V_{YB}\ I_Y \cos(30° + \phi)$$

But

$$I_R = I_Y = I_L$$

$$V_{RB} = V_{YB} = V_L$$

$$W_1 = V_L I_L \cos(30° - \phi)$$

$$W_2 = V_L I_L \cos(30° + \phi)$$

$$W_1 + W_2 = V_L I_L [\cos(30° + \phi) + \cos(30° - \phi)] = V_L I_L (2\cos 30° \cos\phi) = \sqrt{3} V_L I_L \cos\phi$$

Thus, the sum of two wattmeter readings gives three-phase power.

8.13.5 Measurement of Power Factor by Two-wattmeter Method

1. Lagging Power Factor

$$W_1 = V_L I_L \cos(30° - \phi)$$

$$W_2 = V_L I_L \cos(30° + \phi)$$

$$\therefore W_1 > W_2$$

$$W_1 + W_2 = \sqrt{3} V_L I_L \cos\phi$$

$$W_1 - W_2 = V_L I_L [\cos(30° - \phi) - \cos(30° + \phi) = V_L I_L \sin\phi$$

$$\frac{W_1 - W_2}{W_1 + W_2} = \frac{V_L I_L \sin\phi}{\sqrt{3} V_L I_L \cos\phi}$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$\phi = \tan^{-1}\left(\sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}\right)$$

$$\text{pf} = \cos\phi = \cos\left\{\tan^{-1}\left(\sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}\right)\right\}$$

2. Leading Power Factor

Figure 8.32 shows the phasor diagram of a balanced star-connected capacitive load.

$$W_1 = V_L I_L \cos(30° + \phi)$$

$$W_2 = V_L I_L \cos(30° - \phi)$$

$$\therefore W_1 < W_2$$

$$W_1 + W_2 = \sqrt{3} V_L I_L \cos\phi$$

$$W_1 - W_2 = -V_L I_L \sin\phi$$

$$\tan\phi = -\sqrt{3}\frac{(W_1 - W_2)}{(W_1 + W_2)}$$

$$\phi = \tan^{-1}\left(-\sqrt{3}\frac{(W_1 - W_2)}{(W_1 + W_2)}\right)$$

$$\text{pf} = \cos\phi = \cos\left\{\tan^{-1}\left(-\sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}\right)\right\}$$

Fig. 8.32 *Phasor diagram*

Example 8.34 *Two wattmeters are used to measure power in a three-phase balanced load. Find the power factor if (a) two readings are equal and positive, (b) two readings are equal and opposite, and (c) one wattmeter reads zero.*

Solution

(a) Power factor if two readings are equal and positive

$$W_1 = W_2$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\,(0) = 0$$

$$\phi = 0°$$

$$\text{Power factor} = \cos\phi = \cos(0°) = 1$$

(b) Power factor if two readings are equal and opposite

$$W_1 = -W_2$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2} = \infty$$

$$\phi = 90°$$

$$\text{Power factor} = \cos\phi = \cos(90°) = 0$$

(c) Power factor if one wattmeter reads zero

$$W_2 = 0$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\left(\frac{W_1}{W_1}\right) = \sqrt{3}$$

$$\phi = 60°$$

$$\text{Power factor} = \cos\phi = \cos(60°) = 0.5$$

Example 8.35 *What will be the relation between readings on the wattmeter connected to measure power in a three-phase balanced circuit with (a) unity power factor, (b) zero power factor, and (c) power factor = 0.5.*

Solution

(a) Relation between wattmeter readings with power factor = 1

$$\cos\phi = 1$$

$$\phi = 0°$$

$$\tan\phi = \tan(0°) = 0$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$0 = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$W_1 = W_2$$

(b) Relation between wattmeter readings with power factor = 0

$$\cos\phi = 0$$

$$\phi = 90°$$

$$\tan\phi = \tan(90°) = \infty$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$W_1 + W_2 = 0$$

$$W_1 = -W_2$$

(c) Relation between wattmeter readings with power factor = 0.5

$$\cos\phi = 60°$$

$$\tan\phi = \tan(60°) = 1.732$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$1.732 = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$W_1 - W_2 = W_1 + W_2$$

$$W_2 = 0$$

Example 8.36 *In a balanced three-phase system, the power is measured by two-wattmeter method and the ratio of two-wattmeter readings is 4:1. The load is inductive. Determine the load power factor.*

Solution

$$\frac{W_1}{W_2} = \frac{4}{1}$$

Load power factor

$$W_1 = 4\,W_2$$

$$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\,\frac{(3W_2)}{(5W_2)} = \sqrt{3}\left(\frac{3}{5}\right) = 1.039$$

$$\phi = 46.1°$$

$$\text{pf} = \cos\phi = \cos(46.1°) = 0.0693 \text{ (lagging)}$$

Example 8.37 *Find the power and power factor of the balanced circuit in which the wattmeter readings are 5 kW and 0.5 kW, the latter being obtained after the reversal of the current coil terminals of the wattmeter.*

Solution $W_1 = 5$ kW, $W_2 = 0.5$ kW

(a) Power

When the latter reading is obtained after the reversal of the current coil terminals of the wattmeter,

$$W_1 = 5 \text{ kW}$$

$$W_2 = -0.5 \text{ kW}$$

$$\text{Power} = W_1 + W_2 = 5 + (-0.5) = 4.5 \text{ kW}$$

(b) Power factor

$$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\,\frac{(5+0.5)}{(5-0.5)} = 2.12$$

$$\phi = 64.72°$$

$$\text{Power factor} = \cos\phi = \cos(64.72°) = 0.43$$

Example 8.38 *Two wattmeters connected to measure the input to a balanced, three-phase circuit indicate 2000 W and 500 W respectively. Find the power factor of the circuit (a) when both readings are positive and (b) when the latter is obtained after reversing the connection to the current coil of one instrument.*

Solution $W_1 = 2000$ W, $W_2 = 500$ W

(a) Power factor of the circuit when both readings are positive

$$W_1 = 2000 \text{ W}$$

$$W_2 = 500 \text{ W}$$

$$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\,\frac{(2000-500)}{(2000+500)} = 1.039$$

$$\phi = 46.102°$$

$$\text{Power factor} = \cos\phi = \cos(46.102°) = 0.693$$

(b) Power factor of the circuit when the latter reading is obtained after reversing the connection to the current coil of one instrument

$$W_1 = 2000 \text{ W}$$
$$W_2 = -500 \text{ W}$$
$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\frac{(2000+500)}{(2000-500)} = 2.887$$
$$\phi = 70.89°$$
$$\text{Power factor} = \cos\phi = \cos(70.89°) = 0.33$$

Example 8.39 *A three-phase, 10 kVA load has a power factor of 0.342. The power is measured by the two-wattmeter method. Find the reading of each wattmeter when the (a) power factor is leading, and the (b) power factor is lagging.*

Solution

$$S = 10 \text{ kVA}, \quad \text{pf} = 0.342$$
$$S = \sqrt{3}\, V_L I_L$$
$$10 \times 10^3 = \sqrt{3}\, V_L I_L$$
$$V_L I_L = 5.77 \text{ kVA}$$
$$\cos\phi = 0.342$$
$$\phi = 72°$$

(a) Reading of each wattmeter when the power factor is leading

$$W_1 = V_L I_L \cos(30° + \phi) = 5.77 \times 10^3 \cos(30° + 70°) = -1 \text{ kW}$$
$$W_2 = V_L I_L \cos(30° - \phi) = 5.77 \times 10^3 \cos(30° - 70°) = 4.42 \text{ kW}$$

(b) Reading of each wattmeter when the power factor is lagging

$$W_1 = V_L I_L \cos(30° - \phi) = 4.42 \text{ kW}$$
$$W_2 = V_L I_L \cos(30° + \phi) = -1 \text{ kW}$$

Example 8.40 *A three-phase, star-connected load draws a line current of 20 A. The load kVA and kW are 20 and 11 respectively. Find the readings on each of the two wattmeters used to measure the three-phase power.*

Solution

$$I_L = 20 \text{ A}, \quad S = 20 \text{ kVA}, \quad P = 11 \text{ kW}$$
$$S = \sqrt{3}\, V_L I_L$$
$$20 \times 10^3 = \sqrt{3}\, V_L I_L$$
$$V_L I_L = 11.55 \text{ kVA}$$
$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$11 \times 10^3 = 20 \times 10^3 \times \cos\phi$$
$$\cos\phi = 0.55$$
$$\phi = 56.63°$$
$$W_1 = V_L I_L \cos(30° - \phi) = 11.55 \times 10^3 \times \cos(30° - 56.63°) = 10.32 \text{ kW}$$
$$W_2 = V_L I_L \cos(30° + \phi) = 11.55 \times 10^3 \times \cos(30° + 56.63°) = 0.68 \text{ kW}$$

Example 8.41 *Calculate the total power and readings of the two wattmeters connected to measure power in three-phase balanced load, if the reactive power is 15 kVAR and load pf is 0.8 lagging.*

Solution $Q = 15 \text{ kVAR}, \quad \text{pf} = 0.8 \text{ (lagging)}$

(a) Reading of the two wattmeters

$$\cos\phi = 0.8$$
$$\phi = 36.87°$$
$$Q = \sqrt{3}\, V_L I_L \sin\phi$$
$$15 \times 10^3 = \sqrt{3}\, V_L I_L \sin(36.87°)$$
$$V_L I_L = 14.43 \text{ kVA}$$
$$W_1 = V_L\, I_L \cos(30° - \phi) = 14.43 \times 10^3 \times \cos(30° - 36.87°) = 14.03 \text{ kW}$$
$$W_2 = V_L\, I_L \cos(30° + \phi) = 14.43 \times 10^3 \times \cos(30° + 36.87°) = 5.67 \text{ kW}$$

(b) Total power

$$P = W_1 + W_2 = 14.03 + 5.67 = 19.7 \text{ kW}$$

Example 8.42 *Two wattmeters are connected to measure power in a three-phase circuit. The reading of one of the wattmeters is 5 kW when the load power factor is unity. If the power factor of the load is changed to 0.707 lagging without changing the total input power, calculate the readings of the two wattmeters.*

Solution $W_1 = 5 \text{ kW}$

(a) When the power factor is unity,

$$W_1 = W_2 = 5 \text{ kW}$$
$$\text{Power} = W_1 + W_2 = 5 + 5 = 10 \text{ kW} \qquad \ldots\text{(i)}$$

(b) When the power factor is changed to 0.707 lagging,

$$\text{pf} = \cos\phi = 0.707 \text{ (lagging)}$$
$$\phi = 45°$$
$$\tan\phi = \tan(45°) = 1$$
$$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2}$$
$$1 = \sqrt{3}\,\frac{W_1 - W_2}{10}$$
$$W_1 - W_2 = \frac{10}{\sqrt{3}} = 5.77 \text{ kW} \qquad \ldots\text{(ii)}$$

Solving Eq. (i) and (ii),

$$W_1 = 7.89 \text{ kW}$$
$$W_2 = 2.11 \text{ kW}$$

Example 8.43 *A three-phase RYB system has effective line voltage of 173.2 V. Wattmeters in line R and Y read 301 W and 1327 W respectively. Find the impedance of the balanced star-connected load.*

Solution $V_L = 173.2 \text{ V}, \quad W_1 = 301 \text{ W}, \quad W_2 = 1327 \text{ W}$

If the load is capacitive and pf is leading then $W_1 < W_2$.

$$\tan\phi = -\sqrt{3}\frac{W_1 - W_2}{W_1 + W_2} = -\sqrt{3}\frac{(301-1327)}{(301+1327)} = -1.09$$

$$\phi = 47.47°$$

$$P = W_1 + W_2 = 301 + 1327 = 1628 \text{ W}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi$$

$$1628 = \sqrt{3} \times 173.2 \times I_L \times \cos(47.47°)$$

$$I_L = 8.03 \text{ A}$$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{173.2}{\sqrt{3}} = 100 \text{ V}$$

$$I_{ph} = I_L = 8.03 \text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{100}{8.03} = 12.45\ \Omega$$

Example 8.44 *Three coils each with a resistance of 10 Ω and reactance of 10 Ω are connected in star across a three-phase, 50 Hz, 400 V supply. Calculate (a) line current, and (b) readings on the two wattmeters connected to measure the power.*

Solution $R = 10\ \Omega, \quad X_L = 10\ \Omega, \quad V_L = 400 \text{ V}$

For a star-connected load,

(a) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$\mathbf{Z}_{ph} = R + jX_L = 10 + j10 = 14.14\ \angle 45°\ \Omega$$

$$Z_{ph} = 14.14\ \Omega$$

$$\phi = 45°$$

$$\text{Power factor} = \cos\phi = \cos(45°) = 0.707 \text{ (lagging)}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{14.14} = 16.33 \text{ A}$$

$$I_L = I_{ph} = 16.33 \text{ A}$$

(b) Readings on the two wattmeters

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 16.33 \times 0.707 = 7998.83 \text{ W}$$

$$W_1 + W_2 = 7998.83 \text{ W} \qquad \text{...(i)}$$

Also,

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2}$$

$$\tan 45° = \sqrt{3}\frac{W_1 - W_2}{7998.83}$$

$$W_1 - W_2 = 4618.13 \text{ W} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii),

$$W_1 = 6308.48 \text{ W}$$

$$W_2 = 1690.35 \text{ W}$$

Example 8.45 *Three coils each having a resistance of 20 Ω and reactance of 15 Ω are connected in (a) star, and (b) delta, across a three-phase, 400 V, 50 Hz supply. Calculate in each case, the readings on two wattmeters connected to measure the power input.*

Solution $R = 20\ \Omega, \quad X_L = 15\ \Omega, \quad V_L = 400\ \text{V}$

(a) Readings on two wattmeters for a star-connected load

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94\ \text{V}$$

$$\mathbf{Z}_{ph} = 20 + j\,15 = 25\angle 36.87°\ \Omega$$

$$Z_{ph} = 25\ \ \Omega$$

$$\phi = 36.87°$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{25} = 9.24\ \text{A}$$

$$I_L = I_{ph} = 9.24\ \text{A}$$

$$W_1 = V_L\, I_L \cos(30° - \phi) = 400 \times 9.24 \times \cos(30° - 36.87°) = 3669.46\ \text{W}$$

$$W_2 = V_L\, I_L \cos(30° + \phi) = 400 \times 9.24 \times \cos(30° + 36.87°) = 1451.86\ \text{W}$$

(b) Readings on two wattmeters for a delta-connected load

$$V_{ph} = V_L = 400\ \text{V}$$

$$Z_{ph} = 25\ \Omega$$

$$\phi = 36.87°$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{25} = 16\ \text{A}$$

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 16 = 27.72\ \text{A}$$

$$W_1 = V_L I_L \cos(30° - \phi) = 400 \times 27.72 \times \cos(30° - 36.87°) = 11008.39\ \text{W}$$

$$W_2 = V_L I_L \cos(30° + \phi) = 400 \times 27.72 \times \cos(30° + 36.87°) = 4355.57\ \text{W}$$

Example 8.46 *Two wattmeters connected to measure three-phase power for star-connected load reads 3 kW and 1 kW. The line current is 10 A. Calculate (a) line and phase voltage (b) resistance and reactance per phase.*

Solution $W_1 = 3\ \text{kW}, \quad W_2 = 1\ \text{kW}, \quad I_L = 10\ \text{A}$

(a) Line and phase voltage

$$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\,\frac{(3-1)}{(3+1)} = 0.866$$

$$\phi = 40.89°$$

$$\text{Power factor} = \cos\phi = \cos(40.89°) = 0.756$$

$$P = W_1 + W_2 = 3 + 1 = 4\ \text{kW}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$4\times10^3 = \sqrt{3}\times V_L \times 10\times 0.756$$
$$V_L = 305.48 \text{ V}$$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{305.48}{\sqrt{3}} = 176.37 \text{ V}$$

(b) Resistance and reactance per phase

$$I_{ph} = I_L = 10 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{176.37}{10} = 17.637\ \Omega$$
$$R = Z_{ph}\cos\phi = 17.637\times 0.756 = 13.33\ \Omega$$
$$X_L = Z_{ph}\sin\phi = 17.637\times\sin(40.89°) = 11.55\ \Omega$$

Example 8.47 *The power input to a 2000 V, 50 Hz, three-phase motor running on full load at an efficiency of 90% is measured by two wattmeters which indicate 300 kW and 100 kW respectively. Calculate the (a) input power, (b) power factor, and (c) line current.*

Solution $V_L = 2000$ V, $\eta = 90\%$, $W_1 = 300$ kW, $W_2 = 100$ kW

(a) Input power

$$P = W_1 + W_2 = 300 + 100 = 400 \text{ kW}$$

(b) Power factor

$$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2} = \sqrt{3}\,\frac{(300-100)}{(300+100)} = 0.866$$
$$\phi = 40.89°$$
$$\text{pf} = \cos\phi = \cos(40.89°) = 0.76 \text{ (lagging)}$$

(c) Line current

$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$400\times10^3 = \sqrt{3}\times 2000\times I_L \times 0.76$$
$$I_L = 151.93 \text{ A}$$

Example 8.48 *A three-phase, 400 V, 50 Hz induction motor has a full load output of 14.9 kW at which the efficiency and power factor are 0.88 and 0.8 respectively. What is the line current? Find the readings on the two wattmeters connected to measure the power input to the motor.*

Solution $V_L = 400$ V, $f = 50$ Hz, $P_o = 14.9$ kW, pf = 0.8, $\eta = 0.88$

(a) Line current

$$\eta = \frac{P_o}{P_i}$$
$$0.88 = \frac{14.9\times10^3}{P_i}$$
$$P_i = 16.93 \text{ kW}$$

$$\text{pf} = \cos\phi = 0.8$$
$$\phi = 36.87°$$
$$P_i = \sqrt{3}\, V_L I_L \cos\phi$$
$$16.93 \times 10^3 = \sqrt{3} \times 400 \times I_L \times 0.8$$
$$I_L = 30.55 \text{ A}$$

(b) Readings on the two wattmeters

$$W_1 = V_L I_L \cos(30° - \phi) = 400 \times 30.55 \times \cos(30° - 36.87°) = 12.13 \text{ kW}$$
$$W_2 = V_L I_L \cos(30° + \phi) = 400 \times 30.55 \times \cos(30° + 36.87°) = 4.8 \text{ kW}$$

Example 8.49 *A three-phase, 220 V, 50 Hz, 11.2 kW induction motor has a full load efficiency of 88 per cent and draws a line current of 38 A under full load, when connected to a three-phase, 220 V supply. Determine power factor at which the motor is operating. Find the reading on two wattmeters connected in the circuit to measure the input to the motor.*

Solution $V_L = 220$ V, $P_o = 11.2$ kW, $\eta = 88\%$, $I_L = 38$ A

(a) Power factor at which the motor is operating

$$\eta = \frac{P_o}{P_i}$$
$$0.88 = \frac{11.2 \times 10^3}{P_i}$$
$$P_i = 12.73 \text{ kW}$$

But

$$P_i = \sqrt{3}\, V_L I_L \cos\phi$$
$$12.73 \times 10^3 = \sqrt{3} \times 220 \times 38 \times \cos\phi$$
$$\text{pf} = \cos\phi = 0.88 \text{ (lagging)}$$

(b) Reading on two wattmeters

$$\phi = 28.36°$$
$$W_1 = V_L I_L \cos(30° - \phi) = 220 \times 38 \times \cos(30° - 28.36°) = 8356.58 \text{ W}$$
$$W_2 = V_L I_L \cos(30° + \phi) = 220 \times 38 \times \cos(30° + 28.36°) = 4385.49 \text{ W}$$

Example 8.50 *Find the reading of a wattmeter when the network shown in Fig. 8.33 is connected to a symmetrical 440 V, 3-phase supply. The phase sequence is RYB.*

Fig. 8.33

Solution

$$V_L = 440 \text{ V}$$

Let

$$\mathbf{V}_{RY} = 440\angle 0° \text{V}$$

$$\mathbf{V}_{YB} = 440\angle -120° \text{V}$$

$$\mathbf{V}_{BR} = 440\angle -240° \text{V}$$

$$\mathbf{Z}_{RY} = -j53 = 53\angle -90° \ \Omega$$

$$\mathbf{Z}_{BR} = 50 + j40 = 64.03\angle 38.66° \ \Omega$$

$$\mathbf{I}_{RY} = \frac{V_{RY}}{\mathbf{Z}_{RY}} = \frac{440\angle 0°}{53\angle -90°} = 8.3\angle 90° \text{ A}$$

$$\mathbf{I}_{BR} = \frac{V_{BR}}{\mathbf{Z}_{BR}} = \frac{440\angle -240°}{64.03\angle 38.66°} = 6.87\angle 81.34° \text{ A}$$

$$\mathbf{I}_R = \mathbf{I}_{RY} - \mathbf{I}_{BR} = 8.3\angle 90° - 6.87\angle 81.34° = 1.83\angle 124.44° \text{ A}$$

Fig. 8.34

The phasor diagram of the system is shown in Fig. 8.34.
From the phasor diagram, it is clear that angle between V_{YB} and I_R is 115.56°

$$W = V_{YB}\ I_R \cos(115.56°) = 440 \times 1.83 \times \cos(115.56°) = 347.41 \text{ W}$$

Exercises

8.1 Three coils, each of 5 Ω resistance, and 6 Ω inductive reactance are connected in closed delta and supplied from a 440 V, three-phase system. Calculate the line and phase currents, the power factor of the system and the intake in watts.
[97.58 A, 56.33 A, 0.64 (lagging), 47.61 kW]

8.2 Three coils, each having a resistance of 10 Ω and inductance of 0.02 H are connected (i) in star, (ii) in delta to a three-phase, 50 Hz supply, the line voltage being 500 volts. Calculate for each case the line current and the total power taken from the supply.
[(a) star : 24.46 A, 17.94 kW,
(b) delta : 73.39 A, 53.83 kW]

8.3 A balanced delta-connected load of (8 + *j*6) ohms per phase is connected to a three-phase, 230 V supply with phase sequence *R-Y-B*. Find the line current, power factor, power, reactive volt-amperes and the total volt-amperes. Draw the phasor diagram.
[39.85 A, 0.8 (lagging),
12.74 kW, 9.52 kVAR, 15.86 kVA]

8.4 A balanced star-connected load with (6 + *j*8) Ω per phase is connected to a three-phase, 440 V supply. Find the line current, power factor, power, reactive volt-amperes and volt-amperes total.
[25.404 A, 0.6 (lagging),
11.616 kW, 15.488 kVAR, 19.360 kVA]

8.5 Calculate the active and reactive components of the current in each phase of a star-connected 5000 V, three-phase, alternator supplying 3000 kW at a power factor of 0.8.
[250 A, 150 A]

8.6 Three similar coils, connected in star, take a total power of 1.5 kW at a power factor of 0.2 lagging from a three-phase, 440 V, 50 Hz supply. Calculate the resistance and inductance of each coil. [5.16 Ω, 0.08 H]

8.7 A balanced three-phase load connected in delta draws a power of 10 kW at 440 V at a power factor of 0.6 lead. Find the values of the circuit elements and the reactive volt-amperes drawn.
[20.91 Ω, 27.88 Ω, 13.33 kVAR]

8.8 A balanced star-connected load, connected to a 400 V, 50 Hz, three-phase ac supply draws a phase current of 50 A at 0.6 power factor lagging. Calculate (a) phase voltage, (b) total power, and (c) parameters in the star-connected load.

[230.94 V, 20.84 kW, (2.77 + j3.7) Ω]

8.9 Three equal star-connected inductors, consume 8 kW power at 0.8 power factor when connected to 415 V, three-phase, three-wire, 50 Hz supply. Estimate the load parameters per phase and determine the line currents.

[6.2 Ω, 0.0386 H, 13.83 A]

8.10 A balanced three-phase, star-connected load of 150 kW takes a leading current of 100 A with a line voltage of 1100 V, 50 Hz. Find the circuit constant of the load per phase.

[5 Ω, 813 μF]

8.11 Three pure elements connected in star draw x kVAR. What will be the value of elements that will draw the same kVAR, when connected in delta across the same supply?

[$Z_\Delta = 3Z_Y$]

8.12 A balanced wye-connected load with (10 + j20) ohms per phase is connected to a three-phase, 400 V supply. Determine the voltage across, current through and power dissipated in each resistor. Also, determine the total power.

[103.2 V, 10.32 V, 1067 W, 3201 W]

8.13 A delta-connected three-phase load is supplied from a 3- phase, 400 volts balanced supply system. The line current is 20 A and power taken by the load is 10 kW. Find (a) impedance in each branch, (b) line current, power factor and power consumed if the same load is connected in star.

[(24.95 + j24.05) Ω,
6.66 A, 0.72 (lagging), 3323.21 W]

8.14 A balanced star-connected load is supplied from a symmetrical three-phase, 400 V system. The current in each phase is 30 A and lags 30° behind the phase voltage. Find (a) phase voltage, (b) the circuit elements, and (c) draw the vector diagram showing the currents and the voltages.

[230.94 V, 6.67 Ω, 3.849 Ω]

8.15 A three-phase, delta-connected load having a (3 + j4) ohms per phase is connected across a 230 V, three-phase source. Calculate the magnitude of the line current.

[76.21 A]

8.16 A 220 V, three-phase voltage is applied to a balanced delta-connected load. The rms value of the phase current is $20\angle -30°$A. Determine

(i) magnitude and phase of the line current
(ii) total power received by the three-phase load
(iii) value of the resistive portion of the phase impedance

Also, draw the phasor diagram showing clearly the line voltages, phase current and line currents.

[$34.65\angle -60°$A, 11.43 kW, 9.53 Ω]

8.17 A three-phase, 37.3 kW, 440 V, 50 Hz induction motor operates on full load with an efficiency of 89% and at a power factor of 0.85 lagging. Calculate total kVA rating of capacitance required to raise power factor at 0.95 lagging. What will be the value of capacitance/phase if capacitors are (a) delta connected (b) star connected?

[12.19 kVA, 66.8 μF, 200.4 μF]

8.18 Three coils each having impedance (4 + j3) ohms are connected in star to a 440 V, three-phase, 50 Hz supply. Calculate the line current and active power. Now if three pure capacitors, each of C farads, connected in delta, are connected across the same supply, it is found that the total power factor of the circuit becomes 0.96 lag. Find the value of C. Also, find the total line current.

[50.8 A, 30.976 kW, 77.75 μF, 42.34 A]

8.19 Three identical coils each having an resistance of 8 Ω and inductance of 0.02 H are connected in (i) star, and (ii) delta across a 3ϕ, 400 V, 50 Hz supply. Draw a neat phasor diagram and calculate the reading of two wattmeters connected to measure power. Also, calculate pf of the circuit.

[8.99 kW, 3.3 kW, 0.7866 (lagging),
26.98 kW, 10.14 kW, 0.786 (lagging)]

8.20 Three identical coils each having a reactance of 20 Ω and resistance of 10 Ω are connected in (i) star, (ii) delta across a 440 V, three-phase line. Calculate for each method of connection the line current and readings of each of the two wattmeters connected.

[11.36 A, 4.17 kW, – 299.07 W, 34.08 A, 12.52 kW, – 897.14 W]

8.21 A 3-phase motor load has a pf of 0.397 lagging. Two wattmeters connected to measure power show the input as 30 kW. Find the reading on each wattmeter.

[35 kW, – 5 kW]

8.22 Each of the wattmeters connected to measure the input to a 3-phase induction motor reads 10 kW. If the power factor of the motor be changed to 0.866 lagging, determine the readings of the two wattmeters, the total input power remaining unchanged.

[6.67 kW, 13.33 kW]

8.23 A 3ϕ, star-connected load draws a line current of 25 A. The load kVA and kW are 20 and 16 respectively. Find the readings on each of the two wattmeters used to measure the 3ϕ power.

[11.46 kW, 4.54 kW]

8.24 Three similar coils are star-connected to a 3ϕ, 50 Hz supply. The line current taken is 25 A and the two wattmeters connected to measure the input indicate 5.185 kW and 10.37 kW respectively. Calculate (a) the line and phase voltages, and (b) the resistance and reactance of each coil.

[415 V, 240 V, 5.31 Ω, 4.8 Ω]

8.25 A three-phase, 500 V motor load has a power factor of 0.4. Two wattmeters connected to measure power show the input to be 30 kW. Find the reading on each instrument.

[35 kW, – 5 kW]

8.26 The power in a three-phase circuit is measured by two wattmeters. If the total power is 100 kW and power factor is 0.66 leading, what will be the reading of each wattmeter?

[17.26 kW, 82.74 kW]

8.27 Two wattmeters are connected to measure the input to a 400 V, three-phase connected motor outputting 24.4 kW at a power factor of 0.4 lag and 80% efficiency. Calculate (i) resistance and reactance of motor per phase, (ii) reading of each wattmeter.

[2.55 Ω, 5.58 Ω, 34915 W, – 4850 W]

8.28 In a balanced three-phase, 400 V circuit, the line current is 115.5 A. When power is measured by the two wattmeter method, one meter reads 40 kW and the other, zero. What is the power factor of the load? If the power factor were unity and the line current the same, what would be the reading of each wattmeter?

[0.5, 40 kW, 40 kW]

8.29 A 440 V, three-phase, delta-connected induction motor has an output of 14.92 kW at pf of 0.82 and efficiency of 85%. Calculate the readings on each of the two wattmeters connected to measure the input. If another star-connected load of 10 kW at 0.85 pf lagging is added in parallel to the motor, what will be the current drawn from the line and the power taken from the line?

[12.35 kW, 5.26 kW, 43.56 A, 27.6 kW]

8.30 Balanced delta-connected impedances, each of $10\angle 30°\ \Omega$ are connected across three-phase 400 V mains. Determine the two-wattmeter readings if the current coils of the two wattmeters are connected in lines *R* and *Y* and the pressure coils are connected between *R* and *B* and *Y* and *B* lines respectively.

[27.7 kW, 13.86 kW]

8.31 A balanced star-connected load, each phase having a resistance of 10 Ω and the inductive reactance of 30 Ω is connected to 400 V, 50 Hz supply. The phase rotation is *R*, *Y* and *B*. Wattmeters connected to read total power have their currents coils in the red and blue

lines respectively. Calculate the reading on each wattmeter.

[2190 W, – 583 W]

8.32 A balanced star-connected load is supplied from a symmetrical three-phase 400 V, 50 Hz supply system. The current in each phase is 20 A and lags behind its phase voltage by an angle of $2\pi/9$ radians. Calculate line voltage, phase voltage, current in each phase, load parameter, power in each phase, total power, readings of the wattmeters connected in the load circuit to measure the total power. Draw a neat circuit diagram and vector/phase diagram.

[440 V, 254.034 V, 20 A, 9.73 Ω, 0.026 H, 3.892 kW, 11.676 kW, 8.666 kW, 3.009 kW]

Objective-Type Questions

Choose the correct alternative in the following questions:

8.1 In a three-phase system, voltages differ in phase by
(a) 30° (b) 60°
(c) 90° (d) 120°

8.2 The rated voltage of a three-phase power system is given as
(a) rms phase voltage
(b) peak phase voltage
(c) rms line to line voltage
(d) peak line to line voltage

8.3 Total instantaneous power supplied by a three-phase ac supply to a balanced *R-C* load is
(a) zero
(b) constant
(c) pulsating with zero average
(d) pulsating with non-zero average

8.4 Which of the following equations is valid for a three-phase, balanced star-connected system?
(a) $I_R + I_Y + I_B = 0$
(b) $I_R + I_B = I_Y$
(c) $I_R + I_Y - I_B = 0$
(d) $\dfrac{V_R + V_B + V_Y}{Z} = I_N$

8.5 A three-phase load is balanced if all the three phases have the same
(a) impedance
(b) power factor
(c) impedance and power factor
(d) none of the above

8.6 The phase sequence of a three-phase system shown in Fig. 8.35 is

Fig. 8.35

(a) *RYB* (b) *RBY*
(c) *BRY* (d) *YBR*

8.7 In a 3-phase system, $\mathbf{V}_{YN} = 100\angle{-120°}$ V and $\mathbf{V}_{BN} = 100\angle 120°$ V. Then $\mathbf{V}_{YB}$ will be
(a) $170\angle 90°$ V
(b) $173\angle -90°$ V
(c) $200\angle 60°$ V
(d) none of the above.

8.8 If a balanced delta load has an impedance of $(6 + j9)$ ohms per phase then the impedance of each phase in the equivalent star load is
(a) $(6 + j9)$ ohms
(b) $(2 + j3)$ ohms
(c) $(2 + j8)$ ohms
(d) $(3 + j4.5)$ ohms

8.9 Three equal impedances are first connected in delta across a three-phase balanced supply. If the same impedances are connected in star across the same supply
(a) phase current will be one-third
(b) line current will be one-third
(c) power consumed will be one-third
(d) none of the above.

8.10 Three identical resistors connected in star carry a line current of 12 A. If the same resistors are connected in delta across the same supply, the line current will be

(a) 12 A (b) 4 A
(c) 8 A (d) 36 A

8.11 Three delta-connected resistors absorb 60 kW when connected to a 3-phase line. If the resistors are connected in star, the power absorbed is

(a) 60 kW (b) 20 kW
(c) 40 kW (d) 180 kW

8.12 The power consumed in the star-connected load shown in Fig. 8.36 is 690 W. The line current is

Fig. 8.36

(a) 2.5 A (b) 1 A
(c) 1.725 A (d) none of the above

8.13 If the three-phase balanced source in Fig. 8.37 delivers 1500 W at a leading power factor of 0.844 then the value of Z_L (in ohms) is approximately

Fig. 8.37

(a) $90\angle 32.44°$ (b) $80\angle 32.44°$
(c) $80\angle -32.44°$ (d) $90\angle -32.44°$

8.14 If one of the resistors in Fig. 8.38 is open-circuited, power consumed in the circuit is

Fig. 8.38

(a) 8000 W (b) 4000 W
(c) 16000 W (d) none of the above

8.15 For the three-phase circuit shown in Fig. 8.39, the ratio of the currents $I_R : I_Y : I_B$ is given by

Fig. 8.39

(a) $1:1:\sqrt{3}$ (b) $1:1:2$
(c) $1:1:0$ (d) $1:1:\sqrt{\frac{3}{2}}$

8.16 The phase sequence *RYB* denotes that

(a) emf of phase *Y* lags behind that of phase *R* by 120°
(b) emf of phase *Y* leads that of phase *R* by 120°
(c) emf of phase *Y* and phase *R* are in phase
(d) none of the above

8.17 In the two wattmeter method of measurement, if one of the wattmeters reads zero, then power factor will be

(a) zero (b) unity
(c) 0.5 (d) 0.866

8.18 Two wattmeters, which are connected to measure the total power on a three-phase system, supplying a balanced load, read 10.5 kW and −2.5 kW, respectively. The total power and the power factor, respectively are

(a) 13 kW, 0.334

(b) 13 kW, 0.684

(c) 8 kW, 0.52

(d) 8 kW, 0.334

8.19 The minimum number of wattmeter(s) required to measure three-phase, three-wire balanced or unbalanced power is

(a) 1 (b) 2

(c) 3 (d) 4

8.20 One of the two wattmeters has read zero in the two-wattmeter method of power measurement. This indicated that the load phase angle is

(a) 0° (b) 30°

(c) 60° (d) 90°

Answers to Objective-Type Questions

8.1 (d)	8.2 (c)	8.3 (b)	8.4 (a)	8.13 (d)	8.14 (a)	8.15 (a)	8.16 (a)
8.5 (c)	8.6 (b)	8.7 (b)	8.8 (b)	8.17 (c)	8.18 (d)	8.19 (b)	8.20 (c).
8.9 (c)	8.10 (d)	8.11 (b)	8.12 (b)				

9

Network Topology

9.1 INTRODUCTION

The purpose of network analysis is to find voltage across and current through all the elements. When the network is complicated and has a large number of nodes and closed paths, network analysis can be done conveniently by using 'Network Topology'. This theory does not make any distinction between different types of physical elements of the network but makes the study based on a geometric pattern of the network. The basic elements of this theory are nodes, branches, loops and meshes.

Node It is defined as a point at which two or more elements have a common connection.

Branch It is a line connecting a pair of nodes, the line representing a single element or series connected elements.

Loop Whenever there is more than one path between two nodes, there is a circuit or loop.

Mesh It is a loop which does not contain any other loops within it.

9.2 GRAPH OF A NETWORK

A linear graph is a collection of nodes and branches. The nodes are joined together by branches.

The graph of a network is drawn by first marking the nodes and then joining these nodes by lines which correspond to the network elements of each branch. All the voltage and current sources are replaced by their internal impedances. The voltage sources are replaced by short circuits as their internal impedances are zero whereas current sources are replaced by open circuits as their internal impedances are infinite. Nodes and branches are numbered. Figure 9.1 shows a network and its associated graphs.

Each branch of a graph may be given an orientation or a direction with the help of an arrow head which represents the assigned reference direction for current. Such a graph is then referred to as a directed or oriented graph.

Branches whose ends fall on a node are said to be incident at that node. Degree of a node is defined as the number of branches incident to it. Branches 2, 3 and 4 are incident at Node 2 in Fig. 9.1(c). Hence, the degree of Node 2 is 3.

Fig. 9.1 *Network and its graphs*

9.3 DEFINITIONS ASSOCIATED WITH A GRAPH

1. Planar Graph A graph drawn on a two-dimensional plane is said to be planar if two branches do not intersect or cross at a point which is other than a node. Figure 9.2 shows such graphs.

Fig. 9.2 *Planar graphs*

2. Non-planar Graph A graph drawn on a two-dimensional plane is said to be non-planar if there is intersection of two or more branches at another point which is not a node. Figure 9.3 shows non-planar graphs.

Fig. 9.3 *Non-planar graphs*

3. Sub-graph It is a subset of branches and nodes of a graph. It is a proper sub-graph if it contains branches and nodes less than those on a graph. A sub-graph can be just a node or only one branch of the graph. Figure 9.4 shows a graph and its proper sub-graph.

Fig. 9.4 *(a) Graph (b) Proper sub-graph*

4. *Path* It is an improper sub-graph having the following properties:

1. At two of its nodes called terminal nodes, there is incident only one branch of sub-graph.
2. At all remaining nodes called internal nodes, there are incident two branches of a graph.

In Fig. 9.5, branches 2, 5 and 6 together with all the four nodes, constitute a path.

Fig. 9.5 *Path*

5. *Connected Graph* A graph is said to be connected if there exists a path between any pair of nodes. Otherwise, the graph is disconnected.

6. *Rank of a Graph* If there are n nodes in a graph, the rank of the graph is $(n - 1)$.

7. *Loop or Circuit* A loop is a connected sub-graph of a connected graph at each node of which are incident exactly two branches. If two terminals of a path are made to coincide, it will result in a loop or circuit. Figure 9.6 shows two loops.

Fig. 9.6 *Loops*

Loops of a graph have the following properties:

1. There are at least two branches in a loop.
2. There are exactly two paths between any pair of nodes in a circuit.
3. The maximum number of possible branches is equal to the number of nodes.

8. *Tree* A tree is a set of branches with every node connected to every other node in such a way that removal of any branch destroys this property.

Alternately, a tree is defined as a connected sub-graph of a connected graph containing all the nodes of the graph but not containing any loops.

Branches of a tree are called twigs. A tree contains $(n - 1)$ twigs where n is the number of nodes in the graph. Figure 9.7 shows a graph and its trees.

Fig. 9.7 *Graph and its trees*

Trees have the following properties:

1. There exists only one path between any pair of nodes in a tree.
2. A tree contains all nodes of the graph.
3. If n is the number of nodes of the graph, there are $(n - 1)$ branches in the tree.
4. Trees do not contain any loops.
5. Every connected graph has at least one tree.
6. The minimum terminal nodes in a tree are two.

9. *Co-tree* Branches which are not on a tree are called links or chords. All links of a tree together constitute the compliment of the corresponding tree and is called the co-tree.

A co-tree contains $b - (n - 1)$ links where b is the number of branches of the graph.
In Fig. 9.7 (b) and (c) the links are {2, 3, 6} and {1, 4, 6} respectively.

Example 9.1 *Draw directed graph of the networks shown in Fig. 9.8.*

Fig. 9.8

Solution For drawing the directed graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace the voltage source by a short-circuit,
3. assume directions of branch currents, and
4. number all the nodes and branches.

The directed graph for the two networks are shown in Fig. 9.9.

Fig. 9.9

Example 9.2 *Figure 9.10 shows a graph of the network. Show all the trees of this graph.*

Fig. 9.10

Solution A graph has many trees. A tree is a connected sub-graph of a connected graph containing all the nodes of the graph but not containing any loops. Figure 9.11 shows various trees of the given graph.

Fig. 9.11

9.4 INCIDENCE MATRIX

A linear graph is made up of nodes and branches. When a graph is given, it is possible to tell which branches are incident at which nodes and what are its orientations relative to the nodes.

9.4.1 Complete Incidence Matrix (A_a)

For a graph with n nodes and b branches, the complete incidence matrix is a rectangular matrix of order $n \times b$. Elements of this matrix have the following values:

$a_{ij} = 1$, if branch j is incident at node i and is oriented away from node i.
$= -1$, if branch j is incident at node i and is oriented towards node i.
$= 0$, if branch j is not incident at node i.

For the graph shown in Fig. 9.12, branch 1 is incident at nodes 1 and 4. It is oriented away from Node 1 and oriented towards Node 4. Hence, $a_{11} = 1$ and $a_{41} = -1$. Since branch 1 is not incident at nodes 2 and 3, $a_{21} = 0$ and $a_{31} = 0$. Similarly, other elements of the complete incidence matrix are written.

Nodes ↓	Branches → 1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	−1	1	−1	0	0
3	0	0	0	1	1	−1
4	−1	0	−1	0	−1	0

Fig. 9.12 *Graph*

The complete incidence matrix is

$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

It is seen from the matrix A_a that the sum of the elements in any column is zero. Hence, any one row of the complete incidence matrix can be obtained by the algebraic manipulation of other rows.

9.4.2 Reduced Incidence Matrix (A)

The reduced incidence matrix A is obtained from the complete incidence matrix A_a by eliminating one of the rows. It is also called *incidence matrix*. It is of order $(n - 1) \times b$.

Eliminating the third row of matrix A_a,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

When a tree is selected for the graph as shown in Fig. 9.13, the incidence matrix is obtained by arranging a column such that the first $(n - 1)$ column corresponds to twigs of the tree and the last $b - (n - 1)$ branches corresponds to the links of the selected tree.

Fig. 9.13 *Tree*

$$A = \begin{array}{c} \text{Twigs} \quad\quad \text{Links} \\ \begin{array}{cccccc} 2 & 3 & 4 & 1 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

The matrix A can be subdivided into submatrices A_t and A_l.

$$A = [A_t : A_l]$$

Where A_t the is twig matrix and A_l is the link matrix.

9.4.3 Number of Possible Trees of a Graph

Let the transpose of the reduced incidence matrix A be A^T. It can be shown that the number of possible trees of a graph will be given by

$$\text{Number of possible trees} = |AA^T|$$

For the graphs shown in Fig. 9.12, the reduced incidence matrix is given by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

Then transpose of this matrix will be

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Hence, number of all possible trees of the graph

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$|AA^T| = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3(9-1) + (1)(-3-1) - 1(1+3) = 16$$

Thus, 16 different trees can be drawn.

9.5 LOOP MATRIX OR CIRCUIT MATRIX

When a graph is given, it is possible to tell which branches constitute which loop or circuit. Alternately, if a loop matrix or circuit matrix is given, we can reconstruct the graph.

For a graph having n nodes and b branches, the loop matrix B_a is a rectangular matrix of order b columns and as many rows as there are loops.

Its elements have the following values:

$b_{ij} = 1$, if branch j is in loop i and their orientations coincide.
$= -1$, if branch j is in loop i and their orientations do not coincide.
$= 0$, if branch j is not in loop i.

A graph and its loops are shown in Fig. 9.14.

Fig. 9.14 *Graph*

All the loop currents are assumed to be flowing in a clockwise direction.

Loops ↓	Branches → 1	2	3	4	5	6
1	−1	1	1	0	0	0
2	0	0	−1	−1	1	0
3	0	−1	0	1	0	1
4	−1	1	0	−1	1	0
5	−1	0	0	0	1	1
6	0	−1	−1	0	1	1
7	−1	0	1	1	0	1

$$B_a = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

9.5.1 Fundamental Circuit (Tieset) and Fundamental Circuit Matrix

When a graph is given, first select a tree and remove all the links. When a link is replaced, a closed loop or circuit is formed. Circuits formed in this way are called fundamental circuits or f-circuits or tiesets.

Orientation of an f-circuit is given by the orientation of the connecting link. The number of f-circuits is same as the number of links for a graph. In a graph having b branches and n nodes, the number of f-circuits or tiesets will be $(b - n + 1)$. Figure 9.15 shows a tree and f-circuits (tiesets) for the graph shown in Fig. 9.14.

tieset 1: {1, 2, 3}
tieset 5: {5, 3, 4}
tieset 6: {6, 2, 4}

Fig. 9.15 *Tree and f-circuits*

Here, $b = 6$ and $n = 4$.

$$\text{Number of tiesets} = b - n + 1 = 6 - 4 + 1 = 3$$

f-circuits are shown in Fig. 9.15. The orientation of each *f*-circuit is given by the orientation of the corresponding connecting link.

The branches 1, 2 and 3 are in the tieset 1. Orientation of tieset 1 is given by orientation of branch 1. Since the orientation of branch 1 coincides with orientation of tieset 1, $b_{11} = 1$. The orientations of branches 2 and 3 do not coincide with the orientation of tieset 1, Hence, $b_{12} = -1$ and $b_{13} = -1$. The branches 4, 5 and 6 are not in tieset 1. Hence, $b_{14} = 0$, $b_{15} = 0$ and $b_{16} = 0$. Similarly, other elements of the tieset matrix are written. Then, the tieset schedule will be written as

Tiesets ↓	Branches → 1	2	3	4	5	6
1	1	−1	−1	0	0	0
5	0	0	−1	−1	1	0
6	0	−1	0	1	0	1

Hence, an *f*-circuit matrix or tieset matrix will be given as

$$B = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Usually, the *f*-circuit matrix B is rearranged so that the first $(n - 1)$ columns correspond to the twigs and $b - (n - 1)$ columns to the links of the selected tree.

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Twigs} & & & \text{Links} & \\ 2 & 3 & 4 & 1 & 5 & 6 \end{array} \\ B = & \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix B can be partitioned into two matrices B_t and B_l.

$$B = [B_t : B_l] = [B_t : U]$$

where B_t is the twig matrix, B_l is the link matrix and U is the unit matrix.

9.5.2 Orthogonal Relationship between Matrix *A* and Matrix *B*

For a linear graph, if the columns of the two matrices A_a and B_a are arranged in the same order, it can be shown that

$$A_a B_a^T = 0$$

or

$$B_a A_a^T = 0$$

The above equations describe the orthogonal relationship between the matrices A_a and B_a.

If the reduced incidence matrix A and the f-circuit matrix B are written for the same tree, it can be shown that

$$A B^T = 0$$

or

$$B A^T = 0$$

These two equations show the orthogonal relationship between matrices A and B.

9.6 CUTSET MATRIX

Consider a linear graph. By removing a set of branches without affecting the nodes, two connected sub-graphs are obtained and the original graph becomes unconnected. The removal of this set of branches which results in cutting the graph into two parts are known as a *cutset*. The cutset separates the nodes of the graph into two groups, each being in one of the two groups.

Fig. 9.16 *Graph*

Figure 9.16 shows a graph.

Branches 1, 3 and 4 will form a cutset. This set of branches separates the graph into two parts. One having an isolated node 4 and other part having branches 2 and 5 and nodes 1, 2 and 3.

Similarly, branches 1 and 2 will form a cutset. Each branch of the cutset has one of its terminals incident at a node in one part and its other end incident at other nodes in the other parts. The orientation of a cutset is made to coincide with orientation of defining branch.

For a graph having n nodes and b branches, the cutset matrix Q_a is a rectangular matrix of order b columns and as many rows as there are cutsets. Its elements have the following values:

$q_{ij} = 1$, if the branch j is in the cutset i and the orientation coincide.
$= -1$, if the branch j is in the cutset i and the orientations do not coincide.
$= 0$, if the branch j is not in the cutset i.

Figure 9.17 shows a directed graph and its cutsets.

Cutset 1: {1, 2, 6}
Cutset 2: {2, 3, 4}
Cutset 3: {3, 1, 5}
Cutset 4: {4, 5, 6}
Cutset 5: {5, 2, 3, 6}
Cutset 6: {6, 1, 3, 4}

Cutsets ↓	Branches → 1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	1	−1	1	0	0
3	1	0	1	0	1	0
4	0	0	0	1	1	−1
5	0	−1	1	0	1	−1
6	1	0	1	−1	0	1

Fig. 9.17 *Directed graph*

For the cutset 2, which cuts the branches 2, 3 and 4 and is shown by a dotted circle, the entry in the cutset schedule for the branch 2 is 1, since the orientation of this cutset is given by the orientation of the branch 2 and hence it coincides. The entry for branch 3 is −1 as orientation of branch 3 is opposite to that of cutset 2,

i.e., branch 2 goes into cutset while branch 3 goes out of cutset. The entry for branch 4 is 1 as the branch 2 and the branch 4 go into the cutset. Thus their orientations coincide.
Hence, the cutset matrix Q_a is given as

$$Q_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

9.6.1 Fundamental Cutset and Fundamental Cutset Matrix

When a graph is given, first select a tree and note down its twigs. When a twig is removed from the tree, it separates a tree into two parts (one of the separated part may be an isolated node). Now, all the branches connecting one part of the disconnected tree to the other along with the twig removed, constitutes a cutset. This set of branches is called a fundamental cutset or *f*-cutset. A matrix formed by these *f*-cutsets is called an *f*-cutset matrix. The orientation of the *f*-cutset is made to coincide with the orientation of the defining branch, i.e., twig. The number of *f*-cutsets is the same as the number of twigs for a graph.

Figure 9.18 shows a graph, selected tree and *f*-cutsets corresponding to the selected tree.

f-cutset 2: {2, 1, 6}
f-cutset 3: {3, 1, 5}
f-cutset 4: {4, 5, 6}

Fig. 9.18 *Graph, selected tree and f-cutsets*

The branches 2, 1, and 6 are in the *f*-cutset 2. Orientation of *f*-cutset 2 is given by orientation of the branch 2 which is moving away from the *f*-cutset 2. Since the orientations of branches 2, 1 and 6 coincide with the orientation of the *f*-cutset 2, $q_{11} = 1$, $q_{12} = 1$ and $q_{16} = 0$. The branches 3, 4 and 5 are not in the *f*-cutset 2. Hence, $q_{13} = 0$, $q_{14} = 0$ and $q_{15} = 0$.

Similarly, other elements of the *f*-cutset matrix are written.

The cutset schedule is

f-cutsets ↓	Branches → 1	2	3	4	5	6
2	1	1	0	0	0	1
3	1	0	1	0	1	0
4	0	0	0	1	1	−1

Hence, the *f*-cutset matrix Q is given by

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

The *f*-cutset matrix Q is rearranged so that the first $(n - 1)$ columns correspond to twigs and $b - (n - 1)$ columns to links of the selected tree.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{Twigs} & & & \text{Links} & & \\ 2 & 3 & 4 & 1 & 5 & 6 \end{array} \\
Q = & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}
\end{array}
$$

The matrix Q can be subdivided into matrices Q_t and Q_l.

$$Q = [Q_t : Q_l] = [U : Q_l]$$

where Q_t is the twig matrix, Q_l is the link matrix and U is the unit matrix.

9.6.2 Orthogonal Relationship between Matrix *B* and Matrix Q

For a linear graph, if the columns of two matrices B_a and Q_a are arranged in the same order, it can be shown that

$$Q_a B_a^T = 0$$

or

$$B_a Q_a^T = 0$$

If the *f*-circuit matrix B and the *f*-cutset matrix Q are written for the same selected tree, it can be shown that

$$B Q^T = 0$$

or

$$Q B^T = 0$$

These two equations show the orthogonal relationship between matrices A and B.

9.7 RELATIONSHIP AMONG SUBMATRICES OF A, B AND Q

Arranging the columns of matrices A, B and Q with twigs for a given tree first and then the links, we get the partitioned forms as

$$A = [A_t : A_l]$$
$$B = [B_t : B_l] = [B_t : U]$$
$$Q = [Q_t : Q_l] = [U : Q_l]$$

From the orthogonal relation, $AB^T = 0$,

$$AB^T = [A_t : A_l] \begin{bmatrix} B_t^T \\ \dots \\ B_l^T \end{bmatrix}$$

$$A_t B_t^T + A_l B_l^T = 0$$

$$A_t B_t^T = -A_l B_l^T$$

Since A_t is non-singular, i.e., $|A| \neq 0$, A_t^{-1} exists.
Premultiplying with A_t^{-1},

$$B_t^T = -A_t^{-1} A_l B_l^T$$

$$B_t = -B_l (A_t^{-1} \cdot A_l)^T$$

Since B_l is a unit matrix

$$B_t = -(A_t^{-1} \cdot A_l)^T$$

Hence, matrix B is written as

$$B = [-(A_t^{-1} \cdot A_l)^T : U] \qquad \ldots(9.1)$$

We know that

$$AB^T = 0$$

$$A_l B_l^T = -A_t B_t^T$$

Postmultiplying with $(B_l^T)^{-1}$,

$$A_l = -A_t B_t^T (B_l^T)^{-1} = -A_t B_t^T (B_l^{-1})^T = -A_t (B_l^{-1} \cdot B_t)^T$$

Hence matrix A can be written as

$$\begin{aligned} A &= [A_l : -A_t (B_l^{-1} \cdot B_t)^T] \\ &= A_t [U : -(B_l^{-1} \cdot B_t)^T] \end{aligned} \qquad \ldots(9.2)$$

Similarly we can prove that

$$Q = [U : -(B_l^{-1} \cdot B_t)^T] \qquad \ldots(9.3)$$

From Eqs (9.2) and (9.3), we can write

$$A = A_t Q$$

$$Q = A_t^{-1} A = A_t^{-1} [A_t : A_l] = [U : A_t^{-1} A_l]$$

We have shown that

$$B_t = -(A_t^{-1} \cdot A_l)^T$$

$$B_t^T = -(A_t^{-1} \cdot A_l)$$

Hence, Q can be written as

$$Q = [U : -B_t^T]$$

$$Q_1 = -B_t^T$$

Example 9.3 *For the circuit shown in Fig. 9.19, draw the oriented graph and write the (a) incidence matrix, (b) tieset matrix, and (c) f-cutset matrix.*

Fig. 9.19

Solution For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace the voltage source by short circuit and the current source by an open circuit,
3. assume the directions of branch currents arbitrarily, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 9.20.

Fig. 9.20

(a) Incidence Matrix (A)

Nodes ↓	Branches → 1	2	3	4
1	−1	0	−1	1
2	0	1	1	−1
3	1	−1	0	0

$$A_a = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Eliminating the third row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

(b) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 9.21.

Links: {3, 4}
Tieset 3: {3, 1, 2}
Tieset 4: {4, 1, 2}

$$B = \begin{array}{c} \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 9.21

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.22.

Twigs: {1, 2}
f-cutset 1: {1, 3, 4}
f-cutset 2: {2, 3, 4}

$$Q = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

Fig. 9.22

Example 9.4 *For the network shown in Fig. 9.23, draw the oriented graph and write the (a) incidence matrix, (b) tieset matrix, and (c) f-cutset matrix.*

Fig. 9.23

Solution For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace all voltage sources by short circuits and current source by an open circuit,
3. assume directions of branch currents arbitrarily, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 9.24.

Fig. 9.24

(a) Incidence Matrix (A)

Nodes ↓	Branches → 1	2	3	4	5	6	7
1	1	0	0	1	0	1	0
2	−1	−1	1	0	0	0	0
3	0	1	0	0	1	0	1
4	0	0	−1	−1	−1	0	0
5	0	0	0	0	0	−1	−1

$$A_a = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Eliminating the last row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

(b) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 9.25.

Links: {2, 4, 7}
Tieset 2: {2, 3, 5}
Tieset 4: {4, 1, 3}
Tieset 7: {7, 6, 1, 3, 5}

$$B = \begin{array}{c} \\ 2 \\ 4 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.25

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.26.

Twigs: {1, 3, 5, 6}
f-cutset 1: {1, 4, 7}
f-cutset 3: {3, 4, 7, 2}
f-cutset 5: {5, 2, 7}
f-cutset 6: {6, 7}

$$Q = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.26

Example 9.5 *For the circuit shown in Fig. 9.27, draw the oriented graph and write (a) incidence matrix, (b) tieset matrix, and (c) cutset matrix.*

Fig. 9.27

Solution For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace voltage source by short circuit and current source by an open circuit,
3. assume directions of branch currents arbitrarily, and
4. number the nodes and branches.

Fig. 9.28

The oriented graph is shown in Fig. 9.28.

(a) Incidence Matrix (A)

Nodes ↓	Branches → 1	2	3	4
1	−1	1	0	−1
2	0	0	1	1
3	1	−1	−1	0

$$A_a = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

Eliminating the third row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(b) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 9.29.

Links: {1, 3}
Tieset 1: {1, 2}
Tieset 3: {3, 2, 4}

$$B = \begin{array}{c} \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

Fig. 9.29

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.30.

Twigs: {2, 4}
f-cutset 2: {2, 1, 3}
f-cutset 4: {4, 3}

$$Q = \begin{array}{c} \\ 2 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.30

Example 9.6 *For the circuit shown in Fig. 9.31, (a) draw its graph, (b) draw its tree, and (c) write the fundamental cutset matrix.*

Fig. 9.31

Fig. 9.32

Solution

(a) For drawing the oriented graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace the current source by an open circuit,
3. assume directions of branch currents, and
4. number all the nodes and branches.

The oriented graph is shown in Fig. 9.32.

(b) Tree

The oriented graph and its selected tree are shown in Fig. 9.33.

Twigs: {2, 4, 5}
f-cutset 2: {2, 1, 3}
f-cutset 4: {4, 3, 6}
f-cutset 5: {5, 1, 6}

Fig. 9.33

(c) Fundamental Cutset Matrix (Q)

$$Q = \begin{array}{c} \\ 2 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 9.7 *The graph of a network is shown in Fig. 9.34. Write the (a) incidence matrix, (b) tieset matrix, and (c) f-cutset matrix.*

Fig. 9.34

Solution

(a) Incidence Matrix (A)

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

The incidence matrix A is obtained by eliminating any row from the matrix A_a.

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

(b) Tieset, Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 9.35.

Links: {1, 2, 3}
Tieset 1: {1, 4, 6}
Tieset 2: {2, 4, 5}
Tieset 3: {3, 5, 6}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

Fig. 9.35

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.36.

Twigs: {4, 5, 6}
f-cutset 4: {4, 1, 2}
f-cutset 5: {5, 2, 3}
f-cutset 6: {6, 1, 3}

$$Q = \begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 9.36

Example 9.8 *For the graph shown in Fig. 9.37, write the incidence matrix, tieset matrix and f-cutset matrix.*

Fig. 9.37

Solution

(a) Incidence Matrix (A)

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

The incidence matrix A is obtained by eliminating any row from the matrix A_a.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 \end{bmatrix}$$

(b) Tieset Matrix (B)

Links: {5, 6, 7}
Tieset 5: {5, 3, 4}
Tieset 6: {6, 1, 2, 3, 4}
Tieset 7: {7, 1, 2, 3, 4}

$$B = \begin{array}{c} \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.38.

Twigs: {1, 2, 3, 4}
f-cutset 1: {1, 6, 7}
f-cutset 2: {2, 6, 7}
f-cutset 3: {3, 5, 6, 7}
f-cutset 4: {4, 5, 6, 7}

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.38

Example 9.9 *For the graph shown in Fig. 9.39, write the incidence matrix, tieset matrix and f-cutset matrix.*

Fig. 9.39

Solution

(a) Incidence Matrix (A)

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

The incidence matrix is obtained by eliminating any one row.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Tieset Matrix (B)

Links: {2, 4, 6, 8}
Tieset 2: {2, 7, 5, 1}
Tieset 4: {4, 5, 7, 3}
Tieset 6: {6, 5, 1}
Tieset 8: {8, 7, 3}

$$B = \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

(c) f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.40.

Fig. 9.40

Twigs: {1, 3, 5, 7}
f-cutset 1: {1, 6, 2}
f-cutset 3: {3, 4, 8}
f-cutset 5: {5, 4, 6, 2}
f-cutset 7: {7, 2, 8, 4}

$$Q = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 9.10 *How many trees are possible for the graph of the network of Fig. 9.41.*

Fig. 9.41

Solution To draw the graph,

1. replace all resistors, inductors and capacitors by line segments,
2. replace voltage source by short circuit and current source by an open circuit,
3. assume directions of branch currents arbitrarily, and
4. number all the nodes and branches.

Fig. 9.42

The oriented graph is shown in Fig. 9.42.
The complete Incidence Matrix (A_a) is written as

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

The reduced incidence matrix A is obtained by eliminating the last row from matrix A_a.

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

The number of possible trees $= \left|AA^T\right| = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 6 - 1 = 5.$

Example 9.11 *Draw the oriented graph from the complete incidence matrix given below;*

Nodes ↓	Branches → 1	2	3	4	5	6	7	8
1	1	0	0	0	1	0	0	1
2	0	1	0	0	−1	1	0	0
3	0	0	1	0	0	−1	1	−1
4	0	0	0	1	0	0	−1	0
5	−1	−1	−1	−1	0	0	0	0

Fig. 9.43

Solution First, note down the nodes 1, 2, 3, 4, 5 as shown in Fig. 9.43. From the complete incidence matrix, it is clear that the branch number 1 is between nodes 1 and 5 and it is going away from node 1 and towards node 5 as the entry against node 1 is 1 and that against 5 is –1. Hence, connect the nodes 1 and 5 by a line, point the arrow towards 5 and call it branch 1 as shown in Fig. 9.43. Similarly, draw the other oriented branches.

Example 9.12 *The reduced incidence matrix of an oriented graph is given below. Draw the graph.*

$$A = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution First, writing the complete incidence matrix from the matrix A such that the sum of all entries in each column of A_a will be zero, we have

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 9.44

Now, the oriented graph can be drawn with matrix A_a as shown in Fig. 9.44.

Example 9.13 *The reduced incidence matrix of an oriented graph is*

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Draw the graph. (b) How many trees are possible for this graph? (c) Write the tieset and cutset matrices.

Solution

(a) First, writing the complete incidence matrix A_a such that the sum of all the entries in each column of A_a is zero, we have

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Fig. 9.45

Now, the oriented graph can be drawn with the matrix A_a, as shown in Fig. 9.45.

(b) The number of possible trees = $|AA^T|$

$$AA^T = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left|AA^T\right| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(6-1)+1(-2) = 8$$

The number of possible trees = 8.

(c) Tieset Matrix (B)

The oriented graph, selected tree and tiesets are shown in Fig. 9.46.

Links: {1, 2}
Tieset 1: {1, 4, 5}
Tieset 2: {2, 3, 4}

$$B = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 9.46

f-cutset Matrix (Q)

The oriented graph, selected tree and f-cutsets are shown in Fig. 9.47

Twigs: {3, 4, 5}
f-cutset 3: {3, 2}
f-cutset 4: {4, 2, 1}
f-cutset 5: {5, 1}

$$Q = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 9.47

Example 9.14 *The fundamental cutset matrix of a network is given as follows;*

Twigs			*Links*		
a	*c*	*e*	*b*	*d*	*f*
1	0	0	1	0	1
0	1	0	0	1	1
0	0	1	1	1	1

Draw the oriented graph.

Solution No. of links $l = b - n + 1$

No. of nodes $n = b - l + 1 = 6 - 3 + 1 = 4$

f-cutsets are written as,

f-cutsets a: $\{a, b, f\}$

f-cutsets c: $\{c, d, f\}$

f-cutsets e: $\{e, b, d, f\}$

The oriented graph is drawn as shown in Fig. 9.48.

Twigs: {a, c, e}
Links: {b, d, f}

Fig. 9.48

Example 9.15 *Draw the oriented graph of a network with the f-cutset matrix as shown below:*

Twigs				*Links*		
1	2	3	4	5	6	7
1	0	0	0	–1	0	0
0	1	0	0	1	0	1
0	0	1	0	0	1	1
0	0	0	1	0	1	0

Solution No. of links $l = b - n + 1$

No. of nodes $n = b - l + 1 = 7 - 3 + 1 = 5$

f-cutsets are written as

f-cutset 1: {1, 5}

f-cutset 2: {2, 5, 7}

f-cutset 3: {3, 6, 7}

f-cutset 4: {4, 6}

Then oriented graph can be drawn as shown in Fig. 9.49.

Fig. 9.49

9.8 KIRCHHOFF'S VOLTAGE LAW

KVL states that if v_k is the voltage drop across the kth branch, then

$$\sum v_k = 0$$

the sum being taken over all the branches in a given loop. If l is the number of loops or *f*-circuits, then there will be l number of KVL equations, one for each loop. The KVL equation for the *f*-circuit or loop 'l' can be written as

$$\sum_{k=1}^{b} b_{ik}\, v_k = 0 \qquad (k = 1, 2, \ldots, l)$$

where b_{ik} is the elements of the tieset matrix B, b being the number of branches. The set of l KVL equations can be written in matrix form.

$$BV_b = 0$$

where $V_b = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix}$ is a column vector of branch voltages.

and B is the fundamental circuit matrix.

9.9 KIRCHHOFF'S CURRENT LAW

KCL states that if i_k is the current in the kth branch then at a given node

$$\sum i_k = 0$$

the sum being taken over all the branches incident at a given node. If there are 'n' nodes, there will 'n' such equations, one for each node

$$\sum_{k=1}^{b} a_{ik}\, i_k = 0 \qquad (k = 1, 2, \ldots, n)$$

so that set of n equations can be written in matrix form.

$$A_a\, I_b = 0 \qquad \ldots(9.4)$$

where $I_b = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix}$ is a column vector of branch currents.

and A_a is the complete incidence matrix.
If one node is taken as reference node or datum node, we can write the Eq. (9.4) as,

$$AI_b = 0 \qquad \ldots(9.5)$$

where A is the incidence matrix of order $(n-1) \times b$.

We know that $A = A_t\, Q$

Equation (9.5) can be written as

$$A_t\, QI_b = 0$$

Premultiplying with A_t^{-1},

$$A_t^{-1} A_t\, Q\, I_b = A_t^{-1} \cdot 0$$

$$I\, Q\, I_b = 0$$

$$Q\, I_b = 0$$

where Q is the f-cutset matrix.

9.10 RELATION BETWEEN BRANCH VOLTAGE MATRIX V_b, TWIG VOLTAGE MATRIX V_t AND NODE VOLTAGE MATRIX V_n

We know that $BV_b = 0$

$$[B_t : B_l] \begin{bmatrix} V_t \\ \cdots \\ V_l \end{bmatrix} = 0$$

$$B_t\, V_t + B_l\, V_l = 0$$

$$B_l\, V_l = -B_t\, V_t$$

Premultiplying with B_l^{-1}.

$$V_l = -B_l^{-1} B_t\, V_t = -(B_l^{-1} B_t) V_t$$

Now

$$V_b = \begin{bmatrix} V_t \\ \cdots \\ V_l \end{bmatrix}$$

$$= \begin{bmatrix} V_t \\ \cdots \\ -\left(B_l^{-1} B_t\right) V_t \end{bmatrix} = \begin{bmatrix} U \\ \cdots \\ -\left(B_l^{-1} B_t\right) \end{bmatrix} \cdot V_t \qquad \ldots(9.6)$$

Also,
$$V_b = Q^T V_t$$
$$Q = A_t^{-1} A$$
$$Q^T = A^T (A_t^{-1})^T = A^T (A_t^T)^{-1}$$

Hence Eq. (9.6) can be written as
$$V_b = A^T (A_t^T)^{-1} V_t = A^T \left\{ (A_t^T)^{-1} V_t \right\} = A^T V_n$$

where $V_n = (A_t^T)^{-1} V_t$ is node voltage matrix.

9.11 RELATION BETWEEN BRANCH CURRENT MATRIX I_b AND LOOP CURRENT MATRIX I_l

We know that, $A\, I_b = 0$
$$[A_t : A_t] \begin{bmatrix} I_t \\ \dots \\ I_l \end{bmatrix} = 0$$
$$A_t\, I_t + A_l\, I_l = 0$$
$$A_t\, I_t = -A_l\, I_l$$

Premultiplying with A_t^{-1},
$$I_t = -A_t^{-1} A_l\, I_l = -(A_t^{-1} A_l)\, I_l$$
$$I_b = \begin{bmatrix} I_t \\ \dots \\ I_l \end{bmatrix} = \begin{bmatrix} -(A_t^{-1} A_l)\, I_l \\ \dots \\ I_l \end{bmatrix} = \begin{bmatrix} -(A_t^{-1} A_l) \\ \dots \\ U \end{bmatrix} \cdot I_l$$

Now
$$I_b = B^T I_l$$

9.12 NETWORK EQUILIBRIUM EQUATION

9.12.1 KVL Equation

1. If there is a voltage source v_{sk} in the branch k having impedance z_k and carrying current i_k, as shown in Fig. 9.50,
$$v_k = z_k\, i_k - v_{sk} \qquad (k = 1, 2, \dots, b)$$
In matrix form,
$$V_b = Z_b\, I_b - V_s$$

Fig. 9.50 *Circuit diagram*

where Z_b is the branch impedance matrix, I_b is the column vector of branch currents and V_s is the column vector of source voltages. Hence, KVL equation can be written as
$$B V_b = 0$$
$$B(Z_b\, I_b - V_s) = 0$$
$$B\, Z_b\, I_b = B V_s$$

Also,
$$I_b = B^T I_l$$
$$B Z_b B^T I_l = B V_s$$
$$Z I_l = E$$
where
$$E = B V_s$$
and
$$Z = B Z_b B^T$$
The matrix Z is called *loop impedance matrix*.

2. If there is a voltage source in series with an impedance and a current source in parallel with the combination as shown in Fig. 9.51,

Fig. 9.51 *Circuit diagram*

$$i_k = \frac{(v_k + v_{sk})}{zk} - i_{sk}$$
$$v_k = z_k\, i_k + z_k\, i_{sk} - v_{sk}$$
In matrix form,
$$V_b = Z_b I_b + Z_b I_s - V_s$$
KVL equation is $B V_b = 0$.
$$B V_b = B(Z_b I_b + Z_b I_s - V_s) = 0$$
$$B Z_b I_b = B V_s - B Z_b I_s$$
Now
$$I_b = B^T I_l$$
$$B Z_b B^T I_l = B V_s - B Z_b I_s$$
$$Z I_l = B V_s - B Z_b I_s$$
where $Z = B Z_b B^T$ is the loop impedance matrix. This is the generalised KVL equation.

9.12.2 KCL Equation

1. If the branch k contains an input current source i_{sk} and an admittance y_k as shown in Fig. 9.52,

Fig. 9.52 *Circuit diagram*

$$i_k = y_k\, v_k - i_{sk} \qquad (k = 1, 2, \ldots, b)$$
In the matrix form,
$$I_b = Y_b V_b - I_s$$
where Y_b is the branch admittance matrix.
Hence KCL equation is given by,
$$A I_b = 0$$
$$A(Y_b V_b - I_s) = 0$$
$$A Y_b V_b = A I_s$$
Also
$$V_b = A^T V_n$$
$$A Y_b A^T V_n = A I_s$$
$$Y V_n = I$$
where
$$Y = A Y_b A^T$$
and
$$I = A I_s$$

The matrix Y is called admittance matrix. This is the KCL equation in matrix form.
In terms of *f*-cutset matrix, the KCL equation can be written as

$$Q\,I_b = 0$$
$$Q\,(Y_b\,V_b - V_s) = 0$$
$$Q\,Y_b\,V_b = Q\,I_s$$

Also
$$V_b = Q^T\,V_t$$

$$Q\,Y_b\,Q^T\,V_t = Q\,I_s$$
$$Y\,V_t = I$$

where
$$Y = Q\,Y_b\,Q^T$$

and
$$I = Q\,I_s$$

This is the KCL equation in matrix form.

2. If there is a voltage source in series with an impedance and a current source in parallel with the combination as shown in Fig. 9.53,

Fig. 9.53 *Circuit diagram*

$$y_k = \frac{1}{z_k}$$
$$i_k = y_k\,v_k + y_k\,v_{sk} - i_{sk}$$

In matrix form,

$$I_b = Y_b\,V_b + Y_b\,V_s - I_s$$

KCL equation will be given by,

$$A\,I_b = 0$$
$$A\,(Y_b\,V_b + Y_b\,V_s - I_s) = 0$$
$$A\,Y_b\,V_b = A\,I_s - A\,Y_b\,V_s$$

Also
$$V_b = A^T\,V_n$$

$$A\,Y_b\,A^T\,V_n = A\,I_s - A\,Y_b\,V_s$$
$$Y\,V_n = A\,I_s - A\,V_b\,V_s$$

where $Y = A\,Y_b\,A^T$ is the node admittance matrix. This is a generalised KCL equation.
In terms of *f*-cutset matrix, the KCL equation can be written as

$$Q\,I_b = 0$$
$$Q\,(Y_b\,V_b + Y_b\,V_s - I_s) = 0$$
$$Q\,Y_b\,V_b = Q\,I_s - Q\,Y_b\,V_s$$

Also
$$V_b = Q^T\,V_t$$

$$Q\,Y_b\,Q^T\,V_t = Q\,I_s - Q\,Y_b\,V_s$$
$$Y\,V_t = Q\,I_s - Q\,Y_b\,V_s$$

This is a generalised KCL equation.

Note

(i) For a graph having b branches, the branch impedance matrix Z_b is a square matrix of order b, having branch impedances as diagonal elements and the mutual impedances between the branches as non-diagonal elements. For a network having no mutual impedances, only diagonal elements will be present in the branch impedance matrix.

(ii) For a graph having b branches, the branch admittance matrix Y_b is a square matrix of order b, having branch admittances as diagonal elements and the mutual admittances between the branches as non-diagonal elements. For a network having no mutual admittances, only diagonal elements will be present in the branch admittance matrix.

(iii) For a graph having b branches, the voltage source matrix or vector V_s is a rectangular matrix of order $b \times 1$, having the value of the voltage source in the particular branch. The value will be positive if there is a voltage rise in the direction of current and will be negative if there is a voltage fall in the direction of current.

(iv) For a graph having b branches, the current source matrix or vector I_s is a rectangular matrix of order $b \times 1$, having the value of the current source in the particular parallel branch. The value will be positive if the direction of the current source and the corresponding parallel branch current are not same. The value will be negative if the directions of the current source and corresponding parallel branch current are same.

Example 9.16 *Write the incidence matrix of the graph of Fig. 9.54 and express branch voltages in terms of node voltages. Write the tieset matrix and express branch currents in terms of loop currents.*

Fig. 9.54

Solution

(a) Incidence Matrix

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The incidence matrix is obtained by eliminating any one row.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

(b) Branch voltages in terms of node voltages

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \\ V_{n_3} \\ V_{n_4} \end{bmatrix}$$

(c) Tieset Matrix

Selected tree and tiesets are shown in Fig. 9.55.

Links: {1, 4, 6, 8}
Tieset 1: {1, 2, 5}
Tieset 4: {4, 3, 7}
Tieset 6: {6, 2, 3}
Tieset 8: {8, 2, 3, 5}

$$B = \begin{array}{c} \\ 1 \\ 4 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 9.55

(d) Branch currents in terms of loop currents

$$I_b = B^T I_l$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_4} \\ I_{l_6} \\ I_{l_8} \end{bmatrix}$$

Example 9.17 *Branch current and loop-current relationships are expressed in matrix form as*

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \\ I_{l_4} \end{bmatrix}$$

Draw the oriented graph.

Solution Writing the equation in matrix form,

$$I_b = B^T I_l$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \therefore B = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 9.56

From tieset matrix,

No. of links $l = 4$

No of branches $b = 8$

No of nodes $n = b - l + 1 = 8 - 4 + 1 = 1$

The oriented graph is shown in Fig. 9.56.

Example 9.18 *For the given graph shown in Fig. 9.57, write down the basic tieset matrix and taking a tree of branches 2, 4, 5, write down KVL equations from the matrix.*

Fig. 9.57

Solution Selecting branches 2, 4, and 5 as the tree as shown in Fig. 9.58,

Fig. 9.58

Links: {1, 3, 6}
Tieset 1: {1, 5, 4}
Tieset 3: {3, 2, 5}
Tieset 6: {6, 2, 5, 4}

$$B = \begin{array}{c} \\ 1 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

The KVL equation in matrix form is given by

$$BV_b = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = 0$$

$$V_1 - V_4 + V_5 = 0$$

$$V_2 + V_3 + V_5 = 0$$

$$V_2 - V_4 + V_5 + V_6 = 0$$

Example 9.19 *Obtain the f-cutset matrix for the graph shown in Fig. 9.59 taking 1, 2, 3, 4 as tree branches. Write down the network equations from the f-cutset matrix.*

Fig. 9.59

Solution Twigs: {1, 2, 3, 4}
f-cutset 1 : {1, 6, 7, 8}
f-cutset 2 : {2, 5, 6, 7, 8}
f-cutset 3 : {3, 5, 6}
f-cutset 4 : {4, 6, 7}

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

The KCL equation in matrix form is given by

$$QI_b = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix} = 0$$

$$I_1 + I_6 - I_7 + I_8 = 0$$
$$I_2 + I_5 - I_6 + I_7 - I_8 = 0$$
$$I_3 + I_5 - I_6 = 0$$
$$I_4 + I_6 - I_7 = 0$$

Example 9.20 *The reduced incidence matrix of a graph is given as*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Express branch voltages in terms of node voltages.

Solution For the given graph,
No of branches $b = 5$
No of nodes $n = 3$

Branch voltages can be expressed in terms of node voltages by

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \\ V_{n_3} \end{bmatrix}$$

$$V_1 = V_{n_1} - V_{n_2}$$
$$V_2 = -V_{n_2}$$
$$V_3 = -V_{n_2} + V_{n_3}$$
$$V_4 = -V_{n_3}$$
$$V_5 = -V_{n_1}$$

Example 9.21 *The fundamental cutset matrix of a graph is given as*

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Express branch voltages in terms of twig voltages.

Solution For the given graph,
No. of branches $b = 5$
No. of twigs = 3

Branch voltages are expressed in terms of twig voltages by

$$V_b = Q^T V_t$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_{t_1} \\ V_{t_2} \\ V_{t_3} \end{bmatrix}$$

$$V_1 = -V_{t_1} + V_{t_3}$$
$$V_2 = V_{t_1}$$
$$V_3 = V_{t_2}$$
$$V_4 = V_{t_3}$$
$$V_5 = -V_{t_1} - V_{t_2}$$

Example 9.22 *For this network shown in Fig. 9.60, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL. Calculate the loop currents and branch currents.*

Fig. 9.60

Solution The oriented graph and one of its trees are shown in Fig. 9.61.

Links: {1, 2, 3}
Tieset 1: {1, 4, 5}
Tieset 2: {2, 4, 6}
Tieset 3: {3, 5, 6}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.61

The KVL equation in matrix form is given by

$$B Z_b B^T I_l = B V_s - B Z_b I_s$$

Here, $I_s = 0,$

$$B Z_b B^T I_l = B V_s$$

$$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix}$$

$$B\,Z_b\,B^T = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

$$B\,V_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

The KVL equation in matrix form is given by

$$\begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Solving this matrix equation,

$$I_{l_1} = \frac{6}{7}\text{A}$$

$$I_{l_2} = \frac{4}{7}\text{A}$$

$$I_{l_3} = \frac{4}{7}\text{A}$$

The branch currents are given by

$$I_b = B^T\,I_l$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{6}{7} \\ \frac{4}{7} \\ \frac{4}{7} \end{bmatrix} = \begin{bmatrix} \frac{6}{7} \\ \frac{4}{7} \\ \frac{4}{7} \\ \frac{2}{7} \\ \frac{2}{7} \\ 0 \end{bmatrix}$$

Example 9.23 *For the network shown in Fig. 9.62, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL. Calculate loop currents.*

Fig. 9.62

Solution The oriented graph and its selected tree are shown in Fig. 9.63.

Links: {1, 2, 3}
Tieset 1: {1, 4, 6}
Tieset 2: {2, 5, 6}
Tieset 3: {3, 5, 4}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 9.63

The KVL equation in matrix form is given by

$$B Z_b B^T I_l = B V_s - B Z_b I_s$$

Here, $I_s = 0,$

$$B Z_b B^T I_l = B V_s$$

$$Z_b = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}; V_s = \begin{bmatrix} 12 \\ -6 \\ -8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 0 & 6 & -2 \\ 0 & 0 & 2 & -4 & -6 & 0 \end{bmatrix}$$

$$B Z_b B^T = \begin{bmatrix} 6 & 0 & 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 0 & 6 & -2 \\ 0 & 0 & 2 & -4 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & -2 & -4 \\ -1 & 12 & -6 \\ -4 & -6 & 12 \end{bmatrix}$$

$$BV_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ -6 \\ -8 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ -8 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} 12 & -2 & -4 \\ -2 & 12 & -6 \\ -4 & -6 & 12 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ -8 \end{bmatrix}$$

Solving this matrix equation,

$$I_{l_1} = 0.55\text{A}$$
$$I_{l_2} = -0.866\text{A}$$
$$I_{l_3} = -0.916\text{A}$$

Example 9.24 *For the network shown in Fig. 9.64, draw the oriented graph. Write the tieset schedule and hence obtain the equilibrium equation on loop basis. Calculate the values of branch currents and branch voltages.*

Fig. 9.64

Solution The oriented graph and one of its trees are shown in Fig. 9.65.

Links: {1, 2, 3}
Tieset 1: {1, 4, 5, 6}
Tieset 2: {2, 4, 5}
Tieset 3: {3, 4}

Fig. 9.65

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = BV_s - B\,Z_b\,I_s$$

$$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}; V_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; I_s = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$BZ_b\,B^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BV_s = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$B\,Z_b\,I_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solving this matrix equation,

$$I_{l_1} = \frac{1}{5}\text{A}$$

$$I_{l_2} = -\frac{3}{5}\text{A}$$

$$I_{l_3} = 1\text{A}$$

The branch currents are given by

$$I_b = B^T I_l + I_s$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ \frac{7}{5} \\ -\frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

The branch voltages are given by

$$V_b = Z_b I_b - V_s$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ \frac{7}{5} \\ -\frac{2}{5} \\ -\frac{1}{5} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ -1 \\ -\frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

Example 9.25 *For the network shown in Fig. 9.66, obtain the loop equation in matrix form.*

Fig. 9.66

Solution The oriented graph and one of its trees are shown in Fig. 9.67.

Links: {1, 4}
Tieset 1: {1, 2, 3}
Tieset 4: {4, 2, 3}

$$B = \begin{array}{c} \\ 1 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.67

The KVL equation in matrix form is given by

$$B Z_b B^T I_l = B V_s - B Z_b I_s$$

$$Z_b = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}; V_s = \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \end{bmatrix}; I_s = \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}$$

$$B Z_b = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & R_2 & R_3 & R_4 \end{bmatrix}$$

$$B Z_b B^T = \begin{bmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & R_2 & R_3 & R_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 + R_3 & R_2 + R_3 \\ R_2 + R_3 & R_2 + R_3 + R_4 \end{bmatrix}$$

$$B V_s = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

$$B Z_b I_s = \begin{bmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & R_2 & R_3 & R_4 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_2 I \\ R_2 I \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} R_1 + R_2 + R_3 & R_2 + R_3 \\ R_2 + R_3 & R_2 + R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_{l_1} \\ I_{l_4} \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} - \begin{bmatrix} R_2 & I \\ R_2 & I \end{bmatrix} = \begin{bmatrix} V - R_2 I \\ -R_2 I \end{bmatrix}$$

Example 9.26 *For the network shown in Fig. 9.68, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL.*

Fig. 9.68

Solution The oriented graph and its selected tree are shown in Fig. 9.69.

Links: {1, 2, 3}
Tieset 1: {1, 4, 6}
Tieset 2: {2, 5, 6}
Tieset 3: {3, 5, 4}

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 9.69

The KVL equation in matrix form is given by

$$B Z_b B^T I_l = B V_s - B Z_b I_s$$

$$Z_b = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & j5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j4 \end{bmatrix}; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}; V_s = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; I_s = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & j5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 5 & 0 & -j4 \\ 0 & 2 & 0 & 0 & j5 & j4 \\ 0 & 0 & 5 & -5 & -j5 & 0 \end{bmatrix}$$

$$B Z_b B^T = \begin{bmatrix} 2 & 0 & 0 & 5 & 0 & -j4 \\ 0 & 2 & 0 & 0 & j5 & j4 \\ 0 & 0 & 5 & -5 & -j5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 7 - j4 & j4 & -5 \\ j4 & 2 + j1 & -j5 \\ -5 & -j5 & 10 + j5 \end{bmatrix}$$

$$B V_s = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$B Z_b I_s = \begin{bmatrix} 2 & 0 & 0 & 5 & 0 & -j4 \\ 0 & 2 & 0 & 0 & j5 & j4 \\ 0 & 0 & 5 & -5 & -j5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

Hence, the KCL equation in matrix form is given by

$$\begin{bmatrix} 7-j4 & j4 & -5 \\ j4 & 2+j1 & -j5 \\ -5 & -j5 & 10+j5 \end{bmatrix}\begin{bmatrix} I_{l_1} \\ I_{l_2} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}$$

Example 9.27 *For the network shown in Fig. 9.70, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL.*

Fig. 9.70

Solution The branch currents are so chosen that they assume the direction out of the dotted terminals. Because of this choice of current direction, the mutual inductance is positive. The oriented graph and its selected tree are shown in Fig. 9.71.

Links: {1, 3}
Tieset 1: {1, 2}
Tieset 3: {3, 2}

$$B = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.71

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = BV_s - B\,Z_b\,I_s$$

Here, $I_s = 0,$

$$B\,Z_b\,B^T\,I_l = BV_s$$

$$Z_b = \begin{bmatrix} j5 & 0 & j5.66 \\ 0 & 3-j4 & 0 \\ j5.66 & 0 & 5+j10 \end{bmatrix};\ B^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix};\ V_s = \begin{bmatrix} 50\angle 0^\circ \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} j5 & 0 & j5.66 \\ 0 & 3-j4 & 0 \\ j5.66 & 0 & 5+j10 \end{bmatrix} = \begin{bmatrix} j5 & 3-j4 & j5.66 \\ j5.66 & -3+j4 & 5+j10 \end{bmatrix}$$

$$B\,Z_b B^T = \begin{bmatrix} j5 & 3-j4 & j5.66 \\ j5.66 & -3+j4 & 5+j10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3+j1 & -3+j9.66 \\ -3+j9.66 & 8+j6 \end{bmatrix}$$

$$BV_s = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 50\angle 0^\circ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by

$$\begin{bmatrix} 3+j1 & -3+j9.66 \\ -3+j9.66 & 8.j6 \end{bmatrix}\begin{bmatrix} I_{l_1} \\ I_{l_2} \end{bmatrix} = \begin{bmatrix} 50\angle 0^\circ \\ 0 \end{bmatrix}$$

Example 9.28 *For the network shown in Fig. 9.72, write down the tieset matrix and obtain the network equilibrium equation in matrix form using KVL.*

Fig. 9.72

Solution The branch currents are so chosen that they assume the direction out of the dotted terminals. Because of this choice of current direction, the mutual inductance is positive. The oriented graph and its selected tree are shown in Fig. 9.73.

Links: {1, 3}
Tieset 1: {1, 2}
Tieset 3: {3, 2}

$$B = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.73

The KVL equation in matrix form is given by

$$B\,Z_b\,B^T\,I_l = B\,V_s - B\,Z_b\,I_s$$

Here, $I_s = 0,$

$$B\,Z_b\,B^T\,I_l = B\,V_s$$

$$Z_b = \begin{bmatrix} 3+j4 & j3 & 0 \\ j3 & j5 & 0 \\ 0 & 0 & -j8 \end{bmatrix};\ B^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix};\ V_s = \begin{bmatrix} 50\angle 45^\circ \\ 0 \\ 0 \end{bmatrix}$$

$$B\,Z_b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3+j4 & j3 & 0 \\ j3 & j5 & 0 \\ 0 & 0 & -j8 \end{bmatrix} = \begin{bmatrix} 3+j7 & j8 & 0 \\ -j3 & -j5 & -j8 \end{bmatrix}$$

$$B\,Z_b\,B^T = \begin{bmatrix} 3+j7 & j8 & 0 \\ -j3 & -j5 & -j8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}$$

$$B\,V_s = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 50\angle 45^\circ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 50\angle 45^\circ \\ 0 \end{bmatrix}$$

Hence, the KVL equation in matrix form is given by,

$$\begin{bmatrix} 3+j15 & -j8 \\ -j8 & -j3 \end{bmatrix}\begin{bmatrix} I_{l_1} \\ I_{l_3} \end{bmatrix} = \begin{bmatrix} 50\angle 45^\circ \\ 0 \end{bmatrix}$$

Example 9.29 *For the network shown in Fig. 9.74, obtain branch voltages using KCL equation on node basis.*

Fig. 9.74

Solution The oriented graph is shown in Fig. 9.75.

Fig. 9.75

The complete incidence matrix for the graph is

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Eliminating the last row from the matrix A_a, we get the incidence matrix A.

$$A = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

The KCL equation in matrix form is given by

$$AY_b A^T V_n = AI_s - AY_b V_s$$

$$I_s = 0\,,$$

Here,
$$AY_b A^T V_n = -AY_b V_s$$

$$Y_b = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}; A^T = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 0 \\ 12 \\ 0 \\ 24 \\ 0 \end{bmatrix}$$

$$AY_b = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$AY_b A^T = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{17}{24} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{23}{40} \end{bmatrix}$$

$$AY_b V_s = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 12 \\ 0 \\ 24 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Hence, the KCL equation in matrix form is given by

$$\begin{bmatrix} \frac{17}{24} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{23}{40} \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \end{bmatrix} = -\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Solving this matrix equation,

$$V_{n_1} = 3.96\,\text{V}$$
$$V_{n_2} = -9.57\,\text{V}$$

Branch voltages are given by,

$$V_b = A^T V_n$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3.96 \\ -9.57 \end{bmatrix} = \begin{bmatrix} 3.96 \\ -3.96 \\ 13.53 \\ -9.57 \\ -9.57 \end{bmatrix}$$

Example 9.30 *For the network shown in Fig. 9.76, write down the f-cutset matrix and obtain the network equilibrium equation in matrix form using KCL.*

Fig. 9.76

Solution The oriented graph and its selected tree are shown in Fig. 9.77.

Twigs: {2, 4}
f-cutset 2: {2, 1, 3}
f-cutset 4: {4, 3}

$$Q = \begin{array}{c} \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.77

The KCL equation in matrix form is given by

$$Q Y_b Q^T V_t = Q I_s - Q Y_b V_s$$

$$Y_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; I_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}; V_s = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}; Q^T = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$Q Y_b = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$Q Y_b Q^T = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Q\,I_s = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$Q\,Y_b\,V_s = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Hence, the KCL equation is given by

$$\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} V_{t_2} \\ V_{t_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -10 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

Solving this matrix equation,

$$V_{t_2} = 4.4\,\text{V}$$
$$V_{t_4} = 3.2\,\text{V}$$

Example 9.31 *Calculate the twig voltages using KCL equation for the network shown in Fig. 9.78.*

Fig. 9.78

Solution The oriented graph and one of the trees are shown in Fig. 9.79.

Twigs: {1, 2, 3}
f-cutset 1: {1, 4, 5, 6}
f-cutset 2: {2, 4, 5}
f-cutset 3: {3, 4, 6}

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 9.79

The network equilibrium equation on node basis can be written as

$$Q\,Y_b\,Q^T\,V_t = QI_s - QY_b\,V_s$$

Here, $I_s = 0,$

$$Q\,Y_b\,Q^T V_t = -Q\,Y_b\,V_s$$

$$Y_b = \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}; Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; V_s = \begin{bmatrix} 910 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$QY_b = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & 0 & -0.1 & -0.5 & 0.1 \\ 0 & 0.2 & 0 & -0.1 & -0.5 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0.1 \end{bmatrix}$$

$$QY_bQ^T = \begin{bmatrix} 0.2 & 0 & 0 & -0.1 & -0.5 & 0.1 \\ 0 & 0.2 & 0 & -0.1 & -0.5 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.6 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

$$QY_bV_s = \begin{bmatrix} 0.2 & 0 & 0 & -0.1 & -0.5 & 0.1 \\ 0 & 0.2 & 0 & -0.1 & -0.5 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 910 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 182 \\ 0 \\ 0 \end{bmatrix}$$

Hence, KCL equation can be written as,

$$\begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.6 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} v_{t_1} \\ v_{t_2} \\ v_{t_3} \end{bmatrix} = \begin{bmatrix} -182 \\ 0 \\ 0 \end{bmatrix}$$

Solving this matrix equation,

$$v_{t_1} = -460\,\text{V}$$
$$v_{t_2} = 320\,\text{V}$$
$$v_{t_3} = 200\,\text{V}$$

Example 9.32 *For the network shown in Fig. 9.80, obtain equilibrium equation on node basis.*

Fig. 9.80

Solution The oriented graph and its selected tree are shown in Fig. 9.81.

Twigs: {1, 3}
f-cutset 1: {1, 2}
f-cutset 3: {3, 2, 4}

$$Q = \begin{array}{c} \\ 1 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.81

The KCL equation in matrix form is given by

$$Q Y_b Q^T V_t = Q I_s - Q Y_b V_s$$

Here, $V_s = 0,$

$$Q Y_b Q^T V_t = Q I_s$$

$$Y_b = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}; Q^T = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}; I_s = \begin{bmatrix} -10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q Y_b = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 0 & 0 \\ 0 & -5 & 5 & 10 \end{bmatrix}$$

$$Q Y_b Q^T = \begin{bmatrix} 5 & -5 & 0 & 0 \\ 0 & -5 & 5 & 10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 20 \end{bmatrix}$$

$$Q I_s = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Hence, KCL equation will be written as

$$\begin{bmatrix} 10 & 5 \\ 5 & 20 \end{bmatrix}\begin{bmatrix} v_{t_1} \\ v_{t_3} \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Solving this matrix equation,

$$v_{t_1} = -\frac{8}{7}\text{V}$$

$$v_{t_3} = \frac{2}{7}\text{V}$$

Example 9.33 *For the network shown in Fig. 9.82, write down the f-cutset matrix and obtain the network equilibrium equation in matrix form using KCL and calculate v.*

Fig. 9.82

Solution The oriented graph and its selected tree are shown in Fig. 9.83
Since voltage v is to be determined, Branch 2 is chosen as twig,

Twigs: {2, 4}
f-cutset 2: {2, 1, 3}
f-cutset 3: {4, 3}

$$Q = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Fig. 9.83

The KCL equation in matrix form is given by

$$Q Y_b Q^T V_t = Q I_s - Q Y_b V_s$$

$$Y_b = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}; Q^T = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}; I_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2v \end{bmatrix}; V_s = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q Y_b = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix}$$

$$Q Y_b Q^T = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$Q I_s = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2v \end{bmatrix} = \begin{bmatrix} 0 \\ -2v \end{bmatrix}$$

$$Q Y_b V_s = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$Q I_s - Q Y_b V_s = \begin{bmatrix} 1 \\ -2v \end{bmatrix}$$

Hence, the KCL equation can be written as

$$\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} v_{t_2} \\ v_{t_4} \end{bmatrix} = \begin{bmatrix} 1 \\ -2v \end{bmatrix}$$

From Fig. 9.82, $v_{t_2} = v$

Solving this matrix equation,

$$v_{t_2} = 0.44 \text{ V}$$
$$v_{t_4} = 0.66 \text{ V}$$
$$v = v_{t_2} = 0.44 \text{ V}$$

Example 9.34 *For the network shown in Fig. 9.84, write down the f-cutset matrix and obtain the network equilibrium equation in matrix form using KCL and calculate v.*

Fig. 9.84

Solution The voltage and current sources are converted into accompanied sources by source–shifting method as shown in Fig. 9.85.

Fig. 9.85

The oriented graph and its selected tree are shown in Fig. 9.86.

Twigs: {1, 2}
f-cutset 1: {1, 4}
f-cutset 2: {2, 3}

$$Q = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Fig. 9.86

The KCL equation in the matrix form is given by

$$Q Y_b Q^T V_t = Q I_s - Q Y_b V_s$$

$$Y_b = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}; Q^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}; \; I_s = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}; V_s = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q Y_b = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & 0 \end{bmatrix}$$

$$Q Y_b Q^T = \begin{bmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix}$$

$$Q I_s = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$$Q Y_b V_s = \begin{bmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$$

$$Q I_s - Q Y_b V_s = \begin{bmatrix} 0.25 \\ -1 \end{bmatrix}$$

Hence, the KCL equation can be written as

$$\begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix} \begin{bmatrix} v_{t_1} \\ v_{t_2} \end{bmatrix} = \begin{bmatrix} 0.25 \\ -1 \end{bmatrix}$$

Solving this matrix equation,

$$v_{t_1} = 0.33 \text{ V}$$
$$v_{t_2} = -1.33 \text{ V}$$

From Fig. 9.85, $v = 1 + v_{t_2} = -0.33 \text{ V}$

9.13 DUALITY

Two networks are said to be the dual of each other when the mesh equations of one network are the same as the node equations of the other. Kirchhoff's voltage law and current law are same, word for word, with voltage substituted for current, independent loop for independent node pair, etc. Similarly, two graphs are said to be dual of each other if the incidence matrix of any one of them is equal to the circuit matrix of the other. Only planar networks have duals.

Table 9.1 *Conversion for dual electrical circuits*

Loop basis	Node basis
Current	Voltage
Resistance	Conductance
Inductance	Capacitance
Branch current	Branch voltage
Mesh	Node
Short circuit	Open circuit
Parallel path	Series path

The following steps are involved in constructing the dual of a network:

1. Place a node inside each mesh of the given network. These internal nodes correspond to the independent nodes in the dual network.
2. Place a node outside the given network. The external node corresponds to the datum node in the dual network.
3. Connect all internal nodes in the adjacent mesh by dashed lines crossing the common branches. Elements which are the duals of the common branches will form the branches connecting the corresponding independent node in the dual network.
4. Connect all internal nodes to the external node by dashed lines corresponding to all external branches. Duals of these external branches will form the branches connecting independent nodes and the datum node.
5. A clockwise current in a mesh corresponds to a positive polarity (with respect to the datum node) at the dual independent node.
6. A voltage rise in the direction of a clockwise mesh current corresponds to a current flowing towards the dual independent node.

Example 9.35 *Draw the dual of the network shown in Fig. 9.87.*

Fig. 9.87

Solution The following steps are involved in constructing the dual of the network as shown in Fig. 9.88.

1. Place a node inside each mesh.
2. Place a node outside the mesh which will correspond to the datum node.
3. Connect two internal nodes through a dashed line. The element which is dual of the common branch (here capacitance) will form the branch connecting the corresponding independent node in the dual network.
4. Connect all internal nodes to the external node by dashed lines crossing all the branches. The dual of these branches will form the branches connecting the independent node and datum.

Fig. 9.88

The dual network is shown in Fig. 9.89.

Fig. 9.89

Example 9.36 *Draw the dual of the network of Fig. 9.90.*

Fig. 9.90

Solution For drawing the dual network, proceed in the same way as in Example 9.35 as shown in Fig. 9.91.

Fig. 9.91

The dual network is shown in Fig. 9.92.

Fig. 9.92

Example 9.37 *Draw the dual of the network shown in Fig. 9.93.*

Fig. 9.93

Solution For drawing the dual network, proceed in the same way as in Example 9.35, as shown in Fig. 9.94.

Fig. 9.94

The dual network is shown in Fig. 9.95.

Fig. 9.95

Example 9.38 *Draw the dual of the network shown in Fig. 9.96.*

Fig. 9.96

Solution For drawing the dual network, proceed in the same way as in Example 9.35 as shown in Fig. 9.97.

Fig. 9.97

The dual network is shown in Fig. 9.98.

Fig. 9.98

Example 9.39 *Draw the dual of the network shown in Fig. 9.99.*

Fig. 9.99

Solution For drawing the dual network, proceed in the same way as in Example 9.35 as shown in Fig. 9.100.

Fig. 9.100

The dual network is shown in Fig. 9.101.

Fig. 9.101

Example 9.40 *Draw the dual of the network shown in Fig. 9.102.*

Fig. 9.102

Solution For drawing the dual network, proceed in the same way as in Example 9.35 as shown in Fig. 9.103.

Fig. 9.103

The dual network is shown in Fig. 9.104.

Fig. 9.104

Exercises

9.1 For the networks shown in Fig. 9.105–9.108, write the incidence matrix, tieset matrix and *f*-cutset matrix.

(i)

Fig. 9.105

(ii)

Fig. 9.106

(iii)

Fig. 9.107

(iv)

Fig. 9.108

9.2 For the graph shown in Fig. 9.109, write the incidence matrix, tieset matrix and *f*-cutset matrix.

Fig. 9.109

9.3 The incidence matrix is given as follows:

Branches →

1	2	3	4	5	6	7	8
–1	–1	0	0	0	0	1	0
0	1	1	0	1	0	0	0
0	0	–1	–1	0	1	0	0
1	0	0	1	0	0	0	1

Draw oriented graph and write tieset matrix.

9.4 The incidence matrix is given below:

Branches →

1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	0	1	0	0
0	–1	–1	0	0	0	–1	0	0	–1
–1	1	0	0	0	0	0	–1	–1	1
1	0	0	0	–1	–1	1	0	0	0

Draw the oriented graph.

9.5 For the network shown in Fig. 9.110, draw the oriented graph and obtain the tieset matrix. Use this matrix to calculate the current *i*.

Fig. 9.110

[0.91 A]

9.6 Draw the dual networks for the circuits shown in Fig. 9.111–9.115.

(i)

Fig. 9.111

(ii)

Fig. 9.112

(iii)

Fig. 9.113

(iv)

Fig. 9.114

(v)

Fig. 9.115

9.7 Using the principles of network topology, write the loop/node equation in matrix form for the network shown in Fig. 9.116.

Fig. 9.116

Objective-Type Questions

9.1 The number of independent loops for a network with *n* nodes and *b* branches is
(a) n − 1
(b) b − n
(c) b − n + 1
(d) independent of the number of nodes

9.2 A network has 7 nodes and 5 independent loops. The number of branched in the network is
(a) 13 (b) 12
(c) 11 (d) 10

9.3 Identify which of the following is NOT a tree of the graph shown in Fig. 9.117.

Fig. 9.117

(a) *begh* (b) *defg*
(c) *adfg* (d) *aegh*

9.4 The minimum number of equations required to analyze the circuit shown in Fig. 9.118 is

Fig. 9.118

(a) 3 (b) 4
(c) 6 (d) 7

9.5 Consider the network graph shown in Fig. 9.119.

Fig. 9.119

Which one of the following is NOT a tree of this graph?

9.6 Figure 9.120 below shows a network and its graph is drawn aside. A proper tree chosen for analyzing the network will contain the edges.

Fig. 9.120

(a) *ab, bc, ad* (b) *ab, bc, ca*
(c) *ab, bd, ca?* (d) *ac, bd, ad*

9.7 The graph of an electrical network has n nodes and b branches. The number of links with respect to the choice of a tree is given by

(a) $b - n + 1$ (b) $b + n$
(c) $n - b + 1$ (d) $n - 2b - 1$.

9.8 In the graph shown in Fig. 9.121, one possible tree is formed by the branches 4, 5, 6, 7. Then one possible fundamental cutset is

Fig. 9.121

(a) 1, 2, 3, 8 (b) 1, 2, 5, 6
(c) 1, 5, 6, 8 (d) 1, 2, 3, 7, 8

9.9 Which one of the following represents the total number of trees in the graph given in Fig. 9.122?

Fig. 9.122

(a) 4 (b) C
(c) 5 (d) 8

9.10 Which one of the following is a cutset of the graph shown in Fig. 9.123?

Fig. 9.123

(a) 1, 2, 3 and 4 (b) 2, 3, 4 and 6
(c) 1, 4, 5 and 6 (d) 1, 2, 4 and 5

Answers to Objective-Type Questions

9.1. (c)	9.2. (c)	9.3. (c)	9.4. (b)	9.5. (b)
9.6. (d)	9.7. (a)	9.8. (d)	9.9. (d)	9.10. (d)

10

Transient Analysis

10.1 INTRODUCTION

Whenever a network containing energy storage elements such as inductor or capacitor is switched from one condition to another, either by change in applied source or change in network elements, the response current and voltage change from one state to the other state. The time taken to change from an initial steady state to the final steady state is known as the *transient period*. This response is known as *transient response* or *transients*. The response of the network after it attains a final steady value is independent of time and is called the steady-state response. The complete response of the network is determined with the help of a differential equation.

10.2 INITIAL CONDITIONS

In solving the differential equations in the network, we get some arbitary constant. Initial conditions are used to determine these arbitrary constants. It helps us to know the behaviour of elements at the instant of switching.

To differentiate between the time immediately before and immediately after the switching, the signs '–' and '+' are used. The conditions existing just before switching are denoted as $i\ (0^-)$, $v\ (0^-)$, etc. Conditions just after switching are denoted as $i\ (0^+)$, $v\ (0^+)$.

Sometimes conditions at $t = \infty$ are used in the evaluation of arbitrary constants. These are known as *final conditions*.

In solving the problems for initial conditions in the network, we divide the time period in the following ways:

1. Just before switching (from $t = -\infty$ to $t = 0^-$)
2. Just after switching (at $t = 0^+$)
3. After switching (for $t > 0$)

If the network remains in one condition for a long time without any switching action, it is said to be under steady-state condition.

1. Initial Conditions for the Resistor For a resistor, current and voltage are related by $v(t) = Ri(t)$. The current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, the voltage will change instantaneously if the current changes instantaneously.

2. Initial Conditions for the Inductor For an inductor, current and voltage are related by,

$$v(t) = L\frac{di}{dt}$$

Voltage across the inductor is proportional to the rate of change of current. It is impossible to change the current through an inductor by a finite amount in zero time. This requires an infinite voltage across the inductor. An inductor does not allow an abrupt change in the current through it.

The current through the inductor is given by,

$$i(t) = \frac{1}{L}\int_0^t v(t)dt + i(0)$$

where $i(0)$ is the initial current through the inductor.

If there is no current flowing through the inductor at $t = 0^-$, the inductor will act as an open circuit at $t = 0^+$. If a current of value I_0 flows through the inductor at $t = 0^-$, the inductor can be regarded as a current source of I_0 ampere at $t = 0^+$.

3. Initial Conditions for the Capacitor For the capacitor, current and voltage are related by,

$$i(t) = C\frac{dv(t)}{dt}$$

Current through a capacitor is proportional to the rate of change of voltage. It is impossible to change the voltage across a capacitor by a finite amount in zero time. This requires an infinite current through the capacitor. A capacitor does not allow an abrupt change in voltage across it.

The voltage across the capacitor is given by,

$$v(t) = \frac{1}{C}\int_0^t i(t)dt + v(0)$$

where $v(0)$ is the initial voltage across the capacitor.

If there is no voltage across the capacitor at $t = 0^-$, the capacitor will act as a short circuit at $t = 0^+$. If the capacitor is charged to a voltage V_0 at $t = 0^-$, it can be regarded as a voltage source of V_0 volt at $t = 0^+$. These conditions are summarized in Fig. 10.1.

Fig. 10.1 *Initial conditions*

Similarly, we can draw the chart for final conditions as shown in Fig. 10.2

Fig. 10.2 *Final conditions*

4. Procedure for Evaluating Initial Conditions

(a) Draw the equivalent network at $t = 0^-$. Before switching action takes place, i.e., for $t = -\infty$ to $t = 0^-$, the network is under steady-state conditions. Hence, find the current flowing through the inductors $i_L(0^-)$ and voltage across the capacitor $v_C(0^-)$.

(b) Draw the equivalent network at $t = 0^+$, i.e., immediately after switching. Replace all the inductors with open circuits or with current sources $i_L(0^+)$ and replace all capacitors by short circuits or voltage sources $v_C(0^+)$. Resistors are kept as it is in the network.

(c) Initial voltages or currents in the network are determined from the equivalent network at $t = 0^+$.

(d) Initial conditions, i.e., $\frac{di}{dt}(0^+), \frac{dv}{dt}(0^+), \frac{d^2i}{dt^2}(0^+), \frac{d^2v}{dt^2}(0^+)$ are determined by writing integro-differential equations for the network for $t > 0$, i.e., after the switching action by making use of initial condition.

Example 10.1 *In the given network of Fig. 10.3, the switch is closed at t = 0. With zero current in the inductor, find i,* $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

Fig. 10.3

Solution

At $t = 0^-$, no current flows through the inductor.

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.4.

At $t = 0^+$, the inductor acts as an open circuit.

$$i(0^+) = 0$$

Fig. 10.4

For $t > 0$, the network is shown in Fig. 10.5.
Writing the KVL equation for $t > 0$,

$$100 - 10i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$

$$\frac{di}{dt} = 100 - 10i \qquad \ldots\text{(ii)}$$

Fig. 10.5

At $t = 0^+$, $$\frac{di}{dt}(0^+) = 100 - 10i(0^+) = 100 - 10(0) = 100 \text{ A/s}$$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -10\frac{di}{dt}$$

At $t = 0^+$, $$\frac{d^2i}{dt^2}(0^+) = -10\frac{di}{dt}(0^+) = -10(100) = -1000 \text{ A/s}^2$$

Example 10.2 *In the network of Fig. 10.6, the switch is closed at t = 0. With the capacitor uncharged, find value for i,* $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at t = 0⁺.*

Fig. 10.6

Solution

At $t = 0^-$, the capacitor is uncharged.

$$v_C(0^-) = 0$$
$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.7.
At $t = 0^+$, the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$
$$i(0^+) = \frac{100}{1000} = 0.1 \text{ A}$$

Fig. 10.7

For $t > 0$, the network is shown in Fig. 10.8.
Writing the KVL equation for $t > 0$,

$$100 - 1000i - \frac{1}{1\times10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

Fig. 10.8

Differentiating Eq. (i),

$$0 - 1000\frac{di}{dt} - 10^6 i = 0$$

$$\frac{di}{dt} = -\frac{10^6}{1000}i \qquad \ldots\text{(ii)}$$

At $t = 0^+$,

$$\frac{di}{dt}(0^+) = -\frac{10^6}{1000}i(0^+) = -\frac{10^6}{1000}(0.1) = -100 \text{ A/s}$$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -\frac{10^6}{1000}\frac{di}{dt}$$

At $t = 0^+$,

$$\frac{d^2i}{dt^2}(0^+) = -\frac{10^6}{1000}\frac{di}{dt}(0^+) = -\frac{10^6}{1000}(-100) = 10^5 \text{ A/s}^2$$

Example 10.3 *In the network shown in Fig. 10.9, the switch is closed. Assuming all initial conditions as zero, find i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.*

Fig. 10.9

Solution

At $t = 0^-$,

$$i(0^-) = 0$$

$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.10.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i(0^+) = 0$$

$$v_C(0^+) = 0$$

Fig. 10.10

For $t > 0$, the network is shown in Fig. 10.11.
Writing the KVL equation for $t > 0$,

$$10 - 10i - 1\frac{di}{dt} - \frac{1}{10\times10^{-6}}\int_0^t i\,dt = 0 \quad \ldots\text{(i)}$$

Fig. 10.11

At $t = 0^+$, $\quad 10 - 10i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$

$$\frac{di}{dt}(0^+) = 10\text{A/s}$$

Differentiating Eq. (i),

$$0 - 10\frac{di}{dt} - \frac{d^2i}{dt^2} - \frac{1}{10\times10^{-6}}i = 0$$

At $t = 0^+$,

$$0 - 10\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) - \frac{1}{10^{-5}}i(0^+) = 0$$

$$\frac{d^2i}{dt^2}(0^+) = -10\times10 = -100\ \text{A/s}^2$$

Example 10.4 *In the network shown in Fig. 10.12, at t = 0, the switch is opened. Calculate* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

Fig. 10.12

Solution At $t = 0^-$, the switch is closed. Hence, no current flows through the inductor.

$$i_L(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.13.
At $t = 0^+$, the inductor acts as an open circuit.

$$i_L(0^+) = 0$$

$$v(0^+) = 100\times1 = 100\ \text{V}$$

Fig. 10.13

For $t > 0$, the network is shown in Fig. 10.14.
Writing the KCL equation for $t > 0$,

$$\frac{v}{100} + \frac{1}{1}\int_0^t v\,dt = 1 \qquad \ldots\text{(i)}$$

Fig. 10.14

Differentiating Eq. (i),

$$\frac{1}{100}\frac{dv}{dt} + v = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,
$$\frac{dv}{dt}(0^+) = -100v(0^+) = -100 \times 100 = -10000 \text{ V/s}$$

Differentiating Eq. (ii),

$$\frac{1}{100}\frac{d^2v}{dt^2} + \frac{dv}{dt} = 0$$

At $t = 0^+$,
$$\frac{d^2v}{dt^2}(0^+) = -100\frac{dv}{dt}(0^+) = -100 \times (-10^4) = 10^6 \text{ V/s}^2$$

Example 10.5 *In the given network of Fig. 10.15, the switch is opened at t = 0. Solve for* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

Fig. 10.15

Solution At $t = 0^-$, switch is closed. Hence, the voltage across the capacitor is zero.

$$v(0^-) = v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.16.
At $t = 0^+$, the capacitor acts as a short circuit.

$$v(0^+) = v_C(0^+) = 0$$

Fig. 10.16

For $t > 0$, the network is shown in Fig. 10.17.
Writing the KCL equation for $t > 0$,

$$\frac{v}{1000} + 10^{-6}\frac{dv}{dt} = 10 \qquad \ldots\text{(i)}$$

Fig. 10.17

At $t = 0^+$,
$$\frac{v(0^+)}{1000} + 10^{-6}\frac{dv}{dt}(0^+) = 10$$

$$\frac{dv}{dt}(0^+) = \frac{10}{10^{-6}} = 10\times10^6 \text{ V/s}$$

Differentiating Eq. (i),

$$\frac{1}{1000}\frac{dv}{dt} + 10^{-6}\frac{d^2v}{dt^2} = 0$$

At $t = 0^+$,
$$\frac{1}{1000}\frac{dv}{dt}(0^+) + 10^{-6}\frac{d^2v}{dt^2}(0^+) = 0$$

$$\frac{d^2v}{dt^2}(0^+) = -\frac{1}{1000\times10^{-6}}\times10\times10^6 = -10\times10^9 \text{ V/s}^2$$

Example 10.6 *For the network shown in Fig. 10.18, the switch is closed at t = 0, determine* v, $\frac{dv}{dt}$ *and* $\frac{d^2v}{dt^2}$ *at* $t = 0^+$.

Fig. 10.18

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.19.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a shot circuit.

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

Fig. 10.19

For $t > 0$, the network is shown in Fig. 10.20. Writing the KCL equation for $t > 0$,

$$\frac{v}{2} + \frac{1}{1}\int_1^t v\,dt + 0.5\times10^{-6}\frac{dv}{dt} = 10 \qquad \ldots\text{(i)}$$

Fig. 10.20

At $t = 0^+$,
$$\frac{v(0^+)}{2} + 0 + 0.5\times10^{-6}\frac{dv}{dt}(0^+) = 10$$

$$\frac{dv}{dt}(0^+) = 20\times10^6 \text{ V/s}$$

Differentiating Eq. (i),

$$\frac{1}{2}\frac{dv}{dt} + v + 0.5\times10^{-6}\frac{d^2v}{dt^2} = 0$$

At $t = 0^+$,

$$\frac{1}{2}\frac{dv}{dt}(0^+) + v(0^+) + 0.5\times10^{-6}\frac{d^2v}{dt^2}(0^+) = 0$$

$$\frac{d^2v}{dt^2}(0^+) = -20\times10^{12}\ \text{V/s}^2$$

Example 10.7 *In the network shown in Fig. 10.21, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of* $i, \frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t = 0^+$.

Fig. 10.21

Solution At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.22

$$i(0^-) = \frac{20}{10} = 2\ \text{A}$$

At $t = 0^+$, the network is shown in Fig. 10.23.

At $t = 0^+$, the inductor acts as a current source of 2 A.

Fig. 10.23

$$i(0^+) = 2\ \text{A}$$

For $t > 0$, the network is shown in Fig. 10.24.

Writing the KVL equation for $t > 0$,

Fig. 10.24

$$-20i - 10i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$

At $t = 0^+$,

$$-30i(0^+) - \frac{di}{dt}(0^+) = 0$$

$$\frac{di}{dt}(0^+) = -30\times2 = -60\ \text{A/s}$$

Differentiating Eq. (i),

$$-30\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$,
$$-30\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$
$$\frac{d^2i}{dt^2}(0^+) = 1800 \text{ A/s}^2$$

Example 10.8 *In the network shown in Fig.10.25, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.*

Fig. 10.25

Solution At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

Fig. 10.26

$$v_C(0^-) = 30 \text{ V}$$
$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.27.
At $t = 0^+$, the capacitor acts as a voltage source of 30 V.

Fig. 10.27

$$v_C(0^+) = 30 \text{ V}$$
$$i(0^+) = -\frac{30}{30} = -1 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.28.
Writing the KVL equation for $t > 0$,

Fig. 10.28

$$-10i - 20i - \frac{1}{1\times10^{-6}}\int_0^t i\,dt - 30 = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$-30\frac{di}{dt} - 10^6 i = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,
$$-30\frac{di}{dt}(0^+) - 10^6 i(0^+) = 0$$
$$\frac{di}{dt}(0^+) = \frac{10^6(-1)}{30} = 0.33\times10^5 \text{ A/s}$$

Differentiating Eq. (ii),

$$-30\frac{d^2i}{dt^2}-10^6\frac{di}{dt}=0$$

At $t=0^+$,
$$-30\frac{d^2i}{dt^2}(0^+)-10^6\frac{di}{dt}(0^+)=0$$

$$\frac{d^2i}{dt^3}(0^+)=-\frac{10^6\times0.33\times10^5}{30}=-1.1\times10^9\ \text{A/s}^2$$

Example 10.9 *In the network shown in Fig. 10.29, the switch is changed from the position 1 to the position 2 at t = 0, steady condition having reached before switching. Find the values of* i, $\frac{di}{dt}$ *and* $\frac{d^2i}{dt^2}$ *at* $t=0^+$.

Fig. 10.29

Solution At $t = 0^-$, the network attains steady state. Hence, the capacitor acts as an open circuit.

Fig. 10.30

$$v_C(0^-)=40\ \text{V}$$

$$i(0^-)=0$$

At $t=0^+$, the network is shown in Fig. 10.31.

At $t = 0^+$, the capacitor acts as a voltage source of 40 V and the inductor acts as an open circuit.

Fig. 10.31

$$v_C(0^+)=40\ \text{V}$$

$$i(0^+)=0$$

For $t>0$, the network is shown in Fig. 10.32.
Writing the KVL equation fo $t>0$,

Fig. 10.32

$$-1\frac{di}{dt}-20i-\frac{1}{1\times10^{-6}}\int_0^t i\,dt-40=0 \qquad \ldots\text{(i)}$$

At $t=0^+$,
$$-\frac{di}{dt}(0^+)-20i(0^+)-0-40=0$$

$$\frac{di}{dt}(0^+)=-40\ \text{A/s}$$

Differentiating Eq. (i),

$$-\frac{d^2i}{dt^2}-20\frac{di}{dt}-10^6 i-0=0$$

At $t=0^+$,

$$-\frac{d^2i}{dt^2}(0^+)-20\frac{di}{dt}(0^+)-10^6 i(0^+)=0$$

$$\frac{d^2i}{dt^2}(0^+)=800\ \text{A/s}^2$$

Example 10.10 *In the network of Fig. 10.33, the switch is changed from the position 'a' to 'b' at $t=0$. Solve for i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t=0^+$.*

Fig. 10.33

Solution At $t=0^-$, the network attains steady condition. Hence, the inductor acts as a short circuit.

$$i(0^-)=\frac{100}{1000}=0.1\ \text{A}$$

$$v_C(0^-)=0$$

Fig. 10.34

At $t=0^+$, the network is shown in Fig. 10.35.

At $t=0^+$, the inductor acts as a current source of 0.1 A and the capacitor acts as a short circuit.

$$i(0^+)=0.1\ \text{A}$$

$$v_C(0^+)=0$$

Fig. 10.35

For $t>0$, the network is shown in Fig. 10.36.

Writing the KVL equation for $t>0$,

$$-\frac{1}{0.1\times10^{-6}}\int_0^t i\,dt-1000i-1\frac{di}{dt}=0 \qquad \ldots\text{(i)}$$

Fig. 10.36

At $t = 0^+$,
$$-0 - 1000i(0^+) - \frac{di}{dt}(0^+) = 0$$
$$\frac{di}{dt}(0^+) = -1000\,i(0^+) = -1000 \times 0.1 = -100 \text{ A/s}$$

Differentiating Eq. (i),
$$-\frac{1}{10^{-7}}i - 1000\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$,
$$-10^7 i(0^+) - 1000\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$
$$\frac{d^2i}{dt^2}(0^+) = -10^7(0.1) - 1000(-100) = -9 \times 10^5 \text{ A/s}^2$$

Example 10.11 *The network of Fig. 10.37 attains steady-state with the switch closed. At $t = 0$, the switch is opened. Find the voltage across the switch v_K and $\frac{dv_K}{dt}$ at $t = 0^+$.*

Fig. 10.37

Solution: At $t = 0^-$, the network is shown in Fig. 10.38. At $t = 0^-$, the network attains steady-state condition. The capacitor acts as an open circuit and the inductor acts as a short circuit.

Fig. 10.38

$$i(0^-) = \frac{2}{1} = 2 \text{ A}$$
$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.39.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as a current source of 2 A.

Fig. 10.39

$$i(0^+) = 2 \text{ A}$$
$$v_C(0^+) = 0$$
$$v_K(0^+) = 0$$

Also,
$$v_K = \frac{1}{C}\int i\,dt$$
$$\frac{dv_K}{dt} = \frac{i}{C}$$

At $t = 0^+$,
$$\frac{dv_K}{dt}(0^+) = \frac{i(0^+)}{C} = \frac{2}{0.5} = 4 \text{ A/s}$$

Example 10.12 *In the network shown in Fig. 10.40, assuming all initial conditions as zero, find* $i_1(0^+), i_2(0^+), \frac{di_1}{dt}(0^+), \frac{di_2}{dt}(0^+), \frac{d^2i_1}{dt^2}(0^+)$ *and* $\frac{d^2i_2}{dt^2}(0^+)$.

Fig. 10.40

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.41.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

Fig. 10.41

$$i_1(0^+) = \frac{V}{R_1}$$
$$i_2(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.42.

Writing the KVL equations for two meshes for $t > 0$,

Fig. 10.42

$$V - R_1 i_1 - \frac{1}{C}\int_0^t (i_1 - i_2)dt = 0 \qquad \ldots\text{(i)}$$

and
$$-\frac{1}{C}\int (i_2 - i_1)dt - R_2 i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

From Eq. (ii), at $t = 0^+$,

$$-\frac{1}{C}\int_0^{0^+}(i_2 - i_1)dt - R_2 i_2(0^+) - L\frac{di_2}{dt}(0^+) = 0$$

$$\frac{di_2}{dt}(0^+) = 0$$

Differentiating Eq. (i),

$$0 - R_1\frac{di_1}{dt} - \frac{1}{C}(i_1 - i_2) = 0 \qquad \ldots\text{(iii)}$$

At $t = 0^+$,

$$0 - R_1\frac{di_1}{dt}(0^+) - \frac{1}{C}i_1(0^+) + \frac{1}{C}i_2(0^+) = 0$$

$$R_1\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{V}{R_1} = 0$$

$$\frac{di_1}{dt}(0^+) = -\frac{V}{R_1^2 C}$$

Differentiating Eq. (iii),

$$-R_1\frac{d^2 i_1}{dt^2} - \frac{1}{C}\frac{di_1}{dt} + \frac{1}{C}\frac{di_2}{dt} = 0$$

At $t = 0^+$,

$$-R_1\frac{d^2 i_1}{dt^2}(0^+) - \frac{1}{C}\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{di_2}{dt}(0^+) = 0$$

$$\frac{d^2 i_1}{dt^2}(0^+) = \frac{V}{R_1^3 C^2}$$

Differentiating Eq. (ii),

$$-\frac{1}{C}(i_2 - i_1) - R_2\frac{di_2}{dt} - L\frac{d^2 i_2}{dt^2} = 0$$

At $t = 0^+$,

$$\frac{d^2 i_2}{dt^2}(0^+) = -\frac{R_2}{L}\frac{di_2}{dt}(0^+) - \frac{1}{LC}[i_2(0^+) - i_1(0^+)] = \frac{V}{R_1 LC}$$

Example 10.13 *In the network shown in Fig. 10.43, assuming all initial conditions as zero, find* $i_1, i_2, \frac{di_1}{dt}, \frac{di_2}{dt}, \frac{d^2 i_1}{dt^2}$ *and* $\frac{d^2 i_2}{dt^2}$ *at* $t = 0^+$.

Fig. 10.43

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$
$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.44.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as an open circuit.

Fig. 10.44

$$i_1(0^+) = \frac{V}{R_1}$$
$$i_2(0^+) = 0$$
$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.45.

Fig. 10.45

Writing the KVL equation for $t > 0$,

$$V - \frac{1}{C}\int_0^t i_1\, dt - R_1(i_1 - i_2) = 0 \qquad \ldots\text{(i)}$$

and
$$-R_1(i_2 - i_1) - R_2 i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

From Eq. (ii),

$$\frac{di_2}{dt} = \frac{1}{L}[R_1 i_1 - (R_1 + R_2)i_2] \qquad \ldots\text{(iii)}$$

At $t = 0^+$,
$$\frac{di_2}{dt}(0^+) = \frac{1}{L}[R_1 i_1(0^+) - (R_1 + R_2)i_2(0^+)] = \frac{1}{L}\left[R_1\frac{V}{R_1} - (R_1 + R_2)0\right] = \frac{V}{L}$$

Differentiating Eq. (i),

$$0 - \frac{i_1}{C} - R_1\frac{di_1}{dt} + R_1\frac{di_2}{dt} = 0$$
$$\frac{di_1}{dt} = \frac{di_2}{dt} - \frac{i_1}{R_1 C} \qquad \ldots\text{(iv)}$$

At $t = 0^+$,
$$\frac{di_1}{dt}(0^+) = \frac{di_2}{dt}(0^+) - \frac{i_1(0^+)}{R_1 C} = \frac{V}{L} - \frac{V}{R_1^2 C}$$

Differentiating Eq. (iii),

$$\frac{d^2 i_2}{dt^2} = \frac{1}{L}\left[R_1\frac{di_1}{dt} - (R_1 + R_2)\frac{di_2}{dt}\right]$$

At $t = 0^+$,
$$\frac{d^2 i_2}{dt^2}(0^+) = -V\left(\frac{1}{R_1 LC} + \frac{R_2}{L^2}\right)$$

Differentiating Eq. (iv),

$$\frac{d^2 i_1}{dt^2} = \frac{d^2 i_2}{dt^2} - \frac{1}{R_1 C}\frac{di_1}{dt}$$

At $t = 0^+$, $\frac{d^2 i_1}{dt^2}(0^+) = \frac{d^2 i_2}{dt^2}(0^+) - \frac{1}{R_1 C}\frac{di_1}{dt}(0^+) = -\frac{V}{R_1 LC} - \frac{VR_2}{L^2} - \frac{1}{R_1 C}\left(\frac{V}{L} - \frac{V}{R_1^2 C}\right) = \frac{V}{R_1^3 C^2} - \frac{2V}{R_1 LC} - \frac{VR_2}{L^2}$

Example 10.14 *In the network shown in Fig. 10.46, a steady state is reached with the switch open. At t = 0, the switch is closed. For the element values given, determine the value of v_a (0^-), v_b (0^-), v_a (0^+) and v_b (0^+).*

Fig. 10.46

Solution At $t = 0^-$, the network is shown in Fig. 10.47.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.47

$$i_L(0^-) = \frac{5}{(10 \parallel 30)} = \frac{5}{7.5} = \frac{2}{3}\text{ A}$$

$$v_b(0^-) = 0$$

$$v_a(0^-) = 5 \times \frac{20}{30} = 3.33\text{ V}$$

At $t = 0^+$, the network is shown in Fig. 10.48.
At $t = 0^+$, the inductor acts as a current source of $\frac{2}{3}$ A.

Fig. 10.48

$$i_L(0^+) = \frac{2}{3}\text{ A}$$

Writing the KCL equations at $t = 0^+$,

$$\frac{v_a(0^+) - 5}{10} + \frac{v_a(0^+)}{10} + \frac{v_a(0^+) - v_b(0^+)}{20} = 0$$

and
$$\frac{v_b(0^+)-v_a(0^+)}{20}+\frac{v_b(0^+)-5}{10}+\frac{2}{3}=0$$

Solving these two equations,

$$v_a(0^+)=1.9\text{ V}$$
$$v_b(0^+)=-0.477\text{ V}$$

Example 10.15 *In the accompanying Fig. 10.49 is shown a network in which a steady state is reached with switch open. At t = 0, switch is closed. Determine $v_a(0^-)$, $v_a(0^+)$, $v_b(0^-)$ and $v_b(0^+)$.*

Fig. 10.49

Solution At $t=0^-$, the network is shown in Fig. 10.50.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_a(0^-)=5\text{ V}$$
$$v_b(0^-)=5\text{ V}$$

Fig. 10.50

At $t=0^+$, the network is shown in Fig. 10.51.

At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$v_b(0^+)=5\text{ V}$$

Writing the KCL equation at $t = 0^+$,

$$\frac{v_a(0^+)-5}{10}+\frac{v_a(0^+)}{10}+\frac{v_a(0^+)-5}{20}=0$$

$$0.25v_a(0^+)=0.75$$

$$v_a(0^+)=3\text{ V}$$

Fig. 10.51

Example 10.16 *The network shown in Fig. 10.52 has two independent node pairs. If the switch is opened at t = 0. Find v_1, v_2, $\frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$ at $t = 0^+$.*

Fig. 10.52

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$
$$v_C(0^-) = v_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.53.
At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i_L(0^+) = 0$$
$$v_1(0^+) = R_1\, i(0^+)$$
$$v_2(0^+) = 0$$

Fig. 10.53

For $t > 0$, the network is shown in Fig. 10.54.
Writing the KCL equation at Node 1 for $t > 0$,

$$\frac{v_1}{R_1} + \frac{1}{L}\int_0^t (v_1 - v_2)dt = i(t) \qquad \ldots\text{(i)}$$

Fig. 10.54

Differentiating Eq. (i),

$$\frac{1}{R_1}\frac{dv_1}{dt} + \frac{1}{L}(v_1 - v_2) = \frac{di}{dt}$$

At $t = 0^+$,
$$\frac{dv_1}{dt}(0^+) = R_1\left[\frac{di}{dt}(0^+) - \frac{1}{L}R_1 i(0^+)\right]$$

Writing the KCL equation at Node 2 for $t > 0$,

$$\frac{1}{L}\int_0^t (v_2 - v_1)dt + \frac{v_2}{R_2} + C\frac{dv_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,
$$0 + \frac{v_2(0^+)}{R_2} + C\frac{dv_2}{dt}(0^+) = 0$$
$$\frac{dv_2}{dt}(0^+) = 0$$

Example 10.17 *In the network shown in Fig. 10.55, the switch is closed at $t = 0$, with zero capacitor voltage and zero inductor current. Solve for $v_1, v_2, \frac{dv_1}{dt}, \frac{dv_2}{dt}$ and $\frac{d^2v_2}{dt^2}$ at $t = 0^+$.*

Fig. 10.55

Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$v_C(0^-) = 0$$

$$v_1(0^-) = 0$$

$$v_2(0^-) = 0$$

$$i_L(0^-) = 0$$

$$i_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.56.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$

$$v_1(0^+) = 0$$

$$v_2(0^+) = 0$$

$$i_L(0^+) = 0$$

$$i_C(0^+) = \frac{V}{R_1}$$

Fig. 10.56

For $t > 0$, the network is shown in Fig. 10.57.

Writing the KVL equation for $t > 0$,

$$v_C(t) = v_1(t) + v_2(t) \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$\frac{dv_C}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt} \qquad \ldots\text{(ii)}$$

Fig. 10.57

Now,
$$v_C = \frac{1}{C}\int_0^t i_C\, dt \qquad \text{...(iii)}$$

$$\frac{dv_C}{dt} = \frac{i_C}{C}$$

At $t = 0^+$,
$$\frac{dv_C}{dt}(0^+) = \frac{i_C(0^+)}{C} = \frac{V}{R_1 C}\ \text{V/s}$$

Also
$$v_1 = L\frac{di_L}{dt} \qquad \text{...(iv)}$$

$$\frac{di_L}{dt} = \frac{v_1}{L} \qquad \text{...(v)}$$

At $t = 0^+$,
$$\frac{di_L}{dt}(0^+) = \frac{v_1(0^+)}{L} = 0$$

Also,
$$v_2 = R_2 i_L \qquad \text{...(vi)}$$

$$\frac{dv_2}{dt} = R_2\frac{di_L}{dt} \qquad \text{...(vii)}$$

At $t = 0^+$,
$$\frac{dv_2}{dt}(0^+) = R_2\frac{di_L}{dt}(0^+) = 0$$

$$\frac{dv_C}{dt}(0^+) = \frac{dv_1}{dt}(0^+) + \frac{dv_2}{dt}(0^+)$$

$$\frac{dv_1}{dt}(0^+) = \frac{V}{R_1 C}\ \text{V/s}$$

Differentiating Eq. (vii),

$$\frac{d^2 v_2}{dt^2} = R_2\frac{d^2 i_L}{dt^2}$$

At $t = 0^+$,
$$\frac{d^2 v_2}{dt^2}(0^+) = R_2\frac{d^2 i_L}{dt}(0^+)$$

Differentiating Eq. (v),

$$\frac{d^2 i_L}{dt^2} = \frac{1}{L}\frac{dv_1}{dt}$$

At $t = 0^+$,
$$\frac{d^2 i_L}{dt^2}(0^+) = \frac{1}{L}\frac{dv_1}{dt}(0^+) = \frac{1}{L}\frac{V}{R_1 C}$$

$$\frac{d^2 v_2}{dt^2}(0^+) = \frac{R_2 V}{R_1 L C}\ \text{V/s}^2$$

Example 10.18 *In the network shown in Fig. 10.58, a steady state is reached with switch open. At t = 0, switch is closed. Find the three loop currents at t = 0⁺.*

Fig. 10.58

Solution At $t = 0^-$, the network is shown in Fig. 10.59.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor act as a short circuit and the capacitors act as open circuits.

Fig. 10.59

$$i_{4\Omega}(0^-) = i_1(0^-) = \frac{6}{6} = 1 \text{ A}$$

$$i_2(0^-) = 0$$

$$i_3(0^-) = 0$$

$$v_1(0^-) + v_2(0^-) = 6 \times \frac{4}{6} = 4 \text{ V}$$

Since the charges on capacitors are equal when connected in series,

$$Q_1 = Q_2$$

$$C_1 v_1 = C_2 v_2$$

$$\frac{v_1(0^-)}{v_2(0^-)} = \frac{C_2}{C_1} = \frac{1}{0.5} = 2$$

$$v_1(0^-) = \frac{8}{3} \text{ V}$$

and

$$v_2(0^-) = \frac{4}{3} \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 10.60.

At $t = 0^+$, the inductor is replaced by a current source of 1 A and the capacitors are replaced by a voltage source of $\frac{8}{3}$ V and $\frac{4}{3}$ V respectively.

Fig. 10.60

$$v_1(0^+) = \frac{8}{3}\text{ V}$$

$$v_2(0^+) = \frac{4}{3}\text{ V}$$

At $t = 0^+$,
$$6 - 2i_1(0^+) - \frac{8}{3} - \frac{4}{3} = 0$$

$$i_1(0^+) = 1\text{ A}$$

Now,
$$i_1(0^+) - i_3(0^+) = 1$$

$$i_3(0^+) = 0$$

Writing the KVL equation for Mesh 2,

$$-4[i_2(0^+) - i_1(0^+)] - \frac{8}{3} = 0$$

$$-4i_2(0^+) + 4 - \frac{8}{3} = 0$$

$$i_2(0^+) = \frac{1}{3}\text{ A}$$

Example 10.19 *In the network shown in Fig. 10.61, the switch K is closed at $t = 0$ connecting a voltage $V_0 \sin \omega t$ to the parallel RL-RC circuit. Find (a) $i_1(0^+)$ and $i_2(0^+)$ (b) $\frac{di_1}{dt}(0^+)$ and $\frac{di_2}{dt}(0^+)$.*

Fig. 10.61

Solution At $t = 0^-$, no current flows in the inductor and there is no voltage across the capacitor.

$$v_C(0^-) = 0$$

$$i_1(0^-) = 0$$

$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.62.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit. The voltage source $V_0 \sin \omega t$ acts as a short circuit.

$$i_1(0^+) = 0$$

$$i_2(0^+) = 0$$

$$v_C(0^+) = 0$$

Fig. 10.62

For $t > 0$, the network is shown in Fig. 10.63.
Writing the KVL equation for $t > 0$,

$$V_0 \sin \omega t - R\, i_1 - \frac{1}{C}\int i_1\, dt = 0 \qquad \ldots\text{(i)}$$

and

$$V_0 \sin \omega t - R\, i_2 - L\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

Fig. 10.63

Differentiating Eq. (i),

$$V_0\, \omega \cos \omega t - R\frac{di_1}{dt} - \frac{i_1}{C} = 0$$

$$\frac{di_1}{dt} = \frac{V_0 \omega}{R}\cos \omega t - \frac{i_1}{RC} \qquad \ldots\text{(iii)}$$

At $t = 0^+$,

$$\frac{di_1}{dt}(0^+) = \frac{V_0 \omega}{R}\cos \omega t\Big|_{t=0^+} - \frac{i_1(0^+)}{RC} = \frac{V_0 \omega}{R}$$

From Eq. (ii),

$$\frac{di_2}{dt} = \frac{V_0}{L}\sin \omega t - \frac{R}{L} i_2$$

At $t = 0^+$,

$$\frac{di_2}{dt}(0^+) = \frac{V_0}{L}\sin \omega t\Big|_{t=0^+} - \frac{R}{L} i_2(0^+) = 0$$

Example 10.20 *In the network of Fig. 10.64, the switch K is changed from 'a' to 'b' at $t = 0$ (a steady state having been established at the position a). Find i_1, i_2 and i_3 at $t = 0^+$.*

Fig. 10.64

Solution At $t = 0^-$, the network is shown in Fig. 10.65.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitors act as open circuits and inductors act as short circuits.

Fig. 10.65

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$i_3(0^-) = 0$$
$$v_{C_3}(0^-) = V$$
$$v_{C_2}(0^-) = 0$$
$$v_{C_1}(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.66.

At $t = 0^+$, the capacitor C_3 acts as a voltage source of V volts and capacitors C_1 and C_2 act as short circuts. The inductors act as open circuits.

Fig. 10.66

$$i_1(0^+) = i_2(0^+) = -\frac{V}{R_1 + R_2 + R_3}$$

$$i_3(0^+) = 0$$

Example 10.21 *In the network of Fig. 10. 67, the switch K_1 has been closed for a long time prior to $t = 0$. At $t = 0$, the switch K_2 is closed. Find $v_C(0^+)$ and $i_C(0^+)$.*

Fig. 10.67

Solution At $t = 0^-$, the network is shown in Fig. 10.68. At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

Fig. 10.68

$$i_1(0^-) = 0$$
$$i_2(0^-) = 0$$
$$i_C(0^-) = 0$$
$$v_C(0^-) = 10 \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 10.69.

At $t = 0^+$, the capacitor acts as a voltage source of voltage V.

$$v_C(0^+) = 10 \text{ V}$$

Writing the KVL equation at $t = 0^+$,

$$10 - 10\, i_1(0^+) - 10 = 0$$

and

$$10 - 10\, i_2(0^+) = 0$$

$$i_1(0^+) = 0$$

$$i_2(0^+) = -1 \text{ A}$$

$$i_1(0^+) = i_C(0^+) + i_2(0^+)$$

$$i_C(0^+) = 1 \text{ A}$$

Fig. 10.69

Example 10.22 *In the network shown in fig. 10.70, a steady state is reached with the switch open. At t = 0, the switch is closed. Determine* $v_C(0^-), i_1(0^+), i_2(0^+), \frac{di_1}{dt}(0^+)$ *and* $\frac{di_2}{dt}(0^+)$.

Fig. 10.70

Solution At $t = 0^-$, the network is shown in Fig. 10.71.

At $t = 0^-$, the network is in steady-state. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

$$v_C(0^-) = 100 \times \frac{20}{20+10} = 66.67 \text{ V}$$

$$i_1(0^-) = \frac{66.67}{20} = 3.33 \text{ A}$$

$$i_2(0^-) = 0$$

Fig. 10.71

At $t = 0^+$, the network is shown in Fig. 10.72.

At $t = 0^+$, the inductor acts as a current source of 3.33 A and the capacitor acts as a voltage source of 66.67 V.

Fig. 10.72

$$v_C(0^+) = 66.67 \text{ V}$$
$$i_1(0^+) = 3.33 \text{ A}$$
$$i_2(0^+) = \frac{100 - 66.67}{20} = 1.67 \text{ A}$$

For $t > 0^-$, the network is shown in Fig. 10.73.

Writing the KVL equations for $t > 0$,

$$100 - 20i_1 - 1\frac{di_1}{dt} = 0 \qquad \ldots\text{(i)}$$

Fig. 10.73

and
$$100 - 20\, i_2 - \frac{1}{10^{-6}}\int i_2\, dt - 66.67 = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,
$$\frac{di_1}{dt}(0^+) = 100 - 20i_1(0^+) = 100 - 20\,(3.33) = 33.3 \text{ A/s}$$

Differentiating Eq. (ii),

$$0 - 20\frac{di_2}{dt} - 10^6 i_2 = 0$$

At $t = 0^+$,
$$20\frac{di_2}{dt}(0^+) = -10^6\, i_2(0^+)$$
$$\frac{di_2}{dt}(0^+) = -\frac{10^6}{20} \times 1.67 = -83500 \text{ A/s}^2$$

10.3 RESISTOR-INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. 10.74. The switch is closed at time $t = 0$. The inductor in the circuit is initially un-energised.

Fig. 10.74 *Series RL circuit*

Applying KVL to the circuit for $t > 0$,

$$V - Ri - L\frac{di}{dt} = 0$$

This is a linear differential equation of first order. It can be solved if the variables can be separated.

$$(V - Ri)\, dt = L\, di$$
$$\frac{L\, di}{V - Ri} = dt$$

Integrating both the sides,

$$-\frac{L}{R} l_n(V - Ri) = t + k$$

where l_n denotes that the logarithm is of base e and k is an arbitrary constant. k can be evaluated from the initial condition. In the circuit, the switch is closed at $t = 0$, i.e., just before closing the switch, the current in the inductor is zero. Since the inductor does not allow sudden change in current, at $t = 0^+$, just after the switch is closed, the current remains zero.

Setting $i = 0$ at $t = 0$,

$$-\frac{L}{R} l_n V = k$$

$$-\frac{L}{R} l_n (V - Ri) = t - \frac{L}{R} l_n V$$

$$-\frac{L}{R}\left[l_n (V - Ri) - l_n V\right] = t$$

$$\frac{V - Ri}{V} = e^{-\frac{R}{L}t}$$

$$V - Ri = Ve^{-\frac{R}{L}t}$$

$$Ri = V - Ve^{-\frac{R}{t}t}$$

$$i = \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

The complete response is composed of two parts, the steady-state response or forced response or zero state response $\frac{V}{R}$ and transient response or natural response or zero input response $\frac{V}{R} e^{-\frac{R}{L}t}$.

Fig. 10.75 *Current response of series RL circuit*

The natural response is a characteristic of the circuit. Its form may be found by considering the source-free circuit. The forced response has the characteristics of forcing function, i.e., applied voltage. Thus, when the switch is closed, response reaches the steady-state value after some time interval as shown in Fig. 10.75.

Here, the transient period is defined as the time taken for the current to reach its final or steady state value from its initial value.

The term $\frac{L}{R}$ is called time constant and is denoted by T.

$$T = \frac{L}{R}$$

At one time constant, the current reaches 63.2 per cent of its final value $\frac{V}{R}$.

$$i(T) = \frac{V}{R} - \frac{V}{R} e^{-\frac{1}{T}t} = \frac{V}{R} - \frac{V}{R} e^{-1} = \frac{V}{R} - 0.368\frac{V}{R} = 0.632\frac{V}{R}$$

Similarly,

$$i(2T) = \frac{V}{R} - \frac{V}{R}e^{-2} = \frac{V}{R} - 0.135\frac{V}{R} = 0.865\frac{V}{R}$$

$$i(3T) = \frac{V}{R} - \frac{V}{R}e^{-3} = \frac{V}{R} - 0.0498\frac{V}{R} = 0.950\frac{V}{R}$$

$$i(5T) = \frac{V}{R} - \frac{V}{R}e^{-5} = \frac{V}{R} - 0.0067\frac{V}{R} = 0.993\frac{V}{R}$$

After 5 time constants, the current reaches 99.33 per cent of its final value. The voltage across resistor is

$$v_R = Ri = R \times \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= V\left(1 - e^{-\frac{R}{L}t}\right) \qquad \text{for } t > 0$$

Fig. 10.76 *Voltage response of series RL circuit*

Similarly, voltage across inductor is

$$v_L = \frac{di}{dt} = L\frac{V}{R}\frac{d}{dt}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$= Ve^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

Note:

1. Consider a homogeneous equation,

$$\frac{di}{dt} + Pi = 0 \qquad \text{where } P \text{ is a constant.}$$

   The solution of this equation is given by,

$$i\,(t) = ke^{-Pt}$$

   The value of k is obtained by putting $t = 0$ in the equation for $i\,(t)$.

2. Consider a non-homogeneous equation,

$$\frac{di}{dt} + Pi = Q$$

   where P is a constant and Q may be a function of the independent variable t or a constant.
   The solution of this equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\,e^{-Pt}$$

   The value of k is obtained by putting $t = 0$ in the equation of $i(t)$.

Example 10.23 *In the network of Fig. 10.77, the switch is initially at the position 1. On the steady state having reached, the switch is changed to the position 2. Find current i(t).*

Fig. 10.77

Solution At $t = 0^-$, the network is shown in Fig. 10.78.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.78

$$i(0^-) = \frac{V}{R_1}$$

Since the inductor does not allow sudden change in current,

$$i(0^+) = \frac{V}{R_1}$$

For $t > 0$, the network is shown in Fig. 10.79.

Writing the KVL equation for $t > 0$,

Fig. 10.79

$$-R_2 i - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{(R_1 + R_2)}{L} i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = \frac{R_1 + R_2}{L}$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt}$$

$$i(t) = k\,e^{-\left(\frac{R_1+R_2}{L}\right)t}$$

At $t = 0$, $i(0) = \frac{V}{R_1}$

$$\frac{V}{R_1} = k\,e^0 = k$$

$$i(t) = \frac{V}{R_1} e^{-\left(\frac{R_1+R_2}{L}\right)t} \qquad \text{for } t > 0$$

Example 10.24 *In the network shown in Fig. 10.80, the switch is closed at t = 0, a steady state having previously been attained. Find the current i (t).*

Fig. 10.80

Solution At $t = 0^-$, the network is shown in Fig. 10.81.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.81

$$i(0^-) = \frac{V}{R_1 + R_2}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = \frac{V}{R_1 + R_2}$$

For $t > 0$, the network is shown in Fig. 10.82.

Writing the KVL equation for $t > 0$,

Fig. 10.82

$$V - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R_1}{L} i = \frac{V}{L}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = \frac{R_1}{L},\ Q = \frac{V}{L}$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Qe^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-\frac{R_1}{L}t}\int \frac{V}{L} e^{-\frac{R_1}{L}t}\, dt + k\, e^{-\frac{R_1}{L}t}$$

$$= \frac{V}{R_1} + k\, e^{-\frac{R_1}{L}t}$$

At $t = 0$, $i(0) = \dfrac{V}{R_1 + R_2}$

$$\frac{V}{R_1 + R_2} = \frac{V}{R_1} + k$$

$$k = -\frac{VR_2}{R_1(R_1 + R_2)}$$

$$i(t) = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)} e^{-\frac{R_1}{L}t}$$

$$= \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2} e^{-\frac{R_1}{L}t}\right) \quad \text{for } t > 0$$

Example 10.25 *In the network of Fig. 10.83, a steady state is reached with the switch K open. At $t = 0$, the switch K is closed. Find the current $i(t)$ for $t > 0$.*

Fig. 10.83

Solution At $t = 0^-$, the network is shown in Fig. 10.84.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit

$$i(0^-) = \frac{20 + 10}{30 + 20} = 0.6 \text{ A}$$

Fig. 10.84

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 0.6 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.85.

Writing the KVL equation for $t > 0$.

$$10 - 20i - \frac{1}{2}\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 40i = 20$$

Fig. 10.85

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 40, \quad Q = 20$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-40t} \int 20\, e^{40t} dt + k\, e^{-40t}$$

$$= \frac{20}{40} + k\, e^{-40t}$$

$$= 0.5 + k\, e^{-40t}$$

At $t = 0, i(0) = 0.6$ A

$$0.6 = 0.5 + k$$

$$k = 0.1$$

$$i(t) = 0.5 + 0.1\, e^{-40t} \qquad \text{for } t > 0$$

Example 10.26 *The network of Fig. 10.86 is under steady state with switch at the position 1. At t = 0, switch is moved to position 2. Find i (t).*

Fig. 10.86

Solution At $t = 0^-$, the network is shown in Fig. 10.87.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.87

$$i(0^-) = \frac{50}{40} = 1.25 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.25 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.88.

Writing the KVL equation for $t > 0$,

Fig. 10.88

$$10 - 40i - 20 \times 10^{-3} \frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2000i = 500$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 2000, \quad Q = 500$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-2000t} \int 500\, e^{2000t} + k\, e^{-2000t} \\ &= \frac{500}{2000} + k\, e^{-2000t} \\ &= 0.25 + k\, e^{-2000t} \end{aligned}$$

At $t = 0$, $i(0) = 1.25$ A

$$\begin{aligned} 1.25 &= 0.25 + k \\ k &= 1 \\ i(t) &= 0.25 + e^{-2000t} \qquad \text{for } t > 0 \end{aligned}$$

Example 10.27 *In the network of Fig. 10.89, the switch is moved from 1 to 2 at t = 0. Determine i(t).*

Fig. 10.89

Solution At $t = 0^-$, the network is shown in Fig. 10.90.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.90

$$i(0^-) = \frac{20}{5} = 4 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 4 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.91.

Writing the KVL equation for $t > 0$,

Fig. 10.91

$$40 - 2i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 4i = 80$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 4, \quad Q = 80$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-4t} \int 80\, e^{4t}\, dt + k\, e^{-4t}$$

$$= \frac{80}{4} + k\, e^{-4t}$$

$$= 20 + k\, e^{-4t}$$

At $t = 0, i(0) = 4$ A

$$4 = 20 + k$$

$$k = -16$$

$$i(t) = 20 - 16\, e^{-4t} \qquad \text{for } t > 0$$

Example 10.28 *For the network shown in Fig. 10.92, steady state is reached with the switch closed. The switch is opened at t = 0. Obtain expressions for i_L (t) and v_L (t).*

Fig. 10.92

Solution At $t = 0^-$, the network is shown in Fig. 10.93.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = \frac{15}{100} = 0.15 \text{ A}$$

Fig. 10.93

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 0.15 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.94.

Writing the KVL equation for $t > 0$,

$$-3000 i_L - 90 \times 10^{-3} \frac{di_L}{dt} = 0$$

$$\frac{di_L}{dt} + 33.33 \times 10^3\, i_L = 0$$

Fig. 10.94

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 33.33 \times 10^3$$

The solution of this differential equation is given by,

$$i_L(t) = k\,e^{-Pt}$$

$$i_L(t) = k\,e^{-33.33\times10^3 t}$$

At $t = 0$, $i_L(0) = 0.15$ A

$$0.15 = k$$

$$i_L(t) = 0.15\,e^{-33.33\times10^3 t} \qquad \text{for } t > 0$$

Also,

$$v_L(t) = L\frac{di_L}{dt}$$

$$= 90\times10^{-3}\frac{d}{dt}(0.15\,e^{-33.33\times10^3 t})$$

$$= -90\times10^{-3}\times0.15\times33.33\times10^3\times e^{-33.33\times10^3 t}$$

$$= -450\,e^{-33.33\times10^3 t} \qquad \text{for } t > 0$$

Example 10.29 *In the network of Fig. 10.95, the switch is open for a long time and it closes at t = 0. Find i (t).*

Fig. 10.95

Solution At $t = 0^-$, the network is shown in Fig. 10.96.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

Fig. 10.96

$$i(0^-) = \frac{50}{10+10} = 2.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i\,(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.97.

For $t > 0$, representing the network to the left of the inductor by Thevenin's equivalent network,

$$V_{eq} = 50\times\frac{10}{10+10} = 25 \text{ V}$$

$$R_{eq} = (10 \,||\, 10) + 10 = 15\ \Omega$$

Fig. 10.97

For $t > 0$, Thevenin's equivalent network is shown in Fig. 10.98.
Writing the KVL equation for $t > 0$,

$$25 - 15i - 0.1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 150i = 250$$

Fig. 10.98

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 150, \quad Q = 250$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-150t}\int 250\, e^{150t}\, dt + k\, e^{-Pt}$$

$$= \frac{250}{150} + k\, e^{-150t}$$

$$= 1.667 + k\, e^{-150t}$$

At $t = 0$, $i(0) = 2.5$ A

$$2.5 = 1.667 + k$$

$$k = 0.833$$

$$i(t) = 1.667 + 0.833\, e^{-150t} \qquad \text{for } t > 0$$

Example 10.30 *In Fig. 10.99, the switch is closed at t = 0. Find i (t) for t > 0.*

Fig. 10.99

Solution At $t = 0^-$, $\qquad i(0^-) = 0$

Since current through inductor cannot change instantaneously,

$$i(0^+) = 0$$

For $t > 0$, simplifying the network by source-transformation technique as shown in Fig. 10.100.

Fig. 10.100

Writing the KVL equation for $t > 0$,

$$6.67 - 2.67i - 1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2.67i = 6.67$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 2.67,\ Q = 6.67$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt}$$

$$= e^{-2.67t}\int 6.67\,e^{2.67t}\,dt + k\,e^{-2.67t}$$

$$= \frac{6.67}{2.67} + k\,e^{-2.67t}$$

$$= 2.5 + k\,e^{-2.67t}$$

At $t = 0$, $i\,(0) = 0$

$$0 = 2.5 + k$$

$$k = -2.5$$

$$i(t) = 2.5 - 2.5\,e^{-2.67t}$$

$$= 2.5(1 - e^{-2.67t}) \qquad \text{for } t > 0$$

Example 10.31 *Find the current i (t) for t > 0.*

Fig. 10.101

Solution At $t = 0^-$, the inductor acts as a short circuit. Simplifying the network as shown in Fig. 10.102.

Fig. 10.102

$$i(0^-) = 25 \times \frac{140}{140+60} = 17.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 17.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.103.

Fig. 10.103

Simplifying the network by source transformation as shown in Fig. 10.104,

Fig. 10.104

Writing the KVL equation for $t > 0$,

$$-15i - 0.3\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 50i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 50$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-50t}$$

At $t = 0$, $i\,(0) = 17.5$ A

$$k = 17.5$$

$$i(t) = 17.5\,e^{-50t} \qquad \text{for } t > 0$$

Example 10.32 *In the network of Fig. 10.105, the switch is in position 'a' for a long time. At t = 0, the switch is moved from a to b. Find v_2 (t). Assume that the initial current in the 2 H inductor is zero.*

Fig. 10.105

Solution At $t = 0^-$, the switch is in the position *a*. The network has attained steady-state condition. Hence, the inductor acts as a short circuit.
Current through the 1 H inductor is given by

$$i(0^-) = \frac{1}{1} = 1\text{ A}$$

$$v_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1\text{ A}$$

$$v_2(0^+) = -1 \times \frac{1}{2} = -0.5\text{ V}$$

For $t > 0$, the network is shown in Fig. 10.106.
Writing the KCL equation for $t > 0$,

$$\frac{1}{1}\int_0^t v_2 dt + 1 + \frac{v_2}{\frac{1}{2}} + \frac{1}{2}\int_0^t v_2 dt = 0 \qquad \ldots\text{(i)}$$

Fig. 10.106

Differentiating Eq. (i),

$$v_2 + 2\frac{dv_2}{dt} + \frac{1}{2}v_2 = 0$$

$$\frac{dv_2}{dt} + \frac{3}{4}v_2 = 0$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = 0$,

$$P = \frac{3}{4}$$

The solution of this differential equation is given by,

$$v_2(t) = K\,e^{-Pt} = k\,e^{-\frac{3}{4}t}$$

At $t = 0$, $v_2\,(0) = -0.5$ V

$$-0.5 = k\,e^{\circ}$$

$$k = -0.5$$

$$v_2(t) = -0.5\,e^{-\frac{3}{4}t} \qquad \text{for } t > 0$$

Example 10.33 *In the network shown in Fig. 10.107, a steady-state condition is achieved with switch open. At t = 0 switch is closed. Find v_a (t).*

Fig. 10.107

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = 0$$

$$v_a(0^-) = 3 \times \frac{5}{10+5} = 1\text{ V}$$

Since current through inductor cannot change instantaneously,

$$i_L(0^+) = 0$$

$$v_a(0^+) = 1\text{ V}$$

For $t > 0$, the network is shown in Fig. 10.108.

Writing the KCL equation for $t > 0$,

$$\frac{1}{0.5}\int_0^t v_a\,dt + \frac{v_a}{5} + \frac{v_a - 3}{10} = 0$$

Fig. 10.108

Differentiating Eq. (i),

$$2v_a + 0.2\frac{dv_a}{dt} + 0.1\frac{dv_a}{dt} = 0$$

$$\frac{dv_a}{dt} + \frac{20}{3}v_a = 0$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = 0$,

$$P = \frac{20}{3}$$

The solution of this differential equation is given by,

$$v_a(t) = k\,e^{-Pt} = k\,e^{-\frac{20}{3}t}$$

At $t = 0$, $v_a(0) = 1$ V

$$1 = k$$

$$v_a(t) = e^{-\frac{20}{3}t} \qquad \text{for } t > 0$$

Example 10.34 *In the network of Fig. 10.109, determine currents $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$.*

Fig. 10.109

Solution At $t = 0^-$,

$$i_1(0^-) = i_2(0^-) = 0$$

At $t = 0^+$,

$$i_1(0^+) = 0$$

$$i_2(0^+) = \frac{100}{15} = 6.67 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.110.
Writing the KVL equations for $t > 0$,

$$100 - 10(i_1 + i_2) - 5i_1 - 0.01\frac{di_1}{dt} = 0 \qquad \ldots\text{(i)}$$

and

$$100 - 10(i_1 + i_2) - 5i_2 = 0 \qquad \ldots\text{(ii)}$$

Fig. 10.110

From Eq. (ii),

$$i_2 = \frac{100 - 10i_1}{15}$$

Substituting in Eq. (i),

$$\frac{di_1}{dt} + 833i_1 = 3333$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 833, Q = 3333$$

The solution of this differential equation is given by,

$$i_1(t) = e^{-Pt}\int Q\,e^{Pt}dt + k\,e^{-Pt}$$

$$= e^{-833t}\int 3333\,e^{833t}dt + k\,e^{-833t}$$

$$= \frac{3333}{833} + k\,e^{-833t}$$

$$= 4 + k\,e^{-833t}$$

At $t = 0$, $i_1(0) = 0$

$$0 = 4 + k$$

$$k = -4$$

$$i_1(t) = 4 - 4\,e^{-833t}$$

$$= 4(1 - e^{-833t}) \qquad \text{for } t > 0$$

$$i_2(t) = \frac{100 - 10i_1}{15}$$

$$= \frac{100 - 10(4 - 4\,e^{-833t})}{15}$$

$$= 4 + 2.67\,e^{-833t} \qquad \text{for } t > 0$$

Example 10.35 *The switch in the network shown in Fig. 10.111 is closed at $t = 0$. Find $v_2(t)$ for all $t > 0$. Assume zero initial current in the inductor.*

Fig. 10.111

Solution At $t = 0^-$,

$$i_1(0^-) = 0$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_2(0^+) = 0$$

$$i_1(0^+) = \frac{10}{30 + 10} = 0.25 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.112.
Writing the KVL equations for $t > 0$,

$$10 - 30(i_1 + i_2) - 10\,i_1 = 0 \qquad \ldots\text{(i)}$$

and $$10 - 30(i_1 + i_2) - 0.2\frac{di_2}{dt} = 0 \qquad \ldots\text{(ii)}$$

Fig. 10.112

From Eq. (i),

$$i_1 = \frac{10 - 30\,i_2}{40} = 0.25 - 0.75\,i_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) into Eq. (ii),

$$\frac{di_2}{dt} + 37.5\,i_2 = 2.5$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 37.5,\ Q = 2.5$$

The solution of this differential equation is given by,

$$\begin{aligned} i_2(t) &= e^{-Pt}\int Q\,e^{Pt}dt + k\,e^{-Pt} \\ &= e^{-37.5t}\int 2.5\,e^{37.5t}dt + k\,e^{-37.5t} \\ &= \frac{2.5}{37.5} + k\,e^{-37.5t} \\ &= 0.067 + k\,e^{-37.5t} \end{aligned}$$

At $t = 0,\ i_2(0) = 0$

$$\begin{aligned} 0 &= 0.067 + k \\ k &= -0.067 \\ i_2(t) &= 0.067 - 0.067\,e^{-37.5t} \\ v_2(t) &= 0.2\frac{di_2}{dt} \\ &= 0.2\frac{d}{dt}(0.067 - 0.067\,e^{-37.5t}) \\ &= 0.5\,e^{-37.5t} \qquad \text{for } t > 0 \end{aligned}$$

Example 10.36 *For the network shown in Fig. 10.113, find the current i(t) when the switch is changed from the position 1 to 2 at t = 0.*

Fig. 10.113

Solution At $t = 0^-$, the network is shown in Fig. 10.114. At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{500}{40+60} = 5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 5 \text{ A}$$

Fig. 10.114

For $t > 0$, the network is shown in Fig. 10.115. Writing the KVL equation for $t > 0$,

$$10i - 60i - 0.4\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 125i = 0$$

Fig. 10.115

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 125$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-125t}$$

At $t = 0, i(0) = 5$ A

$$5 = k$$

$$i(t) = 5\,e^{-125t} \qquad \text{for } t > 0$$

Example 10.37 *For the network shown in Fig. 10.116, find the current in the 20 Ω resistor when the switch is opened at $t = 0$.*

Fig. 10.116

Solution At $t = 0^-$, the network is shown in Fig. 10.117.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = i_2(0^-)$$

Fig. 10.117

Writing the KVL equations at $t = 0^-$,

$$50 - 30(i_1 - i_2) - 10i_2 = 0$$
$$10i_2 - 30(i_2 - i_1) - 20i_2 = 0$$

Solving these equations,

$$i_1(0^-) = 3.33 \text{ A}$$
$$i_2(0^-) = 2.5 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i_2(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.118.

Fig. 10.118

Writing the KVL equation for $t > 0$,

$$10i_2 - 30i_2 - 20i_2 - 2\frac{di_2}{dt} = 0$$
$$\frac{di_2}{dt} + 20i_2 = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 20$$

The solution of this differential equation is given by,

$$i_2(t) = k\,e^{-Pt} = k\,e^{-20t}$$

At $t = 0, i_2(0) = 2.5$ A

$$2.5 = k$$
$$i_2(t) = 2.5\,e^{-20t} \qquad \text{for } t > 0$$

Example 10.38 *In the network of Fig. 10.119, an exponential voltage $v(t) = 4\,e^{-3t}$ is applied at $t = 0$. Find the expression for current $i(t)$. Assume zero current through inductor at $t = 0$.*

Fig. 10.119

Solution At $t = 0^-$, $\qquad i(0^-) = 0$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 0$$

Writing the KVL equation for $t > 0$,

$$4e^{-3t} - 0.5i - 0.25\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 2i = 16e^{-3t}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 2, \quad Q = 16\,e^{-3t}$$

The solution of this differential equation is given by,

$$\begin{aligned} i(t) &= e^{-Pt}\int Q e^{Pt}\,dt + k\,e^{-Pt} \\ &= e^{-2t}\int 16e^{-3t}\,e^{2t}\,dt + +k\,e^{-2t} \\ &= 16\,e^{-2t}\int e^{-t}\,dt + k\,e^{-2t} \\ &= -16\,e^{-3t} + k\,e^{-2t} \end{aligned}$$

At $t = 0, i(0) = 0$

$$0 = -16 + k$$

$$k = 16$$

$$i(t) = -16\,e^{-3t} + 16\,e^{-2t} \qquad \text{for } t > 0$$

Example 10.39 *For the network shown in Fig. 10.120, a sinusoidal voltage source $v = 150\sin(500t + \theta)$ volts is applied at a time when $\theta = 0$. Find the expression for the current $i(t)$.*

Fig. 10.120

Solution

Writing the KVL equation for $t > 0$,

$$150\sin(500t + \theta) - 50i - 0.2\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 250i = 750\sin(500t + \theta)$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 250, \quad Q = 750\sin(500t + \theta)$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt}$$

$$= e^{-250t}\int 750\sin(500t+\theta)\,e^{250t} + k\,e^{-250t}$$

$$= 750\,e^{-250t}\left[\frac{e^{250t}}{(250)^2+(500)^2}\{250\sin(500t+\theta)-500\cos(500t+\theta)\}\right] + k\,e^{-250t}$$

$$= 0.6\sin(500t+\theta) - 1.2\cos(500t+\theta) + k\,e^{-250t}$$

Let $\quad A\cos\phi = 0.6$

and $\quad A\sin\phi = 1.2$

$$A^2\cos^2\phi + A^2\sin^2\phi = (0.6)^2 + (1.2)^2 = 1.8$$

$$A = 1.342$$

and $\quad \phi = \tan^{-1}\left(\frac{1.2}{0.6}\right) = 63.43°$

$$i(t) = 1.342\cos(63.43°)\sin(500t+\theta) - 1.342\sin(63.43°)\cos(500t+\theta) + k\,e^{-250t}$$

$$i(t) = 1.342\sin(500t+\theta-63.43°) + k\,e^{-250t}$$

At $t = 0, \theta = 0, i(0) = 0$

$$0 = 1.342\sin(-63.43°) + k$$

$$k = 1.2$$

$$i(t) = 1.342\sin(500t+\theta-63.43°) + 1.2\,e^{-250t} \qquad \text{for } t > 0$$

Example 10.40 *For the network shown in Fig. 10.121, find the transient current when the switch is moved from the position 1 to 2 at $t = 0$. The network is in steady state with the switch in the position 1. The voltage applied to the network is $v = 150\cos(200t + 30°)\,V$.*

Fig. 10.121

Solution At $t = 0^-$, the network is shown in Fig 10.122.

At $t = 0^-$ the network attains steady-state condition.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{150\angle 30°}{200 + j\times 200\times 0.5} = 0.67\angle 3.43°\ \text{A}$$

Fig. 10.122

The steady-state current passing through the network when the switch is in the position 1 is

$$i = 0.67\cos(200t + 3.43°) \qquad \ldots(i)$$

For $t > 0$, the network is shown in Fig. 10.123.
Writing the KVL equation for $t > 0$,

Fig. 10.123

$$-200\,i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 400\,i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 400$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-pt} = k\,e^{-400t} \qquad \ldots(ii)$$

From Eqs (i) and (ii),

$$0.67\cos(200t + 3.43°) = k\,e^{-400t}$$

$$0.67\cos(3.43°) = k$$

At $t = 0$,

$$k = 6.67$$

$$i(t) = 0.67\,e^{-400t} \qquad \text{for } t > 0$$

10.4 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 10.124. The switch is closed at time $t = 0$. The capacitor is initially uncharged. Applying KVL to the circuit for $t > 0$,

Fig. 10.124 *Series RC circuit*

$$V - Ri - \frac{1}{C}\int_0^1 i\,dt = 0$$

Differentiating the above equation,

$$0 - R\frac{di}{dt} - \frac{i}{C} = 0$$

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

This is a linear differential equation of first order. The variables may be separated to solve the equation.

$$\frac{di}{i} = -\frac{dt}{RC}$$

Integrating both the sides,

$$l_n i = -\frac{1}{RC}t + k$$

The constant k can be evaluated from initial condition. In the circuit shown, the switch is closed at $t = 0$. Since the capacitor never allows sudden change in voltage, it will act as short circuit at $t = 0^+$. Hence, current in the circuit at $t = 0^+$ is $\frac{V}{R}$.

Setting $i = \frac{V}{R}$ at $t = 0$,

$$l_n \frac{V}{R} = k$$

$$l_n i = -\frac{1}{RC} t + l_n \frac{V}{R}$$

$$l_n i - l_n \frac{V}{R} = -\frac{1}{RC} t$$

$$l_n \frac{i}{\left(\frac{V}{R}\right)} = -\frac{1}{RC} t$$

$$\frac{i}{\frac{V}{R}} = e^{-\frac{1}{RC}t}$$

$$i = \frac{V}{R} e^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

When the switch is closed, the response decays with time as shown in Fig. 10.125(a).

The term RC is called time constant and is denoted by T.

$$T = RC$$

After 5 time constants, the current drops to 99 per cent of its initial value.

The voltage across the resistor is

$$v_R = Ri = R\frac{V}{R} e^{-\frac{1}{RC}t}$$

$$= Ve^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

Similarly, the voltage across the capacitor is

$$v_C = \frac{1}{C}\int_0^t i\, dt$$

$$= \frac{1}{C}\int_0^t \frac{V}{R} e^{-\frac{1}{RC}t}$$

$$= -Ve^{-\frac{1}{RC}t} + k$$

Fig. 10.125(a) *Current response of series RC circuit*

Fig. 10.125(b) *Voltage response of series RC circuit*

At $t = 0$, $v_C(0) = 0$

$$k = V$$

Hence,
$$v_C = V\left(1 - e^{-\frac{1}{RC}t}\right) \quad \text{for } t > 0$$

Example 10.41 *The switch in the circuit of Fig. 10.126 is moved from the position 1 to 2 at t = 0. Find $v_C(t)$.*

Fig. 10.126

Solution At $t = 0^-$, the network is shown in Fig. 10.127.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

Fig. 10.127

$$v_C(0^-) = 100 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 100 \text{ V}$$

For $t > 0$, the network is shown in Fig. 10.128.

Fig. 10.128

Writing the KCL equation for $t > 0$,

$$1 \times 10^{-6}\frac{dv_C}{dt} + \frac{v_C + 50}{5000} = 0$$

$$\frac{dv_C}{dt} + 200v_C = 10^4$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 200, \quad Q = 10^4$$

Solution of this differential equation is given by,

$$v_C(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-200t}\int 10^4\, e^{200t}\, dt + k\, e^{-200t}$$

$$= \frac{10^4}{200} + k\, e^{-200t}$$

$$= -50 + k\, e^{-200t}$$

At $t = 0$, $v_C(0) = 100$ V

$$100 = -50 + k$$
$$k = 150$$
$$v_C(t) = -50 + 150\, e^{-200t} \qquad \text{for } t > 0$$

Example 10.42 *In the network shown in Fig. 10.129, the switch closes at t = 0. The capacitor is initially uncharged. Find v_C (t) and i_C (t).*

Fig. 10.129

Solution At $t = 0^-$, the capacitor is uncharged. Hence, it acts as a short circuit.

$$v_C(0\ ) = 0$$
$$i_C(0\ ) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.130.

Since voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

At $t = 0^+$, $\quad i_T(0^+) = \left[\dfrac{10}{9\text{ k} + (4\text{ k} \,\|\, 1\text{ k})}\right] = \dfrac{10}{9.8\text{ k}} = 1.02\text{ mA}$

$$i_C(0^+) = 1.02\text{ m} \times \frac{1\text{ k}}{1\text{ k} + 4\text{ k}} = 0.204\text{ mA}$$

Fig. 10.130

For $t > 0$, the network is shown in Fig. 10.131.

For $t > 0$, representing the network to the left of the capacitor by Thevenin's equivalent network,

$$V_{eq} = 10 \times \frac{1\text{ k}}{9\text{ k} + 1\text{ k}} = 1\text{ V}$$
$$R_{eq} = (9\text{ k} \,\|\, 1\text{ k}) + 4\text{ k} = 4.9\text{ k}\Omega$$

Fig. 10.131

For $t > 0$, Thevenin's equivalent network is shown in Fig. 10.132.
Writing the KCL equation for $t > 0$,

$$3 \times 10^{-6} \frac{dv_C}{dt} + \frac{v_C - 1}{4.9 \times 10^3} = 0$$
$$\frac{dv_C}{dt} + 68.02\, v_C = 68.02$$

Fig. 10.132

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 68.02, \quad Q = 68.02$$

The solution of this differential equation is given by,

$$v_C(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-68.02t}\int 68.02\, e^{68.02t}\, dt + k\, e^{-68.02t}$$

$$= 1 + k\, e^{-68.02t}$$

At $t = 0, v_C(0) = 0$

$$0 = 1 + k$$

$$k = -1$$

$$v_C(t) = 1 - e^{-68.02t} \qquad \text{for } t > 0$$

$$i_C(t) = C\frac{dv_C}{dt}$$

$$= 3\times10^{-6}\frac{d}{dt}(1 - e^{-68.02t})$$

$$= 3\times10^{-6}\times68.02e^{-68.02t}$$

$$= 204.06\times10^{-6}e^{-68.02t} \qquad \text{for } t > 0$$

Example 10.43 *For the network shown in Fig. 10.133, the switch is open for a long time and closes at t = 0. Determine v_C (t).*

Fig. 10.133

Solution At $t = 0^-$, the network is shown in Fig. 10.134.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 1200 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 1200 \text{ V}$$

Fig. 10.134

For $t > 0$, the network is shown in Fig. 10.135.
Writing the KCL equation for $t > 0$,

Fig. 10.135

$$50\times10^{-6}\frac{dv_C}{dt}+\frac{v_C}{300}+\frac{v_C-1200}{100}=0$$

$$\frac{dv_C}{dt}+266.67\,v_C=0.24\times10^6$$

Comparing with the differential equation $\frac{dv}{dt}+Pv=Q$,

$$P=266.67,\quad Q=0.24\times10^6$$

The solution of this differential equation is given by,

$$v_C(t)=e^{-Pt}\int Q\,e^{Pt}\,dt+k\,e^{-Pt}$$

$$=e^{-266.67t}\int 0.24\times10^6\,e^{266.67t}\,dt+k\,e^{-266.67t}$$

$$=\frac{0.24\times10^6}{266.67}+k\,e^{-266.67t}$$

$$=900+k\,e^{-266.67t}$$

At $t=0, v_C(0)=1200$ V

$$1200=900+k$$

$$k=300$$

$$v_C(t)=900+300\,e^{-266.67t}\qquad \text{for } t>0$$

Example 10.44 *In Fig. 10.136, the switch is closed at t = 0 Find v_C (t) for t > 0.*

Fig. 10.136

Solution At $t=0^-$, $\quad v_C(0^-)=0$
Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+)=0$$

Since the resistor of 2 Ω is connected in parallel with the voltage source of 5 V, it becomes redundant.

For $t > 0$, the network is as shown in Fig. 10.137.
Writing KCL equation for $t > 0$,

Fig. 10.137

$$\frac{v_C-5}{100}+1\frac{dv_C}{dt}=0$$

$$100\frac{dv_C}{dt}+v_C=5$$

$$\frac{dv_C}{dt} + 0.01v_C = 0.05$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 0.01, \quad Q = 0.05$$

The solution of this differential equation is given by,

$$v_C(t) = e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt}$$

$$= e^{-0.01}\int 0.05\,e^{0.01t}\,dt + k\,e^{-0.01t}$$

$$= \frac{0.05}{0.01} + k\,e^{-0.01t}$$

$$= 5 + k\,e^{-0.01t}$$

At $t = 0, v_C(0) = 0$

$$0 = 5 + k$$

$$k = -5$$

$$v_C(t) = 5 - 5e^{-0.01t}$$

$$= 5(1 - e^{-0.01t}) \qquad \text{for } t > 0$$

Example 10.45 *In the network shown, the switch is shifted to position b at t = 0. Find v (t) for t > 0.*

Fig. 10.138

Solution At $t = 0^-$, the network is shown in Fig. 10.139.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 5 \text{ V}$$

$$v(0^-) = 0$$

Fig. 10.139

At $t = 0^+$, the network is shown in Fig. 10.140.

At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$i(0^+) = -\frac{5}{4} = -1.25 \text{ A}$$

$$v(0^+) = -1.25 \times 2 = -2.5 \text{ V}$$

Fig. 10.140

For $t > 0$, the network is shown in Fig. 10.141.
Writing the KVL equation for $t > 0$,

$$-2i - 5 - \frac{1}{\frac{1}{4}}\int_0^t i\,dt - 2i = 0 \qquad \ldots\text{(i)}$$

Fig. 10.141

Differentiating Eq. (i),

$$-4\frac{di}{dt} - 4i = 0$$

$$\frac{di}{dt} + i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-t}$$

At $t = 0, i(0) = -1.25$ A

$$k = -1.25$$

$$i(t) = -1.25\,e^{-t} \quad \text{for } t > 0$$

$$v(t) = 2i(t)$$

$$= -2.5e^{-t} \quad \text{for } t > 0$$

Example 10.46 *In the network of Fig. 10.142, the switch is open for a long time and at t = 0, it is closed. Determine v_2 (t).*

Fig. 10.142

Solution At $t = 0^-$, the switch is open.

$$v_2(0^-) = 0$$

Since voltage across capacitor cannot change instantaneously,

$$v_2(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.143.

Fig 10.143

Writing KCL equation for $t > 0$,

$$\frac{v_2}{\frac{1}{2}} + 0.3\frac{dv_2}{dt} + \frac{v_2 - 6}{0.25} = 0$$

$$\frac{dv_2}{dt} + 20v_2 = 80$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 20, \quad Q = 80$$

The solution of this differential equation is given by,

$$v(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-20t}\int 80\, e^{20t}\, dt + k\, e^{-20t}$$

$$= \frac{80}{20} + k\, e^{-20t}$$

$$v_2 t = 4 + k\, e^{-20t}$$

At $t = 0, v_2(0) = 0$

$$0 = 4 + k$$

$$k = -4$$

$$v_2(t) = 4 - 4\, e^{-20t}$$

$$= 4(1 - e^{-20t}) \qquad \text{for } t > 0.$$

Example 10.47 *The switch is moved from the position a to b at t = 0, having been in the position a for a long time before t = 0. The capacitor C_2 is uncharged at t = 0. Find i (t) and v_2 (t) for t > 0.*

Fig. 10.144

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor C_1 acts as an open circuit and it will charge to V_0 volt.

$$v_{C_1}(0^-) = V_0$$

$$v_{C_2}(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_{C_1}(0^+) = V_0$$

$$v_{C_2}(0^+) = 0$$

$$i(0^+) = \frac{V_0}{R_1}$$

For $t > 0$, the network is shown in Fig. 10.145.
Writing the KVL equation for $t > 0$,

$$V_0 - \frac{1}{C_1}\int_0^t i\,dt - R_1 i - \frac{1}{C_2}\int_0^t i\,dt = 0 \qquad \text{...(i)}$$

Fig. 10.145

Differentiating Eq. (i),

$$-\frac{i}{C_1} - R_1\frac{di}{dt} - \frac{i}{C_2} = 0$$

$$\frac{di}{dt} + \frac{1}{R_1}\left(\frac{C_1 + C_2}{C_1 C_2}\right) i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = \frac{1}{R_1}\left(\frac{C_1 + C_2}{C_1 C_2}\right)$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

At $t = 0$, $i(0) = \frac{V_0}{R_1}$

$$k = \frac{V_0}{R_1}$$

$$i(t) = \frac{V_0}{R_1} e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

$$= \frac{V_0}{R_1} e^{-\frac{1}{R_1 C}t} \qquad \text{where, } C = \frac{C_1 C_2}{C_1 + C_2}$$

$$v_2(t) = \frac{1}{C_2}\int_0^t i\,dt$$

$$= \frac{1}{C_2}\int_0^t \frac{V_0}{R_1} e^{-\frac{t}{R_1 C}}\,dt$$

$$= \frac{V_0}{R_1 C_2} R_1 C\left(1 - e^{-\frac{1}{R_1 C}t}\right)$$

$$= \frac{V_0 C_1}{C_1 + C_2}\left(1 - e^{-\frac{t}{R_1 C}t}\right) \qquad \text{for } t > 0$$

Example 10.48 *For the network shown in Fig.10.146, the switch is opened at $t = 0$. Find v (t) for $t > 0$.*

Fig. 10.146

Solution At $t = 0^-$, the network is shown in Fig. 10.147.

Fig. 10.147

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 0$$

Writing the KCL equation at $t = 0^-$,

$$\frac{v(0^-)}{1} + \frac{v(0^-)}{\frac{1}{2}} = 10$$

$$3v(0^-) = 10$$

$$v(0^-) = 3.33 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = v(0^+) = 3.33 \text{ V}$$

For $t > 0$, the network is shown in Fig. 10.148.

Fig. 10.148

Writing the KCL equation for $t > 0$,

$$\frac{1}{2}\frac{dv}{dt} + \frac{v}{1} = 10$$

$$\frac{dv}{dt} + 2v = 20$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 2, \quad Q = 20$$

The solution of this differential equation is given by,

$$v(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-2t}\int 20\, e^{-2t}\, dt + k\, e^{-2t}$$

$$= \frac{20}{2} + k\, e^{-2t}$$

$$= 10 + k\, e^{-2t}$$

At $t = 0$, $v(0) = 3.33$ V

$$3.33 = 10 + k$$
$$k = 6.67$$
$$v(t) = 10 + 6.67\, e^{-2t}$$

Example 10.49 *For the network shown in Fig. 10.149, find the current i(t) when the switch is opened at $t = 0$.*

Fig. 10.149

Solution At $t = 0^-$, the network is shown in Fig. 10.150.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as open circuit.

Fig. 10.150

$$i(0^-) = \frac{100}{10+10} = 5 \text{ A}$$

$$v_C(0^-) = 100 - 10i(0^-) - 5i(0^-) = 100 - 10(5) - 5(5) = 25 \text{ V}$$

At $t = 0^+$, the network is shown in Fig. 10.151.

Fig. 10.151

$$25 + 5i(0^-) - 10i(0^+) = 0$$
$$i(0^+) = 5 \text{ A}$$

For $t > 0$, the network is shown in Fig. 10.152.

$$25 - \frac{1}{4\times10^{-6}}\int_0^t i\, dt + 5i - 10i = 0 \qquad \ldots\text{(i)}$$

Fig. 10.152

Differentiating Eq. (i),

$$0 - 0.25\times10^6\, i + 5\frac{di}{dt} - 10\frac{di}{dt} = 0$$
$$\frac{di}{dt} + 50000\, i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 50000$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-50000t}$$

At $t = 0$, $i(0) = 5$ A

$$5 = k$$

$$i(t) = 5\,e^{-50000t} \qquad \text{for } t > 0$$

Example 10.50 *For the network shown in Fig. 10.153, find the current i(t) when the switch is opened at $t = 0$.*

Fig. 10.153

Solution At $t = 0^-$, the network is shown in Fig. 10.154. At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

Writing the KVL equation at $t = 0^-$,

$$100 - 10\,i(0^-) - 20\,i(0^-) - 20\,i(0^-) = 0$$

$$i(0^-) = 2 \text{ A}$$

Also, $$20\,i(0^-) + 20\,i(0^-) - 0 - v_C(0^-) = 0$$

$$v_C(0^-) = 40\,i(0^-) = 40(2) = 80 \text{ V}$$

Fig. 10.154

At $t = 0^+$, the network is shown in Fig. 10.155.

From Fig. 10.155, $i(0^+) = -i_2(0^+)$

$$20\,i(0^+) - 20\,i_2(0^+) - 10\,i_2(0^+) - 80 = 0$$

$$20\,i(0^+) + 20\,i(0^+) + 10\,i(0^+) - 80 = 0$$

$$i(0^+) = 1.6 \text{ A}$$

$$v_C(0^+) = 80 \text{ V}$$

Fig. 10.155

For $t > 0$, the network is shown in Fig. 10.156.

From Fig. 10.156, $i(t) = -i_2(t)$

Writing the KVL equation for $t > 0$,

$$20i - 20i_2 - 10i_2 - \frac{1}{2\times10^{-6}}\int_0^t i_2\,dt - 80 = 0$$

Fig. 10.156

$$20i + 20i_2 + 10i_2 + \frac{1}{2\times10^{-6}}\int_0^t i\,dt - 80 = 0$$

$$50i + \frac{1}{2\times10^{-6}}\int i\,dt - 80 = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$50\frac{di}{dt} + 5\times10^5 i = 0$$

$$\frac{di}{dt} + 1\times10^4 i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1\times10^4$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-1\times10^4 t}$$

At $t = 0$, $i(0) = 1.6$ A

$$1.6 = k$$

$$i(t) = 1.6\,e^{-1\times10^4 t} \qquad \text{for } t > 0$$

Example 10.51 *In the network of Fig. 10.157, an exponential voltage $4e^{-5t}$ is applied at time $t = 0$. Find the expression for current i(t). Assume zero voltage across the capacitor at $t = 0$.*

Fig. 10.157

Solution At $t = 0^-$,

$$v_C(0^-) = 0$$

$$i(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 10.158.
Since voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

$$i(0^+) = \frac{4}{0.2} = 20 \text{ A}$$

Fig. 10.158

Writing the KVL equation for $t > 0$,

$$4e^{-5t} - 0.2i - \frac{1}{1}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$-20\,e^{-5t} - 0.2\frac{di}{dt} - i = 0$$

$$\frac{di}{dt} + 5i = -100\,e^{-5t}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 5, \quad Q = -100\,e^{-5t}$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt}$$

$$= e^{-5t}\int -100\,e^{-5t}\,e^{5t}\,dt + k\,e^{-5t}$$

$$= -100t\,e^{-5t} + k\,e^{-5t}$$

At $t = 0$, $i(0) = 20$ A

$$20 = k$$

$$i(t) = -100\,t\,e^{-5t} + 20\,e^{-5t} \qquad \text{for } t > 0$$

Example 10.52 *In the network shown in Fig. 10.159, the switch is closed at $t = 0$ connecting a source e^{-t} to the network. At $t = 0$, $v_C(0) = 0.5$ V. Determine $v(t)$.*

Fig. 10.159

Solution At $t = 0^-$, $\qquad v(0) = v_C(0^-) = 0.5$ V

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = v_C(0^+) = 0.5 \text{ V}$$

Writing the KCL equation for $t > 0$,

$$\frac{v - e^{-t}}{1} + \frac{v}{\frac{1}{2}} + 1\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + 3v = e^{-t}$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 3, \quad Q = e^{-t}$$

The solution of this differential equation is given by,

$$v(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$
$$= e^{-3t}\int e^{-t}\, e^{3t}\, dt + k\, e^{-3t}$$
$$= e^{-3t}\int e^{2t}\, dt + k\, e^{-3t}$$
$$= \frac{1}{2}e^{-t} + k\, e^{-3t}$$

At $t = 0$, $\quad v(0) = 0.5$ V

$$0.5 = \frac{1}{2} + k$$
$$k = 0$$
$$v(t) = 0.5\, e^{-t}$$

Example 10.53 *In the network shown in Fig. 10.160, a sinusoidal voltage $v = 100\sin(500t + \theta)$ volts is applied to the circuit at a time corresponding to $\theta = 45°$. Obtain the expression for the current $i(t)$.*

Fig. 10.160

Solution

Writing the KVL equation for $t > 0$,

$$100\sin(500t + 45°) - 15i - \frac{1}{100\times 10^{-6}}\int_0^t i\, dt = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$(100)(500)\cos(500t + 45°) - 15\frac{di}{dt} - 10^4 i = 0$$
$$\frac{di}{dt} + 666.67i = 3333.33\cos(500t + 45°)$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 666.67,\ Q = 3333.33\cos(500t + 45°)$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt}\int Q\, e^{Pt} dt + k\, e^{-Pt}$$
$$= e^{-666.67t}\int 3333.33\cos(500t + 45°)\, e^{666.67t} + k\, e^{-666.67t}$$

$$= 3333.33\, e^{-666.67t}\left[\frac{e^{666.67t}}{(666.67)^2+(500)^2}\{666.67\cos(500t+45°)+500\sin(500t+45°)\}\right]+k\,e^{-666.67t}$$

$$= 3.2\cos(500t+45°)+2.4\sin(500t+45°)+k\,e^{-666.67t}$$

Let $\quad A\sin\phi = 3.2$

and $\quad A\cos\phi = 2.4$

$$A^2\sin^2\phi + A^2\cos^2\phi = (3.2)^2+(2.4)^2 = 16$$

$$A = 4$$

and $\quad \phi = \tan^{-1}\left(\frac{3.2}{2.4}\right) = 53.13°$

$$i(t) = 4\sin(53.13°)\cos(500t+45°)+4\cos(53.13°)\sin(500t+45°)+k\,e^{-666.67t}$$

$$= 4\sin(500t+98.13°)+k\,e^{-666.67t}$$

Putting $t = 0$, in Eq. (i),

$$100\sin(45°)-15\,i(0)-0 = 0$$

$$i(0) = 4.71$$

$$4.71 = 4\sin(98.13°)+k$$

$$k = 0.75$$

$$i(t) = 4\sin(500t+98.13°)+0.75\,e^{-666.67t} \qquad \text{for } t>0$$

Example 10.54 *In the network of Fig. 10.161, the switch is moved from the position 1 to 2 at $t = 0$. The switch is in the position 1 for a long time. Initial charge on the capacitor is 7×10^{-4} coulombs. Determine the current expression i(t), when $\omega = 1000$ rad/s.*

Fig. 10.161

Solution At $t = 0^-$, the network is shown in Fig. 10.162. At $t = 0^-$, the network attains steady-state condition.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100\angle 30°}{50 - j\dfrac{1}{1000\times20\times10^{-6}}} = 1.41\angle 75°\ \text{A}$$

Fig. 10.162

The steady-state current passing through the network when the switch is in the position 1 is

$$i = 1.41\sin(1000t+75°) \qquad \ldots\text{(i)}$$

For $t > 0$, the network is as shown in Fig. 10.163.
Writing the KVL equation for $t > 0$,

$$-50i - 50i - \frac{1}{20\times10^{-6}}\int_0^t i\,dt - v_C(0) = 0 \qquad \ldots\text{(ii)}$$

Fig. 10.163

Differentiating Eq. (ii),

$$-50\frac{di}{dt} - 50\frac{di}{dt} - \frac{1}{20\times10^{-6}}i = 0$$

$$\frac{di}{dt} + 500i = 0$$

Comparing with differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 500$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-500t} \qquad \ldots\text{(iii)}$$

From Eqs (i) and (ii),

$$1.41\sin(1000t + 75°) = k\,e^{-500t}$$

At $t = 0$,

$$1.41\sin(75°) = k$$

$$k = 1.36$$

$$i(t) = 1.36\,e^{-500t} \quad \text{for } t > 0$$

10.5 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit as shown in Fig. 10.164. The switch is closed at time $t = 0$. The capacitor and inductor are initially uncharged.

Applying KVL to the circuit for $t > 0$,

$$V - Ri - L\frac{di}{dt} - \frac{1}{C}\int_0^t i\,dt = 0$$

Fig. 10.164 *Series RLC circuit*

Differentiating the above equation,

$$0 - R\frac{di}{dt} - L\frac{d^2i}{dt^2} - \frac{1}{C}i = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is a second-order differential equation. The auxiliary equation or characteristic equation will be given by,

$$s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$$

Let s_1 and s_2 be the roots of the equation.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where $$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and $$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

The solution of the above second-order differential equation will be given by,

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation.
Now depending upon the values of α and ω_0, we have 3 cases of the response.

Case I *When* $\alpha > \omega_0$

i.e., $$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

The roots are real and unequal and it gives an overdamped response.
In this case, the solution is given by,

$$i = e^{-\alpha t}(k_1 e^{\beta t} + k_2 e^{-\beta t})$$

or $$i = k_1 e^{s_1 t} + k_2 e^{s_2 t} \quad \text{for } t > 0$$

The current curve for an overdamped case is shown in Fig. 10.165.

Fig. 10.165 *Overdamped response*

Case II *When* $\alpha = \omega_0$

i.e., $$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

The roots are real and equal and it gives a critically damped response.
In this case the solution is given by,

$$i = e^{-\alpha t}(k_1 + k_2 t) \quad \text{for } t > 0$$

The current curve for critically damped case is shown in Fig. 10.166.

Fig. 10.166 *Critically damped response*

Case III *When* $\alpha < \omega_0$

i.e., $$\frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

The roots are complex conjugate and it gives an underdamped response.

In this case, the solution is given by,

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 + \omega_0^2}$$

Let $$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$$

where $$j = \sqrt{-1}$$

and $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Hence, $$i(t) = e^{-\alpha t}(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t})$$

$$= e^{-\alpha t}\left\{(k_1 + k_2)\left[\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right] + j(k_1 - k_2)\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{j2}\right]\right\}$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \quad \text{for } t > 0$$

The current curve for an underdamped case is shown in Fig. 10.167.

Fig. 10.167 *Underdamped response*

Example 10.55 *In the network of Fig. 10.168, the switch is closed at $t = 0$. Obtain the expression for current $i(t)$ for $t > 0$.*

Fig. 10.168

Solution At $t = 0^-$, the switch is open.

$$i(0^-) = 0$$

$$v_C(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i(0^+) = 0$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.169.

Writing the KVL equation for $t > 0$,

$$20 - 9i - 1\frac{di}{dt} - \frac{1}{0.05}\int_0^t i\,dt = 0 \qquad \ldots\text{(i)}$$

Fig. 10.169

Differentiating Eq. (i),

$$0 - 9\frac{di}{dt} - \frac{d^2i}{dt^2} - 20i = 0$$

$$\frac{d^2i}{dt^2} + 9\frac{di}{dt} + 20i = 0$$

$$(D^2 + 9D + 20)i = 0$$

$$D_1 = -4,\ D_2 = -5$$

The solution of this differential equation is given by,

$$i(t) = k_1 e^{-4t} + k_2 e^{-5t} \qquad \ldots\text{(ii)}$$

Differentiating Eq. (ii),

$$\frac{di}{dt} = -4k_1 e^{-4t} - 5k_2 e^{-5t} \qquad \ldots\text{(iii)}$$

At $t = 0, i(0) = 0$

$$0 = k_1 + k_2 \qquad \ldots\text{(iv)}$$

$$\frac{di}{dt}(0) = -4k_1 - 5k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$20 - 9i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 20 - 9i(0^+) = 20 \text{ A/s}$$

From Eq. (v),

$$20 = -4k_1 - 5k_2 \qquad \ldots\text{(vi)}$$

Solving Eqs (iv) and (vi),

$$k_1 = 20$$

$$k_2 = 20$$

$$i(t) = 20e^{-4t} - 20e^{-5t} \qquad \text{for } t > 0$$

Example 10.56 *In the network shown in Fig. 10.170, the switch is moved from the position 1 to 2 at $t = 0$. The switch is in the position 1 for a long time. Determine the expression for the current $i(t)$.*

Fig. 10.170

Solution At $t = 0^-$, the network is shown in Fig. 10.171.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

Fig. 10.171

$$v_C(0^-) = 20 \text{ V}$$

$$i(0^-) = 0$$

Since the current through the inductor and the voltage across the capacitor cannot change instaneously,

$$v_C(0^+) = 20 \text{ V}$$

$$i(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.172.

Writing the KVL equation for $t > 0$,

Fig. 10.172

$$20 - 10i - 2\frac{di}{dt} - \frac{1}{3}\int_0^t i\,dt - 20 = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$0 - 10\frac{di}{dt} - 2\frac{d^2i}{dt^2} - \frac{1}{3}i - 0 = 0$$

$$\frac{d^2i}{dt^2} + 5\frac{di}{dt} + \frac{1}{6}i = 0$$

$$\left(D^2 + 5D + \frac{1}{6}\right)i = 0$$

$$D_1 = -0.03,\ D_2 = -4.97$$

The solution of this differential equation is given by,

$$i(t) = k_1\, e^{-0.03t} + k_2\, e^{-4.97t} \qquad \ldots\text{(ii)}$$

Differentiating Eq. (ii),

$$\frac{di}{dt} = -0.03k_1\, e^{-0.03t} - 4.97k_2\, e^{-4.97t} \qquad \ldots\text{(iii)}$$

At $t = 0, i(0) = 0$

$$0 = k_1 + k_2 \qquad \ldots\text{(iv)}$$

$$\frac{di}{dt}(0) = -0.03\, k_1 - 4.97\, k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$20 - 10\, i(0^+) - 2\frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = \frac{20 - 10\, i(0^+)}{2} = 10 \text{ A/s}$$

From Eq. (v),

$$10 = -0.03\, k_1 - 4.97\, k_2 \qquad \ldots\text{(vi)}$$

Solving Eqs (iv) and (vi),

$$k_1 = 2.02$$

$$k_2 = 2.02$$

$$i(t) = 2.02\, e^{-0.03t} - 2.02\, e^{-4.97t} \quad \text{for } t > 0$$

Example 10.57 *In the network of Fig. 10.173, the switch is closed and a steady state is reached in the network. At $t = 0$, the switch is opened. Find the expression for the current $i_2(t)$ in the inductor.*

Fig. 10.173

Solution At $t = 0^-$, the network is shown in Fig. 10.174.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

Fig. 10.174

$$i_2(0^-) = \frac{100}{10} = 10 \text{ A}$$

$$v_C(0^-) = 0$$

Since current through the inductor and voltage across capacitor cannot change instantaneously,

$$i_2(0^+) = 10 \text{ A}$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.175.
Writing the KVL equation for $t > 0$,

$$-1\frac{di_2}{dt} - \frac{1}{10 \times 10^{-6}} \int_0^t i\, dt = 0 \qquad \ldots\text{(i)}$$

Fig. 10.175

Differentiating Eq. (i),

$$-\frac{d^2 i_2}{dt^2} - 10^5 i = 0$$

$$\frac{d^2 i_2}{dt^2} + 10^5 i = 0$$

$$(D^2 + 10^5) i = 0$$

$$D_1 = j\,316,\ D_2 = -j\,316$$

The solution of this differential equation is given by,

$$i_2(t) = k_1 \cos 316t + k_2 \sin 316t \qquad \ldots\text{(ii)}$$

Differentiating Eq. (ii),

$$\frac{di_2}{dt} = -316\, k_1 \sin 316t + 316\, k_2 \cos 316t \qquad \ldots\text{(iii)}$$

At $t = 0, i_2(0) = 10$ A

$$10 = k_1 \qquad \ldots\text{(iv)}$$

$$\frac{di_2}{dt}(0) = 316\, k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$-\frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 0$$

From Eq. (v),

$$0 = 316\, k_2$$

$$k_2 = 0$$

$$i_2(t) = 10 \cos 316t \quad \text{for } t > 0$$

Example 10.58 *In the network of Fig. 10.176, capacitor C has an initial voltage $v_c(0^-)$ of 10 V and at the same instant, current in the inductor L is zero. The switch is closed at time $t = 0$. Obtain the expression for the voltage $v(t)$ across the inductor L.*

Fig. 10.176

Solution At $t = 0^-$,

$$i_L(0^-) = 0$$

$$v(0^-) = v_C(0^-) = 10 \text{ V}$$

Since current through the inductor and voltage across capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$

$$v(0^+) = v_C(0^+) = 10 \text{ V}$$

For $t > 0$, the network is shown in Fig. 10.177.
Writing the KCL equation for $t > 0$,

$$1\frac{dv}{dt} + \frac{v}{\frac{1}{4}} + \frac{1}{\frac{1}{2}}\int_0^t v\, dt = 0 \qquad \ldots\text{(i)}$$

Fig. 10.177

Differentiating Eq. (i),

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 2v = 0$$

$$(D^2 + 4D + 2)v = 0$$

$$D_1 = -1,\ D_2 = -3$$

The solution of this differential equation is given by,

$$v(t) = k_1 e^{-t} + k_2 e^{-3t} \qquad \ldots\text{(ii)}$$

Differentiating Eq. (iii),

$$\frac{dv}{dt} = -k_1 e^{-t} - 3k_2 e^{-3t} \qquad \ldots\text{(iii)}$$

At $t = 0$, $v(0) = 10$ V

$$10 = k_1 + k_2 \qquad \ldots\text{(iv)}$$

$$\frac{dv}{dt}(0) = -k_1 - 3\,k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$\frac{dv}{dt}(0^+) + 4v(0) + 0 = 0$$

$$\frac{dv}{dt}(0^+) = -40 \text{ V/s}$$

From Eq. (v),

$$-40 = -k_1 - 3\,k_2 \qquad \ldots\text{(vi)}$$

Solving Eqs (iv) and (vi),

$$k_1 = -5$$

$$k_2 = 15$$

$$v(t) = -5\,e^{-t} + 15\,e^{-3t} \qquad \text{for } t > 0$$

Example 10.59 *In the network of Fig. 10.178, the switch is opened at $t = 0$ obtain the expression for $v(t)$. Assume zero initial conditions.*

Fig. 10.178

Solution At $t = 0^-$,

$$i_L(0^-) = 0$$

$$v(0^-) = v_C(0^-) = 0$$

Since current through the inductor and voltage across the capacitor can not change instantaneously,

$$i_L(0^+) = 0$$

$$v(0^+) = v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 10.179.
Writing the KCL equation for $t > 0$,

$$\frac{v}{0.5} + \frac{1}{0.5}\int_0^t v\,dt + 1\frac{dv}{dt} = 2 \qquad \ldots\text{(i)}$$

Fig. 10.179

Differentiating Eq. (i),

$$2\frac{dv}{dt} + 2v + \frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + 2v = 0$$

$$(D^2 + 2D + 2)v = 0$$

$$D_1 = -1 + j1, \quad D_2 = -1 - j1$$

The solution of this differential equation is given by,

$$v(t) = e^{-t}(k_1 \cos t + k_2 \sin t) \qquad \ldots \text{(ii)}$$

Differentiating Eq. (ii),

$$\frac{dv}{dt} = -e^{-t}(k_1 \cos t + k_2 \sin t) + e^{-t}(-k_1 \sin t + k_2 \cos t)$$

$$= e^{-t}[-k_1(\cos t + \sin t) + k_2(\cos t - \sin t)] \qquad \ldots\text{(iii)}$$

At $t = 0$, $v(0) = 0$

$$0 = k_1 \qquad \ldots\text{(iv)}$$

$$\frac{dv}{dt}(0) = -k_1 + k_2 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$2v(0) + 0 + \frac{dv}{dt}(0) = 2$$

$$\frac{dv}{dt}(0) = 2\ V/s$$

From Eq. (v),

$$2 = -k_1 + k_2 \qquad \ldots(\text{vi})$$

Solving Eq. (iv) and (vi),

$$k_1 = 0$$
$$k_2 = 2$$
$$v(t) = 2\,e^{-t}\sin t \qquad \text{for } t > 0$$

Example 10.60 *The network shown in Fig. 10.180, a sinusoidal voltage $v = 150\ \sin(200t + \phi)$ is applied at $\phi = 30°$. Determine the current $i(t)$.*

Fig. 10.180

Solution Writing the KVL equation for $t > 0$,

$$150\sin(200t + 30°) - 10i - 0.5\frac{di}{dt} - \frac{1}{200\times10^{-6}}\int_0^t i\,dt = 0 \qquad \ldots(\text{i})$$

Differentiating Eq. (i),

$$30000\cos(200t + 30°) - 10\frac{di}{dt} - 0.5\frac{d^2i}{dt^2} - 5000i = 0$$

$$\frac{d^2i}{dt^2} + 20\frac{di}{dt} + 10000i = 60000\cos(200t + 30°)$$

$$(D^2 + 20D + 10000)i = 60000\cos(200t + 30°) \qquad \ldots(\text{ii})$$

The roots of the characteristic equation are

$$D_1 = -10 + j\,99.5, \quad D_2 = -10 - j\,99.5$$

The complimentary function is

$$i_C = e^{-10t}(K_1\cos 99.5t + K_2\sin 99.5t)$$

Let the particular function be

$$i_P = A\cos(200t + 30°) + B\sin(200t + 30°)$$
$$i_P' = -200A\sin(200t + 30°) + 200B\cos(200t + 30°)$$
$$i_P'' = -40000A\cos(200t + 30°) - 40000B\sin(200t + 30°)$$

Substituting these values in Eq. (ii),

$$-40000\,A\cos(200t+30°)-40000\,B\sin(200t+30°)+20[-200\,A\sin(200t+30°)+200\,B\cos(200t+30°)]$$
$$+10000[A\cos(200t+30°)+B\sin(200t+30°)]=60000\cos(200t+30°)$$

$$(-40000\,B-4000\,A+10000\,B)\sin(200t+30°)+(-40000\,A+4000\,B+10000\,A)\cos(200t+30°)$$
$$=60000\cos(200t+30°)$$

Equating the coefficients,

$$-40000\,B-4000\,A+10000\,B=0$$
$$-40000\,A+4000\,B+10000\,A=60000$$

Solving these equations,

$$A=-1.97$$
$$B=0.26$$
$$i_P=-1.97\cos(200t+30°)+0.26\sin(200t+30°)$$

Let $\quad A\sin\phi=-1.97$

and $\quad A\cos\phi=0.26$

$$A^2\sin^2\phi+A^2\cos^2\phi=(-1.97)^2+(0.26)^2=3.95$$
$$A=1.987$$

and $$\phi=\tan^{-1}\left(\frac{-1.97}{0.26}\right)=-82.48°$$

$$i_P=1.987\sin(-82.48°)\cos(200t+30°)+1.987\cos(-82.48°)\sin(200t+30°)$$
$$=1.987\sin(200t+30°-82.48°)$$
$$=1.987\sin(200t-52.48°)$$

The solution of the differential equation is given by,

$$i(t)=e^{-10t}(k_1\cos 99.5t+k_2\sin 99.5t)+1.987\sin(200t-52.48°) \qquad \ldots\text{(iii)}$$

Differentiating Eq. (iii),

$$\frac{di}{dt}=e^{-10t}(-99.5\,k_1\sin 99.5t+99.5\,k_2\cos 99.5t)$$
$$-10\,e^{-10t}(k_1\cos 99.5t+k_2\sin 99.5t)+(1.987)(200)\cos(200t-52.48°)$$

At $t=0,\quad i(0)=0$

$$0=k_1+1.987\sin(-52.48°)$$
$$k_1=1.58 \qquad \ldots\text{(iv)}$$

$$\frac{di}{dt}(0)=99.5\,k_2-10\,k_1+(1.987)(200)\cos(-52.48°)$$
$$=99.5\,k_2-10(1.58)+242.03$$
$$=99.5\,k_2+226.23 \qquad \ldots\text{(v)}$$

Putting $t = 0$ in Eq. (i),

$$150\sin(30°) - 10(0) - 0.5\frac{di}{dt}(0) - 0 = 0$$

$$\frac{di}{dt}(0) = 150 \text{ A/s}$$

From Eq. (v),

$$150 = 99.5\,k_2 + 226.23$$

$$k_2 = -0.77$$

$$i(t) = e^{-10t}(1.58\cos 99.5t - 0.77\sin 99.5t) + 1.987\sin(200t - 52.48°) \quad \text{for } t > 0$$

Example 10.61 *The switch in the network of Fig. 10.181 is opened at t = 0. Find i (t) for t > 0 if,*

(a) $L = \frac{1}{2}$ *H and C = 1 F* *(b) L = 1 H and C = 1 F* *(c) L = 5 H and C = 1 F*

Fig. 10.181

Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an pen circuit.

$$v_C(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

$$i(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 2 \text{ V}$$

$$i(0^+) = 0$$

Case I *When* $R = 2\ \Omega$, $L = \frac{1}{2}$ *H*, $C = 1$ *F*

$$\alpha = \frac{R}{2L} = \frac{2}{2 \times \frac{1}{2}} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times 1}} = \frac{1}{\sqrt{0.5}} = 1.414$$

$$\alpha > \omega_0$$

This indicates an overdamped case.

$$i(t) = A_1 e^{s_1 t} - A_2 e^{s_2 t}$$

where, $$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2 - \sqrt{4-2} = -2 - \sqrt{2} = -3.414$$

and $$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2 + \sqrt{2} = -0.586$$

$$i(t) = k_1 e^{-3.414t} + k_2 e^{-0.586t}$$

At $t = 0$, $i(0) = 0$

$$k_1 + k_2 = 0 \qquad \ldots\text{(i)}$$

Also $v_L(0^+) + v_C(0^+) + v_R(0^+) = 0$

$$v_L(0^+) = -v_R(0^+) - v_C(0^+) = -2i(0^+) - v_C(0^+) = -2 \text{ V} \qquad \ldots\text{(ii)}$$

$$v_L(0^+) = L\frac{di}{dt}(0^+)$$

$$\frac{di}{dt}(0^+) = \frac{v_L(0^+)}{L} = -\frac{2}{0.5} = -4 \text{ A/s}$$

Differentiating the equation of $i(t)$ and putting the condition at $t = 0$,

$$-3.414\, k_1 - 0.586\, k_2 = -4 \qquad \ldots\text{(iii)}$$

Solving Eqs (i) and (iii), we get

$$k_1 = 1.414 \quad \text{and} \quad k_2 = -1.414$$

$$i(t) = 1.414(e^{-3.414t} - e^{-0.586t}) \qquad \text{for } t > 0$$

Case II *When* $R = 2\ \Omega$, $L = 1\ H$, $C = 1\ F$

$$\alpha = \frac{R}{2L} = \frac{2}{2\times 1} = \frac{2}{2} = 1$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1}} = 1$$

$$\alpha = \omega_0$$

This indicates a critically damped case.

$$i(t) = e^{-\alpha t}(k_1 + k_2 t) = e^{-t}(k_1 + k_2 t)$$

At $t = 0$, $i(0) = 0$

$$k_1 = 0$$

Also, $$v_L(0^+) = L\frac{di}{dt}(0^+)$$

$$\frac{di}{dt}(0^+) = \frac{v_L(0^+)}{L} = -\frac{2}{1} = -2 \text{ A/s}$$

Differentiating the equation of $i(t)$ and putting the condition at $t = 0$,

$$\left.\frac{di}{dt}\right|_{t=0} = -k_1 + k_2 = -2$$

$$k_2 = -2$$

$$i(t) = -2t\,e^{-t} \qquad \text{for } t > 0$$

Case III *When* $R = 2\ \Omega$, $L = 5\ H$, $C = 1\ F$

$$\alpha = \frac{R}{2L} = \frac{2}{10} = 0.2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5}} = 0.447$$

$$\alpha < \omega_0$$

This indicates an underdamped case.

$$i\,(t) = e^{-\alpha t}\,(B_1 \cos\,\omega_d\,t + B_2 \sin\,\omega_d\,t)$$

where,

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{(0.447)^2 - (0.2)^2} = 0.4$$

$$s_{1,2} = -\alpha \pm j\omega_d = -0.2 \pm j0.4$$

$$i(t) = e^{-0.2t}(B_1 \cos 0.4t + B_2 \sin 0.4t)$$

Applying the initial condition,

$$i(0^+) = 0$$

and

$$\frac{di}{dt}(0^+) = -\frac{v_L(0^+)}{L} = -\frac{2}{5}$$

$$B_1 = i(0) = 0$$

$$B_2 = -1$$

$$i(t) = -e^{-0.2t} \sin 0.4t \qquad \text{for } t > 0$$

Exercises

10.1 The switch in Fig. 10.182 is moved from the position a to b at $t = 0$, the network having been in steady state in the position a. Determine $i_1(0^+)$, $i_2(0^+)$, $i_3(0^+)$, $\frac{di_2}{dt}(0^+)$ and $\frac{di_3}{dt}(0^+)$.

Fig. 10.182

[1.66 A, 5 A, –3.33 A, –3.33 A/s, 2.22 A/s]

10.2 The switch K is closed at $t = 0$ in the network shown in Fig. 10.183. Determine $i(0^+)$, $\frac{di}{dt}(0^+)$ and $\frac{d^2i}{dt^2}(0^+)$.

Fig. 10.183

$$\left[0, \frac{1}{2}V_0, -V_0\right]$$

10.3 In the network of Fig. 10.184, the switch K is closed at $t = 0$. At $t = 0^-$, all capacitor voltages and inductor currents are zero. Find $v_1, \frac{dv_1}{dt}, v_2, \frac{dv_2}{dt}, v_3$ and $\frac{dv_3}{dt}$ at $t = 0^+$.

Fig. 10.184

$$\left[0, \frac{1}{C}\frac{v(0^+)}{R_1}, 0, 0, 0, 0\right]$$

10.4 In the network at Fig. 10.185, the capacitor C_1 is charged to voltage 1000 V and the switch K is closed at $t = 0$. Find $\frac{d^2 i_2}{dt^2}$ at $t = 0^+$.

Fig. 10.185

$$\left[\frac{17}{400000}\text{A/s}^2\right]$$

10.5 In the network shown in Fig. 10.186, switch is closed at $t = 0$. Obtain the current $i_2(t)$.

Fig. 10.186

$[i_2(t) = 5\, e^{-100000t}]$

10.6 The network shown in Fig. 10.187 is under steady-state when the switch is closed. At $t = 0$, it is opened. Obtain an expression for $i(t)$.

Fig. 10.187

$[i(t) = 2.857\, e^{-2\times10^6\, t}]$

10.7 The switch in Fig. 10.188 is open for a long time and closes at $t = 0$. Determine $i(t)$ for $t > 0$.

Fig. 10.188

$[i(t) = 25(1 - e^{-4t})]$

10.8 In the network shown in Fig. 10.189, the steady state is reached with the switch open. At $t = 0$, the switch is closed Find $v_C(t)$ for $t > 0$.

Fig. 10.189

$[v_C(t) = 5e^{-20t}]$

10.9 The circuit shown in Fig. 10.190 has acquired steady state before switching at $t = 0$.

(i) Obtain $v_C(0^+)$, $v_C(0^-)$, $i(0^+)$ and $i(0^-)$.
(ii) Obtain time constant for $t > 0$.
(iii) Find current $i(t)$ for $t > 0$.

Fig. 10.190

[(i) 5 V, 5 V, 1 mA, 0, (ii) 0.01 s, (iii) $e^{-100\,t}$ mA]

10.10 In the network shown in Fig. 10.191, the switch is initially at the position 1 for a long time. At $t = 0$, the switch is changed to the position 2. Find current $i\,(t)$ for $t > 0$.

Fig. 10.191

[$i\,(t) = 2\,e^{-30t}$]

10.11 In the network shown in Fig. 10.192, the switch is closed at $t = 0$. Find $v\,(t)$ for $t > 0$.

Fig. 10.192

[$v(t) = e^{-t}$]

10.12 In the network shown in Fig. 10.193, the switch is in the position 1 for a long time and at $t = 0$, the switch is moved to the position . Find $v\,(t)$ for $t > 0$.

Fig. 10.193

[$v\,(t) = -0.5\,e^{-\frac{3}{4}t}$]

10.13 In Fig. 10.194, the switch is open until time $t = 100$ seconds and is closed for all times thereafter. Find $v\,(t)$ for all times greater than 100 if $v\,(100) = -3$ V.

Fig. 10.194

$$\left[\, v(t) = 5 - 8e^{-\frac{(t-100)}{160}} \,\right]$$

10.14 A series *RL* circuit shown in Fig. 10.195 has a constant voltage V applied at $t = 0$. At what time does $v_R = v_L$.

Fig. 10.195

[0.0693 s]

10.15 In the circuit shown in Fig. 10.196, at time $t = 0$, the voltage across the capacitor is zero and the switch is moved to the position y. The switch is kept at position y for 20 seconds and then moved to position z and kept in that position thereafter. Find the voltage across the capacitor at $t = 30$ seconds.

Fig. 10.196

[0]

10.16 Determine whether *RLC* series circuit shown in Fig 10.197 is underdamped, overdamped or

critically damped. Also, find $v_L(0^+)$, $\frac{di}{dt}(0^+)$ and $i(\infty)$.

Fig. 10.197

[critically damped, 200 V, 2000 A/s, 0]

10.17 Determine whether *RLC* circuit of Fig 10.198 is underdamped, overdamped or critically damped. Also find $v_L(0^+)$, $\frac{di}{dt}(0^+)$, $\frac{d^2v}{dt^2}(0^+)$ if $v(t) = u(t)$.

Fig. 10.198

[underdamped 1 V, 1 A/s, 2 V/s^2]

Objective-Type Questions

10.1 The voltages v_{C_1}, v_{C_2} and v_{C_3} across the capacitors in the circuit in Fig. 10.199 under steady state are respectively

Fig. 10.199

(a) 80 V, 32 V, 48 V
(b) 80 V, 48 V, 32 V
(c) 20 V, 8 V, 12 V
(d) 20 V, 12 V, 8 V

10.2 In the circuit of Fig. 10.200, the voltage $v(t)$ is

Fig. 10.200

(a) $e^{at} - e^{bt}$
(b) $e^{at} + e^{bt}$
(c) $ae^{dt} - be^{bt}$
(d) $ae^{at} + be^{bt}$

10.3 The differential equation for the current $i(t)$ in the circuit of Fig. 10.201 is

Fig. 10.201

(a) $2\frac{d^2i}{dt^2} + 2\frac{di}{dt} + i(t) = \sin t$

(b) $\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 2i(t) = \cos t$

(c) $2\frac{d^2i}{dt^2} + 2\frac{di}{dt} + i(t) = \cos t$

(d) $\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 2i(t) = \sin t$

10.4 At $t = 0^+$, the current i_1 in Fig. 10.202 is

Fig. 10.202

(a) $-\dfrac{V}{2R}$ (b) $-\dfrac{V}{R}$

(c) $-\dfrac{V}{4R}$ (d) zero

10.5 For the circuit shown in Fig. 10.203, the time constant $RC = 1$ ms. The input voltage is $v_i(t) = \sqrt{2}\sin 10^3 t$. The output voltage $v_0(t)$ is equal to

Fig. 10.203

(a) $\sin(10^3 - 45°)$ (b) $\sin(10^3 t + 45°)$

(c) $\sin(10^3 t - 53)$ (d) $\sin(10^3 t + 53°)$

10.6 For the *RL* circuit shown in Fig. 10.204, the input voltage $v_i(t) = u(t)$. The current $i(t)$ is

Fig. 10.204

(a)

(b)

(c)

(d)

Fig. 10.205

10.7 The condition on *R*, *L* and *C* such that the step response $v(t)$ in Fig. 10.206 has no oscillations is

Fig. 10.206

(a) $R \geq \dfrac{1}{2}\sqrt{\dfrac{L}{C}}$ (b) $R \geq \sqrt{\dfrac{L}{C}}$

(c) $R \geq 2\sqrt{\dfrac{L}{C}}$ (d) $R = \dfrac{1}{\sqrt{LC}}$

10.8 The switch *S* in Fig. 10.207 closed at $t = 0$. If $v_2(0) = 10$ V and $v_g(0) = 0$ respectively, the voltages across capacitors in steady state will be

Fig. 10.207

(a) $v_2(\infty) = v_1(\infty) = 0$

(b) $v_2(\infty) = 2$ V, $v_1(\infty) = 8$ V

(c) $v_2(\infty) = v_1(\infty) = 8$ V

(d) $v_2(\infty) = 8$ V, $v_1(\infty) = 2$ V

10.9 The time constant of the network shown in Fig. 10.208 is

Fig. 10.208

(a) $2\,RC$ (b) $3\,RC$

(c) $\frac{1}{2}RC$ (d) $\frac{2}{3}RC$

10.10 In the series *RC* circuit shown in Fig. 10.209, the voltage across *C* starts increasing when the dc source is switched on. The rate of increase of voltage across *C* at the instant just after the switch is closed i.e., at $t = 0^+$ will be

Fig. 10.209

(a) zero (b) infinity

(c) RC (d) $\frac{1}{RC}$

10.11 The $v - i$ characteristic as seen from the terminal pair ($A - B$) of the network of Fig. 10.210(a) is shown in Fig. 10.210(b). If an inductance of value 6 mH is connected across the terminal pair, the time constant of the system will be

Fig. 10.210

(a) 3 μs

(b) 12 s

(c) 32 s

(d) unknown, unless actual network is specified

10.12 In the network shown in Fig. 10.211, the circuit was initially in the steady-state condition with the switch *K* closed. At the instant when the switch is opened, the rate of decay of current through inductance will be

Fig. 10.211

(a) zero (b) 0.5 A/s

(c) 1 A/s (d) 2 A/s

10.13 A step function voltage is applied to an *RLC* series circuit having $R = 2\ \Omega$, $L = 1$ H and $C = 1$ F. The transient current response of the circuit would be

(a) over damped (b) critically damped

(c) under damped (d) none of these

Answers to Objective-Type Questions

10.1 (b)	10.2 (d)	10.3 (c)	10.4 (d)	10.5 (a)	10.6 (b)	10.7 (c)
10.8 (d)	10.9 (d)	10.10 (d)	10.11 (a)	10.12 (d)	10.13 (b)	

11

Laplace Transform and Its Application

11.1 INTRODUCTION

Time-domain analysis is the conventional method of analysing a network. For a simple network with first-order differential equation of network variable, this method is very useful. But as the order of network variable equation increases, this method of analysis becomes very tedious. For such applications, frequency domain analysis using Laplace transform is very convenient. Time-domain analysis, also known as *classical method,* is difficult to apply to a differential equation with excitation functions which contain derivatives. Laplace transform methods prove to be superior. The Laplace transform method has the following advantages:

(1) Solution of differential equations is a systematic procedure.
(2) Initial conditions are automatically incorporated.
(3) It gives the complete solution, i.e., both complementary and particular solution in one step.

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

11.2 LAPLACE TRANSFORMATION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t$$

where s is the complex frequency variable.

$$s = \sigma + j\omega$$

The function $f(t)$ must satisfy the following condition to possess a Laplace transform,

$$\int_0^\infty |f(t)|\, e^{-\sigma t}\, \mathrm{d}t < \infty$$

where σ is real and positive.
The inverse Laplace transform $L^{-1}\left\{F(s)\right\}$ is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

11.3 LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

1. Constant Function k

The Laplace transform of a constant function is

$$L\{k\} = \int_0^\infty k e^{-st} dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}$$

2. Function t^n

The Laplace transform of $f(t)$ is

$$L\{t^n\} = \int_0^\infty t^n e^{-st} dt$$

Putting $st = x,\ dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{dx}{s} = s^{\frac{1}{n+1}}\int_0^\infty x^n e^{-x} dx = \frac{\sqrt{n+1}}{s^{n+1}},\ s > 0,\ n+1 > 0$$

If n is a positive integer, $\sqrt{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. Unit-Step Function

The unit-step function (Fig 11.1) is defined by the equation,

$$u(t) = 1 \qquad t > 0$$
$$\quad\ = 0 \qquad t < 0$$

The Laplace transform of unit step function is

$$L\left\{u(t)\right\} = \int_0^\infty 1\cdot e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Fig. 11.1 *Unit-step function*

4. Delayed or Shifted Unit-Step Function

The delayed or shifted unit-step function (Fig 11.2) is defined by the equation

$$u\,(t-a) = 1 \qquad t > a$$
$$\quad\ = 0 \qquad t < a$$

The Laplace transform of $u\,(t-a)$ is

$$L\left\{u(t-a)\right\} = \int_a^\infty 1\cdot e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

Fig. 11.2 *Shifted unit-step function*

5. *Unit-Ramp Function*

The unit-ramp function (Fig 11.3) is defined by the equation

$$r(t) = t \qquad t > 0$$
$$= 0 \qquad t < 0$$

The Laplace transform of the unit-ramp function is

$$L\{r(t)\} = \int_0^\infty t\, e^{-st} = \frac{1}{s^2}$$

Fig. 11.3 *Unit-ramp function*

6. *Delayed Unit-Ramp Function*

The delayed unit-ramp function (Fig 11.4) is defined by the equation

$$r\,(t-a) = t \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $r\,(t-a)$ is

$$L\{r(t-a)\} = \int_a^\infty t\, e^{-st}\, dt = \frac{e^{-as}}{s^2}$$

Fig. 11.4 *Delayed unit-ramp function*

7. *Unit-Impulse Function*

The unit-impulse function (Fig 11.5) is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad t = 0$$

The Laplace transform of the unit-impulse function is

$$L\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = 1$$

Fig. 11.5 *Unit-impulse function*

8. *Exponential Function (e^{at})*

The Laplace transform of the exponential function (Fig 11.6) is

$$L\{e^{at}\} = \int_0^\infty e^{at}\, e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

Fig. 11.6 *Exponential function*

9. *Sine Function*

We know that $\sin \omega t = \dfrac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$.

The Laplace transform of the sine function is

$$L\{\sin \omega t\} = L\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

10. *Cosine Function*

We know that $\cos \omega t = \dfrac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$.

The Laplace transform of the cosine function is

$$L\{\cos \omega t\} = L\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

11. Hyperbolic sine function

We know that $\sinh \omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic sine function is

$$L\{\sinh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} - e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} - L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2 - \omega^2}$$

12. Hyperbolic cosine function

We know that $\cosh \omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic cosine function is

$$L\{\cosh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} + e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} + L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s-\omega}\right] = \frac{s}{s^2 - \omega^2}$$

13. Exponentially Damped Function

Laplace transform of an exponentially damped function $e^{-at} f(t)$ is

$$L\left\{e^{-at} f(t)\, dt\right\} = \int_{0^-}^{\infty} f(t) e^{-at}\, e^{-st}\, dt = \int_{0^-}^{\infty} f(t) e^{-(s+a)t}\, dt = F(s+a)$$

Thus, the transform of the function $e^{-at} f(t)$ is obtained by putting $(s+a)$ in place of s in the transform of $f(t)$.

$$L\left\{e^{-at} \sin \omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \sinh \omega t\right\} = \frac{\omega}{(s+a)^2 - \omega^2}$$

$$L\left\{e^{-at} \cos \omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \cosh \omega t\right\} = \frac{s+a}{(s+a)^2 - \omega^2}$$

11.4 PROPERTIES OF LAPLACE TRANSFORM

11.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$ where a and b are constants.

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^{\infty} \{af_1(t) + bf_2(t)\} e^{-st} dt = a\int_0^{\infty} f_1(t) e^{-st} dt + b\int_0^{\infty} f_2(t) e^{-st}\, dt = aF_1(s) + bF_2(s)$$

Example 11.1 *Find the Laplace transform of $4t^2 + \sin 3t + e^{2t}$.*

Solution $L\{4t^2 + \sin 3t + e^{2t}\} = 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\} = 4 \cdot \frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2} = \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$

Example 11.2 *Find the Laplace transform of $t^2 - e^{-2t} + \cosh^2 3t$.*

Solution $L\{t^2 - e^{-2t} + \cosh^2 3t\} = L\{t^2\} - L\{e^{-2t}\} + L\{\cosh^2 3t\} = L\{t^2\} - L\{e^{-2t}\} + \frac{1}{2}L\{1 + \cosh 6t\}$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2s} + \frac{s}{2(s^2 - 36)}$$

Example 11.3 *Find the Laplace transform of* $(\sin 2t - \cos 2t)^2$.

Solution $L\{(\sin 2t - \cos 2t)^2\} = L\{\sin^2 2t + \cos^2 2t - 2\cos 2t \sin 2t\} = L\{1 - \sin 4t\} = L\{1\} - L\{\sin 4t\}$

$$= \frac{1}{s} - \frac{4}{s^2 + 16}$$

Example 11.4 *Find the Laplace transform of* $\cos(\omega t + b)$.

Solution $L\{\cos(\omega t + b)\} = L\{\cos \omega t \cos b - \sin \omega t \sin b\} = \cos b L\{\cos \omega t\} - \sin b L\{\sin \omega t\}$

$$= \cos b \cdot \frac{s}{s^2 + \omega^2} - \sin b \cdot \frac{\omega}{s^2 + \omega^2}$$

11.4.2 Time Scaling

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$L\{f(at)\} = \int_0^\infty f(at) e^{-st}\, dt$$

Putting $at = x,\ dt = \frac{dx}{a}$

$$L\{(f(at)\} = \int_0^\infty f(x)\, e^{-s\left(\frac{x}{a}\right)} \frac{dx}{a} = \frac{1}{a}\int_0^\infty f(x)\, e^{-\left(\frac{s}{a}\right)x}\, dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Example 11.5 *If* $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$, *find* $L\{f(2t)\}$.

Solution

$$L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$$

By time-scaling property,

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right) = \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Example 11.6 *If* $L\{f(t)\} = \frac{2}{s^3} e^{-s}$, *find* $L\{f(3t)\}$.

Solution

$$L\{f(t)\} = \frac{2}{s^3} e^{-s}$$

By time-shifting property,

$$L\{f(3t)\} = \frac{1}{3}\frac{2}{\left(\frac{s}{3}\right)^3} e^{-\frac{s}{3}} = \frac{1}{3}\frac{54}{s^3} e^{-\frac{s}{3}} = \frac{18}{s^3} e^{-\frac{s}{3}}$$

11.4.3 Frequency-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)t\, e^{-st}\ d$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st}\ dt = \int_0^\infty f(t) e^{-(s+a)t}\ dt = F(s+a)$$

Example 11.7 *Find the Laplace transform of $e^{-3t} t^4$.*

Solution

$$L\{t^4\} = \frac{4!}{s^5}$$

By frequency-shifting theorem,

$$L\{e^{-3t} t^4\} = \frac{4!}{(s+3)^5}$$

Example 11.8 *Find the Laplace transform of $(t+1)^2 e^t$.*

Solution

$$L\{(t+1)^2\} = L\{t^2 + 2t + 1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

By frequency-shifting theorem,

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

Example 11.9 *Find the Laplace transform of $e^{4t} sin^3 t$.*

$$L\{\sin^3 t\} = \frac{1}{4} L\{3\sin t - \sin 3t\} = \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

Solution By frequency-shifting theorem,

$$L\{e^{4t} \sin^3 t\} = \frac{3}{4[(s-4)^2+1]} - \frac{3}{4[(s-4)^2+9]} = \frac{3}{4(s^2-8s+17)} - \frac{3}{4(s^2-8s+25)} = \frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

Example 11.10 *Find the Laplace transform of cosh at cos at.*

Solution

$$\cosh at \cos at = \left(\frac{e^{at} + e^{-at}}{2}\right)\cos at = \frac{1}{2}(e^{at} \cos at + e^{-at} \cos at)$$

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\cosh at \cos at\} = \frac{1}{2} L\{e^{at} \cos at + e^{-at} \cos at\}$$

By frequency-shifting theorem,

$$L\{\cosh at \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2 + a^2} + \frac{s+a}{(s+a)^2 + a^2}\right] = \frac{1}{2}\left[\frac{s-a}{s^2 + 2a^2 - 2as} + \frac{s+a}{s^2 + 2a^2 + 2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2 + 2a^2 + 2as) + (s+a)(s^2 + 2a^2 - 2as)}{(s^2 + 2a^2)^2 - 4a^2 s^2}\right] = \frac{s^3}{s^4 + 4a^4}$$

11.4.4 Time-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\} = e^{-as}F(s)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt$$

$$L\{f(t-a)\} = \int_0^\infty f(t-a)\, e^{-st}\, dt$$

Putting $t - a = x, \quad dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\} = \int_0^\infty f(x)e^{-s(a+x)}\, dx = e^{-as}\int_0^\infty f(x)e^{-sx}\, dx = e^{-as}\int_0^\infty f(t)\, e^{-st}\, dt = e^{-as}F(s)$$

Example 11.11 *Find the Laplace transform of* $\cos(t-a) \quad t > a.$

Solution Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By time-shifting theorem,

$$L\{\cos(t-a)\} = e^{-as}\frac{s}{s^2+1}$$

Example 11.12 *Find the Laplace transform of* $e^{t-a} \quad t > a.$

Solution Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By time-shifting theorem,

$$L\{e^{t-a}\} = e^{-as}\frac{1}{s-1}$$

Example 11.13 *Find the Laplace transform of* $\sin\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}.$

Solution Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By time-shifting theorem,

$$L\left\{\sin\left(t - \frac{\pi}{4}\right)\right\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

Example 11.14 *Find the Laplace transform of* $(t-1)^3 \quad t > 1.$

Solution Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By time-shifting theorem,

$$L\left\{(t-1)^3\right\} = e^{-s}\frac{3!}{s^4}$$

11.4.5 Multiplication by *t* (Frequency-Differentiation Theorem)

If $L\{f(t)\} = F(s)$ then $L\{t\,f(t)\} = -\dfrac{d}{ds}F(s)$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Differentiating both the sides w.r.t s using DUIS,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}f(t)e^{-st}\,dt$$

$$= \int_0^\infty (-t)\,f(t)e^{-st}\,dt = \int_0^\infty \{-t\,f(t)\}e^{-st}dt = -L\{t\,f(t)\}$$

$$L\{t\,f(t)\} = (-1)\frac{d}{ds}F(s)$$

Example 11.15 *Find the Laplace transform of $t \sin at$.*

Solution

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$L\{t\sin at\} = -\frac{d}{ds}L\{\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

Example 11.16 *Find the Laplace transform of $t \cos^2 t$.*

Solution $L\{\cos^2 t\} = L\left\{\dfrac{1+\cos 2t}{2}\right\} = \dfrac{1}{2}L\{1+\cos 2t\} = \dfrac{1}{2}\left(\dfrac{1}{s}+\dfrac{s}{s^2+4}\right)$

$$L\{t\cos^2 t\} = -\frac{d}{ds}L\{\cos^2 t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+4}\right) = -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

Example 11.17 *Find the Laplace transform of $t \sin^3 t$.*

Solution $L\{\sin^3 t\} = L\left\{\dfrac{3\sin t - \sin 3t}{4}\right\} = \dfrac{1}{4}\left(\dfrac{3}{s^2+1}-\dfrac{1}{s^2+9}\right) = \dfrac{3}{4}\left(\dfrac{1}{s^2+1}-\dfrac{1}{s^2+9}\right)$

$$L\{t\sin^3 t\} = -\frac{d}{ds}L\{\sin^3 t\} = -\frac{3}{4}\frac{d}{ds}\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right) = -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{(s^2+9)^2-(s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right] = \frac{3s}{2}\cdot\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2} = \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

Example 11.18 *Find the Laplace transform of $t \sin 2t \cosh t$.*

Solution $L\{\sin 2t \cosh t\} = L\left\{\sin 2t\left(\frac{e^t + e^{-t}}{2}\right)\right\} = \frac{1}{2} L\{e^t \sin 2t + e^{-t} \sin 2t\}$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right] = \frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}$$

$$L\{t \sin 2t \cosh t\} = -\frac{d}{ds} L\{\sin 2t \cosh t\} = -\frac{d}{ds}\left(\frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}\right) = \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

11.4.6 Division by t (Frequency-Integration Theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^\infty F(s)\, ds = \int_s^\infty \int_0^\infty f(t) e^{-st} dt\, ds$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s) ds = \int_0^\infty \left[\int_s^\infty f(t) e^{-st} ds\right] dt = \int_0^\infty \left[\frac{1}{-t} f(t) e^{-st}\right]_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s) ds$$

Example 11.19 *Find the Laplace transform of $\frac{1-e^{-t}}{t}$.*

Solution $L\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty L\{1-e^{-t}\} ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds = \left|\log s - \log(s+1)\right|_s^\infty$$

$$= \left[\log \frac{s}{s+1}\right]_s^\infty = \log\left[\frac{1}{1+\frac{1}{s}}\right]_s^\infty = \log 1 - \log\left(\frac{1}{1+\frac{1}{s}}\right) = -\log \frac{s}{s+1} = \log \frac{s+1}{s}$$

Example 11.20 *Find the Laplace transform of* $\dfrac{e^{-at}-e^{-bt}}{t}$.

Solution

$$L\{e^{-at}-e^{-bt}\}=\frac{1}{s+a}-\frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty L\{e^{-at}-e^{-bt}\}\,ds=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\left[\log\frac{s+a}{s+b}\right]_s^\infty=\left[\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]_s^\infty=\log 1-\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}=-\log\frac{s+a}{s+b}=\log\frac{s+b}{s+a}$$

Example 11.21 *Find the Laplace transform of* $\dfrac{\sinh t}{t}$.

Solution

$$L\{\sinh t\}=L\left\{\frac{e^{t}-e^{-t}}{2}\right\}=\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$$

$$L\left\{\frac{\sinh t}{t}\right\}=\int_s^\infty L\{\sinh t\}\,ds=\frac{1}{2}\int_s^\infty\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds=\frac{1}{2}\left[\log(s-1)-\log(s+1)\right]_s^\infty=\frac{1}{2}\left[\log\frac{s-1}{s+1}\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right]_s^\infty=\frac{1}{2}\left(\log 1-\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)=-\frac{1}{2}\log\frac{s-1}{s+1}=\frac{1}{2}\log\frac{s+1}{s-1}$$

Example 11.22 *Find the Laplace transform of* $\dfrac{\cos h2t\,\sin 2t}{t}$.

Solution

$$L\left\{\frac{\cosh 2t\sin 2t}{t}\right\}=L\left\{\left(\frac{e^{2t}+e^{-2t}}{2t}\right)\sin 2t\right\}=\frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\}+L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\}=\frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\}=\int_s^\infty L\{\sin 2t\}ds=\int_s^\infty\frac{2}{s^2+4}ds=\left[\tan^{-1}\frac{s}{2}\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}\frac{s}{2}=\cot^{-1}\frac{s}{2}$$

By first shifting theorem,

$$L\left\{\frac{\cosh 2t\sin 2t}{t}\right\}=\frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\}+L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]=\frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right)+\cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

11.4.7 Time-Differentiation Theorem: Laplace Transform of Derivatives

If $L\{f(t)\}=F(s)$ then

$$L\{f'(t)\}=sF(s)-f(0)$$

$$L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$$

In general,

$$L\{f''(t)\} = s''F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)\ldots - f^{(n-1)}(0)$$

Proof

$$L\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integrating by parts,

$$L\{f'(t)\} = \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty (-s)f(t)e^{-st}dt = -f(0) + s\int_0^\infty f(t)e^{-st}dt = -f(0) + sL\{f(t)\}$$

Similarly,

$$L\{f''\ t\} = -f' + sL\{f'\ t\} = -f'(0) + s\left[-f(0) + sL\{f(t)\}\right] = -f'(0) - sf(0) + s^2L\{f(t)\}$$

In general, $L\left\{f^n(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)\ldots - f^{n-1}(0)$

Example 11.23 *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of $f(t) = \dfrac{\sin t}{t}$.*

Solution

$$L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}ds = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cot^{-1}s - \lim_{t\to 0}\frac{\sin t}{t} = s\cot^{-1}s^{-1}$$

Example 11.24 *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of the following function:*

$$f(t) = 3 \qquad 0 \le t < 5$$
$$= 0 \qquad t > 5.$$

Solution $L\{f(t)\} = F(s) = \int_s^\infty e^{-st}f(t)dt = \int_0^5 e^{-st}\cdot 3dt + \int_5^\infty 0\cdot dt = 3\left[\dfrac{e^{-st}}{-s}\right]_0^5 + 0 = \dfrac{-3}{s}(e^{-5s} - 1) = \dfrac{3}{s}(1 - e^{-5s})$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{3}{s}(1 - e^{-5s}) - 3 = -3e^{-5s}$$

Example 11.25 *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of $f(t) = e^{-5t}\sin t$.*

Solution

$$L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\} = \frac{1}{(s+5)^2+1}$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{1}{s^2+10s+26} - e^0\sin 0 = \frac{s}{s^2+10s+26}$$

Example 11.26 *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of the following function:*

$$f(t) = t \qquad 0 \le t < 3$$
$$= 6 \qquad t > 3.$$

Solution $$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^3 e^{-st} t\, dt + \int_3^\infty e^{-st} \cdot 6\, dt = \left[\frac{e^{-st}}{-s} \cdot t\right]_0^3 - \left[\frac{e^{-st}}{s^2}\right]_0^3 + 6\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}e^{-3s} = \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$L'\{f(t)\} = sF(s) = f(0) = \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$$

11.4.8 Time-Integration Theorem: Laplace Transform of Integral

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$

Proof

$$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \int_0^t f(t)\, dt\Big\} e^{-st} dt$$

Integrating by parts,

$$L\left\{\int_0^t f(t)dt\right\} = \left[\int_0^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty \left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{d}{dt}\int_0^t f(t)dt\right)\right] dt = \int_0^\infty \frac{1}{s} f(t)e^{-st} dt = \frac{1}{s} L\{f(t)\} = \frac{F(s)}{s}$$

Example 11.27 *Find the Laplace transform of* $\int_0^t e^{-2t} t^3 dt$.

Solution

$$L\{e^{-2t} t^3\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t} t^3 dt\right\} = \frac{1}{s} L\left\{e^{-2t} t^3\right\} = \frac{6}{s(s+2)^4}$$

Example 11.28 *Find the Laplace transform of* $\int_0^t t \cosh t\, dt$.

Solution $$L\{t \cosh t\} = L\left\{t\left(\frac{e^t + e^{-t}}{2}\right)\right\} = \frac{1}{2} L\{te^t + te^{-t}\} = \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right] = \frac{1}{2} \cdot \frac{2(s^2+1)}{(s^2-1)^2} = \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t \cosh t\, dt\right\} = \frac{1}{s} L\{t \cosh t\} = \frac{s^2+1}{s(s^2-1)^2}$$

Example 11.29 *Find the Laplace transform of the* $\int_0^t te^{-4t} \sin 3t\, dt$.

Solution

$$L\{t \sin 3t\} = -\frac{d}{ds} L\{\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\{te^{-4t} \sin 3t\} = \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\;dt\right\} = \frac{1}{s}L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{s(s^2+8s+25)^2}$$

Example 11.30 *Find the Laplace transform of* $e^{-4t}\int_0^t t\,\sin 3t\,dt$.

Solution

$$L\{t\sin 3t\} = -\frac{d}{ds}L\{\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{[(s+4)^2+9]^2} = \frac{6}{(s^2+8s+25)^2}$$

11.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to\infty}[f'(t)e^{-st}]dt + f(0) = 0 + f(0) = f(0) = \lim_{t\to 0} f(t)$$

11.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to 0}\int_0^\infty f'(t)\,e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to 0}\left[f'(t)e^{-st}\right]dt + f(0) = \int_0^\infty f'(t)dt + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0) = \lim_{t\to\infty} f(t) - f(0) + f(0) = \lim_{t\to\infty} f(t)$$

Example 11.31 *Verify the initial and final value theorems for* $e^{-t}(t+1)^2$.

Solution

$$f(t) = e^{-t}(t+1)^2 = e^{-t}(t^2+2t+1)$$

$$F(s) = \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{2s}{(s+1)^2} + \frac{s}{s+1}$$

$$\lim_{t\to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\frac{2}{s}}{\left(1+\frac{1}{s}\right)^2} + \frac{1}{1+\frac{1}{s}}\right] = 1$$

Hence, the initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = 0$$

$$\lim_{s\to 0} sF(s) = 0$$

Hence, the final value theorem is verified.

Example 11.32 *Verify the initial and final value theorems for* $e^{-t}(t^2+\cos 3t)$.

Solution

$$f(t) = e^{-t}(t^2+\cos 3t)$$

$$F(s) = \frac{2}{(s+1)^3} + \frac{s+1}{(s+1)^2+9}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}$$

$$\lim_{t\to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{1}{s}\right)^2+\frac{9}{s^2}}\right] = 1$$

Hence, the initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(t^2+\cos 3t)e^{-t} = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}\right] = 0$$

Hence, the final value theorem is verified.

Example 11.33 *Find the initial and final values of the function whose Laplace transform is*

$$F(s) = \frac{2s+1}{s^3+6s^2+11s+6}.$$

Solution

$$F(s) = \frac{2s+1}{s^3+6s^2+11s+6}$$

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2s^2+s}{s^3+6s^2+11s+6} = \lim_{s\to\infty} \frac{\frac{2}{s}+\frac{1}{s^2}}{1+\frac{6}{s}+\frac{11}{s^2}+\frac{6}{s^3}} = 0$$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2s^2+s}{s^3+6s^2+11s+6} = 0$$

Example 11.34 *Find the final value of the function whose Laplace transform is* $I(s) = \dfrac{s+6}{s(s+3)}$.

Solution

$$I(s) = \frac{s+6}{s(s+3)}$$

$$I(\infty) = \lim_{s\to 0} \mathrm{s\,I}(s) = \lim_{s\to 0} \frac{s+6}{s+3} = 2$$

11.5 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T(T > 0)$ such that $f(t+T) = f(t)$, for all values of t.

$$f(t+2T) = f(t+T+T) = f(t+T) = f(t)$$

In general, $f(t+nT) = f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T f(t)e^{-st}\,dt$$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_T^\infty f(t)e^{-st}\,dt$$

In the second integral, putting $t = x+T$, $dt = dx$

When

$$t = T, \quad x = 0$$

$$t \to \infty, \quad x \to \infty$$

$$L\{f(t)\} = \int_0^T f(t)e^{-st}\,dt + \int_0^\infty f(x+T)e^{-s(x+T)}\,dx$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-Ts}\int_0^\infty f(x)e^{-sx}\,dx$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-Ts}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^T f(t)e^{-st}\,dt + e^{-Ts}L\{f(t)\}$$

$$(1-e^{-Ts})L\{f(t)\} = \int_0^T f(t)e^{-st}\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt$$

Example 11.35 *Find the Laplace transform of the waveform shown in Fig. 11.7.*

Fig. 11.7

Solution The function $f(t)$ is a periodic function with period T.

$$f(t) = \frac{At}{T} \qquad 0<t<t$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt$$

$$= \frac{1}{1-e^{-Ts}}\int_0^T \frac{At}{T}e^{-st}\,dt$$

$$= \frac{1}{1-e^{-Ts}}\frac{A}{T}\int_0^T te^{-st}\,dt$$

$$= \frac{A}{T(1-e^{-Ts})}\left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^T$$

$$= \frac{A}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{A}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts})\right]$$

$$= \frac{A}{Ts^2} - \frac{Ae^{-Ts}}{s(1-e^{-Ts})}$$

Example 11.36 *Find the Laplace transform of the waveform shown in Fig. 11.8.*

Fig. 11.8

Solution The function $f(t)$ is a periodic function with period 2.

$$f(t) = t \quad 0 < t < 1$$
$$= 0 \quad 1 < t < 2$$

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 t e^{-st}\, dt + \int_1^2 0 \cdot e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-st}}{-s} t - \frac{e^{-st}}{s^2} \right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{s^2(1-e^{-2s})} (1 - e^{-s} - s e^{-s})$$

Example 11.37 *Find the Laplace transform of the waveform shown in Fig. 11.9.*

Fig. 11.9

Solution The function $f(t)$ is a periodic function with period $2a$.

$$f(t) = \frac{t}{a} \qquad 0 < t < a$$
$$= \frac{1}{a}(2a - t) \qquad a < t < 2a$$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_a^a \frac{t}{a} e^{-st} dt + \int_a^{2a} \frac{1}{a}(2a-t) e^{-st}\, dt \right]$$

$$= \frac{1}{a(1-e^{-2as})} \left\{ \left[\frac{e^{-st}}{-s} t - \frac{e^{-st}}{s^2} \right]_0^a + \left[\frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{a(1-e^{-2as})} \left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2} \right)$$

$$= \frac{-2e^{-as} + 1 + e^{-2as}}{as^2(1-e^{-2as})}$$

$$= \frac{(1-e^{-as})^2}{as^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{as^2(1+e^{-as})}$$

$$= \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{as^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)}$$

$$= \frac{\tanh\left(\frac{as}{2}\right)}{as^2}$$

Example 11.38 *Find the Laplace transform of the waveform shown in Fig. 11.10.*

Fig. 11.10

Solution The function $f(t)$ is periodic with period with period $2a$.

$$f(t) = 1 \qquad 0 < t < a$$
$$= -1 \qquad a < t < 2a$$

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} f(t)e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, dt + \int_a^{2a} e^{-st}(-1)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^a + \left[\frac{e^{-st}}{s} \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left(-\frac{e^{-as}}{-s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s} \cdot \frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

Example 11.39 *Find the Laplace transform of the waveform shown in Fig. 11.11.*

Fig. 11.11

Solution The function $f(t)$ is known as a half-sine wave rectifier function with period $\frac{2\pi}{\omega}$.

$$f(t) = a \sin \omega t \quad 0 < t < \frac{\pi}{\omega}$$

$$= 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

The function $f(t)$ is a periodic function.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}} \int_0^{\frac{2\pi}{\omega}} f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left(\int_0^{\frac{\pi}{\omega}} a \sin \omega t\, e^{-st} dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} 0 \cdot e^{-st}\, dt \right)$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{1}{s^2+\omega^2} \cdot e^{-st} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}} \cdot \frac{1}{s^2+\omega^2} \left[e^{-\frac{\pi s}{\omega}} (\omega) + \omega \right]$$

$$= \frac{a\omega\left(1+e^{\frac{-\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2}$$

$$= \frac{a\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2}$$

Example 11.40 *Find the Laplace transform of*

$$f(t) = t^2 \qquad 0 < t < 2$$

if

$$f(t) = f(t+2).$$

Solution The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\, dt = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} \cdot t^2\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[t^2 \cdot \left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left(-4\frac{e^{-2s}}{s} - 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3} + \frac{2}{s^3}\right)$$

$$= \frac{1}{\left(1-e^{-2s}\right)s^3}\left(2 - 2e^{-2s} - 4se^{-2s} - 4s^2e^{-2s}\right)$$

Example 11.41 *Find the Laplace transform of*

$$f(t) = e^t \qquad 0 < t < 2\pi$$

if

$$f(t) = f(t+2\pi).$$

Solution The function $f(t)$ is a periodic function with period 2π.

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} e^t\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{(1-s)t}\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{(1-s)t}}{1-s}\right]_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)2\pi}}{1-s} - \frac{1}{1-s}\right]$$

$$= \frac{e^{(1-s)2\pi}-1}{(1-e^{-2\pi s})(1-s)}$$

Example 11.42 *Find the Laplace transform of the function shown in Fig. 11.12.*

Fig. 11.12

Solution The function $f(t)$ can be represented in terms of Heaviside unit step function.

$$f(t) = [u(t-T) - u(t-2T)] + 2[u(t-2T) - u(t-3T) + 3[u(t-3T) - u(t-4t)] + \ldots\infty$$

$$= u(t-T) + u(t-2T) + u(t-3T) + \ldots\infty$$

$$L\{f(t)\} = L\{u(t-T) + u(t-2T) + u(t-3T) + \ldots\}$$

$$= \frac{1}{s}e^{-Ts} + \frac{1}{s}e^{-2Ts} + \frac{1}{s}e^{-3Ts} + \ldots$$

$$= \frac{1}{s}\left[e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \ldots\right]$$

$$= \frac{e^{-Ts}}{s(1-e^{-Ts})}$$

11.6 WAVEFORM SYNTHESIS

Any waveform can be constructed with unit step, unit ramp and unit impulse functions, etc. We know the Laplace transforms of these special functions. Hence, we can find the Laplace transform of any function in terms of Laplace transform of these functions.

There is another way of synthesising the waveforms. Any function can be expressed in terms of a gate function. The gate function is shown in Fig. 11.13.

This function can be expressed in terms of unit-step functions.

$$f(t) = Au(t) - Au(t-T)$$

Fig. 11.13 *Gate function*

Example 11.43 *Find the Laplace transform of the unit-doublet function.*

Solution The unit-doublet function $\delta'(t)$ is shown in Fig. 11.14.

$$\delta'(t) = \frac{d}{dt}\delta(t)$$

$$L\{\delta'(t)\} = L\left\{\frac{d}{dt}\delta(t)\right\} = sL\{\delta(t)\} = s(1) = s$$

Fig. 11.14

Example 11.44 *Find the Laplace transform of a rectangular pulse shown in Fig. 11.15.*

Fig. 11.15

Solution The rectangular pulse can be constructed from two functions as shown in Fig. 11.16. This function is known as gate function.

Fig. 11.16

$$f(t) = f_1(t) + f_2(t) = u(t) - u(t-T)$$

$$F(s) = L\{u(t)\} - L\{u(t-T)\} = \frac{1}{s} - \frac{1}{s}e^{-Ts} = \frac{1}{s}(1 - e^{-Ts})$$

Example 11.45 *Find the Laplace transform of a sawtooth waveform shown in Fig. 11.17.*

Fig. 11.17

Solution The sawtooth waveform can be constructed from three functions as shown in Fig. 11.18.

Fig. 11.18

$$f(t) = f_1(t) + f_2(t) + f_3(t) = \frac{A}{T}r(t) - \frac{A}{T}r(t-T) - Au(t-T)$$

$$F(s) = \frac{A}{T}L\{r(t)\} - \frac{A}{T}L\{r(t-T)\} - AL\{u(t-T)\} = \frac{A}{Ts^2} - \frac{A}{T}\frac{1}{s^2}e^{-Ts} - \frac{A}{s}e^{-Ts}$$

Example 11.46 *Find the Laplace transform of a triangular waveform shown in Fig. 11.19.*

Fig. 11.19

Solution The triangular waveform can be constructed from three ramp functions as shown in Fig. 11.20.

Fig. 11.20

$$f(t) = f_1(t) + f_2(t) + f_3(t) = \frac{2}{T}r(t) - \frac{4}{T}r\left(t - \frac{T}{2}\right) + \frac{2}{T}r(t-T)$$

$$F(s) = \frac{2}{T}L\{r(t)\} - \frac{4}{T}L\left\{r\left(t - \frac{T}{2}\right)\right\} + \frac{2}{T}L\{r(t-T)\} = \frac{2}{Ts^2} - \frac{4}{Ts^2}e^{-\frac{Ts}{2}} + \frac{2}{Ts^2}e^{-Ts}$$

Example 11.47 *Find the Laplace transform of a trapezoidal pulse shown in Fig. 11.21.*

Fig. 11.21

Solution The trapezoidal waveform can be constructed from four ramp functions as shown in Fig.11.22.

Fig. 11.22

$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) = \frac{1}{a}r(t) - \frac{1}{a}r(t-a) - \frac{1}{a}r(t-3a) + \frac{1}{a}r(t-4a)$$

$$F(s) = \frac{1}{a}L\{r(t)\} - \frac{1}{a}L\{r(t-a)\} - \frac{1}{a}L\{r(t-3a)\} + \frac{1}{a}L\{r(t-4a)\} = \frac{1}{a}\frac{1}{s^2} - \frac{1}{a}\frac{1}{s^2}e^{-as} - \frac{1}{a}\frac{1}{s^2}e^{-3as} + \frac{1}{a}\frac{1}{s^2}e^{-4s}$$

$$= \frac{1}{as^2}(1 - e^{-as} - e^{-3as} + e^{-4as})$$

Example 11.48 *Find the Laplace transform of a sinusoidal waveform shown in Fig. 11.23.*

Fig. 11.23

Solution The waveform can be constructed from two functions as shown in Fig. 11.24.

Fig. 11.24

$$f(t) = f_1(t) + f_2(t) = A\sin\omega t\ u(t) + A\sin\omega t\left(t - \frac{T}{2}\right)u\left(t - \frac{T}{2}\right)$$

where $\omega = \dfrac{2\pi}{T}$

$$F(s) = A\,L\{\sin\omega t\ u(t)\} + AL\left\{\sin\omega t\left(t - \frac{T}{2}\right)u\left(t - \frac{T}{2}\right)\right\} = \frac{A\omega}{s^2+\omega^2} + \frac{A\omega}{s^2+\omega^2}e^{-\frac{Ts}{2}} = \frac{A\omega}{s^2+\omega^2}\left(1 + e^{-\frac{Ts}{2}}\right)$$

Example 11.49 *Find the Laplace transform of the waveform shown in Fig. 11.25.*

Fig. 11.25

Solution The function $f(t)$ can be expressed as sum of four step functions.

$$f(t) = u(t) + u(t-1) - u(t-2) - u(t-3)$$

$$F(s) = L\{u(t)\} + L\{u(t-1)\} - L\{u(t-2)\} - L\{u(t-3)\} = \frac{1}{s} + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} = \frac{1}{s}\left(1 + e^{-s} - e^{-2s} - e^{-3s}\right)$$

Example 11.50 *Determine the Laplace transform of the waveform shown in Fig. 11.26.*

Fig. 11.26

Solution The function $f(t)$ can be expressed as sum of four functions as shown in Fig. 11.27.

Fig. 11.27

$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) = r(t) + r(t-1) + u(t-2) - 2u(t-3)$$

$$F(s) = L\{r(t)\} + L\{r(t-1)\} + L\{u(t-2)\} - 2L\{u(t-3)\} = \frac{1}{s^2} + \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-2s} - \frac{2}{s}e^{-3s}$$

Example 11.51 *Find the Laplace transform of the waveform shown in Fig. 11.28.*

Fig. 11.28

Solution The given parabolic waveform can be constructed from three functions as shown in Fig.11.29.

Fig. 11.29

$$F(t) = f_1(t) + f_2(t) + f_3(t) = 2t^2u(t) - 16r(t-2) - 2(t-4)^2u(t-4)$$

$$F(s) = 2L\{t^2u(t)\} - 16L\{r(t-2)\} - 2L\{(t-4)^2u(t-4)\} = 2\frac{2}{s^3} - 16\frac{1}{s^2}e^{-2s} - 2\frac{2}{s^3}e^{-4s} = \frac{4}{s^3}(1-e^{-4s}) - \frac{16}{s^2}e^{-2s}$$

Example 11.52 *Find the Laplace transform of the waveform as shown in Fig. 11.30.*

Fig. 11.30

Solution The given waveform can be constructed from four functions as shown in Fig. 11.31.

Fig. 11.31

Fig. 11.32

$$f(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) = 3r(t) - 3r(t-1) - u(t-1) - 2u(t-2)$$

$$F(s) = 3L\{r(t)\} - 3L\{r(t-1)\} - L\{u(t-1)\} - 2L\{u(t-2)\} = \frac{3}{s^2} - \frac{3}{s^2}e^{-s} - \frac{1}{s}e^{-s} - \frac{2}{s}e^{-2s}$$

Example 11.53 *Find the Laplace transform of the periodic waveform shown in Fig. 11.33.*

Fig. 11.33

Solution The function $f(t)$ is a periodic function with period 2.

The function $f_1(t)$ can be constructed from three functions by waveform synthesis.

$$f_1(t) = u(t) - 2u(t-1) + u(t-2)$$

$$F_1(s) = L\{u(t)\} - 2L\{u(t-1)\} + L\{u(t-2)\} = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}$$

The Laplace transform of periodic function $f(t)$ is

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-2s}}F_1(s) \\
&= \frac{1}{1-e^{-2s}}\left(\frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}\right) \\
&= \frac{1}{1-e^{-2s}}\left(\frac{1-2e^{-s}+e^{-2s}}{s}\right) \\
&= \frac{(1-e^{-s})^2}{s(1-e^{-s})(1+e^{-s})} \\
&= \frac{1}{s}\frac{1-e^{-s}}{1+e^{-s}} \\
&= \frac{1}{s}\frac{e^{\frac{s}{2}} - e^{-\frac{s}{2}}}{e^{\frac{s}{2}} + e^{-\frac{s}{2}}} \\
&= \frac{1}{s}\tanh\left(\frac{s}{2}\right)
\end{aligned}$$

Example 11.54 *Find the Laplace transform of the waveform shown in Fig. 11.34.*

Fig. 11.34

Solution The function $f(t)$ is a periodic function with period T.
The function $f_1(t)$ can be constructed from three functions by waveform synthesis.

$$f_1(t) = \frac{A}{T}r(t) - \frac{A}{T}r(t-T) - Au(t-T)$$

$$F_1(s) = \frac{A}{T}L\{r(t)\} - \frac{A}{T}L\{r(t-T)\} - AL\{u(t-T)\} = \frac{A}{Ts^2} - \frac{A}{T}\frac{1}{s^2}e^{-Ts} - \frac{A}{s}e^{-Ts}$$

The Laplace transform of the periodic function $f(t)$ is

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}F_1(s) = \frac{1}{1-e^{-Ts}}\left(\frac{A}{Ts^2} - \frac{A}{T}e^{-Ts} - \frac{A}{s}e^{-Ts}\right)$$

Example 11.55 *Find the Laplace transform of periodic waveform shown in Fig. 11.35.*

Fig. 11.35

Solution The function $f(t)$ is a periodic function with period T.
The function $f_1(t)$ can be constructed from two functions by waveform synthesis.

$$f_1(t) = A\sin\omega t\, u(t) + A\sin\omega t\left(t - \frac{T}{2}\right)u\left(t - \frac{T}{2}\right)$$

$$F_1(s) = AL\{\sin\omega t\, u(t)\} + AL\left\{\sin\omega t\left(t - \frac{T}{2}\right)u\left(t - \frac{T}{2}\right)\right\} = A\frac{\omega}{s^2+\omega^2} + A\frac{\omega}{s^2+\omega^2}e^{-\frac{Ts}{2}} = \frac{A\omega}{s^2+\omega^2}\left(1 + e^{-\frac{Ts}{2}}\right)$$

The Laplace transform of the periodic function $f(t)$ is

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1-e^{-Ts}}F_1(s) \\ &= \frac{1}{1-e^{-Ts}}\frac{A\omega}{s^2+\omega^2}\left(1+e^{-\frac{Ts}{2}}\right) \\ &= \frac{A\omega}{s^2+\omega^2}\frac{1+e^{-\frac{Ts}{2}}}{\left(1-e^{-\frac{Ts}{2}}\right)\left(1+e^{-\frac{Ts}{2}}\right)} \\ &= \frac{A\omega}{s^2+\omega^2}\frac{1}{1-e^{-\frac{Ts}{2}}} \end{aligned}$$

11.7 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Partial fraction expansion
(iii) Convolution theorem

11.7.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example 11.56 *Find the inverse Laplace transform of* $\dfrac{s^2 - 3s + 4}{s^3}$.

Solution

$$F(s) = \frac{s^2 - 3s + 4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$

$$L^{-1}\{F(s)\} = 1 - 3t + 2t^2$$

Example 11.57 *Find the inverse Laplace transform of* $\dfrac{3s + 4}{s^2 + 9}$.

Solution

$$F(s) = \frac{3s + 4}{s^2 + 9} = \frac{3s}{s^2 + 9} + \frac{4}{s^2 + 9}$$

$$L^{-1}\{F(s)\} = 3\cos 3t + \frac{4}{3}\sin 3t$$

Example 11.58 *Find the inverse Laplace transform of* $\dfrac{4s + 15}{16s^2 - 25}$.

Solution

$$F(s) = \frac{4s + 15}{16s^2 - 25} = \frac{4s + 15}{16\left(s^2 - \frac{25}{16}\right)} = \frac{1}{4}\frac{s}{s^2 - \frac{25}{16}} + \frac{15}{16}\frac{1}{s^2 - \frac{25}{16}}$$

$$L^{-1}\{F(s)\} = \frac{1}{4}\cosh\frac{5}{4}t + \frac{3}{4}\sinh\frac{5}{4}t$$

Example 11.59 *Find the inverse Laplace transform of* $\dfrac{2s + 2}{s^2 + 2s + 10}$.

Solution

$$F(s) = \frac{2s + 2}{s^2 + 2s + 10} = \frac{2(s + 1)}{(s + 1)^2 + 9}$$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2 + 9}\right\} = 2e^{-t}\cos 3t$$

Example 11.60 *Find the inverse Laplace transform of* $\dfrac{3s+7}{s^2-2s-3}$.

Solution

$$F(s)=\frac{3s+7}{s^2-2s-3}=\frac{3(s-1)+10}{(s-1)^2-4}=3\frac{(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}$$

$$L^{-1}\{F(s)\}=3e^tL^{-1}\left\{\frac{s}{s^2-4}\right\}+10e^tL^{-1}\left\{\frac{1}{s^2-4}\right\}=3e^t\cosh 2t+5e^t\sinh 2t$$

11.7.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in s. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I Factors are linear and distinct,

$$F(s)=\frac{P(s)}{(s+a)(s+b)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+a}+\frac{B}{s+b}$$

Case II Factors are linear and repeated,

$$F(s)=\frac{P(s)}{(s+a)(s+b)^n}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+a}+\frac{B_1}{s+b}+\frac{B_2}{(s+b)^2}+\ldots+\frac{B_n}{(s+b)^n}$$

Case III Factors are quadratic and distinct,

$$F(s)=\frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial-fraction expansion,

$$F(s)=\frac{As+B}{s^2+as+b}+\frac{Cs+D}{s^2+cs+d}$$

Case IV Factors are quadratic are repeated,

$$F(s)=\frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial-fraction expansion,

$$F(s)=\frac{As+B}{s^2+as+b}+\frac{C_1s+D_1}{s^2+cs+d}+\frac{C_2s+D_2}{(s^2+cs+d)^2}+\ldots+\frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Example 11.61 *Find the inverse Laplace transform of* $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

$$F(s)=\frac{s+2}{s(s+1)(s+3)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

$$A=sF(s)\big|_{s=0}=\left.\frac{s+2}{(s+1)(s+3)}\right|_{s=0}=\frac{2}{3}$$

$$B=(s+1)F(s)\big|_{s=-1}=\left.\frac{s+2}{s(s+3)}\right|_{s=-1}=-\frac{1}{2}$$

$$C=(s+3)F(s)\big|_{s=-3}=\left.\frac{s+2}{s(s+1)}\right|_{s=-3}=-\frac{1}{6}$$

$$F(s)=\frac{2}{3}\cdot\frac{1}{s}-\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\}=\frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}=\frac{2}{3}-\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$$

Example 11.62 *Find the inverse Laplace transform of* $\dfrac{s+2}{s^2(s+3)}$.

Solution

$$F(s)=\frac{s+2}{s^2(s+3)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+3}$$

$$\begin{aligned}s+2&=As(s+3)+B(s+3)+Cs^2\\&=As^2+3As+Bs+3B+Cs^2\\&=(A+C)s^2+(3A+B)s+3B\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned}A+C&=0\\3A+B&=1\\3B&=2\end{aligned}$$

Solving these equations,

$$A=\frac{1}{9},\ B=\frac{2}{3},\ C=-\frac{1}{9}$$

$$F(s)=\frac{1}{9}\cdot\frac{1}{s}+\frac{2}{3}\cdot\frac{1}{s^2}-\frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\}=\frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\}+\frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\}=\frac{1}{9}+\frac{2}{3}t-\frac{1}{9}e^{-3t}$$

Example 11.63 *Find the inverse Laplace transform of* $\dfrac{s^2-15s-11}{(s+1)(s-2)^2}$.

Solution

$$F(s)=\frac{5s^2-15s-11}{(s+1)(s-2)^2}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$\begin{aligned}5s^2-15s-11&=A(s-2)^2+B(s+1)(s-2)+C(s+1)\\&=A(s^2-4s+4)+B(s^2-s-2)+C(s+1)\\&=As^2-4As+4A+Bs^2-Bs-2B+Cs+c\\&=(A+B)s^2-(4A+B-C)s+(4A-2B+C)\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned}A+B&=5\\4A+B-C&=15\\4A-2B+C&=-11\end{aligned}$$

Solving these equations,

$$\begin{aligned}A&=1\\B&=4\\C&=-7\end{aligned}$$

$$F(s)=\frac{1}{s+1}+\frac{4}{s-2}-\frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-2}\right\}-7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}=e^{-t}+4e^{2t}-7te^{2t}$$

Example 11.64 *Find the inverse Laplace transform of* $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

$$F(s)=\frac{3s+1}{(s+1)(s^2+2)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+2}$$

$$\begin{aligned}3s+1&=A(s^2+2)+(Bs+C)(s+1)\\&=As^2+2A+Bs^2+Bs+Cs+C\\&=(A+B)s^2+(B+C)s+(2A+C)\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$A+B=0$$
$$B+C=3$$
$$2A+C=1$$

Solving these equations,

$$A=-\frac{2}{3},\ B=\frac{2}{3},\ C=\frac{7}{3}$$

$$F(s)=-\frac{2}{3}\cdot\frac{1}{s+1}+\frac{2}{3}\cdot\frac{s}{s^2+2}+\frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\}=-\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\}+\frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\}=-\frac{2}{3}e^{-t}+\frac{2}{3}\cos\sqrt{2t}+\frac{7}{3\sqrt{2}}\sin\sqrt{2t}$$

Example 11.65 *Find the inverse Laplace transform of* $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

$$F(s)=\frac{s}{(s^2+1)(s^2+4)}=\frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right]=\frac{1}{3}\left[\frac{s}{s^2+1}-\frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\}=\frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\}-L^{-1}\left\{\frac{s}{s^2+4}\right\}\right]=\frac{1}{3}[\cos t-\cos 2t]$$

11.7.3 Convolution Theorem

If $L^{-1}\{F_1(s)\}=f_1(t)$ and $L^{-1}\{F_2(s)\}=f_2(t)$ then $L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t f_1(u)f_2(t-u)du$

where $\int_0^t f_1(u)f_2(t-u)du=f_1(t)*f_2(t)$

Proof

$$F_1(s)\cdot F_2(s)=L\{f_1(t)\}\cdot L\{f_2(t)\}=\int_0^\infty e^{-su}f_1(u)du\cdot\int_0^\infty e^{-sv}f_2(v)dv=\int_0^\infty\int_0^\infty e^{-s(u+v)}f_1(u)f_2(v)du\,dv$$

$$=\int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)}f_2(v)dv\right]du$$

Putting $u+v=t,\ \ dv=dt$

When $\quad v=0,\ \ t=u$

$\quad v\to\infty,\ t\to\infty$

$$F_1(s)\cdot F_2(s)=\int_0^\infty f_1(u)\left[\int_u^\infty e^{-st}f_2(t-u)dt\right]du=\int_0^\infty\int_0^\infty e^{-st}f_1(u)f_2(t-u)dt\,du$$

The region of integration is bounded by the lines $u=0$ and $u=t$. To change the order of integration, draw a vertical strip which starts from line $u=0$ and terminates on the line $u=t$. Hence, u varies from 0 to t and t varies from 0 to ∞.

$$F_1(s)\cdot F_2(s)=\int_0^\infty e^{-st}\int_0^t f_1(u)f_2(t-u)\,du\,dt=L\left\{\int_0^t f_1(u)f_2(t-u)\,du\right\}$$

Fig. 11.36

Hence, $$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)du$$

Note Convolution operation is commutative, i.e.,

$$L\left\{\int_0^t f_1(u) f_2(t-u)du\right\} = L\left\{\int_0^t f_1(t-u) f_2(u)du\right\}$$

Example 11.66 *Find the inverse Laplace transform of* $\dfrac{1}{(s+2)(s-1)}$.

Solution

$$F(s) = \frac{1}{(s+2)(s-1)}$$

Let

$$F_1(s) = \frac{1}{s+2} \qquad F_2(s) = \frac{1}{s-1}$$

$$f_1(t) = e^{-2t} \qquad f_2(t) = e^t$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u} e^{t-u} du = e^t \int_0^t e^{-3u} du = e^t \left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{e^t}{3}(1-e^{-3t})$$

Example 11.67 *Find the inverse Laplace transform of* $\dfrac{1}{s^2(s+1)^2}$.

Solution

$$F(s) = \frac{1}{s^2(s+1)^2}$$

Let

$$F(s) = \frac{1}{(s+1)^2} \qquad F_2(s) = \frac{1}{s^2}$$

$$f_1(t) = te^{-t} \qquad f_2(t) = t$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t ue^{-u}(t-u)\, du = \int_0^t (ut-u^2)e^{-u}\, du = \left[(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right]_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Example 11.68 *Find the inverse Laplace transform of* $\dfrac{1}{(s-2)(s+2)^2}$.

Solution

$$F(s) = \frac{1}{(s-2)(s+2)^2}$$

Let

$$F_1(s) = \frac{1}{(s+2)^2} \qquad F_2(s) = \frac{1}{s-2}$$

$$f_1(t) = te^{-2t} \qquad f_2(t) = e^{2t}$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t ue^{-2u}\, e^{2(t-u)} \int_0^t du = e^{2t} \int_0^t ue - 4u\, du = e^{2t}\left[\frac{ue^{-4u}}{-4} - \frac{e^{-4u}}{16}\right]_0^t = e^{2t}\left(\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right)$$

$$= \frac{e^{2t}}{16} - \frac{te^{-2t}}{4} - \frac{e^{-2t}}{16} = \frac{1}{16}\left(e^{2t} - e^{-2t} - 4t\, e^{-2t}\right)$$

Example 11.69 *Find the inverse Laplace transform of* $\dfrac{1}{s^2(s^2+1)}$.

Solution

$$F(s) = \frac{1}{s^2(s^2+1)}$$

Let

$$F_1(s) = \frac{1}{s^2+1} \qquad F_2(s) = \frac{1}{s^2}$$

$$f_1(t) = \sin t \qquad f_2(t) = t$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\,(t-u)\,du = \left[(t-u)(-\cos u) - \sin u\right]_0^t = t - \sin t$$

Example 11.70 *Find the inverse Laplace transform of* $\dfrac{1}{(s+1)(s^2+1)}$.

Solution

$$F(s) = \frac{1}{(s+1)(s^2+1)}$$

Let

$$F_1(s) = \frac{1}{s^2+1} \qquad F_2(s) = \frac{1}{s+1}$$

$$f_1(t) = \sin t \qquad f_2(t) = e^{-t}$$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\; e^{-(t-u)}\,du = \int_1^t e^{u-t}\sin u\,du = e^{-t}\left[\frac{e^u}{2}(\sin u - \cos u)\right]_0^t = \frac{e^{-t}}{2}[e^t(\sin t - \cos t) + 1]$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$$

11.8 SOLUTION OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.

Example 11.71 *Solve* $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$.

Solution Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$$

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+3} \qquad [\because y(0) = 1]$$

$$(s+2)Y(s) = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

$$Y(s) = \frac{s+4}{(s+2)(s+3)}$$

By partial-fraction expansion,

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = (s+2)Y(s)\big|_{s=-2} = \left.\frac{s+4}{s+3}\right|_{s=-2} = 2$$

$$B = (s+3)Y(s)\big|_{s=-3} = \left.\frac{s+4}{s+2}\right|_{s=-3} = -1$$

$$Y(s) = \frac{2}{s+2} - \frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides.

$$y(t) = 2e^{-2t} - e^{-3t}$$

Example 11.72 *Solve* $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$.

Solution Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s^2} \qquad [\because y(0) = 1, y'(0) = 0]$$

$$(s^2+1)Y(s) = \frac{1}{s^2} + s = \frac{s^3+1}{s^2}$$

$$Y(s) = \frac{s^3+1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{s^2+1-s^2}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \cos t + t - \sin t$$

Example 11.73 *Solve* $y'' + y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$.

Solution Taking Laplace transform of both sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2+s)\,Y(s)=\frac{2}{s^3}+\frac{2}{s^2}+4s+2=\frac{2(1+s)}{s^3}+4s+2$$

$$Y(s)=\frac{2(1+s)}{s^3(s^2+s)}+\frac{4s}{s^2+s}+\frac{2}{s^2+s}=\frac{2}{s^4}+\frac{4}{s+1}+\frac{2}{s}-\frac{2}{s+1}=\frac{2}{s^4}+\frac{2}{s}-\frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{t^3}{3}+2+2e^{-t}$$

Example 11.74 *Solve* $y''+4y=\delta(t),\ y(0)=0,\ y'(0)=0.$

Solution Taking Laplace transform of both the sides,

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=1$$

$$s^2\,Y(s)+4\,Y(s)=1 \qquad [\because y(0)=0,\ y'(0)=0]$$

$$(s^2+4)\,Y(s)=1$$

$$Y(s)=\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{2}\sin 2t$$

Example 11.75 *Solve* $y''+3y'+2y=t\delta(t-1),\ y(0)=0,\ y'(0)=0.$

Solution Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+3\left[sY(s)-y(0)\right]+2Y(s)=e^{-s}$$

$$s^2Y(s)+3s\,Y(s)+2Y(s)=e^{-s} \qquad [\because y(0)=0,\ y'(0)=0]$$

$$(s^2+3s+2)\,Y(s)=e^{-s}$$

$$Y(s)=\frac{e^{-s}}{s^2+3s+2}=\frac{e^{-s}}{(s+1)(s+2)}=e^{-s}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)$$

Taking inverse Laplace transform of both the sides,

$$y(t)=e^{-(t-1)}u(t-1)-e^{-2(t-1)}u(t-1)$$

Example 11.76 *Solve* $y''+4y=u(t-2),\ y(0)=0,\ y'(0)=1.$

Solution Taking Laplace transform of both the sides,

$$\left[s^2\,Y(s)-sy(0)-y'(0)\right]+4Y(s)=\frac{e^{-2s}}{s}$$

$$s^2\,Y(s)-1+4Y(s)=\frac{e^{-2s}}{s} \qquad [\because y(0)=0,\ y'(0)=1]$$

$$(s^2+4)\,Y(s)=\frac{e^{-2s}}{s}+1$$

$$Y(s)=\frac{e^{-2s}}{s(s^2+4)}+\frac{1}{s^2+4}=\frac{e^{-2s}}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)+\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{4}u(t-2)-\frac{1}{4}\cos 2(t-2)u(t-2)+\frac{1}{2}\sin 2t$$

11.9 SOLUTION OF A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transform can also be used to solve two or more simultaneous differential equations. The Laplace transform method transforms the differential equations into algebraic equations.

Example 11.77 *Solve* $\frac{dx}{dt} + y = \sin t.$

$$\frac{dy}{dt} + x = \cos t$$

where $x(0) = 0$ *and* $y(0) = 2$.

Solution Taking Laplace transform of both the equations,

$$sX(s) - x(0) + Y(s) = \frac{1}{s^2+1}$$

$$sX(s) + Y(s) = \frac{1}{s^2+1} \qquad \ldots\text{(i)}$$

and

$$sY(s) - y(0) + X(s) = \frac{s}{s^2+1}$$

$$sY(s) + X(s) = \frac{s}{s^2+1} + 2$$

$$sY(s) + X(s) = \frac{2s^2+s+2}{s^2+1} \qquad \ldots\text{(ii)}$$

Multiplying Eq. (i) by s,

$$s^2X(s) + sY(s) = \frac{s}{s^2+1} \qquad \ldots\text{(iii)}$$

Subtracting Eq. (iii) from Eq. (ii),

$$(s^2-1)X(s) = -2$$

$$X(s) = -\frac{2}{s^2-1} \qquad \ldots\text{(iv)}$$

Substituting $X(s)$ in Eq. (i),

$$Y(s) = \frac{1}{s^2+1} + 2\frac{s}{s^2-1} \qquad \ldots\text{(v)}$$

Taking inverse Laplace transform of Eqs (iv) and (v),

$$x(t) = -2\sinh t$$

and

$$y(t) = \sin t + 2\cosh t$$

Example 11.78 *Solve* $\frac{dx}{dt} - y = e^t$

$$\frac{dy}{dt} + x = \sin t$$

where $x(0) = 1$ *and* $y(0) = 0$.

Solution Taking Laplace transform of both the equations,

$$sX(s) - x(0) - Y(s) = \frac{1}{s-1}$$

$$sX(s) - Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1} \qquad \ldots\text{(i)}$$

and

$$sY(s) - y(0) + X(s) = \frac{1}{s^2+1}$$

$$sY(s) + X(s) = \frac{1}{s^2+1} \qquad \ldots\text{(ii)}$$

Multiplying Eq. (i) by s,

$$s^2X(s) - sY(s) = \frac{s^2}{s-1} \qquad \ldots\text{(iii)}$$

Adding Eqs (ii) and (iii),

$$(s^2+1)X(s) = \frac{1}{s^2+1} + \frac{s^2}{s-1}$$

$$X(s) = \frac{1}{(s^2+1)^2} + \frac{s^2}{(s-1)(s^2+1)}$$

$$= \frac{1}{(s^2+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \qquad \ldots\text{(iv)}$$

Substituting $X(s)$ in Eq. (i),

$$Y(s) = sX(s) - \frac{s}{s-1} = \frac{s}{(s^2+1)} - \frac{s^3}{(s-1)(s^2+1)} - \frac{s}{s-1}$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \qquad \ldots\text{(v)}$$

Taking the inverse Laplace transform of Eqs. (iv) and (v),

$$x(t) = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}(e^t + \cos t + \sin t) = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

and $y(t) = \frac{1}{2}t\sin t - \frac{1}{2}(e^t - \cos t + \sin t) = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$

Example 11.79 *Solve* $\frac{dx}{dt} + 5x - 2y = t$

$$\frac{dy}{dt} + 2x + y = 0$$

where $x(0) = 0$ *and* $y(0) = 0$.

Solution Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) + 5\,X(s) - 2Y(s) = \frac{1}{s^2}$$

$$(s+5)\,X(s) - 2Y(s) = \frac{1}{s^2} \qquad \ldots\text{(i)}$$

and

$$s\,Y(s) - y(0) + 2X(s) + Y(s) = 0$$

$$2X(s) + (s+1)\,Y(s) = 0 \qquad \ldots\text{(ii)}$$

Multiplying Eq. (i) by $\frac{1}{2}(s+1)$,

$$\frac{1}{2}(s+5)(s+1)X(s) - (s+1)Y(s) = \frac{s+1}{2s^2} \qquad \ldots\text{(iii)}$$

Adding Eqs (ii) and (iii),

$$X(s) = \frac{s+1}{s^2(s+3)^2} \qquad \ldots\text{(iv)}$$

Substituting $X(s)$ in Eq. (ii),

$$Y(s) = -\frac{2}{s^2(s+3)^2} \qquad \ldots\text{(v)}$$

Now,

$$X(s) = \frac{s+1}{s^2(s+3)^2}$$

By partial-fraction expansion,

$$X(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$s+1 = A\,s(s+3)^2 + B(s+3)^2 + C(s+3)s^2 + D\,s^2 \qquad \ldots\text{(vi)}$$

$$= As\,(s^2+6s+9) + B(s^2+6s+9) + C(s^3+3s^2) + Ds^2$$

$$= As^3 + 6As^2 + 9As + Bs^2 + 6Bs + 9B + Cs^3 + 3Cs^2 + Ds^2$$

$$= (A+C)s^3 + (6A+B+3C+D)s^2 + (9A+6B)s + 9B$$

Comparing coefficients of s^3, s^2, s^1 and s^0,

$$A + C = 0$$

$$6A + B + 3C + D = 0$$

$$9A + 6B = 1$$

$$9B = 1$$

Solving these equations,

$$A = \frac{1}{27},\ B = \frac{1}{9},\ C = -\frac{1}{27},\ D = -\frac{2}{9}$$

$$X(s) = \frac{1}{27}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s^2} - \frac{1}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$x(t) = \frac{1}{27} + \frac{1}{9}t - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Similarly,

$$Y(s) = \frac{-2}{s^2(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2} = \frac{4}{27}\cdot\frac{1}{s} - \frac{2}{9}\cdot\frac{1}{s^2} - \frac{4}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

11.10 THE TRANSFORMED CIRCUIT

Voltage–current relationships of network elements can also be represented in the frequency domain.

1. Resistor For the resistor, the *v–i* relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig 11.37.

Fig. 11.37 *Resistor*

2. Inductor For the inductor, the *v–i* relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig 11.38.

Fig. 11.38 *Inductor*

3. Capacitor For capacitor, the v–i relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig 11.39.

Fig. 11.39 *Capacitor*

11.11 RESISTOR–INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. 11.40. The switch is closed at time $t = 0$.

Fig. 11.40 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 11.41.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{\dfrac{V}{L}}{s\left(s + \dfrac{R}{L}\right)}$$

Fig. 11.41 *Transformed network*

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \dfrac{R}{L}}$$

$$A = sI(s)\,|_{s=0} = s \times \left.\frac{\dfrac{V}{L}}{s\left(s + \dfrac{R}{L}\right)}\right|_{s=0} = \frac{V}{R}$$

$$B=\left(s+\frac{R}{L}\right)I(s)\Big|_{s=-\frac{R}{L}}=\left(s+\frac{R}{L}\right)\times\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\Bigg|_{s=-\frac{R}{L}}=-\frac{V}{R}$$

$$I(s)=\frac{\frac{V}{R}}{s}+\frac{\left(-\frac{V}{R}\right)}{s+\frac{R}{L}}$$

Taking the inverse Laplace transform,

$$i(t)=\frac{V}{R}-\frac{V}{R}e^{-\frac{R}{L}t}$$

$$=\frac{V}{R}\left[1-e^{-\frac{R}{L}t}\right]\qquad \text{for } t>0$$

Example 11.80 *In the network of Fig. 11.42, the switch is moved from the position 1 to 2 at t = 0, steady-state condition having been established in the position 1. Determine i (t) for t > 0.*

Fig. 11.42

Solution At $t = 0^-$, the network is shown in Fig 11.43. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-)=\frac{10}{1}=10\text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i\,(0^+)=10\text{ A}$$

Fig. 11.43

For $t > 0$, the transformed network is shown in Fig. 11.44.

Applying KVL to the mesh for $t > 0$,

$$-I(s)-I(s)-sI(s)+10=0$$

$$I(s)(s+2)=10$$

$$I(s)=\frac{10}{s+2}$$

Fig. 11.44

Taking inverse Laplace transform,

$$i\,(t)=10e^{-2t}\quad \text{for } t>0$$

Example 11.81 *The network of Fig. 11.45 was initially in the steady state with the switch in the position a. At t = 0, the switch goes from a to b. Find an expression for voltage v (t) for t > 0.*

Fig. 11.45

Solution At $t = 0^-$, the network is shown in Fig 11.46. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor of 2H acts as a short circuit.

Fig. 11.46

$$i(0^-) = \frac{2}{2} = 1 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i\,(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.47.

Fig. 11.47

Applying KCL at the node for $t > 0$,

$$\frac{V(s)+2}{2s} + \frac{V(s)}{1} + \frac{V(s)}{s} = 0$$

$$V(s)\left(1+\frac{3}{2s}\right) = -\frac{1}{s}$$

$$V(s) = \frac{-\frac{1}{s}}{\frac{2s+3}{2s}} = -\frac{2}{2s+3} = -\frac{1}{s+1.5}$$

Taking the inverse Laplace transform,

$$v\,(t) = -\,e^{-1.5\,t} \qquad \text{for } t > 0$$

Example 11.82 *In the network of Fig. 11.48, the switch is opened at t = 0. Find i(t).*

Fig. 11.48

Solution At $t = 0^-$, the network is shown in Fig. 11.49. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the inductor acts as a short circuit.

Fig. 11.49

$$i_T(0^-) = \frac{36}{10+(3\,||\,6)} = \frac{36}{10+2} = 3 \text{ A}$$

$$i_L(0^-) = 3\times\frac{6}{6+3} = 2 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.50.

Applying KVL to the mesh for $t > 0$,

$$-0.2 - 0.1s\, I(s) - 3I(s) - 6I(s) = 0$$

$$0.1sI(s) + 9I(s) = 0.2$$

$$I(s) = \frac{0.2}{0.1s+9} = \frac{2}{s+90}$$

Fig. 11.50

Taking inverse Laplace transform,

$$i(t) = 2e^{-90t} \qquad \text{for } t > 0$$

Example 11.83 *The network shown in Fig. 11.51 has acquired steady-state with the switch closed for t < 0. At t = 0, the switch is opened. Obtain i (t) for t > 0.*

Fig. 11.51

Solution At $t = 0^-$, the network is shown in Fig 11.52. At $t = 0^-$, the switch is closed and the network has acquired steady-state. Hence, the inductor acts as a short circuit.

$$i_T(0^-) = \frac{36}{10 + (4 \parallel 4)} = \frac{36}{10+2} = 3 \text{ A}$$

$$i(0^-) = 3 \times \frac{4}{4+4} = 1.5 \text{ A}$$

Fig. 11.52

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.5 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.53.

Applying KVL to the mesh for $t > 0$,

$$-4I(s) - 4I(s) - 2sI(s) + 3 = 0$$

$$8I(s) + 2sI(s) = 3$$

$$I(s) = \frac{3}{2s+8} = \frac{1.5}{s+4}$$

Fig. 11.53

Taking the inverse Laplace transform,

$$i(t) = 1.5e^{-4t} \qquad \text{for } t > 0$$

Example 11.84 *In the network shown in Fig. 11.54, the switch is closed at t = 0, the steady-state being reached before t = 0. Determine current through inductor of 3 H.*

Fig. 11.54

Solution At $t = 0^-$, the network is shown in Fig. 11.55. At $t = 0^-$, steady-state condition is reached. Hence, the inductor of 2 H acts as a short circuit.

Fig. 11.55

$$i_1(0^-) = \frac{1}{2} \text{ A}$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_1(0^+) = \frac{1}{2} \text{ A}$$

$$i_2(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.56.

Fig. 11.56

Applying KVL to Mesh 1,

$$\frac{1}{s} - 2s\,I_1(s) + 1 - 2[I_1(s) - I_2(s)] = 0$$

$$(2 + 2s)\,I_1(s) - 2I_2(s) = 1 + \frac{1}{s}$$

Applying KVL to Mesh 2,

$$-2\,[I_2(s) - I_1(s)] - 2I_2(s) - 3s\,I_2(s) = 0$$

$$-2I_1(s) + (4 + 3s)\,I_2(s) = 0$$

By Cramer's rule,

$$I_2(s) = \frac{\begin{vmatrix} 2+2s & 1+\frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2+2s & -2 \\ -2 & 4+3s \end{vmatrix}} = \frac{\frac{2}{s}(s+1)}{(2+2s)(4+3s)-4} = \frac{s+1}{s(3s^2+7s+2)} = \frac{s+1}{3s\left(s+\frac{1}{3}\right)(s+2)} = \frac{\frac{1}{3}(s+1)}{s(s+2)\left(s+\frac{1}{3}\right)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+\frac{1}{3}}$$

$$A = s\,I_2(s)\big|_{s=0} = \left.\frac{\frac{1}{3}(s+1)}{(s+2)\left(s+\frac{1}{3}\right)}\right|_{s=0} = \frac{1}{2}$$

$$B = (s+2)I_2(s)\big|_{S=-2} = \left.\frac{\frac{1}{3}(s+1)}{s\left(s+\frac{1}{3}\right)}\right|_{s=-2} = -\frac{1}{10}$$

$$C = \left(s+\frac{1}{3}\right) I_2(s)\Big|_{s=-\frac{1}{3}} = \left.\frac{\frac{1}{3}(s+1)}{s(s+2)}\right|_{s=-\frac{1}{3}} = -\frac{2}{5}$$

$$I_2(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{10}\frac{1}{s+2} - \frac{2}{5}\frac{1}{s+\frac{1}{3}}$$

Taking inverse Laplace transform

$$i_2(t) = \frac{1}{2} - \frac{1}{10}e^{-2t} - \frac{2}{5}e^{-\frac{1}{3}t} \quad \text{for } t > 0$$

Example 11.85 *In the network of Fig. 11.57, the switch is closed at t = 0 with the network previously unenergised. Determine currents $i_1(t)$.*

Fig. 11.57

Solution For $t > 0$, the transformed network is shown in Fig. 11.58.

Fig. 11.58

Applying KVL to Mesh 1,

$$\frac{100}{s} - 10I_1(s) - sI_1(s) - 10\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}$$

Applying KVL to Mesh 2,

$$-10\left[I_2(s) - I_1(s)\right] - s\,I_2(s) - 10\,I_2(s) = 0$$

$$-10\,I_1(s) + (s+20)I_2(s) = 0$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100} = \frac{100(s+20)}{s(s^2+40s+300)} = \frac{100(s+20)}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = s\,I_1(s)\,|_{s=0} = \left.\frac{100(s+20)}{(s+10)(s+30)}\right|_{s=0} = \frac{20}{3}$$

$$B = (s+10)I_1(s)\,|_{s=-10} = \left.\frac{100(s+20)}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30)I_1(s)\,|_{s=-30} = \left.\frac{100(s+20)}{s(s+10)}\right|_{s=-30} = -\frac{5}{3}$$

$$I_1(s) = \frac{20}{3}\frac{1}{s} - \frac{5}{s+10} - \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t}$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{1000}{s}}{(s+20)^2 - 100} = \frac{1000}{s(s^2+40s+300)} = \frac{1000}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = sI_2(s)\,|_{s=0} = \left.\frac{1000}{(s+10)(s+30)}\right|_{s=0} = \frac{10}{3}$$

$$B = (s+10)I_2(s)\,|_{s=-10} = \left.\frac{1000}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30)I_2(s)\,|_{s=-30} = \left.\frac{1000}{s(s+10)}\right|_{s=-30} = \frac{5}{3}$$

$$I_2(s) = \frac{10}{3}\frac{1}{s} - \frac{5}{s+10} + \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t} \qquad \text{for } t > 0$$

11.12 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 11.59. The switch is closed at time $t = 0$.

For $t > 0$, the transformed network is shown in Fig. 11.60.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - \frac{1}{Cs}I(s) = 0$$

Fig. 11.59 *RC circuit*

$$\left(R+\frac{1}{Cs}\right)I(s)=\frac{V}{s}$$

$$I(s)=\frac{\frac{V}{s}}{R+\frac{1}{Cs}}=\frac{\frac{V}{s}}{\frac{RCs+1}{Cs}}=\frac{\frac{V}{R}}{s+\frac{1}{RC}}$$

Fig. 11.60 *Transformed network*

Taking the inverse Laplace transform,

$$i(t)=\frac{V}{R}e^{-\frac{1}{RC}t} \qquad \text{for } t>0$$

Example 11.86 *In the network of Fig. 11.61, the switch is moved from a to b at t = 0. Determine i (t) and v_c (t).*

Fig. 11.61

Solution At $t=0^-$, the network is shown in Fig. 11.62. At $t=0^-$, the network has attained steady-state condition. Hence, the capacitor of 6 F acts as an open circuit.

$$v_{6\,\text{F}}(0^-)=10\text{ V}$$
$$i(0^-)=0$$
$$v_{3\,\text{F}}(0^-)=0$$

Fig. 11.62

Since voltage across the capacitor cannot change instantaneously,

$$v_{6\,\text{F}}(0^+)=10\text{ V}$$
$$v_{3\,\text{F}}(0^+)=0$$

For $t>0$, the transformed network is shown in 11.63.

Applying KVL to the mesh for $t>0$,

$$\frac{10}{s}-\frac{1}{6s}I(s)-I(s)-\frac{1}{3s}I(s)=0$$

$$\frac{1}{6s}I(s)+I(s)+\frac{1}{3s}I(s)=\frac{10}{s}$$

$$I(s)=\frac{10}{s\left(1+\frac{1}{6s}+\frac{1}{3s}\right)}=\frac{60}{6s+3}=\frac{10}{s+0.5}$$

Fig. 11.63

Taking the inverse Laplace transform,

$$i(t)=10e^{-0.5t} \qquad \text{for } t>0$$

Voltage across the 3 F capacitor is given by

$$V_c(s)=\frac{1}{3s}I(s)=\frac{10}{3s(s+0.5)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = sV_c(s)\big|_{s=0} = \frac{10}{3(s+0.5)}\bigg|_{s=0} = \frac{20}{3}$$

$$B = (s+0.5)V_c(s)\big|_{s=-0.5} = \frac{10}{3s}\bigg|_{s=-0.5} = -\frac{20}{3}$$

$$V_c(s) = \frac{20}{3}\frac{1}{s} - \frac{20}{3}\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v_c(t) = \frac{20}{3} - \frac{20}{3}e^{-0.5t}$$

$$= \frac{20}{3}(1 - e^{-0.5t}) \qquad \text{for } t > 0$$

Example 11.87 *The switch in the network shown in Fig. 11.64 is closed at $t = 0$. Determine the voltage cross the capacitor.*

Fig. 11.64

Solution At $t = 0^-$, the capacitor is uncharged.

$$v_c(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.65.

Fig. 11.65

Applying KCL at the node for $t > 0$,

$$\frac{V_c(s) - \frac{10}{s}}{10} + \frac{V_c(s)}{10} + \frac{V_c(s)}{\frac{1}{2s}} = 0$$

$$2sV_c(s) + 0.2V_c(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(2s+0.2)} = \frac{0.5}{s(s+0.1)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.1}$$

$$A = sV_c(s)\big|_{s=0} = \frac{0.5}{s+0.1}\bigg|_{s=0} = \frac{0.5}{0.1} = 5$$

$$B = (s+0.1)V_c(s)\big|_{s=-0.1} = \frac{0.5}{s}\bigg|_{s=-0.1} = -\frac{0.5}{0.1} = -5$$

$$V_c(s) = \frac{5}{s} - \frac{5}{s+0.1}$$

Taking inverse Laplace transform,

$$v_c(t) = 5 - 5e^{-0.1t} \qquad \text{for } t > 0$$

Example 11.88 *In the network of Fig. 11.66, the switch is closed for a long time and at t = 0, the switch is opened. Determine the current through the capacitor.*

Fig. 11.66

Solution At $t = 0^-$, the network is shown in Fig. 11.67. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v_c(0^-) = 0$$

Fig. 11.67

Since voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.68.

Fig. 11.68

Applying KVL to two parallel branches,

$$\frac{2}{s}I_1(s) + I_1(s) = I_2(s)$$

Applying KCL at the node for $t > 0$,

$$\frac{2}{s} = I_1(s) + I_2(s)$$

$$\frac{2}{s}I_1(s) + I_1(s) = \frac{2}{s} - I_1(s)$$

$$\frac{2}{s}I_1(s) + 2I_1(s) = \frac{2}{s}$$

$$I_1(s) = \frac{\frac{2}{s}}{\frac{2}{s}+2} = \frac{1}{s+1}$$

Taking the inverse Laplace transform,

$$i_1(t) = e^{-t} \qquad \text{for } t > 0$$

Example 11.89 *In the network of Fig. 11.69, the switch is moved from a to b, at t = 0. Find v(t).*

Fig. 11.69

Solution At $t = 0^-$, the network is shown in Fig 11.70. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor acts as an open circuit.

Fig. 11.70

$$v(0^-) = 6 \times \frac{2}{4+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 11.71.

Fig. 11.71

Applying KCL at the node for $t > 0$,

$$\frac{V(s)}{6} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{2} = 0$$

$$V(s)\left(\frac{2}{3} + s\right) = 2$$

$$V(s) = \frac{2}{s + \frac{2}{3}}$$

Taking the inverse Laplace transform,

$$v(t) = 2e^{-\frac{2}{3}t} \qquad \text{for } t > 0$$

Example 11.90 *The network shown in Fig. 11.72 has acquired steady-state at t < 0 with the switch open. The switch is closed at t = 0. Determine v (t).*

Fig. 11.72

Solution At $t = 0^-$, the network is shown in Fig 11.73. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor of 1 F acts as an open circuit.

Fig. 11.73

$$v(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 11.74.

Fig. 11.74

Applying KCL at the node for $t > 0$,

$$\frac{V(s) - \frac{4}{s}}{2} + \frac{V(s)}{2} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{\frac{1}{s}} = 0$$

$$2sV(s) + V(s) = \frac{2}{s} + 2$$

$$V(s) = \frac{\frac{2}{s} + 2}{2s+1} = \frac{2s+2}{s(2s+1)} = \frac{2}{s} - \frac{2}{2s+1} = \frac{2}{s} - \frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v(t) = 2 - e^{-0.5t} \qquad \text{for } t > 0$$

11.13 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit shown in Fig. 11.75. The switch is closed at time $t = 0$.

Fig. 11.75 *RLC circuit*

For $t > 0$, the transformed network is shown in Fig. 11.76.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + Ls + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$\left(\frac{LCs^2 + RCs + 1}{Cs}\right)I(s) = \frac{V}{s}$$

Fig. 11.76 *Transformed network*

$$I(s) = \frac{\dfrac{V}{s}}{\dfrac{LCs^2 + RCs + 1}{Cs}} = \frac{\dfrac{V}{L}}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}} = \frac{\dfrac{V}{L}}{(s - s_1)(s - s_2)}$$

where s_1 and s_2 are the roots of the equation $s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0.$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where $\alpha = \dfrac{R}{2L}$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and $\beta = \sqrt{\alpha^2 - \omega_0^2}$

By partial-fraction expansion, of $I(s)$,

$$I(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$$A = (s - s_1)\,I(s)\big|_{s=s_1} = \frac{\dfrac{V}{L}}{s_1 - s_2}$$

$$B = (s - s_2)\,I(s)\big|_{s=s_2} = \frac{\dfrac{V}{L}}{s_2 - s_1} = -\frac{\dfrac{V}{L}}{s_1 - s_2}$$

$$I(s) = \frac{V}{L(s_1 - s_2)}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{L(s_1 - s_2)}\left[e^{s_1 t} - e^{s_2 t}\right] = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation.

Now depending upon the values of s_1 and s_2, we have 3 cases of the response.

Case I When the roots are real and unequal, it gives an overdamped response.

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

$$\alpha > \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}\left(k_1 e^{\beta t} + k_2 e^{-\beta t}\right)$$

or $$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad \text{for } t > 0$$

Case II When the roots are real and equal, it gives a critically damped response.

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

$$\alpha = \omega_0$$

In this case, the solution is given by

$$i(t) = e^{-\alpha t}(k_1 + k_2 t) \qquad \text{for } t > 0$$

Case III When the roots are complex conjugate, it gives an underdamped response.

$$\frac{R}{2L} < \frac{1}{\sqrt{LC}}$$

$$\alpha < \omega_0$$

In this case, the solution is given by

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

where $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Let $$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$$

where $$j = \sqrt{-1}$$

and $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Hence

$$i(t) = e^{-\alpha t}\left(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\left\{\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right\} + j(k_1 - k_2)\left\{\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right\}\right]$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \qquad \text{for } t > 0$$

Example 11.91 *The switch in Fig. 11.77 is opened at time t = 0. Determine the voltage v(t) for t > 0.*

Fig. 11.77

Solution At $t = 0^-$, the network is shown in Fig. 11.78. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

Fig. 11.78

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.79.

Applying KCL at the node for $t > 0$,

$$\frac{V(s)}{0.5} + \frac{V(s)}{0.5s} + \frac{V(s)}{\frac{1}{0.5s}} = \frac{2}{s}$$

$$2V(s) + \frac{2}{s}V(s) + 0.5sV(s) = \frac{2}{s}$$

$$V(s) = \frac{\frac{2}{s}}{\frac{2}{s} + 0.5s + 2} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

Fig. 11.79

Taking inverse Laplace transform,

$$v(t) = 4t\, e^{-2t} \qquad \text{for } t > 0$$

Example 11.92 *In the network of Fig. 11.80, the switch is closed and steady-state is attained. At $t = 0$, switch is opened. Determine the current through the inductor.*

Fig. 11.80

Solution At $t = 0^-$, the network is shown in Fig. 11.81. At $t = 0^-$, the switch is closed and steady-state condition is attained. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit. Current through inductor is same as the current through the resistor.

$$i_L(0^-) = \frac{5}{2.5} = 2 \text{ A}$$

Fig. 11.81

Voltage across the capacitor is zero as it is connected in parallel with a short.

$$v_c(0^-) = 0$$

Since voltage across the capacitor and current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$
$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 11.82. Applying KVL to the mesh for $t > 0$,

Fig. 11.82

$$-\frac{1}{200 \times 10^{-6} s} I(s) - 0.5s\, I(s) + 1 = 0$$

$$0.5s\, I(s) - 1 + 5000 \frac{I(s)}{s} = 0$$

$$I(s) = \frac{1}{0.5s + \frac{5000}{s}} = \frac{2s}{s^2 + 10000}$$

Taking inverse Laplace transform,

$$i(t) = 2 \cos 100t \qquad \text{for } t > 0$$

Example 11.93 *In the network shown in Fig. 11.83, the switch is opened at t = 0. Steady-state condition is achieved before t = 0. Find i(t).*

Fig. 11.83

Solution At $t = 0^-$, the network is shown in Fig 11.84. At $t = 0^-$, the switch is closed and steady-state condition is achieved. Hence, the capacitor acts as an open circuit and the inductor acts as a short circuit.

$$v_c(0^-) = 1 \text{ V}$$
$$i(0^-) = 1 \text{ A}$$

Fig. 11.84

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 1 \text{ V}$$
$$i(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 11.85.

Fig. 11.85

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s} - \frac{1}{s} I(s) - 0.5s\, I(s) + 0.5 - I(s) = 0$$

$$0.5 + \frac{1}{s} = \frac{1}{s} I(s) + 0.5\, s\, I(s) + I(s)$$

$$I(s)\left[1 + \frac{1}{s} + 0.5s\right] = 0.5 + \frac{1}{s}$$

$$I(s) = \frac{s+2}{s^2+2s+2} = \frac{(s+1)+1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Taking the inverse Laplace transform,

$$i(t) = e^{-t}\cos t + e^{-t}\sin t \qquad \text{for } t > 0$$

Example 11.94 *In the network shown in Fig. 11.86, the switch is closed at t = 0. Find the currents $i_1(t)$ and $i_2(t)$ when initial current through the inductor is zero and initial voltage on the capacitor is 4 V.*

Fig. 11.86

Solution For $t > 0$, the transformed network is shown in Fig. 11.87.

Fig. 11.87

Applying KVL to Mesh 1,

$$\frac{10}{s} - I_1(s) - (1+s)\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+2)I_1(s) - (s+1)I_2(s) = \frac{10}{s}$$

Applying KVL to Mesh 2,

$$-(s+1)\left[I_2(s)-I_1(s)\right]-I_2(s)-\frac{1}{s}I_2(s)-\frac{4}{s}=0$$

$$-(s+1)I_1(s)+\left(s+2+\frac{1}{s}\right)I_2(s)=-\frac{4}{s}$$

By Cramer's rule,

$$I_1(s)=\frac{\begin{vmatrix}\frac{10}{s} & -(s+1)\\ -\frac{4}{s} & s+2+\frac{1}{s}\end{vmatrix}}{\begin{vmatrix}s+2 & -(s+1)\\ -(s+1) & s+2+\frac{1}{s}\end{vmatrix}}=\frac{\left(\frac{10}{s}\right)\left(\frac{s^2+2s+1}{s}\right)-(s+1)\left(\frac{4}{s}\right)}{(s+2)\left(\frac{s^2+2s+1}{s}\right)-(s+1)^2}=\frac{\frac{10}{s^2}(s+1)^2-(s+1)\frac{4}{s}}{(s+2)\frac{(s+1)^2}{s}-(s+1)^2}$$

$$=\frac{\frac{10}{s^2}(s+1)-\frac{4}{s}}{(s+2)\frac{(s+1)}{s}-(s+1)}=\frac{3s+5}{s(s+1)}$$

By partial-fraction expansion,

$$I_1(s)=\frac{A}{s}+\frac{B}{s+1}$$

$$A=sI_1(s)\big|_{s=0}=\left.\frac{3s+5}{s+1}\right|_{s=0}=5$$

$$B=(s+1)I_1(s)\big|_{s=-1}=\left.\frac{3s+5}{s}\right|_{s=-1}=-2$$

$$I_1(s)=\frac{5}{s}-\frac{2}{s+1}$$

Taking inverse Laplace transform,

$$i_1(t)=5-2e^{-t} \qquad \text{for } t>0$$

Similarly,

$$I_2(s)=\frac{\begin{vmatrix}s+2 & \frac{10}{s}\\ -(s+1) & -\frac{4}{s}\end{vmatrix}}{\begin{vmatrix}s+2 & -(s+1)\\ -(s+1) & s+2+\frac{1}{s}\end{vmatrix}}=\frac{3s+1}{(s+1)^2}=\frac{3s+3-2}{(s+1)^2}=\frac{3(s+1)-2}{(s+1)^2}=\frac{3}{s+1}-\frac{2}{(s+1)^2}$$

Taking inverse Laplace transform,

$$i_2(t)=3e^{-t}-2te^{-t} \qquad \text{for } t>0$$

11.14 RESPONSE OF RL CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RL* circuit shown in Fig. 11.88. When the switch is closed at $t=0$, $i(0^-)=i(0^+)=0$.

Fig. 11.88 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 11.89.
Applying KVL to the mesh,

$$V(s) - R\,I(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{V(s)}{R + Ls} = \frac{1}{L}\frac{V(s)}{s + \frac{R}{L}}$$

Fig. 11.89 *Transformed network*

(a) When the unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{1}{L}\frac{\frac{1}{s}}{s + \frac{R}{L}}$$

$$= \frac{1}{L}\frac{1}{s\left(s + \frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{1}{L}\left(\frac{A}{s} + \frac{B}{s + \frac{R}{L}}\right)$$

$$A = s\,I(s)\big|_{s=0} = \left.\frac{1}{s + \frac{R}{L}}\right|_{s=0} = \frac{L}{R}$$

$$B = \left(s + \frac{R}{L}\right)I(s)\Big|_{s=-\frac{R}{L}} = \left.\frac{1}{s}\right|_{s=-\frac{R}{L}} = -\frac{L}{R}$$

$$I(s) = \frac{1}{L}\left(\frac{L}{R}\frac{1}{s} - \frac{L}{R}\frac{1}{s + \frac{R}{L}}\right)$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t \qquad \text{for } t > 0$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{1}{L}\frac{1}{s^2\left(s + \frac{R}{L}\right)}$$

By partial-faction expansion,

$$\frac{1}{L}\frac{1}{s^2\left(s + \frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + \frac{R}{L}}$$

$$\frac{1}{L} = As\left(s + \frac{R}{L}\right) + B\left(s + \frac{R}{L}\right) + Cs^2$$

Putting $s = 0$,

$$B = \frac{1}{R}$$

Putting $s = -\frac{R}{L}$,

$$C = \frac{L}{R^2}$$

Comparing coefficients of s^2,

$$A + C = 0$$

$$A = -C = -\frac{L}{R^2}$$

$$I(s) = -\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L}{R^2}\frac{1}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = -\frac{L}{R^2} + \frac{1}{R}t + \frac{L}{R^2}e^{-\left(\frac{R}{L}\right)t}$$

$$= \frac{1}{R}t - \frac{L}{R^2}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(c) When unit impulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{1}{L}\frac{1}{s + \frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{L} e^{-\left(\frac{R}{L}\right)t} \qquad \text{for } t > 0$$

Example 11.95 *At t = 0, unit pulse voltage of unit width is applied to a series RL circuit as shown in Fig. 11.90. Obtain an expression for i(t).*

Fig. 11.90

Solution

$$v(t) = u(t) - u(t-1)$$

$$V(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

For $t > 0$, the transformed network is shown in Fig. 11.91.

Fig. 11.91

Applying KVL to the mesh,

$$V(s) - I(s) - sI(s) = 0$$

$$I(s) = \frac{V(s)}{s+1}$$

$$= \frac{1-e^{-s}}{s(s+1)}$$

$$= \frac{1}{s(s+1)} - \frac{e^{-s}}{s(s+1)}$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1}$$

Taking inverse Laplace transform,

$$i(t) = u(t) - e^{-t}u(t) - u(t-1) + e^{-(t-1)}u(t-1)$$

$$= (1-e^{-t})u(t) - [1-e^{-(t-1)}]u(t-1) \qquad \text{for } t > 0$$

Example 11.96 *For the network shown in Fig. 11.92, determine the current i(t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

Fig. 11.92

Fig. 11.93

Solution For $t > 0$, the transformed network is shown in Fig. 11.93. Applying KVL to the mesh for $t > 0$,

$$\frac{e^{-3s}}{s^2} - 5\,I(s) - 2s\,I(s) = 0$$

$$5\,I(s) + 2s\,I(s) = \frac{e^{-3s}}{s^2}$$

$$I(s) = \frac{e^{-3s}}{s^2(2s+5)} = \frac{0.5\,e^{-3s}}{s^2(s+2.5)}$$

By partial-fraction expansion,

$$\frac{0.5}{s^2(s+2.5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2.5}$$

$$0.5 = As(s+2.5) + B(s+2.5) + Cs^2$$

$$= As^2 + 2.5As + Bs + 2.5B + Cs^2$$

$$= (A+C)s^2 + (2.5A+B)s + 2.5\,B$$

Comparing coefficients of s^2, s and s^0,

$$A + C = 0$$

$$2.5A + B = 0$$

$$2.5B = 0.5$$

Solving these equations,

$$A = -0.08$$

$$B = 0.2$$

$$C = 0.08$$

$$I(s) = e^{-3s}\left(-\frac{0.08}{s} + \frac{0.2}{s^2} + \frac{0.08}{s+2.5}\right)$$

$$= -0.08\frac{e^{-3s}}{s} + 0.2\frac{e^{-3s}}{s^2} + 0.08\frac{e^{-3s}}{s+2.5}$$

Taking inverse Laplace transform,

$$i(t) = -0.08u(t-3) + 0.2r(t-3) + 0.08e^{-2.5(t-3)}u(t-3)$$

Example 11.97 *Determine the expression for $v_L(t)$ in the network shown in Fig. 11.94. Find $v_L(t)$ when (i) $v_s(t) = \delta(t)$, and (ii) $v_s(t) = e^{-t}\,u(t)$.*

Fig. 11.94

Solution For $t > 0$, the transformed network is shown in Fig. 11.95.
By voltage-division rule,

$$V_L(s) = V_s(s) \times \frac{\frac{s}{2}}{\frac{s}{2}+5} = \frac{s}{s+10} V_s(s)$$

Fig. 11.95

(a) For impulse input,

$$V_s(s) = 1$$

$$V_L(s) = \frac{s}{s+10} = \frac{s+10-10}{s+10} = 1 - \frac{10}{s+10}$$

Taking inverse Laplace transform,

$$V_L(t) = \delta(t) - 10e^{-10t}u(t) \qquad \text{for } t > 0$$

(b) For $v_s(t) = e^{-t}u(t)$,

$$V_s(s) = \frac{1}{s+1}$$

$$V_L(s) = \frac{s}{(s+10)(s+1)}$$

By partial-fraction expansion,

$$V_L(s) = \frac{A}{s+10} + \frac{B}{s+1}$$

$$A = (s+10)V_L(s)\big|_{s=-10} = \frac{s}{s+1}\bigg|_{s=-10} = \frac{10}{9}$$

$$B = (s+1)V_L(s)\big|_{s=-1} = \frac{s}{s+10}\bigg|_{s=-1} = -\frac{1}{9}$$

$$V_L(s) = \frac{10}{9}\frac{1}{s+10} - \frac{1}{9}\frac{1}{s+1}$$

Taking inverse Laplace transform,

$$v_L(t) = \frac{10}{9}e^{-10t}u(t) - \frac{1}{9}e^{-t}u(t)$$

$$= \left(\frac{10}{9}e^{-10t} - \frac{1}{9}e^{-t}\right)u(t) \qquad \text{for } t > 0$$

Example 11.98 *For the network shown in Fig. 11.96, determine the current i (t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

Fig. 11.96

Solution For $t > 0$, the transformed network is shown in Fig. 11.97.

Fig. 11.97

Applying KVL to the mesh for $t > 0$,

$$2e^{-3s} - 2I(s) - sI(s) = 0$$

$$2I(s) + sI(s) = 2e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = 2e^{-2(t-3)}u(t-3) \qquad \text{for } t > 0$$

Example 11.99 *Determine the current i(t) in the network shown in Fig. 11.98, when the switch is closed at t = 0.*

Fig. 11.98

Solution For $t > 0$, the transformed network is shown in Fig. 11.99.

Fig. 11.99

Applying KVL to the mesh for $t > 0$,

$$\frac{1250}{s^2+625} - 10I(s) - 5sI(s) = 0$$

$$I(s) = \frac{250}{(s^2+625)(s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+625} + \frac{C}{s+2}$$

$$250 = (As+B)(s+2) + C(s^2+625)$$

$$= (A+C)s^2 + (2A+B)s + (2B+625C)$$

Comparing coefficients,

$$A + C = 0$$

$$2A + B = 0$$

$$2B + 625C = 250$$

Solving the equations,

$$A = -0.397$$

$$B = 0.795$$

$$C = 0.397$$

$$I(s) = \frac{-0.397s + 0.795}{s^2+625} + \frac{0.397}{s+2} = -\frac{0.397s}{s^2+625} + \frac{0.795}{s^2+625} + \frac{0.397}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = -0.397\cos 25t + 0.032\sin 25t + 0.397e^{-2t} \qquad \text{for } t > 0$$

Example 11.100 *Find impulse response of the current i(t) in the network shown in Fig. 11.100.*

Fig. 11.100

Solution The transformed network is shown in Fig. 11.101.

Fig. 11.101

$$Z(s) = \frac{1(2s+1)}{2s+1+1} = \frac{2s+1}{2s+2}$$

$$I_1(s) = \frac{V(s)}{Z(s)} = \frac{1}{\frac{2s+1}{2s+2}} = \frac{2s+2}{2s+1}$$

By current-division rule,

$$I(s) = I_1(s) \times \frac{1}{2s+2} = \frac{1}{2s+2} \times \frac{2s+2}{2s+1} = \frac{1}{2s+1} = \frac{1}{2}\frac{1}{s+0.5}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}e^{-0.5t}u(t) \qquad \text{for } t > 0$$

Example 11.101 *The network shown in Fig. 11.102 is at rest for $t < 0$. If the voltage $v(t) = u(t)\cos t + A\delta(t)$ is applied to the network, determine the value of A so that there is no transient term in the current response i(t).*

Fig. 11.102

$$v(t) = u(t)\cos t + A\delta(t)$$

$$V(s) = \frac{s}{s^2+1} + A$$

Solution For $t > 0$, the transformed network is shown in Fig. 11.103. Applying KVL to the mesh for $t > 0$,

Fig. 11.103

$$V(s) = 2sI(s) + I(s) = \frac{s}{s^2+1} + A$$

$$I(s) = \frac{s + A(s^2+1)}{2\left(s+\frac{1}{2}\right)(s^2+1)} = \frac{K_1}{s+\frac{1}{2}} + \frac{K_2 s + K_3}{s^2+1}$$

The transient part of the response is given by the first term. Hence, for the transient term to vanish, $K_1 = 0$.

$$K_1 = \left(s+\frac{1}{2}\right) I(s)\Big|_{s=-\frac{1}{2}} = \frac{\frac{-1}{2} + A\left(\frac{5}{4}\right)}{2\left(\frac{5}{4}\right)}$$

When $K_1 = 0$

$$\frac{5}{4}A = \frac{1}{2}$$

$$A = \frac{2}{5} = 0.4$$

11.15 RESPONSE OF RC CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RC* circuit as shown in Fig. 11.104.

Fig. 11.104 *RC circuit*

For $t > 0$, the transformed network is shown in Fig. 11.105. Applying KVL to the mesh,

Fig. 11.105 *Transformed network*

$$V(s) - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$I(s) = \frac{V(s)}{\frac{1}{Cs} + R} = \frac{sV(s)}{R\left(s+\frac{1}{RC}\right)}$$

(a) When unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{s \times \frac{1}{s}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R\left(s + \frac{1}{RC}\right)}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R} e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{s \times \frac{1}{s^2}}{R\left(s + \frac{1}{RC}\right)} = \frac{\frac{1}{R}}{s\left(s + \frac{1}{RC}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{RC}}$$

$$A = s\, I(s)\big|_{s=0} = \left.\frac{\frac{1}{R}}{s + \frac{1}{RC}}\right|_{s=0} = C$$

$$B = \left(s + \frac{1}{RC}\right) I(s)\Big|_{s=-\frac{1}{RC}} = \left.\frac{\frac{1}{R}}{s}\right|_{s=-\frac{1}{RC}} = -C$$

$$I(s) = \frac{C}{s} - \frac{C}{s + \frac{1}{RC}}$$

Taking inverse Laplace transform,

$$i(t) = C - Ce^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(c) When unit inpulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{s}{R\left(s + \frac{1}{RC}\right)} = \frac{s + \frac{1}{RC} - \frac{1}{RC}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left(1 - \frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[\delta(t) - \frac{1}{RC}e^{-\frac{1}{RC}t}\right] \qquad \text{for } t > 0$$

Example 11.102 *A rectangular voltage pulse of unit height and T-seconds duration is applied to a series RC network at t = 0. Obtain the expression for the current i(t). Assume the capacitor to be initially uncharged.*

Fig. 11.106

Solution

$$v(t) = u(t) - u(t - T)$$

$$V(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1 - e^{-sT}}{s}$$

Fig. 11.107

For $t > 0$, the transformed network is shown in Fig. 11.107.
Applying KVL to the mesh for $t > 0$,

$$V(s) - RI(s) - \frac{1}{Cs}I(s) = 0$$

$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}} = \frac{\frac{1}{R}s}{s + \frac{1}{RC}}V(s) = \frac{1 - e^{-sT}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left[\frac{1}{s + \frac{1}{RC}} - \frac{e^{-sT}}{s + \frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[e^{-\left(\frac{1}{RC}\right)t}u(t) - e^{-\left(\frac{1}{RC}\right)(t-T)}u(t - T)\right] \qquad \text{for } t > 0$$

Example 11.103 *For the network shown in Fig. 11.108, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

Fig. 11.108

Solution For $t > 0$, the transformed network is shown in Fig. 11.109. Applying KVL to the mesh for $t > 0$,

Fig. 11.109

$$\frac{2e^{-2s}}{s^2} - 3\,I(s) - \frac{1}{s}I(s) = 0$$

$$\left(3 + \frac{1}{s}\right)I(s) = \frac{2e^{-2s}}{s^2}$$

$$I(s) = \frac{2e^{-2s}}{s^2\left(3 + \frac{1}{s}\right)} = \frac{0.67e^{-2s}}{s(s+0.33)}$$

By partial-fraction expansion,

$$\frac{0.67}{s(s+0.33)} = \frac{A}{s} + \frac{B}{s+0.33}$$

$$A = \left.\frac{0.67}{s+0.33}\right|_{s=0} = 2$$

$$B = \left.\frac{0.67}{s}\right|_{s=-0.33} = -2$$

$$I(s) = e^{-2s}\left(\frac{2}{s} - \frac{2}{s+0.33}\right) = 2\frac{e^{-2s}}{s} - 2\frac{e^{-2s}}{s+0.33}$$

Taking inverse Laplace transform,

$$i(t) = 2u(t-2) - 2e^{-0.33(t-2)}u(t-2) \qquad \text{for } t > 0$$

Example 11.104 *For the network shown in Fig. 11.110, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

Fig. 11.110

Solution For $t > 0$, the transformed network is shown in Fig. 11.111.

Applying KVL to the mesh for $t > 0$,

$$1 - 5\,I(s) - \frac{1}{2s}I(s) = 0$$

$$\left(5 + \frac{1}{2s}\right)I(s) = 1$$

$$I(s) = \frac{1}{5 + \frac{1}{2s}}$$

$$= \frac{2s}{10s+1}$$

$$= \frac{0.2s}{s+0.1}$$

$$= \frac{0.2(s+0.1-0.1)}{s+0.1}$$

$$= 0.2\left(1 - \frac{0.1}{s+0.1}\right)$$

$$= 0.2 - \frac{0.02}{s+0.1}$$

Fig. 11.111

Taking inverse Laplace transform,

$$i(t) = 0.2\delta(t) - 0.02\,e^{-0.1t}u(t)$$

Example 11.105 *For the network shown in Fig. 11.112, find the response v_0 (t).*

Fig. 11.112

Solution For $t > 0$, the transformed network is shown in Fig. 11.113.

$$V_s(s) = \frac{1}{2}\frac{s}{s^2+1}$$

By voltage-division rule,

$$V_o(s) = V_s(s) \times \frac{\frac{4}{s}}{2 + \frac{4}{s}} = \frac{2V_s(s)}{s+2} = \frac{s}{(s^2+1)(s+2)}$$

Fig. 11.113

By partial-fraction expansion,

$$V_o(s) = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

$$s = (As+B)(s+2) + c(s^2+1)$$

$$s = (A+C)s^2 + (2A+B)s + (2B+C)$$

Comparing coefficient of s^2, s and s^0,

$$A + C = 0$$
$$2A + B = 1$$
$$2B + C = 0$$

Solving the equations,

$$A = 0.4$$
$$B = 0.2$$
$$C = -0.4$$
$$V_o(s) = \frac{0.4s + 0.2}{s^2 + 1} - \frac{0.4}{s+2} = \frac{0.4s}{s^2+1} + \frac{0.2}{s^2+1} - \frac{0.4}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = 0.4\cos t + 0.2\sin t - 0.4e^{-2t} \qquad \text{for } t > 0$$

Example 11.106 *Find the impulse response of the voltage across the capacitor in the network shown in Fig. 11.114. Also determine response v_c (t) for step input.*

Fig. 11.114

Solution For $t > 0$, the transformed network is shown in Fig. 11.115.

By voltage-division rule,

$$V_c(s) = V(s) \times \frac{\frac{1}{s}}{2 + s + \frac{1}{s}}$$
$$= \frac{V(s)}{s^2 + 2s + 1} = \frac{V(s)}{(s+1)^2}$$

Fig. 11.115

(a) For impulse input,

$$V(s) = 1$$
$$V_c(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = te^{-t}u(t) \qquad \text{for } t > 0$$

(b) For step input,

$$V(s) = \frac{1}{s}$$
$$V_c(s) = \frac{1}{s(s+1)^2}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + Cs$$

$$= A(s^2 + 2s + 1) + B(s^2 + s) + Cs$$

$$= (A+B)s^2 + (2A + B + C)s + A$$

Comparing coefficient of s^2, s^1 and s^0,

$$A = 1$$

$$A + B = 0$$

$$B = -A = -1$$

$$2A + B + C = 0$$

$$C = -2A - B = -2 + 1 = -1$$

$$V_c(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = u(t) - e^{-t}u(t) - te^{-t}u(t)$$

$$= (1 - e^{-t} - te^{-t})ut \qquad \text{for } t > 0$$

Example 11.107 *For the network shown in Fig. 11.116, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

Fig. 11.116

Solution For $t > 0$, the transformed network is shown in Fig. 11.117.

Applying KVL to the mesh for $t > 0$,

$$\frac{5e^{-s}}{s^2} - 5I(s) - sI(s) - \frac{6}{s}I(s) = 0$$

$$5I(s) + sI(s) + \frac{6}{s}I(s) = \frac{5e^{-s}}{s^2}$$

$$I(s) = \frac{5e^{-s}}{s(s^2 + 5s + 6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

Fig. 11.117

By partial-fraction expansion,

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = \left.\frac{1}{(s+3)(s+2)}\right|_{s=0} = \frac{1}{6}$$

$$B = \left.\frac{1}{s(s+2)}\right|_{s=-3} = \frac{1}{3}$$

$$C = \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}$$

$$I(s) = 5e^{-s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right] = \frac{5}{6}\frac{e^{-s}}{s} + \frac{5}{3}\frac{e^{-s}}{s+3} - \frac{5}{2}\frac{e^{-s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}u(t-1) - \frac{5}{2}e^{-2(t-1)}u(t-1) \qquad \text{for } t > 0$$

Example 11.108 *For the network shown in Fig. 11.118, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions.*

Fig. 11.118

Solution For $t > 0$, the transformed network is shown in Fig. 11.119.
Applying KVL to the mesh for $t > 0$,

Fig. 11.119

$$\frac{1}{s^2+1} - 2I(s) - s\,I(s) - \frac{2}{s}I(s) = 0$$

$$\left(2 + s + \frac{2}{s}\right)I(s) = \frac{1}{s^2+1}$$

$$I(s) = \frac{s}{(s^2+1)(s^2+2s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$= As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Cs + Ds^2 + D$$

$$= (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing coefficients of s^3, s^2, s^1 and s^0,

$$A + C = 0$$

$$2A + B + D = 0$$

$$2A + 2B + C = 1$$

$$2B + D = 0$$

Solving these equations,

$$A = 0.2,\ B = 0.4,\ C = -0.2,\ D = -0.8$$

$$I(s) = \frac{0.2s + 0.4}{s^2 + 1} - \frac{0.2s + 0.8}{s^2 + 2s + 2}$$

$$= \frac{0.2s}{s^2 + 1} + \frac{0.4}{s^2 + 1} - \frac{0.2s + 0.2 + 0.6}{(s+1)^2 + (1)^2}$$

$$= \frac{0.2s}{s^2 + 1} + \frac{0.4}{s^2 + 1} - \frac{0.2(s+1)}{(s+1)^2 + 1} - \frac{0.6}{(s+1)^2 + 1}$$

Taking inverse Laplace transform,

$$i(t) = 0.2\cos t + 0.4\sin t - 0.2\,e^{-t}\cos t - 0.6\,e^{-t}\sin t$$

$$= 0.2\cos t + 0.4\sin t - e^{-t}(0.2\cos t + 0.6\sin t) \qquad \text{for } t > 0$$

Example 11.109 *For the network shown in Fig. 11.120, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions in the network elements.*

Fig. 11.120

Solution For $t > 0$, the transformed network is shown in Fig. 11.121.
Applying KVL to the mesh for $t > 0$,

$$\frac{6}{s+2} - 5I(s) - s\,I(s) - \frac{4}{s}I(s) = 0$$

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{6}{s+2}$$

$$I(s) = \frac{6s}{(s+2)(s^2 + 5s + 4)}$$

$$= \frac{6s}{(s+2)(s+1)(s+4)}$$

Fig. 11.121

By partial-fraction expansion,

$$I(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = (s+2)I(s)\,|_{s=-2} = \left.\frac{6s}{(s+1)(s+4)}\right|_{s=-2} = 6$$

$$B = (s+1)I(s)\,|_{s=-1} = \left.\frac{6s}{(s+2)(s+4)}\right|_{s=-1} = -2$$

$$C = (s+4)I(s)\,|_{s=-4} = \left.\frac{6s}{(s+2)(s+1)}\right|_{s=-4} = -4$$

$$I(s) = \frac{6}{s+2} - \frac{2}{s+1} - \frac{4}{s+4}$$

Taking inverse Laplace transform,

$$i(t) = 6e^{-2t}u(t) - 2e^{-t}u(t) - 4\,e^{-4t}u(t) \qquad \text{for } t > 0$$

Example 11.110 *The network shown has zero initial conditions. A voltage $v_i(t) = \delta(t)$ applied to two terminal network produces voltage $v_o(t) = [e^{-2t} + e^{-3t}]\,u(t)$. What should be $v_i(t)$ to give $v_o(t) = t\,e^{-2t}\,u(t)$?*

Fig. 11.122

Solution For $v_i(t) = \delta(t)$,

$$V_i(s) = 1$$

$$v_o(t) = [e^{-2t} + e^{-3t}]u(t)$$

$$V_o(s) = \frac{1}{s+2} + \frac{1}{s+3}$$

System function $H(s) = \dfrac{V_o(s)}{V_i(s)}$

$$= \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)} \qquad \ldots\text{(i)}$$

For $v_o(t) = te^{-2t}u(t)$,

$$V_o(s) = \frac{1}{(s+2)^2}$$

From Eq. (i),

$$V_i(s) = \frac{V_o(s)}{H(s)} = \frac{1}{(s+2)^2} \times \frac{(s+2)(s+3)}{2s+5} = \frac{(s+3)}{2(s+2.5)(s+2)}$$

By partial-fraction expansion,

$$V_i(s) = \frac{A}{s+2} + \frac{B}{s+2.5}$$

$$A = 1$$

$$B = -0.5$$

$$V_i(s) = \frac{1}{s+2} - \frac{0.5}{s+2.5}$$

Taking inverse Laplace transform,

$$v_i(t) = e^{-2t} - 0.5e^{-2.5t} \qquad \text{for } t > 0$$

Example 11.111 *A unit impulse applied to two terminal black box produces a voltage $v_o(t) = 2e^{-t} - e^{-3t}$. Determine the terminal voltage when a current pulse of 1 A height and a duration of 2 seconds is applied at the terminal.*

Fig. 11.123

Solution

$$v_o(t) = 2e^{-t} - e^{-3t}$$

$$V_o(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t) = \delta(t)$,

Fig. 11.124

$$I_s(s) = 1$$

$$V_o(s) = Z(s)\, I_s(s) \qquad \ldots\text{(i)}$$

$$Z(s) = \frac{V_o(s)}{I_s(s)} = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t)$ is a pulse of 1 A height and a duration of 2 seconds then,

$$i_s(t) = u(t) - u(t-2)$$

$$I_s(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

From Eq. (i),

$$V_o(s) = \left[\frac{2}{s+1} - \frac{1}{s+3}\right]\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{2}{s(s+1)} - \frac{1}{s(s+3)} - \frac{2e^{-2s}}{s(s+1)} + \frac{e^{-2s}}{s(s+3)}$$

$$= 2\left[\frac{1}{s} - \frac{1}{s+1}\right] - \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right] - 2e^{-2s}\left[\frac{1}{s} - \frac{1}{s+1}\right] + \frac{e^{-2s}}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

Taking the inverse Laplace transform,

$$v(t) = 2[u(t) - e^{-t}u(t)] - \frac{1}{3}[u(t) - e^{-3t}u(t)] - 2[u(t-2) - e^{-(t-2)}u(t-2)] + \frac{1}{3}[u(t-2) - e^{-3(t-2)}u(t-2)]$$

for $t > 0$

Exercises

11.1 Find $L\{f'(t)\}$ of $f(t) = \left(\dfrac{1-\cos 2t}{t}\right)$

$$\left[s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

11.2 Find Laplace transform of the follwoing function:

$f(t) = t+1 \quad 0 \le t \le 2$

$\quad\;\; = 3 \qquad\quad t > 2$

$$\left[\frac{1}{s}(1-e^{-2s})\right]$$

11.3 For the network shown in Fig 11.125, the switch is closed at $t = 0$. Find the current i1(t) for $t > 0$.

Fig. 11.125

$[i_1(t) = 3 - e^{-25\,t}]$

11.4 Determine the current $i(t)$ in the network of Fig. 11.126, when the switch is closed at $t = 0$. The inductor is initially unenergized.

Fig. 11.126

$[i(t) = 4(1 - e^{-6t})]$

11.5 In the network of Fig. 11.127, after the switch has been in the open position for a long time, it is closed at $t = 0$. Find the voltage across the capacitor.

Fig. 11.127

$[v(t) = 1 + 4\,e^{-10t}]$

11.6 The circuit of Fig. 11.128, has been in the condition shown for a long time. At $t = 0$, switch is closed. Find $v(t)$ for $t > 0$.

Fig. 11.128

$[v(t) = 7.5 + 12.5\,e^{-(4/15)t}]$

11.7 Figure 11.129 shows a circuit which is in the steady-state with the switch open. At $t = 0$, the switch is closed. Determine the current $i\,(t)$. Find its value at $t = 0.114$ μ seconds.

Fig. 11.129

$[i(t) = 0.00857 + 0.01143\,e^{-8.75\times 10^6 t}$, 0.013 A]

11.8 Find $i(t)$ for the network shown in Fig. 11.130.

Fig. 11.130

$[i(t) = 0.125\ e^{-0.308t} + 3.875\ e^{-0.052t}]$

11.9 Determine $v(t)$ in the network of Fig. 11.131 where $i_L(0^-) = 15$ A and $v_c(0^-) = 5$ V.

Fig. 11.131

$[v(t) = 10 - 10e^{-t} + 5e^{-2t}]$

11.10 The network shown in Fig. 11.132 has acquired steady state with the switch at position 1 for $t < 0$. At $t = 0$, the switch is thrown to the position 2. Find $v(t)$ for $t > 0$.

Fig. 11.132

$[v(t) = 4e^{-t} - 2\ e^{-2t}]$

11.11 In the network shown in Fig. 11.133, the switch is closed at $t = 0$. Find current $i_1(t)$ for $t > 0$.

Fig. 11.133

$[i_1(t) = 5 + 5e^{-2t} - 10e^{-3t}]$

11.12 In the network shown in Fig. 11.134, the switch is closed at $t = 0$. Find the current through the 30 Ω resistor.

Fig. 11.134

$[i(t) = 0.1818 - 0.265\ e^{-13.14t} + 0.083\ e^{-41.86t}]$

11.13 The network shown in Fig. 11.135 is in steady state with s_1 closed and s_2 open. At $t = 0$, s_1 is opened and s_2 is closed. Find the current through the capacitor.

Fig. 11.135

$[i(t) = 5 \cos (0.577 \times 10^3\ t)]$

11.14 In the network shown in Fig. 11.136, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

Fig. 11.136

$[i_1(t) = 5\ e^{-0.625t},\ i_2(t) = 1 - e^{-0.625t}]$

11.15 For the network shown in Fig. 11.137, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

Fig. 11.137

$$\left[\begin{array}{l} i_1(t) = 0.101e^{-100.5t} + 10.05e^{-9949.5t} \\ i_2(t) = 5 - 5.05e^{-100.5t} + 0.05e^{-9949.5t} \end{array}\right]$$

11.16 In the network shown in Fig. 11.138, the switch is opened at $t = 0$, the steady state having been established previously. Find $i(t)$ for $t > 0$.

Fig. 11.138

$[i(t) = 1.5124e^{-2.22t} + 3.049e^{-2.5t}]$

11.17 Find the current $i(t)$ in the network of Fig. 11.139, if the switch is closed at $t = 0$. Assume initial conditions to be zero.

Fig. 11.139

$[i(t) = 3 + 0.57e^{-7.14\,t}]$

11.18 In the network shown in Fig. 11.140, find the voltage $v(t)$ for $t > 0$.

Fig. 11.140

$$\left[v(t) = -\frac{6}{5}e^{-t} + \frac{9}{10}e^{-6t} + \frac{3}{10}\cos 2t + \frac{21}{10}\sin 2t\right]$$

11.19 For the network shown in Fig. 11.141, determine $v(t)$ when the input is

(i) an impulse function $[e^{-t}\,u(t)]$

(ii) $i(t) = 4e^{-t}\,u(t)$ $[4t\,e^{-t}\,u(t)]$

Fig. 11.141

11.20 For a unit-ramp input shown in Fig. 11.142, find the response $v_c(t)$ for $t > 0$.

Fig. 11.142

$[v_c(t) = -100\,u(t) + 100e^{-0.01t}\,u(t) + tu(t)]$

Objective-Type Questions

11.1 If the Laplace transform of the voltage across a capacitor of value $\frac{1}{2}$ F is

$$V_c(s) = \frac{1}{s^2 + 1}$$

the value of the current through the capacitor at $t = 0^+$ is

(a) 0 (b) 2 A

(c) $\frac{1}{2}$ A (d) 1 A

11.2 The response of an initially relaxed linear constant parameter network to a unit impulse applied at $t = 0$ is $4\,e^{-2t}\,u(t)$. The response of this network to a unit-step function will be

(a) $2[1 - e^{-2t}]\, u(t)$
(b) $4[e^{-t} - e^{-2t}]\, u(t)$
(c) $\sin 2t$
(d) $(1 - 4\, e^{-4t})\, u(t)$

11.3 The Laplace transform of a unit-ramp function starting at $t = a$ is

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{e^{-as}}{(s+a)^2}$

(c) $\dfrac{e^{-as}}{s^2}$ (d) $\dfrac{a}{s^2}$

11.4 The Laplace transform of $e^{at} \cos \alpha t$ is equal to

(a) $\dfrac{s-\alpha}{(s-\alpha)^2+\alpha^2}$

(b) $\dfrac{s+\alpha}{(s-\alpha)^2+\alpha^2}$

(c) $\dfrac{1}{(s-\alpha)^2}$

(d) none of the above

11.5 The circuit shown in Fig. 11.143 has initial current $i(0^-) = 1$ A through the inductor and an initial voltage $v_c(0^-) = -1$ V across the capacitor. For input $v(t) = u(t)$, the Laplace transform of the current $i(t)$ for $t \geq 0$ is

Fig. 11.143

(a) $\dfrac{s}{s^2+s+1}$ (b) $\dfrac{s+2}{s^2+s+1}$

(c) $\dfrac{s-2}{s^2+s+1}$ (d) $\dfrac{s-2}{s^2+s+1}$

11.6 A square pulse of 3 volts amplitude is applied to an *RC* circuit shown in Fig. 11.144. The capacitor is initially uncharged. The output voltage v_0 at time $t = 2$ seconds is

(a) 3 V (b) –3 V
(c) 4 V (d) –4 V

Fig. 11.144

11.7 A 2 mH inductor with some initial current can be represented as shown in Fig. 11.145. The value of the initial current is

Fig. 11.145

(a) 0.5 A (b) 2 A
(c) 1 A (d) 0

11.8 A current impulse $5\,\delta(t)$ is forced through a capacitor C. The voltage $v_c(t)$ across the capacitor is given by

(a) $5\,t$ (b) $5\,u(t) - C$

(c) $\dfrac{5}{C}t$ (d) $\dfrac{5u(t)}{C}$

11.9 In the circuit shown in Fig. 11.146, it is desired to have a constant direct current $i(t)$ through the ideal inductor L. The nature of the voltage source $v(t)$ must be

Fig. 11.146

(a) a constant voltage
(b) a linearly increasing voltage
(c) an ideal impulse
(d) as exponential increasing voltage

11.10 When a unit-impulse voltage is applied to an inductor of 1 H, the energy supplied by the source is

(a) ∞ (b) 1 J

(c) $\dfrac{1}{2}$ J (d) 0

Answers to Objective-Type Questions

11.1 (c)	11.2 (a)	11.3 (c)	11.4 (a)	11.5 (b)	11.6 (b)
11.7 (a)	11.8 (d)	11.9 (c)	11.10 (c)		

12

Network Functions

12.1 INTRODUCTION

A network function gives the relation between currents or voltages at different parts of the network. It is broadly classified as *driving point* and *transfer function*. It is associated with terminals and ports.

Any network may be represented schematically by a rectangular box. Terminals are needed to connect any network to any other network or for taking some measurements. Two such associated terminals are called *terminal pair* or *port*. If there is only one pair of terminals in the network, it is called a one-port network. If there are two pairs of terminals, it is called a two-port network. The port to which energy source is connected is called the *input port*. The port to which load is connected is known as the *output port*. One such network having only one pair of terminals (1 – 1′) is shown in Fig. 12.1 (a) and is called *one-port network*. Figure 12.1 (b) shows a two-port network with two pairs of terminals. The terminals 1 – 1′ together constitute a port. Similarly, the terminals 2 – 2′ constitute another port.

Fig. 12.1 *(a) One-port network (b) Two-port network*

A voltage and current are assigned to each of the two ports. V_1 and I_1 are assigned to the input port, whereas V_2 and I_2 are assigned to the output port. It is also assumed that currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2 respectively.

12.2 DRIVING-POINT FUNCTIONS

If excitation and response are measured at the same ports, the network function is known as the driving-point function. For a one-port network, only one voltage and current are specified and hence only one network function (and its reciprocal) can be defined.

1. ***Driving-point Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at the same port. It is denoted by $Z(s)$.

$$Z(s) = \frac{V(s)}{I(s)}$$

2. ***Driving-point Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at the same port. It is denoted by $Y(s)$.

$$Y(s) = \frac{I(s)}{V(s)}$$

For a two-port network, the driving-point impedance function and driving-point admittance function at port 1 are

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

Similarly, at port 2,

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$

12.3 TRANSFER FUNCTIONS

The transfer function is used to describe networks which have at least two ports. It relates a voltage or current at one port to the voltage or current at another port. These functions are also defined as the ratio of a response transform to an excitation transform. Thus, there are four possible forms of transfer functions.

1. ***Voltage Transfer Function*** It is defined as the ratio of the voltage transform at one port to the voltage transform at another port. It is denoted by $G(s)$.

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$

$$G_{21}(s) = \frac{V_1(s)}{V_2(s)}$$

2. ***Current Transfer Function*** It is defined as the ratio of the current transform at one port to the current transform at another port. It is denoted by $\alpha(s)$.

$$\alpha_{12}(s) = \frac{I_2(s)}{I_1(s)}$$

$$\alpha_{21}(s) = \frac{I_1(s)}{I_2(s)}$$

3. *Transfer Impedance Function* It is defined as the ratio of the voltage transform at one port to the current transform at another port. It is denoted by $Z(s)$.

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)}$$

$$Z_{21}(s) = \frac{V_1(s)}{I_2(s)}$$

4. *Transfer Admittance Function* It is defined as the ratio of the current transform at one port to the voltage transform at another port. It denoted by Y (s).

$$Y_{12}(s) = \frac{I_2(s)}{V_1(s)}$$

$$Y_{21}(s) = \frac{I_1(s)}{V_2(s)}$$

Example 12. 1 *Determine the driving-point impedance function of a one-port network shown in Fig. 12.2.*

Fig. 12.2

Solution The transformed network is shown in Fig. 12.3.

Fig. 12.3

$$Z(s) = \frac{\frac{1}{Cs}(R+Ls)}{\frac{1}{Cs}+(R+Ls)} = \frac{R+Ls}{LCs^2+RCs+1} = \frac{1}{C}\frac{s+\frac{R}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Example 12.2 *Determine the driving-point impedance of the network shown in Fig. 12.4.*

Fig. 12.4

Solution

$$Z(s) = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{1}{2s}+2s+\frac{1}{2s}} = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{2+4s^2}{2s}} = 2s + \frac{2s+\frac{1}{2s}}{2+4s^2} = \frac{4s+8s^3+2s+\frac{1}{2s}}{2+4s^2} = \frac{16s^4+12s^2+1}{8s^3+4s}$$

Example 12.3 *Determine the driving-point impedance of the network shown in Fig. 12.5.*

Fig. 12.5

Solution

$$Z(s) = \frac{1}{s} + \frac{s\left(\frac{1}{s}+s\right)}{s+\frac{1}{s}+s} = \frac{1}{s} + \frac{(1+s^2)s}{2s^2+1} = \frac{1}{s} + \frac{s+s^3}{2s^2+1} = \frac{s^4+3s^2+1}{2s^3+s}$$

Example 12.4 *Find the driving-point admittance function of the network shown in Fig. 12.6.*

Fig. 12.6

Solution

$$Z(s) = 3s + \frac{1}{5s} + \frac{s\left(\frac{1}{2s}\right)}{s+\frac{1}{2s}} = 3s + \frac{1}{5s} + \frac{s}{2s^2+1} = \frac{30s^4+15s^2+2s^2+1+5s^2}{5s\,(2s^2+1)} = \frac{30s^4+22s^2+1}{5s\,(2s^2+1)}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{5s\,(2s^2+1)}{30s^4+22s^2+1}$$

Example 12.5 *Find the transfer impedance function $Z_{12}(s)$ for the network shown in Fig. 12.7.*

Fig. 12.7

Solution

$$V_2(s) = I_1(s)\frac{R\left(\frac{1}{Cs}\right)}{R+\frac{1}{Cs}}$$

$$\frac{V_2(s)}{I_1(s)} = \frac{R}{RCs+1}$$

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{C\left(s+\frac{1}{RC}\right)}$$

Example 12.6 *Find voltage transfer function of the two-port network shown in Fig. 12.8.*

Fig. 12.8

Solution By voltage division rule,

$$V_2(s) = V_1(s)\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = V_1(s)\frac{1}{RCs+1} = V_1(s)\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

Voltage transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

12.4 ANALYSIS OF LADDER NETWORKS

The network functions of a ladder network can be obtained by a simple method. This method depends upon the relationships that exist between the branch currents and node voltages of the ladder network. Consider the network shown in Fig. 12.9 where all the impedances are connected in series branches and all the admittances are connected in parallel branches.

Fig. 12.9 *Ladder network*

Analysis is done by writing the set of equations. In writing these equations, we begin at the port 2 of the ladder and work towards the port 1.

$$V_b = V_2$$
$$I_b = Y_4\, V_2$$
$$V_a = Z_3\, I_b + V_2 = (Z_3 Y_4 + 1)\, V_2$$
$$I_1 = Y_2\, V_a + I_b = [Y_2\,(Z_3\, Y_4 + 1) + Y_4]\, V_2$$
$$V_1 = Z_1\, I_1 + V_a = [Z_1\,\{Y_2\,(Z_3\, Y_4 + 1) + Y_4\} + (Z_3\, Y_4 + 1)]\, V_2$$

Thus, each succeeding equation takes into account one new impedance or admittance. Except the first two equations, each subsequent equation is obtained by multiplying the equation just preceding it by imittance (either impedance or admittance) that is next down the line and then adding to this product the equation twice preceding it. After writing these equations, we can obtain any network function.

Example 12.7 *For the network shown in Fig. 12.10, determine transfer function* $\frac{V_2}{V_1}$.

Fig. 12.10

Solution The transformed network is shown in Fig. 12.11.

Fig. 12.11

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_a = 1I_b + V_2$$

$$= sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

$$V_1 = 1I_1 + V_a = (s^2 + 2s)V_2 + (s+1)V_2 = (s^2 + 3s + 1)V_2$$

Hence, $$\frac{V_2}{V_1} = \frac{1}{s^2 + 3s + 1}$$

Example 12.8 *For the network shown in Fig. 12.12, determine the voltage transfer function* $\frac{V_2}{V_1}$.

Fig. 12.12

Solution

$$V_b = V_2$$

$$I_b = \frac{V_2}{1} = V_2$$

$$V_a = sI_b + V_2 = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{1} + I_b = (s+1)V_2 + V_2 = (s+2)V_2$$

$$V_1 = sI_1 + V_a = s(s+2)V_2 + (s+1)V_2 = (s^2 + 3s + 1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2 + 3s + 1}$$

Example 12.9 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$ *and* $\frac{V_2}{I_1}$ *for the network shown in Fig. 12.13.*

Fig. 12.13

Solution The transformed network is shown in Fig. 12.14.

Fig. 12.14

$$V_b = V_2$$
$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$
$$V_a = sI_b + V_2 = s(sV_2) + V_2 = (s^2 + 1)V_2$$
$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s^2 + 1)V_2 + sV_2 = (s^3 + 2s)V_2$$
$$V_1 = sI_1 + V_a = s(s^3 + 2s)V_2 + (s^2 + 1)V_2 = (s^4 + 2s^2 + s^2 + 1)V_2 = (s^4 + 3s^2 + 1)V_2$$

Hence,
$$\frac{V_1}{I_1} = \frac{s^4 + 3s^2 + 1}{s^3 + 2s}$$
$$\frac{V_2}{V_1} = \frac{1}{s^4 + 3s^2 + 1}$$
$$\frac{V_2}{V_1} = \frac{1}{s^3 + 2s}$$

Example 12.10 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$, *and* $\frac{V_2}{I_1}$ *for the network in Fig. 12.15.*

Fig. 12.15

Solution The transformed network is shown in Fig. 12.16.

Fig. 12.16

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{2s}} = 2sV_2$$

$$V_a = 3sI_b + V_2 = 3s(2sV_2) + V_2 = (6s^2+1)V_2$$

$$I_1 = \frac{V_a}{\frac{s}{2}} + I_b = \frac{2}{s}(6s^2+1)V_2 + 2sV_2 = \left(\frac{14s^2+2}{s}\right)V_2$$

$$V_1 = \frac{4}{s}I_1 + V_a = \frac{4}{s}\left(\frac{14s^2+2}{s}\right)V_2 + (6s^2+1)V_2 = \left(\frac{6s^4+57s^2+8}{s^2}\right)V_2$$

Hence,

$$\frac{V_1}{I_1} = \frac{6s^4+57s^2+8}{14s^3+2s}$$

$$\frac{V_2}{V_1} = \frac{s^2}{6s^4+57s^2+8}$$

$$\frac{V_2}{I_1} = \frac{s}{14s^2+2}$$

Example 12.11 *For the ladder network of Fig. 12.17, find the driving point-impedance at the 1 – 1′ terminal with 2 – 2′ open.*

Fig. 12.17

Solution The transformed network is shown in Fig. 12.18.

Fig. 12.18

$$V_c = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_b = (s+1)\,I_b + V_2 = (s+1)sV_2 + V_2 = (s^2+s+1)V_2$$

$$I_a = \frac{V_b}{\frac{1}{s}} + I_b = sV_b + I_b = s(s^2+s+1)V_2 + sV_2 = (s^3+s^2+2s)V_2$$

$$V_a = (s+1)\,I_a + V_b = (s+1)(s^3+s^2+2s)V_2 + (s^2+s+1)V_2 = (s^4+2s^3+4s^2+3s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(s^4+2s^3+4s^2+3s+1)V_2 + (s^3+s^2+2s)V_2$$

$$= (s^5+2s^4+5s^3+4s^2+3s)V_2$$

$$V_1 = (s+1)\,I_1 + V_a = (s+1)(s^5+2s^4+5s^3+4s^2+3s)V_2 + (s^4+2s^3+4s^2+3s+1)V_2$$

$$= (s^6+3s^5+8s^4+11s^3+11s^2+6s+1)V_2$$

Hence, $$Z_{11} = \frac{V_1}{I_1} = \frac{s^6+3s^5+8s^4+11s^3+11s^2+6s+1}{s^5+2s^4+5s^3+4s^2+3s}$$

Example 12.12 *Determine the voltage transfer function* $\frac{V_2}{V_1}$ *for the network shown in Fig. 12.19.*

Fig. 12.19

Solution The transformed network is shown in Fig. 12.20.

Fig. 12.20

$$V_c = V_b = V_2$$

$$I_a = I_b + I_c = \frac{V_2}{\frac{1}{s}} + \frac{V_2}{1} = s\,V_2 + V_2 = (s+1)V_2$$

$$V_a = 2s\,I_a + V_2 = 2s(s+1)V_2 + V_2$$

$$= (2s^2+2s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(2s^2+2s+1)V_2 + (s+1)V_2 = (2s^3+2s^2+2s+1)V_2$$

$$V_1 = 1I_1 + V_a = (2s^3+2s^2+2s+1)V_2 + (2s^2+2s+1)V_2 = (2s^3+4s^2+4s+2)V_2$$

Hence, $$\frac{V_2}{V_1} = \frac{1}{2s^3+4s^2+4s+2}$$

Example 12.13 *For the network shown in Fig. 12.21, determine the transfer function* $\frac{I_2}{V_1}$.

Fig. 12.21

Solution The transformed network is shown in Fig. 12.22.

Fig. 12.22

$$V_b = V_a = V_2$$

$$I_1 = I_a + I_b = \frac{V_2}{\frac{1}{2s}} + \frac{V_2}{\frac{1}{6}} = 2sV_2 + 6V_2 = (2s+6)V_2$$

$$V_1 = \left(\frac{\frac{3}{2}\times\frac{3}{2s}}{\frac{3}{2}+\frac{3}{2s}}+1\right)I_1 + V_2 = \left(\frac{9}{6s+6}+1\right)I_1 + V_2 = \frac{(6s+15)}{6s+6}(2s+6)V_2 + V_2 = \left[\frac{(2s+5)(s+3)}{(s+1)}+1\right]V_2$$

$$= \left[\frac{(2s+5)(s+3)+(s+1)}{s+1}\right]V_2 = \left(\frac{2s^2+6s+5s+15+s+1}{s+1}\right)V_2 = \left(\frac{2s^2+12s+16}{s+1}\right)V_2$$

$$= \frac{2(s^2+6s+8)}{s+1}V_2 = \frac{2(s+4)(s+2)}{s+1}V_2$$

Also, $I_2 = -I_b = -6V_2$

$$\frac{I_2}{V_1} = -\frac{3(s+1)}{(s+4)(s+2)}$$

Example 12.14 *For the network shown in Fig. 12.23, compute* $\alpha_{12}(s) = \frac{I_2}{I_1}$ *and* $Z_{12}(s) = \frac{V_2}{I_1}$.

Fig. 12.23

Solution The transformed network is shown in Fig. 12.24.

Fig. 12.24

$$V_a = V_2$$

$$I_2 = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_1 = 1I_2 + V_a = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_1}{\frac{1}{s}} + I_2 = sV_1 + I_2 = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

Hence, $\alpha_{12}(s) = \dfrac{I_2}{I_1} = \dfrac{1}{s+2}$

and $Z_{12}(s) = \dfrac{V_2}{I_1} = \dfrac{1}{s^2 + 2s}$

Example 12.15 *Determine the voltage ratio* $\dfrac{V_2}{V_1}$, *current ratio* $\dfrac{I_2}{I_1}$, *transfer impedance* $\dfrac{V_2}{I_1}$ *and driving- point impedance* $\dfrac{V_1}{I_1}$ *for the network shown in Fig. 12.25.*

Fig. 12.25

Solution The transformed network is shown in Fig. 12.26.

Fig. 12.26

$$V_c = V_b = V_2$$

$$I_2 = \frac{V_2}{1} = V_2$$

$$I_a = I_b + I_2 = \frac{V_2}{2 + \frac{2}{s}} + \frac{V_2}{1} = \frac{s}{2s+2}V_2 + V_2 = \left(\frac{3s+2}{2s+2}\right)V_2$$

$$V_a = 3s\,I_a + V_2 = \frac{3s(3s+2)}{2s+2}V_2 + V_2 = \left(\frac{9s^2 + 8s + 2}{2s+2}\right)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = \frac{s(9s^2+8s+2)}{2s+2}V_2 + \left(\frac{3s+2}{2s+2}\right)V_2 = \left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2$$

$$V_1 = \left(\frac{3\times\frac{3}{s}}{3+\frac{3}{s}}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)\left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2 + \left(\frac{9s^2+8s+2}{2s+2}\right)V_2$$

$$= \left[\frac{27s^3+24s^2+15s+6+9s^3+8s^2+2s+9s^2+8s+2}{(s+1)(2s+2)}\right]V_2 = \frac{(36s^3+41s^2+25s+8)}{(s+1)(2s+2)}V_2$$

$$= \left(\frac{36s^3+41s^2+25s+8}{2s^2+4s+2}\right)V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{2s^2+4s+2}{36s^3+41s^2+25s+8}$$

$$\frac{I_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_1}{I_1} = \frac{36s^3+41s^2+25s+8}{(s+1)(9s^3+8s^2+5s+2)} = \frac{36s^3+41s^2+25s+8}{9s^4+17s^3+13s^2+7s+2}$$

Example 12.16 *For the resistive two-port network of Fig. 12.27, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

Fig. 12.27

Solution The network is redrawn as shown in Fig. 12.28.

Fig. 12.28

$$I_2 = -\frac{V_2}{1} = -V_2$$

$$V_b = -3I_2 = 3V_2$$

$$I_b = \frac{V_b}{1} + \frac{V_b}{3} = \frac{4}{3}V_b = 4V_2$$

$$V_a = 2I_b + V_b = 8V_2 + 3V_2 = 11V_2$$

$$I_a = \frac{V_a}{1} + I_b = 11V_2 + 4V_2 = 15V_2$$

$$V_1 = 2I_a + V_a = 30V_2 + 11V_2 = 41V_2$$

$$I_1 = \frac{V_1}{1} + I_a = 41V_2 + 15V_2 = 56V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{1}{41}$$

$$\frac{V_2}{I_1} = \frac{1}{56}\ \Omega$$

$$\frac{I_2}{V_1} = -\frac{1}{41}\ \mho$$

$$\frac{I_2}{I_1} = -\frac{1}{56}$$

Example 12.17 *Find the network function* $\frac{V_2}{V_1}$ *for the network shown in Fig. 12.29.*

Fig. 12.29

Solution The network is redrawn as shown in Fig. 12.30.

Fig. 12.30

From Fig. 12.30,

$$V_2 = 1\,(3\,I_a) = 3\,I_a$$

Applying KVL to the outermost loop,

$$V_1 - 1\,(I_a) - 1\,(3\,I_a) - 2\,V_1 - 1\,(3\,I_a) = 0$$

$$V_1 = -7\,I_a$$

Hence,

$$\frac{V_2}{V_1} = -\frac{3}{7}$$

Example 12.18 *Find the network function* $\frac{I_2}{I_1}$ *for the network shown in Fig. 12.31.*

Fig. 12.31

Solution The network is redrawn as shown in Fig. 12.32.

Fig. 12.32

From Fig. 12.32,

$$I = I_1 + I_a + I_2 + \frac{I_1}{2}$$

$$= \frac{3}{2} I_1 + I_a + I_2 \qquad \text{...(i)}$$

Applying KVL to the loop *abcda*,

$$-1\,I - 1\,I_a - 2\,I_a = 0$$

$$-I - 3\,I_a = 0$$

$$I + 3\,I_a = 0 \qquad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$\frac{3}{2}I_1 + I_a + I_2 + 3I_a = 0$$

$$\frac{3}{2}I_1 + I_2 + 4I_a = 0 \qquad \text{...(iii)}$$

Applying KVL to the loop *dcefd,*

$$1I_a - 1I_2 - 2\left(I_2 + \frac{I_1}{2}\right) = 0$$

$$I_a - 3I_2 - I_1 = 0$$

$$I_a = 3I_2 + I_1 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (iii),

$$\frac{3}{2}I_1 + I_2 + 4(3I_2 + I_1) = 0$$

$$\frac{3}{2}I_1 + I_2 + 12I_2 + 4I_1 = 0$$

$$\frac{11}{2}I_1 + 13I_2 = 0$$

$$13I_2 = -\frac{11}{2}I_1$$

Hence,

$$\frac{I_2}{I_1} = -\frac{11}{26}$$

12.5 ANALYSIS OF NON-LADDER NETWORKS

The above method is applicable for ladder networks. There are other network configurations to which the technique described is not applicable. Figure 12.33 shows one such network.

Fig. 12.33 *Non-ladder network*

For such a type of network, it is necessary to express the network functions as a quotient of determinants, formulated on KVL and KCL basis.

Example 12.19 *For the resistive bridged T network shown in Fig. 12.34, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

Fig. 12.34

Solution Applying KVL to Mesh 1,

$$V_1 = 1.5I_1 + 0.5I_2 - I_3 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = 0.5I_1 + 2.5I_2 + I_3 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -I_1 + I_2 + 4I_3 \quad \text{...(iii)}$$

Writing these equations in matrix form,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad \text{...(iv)}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} V_1 & 0.5 & -1 \\ 0 & 2.5 & 1 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{V_1(10-1)}{9} = V_1 \quad \text{...(iv)}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 1.5 & V_1 & -1 \\ 0.5 & 0 & 1 \\ -1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{-V_1(2+1)}{9} = -\frac{1}{3}V_1 \quad \text{...(v)}$$

From Fig. 12.34, $V_2 = -1\,(I_2) = -I_2$

From Eq. (v), $V_1 = -3\,I_2$

From Eqs. (iv) and (v), $I_2 = -\frac{1}{3}V_1 = -\frac{1}{3}I_1$

$I_1 = -3I_2$

Hence, $\frac{I_2}{V_1} = -\frac{1}{3}\mho$

$$\frac{I_2}{I_1} = \frac{-\frac{1}{3}V_1}{V_1} = -\frac{1}{3}$$

$$\frac{V_2}{V_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}$$

$$\frac{V_2}{I_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}\,\Omega$$

Example 12.20 *For the network of Fig. 12.35, find Z_{11}, Z_{12} and G_{12}.*

Fig. 12.35

Solution The network can be redrawn as shown in Fig. 12.36. Since the network consists of two identical impedances connected in parallel, the current in I_1 divides equally in each branch.

$$V_1 = (Z_a + Z_b)\frac{I_1}{2}$$

$$Z_{11} = \frac{V_1}{I_1} = \frac{Z_a + Z_b}{2}$$

$$V_2 = Z_b\frac{I_1}{2} - Z_a\left(\frac{I_1}{2}\right) = (Z_b - Z_a)\frac{I_1}{2}$$

$$Z_{12} = \frac{V_2}{I_1} = \frac{Z_b - Z_a}{2}$$

Fig. 12.36

By voltage-division rule,

$$V_2 = \frac{Z_b}{Z_a + Z_b}V_1 - \frac{Z_a}{Z_a + Z_b}V_1 = \frac{Z_b - Z_a}{Z_a + Z_b}V_1$$

$$G_{12} = \frac{V_2}{V_1} = \frac{Z_b - Z_a}{Z_a + Z_b}$$

Example 12.21 *For the network shown in Fig. 12.37, determine Z_{11} (s), G_{12} (s) and Z_{12} (s).*

Fig. 12.37

Solution The transformed network is shown in Fig. 12.38.

Fig. 12.38

Applying KVL to Mesh 1,

$$V_1 = \left(1 + \frac{1}{s}\right) I_1 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{s} I_1 + I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + \left(2 + \frac{1}{s}\right) I_3 = 0$$

$$I_3 = \left(\frac{s}{2s+1}\right) I_1 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eqs. (i) and (ii),

$$V_1 = \left(1 + \frac{1}{s}\right) I_1 - \left(\frac{s}{2s+1}\right) I_1 = \left(\frac{s+1}{s} - \frac{s}{2s+1}\right) I_1 = \left[\frac{s^2 + 3s + 1}{s(2s+1)}\right] I_1$$

$$V_2 = \frac{1}{s} I_1 + \frac{s}{2s+1} I_1 = \left[\frac{s^2 + 2s + 1}{s(2s+1)}\right] I_1$$

Hence,

$$Z_{11}(s) = \frac{V_1}{I_1} = \frac{s^2 + 3s + 1}{s(2s+1)}$$

$$Z_{12}(s) = \frac{V_2}{I_1} = \frac{s^2 + 2s + 1}{s(2s+1)}$$

$$G_{12}(s) = \frac{V_2}{V_1} = \frac{s^2 + 2s + 1}{s^2 + 3s + 1}$$

Example 12.22 *For the network shown in Fig.12.39, find the driving-point admittance Y_{11} and transfer admittance Y_{12}.*

Fig. 12.39

Solution The transformed network is shown in Fig. 12.40.

Fig. 12.40

Applying KVL to Mesh 1,

$$V_1 = \left(\frac{1}{s}+1\right)I_1 + I_2 - \frac{1}{s}I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = I_1 + \left(2+\frac{1}{s}\right)I_2 + \frac{1}{s}I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -\frac{1}{s}I_1 + \frac{1}{s}I_2 + \left(\frac{2}{s}+1\right)I_3 \qquad \text{...(iii)}$$

Writing these equations in matrix from,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta = \begin{vmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = \left(\frac{1}{s}+1\right)\left\{\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right\}-1\left[(1)\left(\frac{2}{s}+1\right)+\frac{1}{s^2}\right]-\frac{1}{s}\left[(1)\left(\frac{1}{s}\right)+\left(\frac{1}{s}\right)\left(2+\frac{1}{s}\right)\right]$$

$$= \frac{s^2+5s+2}{s^2}$$

$$\Delta_1 = \begin{vmatrix} V_1 & 1 & -\frac{1}{s} \\ 0 & 2+\frac{1}{s} & \frac{1}{s} \\ 0 & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = V_1\left\{\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right\} = V_1\left(\frac{2s^2+5s+1}{s^2}\right)$$

$$I_1 = V_1\left(\frac{2s^2+5s+1}{s^2+5s+2}\right)$$

Hence, $$Y_{11} = \frac{I_1}{V_1} = \frac{2s^2+5s+1}{s^2+5s+2}$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{s}+1 & V_1 & -\frac{1}{s} \\ 1 & 0 & \frac{1}{s} \\ -\frac{1}{s} & 0 & \frac{2}{s}+1 \end{vmatrix} = -V_1\left(\frac{2}{s}+1+\frac{1}{s^2}\right) = -V_1\left(\frac{s^2+2s+1}{s^2}\right)$$

$$I_2 = -V_1\left(\frac{s^2+2s+1}{s^2+5s+2}\right)$$

Hence, $$Y_{12} = \frac{I_2}{V_1} = -\frac{s^2+2s+1}{s^2+5s+2}$$

12.6 POLES AND ZEROS OF NETWORK FUNCTIONS

The network function $F(s)$ can be written as ratio of two polynomials.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}{b_m s^m + b_{m-1}\, s^{m-1} + \ldots + b_1\, s + b_0}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are the coefficients of the polynomials $N(s)$ and $D(s)$. These are real and positive for networks of passive elements. Let $N(s) = 0$ have n roots as $z_1, z_2, \ldots\ldots, z_n$ and $D(s) = 0$ have m roots as $p_1, p_2, \ldots\ldots, p_m$. Then $F(s)$ can be written as

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

where $\frac{a_n}{b_m}$ is a constant called *scale factor* and $z_1, z_2, \ldots, z_n, p_1, p_2, \ldots, p_m$ are complex frequencies. When the variable s has the values $z_1, z_2, \ldots, z_n$, the network function becomes zero; such complex frequencies are known as the zeros of the network function. When the variable s has values $p_1, p_2, \ldots, p_m$, the network function becomes infinite; such complex frequencies are known as the poles of the network function. A network function is completely specified by its poles, zeros and the scale factor.

If the poles or zeros are not repeated, then the function is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then the function is said to be having multiple poles or multiple zeros. When $n > m$, then $(n - m)$ zeros are at $s = \infty$, and for $m > n$, $(m - n)$ poles are at $s = \infty$.

The total number of zeros is equal to the total number of poles. For any network function, poles and zeros at zero and infinity are taken into account in addition to finite poles and zeros.

Poles and zeros are critical frequencies. The network function becomes infinite at poles, while the network function becomes zero at zeros. The network function has a finite, non-zero value at other frequencies.

Poles and zeros provide a representation of a network function as shown in Fig. 12.41. The zeros are shown by circles and the poles by crosses. This diagram is referred to as pole-zero plot.

Fig. 12.41 *Pole-zero plot*

12.7 RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR DRIVING-POINT FUNCTIONS [COMMON FACTORS IN N(*s*) AND D(*s*) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real and positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of all poles and zeros, must be negative or zero, i.e., the poles and zeros must lie in left half of s plane.

(4) If the real part of pole or zero is zero, then that pole or zero must be simple.

(5) The polynomials $N(s)$ and $D(s)$ may not have missing terms between those of highest and lowest degree, unless all even or all odd terms are missing.

(6) The degree of $N(s)$ and $D(s)$ may differ by either zero or one only. This condition prevents multiple poles and zeros at $s = \infty$.

(7) The terms of lowest degree in $N(s)$ and $D(s)$ may differ in degree by one at most. This condition prevents multiple poles and zeros at $s = 0$.

12.8 RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR TRANSFER FUNCTIONS [COMMON FACTORS IN N(S) AND D(S) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real, and those for $D(s)$ must be positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of poles must be negative or zero. If the real part is zero, then that pole must be simple.

(4) The polynomial $D(s)$ may not have any missing terms between that of highest and lowest degree, unless all even or all odd terms are missing.

(5) The polynomial $N(s)$ may have terms missing between the terms of lowest and highest degree, and some of the coefficients may be negative.

(6) The degree of $N(s)$ may be as small as zero, independent of the degree of $D(s)$.

(7) For voltage and current transfer functions, the maximum degree of $N(s)$ is the degree of $D(s)$.

(8) For transfer impedance and admittance functions, the maximum degree of $N(s)$ is the degree of $D(s)$ plus one.

Example 12.23 *Test whether the following represent driving-point immittance functions.*

(a) $\dfrac{5s^4+3s^2-2s+1}{s^3+6s+20}$ (b) $\dfrac{s^3+s^2+5s+2}{s^4+6s^3+9s^2}$ (c) $\dfrac{s^2+3s+2}{s^2+6s+2}$

Solution

(a) The numerator and denominator polynomials have a missing term between those of highest and lowest degree and one of the coefficient is negative in numerator polynomial. Hence, the function does not represent a driving-point immittance function.

(b) The term of lowest degree in numerator and denominator polynomials differ in degree by two. Hence, the function does not represent a driving-point immittance function.

(c) The function satisfies all the necessary conditions. Hence, it represents a driving-point immittance function.

Example 12.24 *Test whether the following represent transfer functions.*

(a) $G_{21}=\dfrac{3s+2}{5s^3+4s^2+1}$ (b) $\alpha_{12}=\dfrac{2s^2+5s+1}{s+7}$ (c) $Z_{21}=\dfrac{1}{s^3+2s}$

Solution

(a) The polynomial $D(s)$ has a missing term between that of highest and lowest degree. Hence, the function does not represent a transfer function.

(b) The degree of $N(s)$ is greater than $D(s)$. Hence the function does not represent a transfer function.

(c) The function satisfies all the necessary conditions. Hence, it represents a transfer function.

Example 12.25 *Obtain the pole-zero plot of the following functions.*

(a) $F(s)=\dfrac{s(s+2)}{(s+1)(s+3)}$ (b) $F(s)=\dfrac{s(s+1)}{(s+2)^2(s+3)}$

(c) $F(s)=\dfrac{s(s+2)}{(s+1+j1)(s+1-j1)}$ (d) $F(s)=\dfrac{(s+1)^2(s+5)}{(s+2)(s+3+j2)(s+3-j2)}$

(e) $F(s)=\dfrac{s^2+4}{(s+2)(s^2+9)}$

Solution

(a) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1$ and $s = -3$.
The pole-zero plot is shown in Fig. 12.42.

Fig. 12.42

(b) The function $F(s)$ has zeros at $s = 0$ and $s = -1$ and poles at $s = -2, -2$ and $s = -3$.
The pole-zero plot is shown in Fig. 12.43.

Fig. 12.43

(c) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1 - j1$ and $s = -1 + j1$.
The pole-zero plot is shown in Fig. 12.44.

Fig. 12.44

(d) The function $F(s)$ has zeros at $s = -1, -1$ and $s = -5$ and poles at $s = -2$, $s = -3 + j2$ and $s = -3 - j2$. The pole-zero plot is shown in Fig. 12.45.

Fig. 12.45

(e) The function $F(s)$ has zeros at $s = j2$ and $s = -j2$ and poles at $s = -2$, $s = j3$ and $s = -j3$. The pole-zero plot is shown in Fig. 12.46.

Fig. 12.46

Example 12.26 *Find poles and zeros of the impedance of the network shown in Fig. 12.47 and plot them on the s-plane.*

Fig. 12.47

Solution The transformed network is shown in Fig. 12.48.

$$Z(s) = \frac{1}{s} + \frac{\frac{s}{2}\times 2}{\frac{s}{2}+2} = \frac{1}{s} + \frac{2s}{s+4} = \frac{2s^2+s+4}{s(s+4)} = \frac{2(s^2+0.5s+2)}{s(s+4)}$$

$$= \frac{2(s+0.25+j1.4)(s+0.25-j1.4)}{s(s+4)}$$

Fig. 12.48

The function $Z(s)$ has zeros at $s = -0.25 + j1.4$ and $s = -0.25 - j1.4$ and poles at $s = 0$ and $s = -4$ as shown in Fig. 12.49.

Fig. 12.49

Example 12.27 *Determine the poles and zeros of the impedance function Z (s) in the network shown in Fig. 12.50.*

Fig. 12.50

Solution The transformed network is shown in Fig. 12.51.

Fig. 12.51

$$Z(s)=\frac{1}{2}+\frac{\frac{1}{4s}\times\frac{1}{6}}{\frac{1}{4s}+\frac{1}{6}}=\frac{1}{2}+\frac{1}{4s+6}=\frac{4s+8}{2(4s+6)}=\frac{s+2}{2s+3}=\frac{0.5(s+2)}{s+1.5}$$

The function $Z(s)$ has zero at $s = -2$ and pole at $s = -1.5$.

Example 12.28 *Determine Z(s) in the network shown in Fig. 12.52. Find poles and zeros of Z(s) and plot them on s-plane.*

Fig. 12.52

Solution The transformed network is shown in Fig. 12.53.

Fig. 12.53

$$Z(s) = s + \frac{\frac{20}{s} \times 4}{\frac{20}{s} + 4} = s + \frac{80}{4s+20} = s + \frac{20}{s+5} = \frac{s(s+5)+20}{s+5} = \frac{s^2+5s+20}{s+5}$$

$$= \frac{(s+2.5+j3.71)(s+2.5-j3.71)}{s+5}$$

The function $Z(s)$ has zeros at $s = -2.5 + j3.71$ and $s = -2.5 - j3.71$ and pole at $s = -5$.
The pole-zero diagram is shown in Fig. 12.54.

Fig. 12.54

Example 12.29 *For the network shown in Fig. 12.55, plot poles and zeros of function* $\frac{I_0}{I_i}$.

Fig. 12.55

Solution The transformed network is shown in Fig. 12.56.
By current-division rule,

$$I_0 = I_i \left(\frac{4+2s}{4+2s+\frac{2}{s}} \right)$$

$$\frac{I_0}{I_i} = \frac{s(4+2s)}{4s+2s^2+2} = \frac{s(s+2)}{s^2+2s+1} = \frac{s(s+2)}{(s+1)(s+1)}$$

Fig. 12.56

The function has zeros at $s = 0$ and $s = -2$ and double poles at $s = -1$.
The pole-zero diagram is shown in Fig. 12.57.

Fig. 12.57

Example 12.30 *Draw the pole-zero diagram of* $\frac{I_2}{I_1}$ *for the network shown in Fig. 12.58.*

Fig. 12.58

Solution The transformed network is shown in Fig. 12.59.

Fig. 12.59

By current-division rule,

$$I_2 = I_1 \frac{\dfrac{1}{250\times10^{-6}s}}{\dfrac{1}{250\times10^{-6}s}+10s+200}$$

$$\frac{I_2}{I_1} = \frac{400}{s^2+20s+400} = \frac{400}{(s+10-j17.32)(s+10+j17.32)}$$

The function has no zero and poles at $s = -10 + j17.32$ and $s = -10 - j17.32$.
The pole-zero diagram is shown in Fig. 12.60.

Fig. 12.60

Example 12.31 *For the network shown in Fig. 12.61, draw pole-zero plot of* $\frac{V_c}{V_1}$.

Fig. 12.61

Solution The transformed network is shown in Fig. 12.62.

Fig. 12.62

Applying KVL to the left loop,

$$V_1 - 1I_1 + 2V_c = 0$$

$$I_1 = V_1 + 2V_c$$

Applying KCL at Node C,

$$5I_1 + \frac{V_c}{s} + \frac{V_c}{\frac{2}{s}} = 0$$

$$5(V_1 + 2V_c) + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$5V_1 + 10V_c + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$V_c\left(\frac{20s + 2 + s^2}{2s}\right) = -5V_1$$

$$\frac{V_c}{V_1} = -\frac{10s}{s^2 + 20s + 2} = -\frac{10s}{(s+0.1)(s+19.9)}$$

The function has zero at $s = 0$ and poles at $s = -0.1$ and $s = -19.9$.
The pole-zero diagram is shown in Fig. 12.63.

Fig. 12.63

Example 12.32 *Find the driving point admittance function and draw pole-zero plot for the network shown in Fig. 12.64.*

Fig. 12.64

Solution Applying KVL to the left loop,

$$V_1 - 2\,I_1 + 0.1\,V_2 = 0$$

$$I_1 = \frac{V_1 + 0.1\,V_2}{2} \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$10\,I_1 + \frac{V_2}{0.5s} + \frac{V_2}{1} = 0$$

$$10\,I_1 + \frac{2}{s}V_2 + V_2 = 0$$

$$10\,I_1 + \left(\frac{2}{s} + 1\right)V_2 = 0$$

$$10\,I_1 + \left(\frac{2+s}{s}\right)V_2 = 0$$

$$\left(\frac{s+2}{s}\right)V_2 = -10\,I_1$$

$$V_2 = -\frac{10s}{s+2}\,I_1 \qquad \text{...(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{V_1 + 0.1\left(-\dfrac{10s}{s+2}\right)I_1}{2} = 0.5V_1 + 0.05\left(-\frac{10s}{s+2}\right)I_1$$

$$I_1\left(1 + \frac{0.5s}{s+2}\right) = 0.5V_1$$

Hence,

$$Y_{11} = \frac{I_1}{V_1} = \frac{0.5}{1 + \dfrac{0.5s}{s+2}} = \frac{0.5\,(s+2)}{s + 0.5s + 2} = \frac{0.5s+1}{1.5s+2}$$

The function has zero at $s = -2$ and pole at $s = -1.33$. The pole-zero diagram is shown in Fig 12.65.

Fig. 12.65

Example 12.33 *For the network shown in Fig. 12.66, determine* $\frac{V_2}{I_g}$. *Plot the pole-zero diagram of* $\frac{V_2}{I_g}$.

Fig. 12.66

Solution The transformed network is shown in Fig. 12.67.

Fig. 12.67

$$V_c = V_b = V_2$$

$$I_b = \frac{V_b}{\frac{1}{s}} + \frac{V_c}{1} = sV_2 + V_2 = (s+1)V_2$$

$$V_a = sI_b + V_b = s(s+1)V_2 + V_2 = (s^2 + s + 1)V_2$$

$$I_g = \frac{V_a}{1} + \frac{V_a}{\frac{1}{s}} + I_b = (s^2 + s + 1)V_2 + s(s^2 + s + 1)V_2 + (s+1)V_2 = (s^3 + 2s^2 + 3s + 2)V_2$$

Hence, $$\frac{V_2}{I_g} = \frac{1}{s^3 + 2s^2 + 3s + 2}$$

The function has no zeros. It has poles at $s = -1$, $s = -0.5 + j1.32$ and $s = -0.5 - j1.32$,
The pole-zero diagram is shown in Fig. 12.68.

Fig. 12.68

Example 12.34 *For the transfer function* $H(s) = \frac{V_0}{V_i} = \frac{10}{s^2 + 3s + 10}$, *realise the function using the network shown in Fig. 12.69. Find L and C when R = 5 Ω.*

Fig. 12.69

Solution The transformed network is shown in Fig. 12.70.

Fig. 12.70

Simplifying the network as shown in Fig. 12.71,

Fig. 12.71

$$Z(s) = \frac{R \times \frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{R}{RCs + 1}$$

$$V_0 = V_i \frac{\dfrac{R}{RCs+1}}{Ls + \dfrac{R}{RCs+1}}$$

$$\frac{V_0}{V_i} = \frac{R}{RLC\,s^2 + Ls + R} = \frac{\dfrac{1}{LC}}{s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}} \qquad \text{...(i)}$$

But
$$\frac{V_0}{V_i} = \frac{10}{s^2 + 3s + 10} \qquad \text{...(ii)}$$

and
$$R = 5\ \Omega$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{RC} = 3$$

$$\frac{1}{LC} = 10$$

Solving the above equations,

$$L = 1.5\ \text{H}$$

$$C = \frac{1}{15}\ \text{F}$$

Example 12.35 *Obtain the impedance function Z(s) for which pole-zero diagram is shown in Fig. 12.72.*

Fig. 12.72

Solution The function $Z(s)$ has poles at $s = -1$ and $s = -3$ and zeros at $s = 0$ and $s = -2$.

$$Z(s) = H\frac{s(s+2)}{(s+1)(s+3)} = H\frac{s^2\left(1+\dfrac{2}{s}\right)}{s^2\left(1+\dfrac{1}{s}\right)\left(1+\dfrac{3}{s}\right)}$$

For $s = \infty$,

$$Z(\infty) = H\frac{1}{(1)(1)} = H$$

When
$$Z(\infty) = 1,$$
$$H = 1$$

$$Z(s) = \frac{s\,(s+2)}{(s+1)(s+3)}$$

Example 12.36 *Obtain the admittance function Y(s) for which the pole-zero diagram is shown in Fig. 12.73.*

Fig. 12.73

Solution The function $Y(s)$ has poles at $s = -1 + j1$ and $s = -1 - j1$ and zeros at $s = 0$ and $s = -2$.

$$Y(s) = H\frac{s(s+2)}{(s+1+j1)(s+1-j1)} = H\frac{s(s+2)}{(s+1)^2-(j1)^2} = H\frac{s(s+2)}{s^2+2s+2} = H\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

For $s = \infty$,

$$Y(\infty) = H\frac{(1)}{(1)} = H$$

When $Y(\infty) = 1$,

$$H = 1$$

$$Y(s) = \frac{s(s+2)}{s^2+2s+2}$$

Example 12.37 *A network and its pole-zero configuration are shown in Fig. 12.74. Determine the values of R, L and C if Z (j0) = 1.*

Fig. 12.74

Solution

$$Z(s) = \frac{(Ls+R)\frac{1}{Cs}}{(Ls+R)+\frac{1}{Cs}} = \frac{Ls+R}{LCs^2+RCs+1} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}} \quad \text{...(i)}$$

From the pole-zero diagram, zero is at $s = -3$ and poles are at $s = -1.5 + j\frac{\sqrt{111}}{2}$ and $s = -1.5 - j\frac{\sqrt{111}}{2}$

$$Z(s) = H\frac{s+3}{\left(s+1.5+j\frac{\sqrt{111}}{2}\right)\left(s+1.5-j\frac{\sqrt{111}}{2}\right)}$$

$$Z(s) = H\frac{s+3}{(s+1.5)^2-\left(j\frac{\sqrt{111}}{2}\right)^2} = H\frac{s+3}{s^2+3s+30}$$

When $Z(j0) = 1,$

$$1 = H\left(\frac{3}{30}\right)$$

$$H = 10$$

$$Z(s) = \frac{10(s+3)}{s^2+3s+30} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 3$$

$$\frac{1}{C} = 10$$

$$\frac{1}{LC} = 30$$

Solving the above equations,

$$C = \frac{1}{10}\text{F}$$

$$L = \frac{1}{3}\text{H}$$

$$R = 1\,\Omega$$

Example 12.38 *A network is shown in Fig. 12.75. The poles and zeros of the driving-point function Z(s) of this network are at the following places:*

Poles at $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

Zero at –1

If Z (j0) = 1, determine the values of R, L and C.

Fig. 12.75

Solution

$$Z(s)=\frac{(Ls+R)\dfrac{1}{Cs}}{Ls+R+\dfrac{1}{Cs}}=\frac{Ls+R}{LCs^2+RCs+1}=\frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \quad \text{...(i)}$$

The poles are at $-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$ and zero is at –1.

$$Z(s)=H\frac{s+1}{\left(s+\dfrac{1}{2}+j\dfrac{\sqrt{3}}{2}\right)\left(s+\dfrac{1}{2}-j\dfrac{\sqrt{3}}{2}\right)}=H\frac{s+1}{\left(s+\dfrac{1}{2}\right)^2-\left(j\dfrac{\sqrt{3}}{2}\right)^2}=H\frac{s+1}{s^2+s+1}$$

When $Z(j0)=1,$

$$1=H\frac{(1)}{(1)}$$

$$H=1$$

$$Z(s)=\frac{s+1}{s^2+s+1} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$C=1$$

$$\frac{R}{L}=1$$

$$\frac{1}{LC}=1$$

Solving the above equations,

$$C=1\text{ F}$$

$$L=1\text{ H}$$

$$R=1\ \Omega$$

Example 12.39 *The pole-zero diagram of the driving-point impedance function of the network of Fig. 12.76 is shown below. At dc, the input impedance is resistive and equal to 2 W. Determine the values of R, L and C.*

Fig. 12.76

Solution

$$Z(s)=\frac{(Ls+R)\dfrac{1}{Cs}}{Ls+R+\dfrac{1}{Cs}}=\frac{Ls+R}{LCs^2+RCs+1}=\frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \quad \text{...(i)}$$

From the pole-zero diagram, zero is at $s=-2$ and poles are at $s=-1+j4$ and $s=-1-j4$.

$$Z(s) = H\frac{s+2}{(s+1+j4)(s+1-j4)} = H\frac{s+2}{(s+1)^2-(j4)^2} = H\frac{s+2}{s^2+2s+17}$$

At dc, i.e.,

$$\omega = 0,\ Z(j0) = 2$$

$$2 = H\frac{2}{17}$$

$$H = 17$$

$$Z(s) = 17\frac{s+2}{s^2+2s+17} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 17$$

$$\frac{R}{L} = 2$$

$$\frac{1}{LC} = 17$$

Solving the above equations,

$$C = \frac{1}{17}\text{F}$$

$$L = 1\text{ H}$$

$$R = 2\ \Omega$$

Example 12.40 *The network shown in Fig. 12.77 has the driving-point admittance Y (s) of the form*

$$Y(s) = H\frac{(s-s_1)(s-s_2)}{(s-s_3)}$$

(*a*) *Express s_1, s_2, s_3 in terms of R, L and C.*

(*b*) *When $s_1 = -10 + j10^4$, $s_2 = -10 - j10^4$ and $Y(j0) = 10^{-1}$ mho, find the values of R, L and C and determine the value of s_3.*

Fig. 12.77

Solution

(a)
$$Y(s) = Cs + \frac{1}{Ls+R} = \frac{(Ls+R)\,Cs+1}{Ls+R} = \frac{LCs^2+RCs+1}{Ls+R} = \frac{C\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s+\frac{R}{L}} \qquad \text{...(i)}$$

But
$$Y(s) = \frac{H(s-s_1)(s-s_2)}{(s-s_3)}$$

where
$$s_1, s_2 = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_3 = -\frac{R}{L}$$

(b) When
$$s_1 = -10 + j10^4$$
$$s_2 = -10 - j10^4$$

$$Y(s) = H\frac{(s+10-j10^4)(s+10+j10^4)}{s-s_3} = H\frac{s^2+20s+10^8}{s-s_3} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 20$$
$$s_3 = -20$$
$$Y(s) = H\frac{(s^2+20s+10^8)}{(s+20)}$$

At $s = j0$,

$$Y(j0) = H\frac{(10^8)}{20} = 10^{-1}$$
$$H = 0.02 \times 10^{-6}$$
$$Y(s) = 0.02 \times 10^{-6}\frac{(s^2+20s+10^8)}{(s+20)} \quad \text{...(iii)}$$

Comparing Eq. (iii) with Eq. (i),

$$C = 0.02 \times 10^{-6}\,\text{F} = 0.02\ \mu\text{F}$$
$$\frac{1}{LC} = 10^8$$
$$L = \frac{1}{2}\,\text{H}$$
$$\frac{R}{L} = 20$$
$$R = 10\ \Omega$$

Example 12.41 *A network and pole-zero diagram for driving-point impedance Z(s) are shown in Fig. 12.78. Calculate the values of the parameters R, L, G and C if Z(j0) = 1.*

Fig. 12.78

Solution It is easier to calculate $Y(s)$ and then invert it to obtain $Z(s)$.

$$Y(s) = G + Cs + \frac{1}{Ls+R} = \frac{(G+Cs)(Ls+R)+1}{Ls+R} = \frac{LCs^2 + (GL+RC)\,s + 1 + GR}{Ls+R}$$

$$Z(s) = \frac{1}{Y(s)} = \frac{Ls+R}{LCs^2 + (GL+RC)\,s + 1 + GR} = \frac{\frac{1}{C}\left(s + \frac{R}{L}\right)}{s^2 + \left(\frac{G}{C} + \frac{R}{L}\right)s + \left(\frac{1+GR}{LC}\right)} \quad \text{...(i)}$$

From the pole-zero diagram, zero is at $s = -2$ and poles are at $s = -3 \pm j3$.

$$Z(s) = H\frac{(s+2)}{(s+3-j3)(s+3+j3)} = H\frac{(s+2)}{(s+3)^2 - (j3)^2} = H\frac{s+2}{s^2+6s+18}$$

When $Z(j0) = 1$,

$$1 = H\frac{2}{18}$$

$$H = 9$$

$$Z(s) = \frac{9\,(s+2)}{(s^2+6s+18)} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 9$$

$$\frac{R}{L} = 2$$

$$\frac{G}{C} + \frac{R}{L} = 6$$

$$\frac{1+GR}{LC} = 18$$

Solving the above equation,

$$C = \frac{1}{9}\ \text{F}$$

$$L = \frac{9}{10}\ \text{H}$$

$$G = \frac{4}{9}\ \text{℧}$$

$$R = \frac{9}{5}\ \Omega$$

Example 12.42 *A series R-L-C circuit has its driving-point admittance and pole-zero diagram is shown in Fig. 12.79. Find the values of R, L and C.*

Fig. 12.79

Solution The function $Y(s)$ has poles at $s = -1 + j25$ and $s = -1 - j25$ and zero at $s = 0$.

$$Y(s) = H\frac{s}{(s+1+j25)(s+1-j25)} = H\frac{s}{(s+1)^2-(j25)^2} = H\frac{s}{s^2+2s+626}$$

Scale factor $H = 1$

$$Y(s) = \frac{s}{s^2+2s+626} \quad \text{...(i)}$$

For a series *RLC* circuit,

$$Z(s) = R + Ls + \frac{1}{Cs} = \frac{LCs^2+RCs+1}{Cs} = \frac{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{s}{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} \quad \text{...(ii)}$$

Comparing Eq. (i) with Eq. (ii),

$$L = 1\text{ H}$$

$$\frac{1}{LC} = 626$$

$$C = \frac{1}{626}\text{ F}$$

$$\frac{R}{L} = 2$$

$$R = 2\ \Omega$$

12.9 TIME-DOMAIN BEHAVIOUR FROM THE POLE-ZERO PLOT

The time-domain behaviour of a system can be determined from the pole-zero plot. Consider a network function

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

The poles of this function determine the time-domain behaviour of $f(t)$.The function $f(t)$ can be determined from the knowledge of the poles, the zeros and the scale factor H. Figure 12.80 shows some pole locations and the corresponding time-domain response.

(i) When pole is at origin, i.e., at $s = 0$, the function $f(t)$ represents steady-state response of the circuit i.e., dc value. (Fig. 12.80)

Fig. 12.80 *Pole at origin*

(ii) When pole lies in the left half of the s-plane, the response decreases exponentially. (Fig. 12.81)

Fig. 12.81 *Pole in left half of the s-plane*

(iii) When pole lies in the right half of the s-plane, the response increases exponentially. A pole in the right-half plane gives rise to unbounded response and unstable system. (Fig. 12.82)

Fig. 12.82 *Pole in right half of the s-plane*

(iv) For $s = 0 + j\omega_n$, the response becomes $f(t) = Ae^{\pm j\omega_n t} = A(\cos \omega_n t \pm j \sin \omega_n t)$.The exponential response $e^{\pm j\omega_n t}$ may be interpreted as a rotating phasor of unit length. A positive sign of exponential $e^{j\omega_n t}$ indicates counterclockwise rotation, while a negative sign of exponential $e^{-j\omega_n t}$ indicates clockwise rotation. The variation of exponential function $e^{j\omega_n t}$ with time is thus sinusoidal and hence constitutes the case of sinusoidal steady state. (Fig. 12.83)

Fig. 12.83 *Poles on jω-axis*

(v) For $s = \sigma_n + j\omega_n$, the response becomes $f(t) = Ae^{st} = Ae^{(\sigma_n + j\omega_n t)} = Ae^{\sigma_n t}\, e^{j\omega_n t}$. The response $e^{\sigma_n t}$ is an exponentially increasing or decreasing function. The response $e^{j\omega_n t}$ is a sinusoidal function. Hence, the response of the product of these responses will be over damped sinusoids or under damped sinusoids (Fig. 12.84).

Fig. 12.84 *(a) Complex conjugate poles in left half of the S-plane*
(b) Complex conjugate poles in right half of the S-plane

(vi) The real part s of the pole is the displacement of the pole from the imaginary axis. Since σ is also the damping factor, a greater value of σ (i.e., a greater displacement of the pole from the imaginary axis) means that response decays more rapidly with time. The poles with greater displacement from the real axis correspond to higher frequency of oscillation (Fig. 12.85).

Fig. 12.85 *Nature of response with different positions of poles*

(b)

Fig. 12.85 *(Continued)*

12.9.1 Stability of the Network

Stability of the network is directly related to the location of poles in the s-plane.

(i) When all the poles lie in the left half of the s-plane, the network is said to be stable.
(ii) When the poles lie in the right half of the s-plane, the network is said to be unstable.
(iii) When the poles lie on the $j\omega$ axis, the network is said to be marginally stable.
(iv) When there are multiple poles on the $j\omega$ axis, the network is said to be unstable.
(v) When the poles move away from $j\omega$ axis towards the left half of the s-plane, the relative stability of the network improves.

12.10 GRAPHICAL METHOD FOR DETERMINATION OF RESIDUE

Consider a network function,

$$F(s) = H\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

By partial fraction expansion,

$$F(s) = \frac{K_1}{(s-p_1)} + \frac{K_2}{(s-p_2)} + \cdots + \frac{K_m}{(s-p_m)}$$

The residue K_i is given by

$$K_i = (s-p_i)F(s)\big|_{s\to p_i} = H\frac{(p_i-z_1)(p_i-z_2)\cdots(p_i-z_n)}{(p_i-p_1)(p_i-p_2)\cdots(p_i-p_m)}$$

Each term $(p_i - z_i)$ represents a phasor drawn from zero z_i to pole p_i.
Each term $(p_i - p_k)$, $i \neq k$, represents a phasor drawn from other poles to the pole p_i.

$$K_i = H\frac{\text{Product of phasors (polar form) from each zero to } p_i}{\text{Product of phasors (polar form) from other poles to } p_i}$$

The residues can be obtained by graphical method in the following way:

(1) Draw the pole-zero diagram for the given network function.
(2) Measure the distance from each of the other poles to a given pole.
(3) Measure the distance from each of the other zeros to a given pole.
(4) Measure the angle from each of the other poles to a given pole.
(5) Measure the angle from each of the other zeros to a given pole.
(6) Substitute these values in the required residue equation.

The graphical method can be used if poles are simple and complex. But it cannot be used when there are multiple poles.

Example 12.43 *The current I(s) in a network is given by* $I(s) = \dfrac{2s}{(s+1)(s+2)}$. *Plot the pole-zero pattern in the s-plane and hence obtain i(t).*

Solution Poles are at $s = -1$ and $s = -2$ and zero is at $s = 0$. The pole-zero plot is shown in Fig. 12.86.

By partial-fraction expansion,

$$I(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

Fig. 12.86

The coefficients K_1 and K_2, often referred as residues, can be evaluated from he pole-zero diagram. From Fig. 12.87,

$$K_1 = H\frac{\text{Phasor from zero at origin to pole at } A}{\text{Phasor from pole at } B \text{ to pole at } A} = 2\left(\frac{1\angle 180^\circ}{1\angle 0^\circ}\right) = 2\angle 180^\circ = -2$$

Fig. 12.87

From Fig. 12.88,

$$K_2 = H\frac{\text{Phasor from zero at origin to pole at } B}{\text{Phasor from pole at } A \text{ to pole at } B} = 2\left(\frac{2\angle 180^\circ}{1\angle 180^\circ}\right) = 4$$

Fig. 12.88

$$I(s) = -\frac{2}{s+1} + \frac{4}{s+2}$$

Taking inverse Laplace transform,

$$i\,(t) = -2e^{-t} + 4e^{-2t}$$

Example 12.44 *The voltage V(s) of a network is given by*

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)}$$

Plot its pole-zero diagram and hence obtain v (t).

Solution

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)} = \frac{3s}{(s+2)(s+1+j1)(s+1-j1)}$$

Poles are at $s = -2$ and $s = -1 \pm j1$ and zero is at $s = 0$ as shown in Fig. 12.89.

Fig. 12.89

By partial-fraction expansion,

$$V(s) = \frac{K_1}{s+2} + \frac{K_2}{s+1-j1} + \frac{K_2^*}{s+1+j1}$$

The coefficients K_1, K_2 and K_2^* can be evaluated from the pole-zero diagram.
From Fig. 12.90,

$$K_1 = \frac{3\,(\overline{OA})}{(\overline{BA})\,(\overline{CA})} = 3\left[\frac{2\angle 180°}{(\sqrt{2}\angle -135°)\,(\sqrt{2}\angle 135°)}\right] = 3\sqrt{180°} = -3$$

Fig. 12.90

From Fig. 12.91,

$$K_2 = \frac{3\,(\overline{OB})}{(\overline{AB})\,(\overline{CB})} = 3\left[\frac{(\sqrt{2}\angle 135°)}{(\sqrt{2}\angle 45°)\,(2\angle 90°)}\right] = \frac{3}{2}$$

$$K_2^* = \frac{3}{2}$$

Fig. 12.91

$$V(s) = -\frac{3}{(s+2)} + \frac{\frac{3}{2}}{(s+1-j1)} + \frac{\frac{3}{2}}{(s+1+j1)}$$

Taking inverse Laplace transform,

$$v(t) = -3e^{-2t} + \frac{3}{2}\left[e^{(-1+j1)t} + e^{(-1-j1)t}\right] = -3e^{-2t} + 2\times\frac{3}{2}e^{-t}\left(\frac{e^{j1}+e^{-j1}}{2}\right) = -3e^{-2t} + 3e^{-t}\cos t$$

Example 12.45 *Find the function v(t) using the pole-zero plot of following function:*

$$V(s) = \frac{(s+2)(s+6)}{(s+1)(s+5)}$$

Solution If the degree of the numerator is greater or equal to the degree of the denominator, we have to divide the numerator by the denominator such that the remainder can be expanded into partial fractions.

$$V(s)=\frac{s^2+8s+12}{s^2+6s+5}=1+\frac{2s+7}{s^2+6s+5}=1+\frac{2(s+3.5)}{(s+1)(s+5)}$$

By partial fraction expansion,

$$V(s)=1+\frac{K_1}{s+1}+\frac{K_2}{s+5}$$

K_1 and K_2 can be evaluated from the pole-zero diagram shown in Fig. 12.92 and Fig. 12.93.

Fig. 12.92

Fig. 12.93

From Fig. 12.92

$$K_1=2\left(\frac{2.5\angle 0^\circ}{4\angle 0^\circ}\right)=\frac{5}{4}$$

From Fig. 12.93

$$K_2=2\left(\frac{1.5\angle 180^\circ}{4\angle 180^\circ}\right)=\frac{3}{4}$$

$$V(s)=1+\frac{\frac{5}{4}}{s+1}+\frac{\frac{3}{4}}{s+5}$$

Taking inverse Laplace transform,

$$v(t)=\delta(t)+\frac{5}{4}e^{-t}+\frac{3}{4}e^{-5t}$$

Example 12.46 *The pole-zero plot of the driving-point impedance of a network is shown in Fig. 12.94. Find the time-domain response.*

Fig. 12.94

Solution The function $Z(s)$ has poles at $s=-1+j1$ and $s=-1-j1$ and zero at $s=0$.

$$Z(s) = H\frac{s}{(s+1+j1)(s+1-j1)}$$

Scale factor $H = 5$

$$Z(s) = \frac{5s}{(s+1+j1)(s+1-j1)}$$

By partial fraction expansion,

$$Z(s) = \frac{K_1}{s+1+j1} + \frac{K_1^*}{s+1-j1}$$

Fig. 12.95

The coefficients K_1 and K_1^* can be evaluated from the pole-zero diagram. From Fig. 12.95,

$$K_1 = \frac{5\,(\overline{OA})}{(\overline{BA})} = \frac{5\,(\sqrt{2}\angle 135°)}{2\angle 90°} = 3.54\angle 45°$$

$$K_1^* = 3.54\angle -45°$$

$$Z(s) = \frac{3.54\angle 45°}{s+1+j1} + \frac{3.54\angle -45°}{s+1-j1}$$

Taking inverse Laplace transform,

$$z(t) = 3.54\angle 45°\, e^{(-1-j1)t} + 3.54\angle -45° e^{(-1+j1)t}$$

Example 12.47 *Evaluate amplitude and phase of the network function* $F(s) = \dfrac{4s}{s^2+2s+2}$ *from the pole-zero plot at* $s = j2$.

Solution

$$F(s) = \frac{4s}{s^2+2s+2} = \frac{4s}{(s+1+j1)(s+1-j1)}$$

The pole-zero plot is shown in Fig. 12.96.

At $s = j2$,

Fig. 12.96

$$|F(j2)| = \frac{\text{Product of phasor magnitudes from all zero to } j2}{\text{Product of phasor magnitudes from all poles to } j2} = \frac{2}{(\sqrt{2})\,(\sqrt{10})} = 0.447$$

$$\phi(\omega) = \tan^{-1}\left(\frac{2}{0}\right) - \tan^{-1}\left(\frac{3}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = 90° - 71.56° - 45° = -26.56°$$

Example 12.48 *Using the pole-zero plot, find magnitude and phase of the function*

$$F(s) = \frac{(s+1)(s+3)}{s(s+2)} \; at \; s = j4.$$

Solution

$$F(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

The pole-zero plot is shown in Fig. 12.97.

At $s = j4$,

Fig. 12.97

$$|F(j4)| = \frac{\text{Product of phasor magnitudes from all zeros to } j4}{\text{product of phasor magnitudes from all poles to } j4} = \frac{(5)\,(\sqrt{17})}{(\sqrt{20})\,(4)} = 1.15$$

$$\phi(\omega) = \tan^{-1}\left(\frac{4}{1}\right) + \tan^{-1}\left(\frac{4}{3}\right) - \tan^{1}\left(\frac{4}{0}\right) - \tan^{-1}\left(\frac{4}{2}\right) = 75.96° + 53.13° - 90° - 63.43° = -24.34°$$

Example 12.49 *Plot amplitude and phase response for*

$$F(s) = \frac{s}{s+10}$$

Solution

$$F(j\omega) = \frac{j\omega}{j\omega + 10}$$

$$|F(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + 100}}$$

ω	$\lvert F(j\omega)\rvert$
0	0
10	0.707
100	0.995
1000	1

Fig. 12.98

The amplitude response is shown in Fig. 12.98.

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) = 90° - \tan^{-1}\left(\frac{\omega}{10}\right)$$

ω	$\phi(\omega)$
0	90°
10	45°
100	5.7°
1000	0°

Fig. 12.99

The phase response is shown in Fig. 12.99

Example 12.50 *Sketch amplitude and phase response for* $F(s) = \dfrac{s+10}{s-10}$

Solution

$$F(j\omega) = \frac{j\omega + 10}{j\omega - 10}$$

$$|F(f\omega)| = \frac{\sqrt{\omega^2 + 100}}{\sqrt{\omega^2 + 100}}$$

For all ω, magnitude is unity.

The amplitude response is shown in Fig. 12.100.

Fig. 12.100

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(-\frac{\omega}{10}\right) = 2\tan^{-1}\left(\frac{\omega}{10}\right)$$

The phase response is shown in Fig. 12.101.

ω	$\phi(\omega)$
0	0°
10	90°
100	168.6°
1000	178.9°

Fig. 12.101

Exercises

12.1 Determine the driving-point impedance $\frac{V_1}{I_1}$, transfer impedance $\frac{V_2}{I_1}$ and voltage transfer ratio $\frac{V_2}{V_1}$ for the network shown in Fig. 12.102.

Fig. 12.102

$$\left[\frac{V_1}{I_1} = \frac{7s^2+7s+5}{s2+s+1}; \frac{V_2}{I_1} = \frac{2s}{s^2+s+1}; \frac{V_2}{V_1} = \frac{2s}{7s^2+7s+5}\right]$$

12.2 For the network shown in Fig. 12.103, determine $\frac{V_2}{V_1}$ and $\frac{V_2}{I_1}$.

Fig. 12.103

$$\left[\frac{V_2}{V_1} = \frac{s^2+1}{2s^2+1}; \frac{V_2}{I_1} = \frac{2s^2+2}{s(3s^2+2)}\right]$$

12.3 Find the open-circuit transfer impedance Z_{21} and open-circuit voltage ratio G_{21} for the ladder network shown in Fig. 12.104.

Fig. 12.104

$$\left[Z_{21} = \frac{1}{2s^3+3s}, G_{21} = \frac{1}{4s^4+7s^2+1}\right]$$

12.4 For the two-port network shown in Fig. 12.105, determine Z_{11}, Z_{21} and voltage transfer ratio $G_{21}(s)$.

Fig. 12.105

$$\left[Z_{11} = \frac{2s^3+4s^2+3s+2}{s^2+2s+1}, Z_{21} = \frac{s}{s^2+2s+1}, G_{21} = \frac{s}{2s^3+4s^2+3s+2}\right]$$

12.5 Draw the pole-zero diagram of the following network functions:

(i) $F(s) = \frac{s^2+4}{s^2+6s+4}$

(ii) $F(s) = \frac{5s-12}{s^2+4s+13}$

(iii) $F(s) = \frac{s+1}{(s^2+2s+2)^2}$

(iv) $F(s) = \frac{s(s^2+5)}{s^4+2s^2+1}$

(v) $F(s) = \dfrac{s^2 + s + 2}{s^4 + 5s^3 + 6s^2}$

(vi) $F(s) = \dfrac{s^2 - s}{s^3 + 2s^2 - s - 2}$

(vii) $F(s) = \dfrac{s^2 + 3s + 2}{s^2 + 3s}$

(viii) $F(s) = \dfrac{(s^2 + 4)(s + 1)}{(s^2 + 1)(s^2 + 2s + 5)}$

12.6 For the network shown in Fig. 12.106, draw the pole-zero plot of the impedance function $Z(s)$.

Fig. 12.106

$$Z(s) = \left[\frac{(s + 2.5 - j1.94)(s + 2.5 + j1.94)}{s + 5}\right]$$

12.7 For the network shown in Fig. 12.107, draw the pole-zero plot of driving-point impedance function $Z(s)$.

Fig. 12.107

$$\left[Z(s) = \frac{5(s + 0.01)(s + 0.04)}{s(s + 0.03)}\right]$$

12.8 Find the driving-point impedance of the network shown in Fig. 12.108. Also, find poles and zeros.

Fig. 12.108

$$\left[Z(s) = \frac{1.5s(s^2 + 0.33)}{(s^2 + 1.707)(s^2 + 0.293)}\right]$$

12.9 Find network functions $\dfrac{V_2}{V_1}$ and $\dfrac{V_1}{I_1}$ for the network shown in Fig. 12.109 and plot poles and zeros of $\dfrac{V_2(s)}{V_1(s)}$.

Fig. 12.109

$$\left[\frac{V_2}{V_1} = \frac{1}{2(s^3 + 2s^2 + 2s + 1)}, \frac{V_1}{I_1} = \frac{2(s^3 + 2s^2 + 2s + 1)}{2s^3 + 2s^2 + 2s + 1}\right]$$

12.10 For the network shown in Fig. 12.110, determine $\dfrac{V_1}{I_1}$ *and* $\dfrac{V_2}{I_1}$. Plot the poles and zeros of $\dfrac{V_2}{I_1}$.

Fig. 12.110

$$\left[\frac{V_1}{I_1} = \frac{2s^4 + 5s^2 + 2}{2s^3 + 3s}, \frac{V_2}{I_1} = \frac{2}{2s^3 + 3s}\right]$$

12.11 For the network shown in Fig. 12.111, determine $\frac{V_1}{I_1}$ and $\frac{V_2}{V_1}$. Plot the pole and zeros for $\frac{V_2}{V_1}$.

Fig. 12.111

$$\left[\frac{V_1}{I_1} = \frac{s^4 + 3s^2 + 1}{2s^3 + s}\right]$$

12.12 For the network shown in Fig. 12.112, plot the poles and zeros of transfer impedance function.

Fig. 12.112

$$\left[\frac{V_2}{I_1} = \frac{1}{s+2}\right]$$

12.13 For the network shown in Fig. 12.113, determine $\frac{V_1}{I_1}$ and $\frac{V_2}{V_1}$. Plot the poles and zeros of transfer impedance function.

Fig. 12.113

$$\left[\frac{V_1}{I_1} = \frac{16s^4 + 10s^2 + 1}{8s^3 + 3s}, \frac{V_2}{I_1} = \frac{1}{8s^3 + 3s}, \frac{V_2}{V_1} = \frac{1}{16s^4 + 10s^2 + 1}\right]$$

12.14 Obtain the impedance function for which the pole-zero diagram is shown in Fig. 12.114.

Fig. 12.114

$$\left[Z(s) = \frac{2(s+1)}{s^2 + 2s + 2}\right]$$

12.15 For the network shown in Fig. 12.115, poles and zeros of driving point function $Z(s)$ are,

Poles: $(-1 \pm j4)$; zero: -2

If $Z(j0) = 1$, find the values of R, L and C.

Fig. 12.115

$$\left[1\Omega, 0.5\text{ H}, \frac{2}{17}\text{ F}\right]$$

12.16 For the two-port network shown in Fig. 12.116, find R_1, R_2 and C. $\frac{V_2}{V_1} = \frac{0.2}{s^2 + 3s + 2}$

Fig. 12.116

$$\left[\frac{3}{5}\ \Omega, \frac{1}{15}\ \Omega, 0.5\text{ F}\right]$$

12.17 For the given network function, draw the pole-zero diagram and hence obtain the time domain voltage.

$$V(s) = \frac{5\,(s+5)}{(s+2)(s+7)}$$

$$[v(t) = 3e^{-2t} + 2e^{-7t}]$$

12.18 A transfer function is given by $Y(s) = \dfrac{10s}{(s+5+j15)(s+5-j15)}$. Find time-domain response using graphical method.

$$\left[5.26\,\angle 18.4^\circ\, e^{-(5+j15)t} + 5.26\angle -18.4^\circ\, e^{-(5-j15)t}\right]$$

Objective-Type Questions

12.1 Of the four networks N_1, N_2, N_3 and N_4 of Fig. 12.117, the networks having identical driving-point functions are

(a) N_1 and N_2 (b) N_2 and N_4
(c) N_1 and N_3 (d) N_1 and N_4

Fig. 12.117

12.2 The driving-point impedance $Z(s)$ of a network has the pole-zero locations as shown in Fig. 12.118. If $Z(0) = 3$, then $Z(s)$ is

Fig. 12.118

(a) $\dfrac{3(s+3)}{s^2+2s+3}$ (b) $\dfrac{2(s+3)}{s^2+2s+2}$

(c) $\dfrac{3(s-3)}{s^2-2s-2}$ (d) $\dfrac{2(s-3)}{s^2-2s-3}$

12.3 For the circuit shown in Fig. 12.119, the initial conditions are zero. Its transfer function $H(s) = \dfrac{V_0(s)}{V_1(s)}$ is

Fig. 12.119

(a) $\dfrac{1}{s^2+10^6 s+10^6}$ (b) $\dfrac{10^6}{s^2+10^3 s+10^6}$

(c) $\dfrac{10^3}{s^2+10^3 s+10^6}$ (d) $\dfrac{10^6}{s^2+10^6 s+10^6}$

12.4 In Fig. 12.120, assume that all the capacitors are initially uncharged. If $v_i(t) = 10\,u(t)$, then $v_0(t)$ is given by

Fig. 12.120

(a) $8\,e^{-0.004\,t}$ (b) $8\,(1-e^{-0.004\,t})$

(c) $8\,u(t)$ (d) 8

12.5 A system is represented by the transfer function $\dfrac{10}{(s+1)(s+2)}$. The dc gain of this system is

(a) 1 (b) 2

(c) 5 (d) 10

12.6 Which one of the following is the ratio $\dfrac{V_{24}}{V_{13}}$ of the network shown in Fig. 12.121.

Fig. 12.121

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{3}{4}$ (d) $\dfrac{4}{3}$

12.7 A network has response with time as shown in Fig. 12.122. Which one of the following diagrams represents the location of the poles of this network?

Fig. 12.122

Fig. 12.123

12.8 The transfer function of a low-pass *RC* network is

(a) $(RCs)\,(1+RCs)$ (b) $\dfrac{1}{1+RCs}$

(c) $\dfrac{RCs}{1+RCs}$ (d) $\dfrac{s}{1+RCs}$

12.9 The driving-point admittance function of the network shown in Fig. 12.124 has a

Fig. 12.124

(a) pole at $s = 0$ and zero at $s = \infty$
(b) pole at $s = 0$ and pole at $s = \infty$
(c) pole at $s = \infty$ and zero at $s = 0$
(d) pole at $s = \infty$ and zero at $s = \infty$

12.10 The transfer function $Y_{12}(s) = \dfrac{I_2(s)}{V_1(s)}$ for the network shown in Fig. 12.125 is

Fig. 12.125

(a) $\dfrac{s^2}{s^2+s+1}$ (b) $\dfrac{s}{s+1}$

(c) $\dfrac{1}{s+1}$ (d) $\dfrac{s+1}{s^2+1}$

12.11 As the poles of a network shift away from the x axis, the response

(a) remains constant
(b) becomes less oscillating
(c) becomes more oscillating
(d) none of these

Answers to Objective-Type Questions

1. (c)	2. (b)	3. (d)	4. (c)	5. (c)	6. (a)
7. (d)	8. (b)	9. (a)	10. (a)	11. (c)	

13

Two-Port Networks

13.1 INTRODUCTION

A two-port network has two pairs of terminals, one pair at the input known as *input port* and one pair at the output known as *output port* as shown in Fig. 13.1. There are four variables V_1, V_2, I_1 and I_2 associated with a two-port network. Two of these variables can be expressed in terms of the other two variables. Thus, there will be two dependent variables and two independent variables. The number of possible combinations generated by four variables taken two at a time is ${}^4C_2$, i.e., six. There are six possible sets of equations describing a two-port network.

Fig. 13.1 *Two-port network*

Table 13.1 *Two–port parameters*

Parameter	*Variables*		*Equation*
	Express	*In terms of*	
Open-Circuit Impedance	V_1, V_2	I_1, I_2	$V_1 = Z_{11} I_1 + Z_{12} I_2$ $V_2 = Z_{21} I_1 + Z_{22} I_2$
Short-Circuit Admittance	I_1, I_2	V_1, V_2	$I_1 = Y_{11} V_1 + Y_{12} V_2$ $I_2 = Y_{21} V_1 + Y_{22} V_2$
Transmission	V_1, I_1	V_2, I_2	$V_1 = AV_2 - BI_2$ $I_1 = CV_2 - DI_2$
Inverse Transmission	V_2, I_2	V_1, I_1	$V_2 = A' V_1 - B' I_1$ $I_2 = C' V_1 - D' I_1$
Hybrid	V_1, I_2	I_1, V_2	$V_1 = h_{11} I_1 + h_{12} V_2$ $I_2 = h_{21} I_1 + h_{22} V_2$
Inverse Hybrid	I_1, V_2	V_1, I_2	$I_1 = g_{11} V_1 + g_{12} I_2$ $V_2 = g_{21} V_1 + g_{22} I_2$

13.2 OPEN-CIRCUIT IMPEDANCE PARAMETERS (Z PARAMETERS)

The Z parameters of a two-port network may be defined by expressing two-port voltages V_1 and V_2 in terms of two-port currents I_1 and I_2.

$$(V_1, V_2) = f(I_1, I_2)$$
$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$[V] = [Z][I]$$

The individual Z parameters for a given network can be defined by setting each of the port currents equal to zero.

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$

where Z_{11} is the driving-point impedance with the output port open-circuited. It is also called *open-circuit input impedance*.

Similarly,

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2 = 0}$$

where Z_{21} is the transfer impedance with the output port open-circuited. It is also called *open-circuit forward transfer impedance*.

Case 2 When input port is open-circuited, i.e., $I_1 = 0$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0}$$

where Z_{12} is the transfer impedance with the input port open-circuited. It is also called *open-circuit reverse transfer impedance*.

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0}$$

where Z_{22} is the open-circuit driving-point impedance with the input port open-circuited. It is also called *open circuit output impedance*.

As these impedance parameters are measured with either the input or output port open-circuited, these are called *open-circuit impedance parameters*.

The equivalent circuit of the two-port network in terms of Z parameters is shown in Fig. 13.2.

Fig. 13.2 *Equivalent circuit of the two-port network in terms of Z parameter*

13.2.1 Condition for Reciprocity

A network is said to be reciprocal if the ratio of excitation at one port to response at the other port is same if excitation and response are interchanged.

(a) As shown in Fig. 13.3, voltage V_s is applied at the input port with the output port short-circuited.

Fig. 13.3 *Network for deriving condition for reciprocity*

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2 = -I_2'$$

From the Z-parameter equations,

$$V_s = Z_{11}\ I_1 - Z_{12}\ I_2'$$
$$0 = Z_{21}\ I_1 - Z_{22}\ I_2'$$
$$I_1 = \frac{Z_{22}}{Z_{21}} I_2'$$
$$V_s = Z_{11}\frac{Z_{22}}{Z_{21}} I_2' - Z_{12}\ I_2'$$
$$\frac{V_s}{I_2'} = \frac{Z_{11}\ Z_{22} - Z_{12}\ Z_{21}}{Z_{21}}$$

(b) As shown in Fig. 13.4, voltage V_s is applied at the output port with input port short-circuited.

Fig. 13.4 *Network for deriving condition for reciprocity*

i.e.,

$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1 = -I_1'$$

From the Z-parameter equations,

$$0 = -Z_{11}\ I_1' + Z_{12}\ I_2$$
$$V_s = -Z_{21}\ I_1' + Z_{22}\ I_2$$
$$I_2 = \frac{Z_{11}}{Z_{12}} I_1'$$
$$V_s = -Z_{21} I_1' + Z_{22}\frac{Z_{11}}{Z_{12}} I_1'$$
$$\frac{V_s}{I_1'} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{12}}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_1'} = \frac{V_s}{I_2'}$$

i.e.,

$$Z_{12} = Z_{21}$$

13.2.2 Condition for Symmetry

For a network to be symmetrical, the voltage-to-current ratio at one port should be the same as the voltage-to-current ratio at the other port with one of the ports open-circuited.

(a) When the output port is open-circuited, i.e., $I_2 = 0$
From the Z-parameter equation,

$$V_s = Z_{11}\ I_1$$
$$\frac{V_s}{I_1} = Z_{11}$$

(b) When the input port is open-circuited, i.e., $I_1 = 0$
From the Z-parameter equation,

$$V_s = Z_{22}\ I_2$$
$$\frac{V_s}{I_2} = Z_{22}$$

Hence, for the network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e., $$Z_{11} = Z_{22}$$

Example 13.1 *Test results for a two-port network are (a) $I_1 = 0.1 \angle 0°$ A, $V_1 = 5.2 \angle 50°$ V, $V_2 = 4.1 \angle -25°$ V with Port 2 open-circuited (b) $I_2 = 0.1 \angle 0°$A, $V_1 = 3.1 \angle -80°$ V, $V_2 = 4.2 \angle 60°$V, with Port 1 open-circuited. Find Z parameters.*

Solution

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{5.2\angle 50°}{0.1\angle 0°} = 52 \angle 50° \ \Omega, \qquad Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{3.1\angle -80°}{0.1\angle 0°} = 31 \angle -80° \ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{4.1\angle -25°}{0.1\angle 0°} = 41 \angle -25° \ \Omega, \qquad Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{4.2\angle 60°}{0.1\angle 0°} = 42 \angle 60° \ \Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 52\angle 50° & 31\angle -80° \\ 41\angle -25° & 42\angle 60° \end{bmatrix}$$

Example 13.2 *Find the Z parameters for the network shown in Fig. 13.5.*

Fig. 13.5

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$.

Applying KVL to Mesh 1,

$$V_1 = (Z_1 + Z_2)\, I_1$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_1 + Z_2$$

Also $$V_2 = Z_2\, I_1$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_2$$

Case 2 When the input port is open-circuited, i.e., $I_1 = 0$.

Applying KVL to Mesh 2,

$$V_2 = (Z_2 + Z_3)\, I_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_2 + Z_3$$

Also
$$V_1 = Z_2\, I_2$$
$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_2$$

Hence, the Z-parameters are
$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1+Z_2 & Z_2 \\ Z_2 & Z_2+Z_3 \end{bmatrix}$$

Second Method

The network is redrawn as shown in Fig. 13.6.

Applying KVL to Mesh 1,
$$V_1 = Z_1 I_1 + Z_2(I_1+I_2)$$
$$= (Z_1+Z_2)I_1 + Z_2 I_2 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,
$$V_2 = Z_3 I_2 + Z_2(I_1+I_2)$$
$$= Z_2 I_1 + (Z_2+Z_3)I_2 \quad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,
$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1+Z_2 & Z_2 \\ Z_2 & Z_2+Z_3 \end{bmatrix}$$

Fig. 13.6

Example 13.3 *Find Z-parameter for the network shown in Fig. 13.7.*

Fig. 13.7

Solution The transformed network is shown in Fig. 13.8.
$$Z_1 = 1$$
$$Z_2 = \frac{\left(\frac{1}{2s}\right)(1)}{\frac{1}{2s}+1} = \frac{1}{2s+1}$$
$$Z_3 = s+2$$

Fig. 13.8

From definition of Z-parameters,
$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_1 + Z_2 = 1 + \frac{1}{2s+1} = \frac{2s+2}{2s+1},$$
$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = Z_2 = \frac{1}{2s+1}$$
$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_2 = \frac{1}{2s+1},$$
$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_2 + Z_3 = \frac{1}{2s+1} + s + 2 = \frac{2s^2+5s+3}{2s+1}$$

Example 13.4 *Find Z-parameters for the network shown in Fig. 13.9.*

Fig. 13.9

Solution The network is redrawn as shown in Fig. 13.10.

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2I_1 + 2I_2 + 5I_3 = 0$$

$$I_3 = \frac{2}{5}I_1 - \frac{2}{5}I_2 \qquad \ldots\text{(iii)}$$

Fig. 13.10

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - \frac{4}{5}I_1 + \frac{4}{5}I_2$$

$$= \frac{11}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + \frac{4}{5}I_1 - \frac{4}{5}I_2$$

$$= \frac{4}{5}I_1 + \frac{6}{5}I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{11}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{6}{5} \end{bmatrix}$$

Example 13.5 *Find the Z-parameters for the network shown in Fig. 13.11.*

Fig. 13.11

Solution The transformed network is shown in Fig. 13.12.

Applying KVL to Mesh 1,

$$V_1 = (s+1)I_1 - sI_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = sI_2 + sI_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-sI_1 + sI_2 + (2s+1)I_3 = 0$$

$$I_3 = \frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2 \quad \ldots \text{(iii)}$$

Fig. 13.12

Substituting Eq. (iii) in Eq. (i),

$$V_1 = (s+1)I_1 - s\left(\frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2\right)$$

$$= \left(\frac{s^2+3s+1}{2s+1}\right)I_1 + \left(\frac{s^2}{2s+1}\right)I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = sI_2 + s\left(\frac{s}{2s+1}I_1 - \frac{s}{2s+1}I_2\right)$$

$$= \left(\frac{s^2}{2s+1}\right)I_1 + \left(\frac{s^2+s}{2s+1}\right)I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+3s+1}{2s+1} & \dfrac{s^2}{2s+1} \\ \dfrac{s^2}{2s+1} & \dfrac{s^2+s}{2s+1} \end{bmatrix}$$

Example 13.6 *Find the open-circuit impedance parameters for the network shown in Fig. 13.13. Determine whether the network is symmetrical and reciprocal.*

Fig. 13.13

Solution The network is redrawn as shown in Fig. 13.14.

Applying KVL to Mesh 1,

$$V_1 - 1(I_1 - I_3) - 2(I_1 + I_2) = 0$$

$$V_1 = 3I_1 + 2I_2 - I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 3(I_2 + I_3) - 2(I_1 + I_2) = 0$$

$$V_2 = 2I_1 + 5I_2 + 3I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-4I_3 - 3(I_2 + I_3) - 1(I_3 - I_1) = 0$$

$$I_1 - 3I_2 + 8I_3 = 0$$

$$I_3 = \frac{1}{8}I_1 - \frac{3}{8}I_2 \qquad \ldots\text{(iii)}$$

Fig. 13.14

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 + 2I_2 - \left(\frac{1}{8}I_1 - \frac{3}{8}I_2\right)$$

$$= \frac{23}{8}I_1 + \frac{19}{8}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 5I_2 + 3\left(\frac{1}{8}I_1 - \frac{3}{8}I_2\right)$$

$$= \frac{19}{8}I_1 + \frac{31}{8}I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{23}{8} & \frac{19}{8} \\ \frac{19}{8} & \frac{31}{8} \end{bmatrix}$$

Since $Z_{11} \neq Z_{22}$, the network is not symmetrical.
Since $Z_{12} = Z_{21}$, the network is reciprocal.

13.3 SHORT-CIRCUIT ADMITTANCE PARAMETERS (Y PARAMETERS)

The Y parameters of a two-port network may be defined by expressing the two-port currents I_1 and I_2 in terms of the two-port voltages V_1 and V_2.

$$(I_1, I_2) = f(V_1, V_2)$$

$$I_1 = Y_{11}\,V_1 + Y_{12}\,V_2$$

$$I_2 = Y_{21}\,V_1 + Y_{22}\,V_2$$

In matrix form, we can write

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$[I] = [Y][V]$$

The individual Y parameters for a given network can be defined by setting each of the port voltages equal to zero.

Case 1 When the output port is short-circuited, i.e., $V_2 = 0$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

where Y_{11} is the driving-point admittance with the output port short-circuited. It is also called *short-circuit input admittance*.

Similarly,

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

where Y_{21} is the transfer admittance with the output port short-circuited. It is also called *short-circuit forward transfer admittance*.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

where Y_{12} is the transfer admittance with the input port short-circuited. It is also called *short-circuit reverse transfer admittance*.

Similarly,

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

where Y_{22} is the short-circuit driving-point admittance with the input port short-circuited. It is also called the *short circuit output admittance*.

As these admittance parameters are measured with either input or output port short-circuited, these are called *short-circuit admittance parameters*.

The equivalent circuit of the two-port network in terms of Y parameters is shown in Fig. 13.15.

Fig. 13.15 *Equivalent circuit of the two-port network in terms of Y-parameters*

Condition for Reciprocity

(a) As shown in Fig. 13.16, voltage V_s is applied at input port with the output port short-circuited.

i.e,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2 = -I_2'$$

From the Y-parameter equation,

$$-I_2' = Y_{21}\, V_s$$

$$\frac{I_2}{V_s} = -Y_{21}$$

Fig. 13.16 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 13.17, voltage V_s is applied at output port with the input port short-circuited.

i.e,
$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1 = -I_1'$$

From the Y-parameter equation,
$$-I_1' = Y_{12}\, V_s$$
$$\frac{I_1'}{V_s} = -Y_{12}$$

Fig. 13.17 *Network for deriving condition for reciprocity*

Hence, for the network to be reciprocal,
$$\frac{I_2'}{V_s} = \frac{I_1'}{V_s}$$
i.e,
$$Y_{12} = Y_{21}$$

13.3.2 Condition for Symmetry

(a) When the output port is short-circuited, i.e., $V_2 = 0$.
From the Y-parameter equation,
$$I_1 = Y_{11}\, V_s$$
$$\frac{V_s}{I_1} = \frac{1}{Y_{11}}$$

(b) When the input port is short-circuited, i.e., $V_1 = 0$.
From the Y-parameter equation,
$$I_2 = Y_{22}\, V_s$$
$$\frac{V_s}{I_2} = \frac{1}{Y_{22}}$$

Hence, for the network to be symmetrical,
$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$
i.e.,
$$Y_{11} = Y_{22}$$

Example 13.7 *Test results for a two-port network are*

(a) Port 2 short-circuited: $V_1 = 50\ \angle 0^\circ\, V,\ I_2 = 2.1\ \angle -30^\circ A, I_1 = -1.1\ \angle -20^\circ A$

(b) Port 1 short-circuited: $V_2 = 50\ \angle 0^\circ\, V,\ I_2 = 3\ \angle -15^\circ A, I_1 = -1.1\ \angle -20^\circ A.$
Find Y-parameters.

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{2.1\angle -30^\circ}{50\angle 0^\circ} = 0.042\angle -30^\circ\ \mho, \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-1.1\angle -20^\circ}{50\angle 0^\circ} = -0.022\angle -20^\circ\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-1.1\angle -20^\circ}{50\angle 0^\circ} = -0.022\angle -20^\circ\ \mho, \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3\angle -15^\circ}{50\angle 0^\circ} = 0.06\angle -15^\circ\ \mho$$

Hence, the Y-parameters are
$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.042\angle -30^\circ & -0.022\angle -20^\circ \\ -0.022\angle -20^\circ & 0.06\angle -15^\circ \end{bmatrix}$$

Example 13.8 *Find Y-parameters for the network shown in Fig. 13.18.*

Fig. 13.18

Solution

First Method

Case 1 When the output port is short-circuited, i.e., $V_2 = 0$ as shown in Fig. 13.19,

$$R_{eq} = 1 + \frac{2\times 3}{2+3} = 1 + \frac{6}{5} = \frac{11}{5}\ \Omega$$

Now,
$$V_1 = \frac{11}{5} I_1$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{5}{11}\ \mho$$

Also,
$$I_2 = \frac{2}{5}(-I_1) = -\frac{2}{5}\times\frac{5}{11}V_1 = -\frac{2}{11}V_1$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{2}{11}\ \mho$$

Fig. 13.19

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 13.20,

$$R_{eq} = 3 + \frac{1\times 2}{1+2} = 3 + \frac{2}{3} = \frac{11}{3}\ \Omega$$

Now
$$V_2 = \frac{11}{3} I_2$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3}{11}\ \mho$$

Also
$$I_1 = \frac{2}{3}(-I_2) = -\frac{2}{3}\times\frac{3}{11}V_2 = -\frac{2}{11}V_2$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -\frac{2}{11}\ \mho$$

Fig. 13.20

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

Second Method (Refer Fig. 13.18)

$$I_1 = \frac{V_1 - V_3}{1}$$
$$= V_1 - V_3 \qquad \ldots(i)$$

$$I_2 = \frac{V_2 - V_3}{3}$$

$$= \frac{V_2}{3} - \frac{V_3}{3} \qquad \text{...(ii)}$$

Applying KCL at Node 3,

$$I_1 + I_2 = \frac{V_3}{2} \qquad \text{...(iii)}$$

Substituting Eqs (i) and (ii) in Eq. (iii),

$$V_1 - V_3 + \frac{V_2}{3} - \frac{V_3}{3} = \frac{V_3}{2}$$

$$V_1 + \frac{V_2}{3} = \frac{11}{6} V_3$$

$$V_3 = \frac{6}{11} V_1 + \frac{2}{11} V_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (i),

$$I_1 = V_1 - \frac{6}{11} V_1 - \frac{2}{11} V_2$$

$$= \frac{5}{11} V_1 - \frac{2}{11} V_2 \qquad \text{...(v)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_2 = \frac{V_2}{3} - \frac{1}{3}\left(\frac{6}{11} V_1 + \frac{2}{11} V_2\right)$$

$$= -\frac{2}{11} V_1 + \frac{3}{11} V_2 \qquad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & -\frac{2}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

Example 13.9 *Find Y-parameters of the network shown in Fig. 13.21.*

Fig. 13.21

Solution From Fig. 13.21,

$$I_1 = \frac{V_1 - V_3}{2}$$

$$= \frac{1}{2} V_1 - \frac{1}{2} V_3 \qquad \text{...(i)}$$

$$I_2 = \frac{V_2 - V_3}{2}$$

$$= \frac{1}{2}V_2 - \frac{1}{2}V_3 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$I_1 + I_2 + 3V_2 = 0 \qquad \ldots\text{(iii)}$$

Substituting Eqs (i) and (ii) in Eq. (iii),

$$\frac{V_1 - V_3}{2} + \frac{V_2 - V_3}{2} + 3V_2 = 0$$

$$2V_3 = V_1 + 7V_2$$

$$V_3 = \frac{1}{2}V_1 + \frac{7}{2}V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (i),

$$I_1 = \frac{1}{2}V_1 - \frac{1}{2}\left(\frac{1}{2}V_1 + \frac{7}{2}V_2\right)$$

$$= \frac{1}{4}V_1 - \frac{7}{4}V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_2 = \frac{1}{2}V_2 - \frac{1}{2}\left(\frac{1}{2}V_1 + \frac{7}{2}V_2\right)$$

$$= -\frac{1}{4}V_1 - \frac{5}{4}V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}$$

Example 13.10 *Determine Y-parameters for the network shown in Fig. 13.22. Determine whether the network is symmetrical and reciprocal.*

Fig. 13.22

Solution From Fig. 13.31,

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \ldots\text{(i)}$$

Applying KCL at Node 3,

$$I_1 = \frac{V_3}{2} + \frac{V_3 - V_2}{2}$$

$$= V_3 - \frac{V_2}{2} \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{4} + \frac{V_2 - V_3}{2}$$

$$= \frac{3}{4}V_2 - \frac{V_3}{2} \qquad \ldots\text{(iii)}$$

Substituting Eq. (i) in Eq. (ii),

$$V_1 - V_3 = V_3 - \frac{V_2}{2}$$

$$V_3 = \frac{V_1}{2} + \frac{V_2}{4} \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1 = \frac{V_1}{2} + \frac{V_2}{4} - \frac{V_2}{2}$$

$$= \frac{V_1}{2} - \frac{V_2}{4} \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2 = \frac{3}{4}V_2 - \frac{1}{2}\left(\frac{V_1}{2} + \frac{V_2}{4}\right)$$

$$= -\frac{V_1}{4} + \frac{5V_2}{8} \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{8} \end{bmatrix}$$

Since $Y_{11} \neq Y_{22}$, the network is not symmetrical.
Since $Y_{12} = Y_{21}$, the network is reciprocal.

Example 13.11 *Determine the short-circuit admittance parameters for the network shown in Fig. 13.23.*

Fig. 13.23

Solution The transformed network is shown in Fig. 13.24.
From Fig. 13.24,

$$I_1 = \frac{V_1 - V_3}{1}$$

$$= V_1 - V_3 \qquad \ldots\text{(i)}$$

Fig. 13.24

Applying KCL at Node 3,

$$I_1 = \frac{V_3}{\frac{1}{s}} + \frac{(V_3 - V_2)}{1}$$

$$= (s+1)\,V_3 - V_2 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{\frac{1}{s}} + \frac{(V_2 - V_3)}{1}$$

$$= (s+1)\,V_2 - V_3 \qquad \ldots\text{(iii)}$$

Substituting Eq. (i) in Eq. (ii),

$$V_1 - V_3 = (s+1)\,V_3 - V_2$$

$$(s+2)\,V_3 = V_1 + V_2$$

$$V_3 = \frac{1}{s+2}V_1 + \frac{1}{s+2}V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1 = (s+1)\left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right) - V_2$$

$$= \frac{s+1}{s+2}V_1 - \frac{1}{s+2}V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2 = (s+1)\,V_2 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= -\frac{1}{s+2}V_1 + \frac{s^2+3s+1}{s+2}V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{1}{s+2} \\ -\frac{1}{s+2} & \frac{s^2+3s+1}{s+2} \end{bmatrix}$$

Example 13.12 *Determine Y-parameters for the network shown in Fig. 13.25.*

Fig. 13.25

Solution The transformed network as shown in Fig. 13.26.

From Fig. 13.26,

$$I_1 = \frac{V_1 - V_3}{\frac{2}{s}}$$

$$= \frac{s}{2}V_1 - \frac{s}{2}V_3 \qquad \ldots\text{(i)}$$

Fig. 13.26

Applying KCL at Node 3,

$$\frac{s}{2}(V_1 - V_3) = \frac{V_3}{s} + \frac{s}{2}(V_3 - V_2)$$

$$\frac{s}{2}V_3 + \frac{1}{s}V_3 + \frac{s}{2}V_3 = \frac{s}{2}V_1 + \frac{s}{2}V_2$$

$$V_3 = \frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2 \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{s}{2}V_1 - \frac{s}{2}\left[\frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2\right]$$

$$= \left[\frac{s}{2} - \frac{s^3}{4(s^2+1)}\right]V_1 - \frac{s^3}{4(s^2+1)}V_2$$

$$= \frac{s^3+2s}{4(s^2+1)}V_1 - \frac{s^3}{4(s^2+1)}V_2 \qquad \ldots\text{(iii)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{s} + \frac{s}{2}(V_2 - V_3)$$

$$= \frac{s^2+2}{2s}V_2 - \frac{s}{2}V_3 \qquad \ldots\text{(iv)}$$

Substituting Eq. (ii) in Eq. (iv),

$$I_2 = \frac{s^2+2}{2s}V_2 - \frac{s}{2}\left[\frac{s^2}{2(s^2+1)}V_1 + \frac{s^2}{2(s^2+1)}V_2\right]$$

$$= -\frac{s^3}{4(s^2+1)}V_1 + \left[\frac{s^2+2}{2s} - \frac{s^3}{4(s^2+1)}\right]V_2$$

$$= -\frac{s^3}{4(s^2+1)}V_1 + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iii) and (v) with Y-parameter equation,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{s^3+2s}{4(s^2+1)} & -\frac{s^3}{4(s^2+1)} \\ -\frac{s^3}{4(s^2+1)} & \frac{s^4+6s^2+4}{4s(s^2+1)} \end{bmatrix}$$

Example 13.13 *Obtain Y-parameters of the network shown in Fig. 13.27.*

Fig. 13.27

Solution

Applying KCL at Node 1,

$$I_1 = \frac{V_1 - V_3}{1} + \frac{V_1 - V_2}{\frac{1}{s}}$$

$$= (s+1)\,V_1 - s\,V_2 - V_3 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2 - V_3}{1} + \frac{V_2 - V_1}{\frac{1}{s}}$$

$$= (s+1)\,V_2 - s\,V_1 - V_3 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3}{\frac{1}{s}} + \frac{V_3 - V_1}{1} + \frac{V_3 - V_2}{1} = 0$$

$$(s+2)\,V_3 - V_1 - V_2 = 0$$

$$V_3 = \frac{1}{s+2}V_1 + \frac{1}{s+2}V_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$I_1 = (s+1)\,V_1 - sV_2 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= \left[\frac{(s+1)(s+2)-1}{(s+2)}\right]V_1 - \left[\frac{s(s+2)+1}{(s+2)}\right]V_2$$

$$= \left(\frac{s^2+3s+1}{s+2}\right)V_1 - \left(\frac{s^2+2s+1}{s+2}\right)V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$I_2 = (s+1)V_2 - sV_1 - \left(\frac{1}{s+2}V_1 + \frac{1}{s+2}V_2\right)$$

$$= -\left[\frac{s(s+2)+1}{(s+2)}\right]V_1 + \left[\frac{(s+1)(s+2)-1}{(s+2)}\right]V_2$$

$$= -\left(\frac{s^2+2s+1}{s+2}\right)V_1 + \left(\frac{s^2+3s+1}{s+2}\right)V_2 \qquad \ldots(v)$$

Comparing Eqs (iv) and (v) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+3s+1}{s+2} & -\dfrac{(s^2+2s+1)}{s+2} \\ -\dfrac{(s^2+2s+1)}{s+2} & \dfrac{s^2+3s+1}{s+2} \end{bmatrix}$$

13.4 TRANSMISSION PARAMETERS (ABCD PARAMETERS)

The transmission parameters or chain parameters or *ABCD* parameters serve to relate the voltage and current at the input port to voltage and current at the output port. In equation form,

$$(V_1, I_1) = f(V_2, -I_2)$$

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Here, the negative sign is used with I_2 and not for parameters B and D. The reason the current I_2 carries a negative sign is that in transmission field, the output current is assumed to be coming out of the output port instead of going into the port.

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called transmission matrix.

For a given network, these parameters are determined as follows:

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

where A is the reverse voltage gain with the output port open-circuited.

Similarly,

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

where C is the transfer admittance with the output port open-circuited.

Case 2 When output port is short-circuited, i.e., $V_2 = 0$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$

where B is the transfer impedance with the output port short-circuited.

Similarly,

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

where D is the reverse current gain with the output port short-circuited.

13.4.1 Condition for Reciprocity

(a) As shown in Fig. 13.28, voltage V_s is applied at the input port with the output port short-circuited.

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

From the transmission parameter equations,

$$V_s = B\,I_2'$$
$$\frac{V_s}{I_2'} = B$$

Fig. 13.28 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 13.29, voltage V_s is applied at the output port with the input port short-circuited.

i.e.,

$$V_2 = V_s$$
$$V_1 = 0$$
$$I_1' = -I_1$$

Fig. 13.29 *Network for deriving condition for reciprocity*

From the transmission parameter equations,

$$0 = AV_s - BI_2$$
$$-I_1' = CV_s - DI_2$$
$$I_2 = \frac{A}{B}V_s$$
$$-I_1' = CV_s - \frac{AD}{B}V_s$$
$$\frac{V_s}{I_1'} = \frac{B}{AD - BC}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e.,

$$B = \frac{B}{AD - BC}$$

i.e.,

$$AD - BC = 1$$

13.4.2 Condition for Symmetry

(a) When the output port is open-circuited, i.e., $I_2 = 0$.
From the transmission-parameter equations,

$$V_s = AV_2$$
$$I_1 = CV_2$$
$$\frac{V_s}{I_1} = \frac{A}{C}$$

(b) When the input port is open-circuited, i.e., $I_1 = 0$.
From the transmission parameter equation,

$$CV_s = DI_2$$
$$\frac{V_s}{I_2} = \frac{D}{C}$$

Hence, for network to be symmetrical,

$$\frac{V_s}{I_1} = \frac{V_s}{I_2}$$

i.e., $$A = D$$

Example 13.14 *Find the transmission parameters for the network shown in Fig. 13.30.*

Fig. 13.30

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$.

$$V_1 = 6I_1$$

and $$V_2 = 5I_1$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{6I_1}{5I_1} = \frac{6}{5}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{5}\ \mho$$

Case 2 When the output port is short-circuited, i.e., $V_2 = 0$, as shown in Fig. 13.31,

$$R_{eq} = 1 + \frac{5\times 2}{5+2} = 1 + \frac{10}{7} = \frac{17}{7}\ \Omega$$

Fig. 13.31

Now $$V_1 = \frac{17}{7} I_1$$

and $$I_2 = \frac{5}{7}(-I_1) = -\frac{5}{7} I_1$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{\frac{17}{7} I_1}{-\frac{5}{7} I_1} = \frac{17}{5}\ \Omega$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{7}{5}$$

Hence, the transmission parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Second Method (Refer Fig. 13.37)
Applying KVL to Mesh 1,

$$V_1 = 6I_1 + 5I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 5I_1 + 7I_2 \qquad \ldots\text{(ii)}$$

Hence,

$$5I_1 = V_2 - 7I_2$$

$$I_1 = \frac{1}{5}V_2 - \frac{7}{5}I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 6\left(\frac{1}{5}V_2 - \frac{7}{5}I_2\right) + 5I_2$$

$$= \frac{6}{5}V_2 - \frac{17}{5}I_2 \qquad \ldots\text{(iv)}$$

Comparing Eqs (iii) and (iv) with transmission parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{17}{5} \\ \frac{1}{5} & \frac{7}{5} \end{bmatrix}$$

Example 13.15 *Obtain ABCD parameters for the network shown in Fig 13.32.*

Fig. 13.32

Solution The network is redrawn as shown in Fig 13.33.
Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - I_3 - 2(I_3 + I_2) = 0$$

$$5I_3 = 2I_1 - 2I_2$$

$$I_3 = \frac{2}{5}I_1 - \frac{2}{5}I_2 \qquad \ldots\text{(iii)}$$

Fig. 13.33

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{11}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{4}{5}I_1 + \frac{6}{5}I_2$$

$$\frac{4}{5}I_1 = V_2 - \frac{6}{5}I_2$$

$$I_1 = \frac{5}{4}V_2 - \frac{3}{2}I_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{11}{5}\left(\frac{5}{4}V_2 - \frac{3}{2}I_2\right) + \frac{4}{5}I_2$$

$$= \frac{11}{4}V_2 - \frac{5}{2}I_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{11}{4} & \frac{5}{2} \\ \frac{5}{4} & \frac{3}{2} \end{bmatrix}$$

Example 13.16 *Determine the transmission parameters for the network shown in Fig. 13.34.*

Fig. 13.34

Solution

Applying KCL at Node 1,

$$I_1 = \frac{V_1}{s} + (V_1 - V_2)$$

$$= \frac{s+1}{s}V_1 - V_2 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{\frac{1}{s}} + (V_2 - V_1)$$

$$= (s+1)V_2 - V_1$$

$$V_1 = (s+1)V_2 - I_2 \qquad \ldots\text{(ii)}$$

Substituting Eq. (ii) in Eq. (i),

$$I_1 = \frac{s+1}{s}[(s+1)V_2 - I_2] - V_2$$
$$= \left[\frac{(s+1)^2}{s} - 1\right]V_2 - \frac{s+1}{s}I_2$$
$$= \left(\frac{s^2+s+1}{s}\right)V_2 - \left(\frac{s+1}{s}\right)I_2 \qquad \ldots\text{(iii)}$$

Comparing Eqs (ii) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ \frac{s^2+s+1}{s} & \frac{s+1}{s} \end{bmatrix}$$

Example 13.17 *Find transmission parameters for the two-port network shown in Fig. 13.35.*

Fig. 13.35

Solution Applying KVL to Mesh 1,

$$V_1 = 10I_1 + 25(I_1 - I_3)$$
$$= 35I_1 - 25I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 20(I_2 + I_3)$$
$$= 20I_2 + 20I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-25(I_3 - I_1) + 1.5V_1 - 20(I_2 + I_3) = 0$$
$$-25I_3 + 25I_1 + 1.5(35I_1 - 25I_3) - 20I_2 - 20I_3 = 0$$
$$-25I_3 + 25I_1 + 52.5I_1 - 37.5I_3 - 20I_2 - 20I_3 = 0$$
$$82.5I_3 = 77.5I_1 - 20I_2$$
$$I_3 = 0.94I_1 - 0.24I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 35I_1 - 25(0.94I_1 - 0.24I_2)$$
$$= 11.5I_1 + 6I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 20I_2 + 20(0.94I_1 - 0.24I_2)$$
$$= 18.8I_1 + 15.2I_2 \qquad \ldots\text{(v)}$$

From Eq. (v),

$$I_1 = 0.053V_2 - 0.81I_2 \qquad \ldots\text{(vi)}$$

Substituting Eq. (vi) in Eq. (iv),

$$V_1 = 11.5(0.053\,V_2 - 0.81I_2) + 6I_2$$
$$= 0.61\,V_2 - 3.32I_2 \qquad \ldots\text{(vii)}$$

Comparing Eqs (vi) and (vii) with *ABCD* parameter equations,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0.61 & -3.32 \\ 0.053 & -0.81 \end{bmatrix}$$

13.5 INVERSE TRANSMISSION PARAMETERS (A′B′C′D′ PARAMETERS)

The inverse transmission parameters serve to relate the voltage and current at the outport port to the voltage and current at the input port. In equation form,

$$(V_2, I_2) = f(V_1, -I_1)$$
$$V_2 = A'\,V_1 - B'I_1$$
$$I_2 = C'\,V_1 - D'I_1$$

In matrix form, we can write

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

where matrix $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$ is called the *inverse transmission matrix*.

For a given network, these parameters are determined as follows:

Case 1 When the input port is open-circuited, i.e., $I_1 = 0$

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0}$$

where A' is the forward voltage gain with the input port open-circuited.

Similarly,

$$C' = \left.\frac{I_2}{V_1}\right|_{I_1=0}$$

where C' is the transfer admittance with the input port open-circuited.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$B' = -\left.\frac{V_2}{I_1}\right|_{V_1=0}$$

where B' is the transfer impedance with the input port short-circuited.

Similarly,

$$D' = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

where D' is the forward current gain with the input port short-circuited.

13.5.1 Condition for Reciprocity

(a) As shown in Fig 13.36, voltage V_s is applied at input port with the output port short-circuited.

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2 = -I_2'$$

Fig. 13.36 *Network for deriving condition for reciprocity*

From the inverse transmission parameter equations,

$$0 = A'\,V_s - B'\,I_1$$

$$-I_2' = C'\,V_s - D'\,I_1$$

$$\frac{I_2'}{V_s} = \frac{A'D' - B'C'}{B'}$$

(b) As shown in Fig 13.37, voltage V_s is applied at the output port with the input port short-circuited.

i.e.,

$$V_2 = V_s$$

$$V_1 = 0$$

$$I_1 = -I_1'$$

Fig. 13.37 *Network for deriving condition for reciprocity*

From the inverse transmission parameter equations,

$$V_s = B'I_1'$$

$$I_2 = \frac{A'}{B'}V_s$$

$$\frac{I_1'}{V_s} = \frac{1}{B'}$$

Hence, for the network to be reciprocal,

$$\frac{I_2'}{V_s} = \frac{I_1'}{V_s}$$

i.e.,

$$A'D' - B'C' = 1$$

13.5.2 Condition for Symmetry

The condition for symmetry is obtained from the Z-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 0 = \frac{D'}{C'}$$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 0 = \frac{A'}{C'}$$

For symmetrical network

$$Z_{11} = Z_{22}$$

$$A' = D'$$

Example 13.18 *Find the inverse transmission parameters for the network shown in Fig. 13.38.*

Fig. 13.38

Solution

First method

Case 1 When the input port is open-circuited, i.e., $I_1 = 0$

$$V_1 = 3I_2$$

$$V_2 = 5I_2$$

$$A' = \left.\frac{V_2}{V_1}\right|_{I_1=0} = \frac{5}{3}$$

$$C' = \left.\frac{I_2}{V_1}\right|_{I_1=0} = \frac{1}{3}\ \mho$$

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 13.39.
By current division rule,

Fig. 13.39

$$I_1 = -\frac{3}{4} I_2$$

$$D' = -\left.\frac{I_2}{I_1}\right|_{V_1=0} = \frac{4}{3}$$

$$R_{eq} = 2 + \frac{3\times 1}{3+1} = 2 + \frac{3}{4} = \frac{11}{4}\ \Omega$$

$$V_2 = \frac{11}{4} I_2 = \frac{11}{4}\left(-\frac{4}{3}\right) I_1 = -\frac{11}{3} I_1$$

$$B' = -\left.\frac{V_2}{I_1}\right|_{V_1=0} = \frac{11}{3}\Omega$$

Second Method Refer Fig. 13.38.
Applying KVL to Mesh 1,

$$V_1 = 4I_1 + 3I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 3I_1 + 5I_2 \qquad \text{...(ii)}$$

Hence,

$$3I_2 = V_1 - 4I_1$$

$$I_2 = \frac{1}{3}V_1 - \frac{4}{3}I_1 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 3I_1 + 5\left(\frac{1}{3}V_1 - \frac{4}{3}I_1\right)$$

$$= \frac{5}{3}V_1 - \frac{11}{3}I_1 \qquad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with inverse transmission parameter equations,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{11}{3} \\ \frac{1}{3} & \frac{4}{3} \end{bmatrix}$$

Example 13.19 *Find the inverse transmission parameters for the network shown in Fig. 13.40*

Fig. 13.40

Solution The network is redrawn as shown in Fig 13.41.

Applying KVL to Mesh 1,

$$V_1 = 2I_1 - 2I_3 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2I_1 + 2I_2 + 6I_3 = 0$$

$$I_3 = \frac{1}{3}I_1 - \frac{1}{3}I_2 \quad \ldots\text{(iii)}$$

Fig. 13.41

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 2I_1 - 2\left(\frac{1}{3}I_1 - \frac{1}{3}I_2\right)$$

$$= \frac{4}{3}I_1 + \frac{2}{3}I_2 \quad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{1}{3}I_1 - \frac{1}{3}I_2\right)$$

$$= \frac{2}{3}I_1 + \frac{4}{3}I_2 \quad \ldots\text{(v)}$$

Rewriting Eq. (iv),

$$\frac{2}{3}I_2 = V_1 - \frac{4}{3}I_1$$

$$I_2 = \frac{3}{2}V_1 - 2I_1 \quad \ldots\text{(vi)}$$

Substituting Eq. (vi) in Eq. (v),

$$V_2 = \frac{2}{3}I_1 + \frac{4}{3}\left(\frac{3}{2}V_1 - 2I_1\right)$$

$$= 2V_1 - 2I_1 \quad \ldots\text{(vii)}$$

Comparing Eq. (vi) and (vii) with inverse transmission parameter equations,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} & 2 \end{bmatrix}$$

Example 13.20 *Find the inverse transmission parameters for the network shown in Fig. 13.42.*

Fig. 13.42

Solution The network is redrawn as shown in Fig. 13.43.

Fig. 13.43

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = I_2 + I_3 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2I_1 + I_2 + 4I_3 = 0$$

$$I_3 = \frac{1}{2}I_1 - \frac{1}{4}I_2 \quad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{1}{2}I_1 - \frac{1}{4}I_2\right)$$

$$= 2I_1 + \frac{1}{2}I_2 \quad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = I_2 + \frac{1}{2}I_1 - \frac{1}{4}I_2$$

$$= \frac{1}{2}I_1 + \frac{3}{4}I_2 \quad \ldots\text{(v)}$$

Rewriting Eq. (iv),

$$\frac{1}{2}I_2 = V_1 - 2I_1$$

$$I_2 = 2V_1 - 4I_1 \quad \ldots\text{(vi)}$$

Substituting the Eq. (vi) in Eq. (v),

$$V_2 = \frac{1}{2}I_1 + \frac{3}{4}(2V_1 - 4I_1)$$

$$= \frac{3}{2}V_1 - \frac{5}{2}I_1 \quad \ldots\text{(vii)}$$

Comparing Eqs (vi) and (vii) with inverse transmission parameter equations,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ 2 & 4 \end{bmatrix}$$

13.6 HYBRID PARAMETERS (*h* PARAMETERS)

The hybrid parameters of a two-port network may be defined by expressing the voltage of input port V_1 and current of output port I_2 in terms of current of input port I_1 and voltage of output port V_2.

$$(V_1, I_2) = f(I_1, V_2)$$

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$

$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

In matrix form, we can write

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The individual *h* parameters can be defined by setting $I_1 = 0$ and $V_2 = 0$.

Case 1 When the output port is short-circuited i.e., $V_2 = 0$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

where h_{11} is the short-circuit input impedance.

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

where h_{21} is the short-circuit forward current gain.

Case 2 When the input port is open-circuited, i.e., $I_1 = 0$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

where h_{12} is the open-circuit reverse voltage gain.

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

where h_{22} is the open-circuit output admittance.

Since h parameters represent dimensionally an impedance, an admittance, a voltage gain and a current gain, these are called *hybrid parameters*.

The equivalent circuit of a two-port network in terms of hybrid parameters is shown in Fig 13.44.

Fig. 13.44 *Equivalent circuit of the two-port network in terms of h-parameters*

13.6.1 Condition for Reciprocity

(a) As shown in Fig. 13.45, voltage V_s is applied at the input port and the output port is short-circuited.

i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

From the h-parameter equations,

$$V_s = h_{11}\, I_1$$
$$-I_2' = h_{21}\, I_1$$
$$\frac{V_s}{I_2'} = -\frac{h_{11}}{h_{21}}$$

Fig. 13.45 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 13.46, voltage V_s is applied at the output port with the input port short-circuited.

i.e.,

$$V_1 = 0$$
$$V_2 = V_s$$
$$I_1 = -I_1'$$

Fig. 13.46 *Network for deriving condition for reciprocity*

From the h-parameter equations,

$$0 = h_{11} I_1 + h_{12} V_s$$

$$h_{12} V_s = -h_{11} I_1 = h_{11} I_1'$$

$$\frac{V_s}{I_1'} = \frac{h_{11}}{h_{12}}$$

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e., $$h_{21} = -h_{12}$$

13.6.2 Condition for Symmetry

The condition for symmetry is obtained from the Z-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \left.\frac{h_{11} I_1 + h_{12} V_2}{I_1}\right|_{I_2=0} = h_{11} + h_{12}\frac{V_2}{I_1}$$

But with $I_2 = 0$,

$$0 = h_{21} I_1 + h_{22} V_2$$

$$\frac{V_2}{I_1} = -\frac{h_{21}}{h_{22}}$$

$$Z_{11} = h_{11} - \frac{h_{12} h_{21}}{h_{22}} = \frac{h_{11} h_{22} - h_{12} h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$$

where $$\Delta h = h_{11} h_{22} - h_{12} h_{21}$$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

With $I_1 = 0$,

$$I_2 = h_{22} V_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{1}{h_{22}}$$

For a symmetrical network, $$Z_{11} = Z_{22}$$

i.e., $$\frac{\Delta h}{h_{22}} = \frac{1}{h_{22}}$$

i.e., $$\Delta h = 1$$

i.e., $$h_{11} h_{22} - h_{12} h_{21} = 1$$

Example 13.21 *In the two-port network shown in Fig. 13.47, compute h-parameters from the following data:*

(a) With the output port short-circuited: $V_1 = 25\ V$, $I_1 = 1\ A$, $I_2 = 2\ A$

(b) With the input port open-circuited: $V_1 = 10\ V$, $V_2 = 50\ V$, $I_2 = 2\ A$

Fig. 13.47

Solution

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{25}{1} = 25\ \Omega, \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{10}{50} = 0.2$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{2}{1} = 2, \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2}{50} = 0.04\ \mho$$

Hence, the h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 25 & 0.2 \\ 2 & 0.04 \end{bmatrix}$$

Example 13.22 *Determine hybrid parameters for the network of Fig. 13.48. Determine whether the network is reciprocal.*

Fig. 13.48

Solution

First Method

Case 1 When Port 2 is short-circuited, i.e., $V_2 = 0$ as shown in Fig. 13.49,

$$R_{eq} = 1 + \frac{2\times 2}{2+2} = 2\ \Omega$$

Now,

$$V_1 = 2I_1$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 2\ \Omega$$

Also,

$$I_2 = -I_1 \times \frac{2}{2+2} = -\frac{I_1}{2}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\frac{1}{2}$$

Fig. 13.49

Case 2 When Port 1 is open-circuited, i.e., $I_1 = 0$ as shown in Fig. 13.50,

$$R_{eq} = \frac{(2+2)\times 4}{2+2+4} = 2\ \Omega$$

$$V_1 = 2I_y$$

$$I_y = \frac{I_2}{2}$$

$$V_2 = 4I_x$$

$$I_x = \frac{I_2}{2}$$

Fig. 13.50

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{2I_y}{4I_x} = \frac{2\times\frac{I_2}{2}}{4\times\frac{I_2}{2}} = \frac{1}{2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2I_x}{4I_x} = \frac{1}{2}\ \mho$$

Hence, the h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Second Method (Refer Fig. 13.48)
Applying KVL to Mesh 1,

$$V_1 = 3I_1 - 2I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 4I_2 + 4I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-2\,(I_3 - I_1) - 2I_3 - 4\,(I_3 + I_2) = 0$$

$$8I_3 = 2I_1 - 4I_2$$

$$I_3 = \frac{I_1}{4} - \frac{I_2}{2} \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - 2\left(\frac{I_1}{4} - \frac{I_2}{2}\right)$$

$$= \frac{5}{2}I_1 + I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 4I_2 + 4\left(\frac{I_1}{4} - \frac{I_2}{2}\right)$$

$$= 4I_2 + I_1 - 2I_2$$

$$= I_1 + 2I_2$$

$$I_2 = -\frac{1}{2}I_1 + \frac{1}{2}V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{5}{2}I_1 - \frac{1}{2}I_1 + \frac{1}{2}V_2$$

$$= 2I_1 + \frac{1}{2}V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with h-parameter equations,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Since $h_{12} = -h_{21}$, the network is reciprocal.

Example 13.23 *Find h-parameters for the network shown in Fig. 13.51.*

Fig. 13.51

Solution As solved in Example 13.12, derive the equations for I_1 and I_2 in terms of V_1 and V_2.

$$I_1 = \frac{s^3+2s}{4(s^2+1)}V_1 - \frac{s^3}{4(s^2+1)}V_2 \qquad \text{...(i)}$$

$$I_2 = -\frac{s^3}{4(s^2+1)}V_1 + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2 \qquad \text{...(ii)}$$

From Eq. (i),

$$V_1 = \frac{4(s^2+1)}{s(s^2+2)}I_1 + \frac{s^2}{s^2+2}V_2 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq (ii),

$$I_2 = -\frac{s^3}{4(s^2+1)}\left[\frac{4(s^2+1)}{s(s^2+2)}I_1 + \frac{s^2}{s^2+2}V_2\right] + \frac{s^4+6s^2+4}{4s(s^2+1)}V_2$$

$$= -\frac{s^2}{s^2+2}I_1 + \frac{2(s^2+1)}{s(s^2+2)}V_2 \qquad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with h-parameter equations,

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{4(s^2+1)}{s(s^2+2)} & \dfrac{s^2}{s^2+2} \\ -\dfrac{s^2}{s^2+2} & \dfrac{2(s^2+1)}{s(s^2+2)} \end{bmatrix}$$

13.7 INVERSE HYBRID PARAMETERS (*g* PARAMETERS)

The inverse hybrid parameters of a two-port network may be defined by expressing the current of the input port I_1 and voltage of the output port V_2 in terms of the voltage of the input port V_1 and the current of the output port I_2.

$$(I_1, V_2) = f(V_1, I_2)$$

$$I_1 = g_{11}V_1 + g_{12}I_2$$

$$V_2 = g_{21}V_1 + g_{22}I_2$$

In matrix form, we can write

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

The individual g parameters can be defined by setting $V_1 = 0$ and $I_2 = 0$.

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0}$$

where g_{11} is the open-circuit input admittance.

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

where g_{21} is the open-circuit forward voltage gain.

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

where g_{12} is the short-circuit reverse current gain.

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

where g_{22} is the short-circuit output impedance.

The equivalent circuit of a two-port network in terms of inverse hybrid parameters is shown in Fig. 13.52.

Fig. 13.52 *Equivalent circuit of two-port network in terms of g-parameters*

13.7.1 Condition for Reciprocity

(a) As shown in Fig. 13.53, voltage V_s is applied at the input port and the output port is short-circuited, i.e.,

$$V_1 = V_s$$
$$V_2 = 0$$
$$I_2' = -I_2$$

From the g-parameter equation,

$$g_{21}\, V_s = g_{22}\, I_2'$$
$$\frac{V_s}{I_2'} = \frac{g_{22}}{g_{21}}$$

Fig. 13.53 *Network for deriving condition for reciprocity*

(b) As shown in Fig. 13.54, voltage V_s is applied at the output port with the input port short-circuited, i.e.,

$$V_1 = 0$$
$$V_2 = V_s$$
$$I_1 = -I_1'$$

From the g-parameter equations,

$$-I_1' = g_{12}\, I_2$$
$$V_s = g_{22}\, I_2$$
$$\frac{V_s}{I_1'} = -\frac{g_{22}}{g_{12}}$$

Fig. 13.54 *Network for deriving condition for reciprocity*

Hence, for the network to be reciprocal,

$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

i.e.,

$$g_{21} = -g_{12}$$

13.7.2 Condition for Symmetry

The condition for symmetry is obtained from the Z-parameters.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{1}{g_{11}}$$

Similarly,

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$0 = g_{11}\, V_1 + g_{12}\, I_2$$

$$V_1 = -\frac{g_{12}}{g_{11}} I_2$$

$$V_2 = g_{21}\left(-\frac{g_{12}}{g_{11}}\right) I_2 + g_{22}\, I_2$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{g_{11}g_{22} - g_{12}g_{21}}{g_{11}}$$

For a symmetrical network, $Z_{11} = Z_{22}$.

i.e.,

$$\frac{1}{g_{11}} = \frac{g_{11}g_{22} - g_{12}g_{21}}{g_{11}}$$

i.e.,

$$g_{11}g_{22} - g_{12}g_{21} = 1$$

Table 13.2 *Conditions for reciprocity and symmetry*

Parameter	Condition for Reciprocity	Condition for Symmetry
Z	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
Y	$Y_{12} = Y_{21}$	$Y_{11} = Y_{22}$
T	$AD - BC = 1$	$A = D$
T'	$A'D' - B'C' = 1$	$A' = D'$
h	$h_{12} = -h_{21}$	$h_{11}\, h_{22} - h_{12}\, h_{21} = 1$
g	$g_{12} = -g_{21}$	$g_{11}\, g_{22} - g_{12}\, g_{21} = 1$

Example 13.24 *Find g-parameters for the network shown in Fig. 13.55.*

Fig. 13.55

Solution

First Method

Case 1 When the output port is open-circuited, i.e., $I_2 = 0$

Applying KVL to Mesh 1,

$$V_1 = 4\,I_1$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \frac{1}{4}\ \mho$$

Also

$$V_2 = 3\,I_1$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{3\,I_1}{4\,I_1} = \frac{3}{4}$$

Case 2 When the input port is short-circuited, i.e., $V_1 = 0$ as shown in Fig. 13.56.

By current division rule,

$$I_1 = -\frac{3}{4}\,I_2$$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} = -\frac{3}{4}$$

$$R_{eq} = 2 + \frac{3\times 1}{3+1} = 2 + \frac{3}{4} = \frac{11}{4}\ \Omega$$

$$V_2 = \frac{11}{4}\,I_2$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \frac{11}{4}\ \Omega$$

Fig. 13.56

Second Method (Refer Fig. 13.55).

Applying KVL to Mesh 1,

$$V_1 = 4\,I_1 + 3\,I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 3\,I_1 + 5\,I_2 \qquad \text{...(ii)}$$

Hence,

$$4\,I_1 = V_1 - 3\,I_2$$

$$I_1 = \frac{1}{4}V_1 - \frac{3}{4}I_2 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 3\left(\frac{1}{4}V_1 - \frac{3}{4}I_2\right) + 5\,I_2$$

$$= \frac{3}{4}V_1 + \frac{11}{4}I_2 \qquad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with g-parameter equations,

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} \end{bmatrix}$$

Example 13.25 *Find g-parameters for the network shown in Fig. 13.57.*

Fig. 13.57

Solution The network is redrawn using source transformation as shown in Fig. 13.58.

Applying KVL to Mesh 1,

$$V_1 - 1\,I_1 - 1\,(I_1 + I_2) = 0$$

$$V_1 = 2\,I_1 + I_2 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 5\,I_1 - 1\,I_2 - 1\,(I_2 + I_1) = 0$$

$$V_2 = 6\,I_1 + 2\,I_2 \quad \ldots\text{(ii)}$$

Fig. 13.58

Hence, $2\,I_1 = V_1 - I_2$

$$I_1 = \frac{1}{2}V_1 - \frac{1}{2}I_2 \quad \ldots\text{(iii)}$$

Substituting Eq (iii) in Eq. (ii),

$$V_2 = 6\left(\frac{1}{2}V_1 - \frac{1}{2}I_2\right) + 2\,I_2$$

$$= 3\,V_1 - I_2 \quad \ldots\text{(iv)}$$

Comparing Eqs (iii) and (iv) the *g*- parameter equations,

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

13.8 INTER-RELATIONSHIPS BETWEEN THE PARAMETERS

When it is required to find out two or more parameters of a particular network then finding each parameter will be tedious. But if we find a particular parameter then the other parameters can be found if the inter-relationship between them is known.

13.8.1 Z-parameters in Terms of Other Parameters

1. Z-parameters in Terms of Y-parameters We know that

$$I_1 = Y_{11}\,V_1 + Y_{12}\,V_2$$

$$I_2 = Y_{21}\,V_1 + Y_{22}\,V_2$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix}}{\begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}} = \frac{Y_{22}I_1 - Y_{12}I_2}{Y_{11}Y_{22} - Y_{12}Y_{21}} = \frac{Y_{22}}{\Delta Y}I_1 - \frac{Y_{12}}{\Delta Y}I_2$$

where $\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21}$

Comparing with $V_1 = Z_{11}I_1 + Z_{12}, I_2,$

$$Z_{11} = \frac{Y_{22}}{\Delta Y}$$

$$Z_{12} = -\frac{Y_{12}}{\Delta Y}$$

Also, $$V_2 = \frac{\begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix}}{\Delta Y} = \frac{Y_{11}}{\Delta Y} I_2 - \frac{Y_{21}}{\Delta Y} I_1$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$

$$Z_{22} = \frac{Y_{11}}{\Delta Y}$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y}$$

2. *Z-parameter in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the second equation,

$$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2$$

Comparing with $V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$

$$Z_{21} = \frac{1}{C}$$

$$Z_{22} = \frac{D}{C}$$

Also, $$V_1 = A\left[\frac{1}{C} I_1 + \frac{D}{C} I_2\right] - BI_2 = \frac{A}{C} I_1 + \left[\frac{AD}{C} - B\right] I_2 = \frac{A}{C} I_1 + \left[\frac{AD - BC}{C}\right] I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2,$

$$Z_{11} = \frac{A}{C}$$

$$Z_{12} = \frac{AD - BC}{C}$$

3. *Z-parameters in Terms of A′B′C′D′ Parameters* We know that

$$V_2 = A'V_1 - B'\, I_1$$

$$I_2 = C'V_1 - D'\, I_1$$

Rewriting the second equation,

$$V_1 = \frac{D'}{C'} I_1 + \frac{1}{C'} I_2$$

Comparing with $V_1 = Z_{11}\, I_1 + Z_{12}\, I_2,$

$$Z_{11} = \frac{D'}{C'}$$

$$Z_{12} = \frac{1}{C'}$$

Also,
$$V_2 = A'\left[\frac{D'}{C'} I_1 + \frac{1}{C'} I_2\right] - B' I_1 = \left[\frac{A'D' - B'C'}{C'}\right] I_1 + \frac{A'}{C'} I_2$$

Comparing with
$$V_2 = Z_{21}\ I_1 + Z_{22}\ I_2,$$

$$Z_{21} = \frac{A'D' - B'C'}{C'} = \frac{\Delta T'}{C'}$$

$$Z_{22} = \frac{A'}{C'}$$

4. *Z-parameters in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$
$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

Rewriting the second equation,

$$V_2 = -\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2$$

Comparing with

$$V_2 = Z_{21}\ I_1 + Z_{22}\ I_2,$$

$$Z_{21} = -\frac{h_{21}}{h_{22}}$$

$$Z_{22} = \frac{1}{h_{22}}$$

Also, $$V_1 = h_{11}\, I_1 + h_{12}\left[-\frac{h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2\right] = h_{11}\, I_1 + \frac{h_{12}}{h_{22}} I_2 - \frac{h_{12}h_{21}}{h_{22}} I_1 = \left[\frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}\right] I_1 + \frac{h_{12}}{h_{22}} I_2$$

Comparing with
$$V_1 = Z_{11}\ I_1 + Z_{12}\ I_2,$$

$$Z_{11} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}}$$

$$Z_{12} = \frac{h_{12}}{h_{22}}$$

5. *Z-parameters in Terms of Inverse Hybrid Parameters* We know that

$$I_1 = g_{11}\, V_1 + g_{12}\, I_2$$
$$V_2 = g_{21}\, V_1 + g_{22}\, I_2$$

Rewriting the first equation,

$$V_1 = \frac{1}{g_{11}} I_1 - \frac{g_{12}}{g_{11}} I_2$$

Comparing with
$$V_1 = Z_{11}\, I_1 + Z_{12}\ I_2,$$

$$Z_{11} = \frac{1}{g_{11}}$$

$$Z_{12} = -\frac{g_{12}}{g_{11}}$$

Also
$$V_2 = g_{21}\left[\frac{1}{g_{11}} I_1 - \frac{g_{12}}{g_{11}} I_2\right] + g_{22}\ I_2 = \frac{g_{21}}{g_{11}} I_1 + \left[\frac{g_{11}g_{22} - g_{12}g_{21}}{g_{11}}\right] I_2$$

Comparing with $$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2,$$

$$Z_{21} = \frac{g_{21}}{g_{11}}$$

$$Z_{22} = \frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{11}} = \frac{\Delta g}{g_{11}}$$

13.8.2 *Y*-parameters in Terms of Other Parameters

1. Y-parameters in terms of Z-parameters We know that

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

By Cramer's rule,

$$I_1 = \frac{\begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix}}{\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}} = \frac{Z_{22}\, V_1 - Z_{12}\, V_2}{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}} = \frac{Z_{22}}{\Delta Z} V_1 - \frac{Z_{12}}{\Delta Z} V_2$$

where $$\Delta Z = Z_{11}\, Z_{22} - Z_{12}\, Z_{21}$$

Comparing with $$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2,$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z}$$

$$Y_{12} = -\frac{Z_{12}}{\Delta Z}$$

Also, $$I_2 = \frac{\begin{vmatrix} Z_{11} & V_1 \\ Z_{21} & V_2 \end{vmatrix}}{\Delta Z} = \frac{Z_{11} V_2 - Z_{12} V_1}{\Delta Z} = -\frac{Z_{21}}{\Delta Z} V_1 + \frac{Z_{11}}{\Delta Z} V_2$$

Comparing with $$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2,$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z}$$

$$Y_{22} = \frac{Z_{11}}{\Delta Z}$$

2. Y-parameters in Terms of ABCD Parameters We know that

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Rewriting the first equation,

$$I_2 = -\frac{1}{B} V_1 + \frac{A}{B} V_2$$

Comparing with $$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2,$$

$$Y_{21} = -\frac{1}{B}$$

$$Y_{22} = \frac{A}{B}$$

Also, $$I_1 = CV_2 - D\left[-\frac{1}{B} V_1 + \frac{A}{B} V_2\right] = \frac{D}{B} V_1 + \left[\frac{BC - AD}{B}\right] V_2$$

Comparing with $$I_1 = Y_{11} V_1 + Y_{12} V_2,$$

$$Y_{11} = \frac{D}{B}$$

$$Y_{12} = \frac{BC - AD}{B} = -\frac{AD - BC}{B} = -\frac{\Delta T}{B}$$

3. *Y-parameters in Terms of A′B′C′D′ Parameters* We know that

$$V_2 = A'V_1 - B'I_1$$

$$I_2 = C'V_1 - D'I_1$$

Rewriting the first equation,

$$I_1 = \frac{A'}{B'}V_1 - \frac{1}{B'}V_2$$

Comparing with $$I_1 = Y_{11} V_1 + Y_{12} V_2,$$

$$Y_{11} = \frac{A'}{B'}$$

$$Y_{12} = -\frac{1}{B'}$$

Also, $$I_2 = C'V_1 - D'\left[\frac{A'}{B'}V_1 - \frac{1}{B'}V_2\right] = -\left[\frac{A'D' - B'C'}{B'}\right]V_1 + \frac{D'}{B'}V_2$$

Comparing with $$I_2 = Y_{21} V_1 + Y_{22} V_2,$$

$$Y_{21} = -\left[\frac{A'D' - B'C'}{B'}\right] = -\frac{\Delta T'}{B'}$$

and $$Y_{22} = \frac{D'}{B'}$$

4. *Y-parameters in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

Rewriting the first equation,

$$I_1 = \frac{1}{h_{11}} V_1 - \frac{h_{12}}{h_{11}} V_2$$

Comparing with $$I_1 = Y_{11} V_1 + Y_{12} V_2,$$

$$Y_{11} = \frac{1}{h_{11}}$$

$$Y_{12} = -\frac{h_{12}}{h_{11}}$$

Also $$I_2 = h_{21}\left[\frac{1}{h_{11}}V_1 - \frac{h_{12}}{h_{11}}V_2\right] + h_{22}V_2 = \frac{h_{21}}{h_{11}}V_1 + \left[\frac{h_{11}\,h_{22} - h_{12}\,h_{21}}{h_{11}}\right]V_2$$

Comparing with $$I_2 = Y_{21} V_1 + Y_{22} V_2,$$

$$Y_{21} = \frac{h_{21}}{h_{11}}$$

$$Y_{22} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}} = \frac{\Delta h}{h_{11}}$$

5. Y-parameters in Terms of Inverse Hybrid Parameters We know that

$$I_1 = g_{11} V_1 + g_{12} I_2$$
$$V_2 = g_{21} V_1 + g_{22} I_2$$

Rewriting the second equation,

$$I_2 = -\frac{g_{21}}{g_{22}} V_1 + \frac{1}{g_{22}} V_2$$

Comparing with $I_2 = Y_{21} V_1 + Y_{22} V_2$,

$$Y_{21} = -\frac{g_{21}}{g_{22}}$$

$$Y_{22} = \frac{1}{g_{22}}$$

Also,

$$I_1 = g_{11} V_1 + g_{12}\left[-\frac{g_{21}}{g_{22}} V_1 + \frac{1}{g_{22}} V_2\right] = \left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{22}}\right] V_1 + \frac{g_{12}}{g_{22}} V_2$$

Comparing with $I_1 = Y_{11} V_1 + Y_{12} V_2$,

$$Y_{11} = \frac{g_{11} g_{22} - g_{12} g_{21}}{g_{22}} = \frac{\Delta g}{g_{22}}$$

$$Y_{12} = \frac{g_{12}}{g_{22}}$$

13.8.3 *ABCD* Parameters in Terms of Other Parameters

1. ABCD Parameters in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Rewriting the second equation,

$$I_1 = \frac{1}{Z_{21}} V_2 - \frac{Z_{22}}{Z_{21}} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{1}{Z_{21}}$$

$$D = \frac{Z_{22}}{Z_{21}}$$

Also,

$$V_1 = Z_{11}\left[\frac{1}{Z_{21}} V_2 - \frac{Z_{22}}{Z_{21}} I_2\right] + Z_{12} I_2 = \frac{Z_{11}}{Z_{21}} V_2 - \frac{Z_{22} Z_{11}}{Z_{21}} I_2 + Z_{12} I_2$$

$$= \frac{Z_{11}}{Z_{21}} V_2 - \left[\frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{21}}\right] I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{Z_{11}}{Z_{21}}$$

$$B = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{21}} = \frac{\Delta Z}{Z_{21}}$$

2. *ABCD Parameters in terms of Y-parameters* We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the second equation,

$$V_1 = -\frac{Y_{22}}{Y_{21}} V_2 + \frac{1}{Y_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = -\frac{Y_{22}}{Y_{21}}$$
$$B = -\frac{1}{Y_{21}}$$

Also,

$$I_1 = Y_{11}\left[-\frac{Y_{22}}{Y_{21}} V_2 + \frac{1}{Y_{21}} I_2\right] + Y_{12} V_2 = \left[\frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{21}}\right] V_2 + \frac{Y_{11}}{Y_{21}} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{21}} = -\frac{\Delta Y}{Y_{21}}$$
$$D = -\frac{Y_{11}}{Y_{21}}$$

3. *ABCD Parameters in terms of A′B′C′D′ Parameters* We know that

$$V_2 = A'V_1 - B'I_1$$
$$I_2 = C'V_1 - D'I_1$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} V_2 & B' \\ I_2 & D' \end{vmatrix}}{\begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix}} = \frac{D'}{\Delta T'} V_2 - \frac{B'}{\Delta T'} I_2$$

where $\Delta T' = A'D' - B'C'$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{D'}{\Delta T'}$$
$$B = \frac{B'}{\Delta T'}$$

Also,

$$-I_1 = \frac{\begin{vmatrix} A' & V_2 \\ C' & I_2 \end{vmatrix}}{\Delta T'} = -\frac{C'}{\Delta T'} V_2 + \frac{A'}{\Delta T'} I_2$$
$$I_1 = \frac{C'}{\Delta T'} V_2 - \frac{A'}{\Delta T'} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{C'}{\Delta T'}$$
$$D = \frac{A'}{\Delta T'}$$

4. ABCD Parameters in Terms of Hybrid Parameters We know that

$$V_1 = h_{11}\, I_1 + h_{12}\, V_2$$
$$I_2 = h_{21}\, I_1 + h_{22}\, V_2$$

Rewriting the second equation,

$$I_1 = -\frac{h_{22}}{h_{21}} V_2 + \frac{1}{h_{21}} I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = -\frac{h_{22}}{h_{21}}$$

$$D = -\frac{1}{h_{21}}$$

Also,

$$V_1 = h_{11}\left[\frac{1}{h_{21}} I_2 - \frac{h_{22}}{h_{21}} V_2\right] + h_{12}\, V_2 = \left[\frac{h_{12}\, h_{21} - h_{11}\, h_{22}}{h_{21}}\right] V_2 + \frac{h_{11}}{h_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{h_{12}\, h_{21} - h_{11}\, h_{22}}{h_{21}} = -\frac{\Delta h}{h_{21}}$$

$$B = -\frac{h_{11}}{h_{21}}$$

5. ABCD Parameters in Terms of Inverse Hybrid Parameters We know that

$$I_1 = g_{11}\, V_1 + g_{12}\, I_2$$
$$V_2 = g_{21}\, V_1 + g_{22}\, I_2$$

Rewriting the second equation,

$$V_1 = \frac{1}{g_{21}} V_2 - \frac{g_{22}}{g_{21}} I_2$$

Comparing with $V_1 = AV_2 - BI_2$,

$$A = \frac{1}{g_{21}}$$

$$B = \frac{g_{22}}{g_{21}}$$

Also,

$$I_1 = g_{11}\left[\frac{1}{g_{21}} V_2 - \frac{g_{22}}{g_{21}} I_2\right] + g_{12}\, I_2 = \frac{g_{11}}{g_{21}} V_2 - \left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{21}}\right] I_2$$

Comparing with $I_1 = CV_2 - DI_2$,

$$C = \frac{g_{11}}{g_{21}}$$

$$D = \frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{21}} = \frac{\Delta g}{g_{21}}$$

13.8.4 *A'B'C'D'* Parameters in Terms of Other Parameters

1. A'B'C'D' Parameters in Terms of Z-parameters We know that

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$
$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Rewriting the first equation,

$$I_2 = \frac{1}{Z_{12}} V_1 - \frac{Z_{11}}{Z_{12}} I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{1}{Z_{12}}$$

$$D' = \frac{Z_{11}}{Z_{12}}$$

Also,

$$V_2 = Z_{21} I_1 + Z_{22}\left[\frac{1}{Z_{12}} V_1 - \frac{Z_{11}}{Z_{12}} I_1\right] = Z_{12} I_1 + \frac{Z_{22}}{Z_{12}} V_1 - \frac{Z_{22} Z_{11}}{Z_{12}} I_1$$

$$= \frac{Z_{22}}{Z_{12}} V_1 - \left[\frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{12}}\right] I_1$$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = \frac{Z_{22}}{Z_{12}}$$

$$B' = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{12}} = \frac{\Delta Z}{Z_{12}}$$

2. *A'B'C'D' Parameters in Terms of Y-parameters* We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the first equation,

$$V_2 = -\frac{Y_{11}}{Y_{12}} V_1 + \frac{1}{Y_{12}} I_1$$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = -\frac{Y_{11}}{Y_{12}}$$

$$B' = -\frac{1}{Y_{12}}$$

Also,

$$I_2 = Y_{21} V_1 + Y_{22}\left[-\frac{Y_{11}}{Y_{12}} V_1 + \frac{1}{Y_{12}} I_1\right] = \left[\frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{12}}\right] V_1 + \frac{Y_{22}}{Y_{12}} I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{Y_{12} Y_{21} - Y_{11} Y_{22}}{Y_{12}} = -\frac{\Delta Y}{Y_{12}}$$

$$D' = -\frac{Y_{22}}{Y_{12}}$$

3. *A' B' C' D' Parameters in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

By Cramer's rule,

$$V_2 = \frac{\begin{vmatrix} V_1 & B \\ I_1 & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{D}{\Delta T} V_2 - \frac{B}{\Delta T} I_2$$

where $\Delta T = AD - BC$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = \frac{D}{\Delta T}$$

$$B' = \frac{B}{\Delta T}$$

Also,

$$-I_2 = \frac{\begin{vmatrix} A & V_1 \\ C & I_1 \end{vmatrix}}{\Delta T} = -\frac{C}{\Delta T} V_1 + \frac{A}{\Delta T} I_1$$

$$I_2 = \frac{C}{\Delta T} V_1 - \frac{A}{\Delta T} I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{C}{\Delta T}$$

$$D' = \frac{A}{\Delta T}$$

4. *A′ B′ C′ D′ Parameters in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

Rewriting the first equation,

$$V_2 = \frac{1}{h_{12}} V_1 - \frac{h_{11}}{h_{12}} I_1$$

Comparing with $V_2 = A'V_1 - B'I_1$,

$$A' = \frac{1}{h_{12}}$$

$$B' = \frac{h_{11}}{h_{12}}$$

Also,

$$I_2 = h_{21} I_1 + h_{22}\left[\frac{1}{h_{12}} V_1 - \frac{h_{11}}{h_{12}} I_1\right] = \frac{h_{22}}{h_{12}} V_1 + \left[\frac{h_{11} h_{22} - h_{12} h_{21}}{h_{12}}\right] I_1$$

Comparing with $I_2 = C'V_1 - D'I_1$,

$$C' = \frac{h_{22}}{h_{12}}$$

$$D' = \left[\frac{h_{11} h_{22} - h_{12} h_{21}}{h_{12}}\right] = \frac{\Delta h}{h_{12}}$$

5. *A′ B′ C′ D′ parameters in Terms of Inverse Hybrid Parameters* We know that

$$I_1 = g_{11} V_1 + g_{12} I_2$$

$$V_2 = g_{21} V_1 + g_{22} I_2$$

Rewriting the first equation,

$$I_2 = -\frac{g_{11}}{g_{12}} V_1 + \frac{1}{g_{12}} I_1$$

Comparing with $\quad I_2 = C' V_1 - D' I_1,$

$$C = -\frac{g_{11}}{g_{12}}$$

$$D' = -\frac{1}{g_{12}}$$

Also, $\quad V_2 = g_{21} V_1 + g_{22}\left[-\frac{g_{11}}{g_{12}} V_1 + \frac{1}{g_{12}} I_1\right] = -\left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{12}}\right] V_2 + \frac{g_{22}}{g_{12}} I_1$

Comparing with $\quad V_2 = A' V_1 - B' I_1,$

$$A' = -\left[\frac{g_{11}\, g_{22} - g_{12}\, g_{21}}{g_{12}}\right] = -\frac{\Delta g}{g_{12}}$$

$$B' = \frac{g_{22}}{g_{12}}$$

13.8.5 Hybrid Parameters in Terms of Other Parameters

1. ***Hybrid Parameters in terms of Z-parameters*** We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Rewriting the second equation,

$$I_2 = -\frac{Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2$$

Comparing with $\quad I_2 = h_{21} I_1 + h_{22} V_2,$

$$h_{21} = -\frac{Z_{21}}{Z_{22}}$$

$$h_{22} = \frac{1}{Z_{22}}$$

Also, $\quad V_1 = Z_{11} I_1 + Z_{12}\left[-\frac{Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2\right] = \left[\frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{22}}\right] I_1 + \frac{Z_{12}}{Z_{22}} V_2$

Comparing with $\quad V_1 = h_{11} I_1 + h_{12} V_2,$

$$h_{11} = \frac{Z_{11}\, Z_{22} - Z_{12}\, Z_{21}}{Z_{22}} = \frac{\Delta Z}{Z_{22}}$$

$$h_{12} = \frac{Z_{12}}{Z_{22}}$$

2. ***Hybrid Parameters in terms of Y-parameters*** We know that

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Rewriting the first equation,

$$V_1 = \frac{1}{Y_{11}} I_1 - \frac{Y_{12}}{Y_{11}} V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2,$

$$h_{11} = \frac{1}{Y_{11}}$$

$$h_{12} = -\frac{Y_{12}}{Y_{11}}$$

Also, $$I_2 = Y_{21}\left[\frac{1}{Y_{11}} I_1 - \frac{Y_{12}}{Y_{11}} V_2\right] + Y_{22} V_2 = \left[\frac{Y_{11} Y_{22} - Y_{12} Y_{21}}{Y_{11}}\right] V_2 + \frac{Y_{21}}{Y_{11}} I_1$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2,$

$$h_{21} = \frac{Y_{22}}{Y_{11}}$$

$$h_{22} = \frac{Y_{11} Y_{22} - Y_{12} Y_{21}}{Y_{11}} = \frac{\Delta Y}{Y_{11}}$$

3. *Hybrid Parameters in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

Rewriting the second equation,

$$I_2 = -\frac{1}{D} I_1 + \frac{C}{D} V_2$$

Comparing with $I_2 = h_{21} I_1 + h_{22} V_2,$

$$h_{21} = -\frac{1}{D}$$

$$h_{22} = \frac{C}{D}$$

Also, $$V_1 = AV_2 - B\left[-\frac{1}{D} I_1 + \frac{C}{D} V_2\right] = \frac{B}{D} I_1 + \left[\frac{AD - BC}{D}\right] V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2,$

$$h_{11} = \frac{B}{D}$$

$$h_{12} = \frac{AD - BC}{D} = \frac{\Delta T}{D}$$

4. *Hybrid Parameters in Terms of A′B′C′D′ Parameters* We know that

$$V_2 = A' V_1 - B' I_1$$

$$I_2 = C' V_1 - D' I_1$$

Rewriting the first equation,

$$V_1 = \frac{B'}{A'} I_1 + \frac{1}{A'} V_2$$

Comparing with $V_1 = h_{11} I_1 + h_{12} V_2,$

$$h_{11} = \frac{B'}{A'}$$

$$h_{12} = \frac{1}{A'}$$

Also,
$$I_2 = C'\left[\frac{B'}{A'}I_1 + \frac{1}{A'}V_2\right] - D'I_1 = -\left[\frac{A'D' - B'C'}{A'}\right]I_1 + \frac{C'}{A'}V_2$$

Comparing with
$$I_2 = h_{21}I_1 + h_{22}V_2,$$
$$h_{21} = -\left[\frac{A'D' - B'C'}{A'}\right] = -\frac{\Delta T'}{A'}$$
$$h_{22} = \frac{C'}{A'}$$

5. *Hybrid Parameters in Terms of Inverse Hybrid Parameters* We know that

$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2$$

By Cramer's rule,

$$V_1 = \frac{\begin{vmatrix} I_1 & g_{12} \\ V_2 & g_{22} \end{vmatrix}}{\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}} = \frac{g_{22}}{\Delta g}I_1 - \frac{g_{12}}{\Delta g}V_2$$

where
$$\Delta g = g_{11}g_{22} - g_{12}g_{21}$$

Comparing with
$$V_1 = h_{11}I_1 + h_{12}V_2,$$
$$h_{11} = \frac{g_{22}}{\Delta g}$$
$$h_{12} = -\frac{g_{12}}{\Delta g}$$

Also,
$$I_2 = \frac{\begin{vmatrix} g_{11} & I_1 \\ g_{21} & V_2 \end{vmatrix}}{\Delta g} = -\frac{g_{21}}{\Delta g}I_1 + \frac{g_{11}}{\Delta g}V_2$$

Comparing with
$$I_2 = h_{21}I_1 + h_{22}V_2,$$
$$h_{21} = -\frac{g_{21}}{\Delta g}$$
$$h_{22} = \frac{g_{11}}{\Delta g}$$

13.8.6 Inverse Hybrid Parameters in Terms of Other Parameters

1. *Inverse Hybrid Parameters in Terms of Z-parameters* We know that

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Rewriting the first equation,

$$I_1 = \frac{1}{Z_{11}}V_1 - \frac{Z_{12}}{Z_{11}}I_2$$

Comparing with
$$I_1 = g_{11}V_1 + g_{12}I_2,$$
$$g_{11} = \frac{1}{Z_{11}}$$

$$g_{12} = -\frac{Z_{12}}{Z_{11}}$$

Also,
$$V_2 = Z_{21}\left[\frac{1}{Z_{11}}V_1 - \frac{Z_{12}}{Z_{11}}I_2\right] + Z_{22}\,I_2 = \frac{Z_{21}}{Z_{11}}V_1 + \left[\frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}}\right]I_2$$

Comparing with
$$V_2 = g_{21}V_1 + g_{22}\,I_2,$$

$$g_{21} = \frac{Z_{21}}{Z_{11}}$$

$$g_{22} = \frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}} = \frac{\Delta Z}{Z_{11}}$$

2. *Inverse Hybrid Parameters in Terms of Y-parameters* We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Rewriting the second equation,

$$V_2 = -\frac{Y_{21}}{Y_{22}}V_1 + \frac{1}{Y_{22}}I_2$$

Comparing with
$$V_2 = g_{21}V_1 + g_{22}\,I_2,$$

$$g_{21} = -\frac{Y_{21}}{Y_{22}}$$

$$g_{22} = \frac{1}{Y_{22}}$$

Also,
$$I_1 = Y_{11}V_1 + Y_{12}\left[-\frac{Y_{21}}{Y_{22}}V_1 + \frac{1}{Y_{22}}I_2\right] = \left[\frac{Y_{11}\,Y_{22} - Y_{12}\,Y_{21}}{Y_{22}}\right]V_1 + \frac{Y_{12}}{Y_{22}}I_2$$

Comparing with
$$I_1 = g_{11}V_1 + g_{12}\,I_2,$$

$$g_{11} = \frac{Y_{11}\,Y_{22} - Y_{12}\,Y_{21}}{Y_{22}} = \frac{\Delta Y}{Y_{22}}$$

$$g_{12} = \frac{Y_{12}}{Y_{22}}$$

3. *Inverse Hybrid Parameters in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Rewriting the first equation,

$$V_2 = \frac{1}{A}V_1 + \frac{B}{A}I_2$$

Comparing with
$$V_2 = g_{21}V_1 + g_{22}\,I_2,$$

$$g_{21} = \frac{1}{A}$$

$$g_{22} = \frac{B}{A}$$

Also,
$$I_1 = C\left[\frac{1}{A}V_1 + \frac{B}{A}I_2\right] - DI_2 = \frac{C}{A}V_1 - \left[\frac{AD - BC}{A}\right]I_2$$

Comparing with $$I_1 = g_{11} V_1 + g_{12} I_2,$$

$$g_{11} = \frac{C}{A}$$

$$g_{12} = -\left[\frac{AD - BC}{A}\right] = -\frac{\Delta T}{A}$$

4. *Inverse Hybrid Parameters in Terms of A'B'C'D' Parameters* We know that

$$V_2 = A' V_1 - B' I_1$$

$$I_2 = C' V_1 - D' I_1$$

Rewriting the second equation,

$$I_1 = \frac{C'}{D'} V_1 - \frac{1}{D'} I_2$$

Comparing with $$I_1 = g_{11} V_1 + g_{12} I_2,$$

$$g_{11} = \frac{C'}{D'}$$

$$g_{12} = -\frac{1}{D'}$$

Also, $$V_2 = A' V_1 - B'\left[\frac{C'}{D'} V_1 - \frac{1}{D'} I_2\right] = \left[\frac{A'D' - B'C'}{D'}\right] V_1 + \frac{B'}{D'} I_2$$

Comparing with $$V_2 = g_{21} V_1 + g_{22} I_2,$$

$$g_{21} = \left[\frac{A'D' - B'C'}{D'}\right] = \frac{\Delta T'}{D'}$$

$$g_{22} = \frac{B'}{D'}$$

5. *Inverse Hybrid Parameters in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

By Cramer's rule,

$$I_1 = \frac{\begin{vmatrix} V_1 & h_{12} \\ I_2 & h_{22} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}} = \frac{h_{22}}{\Delta h} V_1 - \frac{h_{12}}{\Delta h} I_2$$

where $$\Delta h = h_{11} h_{22} - h_{12} h_{21}$$

Comparing with $$I_1 = g_{11} V_1 + g_{12} I_2,$$

$$g_{11} = \frac{h_{22}}{\Delta h}$$

$$g_{12} = -\frac{h_{12}}{\Delta h}$$

Also, $$V_2 = \frac{\begin{vmatrix} h_{11} & V_1 \\ h_{21} & I_2 \end{vmatrix}}{\Delta h} = -\frac{h_{21}}{\Delta h} V_1 + \frac{h_{11}}{\Delta h} I_2$$

Comparing with $$V_2 = g_{21} V_1 + g_{22} I_2,$$

$$g_{21} = -\frac{h_{21}}{\Delta h}$$

$$g_{22} = \frac{h_{11}}{\Delta h}$$

Table 13.3 *Inter-relationship between parameters*

$$\Delta X = X_{11} X_{22} - X_{12} X_{21}$$

	In terms of					
	[Z]	**[Y]**	**[T]**	**[T′]**	**[h]**	**[g]**
[Z]	Z_{11} Z_{12}	$\frac{Y_{22}}{\Delta Y}$ $-\frac{Y_{12}}{\Delta Y}$	$\frac{A}{C}$ $\frac{\Delta T}{C}$	$\frac{D'}{C'}$ $\frac{1}{C'}$	$\frac{\Delta h}{h_{22}}$ $\frac{h_{12}}{h_{22}}$	$\frac{1}{g_{11}}$ $-\frac{g_{12}}{g_{11}}$
	Z_{21} Z_{22}	$-\frac{Y_{21}}{\Delta Y}$ $\frac{Y_{11}}{\Delta Y}$	$\frac{1}{C}$ $\frac{D}{C}$	$\frac{\Delta T'}{C'}$ $\frac{A'}{C'}$	$-\frac{h_{21}}{h_{22}}$ $\frac{1}{h_{22}}$	$\frac{g_{21}}{g_{11}}$ $\frac{\Delta g}{g_{11}}$
[Y]	$\frac{Z_{22}}{\Delta Z}$ $-\frac{Z_{12}}{\Delta Z}$	Y_{11} Y_{12}	$\frac{D}{B}$ $-\frac{\Delta T}{B}$	$\frac{A'}{B'}$ $-\frac{1}{B'}$	$\frac{1}{h_{11}}$ $-\frac{h_{12}}{h_{11}}$	$\frac{\Delta g}{g_{22}}$ $\frac{g_{12}}{g_{22}}$
	$-\frac{Z_{21}}{\Delta Z}$ $\frac{Z_{11}}{\Delta Z}$	Y_{21} Y_{22}	$-\frac{1}{B}$ $\frac{A}{B}$	$\frac{\Delta T'}{B'}$ $\frac{D'}{B'}$	$\frac{h_{21}}{h_{11}}$ $\frac{\Delta h}{h_{11}}$	$-\frac{g_{21}}{g_{22}}$ $\frac{1}{g_{22}}$
[T]	$\frac{Z_{11}}{Z_{21}}$ $\frac{\Delta Z}{Z_{21}}$	$-\frac{Y_{22}}{Y_{21}}$ $-\frac{1}{Y_{21}}$	A B	$\frac{D'}{\Delta T'}$ $\frac{B'}{\Delta T'}$	$-\frac{\Delta h}{h_{21}}$ $-\frac{h_{11}}{h_{21}}$	$\frac{1}{g_{21}}$ $\frac{g_{22}}{g_{21}}$
	$\frac{1}{Z_{21}}$ $\frac{Z_{22}}{Z_{21}}$	$-\frac{\Delta Y}{Y_{21}}$ $-\frac{Y_{11}}{Y_{21}}$	C D	$\frac{C'}{\Delta T'}$ $\frac{A'}{\Delta T'}$	$-\frac{h_{22}}{h_{21}}$ $-\frac{1}{h_{21}}$	$\frac{g_{11}}{g_{21}}$ $\frac{\Delta g}{g_{21}}$
[T′]	$\frac{Z_{22}}{Z_{12}}$ $\frac{\Delta Z}{Z_{12}}$	$-\frac{Y_{11}}{Y_{12}}$ $-\frac{1}{Y_{12}}$	$\frac{D}{\Delta T}$ $\frac{B}{\Delta T}$	A' B'	$\frac{1}{h_{12}}$ $\frac{h_{11}}{h_{12}}$	$-\frac{\Delta g}{g_{12}}$ $-\frac{g_{22}}{g_{12}}$
	$\frac{1}{Z_{12}}$ $\frac{Z_{11}}{Z_{12}}$	$-\frac{\Delta Y}{Y_{12}}$ $-\frac{Y_{22}}{Y_{12}}$	$\frac{C}{\Delta T}$ $\frac{A}{\Delta T}$	C' D'	$\frac{h_{22}}{h_{12}}$ $\frac{\Delta h}{h_{12}}$	$-\frac{g_{11}}{g_{12}}$ $-\frac{1}{g_{12}}$
[h]	$\frac{\Delta Z}{Z_{22}}$ $\frac{Z_{12}}{Z_{22}}$	$\frac{1}{Y_{11}}$ $-\frac{Y_{12}}{Y_{11}}$	$\frac{B}{D}$ $\frac{\Delta T}{D}$	$\frac{B'}{A'}$ $\frac{1}{A'}$	h_{11} h_{12}	$\frac{g_{22}}{\Delta g}$ $-\frac{g_{12}}{\Delta g}$
	$-\frac{Z_{21}}{Z_{22}}$ $\frac{1}{Z_{22}}$	$\frac{Y_{21}}{Y_{11}}$ $\frac{\Delta Y}{Y_{11}}$	$-\frac{1}{D}$ $\frac{C}{D}$	$-\frac{\Delta T'}{A'}$ $\frac{C'}{A'}$	h_{21} h_{22}	$-\frac{g_{21}}{\Delta g}$ $\frac{g_{11}}{\Delta g}$
[g]	$\frac{1}{Z_{11}}$ $-\frac{Z_{12}}{Z_{11}}$	$\frac{\Delta Y}{Y_{22}}$ $\frac{Y_{12}}{Y_{22}}$	$\frac{C}{A}$ $-\frac{\Delta T}{A}$	$\frac{C'}{D'}$ $-\frac{1}{D'}$	$\frac{h_{22}}{\Delta h}$ $-\frac{h_{12}}{\Delta h}$	g_{11} g_{12}
	$\frac{Z_{21}}{Z_{11}}$ $\frac{\Delta Z}{Z_{11}}$	$-\frac{Y_{21}}{Y_{22}}$ $\frac{1}{Y_{22}}$	$\frac{1}{A}$ $\frac{B}{A}$	$\frac{\Delta T'}{D'}$ $\frac{B'}{D'}$	$-\frac{h_{21}}{\Delta h}$ $\frac{h_{11}}{\Delta h}$	g_{21} g_{22}

Example 13.26 *The Z parameters of a two-port network are $Z_{11} = 20\ \Omega$, $Z_{22} = 30\ \Omega$, $Z_{12} = Z_{21} = 10\ \Omega$. Find Y and ABCD parameters.*

Solution

$$\Delta Z = Z_{11} Z_{22} - Z_{12} Z_{21} = (20)(30) - (10)(10) = 500$$

***Y*-parameters**

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{30}{500} = \frac{3}{50}\ \mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{10}{500} = -\frac{1}{50}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{10}{500} = -\frac{1}{50}\ \mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{20}{500} = \frac{2}{50}\ \mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{50} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{2}{50} \end{bmatrix}$$

***ABCD* parameters**

$$A = \frac{Z_{11}}{Z_{21}} = \frac{20}{10} = 2, \qquad B = \frac{\Delta Z}{Z_{21}} = \frac{500}{10} = 50$$

$$C = \frac{1}{Z_{21}} = \frac{1}{10} = 0.1, \qquad D = \frac{Z_{22}}{Z_{21}} = \frac{30}{10} = 3$$

Hence, the *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2 & 50 \\ 0.1 & 3 \end{bmatrix}$$

Example 13.27 *Currents I_1 and I_2 entering at Port 1 and Port 2 respectively of a two-port network are given by the following equations:*

$$I_1 = 0.5V_1 - 0.2V_2$$
$$I_2 = -0.2V_1 + V_2$$

Find Y, Z and ABCD parameters for the network.

Solution

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 0.5\ \mho, \qquad Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -0.2\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -0.2\ \mho, \qquad Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 1\ \mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 1 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11} Y_{22} - Y_{12} Y_{21} = (0.5)(1) - (-0.2)(-0.2) = 0.46$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{1}{0.46} = 2.174\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-0.2)}{0.46} = 0.434\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{(-0.2)}{0.46} = 0.434\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{0.5}{0.46} = 1.087\ \Omega$$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 2.174 & 0.434 \\ 0.434 & 1.087 \end{bmatrix}$$

***ABCD* parameters**

$$A = -\frac{Y_{22}}{Y_{21}} = -\frac{1}{-0.2} = 5, \qquad B = -\frac{1}{Y_{21}} = -\frac{1}{-0.2} = 5$$

$$C = -\frac{\Delta Y}{Y_{21}} = -\frac{0.46}{-0.2} = 2.3, \qquad D = -\frac{Y_{11}}{Y_{21}} = -\frac{0.5}{-0.2} = 2.5$$

Hence, the *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2.3 & 2.5 \end{bmatrix}$$

Example 13.28 *Using the relation $Y = Z^{-1}$, show that* $|Z| = \frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right)$.

Solution We know that

$$Y = Z^{-1}$$

i.e.,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{Z_{22}}{\Delta Z} & -\frac{Z_{12}}{\Delta Z} \\ -\frac{Z_{21}}{\Delta Z} & \frac{Z_{11}}{\Delta Z} \end{bmatrix}$$

$$|Z| = Z_{11} Z_{22} - Z_{12} Z_{21}$$

$$\frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right) = \frac{1}{2}\left(\frac{Z_{22}}{\frac{Z_{22}}{\Delta Z}} + \frac{Z_{11}}{\frac{Z_{11}}{\Delta Z}}\right) = \frac{1}{2}(\Delta Z + \Delta Z) = \frac{1}{2}(2\Delta Z) = \Delta Z = Z_{11} Z_{12} - Z_{12} Z_{21}$$

$$|Z| = \frac{1}{2}\left(\frac{Z_{22}}{Y_{11}} + \frac{Z_{11}}{Y_{22}}\right)$$

Example 13.29 *For the network shown in Fig 13.59, find Z and Y-parameters.*

Fig. 13.59

Solution The network is redrawn by source transformation technique as shown in Fig 13.60.

Applying KVL to Mesh 1,

$$V_1 = I_1 - I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2(I_2 + I_3) - 6I_2$$
$$= -4I_2 + 2I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-(I_3 - I_1) - 2I_3 - 2(I_2 + I_3) + 6I_2 = 0$$
$$5I_3 = I_1 + 4I_2$$
$$I_3 = \frac{1}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots\text{(iii)}$$

Fig. 13.60

Substituting Eq. (iii) in Eq. (i),

$$V_1 = I_1 - \frac{1}{5}I_1 - \frac{4}{5}I_2$$
$$= \frac{4}{5}I_1 - \frac{4}{5}I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = -4I_2 + 2\left(\frac{1}{5}I_1 + \frac{4}{5}I_2\right)$$
$$= \frac{2}{5}I_1 - \frac{12}{5}I_2 \qquad \text{...(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{4}{5} \\ \frac{2}{5} & -\frac{12}{5} \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = \left(\frac{4}{5}\right)\left(-\frac{12}{5}\right) - \left(-\frac{4}{5}\right)\left(\frac{2}{5}\right) = -\frac{40}{25} = -\frac{8}{5}$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{-\frac{12}{5}}{-\frac{8}{5}} = \frac{3}{2}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = \frac{-\frac{4}{5}}{-\frac{8}{5}} = -\frac{1}{2}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = \frac{-\frac{2}{5}}{-\frac{8}{5}} = \frac{1}{4}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{\frac{4}{5}}{-\frac{8}{5}} = -\frac{1}{2}\,\mho$$

Hence, the *Y*-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Example 13.30 *Find Z and h-parameters for the network shown in Fig 13.61.*

Fig. 13.61

Solution Applying KVL to Mesh 1,

$$V_1 = 2I_1 + 2(I_1 - I_3)$$
$$= 4I_1 - 2I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2(I_2 + I_3)$$
$$= 4I_2 + 2I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-2(I_3 - I_1) - 4I_1 - 2(I_3 + I_2) = 0$$

$$I_1 + I_2 = -2I_3 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 4I_1 + I_1 + I_2$$

$$= 5I_1 + I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 4I_2 - I_1 - I_2$$

$$= -I_1 + 3I_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

***h*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (5)(3) - (1)(-1) = 15 + 1 = 16$$

$$h_{11} = \frac{\Delta Z}{Z_{22}} = \frac{16}{3}\,\Omega, \qquad h_{12} = \frac{Z_{12}}{Z_{22}} = \frac{1}{3}$$

$$h_{21} = -\frac{Z_{21}}{Z_{22}} = \frac{1}{3}, \qquad h_{22} = \frac{1}{Z_{22}} = \frac{1}{3}\,\mho$$

Hence, the *h*-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{22} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{16}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Example 13.31 *Find Y and Z-parameters for the network shown in Fig 13.62.*

Fig. 13.62

Solution Applying KCL at Node 3,

$$2(V_1 - V_3) = 2V_1 + (V_3 - V_2)$$

$$V_3 = \frac{V_2}{3} \qquad \ldots\text{(i)}$$

Now,

$$I_1 = 2V_1 + (V_3 - V_2)$$

$$= 2V_1 + \frac{V_2}{3} - V_2$$

$$= 2V_1 - \frac{2}{3}V_2 \qquad \ldots\text{(ii)}$$

$$I_2 = 2V_2 + (V_2 - V_3)$$
$$= 3V_2 - \frac{V_2}{3}$$
$$= \frac{8}{3}V_2 \qquad \ldots\text{(iii)}$$

Comparing Eqs (ii) and (iii) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

Z-parameters

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (2)\left(\frac{8}{3}\right) - 0 = \frac{16}{3}$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{\frac{8}{3}}{\frac{16}{3}} = \frac{1}{2}\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{\left(-\frac{2}{3}\right)}{\frac{16}{3}} = \frac{1}{8}\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{0}{\frac{16}{3}} = 0, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{2}{\frac{16}{3}} = \frac{3}{8}\ \Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{3}{8} \end{bmatrix}$$

Example 13.32 *For the network shown in Fig 13.63, find Y and Z-parameters.*

Fig. 13.63

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{2} + \frac{V_1 - 3V_1 - V_2}{1}$$
$$= -\frac{3}{2}V_1 - V_2 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{1} + \frac{V_2 + 3V_1 - V_1}{1} \qquad \ldots\text{(ii)}$$
$$= 2V_1 + 2V_2$$

Comparing Eqs (i) and (ii) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & -1 \\ 2 & 2 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = \left(-\frac{3}{2}\right)(2) - (-1)(2) = -3 + 2 = -1$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{2}{-1} = -2\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-1)}{(-1)} = -1\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{2}{(-1)} = 2\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{-\frac{3}{2}}{-1} = \frac{3}{2}\ \Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & \frac{3}{2} \end{bmatrix}$$

Example 13.33 *Find Z-parameters for the network shown in Fig 13.64. Hence, find Y and h-parameters.*

Fig. 13.64

Solution The network is redrawn by source transformation technique as shown in Fig 13.65.

Applying KVL to Mesh 1,

$$V_1 = 2I_1 + I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 9I_1 + 10I_2 + 1(I_1 + I_2)$$
$$= 10I_1 + 11I_2 \qquad \ldots\text{(ii)}$$

Fig. 13.65

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 10 & 11 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (2)(11) - (1)(10) = 22 - 10 = 12$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{11}{12}\ \mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{1}{12}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{10}{12} = -\frac{5}{6}\ \mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{2}{12} = \frac{1}{6}\ \mho$$

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{11}{12} & -\frac{1}{12} \\ -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

***h*-parameters**

$$h_{11} = \frac{\Delta Z}{Z_{22}} = \frac{12}{11}\,\Omega, \qquad h_{12} = \frac{Z_{12}}{Z_{22}} = \frac{1}{11}$$

$$h_{21} = -\frac{Z_{21}}{Z_{22}} = -\frac{10}{11}, \qquad h_{22} = \frac{1}{Z_{22}} = \frac{1}{11}\,\mho$$

Hence, h-parameters are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{12}{11} & \frac{1}{11} \\ -\frac{10}{11} & \frac{1}{11} \end{bmatrix}$$

Example 13.34 *Find Y and Z-parameters of the network shown in Fig 13.66.*

Fig. 13.66

Solution Applying KCL at Node 1,

$$I_1 + 2V_2 = \frac{V_1}{1} + \frac{3V_1 - V_2}{1}$$

$$I_1 = 4V_1 - 3V_2 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{2} + \frac{V_2 - 2V_1 - V_1}{1}$$

$$= -3V_1 + 1.5V_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Y-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 1.5 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (4)(1.5) - (-3)(-3) = -3$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = -\frac{1.5}{3} = -0.5\,\Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = -\frac{(-3)}{-3} = -1\,\Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{(-3)}{-3} = -1\,\Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = -\frac{4}{3} = -\frac{4}{3}\,\Omega$$

Hence, the Z-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ -1 & -\frac{4}{3} \end{bmatrix}$$

Example 13.35 *Determine Y and Z-parameters for the network shown in Fig 13.67.*

Fig. 13.67

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$
$$= 1.5V_1 - 0.5V_2 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{2} + 3I_1 + \frac{V_2 - V_1}{2}$$
$$= \frac{V_2}{2} + 3(1.5V_1 - 0.5V_2) + \frac{V_2 - V_1}{2}$$
$$= 0.5V_2 + 4.5V_1 - 1.5V_2 + 0.5V_2 - 0.5V_1$$
$$= 4V_1 - 0.5V_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with the *Y*-parameter equation,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ 4 & -0.5 \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (1.5)(-0.5) - (-0.5)(4) = 1.25$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = -\frac{0.5}{1.25} = -0.4\ \Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = \frac{0.5}{1.25} = 0.4\ \Omega$$

$$Z_{21} = -\frac{Y_{21}}{\Delta Y} = -\frac{4}{1.25} = -3.2\ \Omega, \qquad Z_{22} = \frac{Y_{11}}{\Delta Y} = \frac{1.5}{1.25} = 1.2\ \Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix}$$

Example 13.36 *Determine the Y and Z-parameters for the network shown in Fig 13.68.*

Fig. 13.68

Solution Applying KCL at Node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_3}{0.5}$$
$$= 3V_1 - 2V_3 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{0.5} + \frac{V_2 - V_3}{1}$$
$$= 3V_2 - V_3 \qquad \ldots\text{(ii)}$$

Applying KCL at Node 3,

$$\frac{V_3 - V_1}{0.5} + 2V_1 + \frac{V_3 - V_2}{1} = 0$$
$$V_3 = \frac{1}{3}V_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eqs (i) and (ii),

$$I_1 = 3V_1 - \frac{2}{3}V_2 \qquad \ldots\text{(iv)}$$

$$I_2 = 0V_1 + \frac{8}{3}V_2 \qquad \ldots\text{(v)}$$

Comparing Eqs (iv) and (v) with *Y*-parameter equations,

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

***Z*-parameters**

$$\Delta Y = Y_{11}Y_{22} - Y_{12}Y_{21} = (3)\left(\frac{8}{3}\right) - 0 = 8$$

$$Z_{11} = \frac{Y_{22}}{\Delta Y} = \frac{\frac{8}{3}}{8} = \frac{1}{3}\,\Omega, \qquad Z_{12} = -\frac{Y_{12}}{\Delta Y} = \frac{\frac{2}{3}}{8} = \frac{1}{12}\,\Omega$$

$$Z_{21} = -\frac{Y_{12}}{\Delta Y} = \frac{\frac{2}{3}}{8} = \frac{1}{12}\,\Omega, \qquad Z_{22} = \frac{Y_{11}}{Y\Delta} = \frac{3}{8}\,\Omega$$

Hence, the *Z*-parameters are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{3}{8} \end{bmatrix}$$

Example 13.37 *Determine Z and Y-parameters of the network shown in Fig. 13.69.*

Fig. 13.69

Solution Applying KVL to Mesh 1,

$$V_1 - 4I_1 - 0.05\,I_2 = 0$$
$$V_1 = 4I_1 + 0.05I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 2I_2 + 10V_1 = 0$$
$$V_2 = 2I_2 - 10V_1 \qquad \ldots\text{(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$V_2 = 2I_2 - 40I_1 - 0.5I_2$$
$$= -40I_1 + 1.5I_2 \qquad \ldots\text{(iii)}$$

Comparing Eqs (i) and (iii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0.05 \\ -40 & 1.5 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (4)(1.5) - (0.05)(-40) = 8$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{1.5}{8}\,\mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{0.05}{8}\,\mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-40)}{8} = \frac{40}{8}\,\mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{4}{8}\,\mho$$

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1.5}{8} & -\frac{0.05}{8} \\ \frac{40}{8} & \frac{4}{8} \end{bmatrix}$$

Example 13.38 *Determine Z and Y-parameters of the network shown in Fig. 13.70.*

Fig. 13.70

Solution Applying KVL to Mesh 1,

$$V_1 - 1I_1 - 3I_2 - 2(I_1 + I_2) = 0$$
$$V_1 = 3I_1 + 5I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 - 2(I_2 - I_3) - 2(I_1 + I_2) = 0$$
$$V_2 - 2I_2 + 2I_3 - 2I_1 - 2I_2 = 0$$
$$V_2 = 2I_1 + 4I_2 - 2I_3 \qquad \ldots\text{(ii)}$$

Writing equation for Mesh 3,

$$I_3 = 2V_3 \qquad \text{...(iii)}$$

From Fig. 13.70,

$$V_3 = 2(I_1 + I_2)$$
$$I_3 = 2V_3 = 4I_1 + 4I_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$V_2 = -6I_1 - 4I_2 \qquad \text{...(v)}$$

Comparing Eqs (i) and (v) with Z-parameter equations,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -6 & -4 \end{bmatrix}$$

***Y*-parameters**

$$\Delta Z = Z_{11}Z_{22} - Z_{12}Z_{21} = (3)(-4) - (5)(-6) = 18$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = -\frac{4}{18} = -\frac{2}{9}\ \mho, \qquad Y_{12} = -\frac{Z_{12}}{\Delta Z} = -\frac{5}{18}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = -\frac{(-6)}{18} = \frac{1}{3}\ \mho, \qquad Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{3}{18}\ \mho$$

Hence, Y-parameters are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & -\frac{5}{18} \\ \frac{1}{3} & \frac{3}{18} \end{bmatrix}$$

13.9 INTERCONNECTION OF TWO-PORT NETWORKS

We shall now discuss the various types of interconnections of two-port networks, namely, cascade, parallel, series, series-parallel and parallel-series. We shall derive the relation between the input and output quantities of the combined two-port networks.

13.9.1 Cascade Connection

1. ***Transmission Parameter Representation*** Figure 13.71 shows two-port networks connected in cascade. In the cascade connection, the output port of the first network becomes the input port of the second network. Since it is assumed that input and output currents are positive when they enter the network, we have

$$I_1' = -I_2$$

Fig. 13.71 *Cascade Connection*

Let A_1, B_1, C_1, D_1 be the transmission parameters of the network N_1 and A_2, B_2, C_2, D_2 be the transmission parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad \text{...(13.1)}$$

For the network N_2,

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} \quad ...(13.2)$$

Since $V_1' = V_2$ and $I_2' = -I_2$, we can write

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} \quad ...(13.3)$$

Combining Eqs (13.1) and (13.3),

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2' \\ -I_2' \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad ...(13.4)$$

Equation 13.4 shows that the resultant *ABCD* matrix of the cascade connection is the product of the individual *ABCD* matrices.

2. *Inverse Transmission Parameter Representation* Figure 13.72 shows two-port networks connected in cascade. Since it is assumed that input and output currents are positive when they enter the network, we have

Fig. 13.72 *Cascade connection*

Let A_1', B_1', C_1', D_1' be the transmission parameters of the network N_1 and A_2', B_2', C_2', D_2' be the transmission parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} \quad ...(13.5)$$

For the network N_2,

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} V_2' \\ -I_1' \end{bmatrix} \quad ...(13.6)$$

Since $V_1' = V_2$ and $-I_1' = I_2$, we can write

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad ...(13.7)$$

Combining equations (13.5) and (13.7),

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A_2' & B_2' \\ C_2' & D_2' \end{bmatrix} \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} \quad ...(13.8)$$

Equation 13.8 shows that the resultant $A'B'C'D'$ matrix of the cascade connection is the product of the individual $A'B'C'D'$ matrices.

Example 13.39 *Two identical sections of the network shown in Fig. 13.73 are connected in cascade. Obtain the transmission parameters of the overall connection.*

Fig. 13.73

Solution The network is redrawn as shown in Fig. 13.74.

Applying KVL to Mesh 1,

$$V_1 = 3I_1 - I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_2 + 2I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + 2I_2 + 5I_3 = 0$$

$$I_3 = \frac{1}{5}I_1 - \frac{2}{5}I_2 \qquad \ldots\text{(iii)}$$

Fig. 13.74

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 - \left(\frac{1}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{14}{5}I_1 + \frac{2}{5}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iii) in Eq. (ii),

$$V_2 = 2I_2 + 2\left(\frac{1}{5}I_1 - \frac{2}{5}I_2\right)$$

$$= \frac{2}{5}I_1 + \frac{6}{5}I_2$$

$$I_1 = \frac{5}{2}V_2 - 3I_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (v) in Eq. (iv),

$$V_1 = \frac{14}{5}\left(\frac{5}{2}V_2 - 3I_2\right) + \frac{2}{5}I_2$$

$$= 7V_2 - 8I_2 \qquad \ldots\text{(vi)}$$

Comparing the Eqs (vi) and (v) with *ABCD* parameter equations,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix}$$

Hence, transmission parameters of the overall cascaded network are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 2.5 & 3 \end{bmatrix} = \begin{bmatrix} 69 & 80 \\ 25 & 29 \end{bmatrix}$$

Example 13.40 *Determine ABCD parameters for the ladder network shown in Fig. 13.75.*

Fig. 13.75

Solution The above network can be considered as a cascade connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 13.76.

Applying KVL to Mesh 1,

$$V_1 = \left(2 + \frac{1}{2s}\right) I_1 + \frac{1}{2s} I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{2s} I_1 + \left(2s + \frac{1}{2s}\right) I_2 \qquad \ldots\text{(ii)}$$

Fig. 13.76

From Eq. (ii),

$$I_1 = 2s\, V_2 - (4s^2 + 1)\, I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = \left(2 + \frac{1}{2s}\right)[2s\, V_2 - (4s^2 + 1) I_2] + \frac{1}{2s} I_2$$

$$= (4s + 1) V_2 - (8s^2 + 2s + 2) I_2 \qquad \ldots\text{(iv)}$$

Comparing Eqs (iv) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 4s + 1 & 8s^2 + 2s + 2 \\ 2s & 4s^2 + 1 \end{bmatrix}$$

The network N_2 is shown in Fig. 13.77.

Fig. 13.77

Applying KVL to Mesh 1,

$$V_1' = \frac{1}{s+1} I_1' + \frac{1}{s+1} I_2' \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2' = \frac{1}{s+1} I_1' + \frac{1}{s+1} I_2' \qquad \ldots\text{(ii)}$$

From Eq. (ii),

$$I_1' = (s+1)V_2' - I_2' \qquad \ldots\text{(iii)}$$

Also,

$$V_1' = V_2' \qquad \ldots\text{(iv)}$$

Comparing Eqs (iv) and (iii) with *ABCD* parameter equations,

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix}$$

Hence, overall *ABCD* parameters are

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4s+1 & 8s^2+2s+2 \\ 2s & 4s^2+1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} 8s^3+10s^2+8s+3 & 8s^2+2s+2 \\ 4s^3+4s^2+3s+1 & 4s^2+1 \end{bmatrix}$$

13.9.2 Parallel Connection

Figure13.78 shows two-port networks connected in parallel. In the parallel connection, the two networks have the same input voltages and the same output voltages.

Fig. 13.78 *Parallel connection*

Let $Y_{11}', Y_{12}', Y_{21}', Y_{22}'$ be the *Y*-parameters of the network N_1 and $Y_1'', Y_{12}'', Y_{21}'', Y_{22}''$ be the *Y*-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} I_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

For the combined network, $I_1 = I_1' + I_1''$ and $I_2 = I_2' + I_2''$.

Hence,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1' + I_1'' \\ I_2' + I_2'' \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus, the resultant *Y*-parameter matrix for parallel connected networks is the sum of *Y* matrices of each individual two-port networks.

Example 13.41 *Determine Y-parameters for the network shown in Fig. 13.79.*

Fig. 13.79

Solution The above network can be considered as a parallel connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 13.80.

Applying KCL at Node 3,

$$I_1' + I_2' = \frac{V_3}{2} \qquad \ldots\text{(i)}$$

From Fig. 13.80,

$$I_1' = \frac{V_1 - V_3}{2} \qquad \ldots\text{(ii)}$$

$$I_2' = \frac{V_2 - V_3}{2} \qquad \ldots\text{(iii)}$$

Fig. 13.80

Substituting Eqs (ii) and (iii) in Eq (i),

$$\frac{V_1 - V_3}{2} + \frac{V_2 - V_3}{2} = \frac{V_3}{2}$$

$$3V_3 = V_1 + V_2$$

$$V_3 = \frac{V_1}{3} + \frac{V_2}{3} \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1' = \frac{V_1}{2} - \frac{1}{2}\left(\frac{V_1}{3} + \frac{V_2}{3}\right)$$

$$= \frac{1}{3}V_1 - \frac{1}{6}V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2' = \frac{V_2}{2} - \frac{1}{2}\left(\frac{V_1}{3} + \frac{V_2}{3}\right)$$

$$= -\frac{1}{6}V_1 + \frac{1}{3}V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

The network N_2 is shown in Fig. 13.81.

$$I_1'' = -I_2'' = \frac{V_1 - V_2}{3} = \frac{1}{3}V_1 - \frac{1}{3}V_2$$

Fig. 13.81

Hence, the Y-parameters are

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}'+Y_{11}'' & Y_{12}'+Y_{12}'' \\ Y_{21}'+Y_{21}'' & Y_{22}'+Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{1}{3}+\frac{1}{3} & -\frac{1}{6}-\frac{1}{3} \\ -\frac{1}{6}-\frac{1}{3} & \frac{1}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

Example 13.42 *Find Y-parameters for the network shown in Fig. 13.82*

Fig. 13.82

Solution The above network can be considered as a parallel combination of two networks N_1 and N_2. The network N_1 is shown in Fig. 13.83.

Applying KCL at Node 1,

$$I_1' = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$

$$= \frac{3}{2}V_1 - \frac{1}{2}V_2 \qquad \ldots\text{(i)}$$

Fig. 13.83

Applying KCL at Node 2,

$$I_2' = \frac{V_2}{0.5} + \frac{V_2 - V_1}{2}$$

$$= -\frac{1}{2}V_1 + \frac{5}{2}V_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Y-parameter equation,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

The network N_2 is shown in Fig. 13.84.

Fig. 13.84

Applying KCL at Node 3,

$$I_1'' + I_2'' = \frac{V_3}{2} \qquad \text{...(i)}$$

where
$$I_1'' = \frac{V_1 - V_3}{0.5} = 2V_1 - 2V_3$$
$$I_2'' = \frac{V_2 - V_3}{0.5} = 2V_2 - 2V_3$$

Substituting I_1'' and I_2'' in Eq (i),

$$2V_1 - 2V_3 + 2V_2 - 2V_3 = 0.5V_3$$
$$4.5V_3 = 2V_1 + 2V_2$$
$$V_3 = \frac{4}{9}V_1 + \frac{4}{9}V_2 \qquad \text{...(ii)}$$
$$I_1'' = 2V_1 - 2V_3 = 2V_1 - 2\left(\frac{4}{9}V_1 + \frac{4}{9}V_2\right) = \frac{10}{9}V_1 - \frac{8}{9}V_2 \qquad \text{...(iii)}$$

and
$$I_2'' = 2V_2 - 2V_3 = 2V_2 - 2\left(\frac{4}{9}V_1 + \frac{4}{9}V_2\right)$$
$$= -\frac{8}{9}V_1 + \frac{10}{9}V_2 \qquad \text{...(iv)}$$

Comparing Eqs (iii) and (iv) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{10}{9} & -\frac{8}{9} \\ -\frac{8}{9} & \frac{10}{9} \end{bmatrix}$$

Hence, overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{10}{9} & -\frac{1}{2} - \frac{8}{9} \\ -\frac{1}{2} - \frac{8}{9} & \frac{5}{2} + \frac{10}{9} \end{bmatrix} = \begin{bmatrix} \frac{47}{18} & -\frac{25}{18} \\ -\frac{25}{18} & \frac{65}{18} \end{bmatrix}$$

Example 13.43 *Find Y-parameters for the network shown in Fig. 13.85.*

Fig. 13.85

Solution The above network can be considered as a parallel connection of two networks, N_1 and N_2. The network N_1 is shown in Fig. 13.86.

Fig. 13.86

Applying KCL at Node 3,

$$I_1' + I_2' = 2s\,V_3 \qquad \ldots\text{(i)}$$

From Fig. 13.86,

$$I_1' = \frac{V_1 - V_3}{2}$$

$$= \frac{1}{2}V_1 - \frac{1}{2}V_3 \qquad \ldots\text{(ii)}$$

$$I_2' = \frac{V_2 - V_3}{2}$$

$$= \frac{1}{2}V_2 - \frac{1}{2}V_3 \qquad \ldots\text{(iii)}$$

Substituting Eq. (ii) and Eq. (iii) in Eq. (i),

$$\frac{V_1}{2} - \frac{V_3}{2} + \frac{V_2}{2} - \frac{V_3}{2} = (2s)\,V_3$$

$$(2s+1)V_3 = \frac{V_1}{2} + \frac{V_2}{2}$$

$$V_3 = \frac{1}{2(2s+1)}V_1 + \frac{1}{2(2s+1)}V_2 \qquad \ldots\text{(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1' = \frac{V_1}{2} - \frac{1}{2}\left[\frac{1}{2(2s+1)}V_1 + \frac{1}{2(2s+1)}V_2\right]$$

$$= \left(\frac{4s+1}{8s+4}\right)V_1 - \left(\frac{1}{8s+4}\right)V_2 \qquad \ldots\text{(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2' = \frac{V_2}{2} - \frac{1}{2}\left[\frac{1}{2(2s+1)}V_1 - \frac{1}{2(2s+1)}V_2\right]$$

$$= -\left(\frac{1}{8s+4}\right)V_1 + \left(\frac{4s+1}{8s+4}\right)V_2 \qquad \ldots\text{(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}' & Y_{12}' \\ Y_{21}' & Y_{22}' \end{bmatrix} = \begin{bmatrix} \dfrac{4s+1}{8s+4} & -\dfrac{1}{8s+4} \\ -\dfrac{1}{8s+4} & \dfrac{4s+1}{8s+4} \end{bmatrix}$$

The network N_2 is shown in Fig. 13.87.

Fig. 13.87

Applying KCL at Node 3,

$$I_1'' + I_2'' = V_3 \qquad \text{...(i)}$$

From Fig. 13.87,

$$I_1'' = \frac{V_1 - V_3}{\frac{1}{s}}$$

$$= sV_1 - sV_3 \qquad \text{...(ii)}$$

$$I_2' = \frac{V_2 - V_3}{\frac{1}{s}}$$

$$= sV_2 - sV_3 \qquad \text{...(iii)}$$

Substituting Eqs (ii) and (iii) in Eq. (i),

$$sV_1 - sV_3 + sV_2 - sV_3 = V_3$$

$$(2s+1)V_3 = sV_1 + sV_2$$

$$V_3 = \left(\frac{s}{2s+1}\right)V_1 + \left(\frac{s}{2s+1}\right)V_2 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (ii),

$$I_1'' = sV_1 - s\left[\left(\frac{s}{2s+1}\right)V_1 + \frac{s}{(2s+1)}V_2\right]$$

$$= \left[\frac{s(s+1)}{2s+1}\right]V_1 - \left(\frac{s^2}{2s+1}\right)V_2 \qquad \text{...(v)}$$

Substituting Eq. (iv) in Eq. (iii),

$$I_2'' = sV_2 - s\left[\left(\frac{s}{2s+1}\right)V_1 + \left(\frac{s}{2s+1}\right)V_2\right]$$

$$= -\left(\frac{s^2}{2s+1}\right)V_1 + \left[\frac{s(s+1)}{2s+1}\right]V_2 \qquad \text{...(vi)}$$

Comparing Eqs (v) and (vi) with Y-parameter equations,

$$\begin{bmatrix} Y_{11}'' & Y_{12}'' \\ Y_{21}'' & Y_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{s(s+1)}{2s+1} & -\left(\frac{s^2}{2s+1}\right) \\ -\left(\frac{s^2}{2s+1}\right) & \frac{s(s+1)}{2s+1} \end{bmatrix}$$

Hence, the overall Y-parameters of the network are

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21}' & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11}'+Y_{11}'' & Y_{12}'+Y_{12}'' \\ Y_{21}'+Y_{21}'' & Y_{22}'+Y_{22}'' \end{bmatrix} = \begin{bmatrix} \dfrac{4s^2+8s+1}{4(2s+1)} & -\dfrac{(4s^2+1)}{4(2s+1)} \\ -\dfrac{(4s^2+1)}{4(2s+1)} & \dfrac{4s^2+8s+1}{4(2s+1)} \end{bmatrix}$$

13.9.3 Series Connection

Figure 13.88 shows two-port networks connected in series. In a series connection, both the networks carry the same input current. Their output currents are also equal.

Fig. 13.88 *Series connection*

Let $Z_{11}', Z_{12}', Z_{21}', Z_{22}'$ be the Z-parameters of the network N_1 and $Z_{11}'', Z_{12}'', Z_{21}'', Z_{22}''$ be the Z-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} Z_{11}' & Z_{12}'' \\ Z_{21}' & Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} V_1'' \\ V_2'' \end{bmatrix} = \begin{bmatrix} Z_{11}' & Z_{12}'' \\ Z_{21}' & Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

For the combined network $\quad V_1 = V_1' + V_1''$ and $V_2 = V_2' + V_2''$.

Hence,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1'+V_1'' \\ V_2'+V_2'' \end{bmatrix} = \begin{bmatrix} Z_{11}'+Z_{11}'' & Z_{12}'+Z_{12}'' \\ Z_{21}'+Z_{21}'' & Z_{22}'+Z_{22}'' \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus, the resultant Z-parameter matrix for the series-connected networks is the sum of Z matrices of each individual two-port network.

Example 13.44 *Two identical sections of the network shown in Fig. 13.89 are connected in series. Obtain Z-parameters of the overall connection.*

Fig. 13.89

Solution

Applying KVL to Mesh 1,

$$V_1 = 3I_1 + I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = I_1 + 3I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Hence, Z-parameters of the overall connection are

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$$

Example 13.45 *Determine Z-parameters for the network shown in Fig. 13.90.*

Fig. 13.90

Solution The above network can be considered as a series connection of two networks, N_1 and N_2.

The network N_1 is shown in Fig. 13.91.

Fig. 13.91

Applying KVL to Mesh 1,

$$V_1' = \left(Ls + \frac{1}{Cs}\right)I_1 + \left(\frac{1}{Cs}\right)I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2' = \left(\frac{1}{Cs}\right)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11}' & Z_{12}' \\ Z_{21}' & Z_{22}' \end{bmatrix} = \begin{bmatrix} Ls + \dfrac{1}{Cs} & \dfrac{1}{Cs} \\ \dfrac{1}{Cs} & Ls + \dfrac{1}{Cs} \end{bmatrix}$$

The network N_2 is shown in Fig. 13.92.

Fig. 13.92

Applying KVL to Mesh 1,

$$V_1'' = \left(Ls + \frac{1}{Cs}\right)I_1 + (Ls)I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2'' = (Ls)I_1 + \left(Ls + \frac{1}{Cs}\right)I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$\begin{bmatrix} Z_{11}'' & Z_{12}'' \\ Z_{21}'' & Z_{22}'' \end{bmatrix} = \begin{bmatrix} Ls + \dfrac{1}{Cs} & Ls \\ Ls & Ls + \dfrac{1}{Cs} \end{bmatrix}$$

Hence, the overall Z-parameters of the network are,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}' + Z_{11}'' & Z_{12}' + Z_{12}'' \\ Z_{21}' + Z_{21}'' & Z_{22}' + Z_{22}'' \end{bmatrix} = \begin{bmatrix} 2Ls + \dfrac{2}{Cs} & Ls + \dfrac{1}{Cs} \\ Ls + \dfrac{1}{Cs} & 2Ls + \dfrac{2}{Cs} \end{bmatrix} = \left(Ls + \frac{1}{Cs}\right)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

13.9.4 Series-Parallel Connection

Figure 13.93 shows two networks connected in series-parallel. Here, the input ports of two networks are connected in series and the output ports are connected in parallel.

Fig. 13.93 *Series-parallel connection*

Let $h_{11}', h_{12}', h_{21}', h_{22}'$ be the h-parameters of the network N_1 and $h_{11}'', h_{12}'', h_{21}'', h_{22}''$ be the h-parameters of the network N_2.

For the network N_1,

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} V_1'' \\ I_2'' \end{bmatrix} = \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

For the combined network, $V_1 = V_1' + V_1''$ and $I_2 = I_2' + I_2''$

Hence,
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1' + V_1'' \\ I_2' + I_2'' \end{bmatrix} = \begin{bmatrix} h_{11}' + h_{11}'' & h_{12}' + h_{12}'' \\ h_{21}' + h_{21}'' & h_{22}' + h_{22}'' \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Thus, the resultant h-parameter matrix is the sum of h-parameter matrices of each individual two-port networks.

Example 13.46 *Determine h-parameters for the network shown in Fig. 13.94.*

Fig. 13.94

Solution The above network can be considered as a series-parallel connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 13.95.

Fig. 13.95

Applying KVL to Mesh 1,

$$V_1 = 4I_1 + 2I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 4I_2 \qquad \ldots\text{(ii)}$$

Rewriting Eq. (ii)

$$4I_2 = -2I_1 + V_2$$

$$I_2 = -\frac{1}{2}I_1 + \frac{1}{4}V_2 \qquad \ldots\text{(iii)}$$

Substituting Eq. (iii) in Eq. (i),

$$V_1 = 4I_1 + 2\left(-\frac{1}{2}I_1 + \frac{1}{4}V_2\right)$$

$$= 3I_1 + \frac{1}{2}V_2 \qquad \ldots\text{(iv)}$$

Comparing Eqs (iii) and (iv) with h-parameters equations,

$$\begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

For network N_2, h-parameters will be same as the two networks are identical.

$$\begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Hence, the overall h-parameters of the network are

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} h_{11}' & h_{12}' \\ h_{21}' & h_{22}' \end{bmatrix} + \begin{bmatrix} h_{11}'' & h_{12}'' \\ h_{21}'' & h_{22}'' \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} 3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

13.9.5 Parallel–Series Connection

Figure 13.96 shows two networks connected in parallel–series. Here the input ports of two networks are connected in parallel and the output ports are connected in series.

Fig. 13.96 *Parallel–series connection*

Let $g_{11}', g_{12}', g_{21}', g_{22}'$ be the g-parameters of the network N_1 and $g_{11}'', g_{12}'', g_{21}'', g_{22}''$ be the g-parameters of the network N_2,

For the network N_1,

$$\begin{bmatrix} I_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

For the network N_2,

$$\begin{bmatrix} I_1'' \\ V_2'' \end{bmatrix} = \begin{bmatrix} g_{11}'' & g_{12}'' \\ g_{21}'' & g_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

For the combined network, $I_1 = I_1' + I_1''$ and $V_2 = V_2' + V_2''$

Hence,

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1' + I_1'' \\ V_2' + V_2'' \end{bmatrix} = \begin{bmatrix} g_{11}' + g_{11}'' & g_{12}' + g_{12}'' \\ g_{21}' + g_{21}'' & g_{22}' + g_{22}'' \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

Thus, the resultant g-parameter matrix is the sum of the g-parameter matrices of each individual two-port network.

Example 13.47 *Determine the g-parameters for the network shown in Fig. 13.97.*

Fig. 13.97

Solution The above network can be considered as a parallel series connection of two networks N_1 and N_2. The network N_1 is shown in Fig. 13.98.

Applying KVL to Mesh 1,

$$V_1 = 4I_1 - I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = I_2 + I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + I_2 + 5I_3 = 0$$

$$I_3 = \frac{1}{5}I_1 - \frac{1}{5}I_2 \qquad \ldots\text{(iii)}$$

Fig. 13.98

Substituting Eq (iii) in Eq (i),

$$V_1 = 4I_1 - \left(\frac{1}{5}I_1 - \frac{1}{5}I_2\right)$$

$$= \frac{19}{5}I_1 + \frac{1}{5}I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq (iii) in Eq (ii),

$$V_2 = I_2 + \left(\frac{1}{5}I_1 - \frac{1}{5}I_2\right)$$

$$= \frac{1}{5}I_1 + \frac{4}{5}I_2 \qquad \ldots\text{(v)}$$

Rewriting Eq (iv),

$$\frac{19}{5}I_1 = V_1 - \frac{1}{5}I_2$$

$$I_1 = \frac{5}{19}V_1 - \frac{1}{19}I_2 \qquad \ldots\text{(vi)}$$

Substituting Eq (vi) in Eq (v),

$$V_2 = \frac{1}{5}\left(\frac{5}{19}V_1 - \frac{1}{19}I_2\right) + \frac{4}{5}I_2$$

$$= \frac{1}{19}V_1 + \frac{15}{19}I_2 \qquad \ldots\text{(vii)}$$

Comparing Eqs (vi) and (vii) with g-parameters equation,

$$\begin{bmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{bmatrix} = \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix}$$

For the network N_2, g-parameters will be same as the two networks are identical.

$$\begin{bmatrix} g_{11}'' & g_{12}'' \\ g_{21}'' & g_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix}$$

Hence, the overall g-parameters of the network are

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11}' & g_{12}' \\ g_{21}' & g_{22}' \end{bmatrix} + \begin{bmatrix} g_{11}'' & g_{12}'' \\ g_{21}'' & g_{22}'' \end{bmatrix} = \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix} + \begin{bmatrix} \frac{5}{19} & -\frac{1}{19} \\ \frac{1}{19} & \frac{15}{19} \end{bmatrix} = \begin{bmatrix} \frac{10}{19} & -\frac{2}{19} \\ \frac{2}{19} & \frac{30}{19} \end{bmatrix}$$

13.10 *T*-NETWORK

Any two-port network can be represented by an equivalent *T* network as shown in Fig 13.99.
The elements of the equivalent *T* network may be expressed in terms of *Z*-parameters.
Applying KVL to Mesh 1,

$$V_1 = Z_A I_1 + Z_C(I_1 + I_2)$$
$$= (Z_A + Z_C)I_1 + Z_C I_2 \qquad ...(13.9)$$

Applying KVL to Mesh 2,

$$V_2 = Z_B I_2 + Z_C(I_2 + I_1)$$
$$= Z_C I_1 + (Z_B + Z_C)I_2 \qquad ...(13.10)$$

Fig. 13.99 *T-Network*

Comparing Eqs (13.9) and (13.10) with *Z*-parameter equations,

$$Z_{11} = Z_A + Z_C$$
$$Z_{12} = Z_C$$
$$Z_{21} = Z_C$$
$$Z_{22} = Z_B + Z_C$$

Solving the above equations,

$$Z_A = Z_{11} - Z_{12} = Z_{11} - Z_{21}$$
$$Z_B = Z_{22} - ZZ_{21} = Z_{22} - Z_{12}$$
$$Z_C = Z_{12} = Z_{21}$$

13.11 *PI* (π)-NETWORK

Any two-port network can be represented by an equivalent pi (π) network as shown in Fig. 13.100.
Applying KCL at Node 1,

$$I_1 = Y_A V_1 + Y_B(V_1 - V_2)$$
$$= (Y_A + Y_B)V_1 - Y_B V_2 \qquad ...(13.11)$$

Applying KCL at Node 2,

$$I_2 = Y_C V_2 + Y_B(V_2 - V_1)$$
$$= -Y_B V_1 + (Y_B + Y_C)V_2 \qquad ...(13.12)$$

Fig. 13.100 *π-network*

Comparing Eqs (13.11) and (13.12) with *Y*-parameter equations,

$$Y_{11} = Y_A + Y_B$$
$$Y_{12} = -Y_B$$
$$Y_{21} = -Y_B$$
$$Y_{22} = Y_B + Y_C$$

Solving the above equations,

$$Y_A = Y_{11} + Y_{12} = Y_{11} + Y_{21}$$
$$Y_B = -Y_{12} = -Y_{21}$$
$$Y_C = Y_{22} + Y_{12} = Y_{22} + Y_{21}$$

Example 13.48 *The Z-parameters of a two-port network are: $Z_{11} = 10\ \Omega$, $Z_{12} = Z_{21} = 5\ \Omega$, $Z_{22} = 20\ \Omega$. Find the equivalent T-network.*

Solution The *T*-network is shown in Fig 13.101

Fig. 13.101

Applying KVL to Mesh 1,

$$V_1 = (Z_1 + Z_2)I_1 + Z_2 I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = Z_2 I_1 + (Z_2 + Z_3)I_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with Z parameter equations,

$$Z_{11} = Z_1 + Z_2 = 10$$
$$Z_{12} = Z_2 = 5$$
$$Z_{21} = Z_2 = 5$$
$$Z_{22} = Z_2 + Z_3 = 20$$

Solving the above equations,

$$Z_1 = 5\ \Omega$$
$$Z_2 = 5\ \Omega$$
$$Z_3 = 15\ \Omega$$

Example 13.49 *Admittance parameters of a pi network are $Y_{11} = 0.09$ ℧, $Y_{12} = Y_{21} = -0.05$ ℧ and $Y_{22} = 0.07$ ℧. Find the values of R_a, R_b and R_c.*

Solution The pi network is shown in Fig 13.102.

Applying KCL at Node, 1,

$$I_1 = \frac{V_1}{R_a} + \frac{V_1 - V_2}{R_b}$$
$$= \left(\frac{1}{R_a} + \frac{1}{R_b}\right)V_1 - \frac{1}{R_b}V_2 \qquad \ldots\text{(i)}$$

Fig. 13.102

Applying KCL at Node 2,

$$I_2 = \frac{V_2}{R_c} + \frac{V_2 - V_1}{R_b}$$
$$= -\frac{1}{R_b}V_1 + \left(\frac{1}{R_B} + \frac{1}{R_c}\right)V_2 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii) with *Y*-parameter equations,

$$Y_{11} = \frac{1}{R_a} + \frac{1}{R_b} = 0.09$$
$$Y_{12} = -\frac{1}{R_b} = -0.05$$
$$Y_{21} = -\frac{1}{R_b} = -0.05$$
$$Y_{22} = \frac{1}{R_b} + \frac{1}{R_c} = 0.07$$

Solving the above equations,

$$R_a = 25\ \Omega$$
$$R_b = 20\ \Omega$$
$$R_c = 50\ \Omega$$

Example 13.50 *Find the parameters YA, YB and YC of the equivalent p network as shown in Fig. 13.103 to represent a two-terminal pair network for which the following measurements were taken:*

(a) With terminal 2 short-circuited, a voltage of 10 ∠ 0° V applied at terminal pair I resulted in $I_1 = 2.5 \angle 0°$ A and $I_2 = -0.5 \angle 0°$ A.

(b) With terminal 1 short-circuited, the same voltage at terminal pair 2 resulted in $I_2 = 1.5 \angle 0°$ A and $I_1 = -1.1 \angle -20°$ A.

Fig. 13.103

Solution Since measurements were taken with either of the terminal pairs short-circuited, we have to calculate *Y*-parameters first.

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{2.5\angle 0°}{10\angle 0°} = 0.25\ \mho$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-0.5\angle 0°}{10\angle 0°} = -0.05\ \mho$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{1.5\angle 0°}{10\angle 0°} = 0.15\ \mho$$

Applying KCL at Node 1,

$$I_1 = Y_A V_1 + Y_B (V_1 - V_2)$$
$$= (Y_A + Y_B)V_1 - Y_B V_2 \qquad \text{...(i)}$$

Applying KCL at Node 2,

$$I_2 = Y_C V_2 + Y_B (V_2 - V_1)$$
$$= -Y_B V_1 + (Y_B + Y_C)V_2 \qquad \text{...(ii)}$$

Comparing Eqs (i) and (ii) with the *Y*-parameter equation,

$$Y_{11} = Y_A + Y_B = 0.25$$
$$Y_{12} = Y_{21} = -Y_B = -0.05$$
$$Y_{22} = Y_B + Y_C = 0.15$$

Solving the above equation,

$$Y_A = 0.20\ \mho$$
$$Y_B = 0.05\ \mho$$
$$Y_C = 0.10\ \mho$$

Example 13.51 *A network has two input terminals (a, b) and two output terminals (c, d) as shown in Fig. 13.104. The input impedance with c and d open-circuited is (250 + j100) ohms and with c and d short-circuited is (400 + j 300) ohms. The impedance across c and d with a and b open-circuited is 200 ohms. Determine the equivalent T-network parameters.*

Fig. 13.104

Solution The input impedance with *c* and *d* open-circuited is

$$Z_A + Z_C = 250 + j100 \qquad \ldots\text{(i)}$$

The input impedance with *c* and *d* short-circuited is,

$$Z_A + \frac{Z_B Z_C}{Z_B + Z_C} = 400 + j300 \qquad \ldots\text{(ii)}$$

The impedance across *c* and *d* with *a* and *b* open-circuited is

$$Z_B + Z_C = 200 \qquad \ldots\text{(iii)}$$

Subtracting Eq. (i) from (ii),

$$\frac{Z_B Z_C}{Z_B + Z_C} - Z_C = 150 + j200 \qquad \ldots\text{(iv)}$$

From Eq. (iii),

$$Z_B = 200 - Z_C \qquad \ldots\text{(v)}$$

Subtracting the value of Z_B in the equation (iv) and simplifying,

$$Z_C = (100 - j200)\ \Omega \qquad \ldots\text{(vi)}$$

From Eqs (i) and (vi),

$$Z_A = (150 + j300)\ \Omega$$

From Eqs (iii) and (vi),

$$Z_B = (100 + j200)\ \Omega$$

Example 13.52 *Find the equivalent π-network for the T-network shown in Fig. 13.105.*

Fig. 13.105

Solution Figure 13.106 shows T-network and π-network.

Fig. 13.106

For converting a T-network (star network) into an equivalent π-network (delta network), we can use star-delta transformation technique.

$$Z_1 = Z_A + Z_C + \frac{Z_A Z_C}{Z_B} = 2 + 5 + \frac{2 \times 5}{2.5} = 11\,\Omega$$

$$Z_3 = Z_A + Z_B + \frac{Z_A Z_B}{Z_C} = 2 + 2.5 + \frac{2 \times 2.5}{5} = 5.5\,\Omega$$

$$Z_2 = Z_B + Z_C + \frac{Z_B Z_C}{Z_A} = 2.5 + 5 + \frac{2.5 \times 5}{2} = 13.75\,\Omega$$

The equivalent π-network is shown in Fig. 13.107.

Fig. 13.107

Example 13.53 *For the network shown in Fig. 13.108. Find the equivalent T-network.*

Fig. 13.108

Solution Applying KVL to Mesh 1,

$$V_1 = 3I_1 + 2I_2 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 6I_2 + 4I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$13I_3 - I_1 + 4I_2 = 0$$

$$I_3 = \frac{1}{13} I_1 - \frac{4}{13} I_2 \qquad \text{...(iii)}$$

Substituting the Eq. (iii) in Eq. (i),

$$V_1 = 3I_1 + 2I_2 - \frac{1}{13}I_1 + \frac{4}{13}I_2$$

$$= \frac{38}{13}I_1 + \frac{30}{13}I_2 \qquad \ldots\text{(iv)}$$

Substituting the Eq. (iii) in Eq. (ii),

$$V_2 = 2I_1 + 6I_2 + 4\left(\frac{1}{13}I_1 - \frac{4}{13}I_2\right)$$

$$= \frac{30}{13}I_1 + \frac{62}{13}I_2 \qquad \ldots\text{(v)}$$

The *T*-network is shown in Fig. 13.109.

Applying KVL to Mesh 1,

$$V_1 = (Z_A + Z_C)I_1 + Z_C I_2 \qquad \ldots\text{(vi)}$$

Applying KVL to Mesh 2,

$$V_2 = Z_C I_1 + (Z_B + Z_C)I_2 \qquad \ldots\text{(vii)}$$

Fig. 13.109

Comparing Eqs (iv) and (v) with Eqs (vi) and (vii),

$$Z_A + Z_C = \frac{38}{13}$$

$$Z_C = \frac{30}{13}$$

$$Z_B + Z_C = \frac{62}{13}$$

Solving the above equations,

$$Z_A = \frac{8}{13}\Omega$$

$$Z_B = \frac{32}{13}\Omega$$

$$Z_C = \frac{30}{13}\Omega$$

13.12 LATTICE NETWORKS

A lattice network is one of the common two-port networks, shown in Fig. 13.110. It is used in filter sections and is also used as attenuator. This network can be represented in terms of *z*-parameters.

Fig. 13.110 *Lattice network*

The lattice network can be redrawn as a bridge network as shown in Fig. 13.111. This lattice network is symmetric and reciprocal. The current I_1 divides equally between the two arms of the bridge.

When the output port is open-circuited, i.e., $I_2 = 0$

$$V_1 = \frac{I_1}{2}(Z_A + Z_B)$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{Z_A + Z_B}{2}$$

Also

$$V_2 = \frac{I_1}{2}Z_B - \frac{I_1}{2}Z_A = \frac{I_1}{2}(Z_B - Z_A)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{Z_B - Z_A}{2}$$

Fig. 13.111 *Bridge network*

Since the network is symmetric,

$$Z_{11} = Z_{22} = \frac{Z_A + Z_B}{2}$$

$$Z_{12} = Z_{21} = \frac{Z_B - Z_A}{2}$$

Solving the above equations,

$$Z_A = Z_{11} - Z_{12}$$

$$Z_B = Z_{11} + Z_{12}$$

The lattice network can be represented in terms of other two-port network parameters, with the help of inter-relationship formulae of various parameters.

Example 13.54 *Find the lattice equivalent of a symmetrical T network shown in Fig. 13.112.*

Fig. 13.112

Solution Applying KVL to Mesh 1,

$$V_1 = 3I_1 + 2I_2 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 2I_1 + 3I_2 \qquad \text{...(ii)}$$

Comparing Eqs (i) and (ii) with Z-parameter equations,

$$Z_{11} = 3\ \Omega$$

$$Z_{12} = 2\ \Omega$$

$$Z_{21} = 2\ \Omega$$

$$Z_{22} = 3\ \Omega$$

Since $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$, the network is symmetric and reciprocal. The parameters of lattice network are

$$Z_A = Z_{11} - Z_{12} = 3 - 2 = 1\,\Omega$$
$$Z_B = Z_{11} + Z_{12} = 3 + 2 = 5\,\Omega$$

The lattice network is shown in Fig. 13.113.

Fig. 13.113

Example 13.55 *Find the lattice equivalent of a symmetric π-network shown in Fig. 13.114.*

Fig. 13.114

Solution The network is redrawn as shown in Fig. 13.115.

Fig. 13.115

Applying KVL to Mesh 1,

$$V_1 = 10I_1 - 10I_3 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

$$V_2 = 10I_2 + 10I_3 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-10I_1 + 10I_2 + 25I_3 = 0$$

$$I_3 = \frac{2}{5}I - \frac{2}{5}I_2 \qquad \ldots\text{(iii)}$$

Substituting Eq (iii) in Eq (i),

$$V_1 = 10I_1 - 10\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$
$$= 6I_1 + 4I_2 \qquad \ldots\text{(iv)}$$

Substituting Eq (iii) in Eq (ii),

$$V_2 = 10I_2 + 10\left(\frac{2}{5}I_1 - \frac{2}{5}I_2\right)$$
$$= 4I_1 - 6I_2 \qquad \ldots\text{(v)}$$

Comparing the Eqs (iv) and (v) with Z-parameter equations,

$$Z_{11} = 6\ \Omega$$
$$Z_{12} = 4\ \Omega$$
$$Z_{21} = 4\ \Omega$$
$$Z_{22} = 6\ \Omega$$

Since $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$, the network is symmetric and reciprocal. The parameters of lattice network are

$$Z_A = Z_{11} - Z_{12} = 6 - 4 = 2\ \Omega$$
$$Z_B = Z_{11} + Z_{12} = 6 + 4 = 10\ \Omega$$

The lattice network is shown in Fig. 13.116.

Fig. 13.116

13.13 TERMINATED TWO-PORT NETWORKS

13.13.1 Driving-Point Impedance at Input Port

A two-port network is shown in Fig. 13.117. The output port of the network is terminated in load impedance Z_L. The input impedance of this network can be expressed in terms of parameters of two-port network parameters.

Fig. 13.117 *Terminated two-port network*

1. Input Impedance in Terms of Z-parameters We know that

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

From Fig. 13.117,

$$V_2 = -Z_L I_2$$
$$-I_2 Z_L = Z_{21} I_1 + Z_{22} I_2$$
$$I_2 = -\frac{Z_{21}}{Z_{22} + Z_L} I_1$$
$$Z_{in} = \frac{V_1}{I_1} = Z_{11} + Z_{12}\left(-\frac{Z_{21}}{Z_{22} + Z_L}\right) = \frac{Z_{11}Z_{22} + Z_{11}Z_L - Z_{12}Z_{21}}{Z_{22} + Z_L}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{in} = \lim_{Z_L \to \infty} \frac{\dfrac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_L} + Z_{11}}{\dfrac{Z_{22}}{Z_L} + 1} = Z_{11}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{in} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{22}}$$

2. *Input Impedance in Terms of Y-parameters* We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

From Fig. 13.117,

$$V_2 = -Z_L I_2$$
$$I_2 = -\frac{V_2}{Z_L} = -Y_L V_2 \qquad \text{where } Y_L = \frac{1}{Z_L}$$
$$-Y_L V_2 = Y_{21}V_1 + Y_{22}V_2$$
$$V_2 = -\frac{Y_{21}}{Y_{22} + Y_L}V_1$$
$$I_1 = Y_{11}V_1 + Y_{12}\left(-\frac{Y_{21}}{Y_{22} + Y_L}\right)V_1 = Y_{11}V_1 - \frac{Y_{21}Y_{12}}{Y_{22} + Y_L}V_1 = \frac{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{11}Y_L}{Y_{22} + Y_L}V_1$$
$$Z_{in} = \frac{V_1}{I_1} = \frac{Y_{22} + Y_L}{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_{11}Y_L}$$

When output port is open-circuited, i.e., $Y_L = 0$

$$Z_{in} = \frac{Y_{22}}{Y_{11}Y_{22} - Y_{12}Y_{21}}$$

When output port is short-circuited, i.e., $Y_L = \infty$,

$$Z_{in} = \lim_{Y_L \to \infty} \frac{\dfrac{Y_{22}}{Y_L} + 1}{\dfrac{Y_{11}Y_{22} - Y_{12}Y_{21}}{Y_L} + Y_{11}} = \frac{1}{Y_{11}}$$

3. *Input Impedance in Terms of Transmission Parameters* We know that

$$V_1 = AV_2 - BV_2$$
$$I_1 = CV_2 - DI_2$$

From Fig. 13.117,

$$V_2 = -Z_L I_2$$
$$I_1 = -CZ_L I_2 - DI_2 = -(CZ_L + D)I_2$$
$$I_2 = -\frac{I_1}{CZ_L + D}$$
$$V_1 = AZ_L I_2 - B = \left(-\frac{I_1}{CZ_L + D}\right) = \left(\frac{AZ_L + B}{CZ_L + D}\right)I_1$$
$$Z_{in} = \frac{V_1}{I_1} = \frac{AZ_L + B}{CZ_L + D}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{in} = \frac{A}{C}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{in} = \frac{B}{D}$$

4. *Input Impedance in Terms of Hybrid Parameters* We know that

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$
$$V_2 = -Z_L I_2$$
$$I_2 = h_{21}I_1 - h_{22}Z_L I_2$$
$$I_2 = \frac{h_{21}}{1 + h_{22}Z_L} I_L$$
$$V_2 = -\frac{h_{21}Z_L}{1 + h_{22}Z_L} I_1$$

Substituting the value of V_2 in V_1,

$$V_1 = h_{11}I_1 + h_{12}\left[\frac{-h_{21}Z_L}{1 + h_{22}Z_L} I_L\right] = \left[\frac{(h_{11}h_{22} - h_{12}h_{21})Z_L + h_{11}}{1 + h_{22}Z_L}\right] I_1$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{(h_{11}h_{22} - h_{12}h_{21})Z_L + h_{11}}{1 + h_{22}Z_L}$$

If the output port is open-circuited, i.e., $Z_L = \infty$,

$$Z_{in} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}$$

If the output port is short-circuited, i.e., $Z_L = 0$,

$$Z_{in} = h_{11}$$

13.13.2 Driving-Point Impedance at Output Port

A two-port network is shown in Fig. 13.118. The input port is terminated in load impedance Z_L. The output impedance of this network can be expressed in terms of two port network parameters.

Fig. 13.118 *Terminated two-port network*

1. *Output Impedance in terms of Z-parameters* We know that

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

From Fig. 13.118,

$$V_1 = -Z_L I_1$$

$$-I_1 Z_1 = Z_{11} I_1 + Z_{12} I_2$$

$$I_1 = \left(-\frac{Z_{12}}{Z_L + Z_{11}}\right) I_2$$

$$V_2 = Z_{21}\left(-\frac{Z_{12}}{Z_L + Z_{11}}\right) I_2 + Z_{22} I_2 = I_2\left(Z_{22} - \frac{Z_{21}Z_{12}}{Z_L + Z_{11}}\right) = \left(\frac{Z_{11}Z_{22} - Z_{12}Z_{21} + Z_{22}Z_L}{Z_{11} + Z_L}\right) I_2$$

$$Z_0 = \frac{V_2}{I_2} = \frac{Z_{11}Z_{22} - Z_{12}Z_{21} + Z_{22}Z_L}{Z_{11} + Z_L}$$

If the input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = Z_{22}$$

If the input port is short-circuited, i.e., $Z_L = 0$,

$$Z_0 = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{11}}$$

2. *Output Impedance in Terms of Y-parameters* We know that

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

From Fig. 13.118,

$$V_1 = -Z_L I_1$$

$$I_1 = -\frac{V_1}{Z_L} = -Y_L V_1$$

$$-Y_L V_1 = Y_{11}V_1 + Y_{12}V_2$$

$$V_1 = \left(-\frac{Y_{12}}{Y_L + Y_{11}}\right) V_2$$

$$I_2 = Y_{21}\left(-\frac{Y_{12}}{Y_L + Y_{11}}\right) V_2 + Y_{22}V_2 = V_2\left[Y_{22} - \frac{Y_{21}Y_{12}}{Y_L + Y_{11}}\right] = V_2\left[\frac{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_L Y_{22}}{Y_L + Y_{11}}\right]$$

$$Z_0 = \frac{V_2}{I_2} = \frac{Y_L + Y_{11}}{Y_{11}Y_{22} - Y_{12}Y_{21} + Y_L Y_{22}}$$

If input port is open-circuited, i.e., $Y_L = 0$,

$$Z_0 = \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}Y_{21}}$$

If input port is short-circuited, i.e., $Y_L = \infty$,

$$Z_0 = \frac{1}{Y_{22}}$$

3. *Output Impedance in Terms of ABCD Parameters* We know that

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

From Fig. 13.118,

$$V_1 = -Z_L I_1$$

$$\frac{V_1}{I_1} = -Z_L = \frac{AV_2 - BI_2}{CV_2 - DI_2}$$

$$V_2(CZ_L + A) = I_2(DZ_L + B)$$

$$Z_0 = \frac{V_2}{I_2} = \frac{DZ_L + B}{CZ_L + A}$$

If input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = \frac{D}{C}$$

If input port is short-circuited, i.e., $Z_L = 0$,

$$Z_0 = \frac{B}{A}$$

4. *Output Impedance in Terms of h-parameters* We know that

$$V_1 = h_{11}I_1 + h_{12}V_{12}$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

From Fig.13.118,

$$V_1 = -Z_L I_1$$

$$-I_1 Z_L = h_{11}I_1 + h_{22}V_2$$

$$I_1 = \left(-\frac{h_{12}}{h_{11} + Z_L}\right)V_2$$

$$I_2 = h_{21}\left(-\frac{h_{12}}{h_{11} + Z_L}\right)V_2 + h_{22}V_2 = V_2\left[\frac{h_{11}h_{22} - h_{12}h_{21} + h_{22}Z_L}{h_{11} + Z_L}\right]$$

$$Z_0 = \frac{V_2}{I_2} = \frac{h_{11} + Z_L}{h_{11}h_{22} - h_{12}h_{21} + h_{22}Z_L}$$

If input port is open-circuited, i.e., $Z_L = \infty$,

$$Z_0 = \frac{1}{h_{22}}$$

If input port is short-circuited i.e., $Z_L = 0$,

$$Z_0 = \frac{h_{11}}{h_{11}h_{22} - h_{12}h_{21}}$$

Example 13.56 *Measurements were made on a two-terminal network shown in Fig. 13.119.*

Fig. 13.119

(a) With terminal pair 2 open, a voltage of 100 ∠0° V applied to terminal pair 1 resulted in

$$I_1 = 10\ \angle 0°\ A, \quad V_2 = 25\ \angle 0°\ V$$

(b) With terminal pair 1 open, the same voltage applied to terminal pair 2 resulted in

$$I_2 = 20\ \angle 0°\ A, \quad V_1 = 50\ \angle 0°\ V$$

Write mesh equations for this network. What will be the voltage across a 10-Ω resistor connected across Terminal pair 2 if a 100 ∠0° V is connected across terminal pair 1?

Solution Since measurements were done with either of the terminal pairs open-circuited, we have to calculate *Z*-parameters first.

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{100\angle 0°}{10\angle 0°} = 10\ \Omega$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{25\angle 0°}{10\angle 0°} = 2.5\ \Omega$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{100\angle 0°}{20\angle 0°} = 5\ \Omega$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{50\angle 0°}{20\angle 0°} = 2.5\ \Omega$$

Putting these values in *Z*-parameter equations,

$$V_1 = 10I_1 + 2.5I_2 \qquad \ldots\text{(i)}$$

$$V_2 = 2.5I_1 + 5I_2 \qquad \ldots\text{(ii)}$$

When a 10-Ω resistor is connected across terminal pair 1,

$$V_1 = 100\ \angle 0°\ V$$

$$V_2 = -R_L I_2 = -10I_2$$

Substituting values of V_1 and V_2 in Eqs (i) and (ii),

$$100 = 10I_1 + 2.5I_2$$

and

$$-10I_2 = 2.5I_1 + 5I_2$$

$$2.5I_1 = -15I_2$$

$$I_1 = -6I_2$$

$$100 = -60I_2 + 2.5I_2$$

$$I_2 = -\frac{100}{57.5} = -1.74\ \text{A}$$

Voltage across the resistor $= -I_2R_L = -10(-1.74) = 17.4$ V

Example 13.57 *The Z-parameters of a two-port network shown in Fig. 13.120 are $Z_{11} = Z_{22} = 10\ \Omega$, $Z_{21} = Z_{22} = 4\ \Omega$. If the source voltage is 20 V, determine I_1, I_2, V_2 and input impedance.*

Fig. 13.120

Solution

$$V_1 = V_s = 20 \text{ V}$$
$$V_2 = -20I_2$$

The two-port network can be represented in terms of Z-parameters.

$$V_1 = 10I_1 + 4I_2 \quad \ldots\text{(i)}$$
$$V_2 = 4I_1 + 10I_2 \quad \ldots\text{(ii)}$$
$$-20I_2 = 4I_1 + 10I_2$$
$$4I_1 = -30I_2$$
$$I_1 = -7.5I_2$$

Substituting the value of I_1 in Eq. (i),

$$V_1 = 10(-7.5I_2) + 4I_2 = -71I_2$$
$$20 = -71I_2$$
$$I_2 = -0.28 \text{ A}$$
$$I_1 = -7.5(-0.28) = 2.1 \text{ A}$$
$$V_2 = -20(-0.28) = 56 \text{ V}$$

Input impedance $Z_i = \dfrac{V_1}{I_1} = \dfrac{20}{2.1} = 9.52\ \Omega$

Example 13.58 *The Z-parameters of a two-port network shown in Fig. 13.121 are, Z_{11} = 2 Ω, Z_{12} = 1 Ω, Z_{21} = 2 Ω, Z_{22} = 5 Ω. Calculate the voltage ratio $\dfrac{V_2}{V_s}$, current ratio $-\dfrac{I_2}{I_1}$ and input impedance $\dfrac{V_1}{I_1}$.*

Fig. 13.121

Solution The two-port network can be represented in terms of Z-parameters.

$$V_1 = 2I_1 + I_2 \quad \ldots\text{(i)}$$
$$V_2 = 2I_1 + 5I_2 \quad \ldots\text{(ii)}$$

When the 5 Ω resistor is connected across port-2,

$$V_2 = -5I_2 \quad \ldots\text{(iii)}$$

Applying KVL to the input port,

$$V_s - 1I_1 - V_1 = 0$$
$$V_1 = V_s + I_1 \quad \ldots\text{(iv)}$$

Substituting values of V_1 and V_2 in Eqs (i) and (ii),

$$V_s - I_1 = 2I_1 + I_2$$
$$V_s = 3I_1 + I_2 \quad \ldots\text{(v)}$$

and
$$-5I_2 = 2I_1 + 5I_2$$
$$0 = 2I_1 + 10I_2 \qquad \ldots\text{(vi)}$$

Solving Eqs (v) and (vi), we get

$$I_1 = \frac{\begin{vmatrix} V_s & 1 \\ 0 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = \frac{5}{14}V_s$$

$$I_2 = \frac{\begin{vmatrix} 3 & V_s \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = -\frac{1}{14}V_s$$

$$-\frac{I_2}{I_1} = \frac{1}{5}$$

$$V_2 = 2I_1 + 5I_2 = 2\left(\frac{5}{14}V_s\right) + 5\left(-\frac{1}{14}V_s\right) = \frac{5}{14}V_s$$

$$\frac{V_2}{V_s} = \frac{5}{14}$$

$$V_1 = 2I_1 + I_2 = 2\left(\frac{5}{14}V_s\right) - \frac{1}{14}V_s = \frac{9}{14}V_s$$

$$\frac{V_1}{I_1} = \frac{9}{5}\Omega$$

Example 13.59 *The following equations give the voltages V_1 and V_2 at the two ports of a two-port network shown in Fig. 13.122.*

$$V_1 = 5I_1 + 2I_2$$
$$V_2 = 2I_1 + I_2$$

A load resistor of 3 Ω is connected across port 2. Calculate the input impedance.

Fig. 13.122

Solution From Fig. 13.122,

$$V_2 = -3I_2 \qquad \ldots\text{(i)}$$

Substituting Eq. (i) in the given equation,

$$-3I_2 = 2I_1 + I_2$$
$$I_2 = -\frac{I_1}{2} \qquad \text{(ii)}$$

Substituting the Eq. (ii) in the given equation.

$$V_1 = 5I_1 - I_1 = 4I_1$$

Input impedance
$$Z_i = \frac{V_1}{I_1} = 4\ \Omega$$

Example 13.60 *The y-parameters for a two-port network shown in Fig. 13.123 are given as, $Y_{11} = 4$ ℧, $Y_{22} = 5$ ℧, $Y_{12} = Y_{21} = 4$ ℧. If a resistor of 1 Ω is connected across port-1 of the network then find the output impedance.*

Fig. 13.123

Solution The two-port network can be represented in terms of *Y*-parameters.

$$I_1 = 4V_1 + 4V_2 \quad \ldots\text{(i)}$$

$$I_2 = 4V_1 + 5V_2 \quad \ldots\text{(ii)}$$

When the 1-Ω resistor is connected across port-1 of the network,

$$V_1 = -1I_1 = -I_1$$

$$I_1 = -V_1$$

Substituting value of I_1 in Eq (i),

$$-V_1 = 4V_1 + 4V_2$$

$$-5V_1 = 4V_2$$

$$V_1 = -\frac{4}{5}V_2$$

Substituting value of V_1 in Eq (ii),

$$I_2 = 4\left(-\frac{4}{5}V_2\right) + 5V_2 = \frac{9}{5}V_2$$

Output impedance $Z_0 = \dfrac{V_2}{I_2} = \dfrac{5}{9}\,\Omega$

Example 13.61 *The following equation gives the voltage and current at the input port of a two-port network shown in Fig. 13.124.*

$$V_1 = 5V_2 - 3I_2$$

$$I_1 = 6V_2 - 2I_2$$

A load resistance of 5 Ω is connected across the output port. Calculate the input impedance.

Fig. 13.124

Solution From Fig. 13.124,

$$V_2 = -5I_2$$

Substituting the value of V_2 in the given equations,

$$V_1 = 5(-5I_2) - 3I_2 = -28I_2$$

$$I_1 = 6(-5I_2) - 2I_2 = -32I_2$$

Input impedance $Z_i = \dfrac{V_1}{I_1} = \dfrac{-28I_2}{-32I_2} = \dfrac{7}{8}\,\Omega$

Example 13.62 *The ABCD parameters of a two-port network shown in Fig. 13.125 are A = 2.5, B = 4 Ω, C = 1 ℧, D = 2. What must be the input voltage V_1 applied for the output voltage V_2 to be 10 V across the load of 10 Ω connected at Port 2?*

Fig. 13.125

Solution The two-port network can be represented in terms of *ABCD* parameters.

$$V_1 = 2.5\,V_2 - 4I_2 \quad \text{...(i)}$$

$$I_1 = V_2 - 2I_2 \quad \text{...(ii)}$$

When the 10 Ω resistor is connected across Port 2,

$$V_2 = -10I_2 = 10 \quad \text{...(iii)}$$

$$I_2 = -1\text{A}$$

$$V_1 = 2.5(10) - 4(-1) = 29\text{ V}$$

Example 13.63 *The h-parameters of a two-port network shown in Fig. 13.126 are h_{11} = 4 Ω, h_{12} = 1, h_{21} = 1, h_{22} = 0.5 ℧. Calculate the output voltage V_2 when the output port is terminated in a 3 Ω resistance and a 1 V is applied at the input port.*

Fig. 13.126

Solution

$$V_1 = 1\text{ V}$$

$$V_2 = -3I_2$$

The two-port network can be represented in terms of *h*-parameters.

$$V_1 = 4I_1 + V_2 \quad \text{...(i)}$$

$$I_2 = I_1 + 0.5\,V_2 \quad \text{...(ii)}$$

$$I_2 = I_1 + 0.5(-3I_2)$$

$$2.5\,I_2 = I_1$$

Substituting the value of V_1 and I_1 in Eq (i),

$$1 = 4(2.5\,I_2) - 3\,I_2$$

$$1 = 7I_2$$

$$I_2 = \frac{1}{7}\text{A}$$

$$V_2 = -3\left(\frac{1}{7}\right) = -\frac{3}{7}\text{V}$$

Example 13.64 *The h-parameters of a two-port network shown in Fig. 13.127 are h_{11} = 1 Ω, h_{12} = −h_{21} = 2, h_{22} = 1 ℧. The power absorbed by a load resistance of 1 Ω connected across port-2 is 100W. The network is excited by a voltage source of generated voltage V_s and internal resistance 2 Ω. Calculate the value of V_s.*

Fig. 13.127

Solution The two-port network can be represented in terms of *h*-parameters.

$$V_1 = I_1 + 2V_2 \quad \ldots\text{(i)}$$

$$I_2 = -2I_1 + V_2 \quad \ldots\text{(ii)}$$

When the 1 Ω resister is connected across port-2,

$$\frac{V_2^2}{1} = 100$$

$$V_2 = 10 \text{ V}$$

$$V_2 = -1I_2 = 10$$

$$I_2 = -10 \text{ A}$$

Substituting values of I_2 and V_2 in Eq (ii),

$$-10 = -2I_1 + 10$$

$$I_1 = 10 \text{ A}$$

Applying KVL to the input port,

$$V_s - 2I_1 - V_1 = 0$$

$$V_s - 2I_1 - (I_1 + 2V_2) = 0$$

$$V_s - 3I_1 - 2V_2 = 0$$

$$V_s = 3I_1 + 2V_2 = 3(10) + 2(10) = 50 \text{ V}$$

13.14 IMAGE PARAMETERS

If the driving-point impedance at port 1, with impedance Z_{i2} connected across port 2, is Z_{i1} and driving-point impedance at port 2, with impedance Z_{i1} connected across the port 1, is Z_{i2} then Z_{i1} and Z_{i2} are known as image impedances of the network. These are also known as image parameters.

The image parameters can be expressed in terms of *ABCD* parameters. Figure 13.128 shows a two-port network terminated in Z_{i2} at Port 2.

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2$$

$$V_2 = -Z_{i2}I_2$$

$$Z_{i1} = \frac{V_1}{I_1} = \frac{AV_2 - BI_2}{CV_2 - DI_2} = \frac{-AZ_{i2}I_2 - BI_2}{-CZ_{i2}I_2 - DI_2} = \frac{-AZ_{i2} - B}{-CZ_{i2} - D}$$

Fig. 13.128 *Terminated two-port network*

$$= \frac{A Z_{i2} + B}{C Z_{i2} + D} \qquad \text{...(13.13)}$$

Fig. 13.129 *Terminated two-port network*

Similarly, if the two-port network is terminated in Z_{i1} at port 1 as shown in Fig. 13.129 then

$$V_1 = -Z_{i1} I_1$$

$$Z_{i2} = \frac{V_2}{I_2} = \frac{V_1\left(\dfrac{D}{AD-BC}\right) - I_1\left(\dfrac{B}{AD-BC}\right)}{V_1\left(\dfrac{C}{AD-BC}\right) - I_1\left(\dfrac{A}{AD-BC}\right)} \qquad \text{...(from Cramer's rule)}$$

$$= \frac{-Z_{i1}I_1\left(\dfrac{D}{AD-BC}\right) - I_1\left(\dfrac{B}{AD-BC}\right)}{-Z_{i1}I_1\left(\dfrac{C}{AD-BC}\right) - I_1\left(\dfrac{A}{AD-BC}\right)} = \frac{-DZ_{i1} - B}{-CZ_{i1} - A}$$

$$= \frac{DZ_{i1} + B}{CZ_{i1} + A} \qquad \text{...(13.14)}$$

Solving Eqs (13.13) and (13.14),

$$Z_{i1} = \sqrt{\frac{AB}{CD}}$$

$$Z_{i2} = \sqrt{\frac{BD}{AC}}$$

The image impedances Z_{i1} and Z_{i2} do not define a network completely. A third parameter image transfer constant ϕ is also used to define the network. ϕ can also be expressed in terms of *ABCD* parameters. Let the network be terminated in impedance Z_{i2} at Port 2.

$$V_2 = -Z_{i2} I_2$$

$$V_1 = AV_2 - BI_2 = AV_2 - B\left(\frac{V_2}{-Z_{i2}}\right) = \left(A + \frac{B}{Z_{i2}}\right)V_2$$

$$I_1 = CV_2 - DI_2 = -CZ_{i2}I_2 - DI_2 = -(CZ_{i2} + D)I_2$$

$$\frac{V_1}{V_2} = A + \frac{B}{Z_{i2}} = A + B\sqrt{\frac{AC}{BD}} = A + \frac{\sqrt{ABCD}}{D} = \frac{AD + \sqrt{ABCD}}{D}$$

$$-\frac{I_1}{I_2} = CZ_{i2} + D = C\sqrt{\frac{BD}{AC}} + D = \frac{\sqrt{ABCD}}{A} + D = \frac{\sqrt{ABCD} + AD}{A} = \frac{AD + \sqrt{ABCD}}{A}$$

Hence,

$$-\frac{V_1}{V_2}\frac{I_1}{I_2} = \frac{(AD + \sqrt{ABCD})^2}{AD} = (\sqrt{AD} + \sqrt{BC})^2$$

$$\sqrt{-\frac{V_1}{V_2}\frac{I_1}{I_2}} = \sqrt{AD} + \sqrt{BC}$$

$$= \sqrt{AD} + \sqrt{AD - 1} \qquad (\because AD - BC = 1)$$

$$\text{Let } \sqrt{AD} = \cosh\phi, \quad \text{then } \sqrt{AD-1} = \sinh\phi$$

$$\tanh\phi = \frac{\sqrt{AD-1}}{\sqrt{AD}} = \sqrt{\frac{BC}{AD}}$$

$$\phi = \tanh^{-1}\sqrt{\frac{BC}{AD}}$$

Also
$$\sqrt{-\frac{V_1}{V_2}\frac{I_1}{I_2}} = \cosh\phi + \sinh\phi = e^{\phi}$$

$$\phi = \log_e\sqrt{-\frac{V_1}{V_2}\frac{I_1}{I_2}} = \frac{1}{2}\log_e\left(-\frac{V_1}{V_2}\frac{I_1}{I_2}\right)$$

Relationship between Image parameters and the Open-circuit and Short-Circuit Impedances

Open and short-circuit impedance parameters can be expressed in terms of *ABCD* parameters.

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

When the output port is open-circuited, i.e., $I_2 = 0$,

$$V_1 = AV_2$$
$$I_1 = CV_2$$

Hence, the impedance measured at input port with output port open-circuited is

$$Z_{oc_1} = \frac{V_1}{I_1} = \frac{A}{C}$$

When the output port is short-circuited, i.e., $V_2 = 0$,

$$V_1 = -BI_2$$
$$I_1 = -DI_2$$

Hence, the impedance measured at input port with output port short-circuited is

$$Z_{sc_1} = \frac{V_1}{I_1} = \frac{B}{D}$$

Similarly the impedance measured at output port with input port open-circuited and short-circuited are

$$Z_{oc_2} = \frac{D}{C}$$

$$Z_{sc_2} = \frac{B}{A}$$

The ratio of short-circuit to open-circuit impedance at the two ports is

$$\frac{Z_{sc_1}}{Z_{oc_1}} = \frac{Z_{sc_2}}{Z_{oc_2}} = \frac{BC}{AD}$$

The image parameters can also be expressed in terms of open circuit and short-circuit impedances.

$$Z_{i1} = \sqrt{\frac{AB}{CD}} = \sqrt{Z_{oc_1} Z_{sc_1}}$$

$$Z_{i2} = \sqrt{\frac{BD}{AC}} = \sqrt{Z_{oc_2} Z_{sc_2}}$$

$$\phi = \tanh^{-1}\sqrt{\frac{BC}{AD}} = \tanh^{-1}\sqrt{\frac{Z_{sc_1}}{Z_{oc_1}}} = \tanh^{-1}\sqrt{\frac{Z_{sc_2}}{Z_{oc_2}}}$$

Example 13.65 *The ABCD parameters of a two-port network are given as,* $A = \frac{6}{5}, B = \frac{17}{5}, C = \frac{1}{5}, D = \frac{7}{5}$. *Find the image parameters.*

Solution

$$Z_{i1} = \sqrt{\frac{AB}{CD}} = \sqrt{\frac{\left(\frac{6}{5}\right)\left(\frac{17}{5}\right)}{\left(\frac{1}{5}\right)\left(\frac{7}{5}\right)}} = 3.82\ \Omega$$

$$Z_{i2} = \sqrt{\frac{BD}{AC}} = \sqrt{\frac{\left(\frac{17}{5}\right)\left(\frac{7}{5}\right)}{\left(\frac{6}{5}\right)\left(\frac{1}{5}\right)}} = 4.45\ \Omega$$

$$\phi = \tanh^{-1}\sqrt{\frac{BC}{AD}} = \tanh^{-1}\sqrt{\frac{\left(\frac{17}{5}\right)\left(\frac{1}{5}\right)}{\left(\frac{6}{5}\right)\left(\frac{7}{5}\right)}} = \tanh^{-1}\sqrt{\frac{17}{42}} = 0.75°$$

Example 13.66 *Find the image parameters of the network shown in Fig. 13.130.*

Fig. 13.130

Solution

$$Z_{oc_1} = 30 + 10 = 40\ \Omega$$

$$Z_{sc_1} = 30 + \frac{10 \times 20}{10 + 20} = 36.67\ \Omega$$

$$Z_{i1} = \sqrt{Z_{oc_1}\ Z_{sc_1}} = \sqrt{40 \times 36.67} = 38.3\ \Omega$$

$$Z_{oc_2} = 20 + 10 = 30\ \Omega$$

$$Z_{sc_2} = 20 + \frac{10 \times 30}{10 + 30} = 27.5\ \Omega$$

$$Z_{i2} = \sqrt{Z_{oc_2}\ Z_{sc_2}} = \sqrt{30 \times 27.5} = 28.72\ \Omega$$

$$\phi = \tanh^{-1}\sqrt{\frac{Z_{sc_1}}{Z_{oc_1}}} = \tanh^{-1}\sqrt{\frac{36.67}{40}} = 1.91°$$

Example 13.67 *Find image parameters for the lattice two-port network shown in Fig. 13.131.*

Fig. 13.131

Solution The network is redrawn as shown in Fig. 13.132.

$$Z_{oc_1} = Z_{oc_2} = (2+4) \parallel (2+4) = 3\ \Omega$$

$$Z_{sc_1} = Z_{sc_2} = (2 \parallel 4) + (2 \parallel 4) = 2.67\ \Omega$$

$$Z_{i1} = \sqrt{Z_{oc_1}\ Z_{sc_1}} = \sqrt{(3)(2.67)} = 2.83\ \Omega$$

$$Z_{i2} = \sqrt{Z_{oc_2}\ Z_{sc_2}} = \sqrt{(3)(2.67)} = 2.83\ \Omega$$

$$\phi = \tanh^{-1}\sqrt{\frac{Z_{sc_1}}{Z_{oc_1}}} = \tanh^{-1}\sqrt{\frac{2.67}{3}} = 1.76°$$

Fig. 13.132

Exercises

13.1 Determine Z-parameters for the network shown in Fig. 13.133.

Fig. 13.133

$$Z = \begin{bmatrix} \frac{13}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} \end{bmatrix}$$

13.2 Find Z-parameters for the network shown in Fig. 13.134.

Fig. 13.134

$$Z = \begin{bmatrix} \frac{4s^4 + 6s^2 + 1}{4s^3 + s} & \frac{4s^3}{4s^2 + 1} \\ \frac{4s^3}{4s^2 + 1} & \frac{4s^3 + 2s}{4s^2 + 1} \end{bmatrix}$$

13.3 Find Y-parameters of the network shown in Fig 13.135.

Fig. 13.135

$$Y = \begin{bmatrix} 0.36 & -0.033 \\ -0.033 & -0.36 \end{bmatrix}$$

13.4 Find Y-parameters for the network shown in Fig. 13.136.

Fig. 13.136

$$Y = \begin{bmatrix} \dfrac{10s^2+13s+2}{5s+6} & -\dfrac{2}{5s+6} \\ -\dfrac{2}{5s+6} & \dfrac{5s^2+6s+5}{5s+6} \end{bmatrix}$$

13.5 Find Y-parameters for the network shown in Fig. 13.137.

Fig. 13.137

$$Y = \begin{bmatrix} \dfrac{7s+6}{12} & -\dfrac{s}{4} \\ -\dfrac{s}{4} & \dfrac{s^2+4s+2}{4s} \end{bmatrix}$$

13.6 Find Y-parameters for the network shown in Fig. 13.138.

Fig. 13.138

$$Y = \begin{bmatrix} \dfrac{3}{20} & -\dfrac{1}{20} \\ -\dfrac{1}{4} & \dfrac{1}{4} \end{bmatrix}$$

13.7 Show the $ABCD$ parameters of the network shown in Fig. 13.139.

Fig. 13.139

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \dfrac{1+s^2}{s^2} & \dfrac{1+2s^2}{s^3} \\ \dfrac{1}{s} & \dfrac{1+s^2}{s^2} \end{bmatrix}$$

13.8 Find $ABCD$ parameters for the network shown in Fig. 13.140.

Fig. 13.140

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 27 & 206 \\ \dfrac{11}{2} & 42 \end{bmatrix}$$

13.9 For the network shown in Fig. 13.141, determine parameter h_{21}.

Fig. 13.141

$$\left[h_{21} = \frac{-(2+s)}{1+s}\right]$$

13.10 Determine Y and Z-parameters for the network shown in Fig. 13.142.

Fig. 13.142

$$\left[Y_{11} = 1\ \mho,\ Y_{12} = -0.5\ \mho,\ Y_{21} = 1.5\ \mho,\ Y_{22} = 0.5\ \mho\right.$$
$$\left. Z_{11} = \frac{2}{5}\Omega,\ Z_{12} = \frac{2}{5}\Omega,\ Z_{21} = -\frac{6}{5}\Omega,\ Z_{22} = \frac{4}{5}\Omega\right]$$

13.11 For the bridged T, R-C network shown in Fig. 13.143 determine Y-parameters using interconnections of two-port networks.

Fig. 13.143

$$Y = \begin{bmatrix} \dfrac{s^2+8s+8}{2(s+6)} & -\dfrac{(s^2+6s+8)}{2(s+6)} \\ -\dfrac{(s^2+6s+8)}{2(s+6)} & \dfrac{s^2+10s+8}{2(s+6)} \end{bmatrix}$$

13.12 For the network of Fig. 13.144, find Y-parameters using interconnection of two-port networks.

Fig. 13.144

$$Y = \begin{bmatrix} \dfrac{3}{4} & -\dfrac{1}{4} \\ -\dfrac{1}{4} & \dfrac{3}{4} \end{bmatrix}$$

13.13 Two identical sections of the network shown in Fig. 13.145 are connected in parallel. Obtain Y-parameters of the connection.

Fig. 13.145

$$Y = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{5}{4} \end{bmatrix}$$

13.14 Determine Y-parameters using interconnection of two-port networks for the network shown in Fig. 13.146.

Fig. 13.146

$$Y = \begin{bmatrix} 3.1 & -0.9 \\ -0.9 & 3.1 \end{bmatrix}$$

13.15 Determine the transmission parameters of the network shown in Fig. 13.147 using the concept of interconnection of two two-port networks.

Fig. 13.147

$$\begin{bmatrix} 1+3s^2+s^4 & 3s+4s^3+s^5 \\ 2s+s^3 & 1+3s^2+s^4 \end{bmatrix}$$

13.16 Two networks shown in Fig. 13.148 are connected in series. Obtain the Z-parameters of the resulting network.

Fig. 13.148

$$\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$$

13.17 Two identical sections of the network shown in Fig. 13.149 are connected in series-parallel. Determine the h-parameters of the overall network.

Fig. 13.149

$$\begin{bmatrix} \frac{15}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

13.18 The h-parameters of a two-port network shown in Fig. 13.150 are $h_{11} = 2\ \Omega$, $h_{12} = 4$, $h_{21} = -5$, $h_{22} = 2\ \mho$. Determine the supply voltage V_s if the power dissipated in the load resistor of 4 Ω is 25 W and $R_s = 2\ \Omega$.

Fig. 13.150

[58 V]

13.19 The Z-parameters of a two-port network are $Z_{11} = 2.1\ \Omega$, $Z_{12} = Z_{21} = 0.6\ \Omega$, $Z_{22} = 1.6\ \Omega$. A resistor of 2 Ω is connected across port 2. What voltage must be applied at port 1 to produce a current of 0.5 A in the 2 Ω resistor.

[6 V]

13.20 If a two-port network has $Z_{11} = 25\ \Omega$, $Z_{12} = Z_{21} = 20\ \Omega$, $Z_{22} = 50\ \Omega$, find the equivalent T-network.

[10 Ω, 30 Ω, 20 Ω]

Objective-Type Questions

13.1 The open-circuit impedance matrix of the two-port network shown in Fig. 13.151 is

Fig. 13.151

(a) $\begin{bmatrix} -2 & 1 \\ -8 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -8 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

13.2 Two two-port networks are connected in cascade. The combination is to be represented as a single two-port network. The parameters are obtained by multiplying the individual

(a) z-parameter matrix
(b) h-parameter matrix
(c) y-parameter matrix
(d) $ABCD$ parameter matrix

13.3 For a two-port network to be reciprocal

(a) $z_{11} = z_{22}$ (b) $y_{21} = y_{12}$
(c) $h_{21} = -h_{12}$ (d) $AD - BC = 0$

13.4 The short-circuit admittance matrix of a two-port network is $\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$. The two-port network is

(a) non-reciprocal and passive
(b) non-reciprocal and active
(c) reciprocal and passive
(d) reciprocal and active

13.5 A two-port network is shown in Fig. 13.152. The parameter h_{21} for this network can be given by

Fig. 13.152

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$
(c) $-\frac{3}{2}$ (d) $\frac{3}{2}$

13.6 The admittance parameter Y_{12} in the two-port network in Fig. 13.153.

Fig. 13.153

(a) –0.2 mho (b) 0.1 mho
(c) –0.05 mho (d) 0.05 mho

13.7 The Z-parameters Z_{11} and Z_{21} for the two-port network in Fig. 13.154 are,

Fig. 13.154

(a) $-\frac{6}{11}\Omega, \frac{16}{11}\Omega$ (b) $\frac{6}{11}\Omega, \frac{4}{11}\Omega$
(c) $\frac{6}{11}\Omega, -\frac{16}{11}\Omega$ (d) $\frac{4}{11}\Omega, \frac{4}{11}\Omega$

13.8 The impedance parameters Z_{11} and Z_{12} of a two-port network in Fig. 13.155.

Fig. 13.155

(a) 2.75 Ω, 0.25 Ω (b) 3 Ω, 0.5 Ω
(c) 3 Ω, 0.25 Ω (d) 2.25 Ω, 0.5 Ω

13.9 The h parameters of the circuit shown in Fig. 13.156.

Fig. 13.156

(a) $\begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & -1 \\ 1 & 0.05 \end{bmatrix}$

(c) $\begin{bmatrix} 30 & 20 \\ 20 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 1 \\ -1 & 0.05 \end{bmatrix}$

13.10 A two-port network is represented by *ABCD* parameters given by $\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$. If port 2 is terminated by R_L, then the input impedance seen at port 1 is given by

(a) $\dfrac{A + BR_L}{C + DR_L}$ (b) $\dfrac{AR_L + C}{BR_L + D}$

(c) $\dfrac{DR_L + A}{BR_L + C}$ (d) $\dfrac{B + AR_L}{D + CR_L}$

13.11 In the two-port network shown in Fig. 13.157, Z_{12} and Z_{21} are respectively

Fig. 13.157

(a) r_e and βr_0 (b) 0 and $-\beta r_0$

(c) 0 and βr_0 (d) r_e and $-\beta r_0$

13.12 If a two-port network is passive, then we have, with the usual notation, the following relationship for symmetrical network

(a) $h_{12} = h_{21}$
(b) $h_{12} = -h_{21}$
(c) $h_{11} = h_{22}$
(d) $h_{11}h_{22} - h_{12}h_{21} = 1$

13.13 A two-port network is defined by the following pair of equations $I_1 = 2V_1 + V_2$ and $I_2 = V_1 + V_2$. Its impedance parameters (Z_{11}, Z_{12}, Z_{21}, Z_{22}) are given by

(a) 2, 1, 1, 1 (b) 1, –1, –1, 2

(c) 1, 1, 1, 2 (d) 2, –1, –1, 1

13.14 A two-port network has transmission parameters $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. The input impedance of the network at port 1 will be

(a) $\dfrac{A}{C}$ (b) $\dfrac{AD}{BC}$

(c) $\dfrac{AB}{DC}$ (d) $\dfrac{D}{C}$

13.15 A two-port network is symmetrical if

(a) $Z_{11}\,Z_{22} - Z_{12}\,Z_{21} = 1$
(b) $AD - BC = 1$
(c) $h_{11}\,h_{22} - h_{12}\,h_{21} = 1$
(d) $Y_{11}\,Y_{22} - Y_{12}\,Y_{21} = 1$

13.16 For the network shown in Fig. 13.158 admittance parameters are Y_{11} = 8 mho, Y_{12} = Y_{21} = –6 mho and Y_{22} = 6 mho. The value of Y_A, Y_B and Y_C (in mho) will be respectively

(a) 2, 6, –6 (b) 2,6,0

(c) 2,0,6 (d) 2,6,8

Fig. 13.158

13.17 The impedance matrices of two two-port networks are given by $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 15 & 5 \\ 3 & 25 \end{bmatrix}$. If these two networks are connected in series, the impedance matrix of the resulting two-port network will be

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 7 & 28 \end{bmatrix}$

(c) $\begin{bmatrix} 15 & 2 \\ 5 & 3 \end{bmatrix}$ (d) inderminate

13.18 If the π network and T network are equivalent, then the values of R_1, R_2 and R_3 (in ohms) will be respectively

(a) 6, 6, 6 (b) 6, 6, 9
(c) 9, 6, 9 (d) 6, 9, 6

Fig. 13.159

13.19 For a two-port symmetrical bilateral network, if $A = 3$ and $B = 1$, the value of the parameter C will be

(a) 4 (b) 6
(c) 8 (d) 16

13.20 The impedance matrix for the network shown in Fig. 13.160 is

Fig. 13.160

(a) $\begin{bmatrix} 2s+1 & 2s \\ 2s & 2s+\frac{3}{s} \end{bmatrix}$ (b) $\begin{bmatrix} 2s+1 & -2s \\ -2s & 2s+\frac{3}{s} \end{bmatrix}$

(c) $\begin{bmatrix} 2s+1 & 2s \\ -2s & 2s+\frac{3}{s} \end{bmatrix}$ (d) $\begin{bmatrix} 2s+\frac{3}{2} & -2s \\ 2s & 2s+\frac{3}{s} \end{bmatrix}$

13.21 With the usual notations, a two-port resistive network satisfies the conditions $A = D = \frac{3}{2}B = \frac{4}{3}C$. The Z_{11} of the network is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$

(c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Answers to Objective-Type Questions

1. (a)	2. (d)	3. (b) , (c)	4. (b)	5. (a)	6. (c)	7. (c)
8. (a)	9. (d)	10. (d)	11. (b)	12. (d)	13. (b)	14. (a)
15. (c)	16. (c)	17. (b)	18. (a)	19. (c)	20. (a)	21. (b)

14

Fourier Analysis

14.1 INTRODUCTION

Any arbitrary periodic function can be represented by an infinite series of sinusoids of harmonically related frequencies. This representation is known as *Fourier-series representation*. The Fourier representation of periodic functions is extended to nonperiodic functions by letting the fundamental period T tend to infinity, and this Fourier method of representing nonperiodic functions as a function of frequency is called *Fourier transform*. The Fourier representation of functions is also known as *frequency-domain representation*. The Fourier-series representation can be obtained only for periodic functions, but the Fourier-transform technique can be applied to both periodic and nonperiodic functions to obtain the frequency-domain representation of functions. The information about magnitude and phase of various frequency components can be obtained by Fourier analysis. The plot of magnitude versus frequency is called *magnitude spectrum*, and the plot of phase versus frequency is called *phase spectrum*.

14.2 TRIGONOMETRIC FOURIER SERIES

Any periodic function *f(t)* can be expressed into an infinite trigonometric series if it satisfies the following conditions:

1. It is well defined and single valued, except possibly at a finite number of points.
2. It has a finite number of discontinuities in the period T.
3. It has a finite number of positive and negative maxima in the period T.

These conditions are known as *Dirichlet's conditions*.

Any function *f(t)* which is periodic and satisfies the Dirichlet's condition, can be represented in terms of sine and cosine functions.

$$f(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + a_n \cos n\,\omega t + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots + b_n \sin n\,\omega t + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega t + \sum_{n=1}^{\infty} b_n \sin n\,\omega t \qquad \ldots(14.1)$$

where the coefficients a_0, a_n and b_n are given by,

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\,\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\,\omega t\,dt$$

14.2.1 Evaluation of Fourier Coefficients

1. Evaluation of a_0 Integrating Eq. (14.1) w.r.t. t,

$$\int_0^T f(t)\,dt = a_0\int_0^T dt + \int_0^T\left(\sum_{n=1}^{\infty} a_n \cos n\,\omega t\right)dt + \int_0^T\left(\sum_{n=1}^{\infty} b_n \sin n\,\omega t\right)dt$$

$$= a_0\,T + 0 + 0$$

$$\left[\because \int_0^T \sin n\,\omega t\,dt = 0 \text{ for all } n \quad \text{and} \int_0^T \cos n\,\omega t\,dt = 0 \text{ for } n \neq 0\right]$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

2. Evaluation of a_n Multiplying Eq. (14.1) by $\cos n\,\omega t$ and integrating w.r.t t between the limits 0 and T,

$$\int_0^T f(t)\cos n\,\omega t\,dt = \int_0^T a_0 \cos n\,\omega t\,dt + \int_0^T \cos n\,\omega t \sum_{n=1}^{\infty} a_n \cos n\,\omega t\,dt + \int_0^T \cos n\,\omega t \sum_{n=1}^{\infty} b_n \sin n\,\omega t\,dt$$

$$= 0 + \frac{T}{2}a_n + 0$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\,\omega t\,dt$$

3. Evaluation of b_n Multiplying Eq. (14.1) by $\sin n\,\omega t$ and integrating w.r.t t between the limits 0 and T,

$$\int_0^T f(t)\sin n\,\omega t\,dt = \int_0^T a_0 \sin n\,\omega t\,dt + \int_0^T \sin n\,\omega t \sum_{n=1}^{\infty} a_n \cos n\,\omega t\,dt + \int_0^T \sin n\,\omega t \sum_{n=1}^{\infty} b_n \sin n\,\omega t\,dt$$

$$= 0 + 0 + \frac{T}{2}b_n$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\,\omega t\,dt$$

14.2.2 Complex Fourier Spectrum

A Fourier series expansion of a periodic function is equivalent to resolving the function in terms of its components of various frequencies. A periodic function with period T has frequency components of angular

frequencies $\omega, 2\omega, 3\omega, \ldots\ n\omega$ where $\omega = \dfrac{2\pi}{T}$. Thus, the periodic function *f(t)* possesses its spectrum of frequencies, known as *line spectrum*.

The spectrum exists only at $\omega, 2\omega, 3\omega, \ldots$ etc. Thus, the spectrum is not a continuous curve but exists only at some discrete values of ω. The line spectrum is the plot which shows the variation of magnitude and phase of the function w.r.t n.

(i) Magnitude Spectrum It is a plot of magnitude c_n versus $n\omega$ where $c_n = \sqrt{a_n^2 + b_n^2}$.

(ii) Phase Spectrum It is a plot of phase angle ϕ_n versus $n\omega$ where $\phi_n = \tan^{-1}\dfrac{b_n}{a_n}$.

Example 14.1 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.1. Draw the magnitude and phase spectrum.*

Fig. 14.1

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = A \quad 0 < t < \pi$$
$$= 0 \quad \pi < t < 2\pi$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega\, t + \sum_{n=1}^{\infty} b_n \sin n\,\omega t$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} 0\,dt\right]$$

$$= \frac{A}{2\pi}[t]_0^{\pi}$$

$$= \frac{A}{2\pi}(\pi - 0)$$

$$= \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\,\omega t\,dt$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} A\cos n\,t\,dt + \int_{\pi}^{2\pi} 0\,dt\right] \qquad [\because \omega = 1]$$

$$= \frac{A}{\pi}\left[\frac{\sin n\,t}{n}\right]_0^{\pi}$$

$$= \frac{A}{\pi}\left[\frac{1}{n}(\sin n\pi - \sin 0)\right]$$

$$= 0 \qquad [\because \sin n\pi = 0]$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\,\omega t\,dt$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} A\sin n\,t\,dt + \int_{\pi}^{2\pi} 0\,dt\right] \qquad [\because \omega = 1]$$

$$= \frac{A}{\pi}\left[-\frac{\cos n\,t}{n}\right]_0^{\pi}$$

$$= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0\right]$$

$$= \frac{A}{n\pi}[-(-1)^n + 1]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2A}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = \frac{A}{2} + \sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{2A}{n\pi}\sin n\,\omega t$$

$$= \frac{A}{2} + \frac{2A}{\pi}\sin \omega t + \frac{2A}{3\pi}\sin 3\,\omega t + \ldots\ldots$$

The magnitude and phase spectrum are shown in Fig. 14.2.

Fig. 14.2

Example 14.2 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.3. Draw the magnitude and phase spectrum.*

Fig. 14.3

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{2\pi} t \qquad 0 < t < 2\pi$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega\, t + \sum_{n=1}^{\infty} b_n \sin n\,\omega\, t$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{A}{2\pi}\, t\, dt$$

$$= \frac{A}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \frac{A}{4\pi^2}\left[\frac{4\pi^2}{2} - 0\right]$$

$$= \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\,\omega t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{A}{2\pi}\, t\cos n\, t\, dt \qquad [\because \omega = 1]$$

$$= \frac{A}{2\pi^2}\int_0^{2\pi} t\cos n\, t\, dt$$

$$= \frac{A}{2\pi^2}\left[t\left(\frac{\sin n\, t}{n}\right) - (1)\left(-\frac{\cos n\, t}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi^2}\left[2\pi\left(\frac{\sin 2\,n\pi}{n}\right) + \frac{\cos 2\,n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{A}{2\pi^2}\left[0 + \frac{1}{n^2} - \frac{1}{n^2}\right] \qquad \begin{bmatrix} \because \sin 2\,n\pi = 0 \\ \cos 2\,n\pi = 1 \end{bmatrix}$$

$$= 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\,\omega t\,dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{A}{2\pi}\,t\sin n\,dt \qquad [\because \omega = 1]$$

$$= \frac{A}{2\pi^2}\int_0^{2\pi} t\sin n\,t\,dt$$

$$= \frac{A}{2\pi^2}\left[t\left(-\frac{\cos n\,t}{n}\right) - (1)\left(-\frac{\sin n\,t}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi^2}\left[-2\pi\left(\frac{\cos 2\,n\pi}{n}\right) + \frac{\sin 2\,n\pi}{n^2} - \frac{\sin 0}{n^2}\right]$$

$$= \frac{A}{2\pi^2}\left[-\frac{2\pi}{n}\right] \qquad \begin{bmatrix} \because \cos 2\,n\pi = 1 \\ \sin 2\,n\pi = 0 \end{bmatrix}$$

$$= -\frac{A}{n\pi}$$

$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} -\frac{A}{n\,\pi}\sin n\,\omega t$$

$$= \frac{A}{2} - \frac{A}{\pi}\sin \omega t - \frac{A}{2\pi}\sin 2\,\omega t - \ldots\ldots$$

The magnitude and phase spectrum are shown in Fig. 14.4.

Fig. 14.4

Example 14.3 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.5.*

Fig. 14.5

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = A \sin \omega t$$

$$f(t) = A \sin t \qquad 0 < t < \pi$$

$$= 0 \qquad \pi < t < 2\pi$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega\, t + \sum_{n=1}^{\infty} b_n \sin n\,\omega\, t$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A \sin t\, dt + \int_{\pi}^{2\pi} 0\, dt\right]$$

$$= \frac{A}{2\pi}[-\cos t]_0^{\pi}$$

$$= \frac{A}{2\pi}[-\cos \pi + \cos 0]$$

$$= \frac{A}{2\pi}[1+1] \qquad [\because \cos \pi = -1]$$

$$= \frac{A}{\pi}$$

$$a_n = \frac{2}{T}\int_0^T f(t) \cos n\,\omega t\, dt$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} A \sin t \cos n\, t\, dt + \int_{\pi}^{2\pi} 0\, dt\right] \qquad [\because \omega = 1]$$

$$= \frac{A}{2\pi}\left[\int_0^{\pi} \{\sin(n+1)\, t - \sin(n-1)\, t\}\, dt\right] \qquad \left[\begin{array}{l}\because 2\cos A \sin B \\ \quad = \sin(A+B) - \sin(A-B)\end{array}\right]$$

$$= \frac{A}{2\pi}\left[-\frac{\cos(n+1)\, t}{n+1} + \frac{\cos(n-1)\, t}{n-1}\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1}\right]$$

$$= \frac{A}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{A}{2\pi}\left[-\frac{-(-1)^{n}}{n+1} + \frac{-(-1)^{n}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad \begin{bmatrix} (-1)^1 & = -1 \\ (-1)^{-1} & = -1 \end{bmatrix}$$

$$= \frac{A}{2\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{A}{2\pi}\left[(-1)^n\left(\frac{-2}{n^2-1}\right) + \left(\frac{-2}{n^2-1}\right)\right]$$

$$= \frac{A}{2\pi}\left(-\frac{2}{n^2-1}\right)\left[(-1)^n + 1\right]$$

$$= -\frac{A}{\pi(n^2-1)}\left[(-1)^n + 1\right]$$

$$a_n = \begin{cases} -\dfrac{2A}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\,\omega t\,dt$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} A\sin t\,\sin n\,t\,dt + \int_{\pi}^{2\pi} 0\,dt\right] \qquad [\because \omega = 1]$$

$$= \frac{A}{2\pi}\left[\int_0^{\pi}\{\cos(n-1)\,t - \cos(n+1)\,t\}\,dt\right]$$

$$= \frac{A}{2\pi}\left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[\frac{\sin(n-1)\,\pi}{n-1} - \frac{\sin(n+1)\,\pi}{n+1} - \frac{\sin 0}{n-1} + \frac{\sin 0}{n+1}\right]$$

$$= \frac{A}{2\pi}[0] \qquad n \neq 1 \qquad \begin{bmatrix} \sin(n-1)\,\pi & = 0 \\ \sin(n+1)\,\pi & = 0 \end{bmatrix}$$

$$= 0$$

For $n = 1$,

$$b_1 = \frac{2}{2\pi}\int_0^{\pi} A\sin t\,\sin t\,dt$$

$$= \frac{A}{\pi}\int_0^{\pi}\sin^2 t\,dt$$

$$= \frac{A}{\pi}\int_0^{\pi}\frac{1-\cos 2\,t}{2}\,dt$$

$$= \frac{A}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 0}{2}\right]$$

$$= \frac{A}{2\pi}[\pi] \qquad [\because \sin 2\pi = 0]$$

$$= \frac{A}{2}$$

$$f(t) = \frac{A}{\pi} + \sum_{\substack{n=1 \\ n=even}}^{\infty} \left\{-\frac{2A}{\pi(n^2-1)}\right\} \cos n\,\omega t + \frac{A}{2}\sin n\,\omega t$$

$$= \frac{A}{\pi} + \frac{A}{2}\sin \omega t - \frac{2A}{3\pi}\cos 2\,\omega t - \frac{2A}{15\pi}\cos 4\,\omega t - \ldots\ldots$$

14.3 WAVEFORM SYMMETRY

There are four types of waveform symmetry in any function:

(i) Even symmetry
(ii) Odd symmetry
(iii) Half-wave symmetry
(iv) Quarter-wave symmetry

14.3.1 Even Symmetry

A function is said to have even symmetry if $f(t) = f(-t)$ as shown in Fig. 14.6. Even functions are symmetrical about the vertical axis. Examples of even functions are x^2, $\cos x$, $\cosh x$, etc. The product of even and odd function is an odd function, and the product of two even functions is an even function.

Fig. 14.6 *Even functions*

For an even function,

$$f_e(t)\,dt = \int_{-T}^{0} f_e(t)\,dt + \int_{0}^{T} f_e(t)\,dt$$

Let $\qquad t = -x,$

$$\int_{-T}^{T} f_e(t)\,dt = \int_{0}^{T} f_e(-x)\,dx + \int_{0}^{T} f_e(t)\,dt$$

$$= \int_{0}^{T} f_e(x)\,dx + \int_{0}^{T} f_e(t)\,dt \qquad [\therefore f_e(-x) = f_e(x)]$$

$$= 2\int_{0}^{T} f_e(t)\,dt$$

Similarly, for an odd function,

$$\int_{-T}^{T} f_0(t)\,dt = \int_{-T}^{0} f_0(t) + \int_{0}^{T} f_0(t)\,dt$$

$$= -\int_{0}^{T} f_0(x)\,dx + \int_{0}^{T} f_0(t)\,dt$$

$$= -\int_{0}^{T} f_0(t)\,dt + \int_{0}^{T} f_0(t)\,dt$$

$$= 0$$

We know that

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\,\omega t\,dt$$

When *f(t)* is an even function,

$$a_0 = \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt \quad [\because f(t) \text{ and } \cos n\,\omega t \text{ are even functions}]$$

$$b_n = 0 \quad [\because f(t) \text{ is an even function and } \sin n\,\omega t \text{ is an odd functio}$$

14.3.2 Odd Symmetry

A function is said to have odd symmetry if $f(t) = -f(-t)$ as shown in Fig. 14.7. Odd functions are symmetrical about the origin. Examples of odd functions are x, x^3, $\sin x$, etc. The product of an even and odd function is an odd function, and the product of two odd functions is an even function.

Fig. 14.7 *Odd functions*

When *f(t)* is an odd function,

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\,\omega t\,dt$$

14.3.3 Half-Wave Symmetry

A function is said to have half-wave symmetry if $f(t) = -f\left(t \pm \frac{T}{2}\right)$ as shown in Fig. 14.8. The values of *f(t)* are equal and opposite during the interval $0 < t < \frac{T}{2}$ and $\frac{T}{2} < t < T$. Similarly, the values of *f(t)* are also equal and opposite during the interval $-\frac{T}{2} < t < 0$ and $-T < t < -\frac{T}{2}$.

Fig. 14.8 *Functions with half-wave symmetry*

When *f(t)* has half-wave symmetry,

$$a_0 = 0$$

$$a_n = 0 \qquad \text{if } n \text{ is even}$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt \qquad \text{if } n \text{ is odd}$$

$$b_n = 0 \qquad \text{if } n \text{ is even}$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\,\omega t\,dt \qquad \text{if } n \text{ is odd}$$

14.3.4 Quarter-Wave Symmetry

A function is said to have quarter-wave symmetry if it is a combination of even, odd and half-wave symmetry as shown in Fig. 14.9.

(a) Combination of odd and half-wave symmetry

(b) Combination of even and half-wave symmetry

Fig. 14.9 *Functions with quarter-wave symmetry.*

When *f(t)* has quarter-wave symmetry (with combination of even and half-wave symmetry),

$$a_0 = 0$$

$$a_n = 0 \qquad \text{if } n \text{ is even}$$

$$= \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\cos n\,\omega t\,dt \qquad \text{if } n \text{ is odd}$$

$$b_n = 0$$

When *f(t)* has quarter-wave symmetry (with combination of odd and half-wave symmetry),

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = 0 \qquad \text{if } n \text{ is even}$$

$$= \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\sin n\,\omega t\,dt \qquad \text{if } n \text{ is odd}$$

Example 14.4 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.10.*

Fig. 14.10

Solution The waveform has even symmetry.

$$b_n = 0$$

$$f(t) = A \qquad 0 < t < \frac{T}{4}$$

$$= 0 \qquad \frac{T}{4} < t < \frac{T}{2}$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega t$$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\,dt$$

$$= \frac{2}{T}\left[\int_0^{\frac{T}{4}} A\,dt + \int_{\frac{T}{4}}^{\frac{T}{2}} 0\,dt\right]$$

$$= \frac{2}{T}[At]_0^{\frac{T}{4}}$$

$$= \frac{2}{T}[A\frac{T}{4} - 0]$$

$$= \frac{A}{2}$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt$$

$$= \frac{4}{T}\left[\int_0^{\frac{T}{4}} A\cos n\,\omega t\,dt + \int_{\frac{T}{4}}^{\frac{T}{2}} 0\cos n\,\omega t\,dt\right]$$

$$= \frac{4A}{T}\int_0^{\frac{T}{4}} \cos n\,\omega t\,dt$$

$$= \frac{4A}{T}\left[\frac{\sin n\,\omega t}{n\omega}\right]_0^{\frac{T}{4}}$$

$$= \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}t\right)}{n\frac{2\pi}{T}}\right]_0^{\frac{T}{4}} \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

$$= \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right]$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi}\sin\frac{n\pi}{2}\right] \qquad [\because \sin 0 = 0]$$

$$= \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

$$a_n = \begin{cases} \frac{2A}{n\pi} & \text{for } n = 1,5,9,\ldots \\ -\frac{2A}{n\pi} & \text{for } n = 3,7,11,\ldots \\ 0 & \text{for } n = 2,4,6 \end{cases}$$

$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty}\left(\frac{2A}{n\pi}\sin\frac{n\pi}{2}\right)\cos n\,\omega t$$

$$= \frac{A}{2} + \frac{2A}{\pi}\cos\omega t - \frac{2A}{3\pi}\cos 3\,\omega t + \frac{2A}{5\pi}\cos 5\,\omega t - \ldots$$

Example 14.5 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.11.*

Fig. 14.11

Solution The waveform has even symmetry.

$$b_n = 0$$

$$f(t) = \frac{2A}{T}\,t \qquad\qquad 0 < t < \frac{T}{2}$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega t$$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\,dt$$

$$= \frac{2}{T}\int_0^{\frac{T}{2}} \frac{2A}{T}\,t\,dt$$

$$= \frac{4A}{T^2}\left[\frac{t^2}{2}\right]_0^{\frac{T}{2}}$$

$$= \frac{4A}{T^2}\left[\frac{T^2}{8} - 0\right]$$

$$= \frac{A}{2}$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} \frac{2A}{T}\,t\cos n\,\omega t\,dt$$

$$= \frac{8A}{T^2}\int_0^{\frac{T}{2}} t\cos n\,\omega t\,dt$$

$$= \frac{8A}{T^2}\left[t\left(\frac{\sin n\,\omega t}{n\,\omega}\right) - (1)\left(-\frac{\cos n\,\omega t}{n^2\,\omega^2}\right)\right]_0^{\frac{T}{2}}$$

$$= \frac{8A}{T^2}\left[\frac{t\sin\left(n\frac{2\pi}{T}t\right)}{n\frac{2\pi}{T}} + \frac{\cos\left(n\frac{2\pi}{T}t\right)}{n^2\frac{4\pi^2}{T^2}}\right]_0^{\frac{T}{2}} \qquad \left[\therefore\ \omega = \frac{2\pi}{T}\right]$$

$$= \frac{8A}{T^2}\left[\frac{T}{2}\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{2}\right)}{n\frac{2\pi}{T}} + \frac{\cos\left(n\frac{2\pi}{T}\frac{T}{2}\right)}{n^2\frac{4\pi^2}{T^2}} - \frac{1}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= \frac{8A}{T^2}\left[\frac{T^2}{4n\pi}\sin n\pi + \frac{T^2}{4n^2\pi^2}\cos n\pi - \frac{T^2}{4n^2\pi^2}\right]$$

$$= \frac{8A}{T^2}\left[0 + \frac{T^2}{4n^2\pi^2}(-1)^n - \frac{T^2}{4n^2\pi^2}\right] \qquad \left[\because \begin{array}{l} \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{array}\right]$$

$$= \frac{8A}{T^2}\left[\frac{T^2}{4n^2\pi^2}\left\{(-1)^n - 1\right\}\right]$$

$$= \frac{2A}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4A}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = \frac{A}{2} + \sum_{\substack{n=1 \\ n=odd}}^{\infty}\left(-\frac{4A}{n^2\pi^2}\right)\cos n\,\omega t$$

$$= \frac{A}{2} - \frac{4A}{\pi^2}\cos\omega t - \frac{4A}{9\pi^2}\cos 3\omega t - \ldots\ldots$$

Example 14.6 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.12.*

Fig. 14.12

Solution The waveform has even symmetry.

$$b_n = 0$$

$$T = \pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$$

$$f(t) = A\sin\omega t$$
$$= A\sin 2t \quad 0 < t < \pi$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega t$$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} A \sin 2\,t\,dt$$

$$= \frac{2A}{\pi}\left[-\frac{\cos 2\,t}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2A}{2\pi}\left[-\cos 2\frac{\pi}{2} + \cos 0\right]$$

$$= \frac{A}{\pi}\left(-\cos\pi + \cos 0\right) \qquad \left[\begin{array}{ll} \therefore \cos\pi & = -1 \\ \cos 0 & = \ \ 1 \end{array}\right]$$

$$= \frac{A}{\pi}\left[1+1\right]$$

$$= \frac{2A}{\pi}$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} A\sin 2t\cos 2n\,t\,dt$$

$$= \frac{2A}{\pi}\int_0^{\frac{\pi}{2}} \left\{\sin\,(2n+2)\,t - \sin\,(2n-2)\,t\right\} dt \qquad \left[\begin{array}{l} \therefore\ 2\cos A\sin B \\ \quad = \sin\,(A+B) - \sin\,(A-B) \end{array}\right]$$

$$= \frac{2A}{\pi}\left[-\frac{\cos\,(2n+2)\,t}{2n+2} + \frac{\cos\,(2n-2)\,t}{2n-2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2A}{\pi}\left[-\frac{\cos\left\{(2n+2)\frac{\pi}{2}\right\}}{2n+2} + \frac{\cos\left\{(2n-2)\frac{\pi}{2}\right\}}{2n-2} + \frac{\cos 0}{2n+2} - \frac{\cos 0}{2n-2}\right]$$

$$= \frac{A}{\pi}\left[-\frac{\cos\,(n+1)\,\pi}{n+1} + \frac{\cos\,(n-1)\,\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{A}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{A}{\pi}\left[-\frac{-(-1)^n}{n+1}+\frac{-(-1)^n}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right] \qquad \left[\because \begin{array}{l}(-1)^1 = -1\\ (-1)^{-1} = -1\end{array}\right]$$

$$= \frac{A}{\pi}\left[(-1)^n\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\right]$$

$$= \frac{A}{\pi}\left[(-1)^n\left(-\frac{2}{n^2-1}\right)+\left(-\frac{2}{n^2-1}\right)\right]$$

$$= \frac{A}{\pi}\left(-\frac{2}{n^2-1}\right)[(-1)^n+1]$$

$$= -\frac{2A}{\pi(n^2-1)}[(-1)^n+1]$$

$$a_n = \begin{cases} -\dfrac{4A}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = \frac{2A}{\pi} + \sum_{\substack{n=1 \\ n=\text{even}}}^{\infty}\left\{-\frac{4A}{\pi(n^2-1)}\right\}\cos 2n\,\omega t$$

$$= \frac{2A}{\pi} - \frac{4A}{3\pi}\cos 4\,\omega t - \frac{4A}{15\pi}\cos 8\,\omega t - \ldots\ldots$$

Example 14.7 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.13.*

Fig. 14.13

Solution The waveform has odd symmetry.

$$a_0 = 0 \text{ and } a_n = 0$$

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{\pi}t \quad 0 < t < \pi$$

The trigonometric Fourier series is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\,\omega t$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\,\omega t\,dt$$

$$= \frac{4}{2\pi}\int_0^{\pi} \frac{A}{\pi}\,t\sin n\,t\,dt \qquad [\because \omega = 1]$$

$$= \frac{2A}{\pi^2}\int_0^{\pi} t\sin n\,t\,dt$$

$$= \frac{2A}{\pi^2}\left[t\left(-\frac{\cos n\,t}{n}\right) - (1)\left(-\frac{\sin n\,t}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2A}{\pi^2}\left[-\pi\left(\frac{\cos n\pi}{n}\right) + \frac{\sin n\pi}{n^2} - \frac{\sin 0}{n^2}\right]$$

$$= \frac{2A}{\pi^2}\left[-\frac{\pi(-1)^n}{n}\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{2A}{n\pi}(-1)^n$$

$$= \begin{cases} \dfrac{2A}{n\pi} & \text{if } n \text{ is even} \\ -\dfrac{2A}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = \sum_{n=1}^{\infty}\left\{\frac{2A}{n\pi}(-1)^n\right\}\sin n\,\omega t$$

$$= \frac{2A}{\pi}\sin \omega t - \frac{A}{\pi}\sin 2\,\omega t + \frac{2A}{3\pi}\sin 3\,\omega t - \frac{A}{2\pi}\sin 4\,\omega t + \ldots\ldots$$

Example 14.8 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.14.*

Fig. 14.14

Solution The waveform has half-wave symmetry.

$$a_0 = 0$$

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{\pi} t \qquad 0 < t < \pi$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = \sum_{n=1}^{\infty} a_n \cos n\,\omega t + \sum_{n=1}^{\infty} b_n \sin n\,\omega t$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\,\omega t\,dt, \text{ if } n \text{ is odd}$$

$$= \frac{4}{2\pi}\int_0^{\pi} \frac{A}{\pi} t\cos nt\,dt \qquad [\because \omega = 1]$$

$$= \frac{2A}{\pi^2}\int_0^{\pi} t\cos nt\,dt$$

$$= \frac{2A}{\pi^2}\left[t\left(\frac{\sin nt}{n}\right) - (1)\left(-\frac{\cos nt}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2A}{\pi^2}\left[\pi\left(\frac{\sin n\pi}{n}\right) + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2A}{\pi^2}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \left[\begin{array}{l}\because \sin n\pi = 0 \\ \cos n\pi = (-1)^n\end{array}\right]$$

$$= \frac{2A}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = -\frac{4A}{n^2\pi^2} \qquad \text{if } n \text{ is odd}$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\,\omega t\,dt \qquad \text{if } n \text{ is odd}$$

$$= \frac{4}{2\pi}\int_0^{\pi} \frac{A}{\pi} t\sin n\,t\,dt \qquad [\because \omega = 1]$$

$$= \frac{2A}{\pi^2}\int_0^{\pi} t\sin n\,t\,dt$$

$$= \frac{2A}{\pi^2}\left[t\left(-\frac{\cos n\,t}{n}\right) - (1)\left(-\frac{\sin n\,t}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2A}{\pi^2}\left[-\pi\left(\frac{\cos n\pi}{n}\right)+\frac{\sin n\pi}{n^2}-\frac{\sin 0}{n^2}\right]$$

$$= \frac{2A}{\pi^2}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= \frac{2A}{n\pi}(-1)^n$$

$$b_n = -\frac{2A}{n\pi} \qquad \text{if } n \text{ is odd}$$

$$f(t) = \sum_{\substack{n=1\\ n=odd}}^{\infty}\left[\frac{2A}{n^2\pi^2}\left\{(-1)^n-1\right\}\right]\cos n\,\omega t + \sum_{\substack{n=1\\ n=odd}}^{\infty}\left\{\frac{2A}{n\pi}(-1)^n\right\}\sin n\,\omega t$$

$$= \left(-\frac{4A}{\pi^2}\cos\omega t - \frac{4A}{9\pi^2}\cos 2\,\omega t - \ldots\ldots\right)+\left(-\frac{2A}{\pi}\sin\omega t - \frac{2A}{3\pi}\sin 3\,\omega t - \ldots\ldots\right)$$

Example 14.9 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.15.*

Fig. 14.15

Solution The waveform has odd symmetry, half-wave symmetry and quarter-wave symmetry.

$$a_0 = 0 \text{ and } a_n = 0$$

$$f(t) = A \quad 0 < t < \frac{T}{2}$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\,\omega t$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\,\omega t\,dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} A\sin n\,\omega t\,dt$$

$$= \frac{4A}{T}\left[-\frac{\cos n\,\omega t}{n\omega}\right]_0^{\frac{T}{2}}$$

$$= \frac{4A}{T}\left[-\frac{\cos\left(n\frac{2\pi}{T}t\right)}{n\frac{2\pi}{T}}\right]_0^{\frac{T}{2}} \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

$$= \frac{4A}{T}\left[-\frac{1}{n\frac{2\pi}{T}}\left\{\cos\left(n\frac{2\pi}{T}\frac{T}{2}\right)-\cos 0\right\}\right]$$

$$= \frac{4A}{T}\left[-\frac{T}{2n\pi}\left(\cos n\pi - \cos 0\right)\right]$$

$$= -\frac{2A}{n\pi}\left[(-1)^n - 1\right] \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$= \frac{2A}{n\pi}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4A}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = \sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{4A}{n\pi}\sin n\,\omega t$$

$$= \frac{4A}{\pi}\sin \omega t + \frac{4A}{3\pi}\sin 3\,\omega t + \ldots$$

Example 14.10 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.16.*

Fig. 14.16

Solution The waveform has even symmetry, half-wave symmetry and quarter-wave symmetry.

$$a_0 = 0 \quad \text{and} \quad b_n = 0$$

$$f(t) = A \qquad 0 < t < \frac{T}{4}$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = \sum_{n=1}^{\infty} a_n \cos n\,\omega t$$

$$a_n = \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\cos n\,\omega t\,dt, \qquad \text{if } n \text{ is odd}$$

$$= \frac{8}{T}\int_0^{\frac{T}{4}} A\cos n\,\omega t\,dt$$

$$= \frac{8A}{T}\left[\frac{\sin n\,\omega t}{n\,\omega}\right]_0^{\frac{T}{4}}$$

$$= \frac{8A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}t\right)}{n\frac{2\pi}{T}}\right]_0^{\frac{T}{4}} \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

$$= \frac{8A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right]$$

$$= \frac{8A}{T}\left[\frac{T}{2n\pi}\sin\frac{n\pi}{2} - 0\right]$$

$$= \frac{4A}{n\pi}\sin\frac{n\pi}{2}$$

$$a_n = \begin{cases} \dfrac{4A}{n\pi} & \text{for } n = 1, 5, 9, \ldots\ldots\ldots \\ -\dfrac{4A}{n\pi} & \text{for } n = 3, 7, 11, \ldots\ldots\ldots \end{cases}$$

$$f(t) = \left(\sum_{n=1}^{\infty} \frac{4A}{n\pi}\sin n\frac{\pi}{2}\right)\cos n\,\omega t$$

$$= \frac{4A}{\pi}\cos\omega t - \frac{4A}{3\pi}\cos 3\,\omega t + \frac{4A}{5\pi}\cos 5\,\omega t - \ldots\ldots\ldots$$

Example 14.11 *Find the trigonometric Fourier series of the waveform shown in Fig. 14.17.*

Fig. 14.17

Solution The waveform has odd symmetry, half-wave symmetry and quarter-wave symmetry.

$$a_0 = 0 \text{ and } a_n = 0$$

$$f(t) = \frac{4A}{T}t \quad 0 < t < \frac{T}{4}$$

The trigonometric Fourier series of *f(t)* is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\,\omega t$$

$$b_n = \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\sin n\,\omega t\,dt, \qquad \text{if } n \text{ is odd}$$

$$= \frac{8}{T}\int_0^{\frac{T}{4}} \frac{4A}{T}t\sin n\,\omega t\,dt$$

$$= \frac{32A}{T^2}\int_0^{\frac{T}{4}} t\,\sin n\,\omega t\,dt$$

$$= \frac{32A}{T^2}\left[t\left(-\frac{\cos n\,\omega t}{n\,\omega}\right) - (1)\left(-\frac{\sin n\,\omega t}{n^2\,\omega^2}\right)\right]_0^{\frac{T}{4}}$$

$$= \frac{32A}{T^2}\left[-t\frac{\cos\left(n\frac{2\pi}{T}t\right)}{n\frac{2\pi}{T}} + \frac{\sin\left(n\frac{2\pi}{T}t\right)}{n^2\frac{4\pi^2}{T^2}}\right]_0^{\frac{T}{4}} \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

$$= \frac{32A}{T^2}\left[-\frac{T}{4}\frac{\cos\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}} + \frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n^2\frac{4\pi^2}{T^2}} + 0 - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= \frac{32A}{T^2}\left[-\frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{4A}{n\pi}\cos\frac{n\pi}{2} + \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \left[\because \cos\frac{n\pi}{2} = 0 \quad \text{if } n \text{ is odd}\right]$$

$$b_n = \begin{cases} \dfrac{8A}{n^2\pi^2} & \text{for } n = 1, 5, 9, \ldots \\ -\dfrac{8A}{n^2\pi^2} & \text{for } n = 3, 7, 11, \ldots \end{cases}$$

$$f(t) = \sum_{n=1}^{\infty}\left(\frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin n\,\omega t$$

$$= \frac{8A}{\pi^2}\sin\omega t - \frac{8A}{9\pi^2}\sin 3\,\omega t + \frac{8A}{25\pi^2}\sin 5\,\omega t - \ldots$$

14.4 EXPONENTIAL FOURIER SERIES

Any periodic function *f(t)* can be expressed into an infinite series in terms of exponential functions.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t} \qquad \ldots(14.2)$$

where the coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-jn\omega t}\, dt$$

$$\text{or } c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt$$

Evaluation of c_n Multiplying Eq. (14.2) by $e^{-jn\omega t}$ and integrating w.r.t t between the limits 0 and T,

$$\int_0^T f(t)\, e^{-jn\omega t}\, dt = \int_0^T e^{-jn\omega t} \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t}\, dt$$

$$= T\, c_n$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt$$

Magnitude spectrum is the plot of the magnitude c_n versus $n\omega$, and *phase spectrum* is the plot of the phase angle ϕ_n versus $n\omega$.

Example 14.12 *Find the exponential Fourier series of the waveform shown in Fig. 14.18. Draw the magnitude spectrum.*

Fig. 14.18

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = A \qquad 0 < t < \pi$$
$$= -A \quad \pi < t < 2\pi$$

The exponential Fourier series of *f(t)* is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t}$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\, e^{-jnt}\, dt + \int_{\pi}^{2\pi} (-A)\, e^{-jnt}\, dt\right] \quad [\because \omega = 1]$$

$$= \frac{A}{2\pi}\left[\frac{e^{-jnt}}{-jn}\right]_0^{\pi} - \frac{A}{2\pi}\left[\frac{e^{-jnt}}{-jn}\right]_{\pi}^{2\pi}$$

$$= \frac{A}{-j2n\pi}\left[e^{-jn\pi} - e^{0}\right] + \frac{A}{j2n\pi}\left[e^{-j2n\pi} - e^{-jn\pi}\right]$$

$$= \frac{A}{-j2n\pi}\left[(-1)^n - 1\right] + \frac{A}{j2n\pi}\left[1 - (-1)^n\right]$$

$$\left[\begin{array}{ll} \because e^{-jn\pi} & = (-1)^n \\ \quad e^{-j2n\pi} & = 1 \end{array}\right]$$

$$= \frac{2A}{j2n\pi}\left[1 - (-1)^n\right]$$

$$= -j\frac{A}{n\pi}\left[1 - (-1)^n\right] \qquad n \neq 0$$

$$c_n = 0 \qquad \text{if } n \text{ is even}$$

$$= -j\frac{2A}{n\pi} \qquad \text{if } n \text{ is odd}$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi}(-A)\,dt\right]$$

$$= \frac{A}{2\pi}[t]_0^{\pi} - \frac{A}{2\pi}[t]_{\pi}^{2\pi}$$

$$= \frac{A}{2\pi}[\pi - 0] - \frac{A}{2\pi}[2\pi - \pi]$$

$$= \frac{A}{2} - \frac{A}{2}$$

$$= 0$$

$$f(t) = \sum_{\substack{n=-\infty \\ n\ odd}}^{\infty}\left(-j\frac{2A}{n\pi}\right)e^{jn\omega t}$$

$$= \ldots + j\frac{2A}{3\pi}e^{-j3\omega t} + j\frac{2A}{\pi}e^{-j\omega t} - j\frac{2A}{\pi}e^{j\omega t} - j\frac{2A}{3\pi}e^{j3\omega t}\ldots$$

The magnitude spectrum is shown in Fig. 14.19.

Fig. 14.19

Example 14.13 *Find the exponential Fourier series of the waveform shown in Fig. 14.20. Draw the magnitude spectrum.*

Fig. 14.20

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{2\pi} t \qquad 0 < t < 2\pi$$

The exponential Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t}$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{A}{2\pi} t\, e^{-jnt}\, dt \qquad [\because \omega = 1]$$

$$= \frac{A}{4\pi^2}\int_0^{2\pi} t\, e^{-jnt}\, dt$$

$$= \frac{A}{4\pi^2}\left[t\left(\frac{e^{-jnt}}{-jn}\right) - \left\{\frac{e^{-jnt}}{(-jn)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{A}{4\pi^2}\left[2\pi\frac{e^{-j2n\pi}}{-jn} + \frac{e^{-j2n\pi}}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{A}{4\pi^2}\left[\frac{2\pi}{-jn} + \frac{1}{n^2} - \frac{1}{n^2}\right] \qquad \left[\because e^{-j2n\pi} = 1\right]$$

$$= \frac{A}{4\pi^2}\, j\frac{2\pi}{n}$$

$$= j\frac{A}{2\pi n} \qquad n \neq 0$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{A}{2\pi} t\, dt$$

$$= \frac{A}{4\pi^2}\int_0^{2\pi} t\, dt$$

$$= \frac{A}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \frac{A}{2}$$

$$f(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} j\,\frac{A}{2\pi n}\,e^{jn\omega t}$$

$$= \ldots - j\frac{A}{2\pi}e^{-j2\omega t} - j\frac{A}{4\pi}e^{-j\omega t} + \frac{A}{2} + j\frac{A}{2\pi}e^{j\omega t} + j\frac{A}{4\pi}e^{j2\omega t} + \ldots$$

The magnitude spectrum is shown in Fig. 14.21.

Fig. 14.21

Example 14.14 *Find the exponential Fourier series of the waveform shown in Fig. 14.22.*

Fig. 14.22

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = A\sin\omega t$$

$$f(t) = A\sin t \qquad 0 < t < \pi$$

$$= 0 \qquad \pi < t < 2\pi$$

The exponential Fourier series of *f(t)* is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t}$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,\sin t\, e^{-jnt}\, dt + \int_{\pi}^{2\pi} 0\, dt\right] \qquad [\because \omega = 1]$$

$$= \frac{A}{2\pi}\left[\int_0^{\pi} \sin t\, e^{-jnt}\, dt\right]$$

$$= \frac{A}{2\pi}\left[\frac{e^{-jnt}}{1-n^2}(-jn\sin t - \cos t)\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[\frac{e^{-jn\pi}}{1-n^2}(-jn\sin\pi - \cos\pi) - \frac{e^0}{1-n^2}(-jn\sin 0 - \cos 0)\right]$$

$$= \frac{A}{2\pi}\left[\frac{(-1)^n}{1-n^2}(0+1) - \frac{1}{1-n^2}(-1)\right] \qquad \left[\begin{array}{l} \because \sin n\pi = 0 \\ \quad \cos\pi = 0 \\ \quad e^{-jn\pi} = (-1)^n \end{array}\right]$$

$$= \frac{A}{2\pi}\left[\frac{1}{1-n^2}\left\{(-1)^n + 1\right\}\right], \qquad n \neq \pm 1$$

$$= \begin{cases} \dfrac{A}{\pi(1-n^2)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

When $n = \pm 1$, the expression of c_n becomes indeterminate. Hence, L' Hospital's rule is applied, i.e., the numerator and denominator are separately differentiated w.r.t n after which n is allowed to approach ± 1.

$$c_n = \frac{A}{2\pi}\left[\frac{1}{1-n^2}(e^{-jn\pi} + 1)\right]$$

$$= \frac{A}{2\pi}\frac{(e^{-jn\pi}+1)}{1-n^2}$$

$$= \frac{A}{2\pi}\left(\frac{-j\pi e^{-jn\pi}}{-2n}\right)$$

$$= j\frac{A}{4n}e^{-jn\pi}$$

when $n \to 1$, $\quad c_1 = j\dfrac{A}{4}e^{-j\pi} = -j\dfrac{A}{4} \qquad [\because e^{-j\pi} = -1]$

when $n \to -1$, $\quad c_{-1} = -j\dfrac{A}{4}e^{j\pi} = j\dfrac{A}{4} \qquad [\because e^{j\pi} = -1]$

When $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} A \sin t\, dt$$

$$= \frac{A}{2\pi}[-\cos t]_0^{\pi}$$

$$= \frac{A}{2\pi}[-\cos\pi + \cos 0]$$

$$= \frac{A}{2\pi}(1+1) \qquad [\because \cos\pi = (-1)]$$

$$= \frac{A}{\pi}$$

$$f(t) = \frac{A}{\pi} - j\frac{A}{4} + \sum_{\substack{n=-\infty \\ n=\text{even} \\ n\neq 0 \\ n\neq\pm 1}}^{\infty} \frac{A}{\pi(1-n^2)} e^{jn\omega t}$$

$$= \ldots - \frac{A}{3\pi} e^{-j2\omega t} + j\frac{A}{4} + \frac{A}{\pi} - j\frac{A}{4} - \frac{A}{3\pi} e^{j2\omega t} - \ldots$$

Example 14.15 *Find the exponential Fourier series of the waveform shown in Fig. 14.23.*

Fig. 14.23

Solution

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{\pi}t + A \qquad -\pi < t < 0$$

$$= -\frac{A}{\pi}t + A \qquad 0 < t < \pi$$

The exponential Fourier series of *f(t)* is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t}$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(\frac{A}{\pi}t + A\right)e^{-jnt}\,dt + \int_{0}^{\pi}\left(-\frac{A}{\pi}t + A\right)e^{-jnt}\,dt\right]$$

$$= \frac{A}{2\pi}\left[\left(\frac{t}{\pi}+1\right)\frac{e^{-jnt}}{-jn} - \frac{1}{\pi}\cdot\frac{e^{-jnt}}{(-jn)^2}\right]_{-\pi}^{0} + \frac{A}{2\pi}\left[\left(-\frac{t}{\pi}+1\right)\frac{e^{-jnt}}{-jn} - \left(-\frac{1}{\pi}\right)\frac{e^{-jnt}}{-jn^2}\right]_{0}^{\pi}$$

$$= \frac{A}{2\pi}\left[-\frac{1}{jn} + \frac{1}{\pi}\cdot\frac{1}{n^2} - \frac{1}{\pi}\cdot\frac{e^{jn\pi}}{n^2}\right] + \frac{A}{2\pi}\left[\frac{1}{jn} - \frac{1}{\pi}\frac{e^{-jn\pi}}{n^2} + \frac{1}{\pi}\cdot\frac{1}{n^2}\right]$$

$$= \frac{A}{2\pi}\left[\frac{2}{\pi n^2} - \frac{1}{\pi n^2}(e^{jn\pi} + e^{-jn\pi})\right]$$

$$= \frac{A}{2\pi}\left[\frac{2}{\pi n^2} - \frac{1}{\pi n^2}\cdot 2\cos n\pi\right]$$

$$= \frac{A}{n^2\pi^2}\left[1-(-1)^n\right], \qquad n \neq 0 \qquad [\therefore \cos n\pi = (-1)^n]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2A}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

For $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(\frac{A}{\pi}t + A\right)dt + \int_{0}^{\pi}\left(-\frac{A}{\pi}t + A\right)dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{A}{\pi}\frac{t^2}{2} + At\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{A}{\pi}\frac{t^2}{2} + At\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{A}{\pi}\frac{\pi^2}{2} + A\pi\right] + \frac{1}{2\pi}\left[-\frac{A}{\pi}\frac{\pi^2}{2} + A\pi\right]$$

$$= \frac{1}{2\pi}\left[-\frac{A\pi}{2} + A\pi\right] + \frac{1}{2\pi}\left[-\frac{A\pi}{2} + A\pi\right]$$

$$= \frac{1}{2\pi}\left[\frac{A\pi}{2}\right] + \frac{1}{2\pi}\left[\frac{A\pi}{2}\right]$$

$$= \frac{A}{4} + \frac{A}{4}$$

$$= \frac{A}{2}$$

$$f(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ n=\text{odd} \\ n\neq 0}}^{\infty} \frac{2A}{n^2\pi^2}\,e^{jn\omega t}$$

$$= \frac{A}{2} + \frac{2A}{\pi^2}e^{j\omega t} + \frac{2A}{9\pi^2}e^{j3\omega t} + \ldots$$

14.5 AVERAGE VALUE OF A PERIODIC COMPLEX WAVE

Let the Fourier series representation of a periodic complex wave be,

$$v(t) = V_0 + \sum_{n=1}^{n} V_{nm} \sin(n\omega t + \phi_n)$$
$$= V_0 + V_{1m} \sin(\omega t + \phi_1) + V_{2m} \sin(2\omega t + \phi_2) + \ldots\ldots + V_{nm} \sin(n\omega t + \phi_n)$$

The average value of the periodic wave is given by

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\, dt$$

Since the integration of any sine function between $T = 0$ to $T = 2\pi$ is zero,

$$V_{\text{avg}} = V_0 + 0 + 0 + \cdots$$
$$= V_0$$

Hence, the average value of any periodic complex wave is the constant term of the Fourier series representing the periodic wave.

14.6 rms VALUE OF PERIODIC COMPLEX WAVE

Let the Fourier series representation of a periodic complex wave be,

$$v(t) = V_0 + \sum_{n=1}^{n} V_{nm} \sin(\omega t + \phi_n)$$
$$= V_0 + V_{1m} \sin(\omega t + \phi_1) + V_{2m} \sin(2\omega t + \phi_2) \ldots\ldots + V_{nm} \sin(n\omega t + \phi_n)$$

The rms value of the periodic wave is given by,

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\, dt}$$
$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \left[V_0 + V_{1m} \sin(\omega t + \phi_1) + V_{2m} \sin(\omega t + \phi_2) + \cdots + V_{nm} \sin(\omega t + \phi_n)\right]^2 dt}$$

It can be proved that

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_{1m}^2}{2} + \frac{V_{2m}^2}{2} + \cdots + \frac{V_{nm}^2}{2}}$$
$$= \sqrt{V_0^2 + \left(\frac{V_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{V_{2m}}{\sqrt{2}}\right)^2 + \cdots + \left(\frac{V_{nm}}{\sqrt{2}}\right)^2}$$
$$= \sqrt{V_0^2 + V_1^2 + V_2^2 + \ldots + V_n^2}$$

where $V_1, V_2, \ldots, V_n$ are the rms values of the harmonic components of the wave. Hence, the rms value of any periodic complex wave is the square root of sum of squares of rms values of each harmonic component and the square of constant terms.

14.7 POWER SUPPLIED BY COMPLEX WAVE

Let the complex voltage be represented by the equation

$$v(t) = V_0 + \sum_{n=1}^{n} V_{nm} \sin(n\omega t + \phi_n)$$

be applied to a circuit. Let the equation of the resultant current be

$$i(t) = I_0 + \sum_{n=1}^{n} I_{nm} \sin(n\omega t + \phi_n + \psi_n)$$

The average power of the complex wave is given by

$$\begin{aligned}
P_{\text{avg}} &= \frac{1}{2\pi}\int_0^{2\pi} v \cdot i\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\left\{V_0 + \sum_{n=1}^{n} V_{nm}\sin(n\omega t + \phi_n)\right\}\cdot\left\{I_0 + \sum_{n=1}^{n} I_{nm}\sin(n\omega t + \phi_n + \psi_n)\right\}\right]dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[V_0 I_0 + V_0\sum_{n=1}^{n} I_{nm}\sin(n\omega t + \phi_n + \psi_n) + I_0\sum_{n=1}^{n} V_{nm}\sin(n\omega t + \phi_n)\right] \\
&\quad + \left[\left\{\sum_{n=1}^{n} V_{nm}\sin(n\omega t + \phi_n)\right\}\left\{\sum_{n=1}^{n} I_{nm}\sin(n\omega t + \phi_n + \psi_n)\right\}\right]dt \\
&= V_0 I_0 + \frac{V_{1m} I_{1m}}{2}\cos\psi_1 + \frac{V_{2m} I_{2m}}{2}\cos\psi_2 + \ldots\ldots + \frac{V_{nm} I_{nm}}{2}\cos\psi_n \\
&= V_0 I_0 + V_1 I_1 \cos\psi_1 + V_2 I_2 \cos\psi_2 + \ldots\ldots + V_n I_n \cos\psi_n
\end{aligned}$$

Hence, the average power supplied by a complex wave is the sum of the average power supplied by each harmonic component acting independently.

Example 14.16 *A complex wave is given by $i(t) = 10 + 100 \sin\omega t + 40 \sin 5\,\omega t$. Determine the average value and rms value of the complex wave.*

Solution

$$i(t) = 10 + 100\sin\omega t + 40\sin 5\omega t$$

$$I_{\text{avg}} = 10 \text{ A}$$

$$I_{\text{rms}} = \sqrt{(10)^2 + \left(\frac{100}{\sqrt{2}}\right)^2 + \left(\frac{40}{\sqrt{2}}\right)^2} = 76.81 \text{ A}$$

Example 14.17 *A complex wave of 240 V rms value has 20% third harmonic content, 5% fifth harmonic content and 2% seventh harmonic content. Find the rms value of the fundamental and each harmonic.*

Solution Let V_1, V_3, V_5 and V_7 be the rms values of the fundamental and harmonic voltages.

$$V_3 = 0.2\ V_1$$

$$V_5 = 0.05\ V_1$$

$$V_7 = 0.02\ V_1$$

$$V_{\text{rms}} = \sqrt{V_1^2 + V_3^2 + V_5^2 + V_7^2}$$
$$240 = \sqrt{V_1^2 + (0.2V_1)^2 + (0.05V_1)^2 + (0.02V_1)^2}$$
$$V_1 = 235.01 \text{ V}$$
$$V_3 = 0.2 \times 235.01 = 47 \text{ V}$$
$$V_5 = 0.05 \times 235.01 = 11.75 \text{ V}$$
$$V_7 = 0.02 \times 235.01 = 4.7 \text{ V}$$

Example 14.18 *Find the value of R if the average power dissipated in the resistor is 1000 W if the voltage has the following Fourier series*

$$v(t) = 200 \sin\omega t + 100 \sin 3\omega t + 50 \sin 5\omega t$$

Solution

$$V_{\text{rms}} = \sqrt{\left(\frac{200}{\sqrt{2}}\right)^2 + \left(\frac{100}{\sqrt{2}}\right)^2 + \left(\frac{50}{\sqrt{2}}\right)^2} = 162.02 \text{ V}$$
$$P = \frac{V_{\text{rms}}^2}{R}$$
$$1000 = \frac{(162.02)^2}{R}$$
$$R = 26.25\ \Omega$$

Example 14.19 *If $v(t) = 10 + 6\cos(t + 45°) + 1.8\cos(2t - 10)°$ volts and $i(t) = 3 + 1.4\cos(t + 20°) + 0.5\cos 2t$ A, determine the effective value of the voltage and current. Also, calculate the average power.*

Solution

$$v(t) = 10 + 6\cos(t + 45°) + 1.8\cos(2t - 10)°$$
$$V_{\text{eff}} = \sqrt{V_{0m}^2 + \left(\frac{V_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{V_{2m}}{\sqrt{2}}\right)^2} = \sqrt{(10)^2 + \left(\frac{6}{\sqrt{2}}\right)^2 + \left(\frac{1.8}{\sqrt{2}}\right)^2} = 10.94 \text{ V}$$
$$i(t) = 3 + 1.4\cos(t + 20°) + 0.5\cos 2t$$
$$I_{\text{eff}} = \sqrt{I_{om}^2 + \left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{2m}}{\sqrt{2}}\right)^2} = \sqrt{(3)^2 + \left(\frac{1.4}{\sqrt{2}}\right)^2 + \left(\frac{0.5}{\sqrt{2}}\right)^2} = 3.18 \text{ A}$$
$$P_{\text{avg}} = V_{0m}I_{0m} + \frac{V_{1m}I_{1m}}{2}\cos\phi_1 + \frac{V_{2m}I_{2m}}{2}\cos\phi_2$$
$$= 10 \times 3 + \frac{6 \times 1.4}{2}\cos(45° - 20°) + \frac{1.8 \times 5}{2}\cos(-10°)$$
$$= 34.25 \text{ W}$$

Example 14.20 *A series RL circuit with $R = 10\ \Omega$ and $L = 5$ H contains a current $i(t) = 10 \sin 1000t + 5\sin 3000t + 3\sin 5000t$. Find the effective value of voltage and the average power.*

Solution

$$i(t) = 10\sin 1000t + 5\sin 3000t + 3\sin 5000t$$
$$\omega = 1000 \text{ rad/s}$$

For fundamental harmonic,

$$R_1 = 10\ \Omega$$

$$X_{L_1} = \omega L = 1000 \times 5 = 5000\ \Omega$$

$$\mathbf{Z}_1 = 10 + j5000 = 5000\angle 89.89°\ \Omega$$

For third harmonic,

$$R_3 = 10\ \Omega$$

$$X_{L_3} = 3\omega L = 3 \times 1000 \times 5 = 15000\ \Omega$$

$$\mathbf{Z}_3 = 10 + j15000 = 15000\angle 89.96°\ \Omega$$

For fifth harmonic,

$$R_5 = 10\ \Omega$$

$$X_{L_5} = 5\omega L = 5 \times 1000 \times 5 = 25000\ \Omega$$

$$\mathbf{Z}_5 = 10 + j25000 = 25000\angle 89.98°\ \Omega$$

$$\begin{aligned} v(t) &= (10)(5000)\sin(1000t + 89.89°) + (5)(15000)\sin(3000t + 89.96°) \\ &\quad + 3(25000)\sin(5000t + 89.98°) \\ &= 50000\sin(1000t + 89.89°) + 75000\sin(3000t + 89.96°) \\ &\quad + 75000\sin(5000t + 89.98°) \end{aligned}$$

$$V_{\text{eff}} = \sqrt{\left(\frac{V_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{V_{2m}}{\sqrt{2}}\right)^2 + \left(\frac{V_{3m}}{\sqrt{2}}\right)^2} = \sqrt{\left(\frac{50000}{\sqrt{2}}\right)^2 + \left(\frac{75000}{\sqrt{2}}\right)^2 + \left(\frac{75000}{\sqrt{2}}\right)^2} = 82.92\ \text{kV}$$

$$\begin{aligned} P_{\text{avg}} &= \frac{V_{1m}I_{1m}}{2}\cos\phi_1 + \frac{V_{2m}I_{2m}}{2}\cos\phi_2 + \frac{V_{3m}I_{3m}}{2}\cos\phi_3 \\ &= \frac{50000 \times 10}{2}\cos(89.89°) + \frac{75000 \times 5}{2}\cos(89.96°) + \frac{75000 \times 3}{2}\cos(89.98°) \\ &= 650.13\ \text{W} \end{aligned}$$

Example 14.21 *The voltage given by $v(t) = 100\cos 314t + 50\sin(1570t - 30°)$ is applied to a circuit consisting of a resistor of 10 Ω, an inductor of 0.02 H and a capacitor of 50 μF. Determine the instantaneous current through the circuit. Also find the rms value of the voltage and current.*

Solution

$$v(t) = 100\cos 314t + 50\sin(1570t - 30°)$$

$$\omega = 314\ \text{rad/s}$$

For fundamental harmonic,

$$R_1 = 10\ \Omega$$

$$X_{L_1} = \omega L = 314 \times 0.02 = 6.28\ \Omega$$

$$X_{C_1} = \frac{1}{\omega C} = \frac{1}{314 \times 50 \times 10^{-6}} = 63.69\ \Omega$$

$$\mathbf{Z}_1 = 10 + j6.28 - j63.69 = 58.27\angle -80.12°\ \Omega$$

For fifth harmonic,

$$R_5 = 10\ \Omega$$

$$X_{L_5} = 5\omega L = 5 \times 314 \times 0.02 = 31.4\ \Omega$$

$$X_{C_5} = \frac{1}{5\omega C} = \frac{1}{5 \times 314 \times 50 \times 10^{-6}} = 12.74\ \Omega$$

$$\mathbf{Z}_2 = 10 + j31.4 - j12.74 = 21.17\angle 61.81°\ \Omega$$

$$i(t) = \frac{100}{58.27}\cos(314t + 80.12°) + \frac{50}{21.17}\sin(1570t - 30° - 61.81°)$$

$$= 1.72\cos(314t + 80.12°) + 2.36\sin(1570t - 91.81°)$$

$$V_{\text{rms}} = \sqrt{\left(\frac{V_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{V_{2m}}{\sqrt{2}}\right)^2} = \sqrt{\left(\frac{100}{\sqrt{2}}\right)^2 + \left(\frac{50}{\sqrt{2}}\right)^2} = 79.06\ \text{V}$$

$$I_{\text{rms}} = \sqrt{\left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{2m}}{\sqrt{2}}\right)^2} = \sqrt{\left(\frac{1.72}{\sqrt{2}}\right)^2 + \left(\frac{2.36}{\sqrt{2}}\right)^2} = 2.06\ \text{A}$$

14.8 FOURIER TRANSFORM

Any nonperiodic signal can be viewed as a limiting case of a periodic signal where the period of the signal approaches infinity.

Consider a periodic signal *f(t)* with period *T*. The Fourier series representation for this signal is

$$f(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t} \qquad \ldots(14.3)$$

where

$$c_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-jn\omega t}\, dt$$

and

$$\omega = \frac{2\pi}{T}$$

A typical discrete magnitude spectrum is shown in Fig. 14.24.

Fig. 14.24 *Magnitude spectrum*

The spacing between successive harmonics is $\omega = \dfrac{2\pi}{T} = \Delta\omega$ (say)

$$\frac{1}{T} = \frac{\Delta\omega}{2\pi}$$

Hence, Eq. (14.3) can be written as

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t} (\Delta\omega)$$

Considering the limiting case as $T \to \infty, \Delta\omega \to 0,$ the discrete line in the spectrum merges to form a continuous spectrum. The infinite sum becomes an integral. c_n is now defined for all frequencies, rather only for integral multiplies of ω. The integer n has no significance now and in the limit as $T \to \infty, c_n$ becomes a continuous function.

$$F(\omega) = \lim_{T \to \infty} c_n$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \qquad \ldots(14.4)$$

The Fourier series for a periodic signal now becomes the Fourier integral representation for nonperiodic signal over the entire interval given by,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega \qquad \ldots(14.5)$$

The above Eqs (14.4) and (14.5) represent a Fourier transform pair.

Conditions for Existence of Fourier Transform

Fourier transform $F(\omega)$ exists if the function $f(t)$ satisfies the following Dirichlet's conditions:

(i) $f(t)$ is absolutely integrable.
(ii) $f(t)$ has a finite number if maxima and minima.
(iii) $f(t)$ has a finite number of discontinuities.

There are certain functions which are not absolutely intergrable. In such cases, we find the Fourier transform of the function as the limiting value of the Fourier transform of some other function.

14.9 FOURIER TRANSFORMS OF SOME USEFUL FUNCTIONS

1. One-sided Exponential Function One-sided exponential function shown in Fig. 14.25 is defined by the equation

$$f(t) = e^{-at} \qquad 0 < t < \infty$$
$$= 0 \qquad t < 0$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(a+j\omega)t}\, dt$$

Fig. 14.25 *One-sided exponential function*

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^\infty$$

$$= \frac{1}{a+j\omega}$$

$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

The magnitude and phase spectrum are shown in Fig. 14.26.

Fig. 14.26 *(a) Magnitude spectrum, (b) Phase spectrum*

2. *Two-sided Exponential Function* Two-sided exponential function shown in Fig. 14.27 is defined by the equation

$$f(t) = e^{-at} \qquad t > 0$$

$$= e^{at} \qquad t < 0$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^\infty$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

$$|F(\omega)| = \frac{2a}{a^2+\omega^2}, \qquad \phi(\omega) = 0$$

Fig. 14.27 *Two-sided exponential function*

The magnitude spectrum is shown in Fig. 14.28.

Fig. 14.28 *Magnitude Spectrum*

3. Gate Function The gate function shown in Fig. 14.29 is defined by the equation

$$f(t) = A \qquad -\frac{\tau}{2} < t < \frac{\tau}{2}$$

$$= 0 \qquad |t| > \frac{\tau}{2}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-j\omega t}\, dt$$

$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= A\left[\frac{e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}}{-j\omega}\right]$$

$$= \frac{2A}{\omega}\left[\frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j}\right]$$

$$= \frac{2A}{\omega}\sin\left(\omega\frac{\tau}{2}\right)$$

$$= 2A\frac{\tau}{2}\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}}$$

$$= A\tau\,\mathrm{Sa}\left(\omega\frac{\tau}{2}\right)$$

Fig. 14.29 *Gate Function*

The magnitude and phase spectrum are shown in Fig. 14.30.

Fig. 14.30 *(a) Magnitude spectrum (b) Phase spectrum*

4. Impulse Function The impulse function shown in Fig. 14.31 is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad t = 0$$

Fig. 14.31 *Impulse function*

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$

$$= e^{-j\omega(0)} \int_{-\infty}^{\infty} \delta(t)\,dt \left\{ \begin{array}{l} \because \int_{-\infty}^{\infty} f(t)\delta(t) = f(0)\int_{-\infty}^{\infty} \delta(t)dt = f(0)1 = f(0) \\ \text{where } f(0) \text{ is the value of the function } f(t) \text{ at } t = 0. \end{array} \right\}$$

$$= 1$$

The magnitude spectrum is shown in Fig. 14.32.

Fig. 14.32 *Magnitude spectrum*

5. Constant Function The constant function shown in Fig. 14.33 is defined by the equation

$$f(t) = A \qquad -\infty < t < \infty$$

$$\int_{-\infty}^{\infty} A\,dt = \infty$$

Fig. 14.33 *Constant function*

The Fourier transform cannot be obtained directly. Hence, its Fourier transform can be evaluated as limiting value of Fourier transform of gate function.

$$f(t) = \lim_{\tau \to \infty} G(t)$$

$$F(\omega) = \mathbf{F}\{f(t)\} = \mathbf{F}\left\{\lim_{\tau \to \infty} G(t)\right\}$$

$$= \lim_{\tau \to \infty} \mathbf{F}\{G(t)\}$$

$$= \lim_{\tau \to \infty} A\tau \mathrm{Sa}\left(\omega \frac{\tau}{2}\right)$$

$$= A \lim_{\tau \to \infty} \tau \mathrm{Sa}\left(\omega \frac{\tau}{2}\right)$$

$$= 2\pi A \lim_{\frac{\tau}{2}\to\infty} \frac{\tau}{2\pi} Sa\left(\frac{\omega\tau}{2}\right)$$

$$= 2\pi A \lim_{\tau\to\infty} \frac{\tau}{2\pi} Sa\left(\frac{\omega\tau}{2}\right)$$

$$= 2\pi A\, \delta(\omega) \qquad \left[\because \lim_{K\to\infty} \frac{K}{\pi}\mathrm{Sa}(Kt)=\delta(t)\right]$$

Fig. 14.34 *Magnitude spectrum*

The magnitude spectrum is shown in Fig. 14.34.

6. *Signum Function* The signum function shown in Fig. 14.35 is defined by the equation

$$f(t) = 1 \qquad t > 0$$
$$= -1 \qquad t < 0$$

Fig. 14.35 *Signum function*

Since signum function is not absolutely integrable, its Fourier transform is evaluated as limiting value of Fourier transform of an exponential function.

$$f(t) = \lim_{a\to 0} \begin{cases} e^{-at} & t > 0 \\ -e^{at} & t < 0 \end{cases}$$

$$F(\omega) = \lim_{a\to 0}\left[\int_{-\infty}^{0} -e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt\right]$$

$$= \lim_{a\to 0}\left[-\int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt\right]$$

$$= \lim_{a\to 0}\left\{-\left[\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}\right\}$$

$$= \lim_{a\to 0}\left\{-\frac{1}{a-j\omega} + \frac{1}{a+j\omega}\right\}$$

$$= \lim_{a\to 0}\left\{-\frac{a-j\omega+a-j\omega}{a^2+\omega^2}\right\}$$

$$= \lim_{a\to 0} -\frac{2j\omega}{a^2+\omega^2}$$

$$= -\frac{2j\omega}{\omega^2} = \frac{2}{j\omega}$$

$$|F(\omega)| = \frac{2}{\omega}$$

Fig. 14.36 *Exponential function*

The magnitude and phase spectrum are shown in Fig. 14.37.

Fig. 14.37 *(a) Magnitude spectrum (b) Phase spectrum*

7. Unit Step Function The unit step function shown in Fig. 14.38 is defined by the equation

$$u(t) = 1 \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

$u(t)$ is not absolutely integrable.

Fig. 14.38 *Unit step function*

$$u(t) = \frac{1}{2}\{1 + \text{sgn}(t)\}$$

$$F(\omega) = \mathbf{F}\{u(t)\} = \mathbf{F}\left\{\frac{1}{2} + \frac{1}{2}\text{sgn}(t)\right\}$$

$$= \mathbf{F}\left\{\frac{1}{2}\right\} + \frac{1}{2}\mathbf{F}\{\text{sgn}(t)\}$$

$$= 2\pi\left(\frac{1}{2}\right)\delta(\omega) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

The magnitude spectrum is shown in Fig. 14.39.

Fig. 14.39 *Magnitude spectrum*

8. Sine Function The sine function shown in Fig. 14.40 is defined by the equation

$$f(t) = \sin \omega_0 t \qquad -\infty < t < \infty$$

The sine function is not absolutely integrable in the interval $(-\infty, \infty)$. Hence, its Fourier transform is evaluated as a limiting case of some other function.

$$f(t) = \lim_{\tau \to \infty} \sin \omega_0 t \, G(t) \qquad \left\{ -\frac{\tau}{2} < t < \frac{\tau}{2} \right\}$$

$$F(\omega) = \mathbf{F}\left\{ \lim_{\tau \to \infty} \sin \omega_0 t \, G(t) \right\}$$

$$= \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin \omega_0 t \, e^{-j\omega t} \, dt$$

Fig. 14.40 *Sine function*

$$= \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin \omega_0 t \, e^{-j\omega t} \, dt$$

$$= \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} e^{-j\omega t} \, dt$$

$$= \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{e^{-j(\omega-\omega_0)t} - e^{-j(\omega+\omega_0)t}}{2j} dt$$

$$= \lim_{\tau \to \infty} \frac{1}{2j} \left\{ \left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} - \left[\frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \right\}$$

$$= \lim_{\tau \to \infty} \frac{1}{2j} \left\{ \left[\frac{e^{-j(\omega-\omega_0)\frac{\tau}{2}} - e^{j(\omega-\omega_0)\frac{\tau}{2}}}{-j(\omega-\omega_0)} - \frac{e^{-j(\omega+\omega_0)\frac{\tau}{2}} - e^{j(\omega+\omega_0)\frac{\tau}{2}}}{-j(\omega+\omega_0)} \right] \right\}$$

$$= \lim_{\tau \to \infty} \frac{1}{j} \left\{ \left[\frac{e^{j(\omega-\omega_0)\frac{\tau}{2}} - e^{-j(\omega-\omega_0)\frac{\tau}{2}}}{2j(\omega-\omega_0)} - \frac{e^{-j(\omega+\omega_0)\frac{\tau}{2}} - e^{j(\omega+\omega_0)\frac{\tau}{2}}}{2j(\omega+\omega_0)} \right] \right\}$$

$$= \lim_{\tau \to \infty} \frac{1}{j} \left\{ \frac{\sin(\omega-\omega_0)\frac{\tau}{2}}{(\omega-\omega_0)} - \frac{\sin(\omega+\omega_0)\frac{\tau}{2}}{\omega+\omega_0} \right\}$$

$$= \lim_{\tau\to\infty} \frac{\tau}{2j}\left\{\frac{\sin(\omega-\omega_0)\frac{\tau}{2}}{(\omega-\omega_0)\frac{\tau}{2}} - \frac{\sin(\omega+\omega_0)\frac{\tau}{2}}{(\omega+\omega_0)\frac{\tau}{2}}\right\}$$

$$= \lim_{\tau\to\infty} \frac{\tau}{2j}\left\{\text{Sa}(\omega-\omega_0)\frac{\tau}{2} - \text{Sa}(\omega+\omega_0)\frac{\tau}{2}\right\}$$

$$= \frac{\pi}{j}\left[\lim_{\frac{\tau}{2}\to\infty}\left\{\frac{\tau}{2j}\text{Sa}(\omega-\omega_0)\frac{\tau}{2}\right\} - \lim_{\frac{\tau}{2}\to\infty}\left\{\frac{\tau}{2\pi}\text{Sa}(\omega+\omega_0)\frac{\tau}{2}\right\}\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

Fig. 14.41 *Magnitude spectrum*

The magnitude spectrum is shown in Fig. 14.41.

9. Cosine Function The cosine function shown in Fig. 14.42 is defined by the equation

$$f(t) = \cos\omega_0 t \qquad -\infty < t < \infty$$

The cosine function is not absolutely integrable in the interval $(-\infty, \infty)$. Hence, its Fourier transform is calculated as a limiting case of some other function.

Fig. 14.42 *Cosine function*

$$f(t) = \lim_{\tau\to\infty} \cos\omega_0 t\, G(t) \quad \left(-\frac{\tau}{2} < t < \frac{\tau}{2}\right)$$

$$F(\omega) = \mathbf{F}\left\{\lim_{\tau\to\infty} \cos\omega_0 t\, G(t)\right\}$$

$$= \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\omega_0 t\, G(t)\, e^{-j\omega t}\, dt$$

$$= \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos\omega_0 t\, e^{-j\omega t}\, dt$$

$$= \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-j\omega t}\, dt$$

$$= \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t}}{2} dt$$

$$= \lim_{\tau\to\infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{e^{-j(\omega-\omega_0)t}}{2} dt \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{e^{-j(\omega+\omega_0)t}}{2} dt$$

$$= \lim_{\tau\to\infty} \left\{ \left[\frac{e^{-j(\omega-\omega_0)t}}{-2j(\omega-\omega_0)} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} + \left[\frac{e^{-j(\omega+\omega_0)t}}{-2j(\omega+\omega_0)} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \right\}$$

$$= \lim_{\tau\to\infty} \left\{ \frac{e^{-j(\omega-\omega_0)\frac{\tau}{2}} - e^{j(\omega-\omega_0)\frac{\tau}{2}}}{-2j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)\frac{\tau}{2}} - e^{j(\omega+\omega_0)\frac{\tau}{2}}}{-2j(\omega+\omega_0)} \right\}$$

$$= \lim_{\tau\to\infty} \left\{ \frac{e^{j(\omega-\omega_0)\frac{\tau}{2}} - e^{-j(\omega-\omega_0)\frac{\tau}{2}}}{2j(\omega-\omega_0)} + \frac{e^{j(\omega+\omega_0)\frac{\tau}{2}} - e^{-j(\omega+\omega_0)\frac{\tau}{2}}}{2j(\omega+\omega_0)} \right\}$$

$$= \lim_{\tau\to\infty} \left\{ \frac{\sin(\omega-\omega_0)\frac{\tau}{2}}{(\omega-\omega_0)} + \frac{\sin(\omega+\omega_0)\frac{\tau}{2}}{(\omega+\omega_0)} \right\}$$

$$= \lim_{\tau\to\infty} \frac{\tau}{2} \left\{ \frac{\sin(\omega-\omega_0)\frac{\tau}{2}}{(\omega-\omega_0)\frac{\tau}{2}} + \frac{\sin(\omega+\omega_0)\frac{\tau}{2}}{(\omega+\omega_0)\frac{\tau}{2}} \right\}$$

$$= \pi \lim_{\frac{\tau}{2}\to\infty} \frac{\tau}{2\pi} \left\{ \mathrm{Sa}(\omega-\omega_0)\frac{\tau}{2} + \mathrm{Sa}(\omega+\omega_0)\frac{\tau}{2} \right\}$$

$$= \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Fig. 14.43 *Magnitude spectrum*

The magnitude spectrum is shown in Fig. 14.43.

9. Exponential Function $e^{j\omega_0 t}$ The Exponential function $e^{j\omega_0 t}$ is not absolute integrable. We know that

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

$$F(\omega) = \mathbf{F}\left\{e^{j\omega_0 t}\right\}$$

$$= \mathbf{F}\left\{\cos\omega_0 t + j\sin\omega_0 t\right\}$$

$$= \mathbf{F}\left\{\cos\omega_0 t\right\} + j\,\mathbf{F}\left\{\sin\omega_0 t\right\}$$

$$= \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] + j\frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$= 2\pi\delta(\omega-\omega_0)$$

The magnitude spectrum is shown in Fig. 14.44.

Fig. 14.44 *Magnitude spectrum*

14.10 FOURIER TRANSFORM OF PERIODIC FUNCTION

Periodic function exists from $-\infty$ to ∞.

$$\int_{-\infty}^{\infty} f(t)\, dt = \infty$$

i.e. *f(t)* is not absolutely integrable. Strictly speaking, Fourier transform of periodic function does not exist. But its Fourier transform can be evaluated as a limiting value of Fourier transform of some other function. A periodic function *f(t)* with angular frequency ω_0 can be represented in exponential Fourier series as,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-jn\omega_0 t}\, dt \text{ and } \omega_0 = \frac{2\pi}{T}$$

$$\mathbf{F}\{f(t)\} = \mathbf{F}\left\{\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{\infty} c_n \mathbf{F}\left\{e^{jn\omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{\infty} c_n\, 2\pi\, \delta(\omega - n\omega_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} c_n\, \delta(\omega - n\omega_0)$$

Example 14.22 *Find the Fourier transform of periodic gate function shown in Fig. 14.45.*

Fig. 14.45

Solution

$$f(t) = A \qquad -\frac{\tau}{2} < \tau < \frac{\tau}{2}$$

$$= 0 \qquad |t| > \frac{\tau}{2}$$

$$c_n = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= \frac{A}{n\omega_0\tau}\left[\frac{e^{-jn\omega_0\frac{\tau}{2}} - e^{jn\omega_0\frac{\tau}{2}}}{-j}\right]$$

$$= \frac{2A}{n\omega_0\tau}\left[\frac{e^{jn\omega_0\frac{\tau}{2}} - e^{-jn\omega_0\frac{\tau}{2}}}{2j}\right]$$

$$= \frac{2A}{n\omega_0\tau}\sin n\omega_0\frac{\tau}{2}$$

$$= \frac{2A\frac{\tau}{2}}{T}\frac{\sin n\omega_0\frac{\tau}{2}}{n\omega_0\frac{\tau}{2}}$$

$$= \frac{A\tau}{T}\,\mathrm{Sa}\left(n\omega_0\frac{\tau}{2}\right)$$

$$F(\omega) = 2\pi\sum_{n=-\infty}^{\infty} c_n\delta(\omega - n\omega_0)$$

$$= 2\pi\sum_{n=-\infty}^{\infty}\frac{A\tau}{T}\mathrm{Sa}\left(n\omega_0\frac{\tau}{2}\right)\delta(\omega - n\omega_0)$$

$$= \frac{2\pi A\tau}{T}\sum_{n=-\infty}^{\infty}\mathrm{Sa}\left(n\omega_0\frac{\tau}{2}\right)\delta(\omega - n\omega_0)$$

$$= \omega_0\, A\tau\sum_{n=-\infty}^{\infty}\mathrm{Sa}\left(n\omega_0\frac{\tau}{2}\right)\delta(\omega - n\omega_0)$$

The Fourier transform is shown in Fig. 14.46.

Fig. 14.46

Example 14.23 *Find the Fourier transform of an impulse train shown in Fig. 14.47.*

Fig. 14.47

Solution

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} \delta_T(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T} e^{-jn\omega_0 (0)} \delta(0)$$

$$= \frac{1}{T}$$

$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n\, \delta(\omega - n\omega_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T}\, \delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

The Fourier transform is shown in Fig. 14.48.

Fig. 14.48

14.11 PROPERTIES OF FOURIER TRANSFORM

1. Linearity

If $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$ then $a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$

Proof:

$$\mathbf{F}\{a_1 f_1(t) + a_2 f_2(t)\} = \int_{-\infty}^{\infty} \{a_1 f_1(t) + a_2 f_2(t)\} e^{-j\omega t}\, dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1(t)\, e^{-j\omega t}\, dt + a_2 \int_{-\infty}^{\infty} f_2(t)\, e^{-j\omega t}\, dt = a_1 F_1(\omega) + a_2 F_2(\omega)$$

2. Time Scaling

If $f(t) \leftrightarrow F(\omega)$ then $f(at) \leftrightarrow \dfrac{1}{|a|} F\left(\dfrac{\omega}{a}\right)$

Proof:

(a) When $a > 0$

$$\mathbf{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t}\, dt$$

Putting $at = \lambda, t = \dfrac{\lambda}{a}, dt = \dfrac{d\lambda}{a}$

$$\mathbf{F}\{f(at)\} = \int_{-\infty}^{\infty} f(\lambda) e^{-j\left(\frac{\omega}{a}\right)\lambda} \frac{d\lambda}{a} = \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

(b) When $a < 0$

$$\mathbf{F}\{f(at)\} = -\frac{1}{a} F\left(\frac{\omega}{a}\right)$$

Hence, $$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

The above proof shows that compression in time domain is equivalent to expansion in frequency domain and vice versa.

3. Time Shifting

If $f(t) \leftrightarrow F(\omega)$ then $f(t-t_0) \leftrightarrow F(\omega)e^{-j\omega t_0}$

Proof: $$\mathbf{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-j\omega t}\, dt$$

Putting $t - t_0 = \lambda, t = \lambda + t_0, dt = d\lambda$

$$\mathbf{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(\lambda)\, e^{-j(\lambda+t_0)\omega}\, d\lambda = \int_{-\infty}^{\infty} f(\lambda)\, e^{-j\lambda\omega}\, e^{-j\omega t_0}\, d\lambda$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(\lambda)\, e^{-j\omega\lambda}\, d\lambda = F(\omega)\, e^{-j\omega t_0}$$

Similarly, $$f(t+t_0) \leftrightarrow F(\omega)\, e^{j\omega t_0}$$

The above proof shows that shifting in time domain introduces a phase lag $(-\omega t_0)$ in the frequency domain.

Example 14.24 *Find the Fourier transform of the function shown in Fig. 14.49.*

Fig. 14.49

Solution $$f(t) = G\left(t - \frac{\tau}{2}\right)$$

By time-shifting property, $f(t-t_0) \leftrightarrow F(\omega)e^{-j\omega t_0}$

$$\mathbf{F}\{G(t)\} = A\tau\ \mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$$

$$\mathbf{F}\{f(t)\} = A\tau\ \mathrm{Sa}\left(\frac{\omega\tau}{2}\right) e^{-j\omega\frac{\tau}{2}}$$

Example 14.25 *Find the Fourier transform of the function shown in Fig. 14.50.*

Fig. 14.50

Solution $$f(t) = G\left(t + \frac{\tau}{2}\right)$$

By time-shifting property, $f(t + t_0) \leftrightarrow F(\omega)e^{-j\omega t_0}$

$$\mathbf{F}\{G(t)\} = A\tau \text{ Sa}\left(\frac{\omega\tau}{2}\right)$$

$$\mathbf{F}\{f(t)\} = A\tau \text{ Sa}\left(\frac{\omega\tau}{2}\right)e^{j\omega\frac{\tau}{2}}$$

Example 14.26 *Find the Fourier transform of the function shown in Fig.14.51.*

Fig. 14.51

Solution The function $f(t)$ can be viewed as a combination of two functions $f_1(t)$ and $f_2(t)$.

$$f(t) = f_1(t) - f_2(t)$$

$$f_1(t) = G\left(t + \frac{\tau}{2}\right)$$

By time-shifting property,

$$F_1(\omega) = \frac{1}{\tau} \times \tau \text{ Sa}\left(\frac{\omega\tau}{2}\right)e^{j\omega\frac{\tau}{2}} = \text{Sa}\left(\frac{\omega\tau}{2}\right)e^{j\omega\frac{\tau}{2}}$$

$$f_2(t) = G\left(t - \frac{\tau}{2}\right)$$

Fig. 14.52

$$F_2(\omega) = \frac{1}{\tau} \times \tau \text{ Sa}\left(\frac{\omega\tau}{2}\right) e^{-j\omega\frac{\tau}{2}} = \text{Sa}\left(\frac{\omega\tau}{2}\right) e^{-j\omega\frac{\tau}{2}}$$

$$F(\omega) = F_1(\omega) - F_2(\omega)$$

$$= \text{Sa}\left(\frac{\omega\tau}{2}\right) e^{j\omega\frac{\tau}{2}} - \text{Sa}\left(\frac{\omega\tau}{2}\right) e^{-j\omega\frac{\tau}{2}}$$

$$= \text{Sa}\left(\frac{\omega\tau}{2}\right)\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right)$$

$$= 2j \text{ Sa}\left(\frac{\omega\tau}{2}\right)\left(\frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j}\right)$$

$$= 2j \text{ Sa}\left(\frac{\omega\tau}{2}\right)\sin\left(\frac{\omega\tau}{2}\right)$$

$$= 2j\frac{\omega\tau}{2} \text{ Sa}\left(\frac{\omega\tau}{2}\right)\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}}$$

$$= j\omega\tau \text{ Sa}\left(\omega\frac{\tau}{2}\right) \text{Sa}\left(\omega\frac{\tau}{2}\right)$$

$$= j\omega\tau \text{ Sa}^2\left(\frac{\omega\tau}{2}\right)$$

Fig. 14.53

Example 14.27 *Find the Fourier transform of the function shown in Fig. 14.54.*

Fig. 14.54

Solution The function $f(t)$ is considered as a combination of two functions $f_1(t)$ and $f_2(t)$.

$$f(t) = f_2(t) - f_1(t)$$

Now, $$f_2(t) = G\left(t - \frac{\tau}{2}\right)$$

By time-shifting property,

$$\mathbf{F}\{G(t)\} = A\tau \text{ Sa}\left(\frac{\omega\tau}{2}\right)$$

$$\mathbf{F}\{f_2(t)\} = F_2(\omega) = \tau \text{ Sa}\left(\frac{\omega\tau}{2}\right) e^{-j\omega\frac{\tau}{2}}$$

Fig. 14.55

Also,
$$f_1(t) = G\left(t + \frac{\tau}{2}\right)$$

Fig. 14.56

$$F_1(\omega) = \tau\ \text{Sa}\left(\frac{\omega\tau}{2}\right)e^{j\omega\frac{\tau}{2}}$$

$$F(\omega) = F_2(\omega) - F_1(\omega)$$

$$= \tau\ \text{Sa}\left(\frac{\omega\tau}{2}\right)e^{-j\omega\frac{\tau}{2}} - \tau\ \text{Sa}\left(\frac{\omega\tau}{2}\right)e^{j\omega\frac{\tau}{2}}$$

$$= \tau\ \text{Sa}\left(\frac{\omega\tau}{2}\right)\left[e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right]$$

$$= -\tau\ \text{Sa}\left(\frac{\omega\tau}{2}\right)\left[\frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j}\right]$$

$$= 2j\tau\ \text{Sa}\left(\frac{\omega\tau}{2}\right)\sin\left(\frac{\omega\tau}{2}\right)$$

$$= -2j\tau\frac{\omega\tau}{2}\ \text{Sa}\left(\frac{\omega\tau}{2}\right)\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}}$$

$$= -j\omega\tau^2\ \text{Sa}\left(\frac{\omega\tau}{2}\right)\text{Sa}\left(\frac{\omega\tau}{2}\right)$$

$$= -j\omega\tau^2\ \text{Sa}^2\left(\frac{\omega\tau}{2}\right)$$

4. *Frequency Shifting or Frequency Translation*

If $f(t) \leftrightarrow F(\omega)$ then $f(t)\,e^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0)$

Proof:
$$\mathbf{F}\left\{f(t)\,e^{j\omega_0 t}\right\} = \int_{-\infty}^{\infty} f(t)\,e^{j\omega_0 t}\ e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)\,e^{-j(\omega-\omega_0)t}\,dt = F(\omega - \omega_0)$$

This proof shows that if the function *f(t)* is given a time lead ω_0 in the time domains then the spectrum of the signal gets shifted in frequency domain by ω_0.

Example 14.28 *Find the Fourier transform of $f(t)\cos\omega_0 t$.*

Solution
$$f(t)\cos\omega_0 t = f(t)\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] = \frac{1}{2}f(t)e^{j\omega_0 t} + \frac{1}{2}f(t)e^{-j\omega_0 t}$$

$$\mathbf{F}\left\{f(t)\cos\omega_0 t\right\} = \frac{1}{2}\mathbf{F}\left\{f(t)e^{j\omega_0 t} + f(t)e^{-j\omega_0 t}\right\} = \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right]$$

Example 14.29 *Find the Fourier transform of $f(t)\sin\omega_0 t$.*

Solution

$$f(t)\sin\omega_0 t = f(t)\left[\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right] = \frac{1}{2j}\left[f(t)e^{j\omega_0 t} - f(t)e^{-j\omega_0 t}\right]$$

$$\mathbf{F}\left\{f(t)\sin\omega_0 t\right\} = \frac{1}{2j}\mathbf{F}\left\{f(t)e^{j\omega_0 t} - f(t)e^{-j\omega_0 t}\right\} = \frac{1}{2j}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]$$

Example 14.30 *Find Fourier transform of periodic function $e^{j\omega_0 t}$. Hence, find Fourier transform of the periodic function (i) $\sin\omega_0 t$, and (ii) $\cos\omega_0 t$.*

Solution

$$e^{j\omega_0 t} = 1e^{j\omega_0 t}$$

$$\mathbf{F}\{1\} = 2\pi\,\delta(\omega)$$

By frequency shifting property,

$$\mathbf{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\,\delta(\omega-\omega_0)$$

Now,

$$\sin\omega_0 t = \frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$$

$$\mathbf{F}\{\sin\omega_0 t\} = \frac{1}{2j}\mathbf{F}\left\{e^{j\omega_0 t}-e^{-j\omega_0 t}\right\} = \frac{1}{2j}\left[2\pi\,\delta(\omega-\omega_0)-2\pi\,\delta(\omega+\omega_0)\right] = \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

$$\mathbf{F}\{\cos\omega_0 t\} = \frac{1}{2}\mathbf{F}\left\{e^{j\omega_0 t}+e^{-j\omega_0 t}\right\} = \frac{1}{2}\mathbf{F}\left\{e^{j\omega_0 t}+e^{-j\omega_0 t}\right\} = \frac{1}{2}\left[2\pi\,\delta(\omega-\omega_0)+2\pi\,\delta(\omega+\omega_0)\right]$$

$$= \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

5. Time Differentiation

If $f(t)\leftrightarrow F(\omega)$ then $\dfrac{d}{dt}f(t)\leftrightarrow j\omega F(\omega)$

Proof:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

$$\frac{d}{dt}f(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega\right] = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{d}{dt}e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\,F(\omega)e^{j\omega t}\,d\omega$$

Comparing with inverse Fourier transform equation,

$$\frac{d}{dt}f(t)\leftrightarrow j\omega F(\omega)$$

Example 14.31 *Find the Fourier transform of the function shown in Fig. 14.57.*

Fig. 14.57

Solution

$$f(t) = -\frac{1}{\tau}t + 1 \qquad 0 < t < \tau$$

$$= \frac{1}{\tau}t + 1 \qquad -\tau < t < 0$$

Differentiating *f(t)* w.r.t. *t*,

$$\frac{d}{dt}f(t) = -\frac{1}{\tau} \qquad 0 < t < \tau$$

$$= \frac{1}{\tau} \qquad -\tau < t < 0$$

$$\mathbf{F}\left\{\frac{d}{dt}f(t)\right\} = j\omega\tau\ \mathrm{Sa}^2\left(\omega\frac{\tau}{2}\right)$$

Fig. 14.58

By time-differentiation property,

$$\mathbf{F}\left\{\frac{d}{dt}f(t)\right\} = j\omega\, F(\omega)$$

$$F(\omega) = \tau\ \mathrm{Sa}^2\left(\frac{\omega\tau}{2}\right)$$

6. *Frequency Differentiation*

If $f(t) \leftrightarrow F(\omega)$ then $\dfrac{d}{d\omega}F(\omega) \leftrightarrow -jt\ f(t)$

Proof: $$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\ dt$$

Differentiating w.r.t ω,

$$\frac{d}{d\omega}F(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\ dt = \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}e^{-j\omega t}\ dt = \int_{-\infty}^{\infty}(-jt)\,f(t)e^{-j\omega t}\ dt$$

$$\frac{d}{d\omega}\left[F(\omega)\right] \leftrightarrow -jt\ f(t)$$

7. Time Convolution

If $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$ then $f_1(t) * f_2(t) \leftrightarrow F_1(\omega) \cdot F_2(\omega)$

Proof: By definition of convolution,

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

$$\mathbf{F}\{f_1(t) * f_2(t)\} = \int_{-\infty}^{\infty} \{f_1(t) * f_2(t)\} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$

Putting $\quad t - \tau = \lambda, \quad t = \lambda + \tau, \quad dt = d\lambda$

$$\mathbf{F}\{f_1(t) * f_2(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau) f_2(\lambda)\, d\tau\, e^{-j\omega(\lambda+\tau)}\, d\lambda$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau) f_2(\lambda)\, d\tau\, e^{-j\omega\tau}\, e^{-j\omega t}\, d\lambda$$

$$= \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} d\tau \cdot \int_{-\infty}^{\infty} f_2(\lambda) e^{-j\omega\lambda}\, d\lambda$$

$$= F_1(\omega) \cdot F_2(\omega)$$

This proof shows that convolution in time domain is equivalent to multiplication in frequency domain.

8. Frequency Convolution

If $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$ then $F_1(\omega) * F_2(\omega) \leftrightarrow 2\pi f_1(t) \cdot f_2(t)$

Proof: By definition of convolution,

$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(u)\, F_2(\omega - u)\, du$$

$$\mathbf{F}^{-1}\{F_1(\omega) * F_2(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(u) F_2(\omega - u)\, du\, e^{j\omega t}\, d\omega$$

Putting $\quad \omega - u = \lambda, \quad \omega = \lambda + u, \quad d\omega = d\lambda$

$$\mathbf{F}^{-1}\{F_1(\omega) * f_2(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(u) F_2(\lambda)\, du\, e^{j(\lambda+u)t}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(u) F_2(\lambda) e^{j\lambda t}\, e^{jut}\, du\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(u) e^{jut} du \cdot \int_{-\infty}^{\infty} F_2(\lambda) e^{j\lambda t}\, d\lambda$$

$$= f_1(t) \cdot 2\pi f_2(t)$$

$$= 2\pi f_1(t) \cdot f_2(t)$$

This proof shown that convolution in the frequency domain is equivalent to multiplication in the time domain.

14.12 ENERGY DENSITY SPECTRUM

The energy of any function *f(t)* is given by

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt$$

We know that,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

$$E = \int_{-\infty}^{\infty} f(t)\cdot f(t)\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot F(-\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega \text{ where } |F(\omega)|^2 = F(\omega)\cdot F(-\omega)$$

$|F(j\omega)|$ is even function of ω.

$$E = \frac{1}{\pi}\int_{0}^{\infty} |F(\omega)|^2\, d\omega$$

Example 14.32 *A low-pass brick-wall filter is given by*

$$H(\omega) = \begin{cases} 1 & |\omega| < W_1 \\ 0 & |\omega| > W_1 \end{cases}$$

(a) Determine impulse response of the filter and sketch it as a function of time.

(b) It the bandwidth of the brick-wall filter is decreased to $\frac{1}{2}W_1$, *sketch its impulse response.*

Solution (a)

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W_1}^{W_1} 1\cdot e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W_1}^{W_1}$$

$$= \frac{1}{\pi t}\left(\frac{e^{jW_1 t} - e^{-jW_1 t}}{2j}\right)$$

$$= \frac{1}{\pi t}(\sin W_1 t)$$

$$= \frac{W_1}{\pi}\left(\frac{\sin W_1 t}{W_1 t}\right)$$

$$= \frac{W_1}{\pi}\ \text{Sa}\ (W_1 t)$$

Fig. 14.59

The impulse response is shown in Fig 14.59

(b) As we decrease the bandwidth of brick-wall filter, the function $h(t)$ becomes broader and vice versa.

Fig. 14.60

Example 14.33 *Find the output voltage response for the network shown in Fig.14.61.*

Fig. 14.61

Solution The transformed network is shown in Fig. 14.62.

Fig. 14.62

Applying KCL at the node,

$$\frac{1}{1+j\omega} = \frac{V(\omega)}{0.5} + \frac{V(\omega)}{\frac{1}{j\omega}} = (2+j\omega)\,V(\omega)$$

$$V(\omega) = \frac{1}{1+j\omega}\frac{1}{2+j\omega} = \frac{(2+j\omega)-(1+j\omega)}{(1+j\omega)(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

Taking the inverse Fourier transform,

$$v(t) = e^{-t}\,u(t) - 0.5\,e^{-2t}$$

Example 14.34 *The current in a 10 Ω resistor is $i(t) = 10\,e^{-2t}u(t)$. What is the energy associated with the frequency band $0 \le \omega \le 2$ rad/s?*

Solution

$$i(t) = 10\,e^{-2t}$$

$$I(\omega) = \frac{10}{2+j\omega}$$

Energy associated with the given frequency band is

$$E = \frac{10}{\pi}\int_0^2 |I(\omega)|^2\,d\omega$$

$$= \frac{10}{\pi}\int_0^2 \frac{100}{4+\omega^2}\,d\omega$$

$$= \frac{1000}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]_0^2$$

$$= \frac{1000}{2\pi}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1000}{2\pi}\left(\frac{\pi}{4} - 0\right)$$

$$= 125 \text{ J}$$

Exercises

14.1 Find the trigonometric Fourier series of the waveform shown in Fig. 14.63.

Fig. 14.63

$$\left[\frac{4A}{\pi}\sin\omega t + \frac{4A}{3\pi}\sin 3\omega t + \frac{4A}{5\pi}\sin 5\omega t + \ldots\ldots\right]$$

14.2 Find the trigonometric Fourier series of the waveform shown in Fig. 14.64.

Fig. 14.64

$$\left[-\frac{2A}{\pi}\sin\omega t-\frac{A}{\pi}\sin 2\omega t-\frac{2A}{3\pi}\sin 3\omega t-\ldots\ldots\right]$$

14.3 Find the trigonometric Fourier series of the waveform shown in Fig. 14.65.

Fig. 14.65

$$\left[\left(\frac{4A}{\pi^2}\cos\omega t-\frac{4A}{9\pi^2}\cos 3\omega t+\ldots\right)+\left(-\frac{2A}{\pi}\sin\omega t-\frac{2A}{3\pi}\sin 3\omega t-\ldots\right)\right]$$

14.4 Find the trigonometric Fourier series of the waveform shown in Fig. 14.66.

Fig. 14.66

$$\left[\frac{2A}{\pi}-\frac{4A}{3\pi}\cos 2\omega t-\frac{4A}{15\pi}\cos 4\omega t-\frac{4A}{35\pi}\cos 6\omega t-\ldots\ldots\right]$$

14.5 Find the trigonometric Fourier series of the waveform shown in Fig. 14.67.

Fig. 14.67

$$\left[\frac{2A}{\pi}+\frac{4A}{3\pi}\cos 2\omega t-\frac{4A}{15\pi}\cos 4\omega t+\frac{4A}{35\pi}\cos 6\omega t-\ldots\ldots\right]$$

14.6 Find the trigonometric Fourier series of the waveform shown in Fig. 14.68.

Fig. 14.68

$$\left[\frac{8A}{\pi^2}\cos\omega t+\frac{8A}{9\pi^2}\cos 3\omega t+\frac{8A}{25\pi^2}\cos 5\omega t+\ldots\ldots\right]$$

14.7 Find the exponential Fourier series of the waveform shown in Fig. 14.69.

Fig. 14.69

$$\left[\begin{aligned}&\ldots\ldots+\left(\frac{1}{9\pi^2}-j\frac{1}{6\pi}\right)Ae^{-j3\omega t}-j\frac{A}{4\pi}e^{-j2\omega t}\\&-\left(\frac{1}{\pi^2}-j\frac{1}{2\pi}\right)Ae^{-j\omega t}\\&+\frac{1}{4}-\left(\frac{1}{\pi^2}+j\frac{1}{2\pi}\right)Ae^{j\omega t}+j\frac{A}{4\pi}e^{j2\omega t}\\&-\left(\frac{1}{9\pi^2}+j\frac{1}{6\pi}\right)Ae^{j3\omega t}-\ldots\ldots\end{aligned}\right]$$

14.8 Find the exponential Fourier series of the waveform shown in Fig. 14.70.

Fig. 14.70

$$\left[\ldots\ldots - j\frac{A}{2\pi}e^{-j2\omega t} - j\frac{A}{\pi}e^{-j\omega t} + j\frac{A}{\pi}e^{j\omega t} + j\frac{A}{2\pi}e^{j2\omega t} + \ldots\ldots\right]$$

14.9 Find the effective voltage, effective current, and average power supplied to a passive network if the applied voltage is $v(t) = 200 + 100\cos(500t + 30°) + 75\cos(1500t + 60°)$ volts and the resulting current is $i(t) = 3.53\cos(500t + 75°) + 3.55\cos(1500t + 78.45°)$ amperes.

[218.5 V, 3.54 A, 250.8 W]

14.10 A two-element series circuit with $R = 10\ \Omega$ and $L = 20$ mH has current $i = 5\sin 100t + 3\sin 300t + 2\sin 100t$. Find the effective applied voltage and the average power.

[48 V, 190 W]

14.11 A three-element series circuit with $R = 5\ \Omega$, $L = 5\ \mu$H and $C = 50\ \mu$F, has an applied voltage $v = 150\sin 1000t + 10\sin 200t + 75\sin 300t$. Find the effective current and the average power for the circuit.

[16.58 A, 1374 W]

14.12 A three-element circuit consists of $R = 5\ \Omega$ in series with a parallel combination of L and C. At $\omega = 500$ rad/s, $X_L = 2\ \Omega$, $X_C = 8\ \Omega$. Find the total current if the applied voltage is given by $v = 50 + 20\sin 500t + 10\sin 1000t$.

$[i = 10 + 3.53\sin(500t - 28.1°)]$

14.13 Find the Fourier transform of the function shown in Fig. 14.71.

Fig. 14.71

$$\left[\frac{20}{\omega}(\sin 5\omega + \sin 10\omega)\right]$$

14.14 Find the Fourier transform of the function shown in Fig. 14.72.

Fig. 14.72

$$\left[\frac{200}{\omega^2}(\cos 5\omega - \cos 10\omega)\right]$$

Objective-Type Questions

14.1 A periodic function $f(t)$ with time period T repeats itself after half time period $\frac{T}{2}$. The fourier series of $f(t)$ would contain

(a) cosine terms only
(b) odd harmonic terms only
(c) even harmonic terms only
(d) sine terms only

14.2 The fourier series expansion of a periodic function with half-wave symmetry contains only
(a) sine terms (b) cosine terms
(c) odd harmonics (d) even harmonics

14.3 With regard to Fourier transform, tick the correct combination:

Function	Transform
A Real and even	1. Real and even
B Real and odd	2. Real and odd
C. Real even plus imaginary odd	3. Complex and even
D. Complex and even	4. Real
	5. Imaginary and odd

	A	B	C	D
(a)	1	2	3	4
(b)	1	5	3	4
(c)	1	5	4	3
(d)	2	5	4	3

14.4 Which of the following periodic waveforms will have only odd harmonics of sinusoldal waveforms?

(i)

(ii)

(iii)

(iv)

Fig 14.73

Select the correct answer using the codes given below:
(a) (i) and (ii) (b) (i) and (iii)
(c) (i) and (iv) (d) (ii) and (iv)

14.5 If $f(t) = -f(-t)$ and $f(t)$ satisfy the Dirichlet's conditions, then $f(t)$ can be expanded in a fourier series containing
(a) Only sine terms
(b) Only cosine terms
(c) Cosine terms and a constant term
(d) Sine terms and a constant term

14.6 The inverse fourier transform of the function

$$F(\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega) \text{ is}$$

(a) $\sin \omega t$ (b) $\cos \omega t$
(c) $\sin (t)$ (d) $\mu(t)$

14.7 The amplitude of the first odd harmonic of the square wave shown in Fig. 14.74 is equal to

Fig 14.74

(a) $\frac{4A}{\pi}$ (b) $\frac{2A}{3\pi}$ (c) $\frac{A}{\pi}$ (d) 0

14.8 The fourier transform of $f(t) = \cos \omega_0 t$ is

(a) $\frac{1}{2}\delta(f - f_0)$

(b) $\frac{1}{2}\delta(f + f_0)$

(c) $\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]$

(d) $\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$

14.9 A periodic function $f(t)$ is said to have a quarter-wave symmetry, if it possesses

(a) even symmetry at an interval of quarter of a wave
(b) even symmetry and half-wave symmetry only
(c) even or odd symmetry without half-wave symmetry
(d) even or odd symmetry with half-wave symmetry

14.10 Periodic signal that obeys Dirichlet's condition can be represented by

(a) Fourier series
(b) Fourier transform
(c) Inverse Fourier transform
(d) None of these

14.11 The current waveform in a pure resistor of 10 Ω is shown in Fig. 14.75 power dissipated in the resistor is

Fig. 14.75

(a) 7.29 W (b) 52.4 W
(c) 135 W (d) 270 W

Answers to Objective-Type Questions

1. (c) 2. (c) 3. (c) 4. (a) 5. (a) 6. (d) 7. (a)
8. (d) 9. (d) 10. (a) 11. (d)

15

Filters and Attenuators

15.1 INTRODUCTION

Filters are frequency-selective networks that attenuate signals at some frequency and allow others to pass with or without attenuation. A filter is constructed from purely reactive elements. Ideally, filters should produce no attenuation in the desired band, called *pass band* and should provide attenuation at all other frequencies called *attenuation band* or *stop band*. The frequency which separates the pass band and the stop band is called *cut-off frequency*. Filter networks are widely used in communication systems to separate various channels in carrier-frequency telephone circuits.

15.2 CLASSIFICATION OF FILTERS

On the basis of frequency characteristics, filters are classified into four categories:

(i) Low-pass filter
(ii) High-pass filter
(iii) Band-pass filter
(iv) Band-stop filter

A *low-pass filter* allows all frequencies up to a certain cut-off frequency to pass through it and attenuates all the other frequencies above the cut-off frequency.

A *high-pass filter* attenuates all the frequency below the cut-off frequency and allows all other frequencies above the cut-off frequency to pass through it.

A *band-pass filter* allows a limited band of frequencies to pass through it and attenuates all other frequencies below or above the frequency band.

A *band-stop filter* attenuates a limited band of frequencies but allows all other frequencies to pass through it.

15.3 T-NETWORK

Figure 15.1 shows a *T* network.

Fig. 15.1 *T-Network*

15.3.1 Characteristic Impedance

For a *T* network, the value of input impedance when it is terminated by characteristic impedance $\mathbf{Z}_0$, is given by

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_2\left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_0\right)}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0}$$

But $$\mathbf{Z}_{\text{in}} = \mathbf{Z}_0$$

$$\mathbf{Z}_0 = \frac{\mathbf{Z}_1}{2} + \frac{2\mathbf{Z}_2\left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_0\right)}{\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0}$$

$$= \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_2\mathbf{Z}_0}{\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0}$$

$$= \frac{\mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_1\mathbf{Z}_0 + 2\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_2\mathbf{Z}_0}{2(\mathbf{Z}_1 + 2\mathbf{Z}_2 + 2\mathbf{Z}_0)}$$

$$2\mathbf{Z}_1\mathbf{Z}_0 + 4\mathbf{Z}_2\mathbf{Z}_0 + 4\mathbf{Z}_0^2 = \mathbf{Z}_1^2 + 2\mathbf{Z}_1\mathbf{Z}_2 + 2\mathbf{Z}_1\mathbf{Z}_0 + 2\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_2\mathbf{Z}_0$$

$$4\mathbf{Z}_0^2 = \mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2$$

$$\mathbf{Z}_0^2 = \frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2$$

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

Hence, characteristic impedance for symmetrical *T* section is given by,

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$

Characteristic impedance can also be expressed in terms of open-circuit impedance $\mathbf{Z}_{oc}$ and short-circuit impedance $\mathbf{Z}_{sc}$.

Open-circuit impedance $\mathbf{Z}_{oc} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 = \frac{\mathbf{Z}_1 + 2Z_2}{2}$

Short-circuit impedance $\mathbf{Z}_{sc} = \frac{\mathbf{Z}_1}{2} + \frac{\frac{\mathbf{Z}_1}{2}\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + 2\mathbf{Z}_2} = \frac{\mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + 4\mathbf{Z}_2}$

$$\mathbf{Z}_{oc}\mathbf{Z}_{sc} = \left(\frac{\mathbf{Z}_1 + 2\mathbf{Z}_2}{2}\right)\left(\frac{\mathbf{Z}_1^2 + 4\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_1 + 4\mathbf{Z}_2}\right) = \frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2 = \mathbf{Z}_{0T}^2$$

$$\mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$$

15.3.2 Propagation Constant

The propagation constant γ of the network in Fig. 15.2 is given by,

$$\gamma = \log_e \frac{I_1}{I_2}$$

Fig. 15.2 *T-Network*

Applying KVL to Mesh1,

$$V_s - \frac{\mathbf{Z}_1}{2} I_1 - \mathbf{Z}_2 (I_1 - I_2) = 0$$

$$V_s = \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 \right) I_1 - \mathbf{Z}_2 I_2$$

Applying KVL to Mesh 2,

$$-\mathbf{Z}_2 (I_2 - I_1) - \frac{\mathbf{Z}_1}{2} I_2 - \mathbf{Z}_0 I_2 = 0$$

$$-\mathbf{Z}_2 I_1 + \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 \right) I_2 = 0$$

$$\mathbf{Z}_2 I_1 = \left(\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 \right) I_2$$

$$\frac{I_1}{I_2} = \frac{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0}{\mathbf{Z}_2} = e^{\gamma}$$

$$\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_0 = \mathbf{Z}_2 e^{\gamma}$$

$$\mathbf{Z}_0 = \mathbf{Z}_2 (e^{\gamma} - 1) - \frac{\mathbf{Z}_1}{2}$$

The characteristic impedance of the *T* network is given by,

$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2}$$

$$\sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2} = \mathbf{Z}_2 (e^{\gamma} - 1) - \frac{\mathbf{Z}_1}{2}$$

$$\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2 = \mathbf{Z}_2^2 (e^{\gamma} - 1)^2 + \frac{\mathbf{Z}_1^2}{4} - \mathbf{Z}_1 \mathbf{Z}_2 (e^{\gamma} - 1)$$

$$\mathbf{Z}_2^2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 \mathbf{Z}_2 (1 + e^{\gamma} - 1) = 0$$

$$\mathbf{Z}_2^2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 \mathbf{Z}_2 e^{\gamma} = 0$$

$$\mathbf{Z}_2 (e^{\gamma} - 1)^2 - \mathbf{Z}_1 e^{\gamma} = 0$$

$$(e^{\gamma} - 1)^2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2} e^{\gamma}$$

$$e^{2\gamma} - 2e^{\gamma} + 1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2 e^{-\gamma}}$$

$$e^{-\gamma} (e^{2\gamma} - 2e^{\gamma} + 1) = \frac{\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$e^{\gamma} + e^{-\gamma} - 2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_2}$$

Dividing both the sides by 2,

$$\frac{e^{\gamma} + e^{-\gamma}}{2} = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\cosh\gamma = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\sinh\gamma = \sqrt{\cosh^2\gamma - 1} = \sqrt{\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)^2 - 1} = \sqrt{1 + \frac{\mathbf{Z}_1}{\mathbf{Z}_2} + \frac{\mathbf{Z}_1^2}{4\mathbf{Z}_2^2} - 1} = \sqrt{\frac{\mathbf{Z}_1}{\mathbf{Z}_2} + \left(\frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)^2}$$

$$= \frac{1}{\mathbf{Z}_2}\sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^2}{4}} = \frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2}$$

$$\tanh\gamma = \frac{\sinh\gamma}{\cosh\gamma} = \frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)} = \frac{\mathbf{Z}_{0T}}{\mathbf{Z}_2 + \frac{\mathbf{Z}_1}{2}}$$

But $\quad \mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$

and $\quad \mathbf{Z}_{oc} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2$

$$\tanh\gamma = \sqrt{\frac{\mathbf{Z}_{sc}}{\mathbf{Z}_{oc}}}$$

Also, $\quad \sinh\frac{\gamma}{2} = \sqrt{\frac{1}{2}(\cosh\gamma - 1)} = \sqrt{\frac{1}{2}\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2} - 1\right)} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$

15.4 π NETWORK

Figure 15.3 shows a π-network.

Fig. 15.3 *π Network*

15.4.1 Characteristic Impedance

For a π network, the value of input impedance when it is terminated by impedance $\mathbf{Z}_0$, is given by,

$$\mathbf{Z}_{\text{in}} = \frac{2\mathbf{Z}_2\left(\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0}\right)}{\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_2}$$

But
$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_0$$
$$\mathbf{Z}_0 = \frac{2\mathbf{Z}_2\left(\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0}\right)}{\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_0}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_2}$$
$$\mathbf{Z}_0\mathbf{Z}_1 + \frac{2\mathbf{Z}_2\mathbf{Z}_0^2}{2\mathbf{Z}_2 + \mathbf{Z}_0} + 2\mathbf{Z}_0\mathbf{Z}_2 = \frac{2\mathbf{Z}_2(2\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_0\mathbf{Z}_1 + 2\mathbf{Z}_0\mathbf{Z}_2)}{2\mathbf{Z}_2 + \mathbf{Z}_0}$$
$$2\mathbf{Z}_0\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_0^2 + 2\mathbf{Z}_0\mathbf{Z}_2^2 + 4\mathbf{Z}_2\mathbf{Z}_0^2 + 2\mathbf{Z}_2\mathbf{Z}_0^2 = 4\mathbf{Z}_1\mathbf{Z}_2^2 + 2\mathbf{Z}_0\mathbf{Z}_1\mathbf{Z}_2 + 4\mathbf{Z}_0\mathbf{Z}_2^2$$
$$\mathbf{Z}_1\mathbf{Z}_0^2 + 4\mathbf{Z}_2\mathbf{Z}_0^2 = 4\mathbf{Z}_1\mathbf{Z}_2^2$$
$$\mathbf{Z}_0^2(\mathbf{Z}_1 + 4\mathbf{Z}_2) = 4\mathbf{Z}_1\mathbf{Z}_2^2$$
$$\mathbf{Z}_0^2 = \frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}$$
$$\mathbf{Z}_0 = \sqrt{\frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}}$$

Dividing both the sides by $4Z_2$,
$$\mathbf{Z}_0 = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}$$

Hence, characteristic impedance of a symmetrical π network is given by
$$\mathbf{Z}_{0\pi} = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1^2}{4}}}$$

But
$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}$$
$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}}$$

Characteristic impedance can also be expressed in terms of open-circuit impedance $\mathbf{Z}_{oc}$ and short-circuit impedance $\mathbf{Z}_{sc}$.

Open-circuit impedance $\mathbf{Z}_{oc} = \dfrac{2\mathbf{Z}_2(\mathbf{Z}_1 + 2\mathbf{Z}_2)}{2\mathbf{Z}_2 + \mathbf{Z}_1 + 2\mathbf{Z}_2} = \dfrac{2\mathbf{Z}_2(\mathbf{Z}_1 + 2\mathbf{Z}_2)}{\mathbf{Z}_1 + 4\mathbf{Z}_2}$

Short-circuit impedance $\mathbf{Z}_{sc} = \dfrac{2\mathbf{Z}_1\mathbf{Z}_2}{2\mathbf{Z}_2 + \mathbf{Z}_1}$
$$\mathbf{Z}_{oc}\mathbf{Z}_{sc} = \frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$
$$\mathbf{Z}_{0\pi} = \sqrt{\mathbf{Z}_{oc}\mathbf{Z}_{sc}}$$

15.4.2 Propagation Constant

The propagation constant of a symmetrical π network is same as that of a symmetrical T network.

15.5 CHARACTERISTIC OF FILTERS

A filter transmits or passes desired range of frequencies without loss and attenuates all undesired frequencies.

The propagation constant

$$\gamma = \alpha + j\beta$$

where α is attenuation constant and β is the phase constant. We know that

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\sinh\left(\frac{\alpha + j\beta}{2}\right) = \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

Depending upon the type of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, there are two cases:

Case (i) If $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are same type of reactances then $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ and $\sinh\frac{\gamma}{2}$ are real.

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = 0$$

$$\sin\frac{\beta}{2} = 0 \qquad \text{if } \beta = 0 \text{ or } n\pi \text{ where } n = 0, 1, 2, \ldots \quad \left[\because \cosh\frac{\alpha}{2} \text{ cannot be zero}\right]$$

$$\cos\frac{\beta}{2} = 1$$

$$\sinh\frac{\alpha}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\alpha = 2\sinh^{-1}\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

Case (ii) If $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are opposite types of reactances then $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative, i.e. $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < 0$ and $\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$ is purely imaginary.

$$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0$$

$$j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = j\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

$$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}$$

The two conditions of operation of the filter, viz. the pass band and stop band are mathematically obtained from these equations. Both the above equations must be satisfied simultaneously by α and β. Two conditions may arise

(a) $\sinh\frac{\alpha}{2} = 0$ i.e $\alpha = 0$ when $\beta \neq 0$

Since $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative and $\sin\frac{\beta}{2}$ is real,

$$\sin\frac{\beta}{2} = \left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right| \qquad \text{as } \cosh\frac{\alpha}{2} = 1$$

This signifies the region of zero attenuation or pass band which is limited by the upper limit of sine terms.

$$\sin\frac{\beta}{2} = 1$$

$$-1 < \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < 0$$

The phase angle in the pass band is given by,

$$\frac{\beta}{2} = \sin^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

$$\beta = 2\sin^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

(b)

$$\cos\frac{\beta}{2} = 0$$

$$\sin\frac{\beta}{2} = \pm 1$$

Since $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}$ is negative and $\cosh\frac{\alpha}{2}$ is real,

$$\cosh\frac{\alpha}{2} = \left|\sqrt{\frac{\mathbf{Z}_1}{\mathbf{Z}_2}}\right|$$

This signifies a stop band since $\alpha \neq 0$.

$$\frac{\alpha}{2} = \cosh^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

$$\alpha = 2\cosh^{-1}\left|\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right|$$

and

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} < -1$$

15.6 CONSTANT-*k* LOW PASS FILTER

A T or π network is said to be of the constant *k* type if $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are opposite types of reactances satisfying the relation

$$\mathbf{Z}_1\mathbf{Z}_2 = k^2$$

where k is a constant, independent of frequency. k is often referred to as *design impedance* or *nominal impedance* of the constant-k filter. The constant-k, T or π-type filter is also known as the prototype filter because other complex networks can be derived from it. Figure 15.4 shows constant-k, T and π-section filters.

Fig. 15.4 *Constant–k Filter*

In constant-k low pass filter,

$$\mathbf{Z}_1 = j\omega L$$

$$\mathbf{Z}_2 = -j\frac{1}{\omega c} = \frac{1}{j\omega c}$$

1. Nominal Impedance

$$k = \sqrt{\mathbf{Z}_1\mathbf{Z}_2} = \sqrt{(j\omega L)\left(\frac{1}{j\omega c}\right)} = \sqrt{\frac{L}{C}}$$

2. Cut-off Frequency

The cut-off frequencies are obtained when $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$ and $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

(i) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$$

$$\mathbf{Z}_1 = 0$$

$$j\omega L = 0$$

$$\omega = 0$$

$$f = 0$$

(ii) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$$

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$j\omega L = \frac{4j}{\omega C}$$

$$\omega^2 LC = 4$$

$$\omega^2 = \frac{4}{LC}$$

$$\omega = \omega_c = \frac{2}{\sqrt{LC}}$$

$$f = f_c = \frac{1}{\pi\sqrt{LC}}$$

Hence, the pass band starts at $f = 0$ and continues up to the cut-off frequency f_c. All the frequencies above f_c are in the attenuation or stop band.

3. Attenuation Constant

In pass band, $\alpha = 0$

In stop band, $$\alpha = 2\cosh^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\cosh^{-1}\sqrt{\frac{\omega^2 LC}{4}} = 2\cosh^{-1}\sqrt{\frac{\omega^2}{\omega_c{}^2}}$$

$$= 2\cosh^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

The attenuation constant α is zero throughout the pass band but increases gradually from the cut-off frequency. The variation of α is plotted in Fig. 15.5.

Fig. 15.5 *Variation of α with frequency*

4. Phase Constant

In pass band,

$$\beta = 2\sin^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\sin^{-1}\left(\frac{f}{f_c}\right)$$

In stop band, $\beta = \pi$.

The phase constant β is zero at zero frequency and increases gradually through the pass band, reaches π at cut-off frequency f_c and remains at π for all frequency beyond f_c. The variation of β is plotted in Fig. 15.6.

Fig. 15.6 *Variation of β with frequency*

5. Characteristic Impedance

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)} = k\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} = k\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}}$$

The plot of characteristic impedance versus frequency is shown in Fig. 15.7.

Fig. 15.7 *Variation of characteristic impedance with frequency*

The characteristic impedance $\mathbf{Z}_{0T}$ is real when $f < f_c$. At $f = f_c$, $\mathbf{Z}_{0T} = 0$. For $f > f_c$, $\mathbf{Z}_{0T}$ is imaginary in the stop band, rising to infinite reactance at infinite frequency. $\mathbf{Z}_{0\pi}$ is real when $f < f_c$. At $f = f_c$, $\mathbf{Z}_{0\pi}$ is infinite and for $f > f_c$, $\mathbf{Z}_{0\pi}$ is imaginary.

6. Design of Filter

$$k = \sqrt{\frac{L}{C}}$$

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

Solving these two equations,

$$L = \frac{k}{\pi f_c}$$

$$C = \frac{1}{\pi f_c k}$$

The constant-k, T and π section filters are shown in Fig. 15.8.

Fig. 15.8 *Constant-k low-pass filter*

Example 15.1 *Find the nominal impedance, cut-off frequency and pass band for the network shown in Fig. 15.9.*

Fig. 15.9

Solution The network is a low-pass filter.

$$\frac{L}{2} = 25 \text{ mH}, \quad C = 0.2\ \mu\text{F}$$

$$L = 50 \text{ mH}$$

(a) Nominal Impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{50\times10^{-3}}{0.2\times10^{-6}}} = 500\ \Omega$$

(b) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{50\times10^{-3}\times0.2\times10^{-6}}} = 3.18\ \text{kHz}$$

(c) Pass band
The pass band is from zero to 3.18 kHz.

Example 15.2 *A low-pass filter is composed of a symmetrical π section. Each series branch is a 0.02 H inductor and shunt branch is a 2 μF capacitor. Find (a) cut-off frequency, (b) nominal impedance, (c) characteristic impedance at 200 Hz and 2000 Hz, (d) attenuation at 200 Hz and 2000 Hz, and (e) phase shift constant at 200 Hz and 2000 Hz.*

Solution

$$L = 0.02\ \text{H},\quad \frac{C}{2} = 2\ \mu\text{F}$$

$$C = 4\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{0.02\times4\times10^{-6}}} = 1125.4\ \text{Hz}$$

(b) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.02}{4\times10^{-6}}} = 70.71\ \Omega$$

(c) Characteristic impedance at 200 Hz

$$\mathbf{Z}_0 = \frac{k}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}} = \frac{70.71}{\sqrt{1-\left(\frac{200}{1125.4}\right)^2}} = 71.85\ \Omega$$

Characteristic impedance at 2000 Hz

$$\mathbf{Z}_0 = \frac{k}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}} = \frac{70.71}{\sqrt{1-\left(\frac{2000}{1125.4}\right)^2}} = j48.13\ \Omega$$

(d) Attenuation at 200 Hz and 2000 Hz
The 200 Hz frequency lies in the pass band.

$$\alpha = 0$$

At 2000 Hz,

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{2000}{1125.4}\right) = 2.35$$

(e) Phase shift constant at 200 Hz and 2000 Hz
At 200 Hz,

$$\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right) = 2\sin^{-1}\left(\frac{200}{1125.4}\right) = 20.47°$$

2000 Hz frequency lies in the attenuation band.

$$\beta = 180°$$

Example 15.3 *A π section filter network consists of a series arm inductor of 20 mH and two shunt-arm capacitors of 0.16 μF each.Calculate the cut-off frequency, and attenuation and phase shift at 15 kHz. What is the value of the nominal impedance in the pass band?*

Solution $L = 20$ mH, $\dfrac{C}{2} = 0.16$ μF

$C = 0.32$ μF

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{20\times10^{-3}\times0.32\times10^{-6}}} = 3.98 \text{ kHz}$$

(b) Attenuation at 15 kHz

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{15\times10^3}{3.98\times10^3}\right) = 4$$

(c) Phase shift at 15 kHz

15 kHz frequency lies in the attenuation band.

$$\beta = 180°$$

(d) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{20\times10^{-3}}{0.32\times10^{-6}}} = 250\ \Omega$$

Example 15.4 *Each of the two series elements of a T-section low-pass filter consists of an inductor of 60 mH having negligible resistance and a shunt element having a capacitance of 0.2 μF. Calculate (a) the cut-off frequency, (b) nominal impedance, and (c) characteristic impedance at frequencies of 1 kHz and 5 kHz.*

Solution $L = 60$ mH, $C = 0.2$ μF

(a) Cut-off frequency

$$f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{60\times10^{-3}\times0.2\times10^{-6}}} = 2.91 \text{ kHz}$$

(b) Nominal impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{60\times10^{-3}}{0.2\times10^{-6}}} = 547.72\ \Omega$$

(c) Characteristic impedance at 1 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 547.72\sqrt{1-\left(\frac{1\times10^3}{2.91\times10^3}\right)^2} = 514.36\,\Omega$$

Characteristic impedance at 5 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 547.72\sqrt{1-\left(\frac{5\times10^3}{2.91\times10^3}\right)^2} = j765.29\ \Omega$$

Example 15.5 *Design a constant-k low-pass T and π section filters having cut-off frequency of 4 kHz and nominal characteristic impedance of 500 Ω.*

Solution

$$f_c = 4\text{ kHz},\quad k = 500\ \Omega$$

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi\times4\times10^3} = 39.79\text{ mH}$$

$$\text{C} = \frac{1}{\pi f_c k} = \frac{1}{\pi\times4\times10^3\times500} = 0.16\ \mu\text{F}$$

The *T* section consists of an inductor of $\frac{L}{2}$, i.e. 19.9 mH in each series branch and a capacitor of 0.16 μF in the shunt branch as shown in Fig. 15.10 (a). The *π*-section consists of an inductor of 39.79 mH in the series branch and a capacitor of $\frac{C}{2}$, i.e. 0.08 μF in each shunt branch as shown in Fig. 15.10 (b).

Fig. 15.10

Example 15.6 *Design a constant-k, T section low-pass filter is having 2.5 kHz cut-off frequency and nominal impedance of 700 Ω. Also find the frequency at which this filter produces an attenuation of 19.1 dB. Find its characteristic impedances and phase constant at pass band and stop or attenuation band.*

Solution

$$f_c = 2.5\text{ kHz},\quad k = 700\ \Omega$$

$$L = \frac{k}{\pi f_c} = \frac{700}{\pi\times2.5\times10^3} = 89.13\text{ mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi\times2.5\times10^3\times700} = 0.18\ \mu\text{F}$$

The *T*-section filter consists of an inductor of $\frac{L}{2}$, i.e. 44.57 mH in each series arm and a capacitor of 0.18 μF in the shunt arm as shown in Fig. 15.11.

$$\alpha = 19.1 \text{ dB} = 2.197 \text{ nepers}$$

$$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right)$$

$$2.197 = 2\cosh^{-1}\left(\frac{f}{2.5\times10^3}\right)$$

$$f = 4.17 \text{ kHz}$$

Fig. 15.11

β is imaginary in pass band.

In attenuation band, $\beta = 180°$

15.7 CONSTANT-*k* HIGH-PASS FILTER

A constant-*k* high-pass filter is obtained by changing the positions of series and shunt arm of the constant-*k* low-pass filter, Figure 15.12 shows a constant-*k*, *T* and π section, high-pass filter.

Fig. 15.12 *Constant-k high pass filter*

In a constant-*k* high-pass filter

$$\mathbf{Z}_1 = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

$$\mathbf{Z}_2 = j\omega L$$

1. Nominal Impedance

$$k = \sqrt{\mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\left(\frac{1}{j\omega C}\right)(j\omega L)} = \sqrt{\frac{L}{C}}$$

2. Cut-off Frequency

The cut-off frequencies are obtained when $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$ and $\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$

(i) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0$$

$$\mathbf{Z}_1 = 0$$

$$\frac{1}{j\omega C} = 0$$

$$\omega = \infty$$

$$f = \infty$$

(ii) When

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1$$

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$-j\frac{1}{\omega_c} = 4j\omega L$$

$$\omega^2 LC = \frac{1}{4}$$

$$\omega^2 = \frac{1}{4LC}$$

$$\omega = \omega_c = \frac{1}{2\sqrt{LC}}$$

$$f = f_c = \frac{1}{4\pi\sqrt{LC}}$$

Hence, the filter passes all the frequencies beyond f_c. The pass band starts at $f = f_c$ and continues up to infinite frequency. All the frequencies below the cut-off frequency lie in the attenuation or stop band.

3. Attenuation Constant

In pass band, $\alpha = 0$

In stop band, $\alpha = 2\cosh^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\cosh^{-1}\left(\frac{f_c}{f}\right)$

The attenuation constant α decreases gradually to zero at the cut-off frequency and remains at zero through the pass band. The variation of α is plotted in Fig. 15.13.

Fig. 15.13 *Variation of α with frequency*

4. Phase Constant

In pass band, $\beta = 2\sin^{-1}\sqrt{\left|\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right|} = 2\sin^{-1}\left(\frac{f_c}{f}\right)$

In stop band, $\beta = \pi$.

The phase constant β remains at constant value π in the stop band and decreases to $-\pi$ in at f_c and reaches zero value gradually as f increases in the pass band. The variation of β is plotted in Fig. 15.14.

Fig. 15.14 *Variation of β with frequency*

5. Characteristic Impedance

$$\mathbf{Z}_{0T} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)} = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\mathbf{Z}_{0\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{0T}} = \frac{k}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Fig. 15.15 *Variation of characteristic impedance with frequency*

The variation of characteristic impedance is plotted in Fig. 15.15.

6. Design of Filter

$$k = \sqrt{\frac{L}{C}}$$

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

Solving these two equations,

$$L = \frac{k}{4\pi f_c}$$

$$C = \frac{1}{4\pi f_c k}$$

Example 15.7 *Find the characteristic impedance, cut-off frequency and pass band for the network shown in Fig. 15.16.*

Fig. 15.16

Solution The network is a high-pass filter.

$$2C = 0.4\ \mu\text{F}, \quad L = 50\ \text{mH}$$
$$C = 0.2\ \mu\text{F}$$

(a) Characteristic impedance

$$k = \sqrt{\frac{L}{C}} = \sqrt{\frac{50\times10^{-3}}{0.2\times10^{-6}}} = 500\ \Omega$$

(b) Cut-off frequency

$$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{1}{4\pi\sqrt{50\times10^{-3}\times0.2\times10^{-6}}} = 795.77\ \text{Hz}$$

(c) Pass band

The pass band is from 795.77 Hz to infinite frequency.

Example 15.8 *A high-pass filter section is constructed from two capacitors of 1 μF each and a 15 mH inductance. Find (a) cut-off frequency, (b) infinite frequency characteristic impedance, (c) characteristic impedance at 200 Hz and 2000 Hz, (d) attenuation at 200 Hz and 2000 Hz, and (e) phase constant at 200 Hz and 2000 Hz.*

Solution

$$L = 15\ \text{mH}, \quad 2C = 1\ \mu\text{F}$$
$$C = 0.5\ \mu\text{F}$$

(a) Cut-off frequency

$$f_c = \frac{1}{4\pi\sqrt{LC}} = \frac{1}{4\pi\sqrt{15\times10^{-3}\times0.5\times10^{-6}}} = 918.88 \text{ Hz}$$

(b) Infinite-frequency characteristic impedance

At $f = \infty$,
$$\mathbf{Z}_0 = k = \sqrt{\frac{L}{C}} = \sqrt{\frac{1.5\times10^{-3}}{0.5\times10^{-6}}} = 173.21\,\Omega$$

(c) Characteristic impedance at 200 Hz

$$\mathbf{Z}_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 173.21\sqrt{1-\left(\frac{918.88}{2000}\right)^2} = j776.72\,\Omega$$

Characteristic impedance at 2000 Hz

$$\mathbf{Z}_{0T} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 173.21\sqrt{1-\left(\frac{918.88}{2000}\right)^2} = 153.85\,\Omega$$

(d) Attenuation at 200 Hz

$$\alpha = 2\cosh^{-1}\left(\frac{f_c}{f}\right) = 2\cosh^{-1}\left(\frac{918.88}{200}\right) = 4.41$$

The frequency of 2000 Hz lies in the pass band.

$$\alpha = 0$$

(e) Phase constant

The frequency 200 Hz lies in the attenuation band.

$$\beta = \pi \text{ radians}$$

At 2000 Hz,

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) = 2\sin^{-1}\left(\frac{918.88}{2000}\right) = 54.7°$$

Example 15.9 *Design a constant-k high-pass T and π sections filters having a cut-off frequency of 2000 Hz and infinite frequency characteristic impedance of 300 Ω.*

Solution

$$f_c = 2000 \text{ Hz}, \quad k = 300\,\Omega$$

$$L = \frac{k}{4\pi f_c} = \frac{300}{4\pi\times2000} = 11.94 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi\times2000\times300} = 0.13\,\mu\text{F}$$

The *T*-section filter consists of a capacitor of 2*C*, i.e 0.26 μF in each series arm and an inductor of 11.94 mH in shunt arm as shown in Fig. 15.17 (a).

The *π*-section filter consists of a capacitor of 0.13 μF in the series arm and an inductor of 2*L*, i.e. 23.88 mH in each shunt arm as shown in Fig. 15.17 (b).

Fig. 15.17

Example 15.10 *Design a T-section constant-k high-pass filter having a cut-off frequency of 10 kHz and a design impedance of 600 Ω. Find its characteristic impedance and phase constant at 25 kHz.*

Solution $f_c = 10$ kHz, $k = 600\ \Omega$

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 10 \times 10^3} = 4.77 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 10 \times 10^3 \times 600} = 0.013\ \mu\text{F}$$

The T-section filter consists of a capacitor of $2C$, i.e. 0.026 μF in each series arm and an inductor of 4.77 mH in shunt arm as shown in Fig. 15.18.

Fig. 15.18

(a) Characteristic impedance at 25 kHz

$$\mathbf{Z}_0 = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 600\sqrt{1-\left(\frac{10\times10^3}{25\times10^3}\right)^2} = 549.91\ \Omega$$

(b) Phase constant at 25 kHz

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) = 2\sin^{-1}\left(\frac{10\times10^3}{25\times10^3}\right) = 47.16°$$

The frequency of 25 kHz lies in the attenuation band.

In the attenuation band, $\beta = 180° = \pi$ radians

15.8 BAND-PASS FILTER

A band-pass filter attenuates all the frequencies below a lower cutoff frequency and above an upper cut-off frequency. It passes a band of frequencies without attenuation. A band pass filter is obtained by using a low pass filter followed by a high-pass filter.

Figure 15.19 shows a band pass filter. The series arm is a series resonant circuit comprising L_1 and C_1 while its shunt arm is formed by a parallel resonant circuit L_2 and C_2. The resonant frequency of series arm and shunt arm are made equal.

Fig. 15.19 *Band-pass filter*

For series arm,

$$\omega_0 \frac{L_1}{2} = \frac{1}{2\omega_0 C_1}$$

$$\omega_0^2 = \frac{1}{L_1 C_1}$$

For shunt arm,

$$\frac{1}{\omega_0 C_2} = \omega_0 L_2$$

$$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_1 C_1 = L_2 C_2$$

For series arm,

$$\mathbf{Z}_1 = j\omega L_1 - \frac{j}{\omega c_1} = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega c_1}\right)$$

For shunt arm,

$$\mathbf{Z}_2 = \frac{j\omega L_2 \dfrac{1}{j\omega c_2}}{j\omega L_2 + \dfrac{1}{j\omega c_2}} = \frac{j\omega L_2}{1 - \omega^2 L_2 C_2}$$

$$\mathbf{Z}_1\mathbf{Z}_2 = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega c_1}\right)\left(\frac{j\omega L_2}{1 - \omega^2 L_2 C_2}\right) = -\frac{L_2}{C_1}\left(\frac{\omega^2 L_1 C_1 - 1}{1 - \omega^2 L_2 C_2}\right) = \frac{L_2}{C_1} = \frac{L_1}{C_2} = k^2$$

where k is constant.

For constant k filter, at cut-off frequency,

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$\mathbf{Z}_1^2 = -4\mathbf{Z}_1\mathbf{Z}_2 = -4k^2$$

$$\mathbf{Z}_1 = \pm j2k$$

i.e. the value of $\mathbf{Z}_1$ at lower cut-off frequency is equal to the negative value of $\mathbf{Z}_1$ at the upper cut-off frequency.

$$\frac{1}{j\omega_1 C_1} + j\omega_1 L_1 = -\left(\frac{1}{j\omega_2 C_1} + j\omega_2 L_1\right)$$

$$1 - \omega_1^2 L_1 C_1 = \frac{\omega_1}{\omega_2}\left(\omega_2^2 L_1 C_1 - 1\right)$$

But
$$\omega_0^2 = \frac{1}{L_1 C_1}$$
$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{\omega_1}{\omega_2}\left(\frac{\omega_2^2}{\omega_0^2} - 1\right)$$
$$(\omega_0^2 - \omega_1^2)\omega_2 = \omega_1(\omega_2^2 - \omega_0^2)$$
$$\omega_0^2\omega_2 - \omega_1^2\omega_2 = \omega_1\omega_2^2 - \omega_1\omega_0^2$$
$$\omega_0^2(\omega_1 + \omega_2) = \omega_1\omega_2(\omega_2 + \omega_1)$$
$$\omega_0^2 = \omega_1\omega_2$$
$$\omega_0 = \sqrt{\omega_1\omega_2}$$
$$f_0 = \sqrt{f_1 f_2}$$

Thus, resonant frequency is the geometric mean of the cutoff frequencies. The variation of attenuation, phase constant and characteristic impedance with frequency are shown in Fig. 15.20.

Fig. 15.20 *Variation of (a) attenuation, (b) phase constant, and (c) characteristic impedance with frequency for constant–k band-pass filter*

Design of Filter

If the filter is terminated in a load resistance $R = k$ then at lower cut-off frequency,

$$\mathbf{Z}_1 = -2jk$$
$$\frac{1}{j\omega_1 C_1} + j\omega_1 L_1 = -2jk$$
$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2k$$
$$1 - \omega_1^2 L_1 C_1 = 2k\omega_1 C_1$$
$$1 - \frac{\omega_1^2}{\omega_0^2} = 2k\omega_1 C_1 \qquad \left(\because\ \omega_0^2 = \frac{1}{L_1 C_1}\right)$$
$$1 - \left(\frac{f_1}{f_0}\right)^2 = 4\pi k f_1 C_1$$

$$1-\frac{f_1^2}{f_1 f_2}=4\pi k f_1 C_1 \qquad \left(\because f_0=\sqrt{f_1 f_2}\right)$$

$$f_2-f_1=4\pi k f_1 f_2 C_1$$

$$C_1=\frac{f_2-f_1}{4\pi\, k f_1 f_2}$$

$$L_1=\frac{1}{\omega_0^2 C_1}=\frac{4\pi\, k f_1 f_2}{\omega_0^2(f_2-f_1)}=\frac{4\pi\, k f_1 f_2}{4\pi^2\, f_0^2\left(f_2-f_1\right)}=\frac{k}{\pi(f_2-f_1)}$$

For shunt arm,

$$\mathbf{Z}_1\mathbf{Z}_2=\frac{L_2}{C_1}=\frac{L_1}{C_2}=k^2$$

$$L_2=C_1k^2=\frac{(f_2-f_1)k}{4\pi\, f_1 f_2}$$

$$C_2=\frac{L_1}{k^2}=\frac{1}{\pi(f_2-f_1)k}$$

Example 15.11 *In a constant-k band-pass filter, the ratio of capacitances in the shunt and series arms is 100:1. The resonant frequency of both arms is 1000 Hz. Find the bandwidth,*

Solution

$$\frac{C_2}{C_1}=\frac{100}{1}=100, \quad f_0=1000\text{ Hz}$$

$$C_2=\frac{1}{\pi k(f_2-f_1)}$$

$$C_1=\frac{f_2-f_1}{4\pi k f_1 f_2}$$

$$\frac{C_2}{C_1}=\frac{1}{\pi k(f_2-f_1)}\frac{4\pi k f_1 f_2}{f_2-f_1}$$

$$100=\frac{4 f_1 f_2}{(f_2-f_1)^2}$$

$$100=\frac{4 f_0^2}{(f_2-f_1)^2}$$

$$f_2-f_1=\frac{2 f_0}{\sqrt{100}}=\frac{2\times 1000}{\sqrt{100}}=200$$

$$BW=f_2-f_1=200\text{ Hz}$$

Example 15.12 *Design a band pass constant-k filter with cut-off frequency of 4 kHz and 10 kHz and nominal characteristic impedance of 500 Ω.*

Solution

$$f_1=4\text{ kHz}, \quad f_2=10\text{ kHz}, \quad k=500\ \Omega$$

$$L_1=\frac{k}{\pi(f_2-f_1)}=\frac{500}{\pi(10\times 10^3-4\times 10^3)}=26.53\text{ mH}$$

$$C_1=\frac{f_2-f_1}{4\pi k f_1 f_2}=\frac{10\times 10^3-4\times 10^3}{4\pi\times 500\times 4\times 10^3\times 10\times 10^3}=0.024\ \mu\text{F}$$

$$L_2 = \frac{(f_2 - f_1)k}{4\pi f_1 f_2} = \frac{(10\times10^3 - 4\times10^3)\,500}{4\pi\times4\times10^3\times10\times10^3} = 5.97 \text{ mH}$$

$$C_2 = \frac{1}{\pi(f_2 - f_1)k} = \frac{1}{\pi\times(10\times10^3 - 4\times10^3)\times500} = 0.11\,\mu\text{F}$$

The T-section filter consists of series combination of an inductor of $\frac{L_1}{2}$, i.e. 13.27 mH and a capacitor of 2 C_1, i.e. 0.048 μF in each series arm and a parallel combination of an inductor of 5.97 mH and a capacitor of 0.11 μF in shunt arm as shown in Fig. 15.21.

Fig. 15.21

The π-section filter consists of a series combination of an inductor 26.53 mH and a capacitor of 0.024 μF in the series arm and a parallel combination of an inductor of $2L_2$, i.e. 11.94 mH and a capacitor of $\frac{C_2}{2}$, i.e. 0.055 μF in each shunt arm as shown in Fig. 15.22.

Fig. 15.22

15.9 BAND-STOP FILTER

A band-stop filter attenuates a specified band of frequencies and allows all frequencies below and above this band. A band-stop filter is realised by connecting a low-pass filter in parallel with a high-pass filter. Figure 15.23 shows a band-stop filter.

As in the band-pass filter, the series and shunt arms are chosen to resonate at same frequency ω_0.

For series arm,

$$\frac{\omega_0 L_1}{2} = \frac{1}{2\omega_0 C_1}$$

$$\omega_0^2 = \frac{1}{L_1 C_1}$$

Fig. 15.23 *Band-stop filter*

For shunt arm,

$$\omega_0 L_2 = \frac{1}{\omega_0 C_2}$$

$$\omega_0^2 = \frac{1}{L_2 C_2}$$

$$L_1 C_1 = L_2 C_2$$

Similarly,

$$\mathbf{Z}_1 \mathbf{Z}_2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} = k^2$$

and

$$f_0 = \sqrt{f_1 f_2}$$

At cut-off frequencies,

$$\mathbf{Z}_1 = -4\mathbf{Z}_2$$

$$\mathbf{Z}_1 \mathbf{Z}_2 = -4\mathbf{Z}_2^2 = k^2$$

$$\mathbf{Z}_2 = \pm j\frac{k}{2}$$

The variation of attenuation, phase shift and characteristic impedance with frequency are shown in Fig. 15.24.

Fig 15.24 *Variation of (a) attenuation, (b) phase constant, and (c) characteristic impedance with frequency for constant-k band-stop filter.*

Design of Filter

If the load is terminated in load resistance $R = k$ then at lower cut-off frequency,

$$\mathbf{Z}_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = j\frac{k}{2}$$

$$\frac{1}{\omega_1 C_2} - \omega_1 L_2 = \frac{k}{2}$$

$$1 - \omega_1^2 L_2 C_2 = \omega_1 C_2 \frac{k}{2}$$

$$1 - \frac{\omega_1^2}{\omega_0^2} = \frac{k}{2}\omega_1 C_2 \qquad \left(\because \omega_0^2 = \frac{1}{L_2 C_2}\right)$$

$$1 - \left(\frac{f_1}{f_0}\right)^2 = k\pi f_1 C_2$$

$$C_2 = \frac{1}{k\pi\, f_1}\left[1 - \left(\frac{f_1}{f_0}\right)^2\right] = \frac{1}{k\pi}\left(\frac{1}{f_1} - \frac{1}{f_2}\right) = \frac{1}{k\pi}\left(\frac{f_2 - f_1}{f_1 f_2}\right) = \frac{f_2 - f_1}{\pi k\, f_1 f_2}$$

$$L_2 = \frac{1}{\omega_0^2 C_2} = \frac{\pi k f_1 f_2}{\omega_0^2 (f_2 - f_1)} = \frac{\pi k f_1 f_2}{4\pi^2 f_0^2 (f_2 - f_1)} = \frac{k}{4\pi(f_2 - f_1)}$$

$$k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1}$$

$$L_1 = k^2 C_2 = \frac{k(f_2 - f_1)}{\pi f_1 f_2}$$

$$C_1 = \frac{L_2}{k^2} = \frac{1}{4\pi\, k(f_2 - f_1)}$$

Example 15.13 *Design a constant-k band-stop filter having cut-off frequencies at 2000 Hz and 5000 Hz and characteristic resistance of 600 Ω.*

Solution

$$f_1 = 2000 \text{ Hz}, \quad f_2 = 5000 \text{ Hz}, \quad k = 600\ \Omega$$

$$L_1 = \frac{k(f_2 - f_1)}{\pi f_1 f_2} = \frac{600(5000 - 2000)}{\pi \times 2000 \times 5000)} = 57.3 \text{ mH}$$

$$C_1 = \frac{1}{4\pi k(f_2 - f_1)} = \frac{1}{4\pi \times 600 \times (5000 - 2000)} = 0.044\ \mu\text{F}$$

$$L_2 = \frac{k}{4\pi(f_2 - f_1)} = \frac{600}{4\pi \times (5000 - 2000)} = 15.92 \text{ mH}$$

$$C_2 = \frac{f_2 - f_1}{\pi k f_1 f_2} = \frac{5000 - 2000}{\pi \times 600 \times 2000 \times 5000} = 0.16\ \mu\text{F}$$

The *T*-section filter consists of a parallel combination of $\frac{L_1}{2}$, i.e. 28.65 mH and a capacitor of 2 C_1, i.e. 0.088 μF in each series arm and a series combination of an inductor of 15.92 mH and a capacitor of 0.16 μF in shunt arm as shown in Fig. 15.25.

Fig. 15.25

Fig. 15.26

The π-section filter consists of a parallel combination of an inductor of 57.3 mH and a capacitor of 0.044 μF in the series arm and a series combination of an inductor of $2L_2$, i.e. 31.84 mH and a capacitor of $\frac{C_2}{2}$, i.e. 0.08 μF in each shunt arm as shown in Fig. 15.26.

15.10 *m*-DERIVED FILTERS

There are two disadvantages of constant-*k* filters.

(i) The attenuation is not sharp in the stop band.
(ii) The characteristic impedance varies widely in the pass band.

If the constant-*k* filter is regarded as prototype, it is possible to design a filter to have rapid attenuation in the stop band and the same characteristic impedance as the prototype throughout the pass band and stop band. Such a filter is called an *m- derived filter*.

Constant-*k* and *m*-derived *T* sections are shown in Fig. 15.27. The constant-*k* *T*-network is identical to that of the *m*-derived *T*-network except that the series arm is multiplier by *m*.

Fig. 15.27 *Constant-k and m-derived T sections*

For the constant-*k* *T* section,

$$\mathbf{Z}_{OT} = \sqrt{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1 \mathbf{Z}_2}$$

For the *m*-derived *T* section,

$$\mathbf{Z}_{OT} = \sqrt{\frac{m^2 \mathbf{Z}_1^2}{4} + m \mathbf{Z}_1 \mathbf{Z}_2'}$$

Equating the characteristic impedance of the networks,

$$\sqrt{\frac{\mathbf{Z}_1^2}{4}+\mathbf{Z}_1\,\mathbf{Z}_2}=\sqrt{\frac{m^2\mathbf{Z}_1^2}{4}+m\mathbf{Z}_1\,\mathbf{Z}_2'}$$

$$\frac{\mathbf{Z}_1^2}{4}+\mathbf{Z}_1\,\mathbf{Z}_2=\frac{m^2\mathbf{Z}_1^2}{4}+m\mathbf{Z}_1\,\mathbf{Z}_2'$$

$$m\mathbf{Z}_1\,\mathbf{Z}_2'=\frac{\mathbf{Z}_1^2}{4}(1-m^2)+\mathbf{Z}_1\,\mathbf{Z}_2$$

$$\mathbf{Z}_2'=\left(\frac{1-m^2}{4m}\right)\mathbf{Z}_1+\frac{\mathbf{Z}_2}{m}$$

Hence, $\mathbf{Z}_2'$ must be an impedance composed of two series impedances $\left(\frac{1-m^2}{4m}\right)\mathbf{Z}_1$ and $\frac{\mathbf{Z}_2}{m}$. The term $\frac{1-m^2}{4m}$ should be positive to realise the impedance $\mathbf{Z}_2'$ physically, i.e. $0<m<1$. Thus, an m-derived filter can be obtained from constant-k filter by modifying its series and shunt impedances as shown in Fig. 15.28.

Fig. 15.28 *m-derived T-section filter*

Similarly, the constant-k π-section filter can be modified to m-derived π section filter. A constant-k and m-derived π sections are shown in Fig. 15.29.

Fig. 15.29 *(a) Constant-k section (b) m-derived π section*

For the constant-k π section,

$$\mathbf{Z}_{O\pi}=\sqrt{\frac{\mathbf{Z}_1\,\mathbf{Z}_2}{1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}$$

For the m-derived π section,

$$\mathbf{Z}_{O\pi}=\sqrt{\frac{\mathbf{Z}_1'\frac{\mathbf{Z}_2}{m}}{1+\frac{\mathbf{Z}_1'}{4\frac{\mathbf{Z}_2}{m}}}}$$

Equating the characteristic impedance of the networks,

$$\sqrt{\frac{\mathbf{Z}_1\,\mathbf{Z}_2}{1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}}=\sqrt{\frac{\mathbf{Z}_1'\frac{\mathbf{Z}_2}{m}}{1+\frac{\mathbf{Z}_1'}{4\frac{\mathbf{Z}_2}{m}}}}$$

$$\frac{\mathbf{Z}_1\,\mathbf{Z}_2}{1+\dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}} = \frac{\mathbf{Z}_1'\,\dfrac{\mathbf{Z}_2}{m}}{1+\dfrac{\mathbf{Z}_1'}{4\dfrac{\mathbf{Z}_2}{m}}}$$

$$\mathbf{Z}_1\,\mathbf{Z}_2\left(1+\frac{\mathbf{Z}_1'}{4\dfrac{\mathbf{Z}_2}{m}}\right) = \mathbf{Z}_1'\,\frac{\mathbf{Z}_2}{m}\left(1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)$$

$$\mathbf{Z}_1\,\mathbf{Z}_2 + \frac{m\mathbf{Z}_1\,\mathbf{Z}_1'}{4} = \frac{\mathbf{Z}_1\,\mathbf{Z}_2}{m} + \frac{\mathbf{Z}_1\mathbf{Z}_1'}{4m}$$

$$4\mathbf{Z}_1\,\mathbf{Z}_2 + m\,\mathbf{Z}_1\,\mathbf{Z}_1' = \frac{4\,\mathbf{Z}_1'\,\mathbf{Z}_2 + \mathbf{Z}_1\,\mathbf{Z}_1'}{m}$$

$$\mathbf{Z}_1'\left(\frac{\mathbf{Z}_1}{m}+\frac{4\mathbf{Z}_2}{m}-m\mathbf{Z}_1\right) = 4\,\mathbf{Z}_1\,\mathbf{Z}_2$$

$$\mathbf{Z}_1' = \frac{\mathbf{Z}_1\,\mathbf{Z}_2}{\dfrac{\mathbf{Z}_1}{4m}+\dfrac{\mathbf{Z}_2}{4m}-\dfrac{m\mathbf{Z}_1}{4}}$$

$$= \frac{\mathbf{Z}_1\,\mathbf{Z}_2}{\dfrac{\mathbf{Z}_1}{4m}(1-m^2)+\dfrac{\mathbf{Z}_2}{m}} = \frac{\mathbf{Z}_1\,\mathbf{Z}_2\,\dfrac{4m^2}{(1-m^2)}}{m\mathbf{Z}_1+\dfrac{4m^2}{m(1-m)}\mathbf{Z}_2}$$

$$= \frac{m\mathbf{Z}_1\,\dfrac{4m}{1-m^2}\mathbf{Z}_2}{m\mathbf{Z}_1+\dfrac{4m}{1-m^2}\mathbf{Z}_2}$$

Hence, $\mathbf{Z}_1'$ must be an impedance composed of two parallel impedances $m\mathbf{Z}_1$ and $\left(\dfrac{4m}{1-m^2}\right)\mathbf{Z}_2$. Thus, an m-derived filter can be obtained from a constant-k filter by modifying its series and shunt impedances as shown in Fig. 15.30.

Fig. 15.30 *m-derived π-section filter*

15.11 *m*-DERIVED LOW-PASS FILTER

Figure 15.31 shows an m-derived low-pass T and π filter. At a particular frequency, the shunt arm of the T section or series arm of the π section will be in resonance giving a short circuit to the input for the T section and open circuit for the π section causing infinite attenuation.

Fig. 15.31 *m-derived low-pass filter*

Let
$$\mathbf{Z}_1 = j\omega L$$
$$\mathbf{Z}_2 = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

At resonant frequency,

$$\left(\frac{1-m^2}{4m}\right)\omega_0 L = \frac{1}{m\,\omega_0 C}$$
$$\omega_0{}^2 = \frac{4}{(1-m^2)\,LC}$$
$$\omega_0 = \frac{2}{\sqrt{LC(1-m^2)}}$$
$$f_0 = \frac{1}{\pi\sqrt{LC(1-m^2)}} = f_\infty$$

For low pass filter,

$$f_c = \frac{1}{\pi\sqrt{LC}}$$
$$f_\infty = \frac{f_c}{\sqrt{1-m^2}}$$
$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2}$$

For sharp cut-off, f_∞ should be near to f_C. For smaller value of m, f_∞ comes close to f_c. The m-derived T-section low-pass filter is shown in Fig. 15.32.

Fig. 15.32 *m-derived T-section low-pass filter*

Similarly, for m-derived π-section, the inductance and capacitance in the series arm constitute a resonant circuit.

At resonant frequency,

$$m\,\omega_0 L = \frac{1}{\left(\dfrac{1-m^2}{4m}\right)\omega_0 C}$$

$$\omega_0^2 = \frac{4}{LC(1-m^2)}$$

$$\omega_0 = \frac{4}{\sqrt{LC(1-m^2)}}$$

$$f_0 = \frac{1}{\pi\sqrt{LC(1-m^2)}} = f_\infty$$

For low-pass filter,

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$f_\infty = \frac{f_c}{\sqrt{1-m^2}}$$

For both m-derived low-pass networks, $f_\infty > f_c$ for positive values of m $(0 < m < 1)$.

The m-derived π section low-pass filter is shown in Fig. 15.33.

Fig. 15.33 *m-derived π-section low-pass filter*

The variation of attenuation with frequency is shown in Fig. 15.34. It is seen that the attenuation rises abruptly becomes infinite at f_∞ and then decreases for the m-derived filter.

Thus, an m-derived filter gives a sharp cut-off. The decrease in attenuation for frequencies higher than f_∞ can be avoided by connecting an m-derived filter and a constant-k filter in cascade.

The variation of β with frequency is shown in Fig. 15.35.

Fig. 15.34 *Variation of α with frequency*

Fig. 15.35 *Variation of β with frequency*

Fig. 15.36 *Variation in characteristic impedance with frequency.*

The variation of characteristic impedance with frequency is shown in Fig. 15.36. The characteristic impedance of an *m*-derived filter also varies with frequency. The impedance matching can be achieved by using half-section terminations.

Example 15.14 *Design an m-derived T-section low-pass filter having cut-off frequency of 800 Hz, design impedance of 500 Ω and frequency of attenuation of 1000 Hz.*

Solution $k = 500\ \Omega, \quad f_c = 800\ \text{Hz}, \quad f_\infty = 1000\ \text{Hz}$

For a constant-*k* *T*-section low-pass filter

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi \times 800} = 198.94\ \text{mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi \times 800 \times 500} = 0.8\ \mu\text{F}$$

For an *m*-derived *T*-section low-pass filter

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1 - \left(\frac{800}{1000}\right)^2} = 0.6$$

$$\frac{mL}{2} = \frac{0.6 \times 198.94 \times 10^{-3}}{2} = 59.68\ \text{mH}$$

$$mC = 0.6 \times 0.8 \times 10^{-6} = 0.48\ \mu\text{F}$$

$$\left(\frac{1 - m^2}{4m}\right) L = \left[\frac{1 - (0.6)^2}{4(0.6)}\right] \times 198.94 \times 10^{-3} = 53.05\ \text{mH}$$

Fig. 15.37

The *m*-derived *T*-section low-pass filter is shown in Fig. 15.37.

Example 15.15 *Design an m-derived π-section low-pass filter having cut-off frequency of 1500 Hz, design impedance of 500 Ω and infinite attenuation frequency of 2000 Hz.*

Solution $f_C = 1500\ \text{Hz}, \quad k = 500\ \Omega, \quad f_\infty = 2000\ \text{Hz}$

For constant-*k* *π*-section low-pass filter,

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi \times 1500} = 106.1\ \text{mH}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi \times 1500 \times 500} = 0.424\ \mu\text{F}$$

For m-derived π-section low-pass filter,

$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1-\left(\frac{1500}{2000}\right)^2} = 0.66$$

$$\frac{mC}{2} = \frac{0.66 \times 0.424 \times 10^{-6}}{2} = 0.14\ \mu\text{F}$$

$$mL = 0.66 \times 106.1 \times 10^{-3} = 70.03\ \text{mH}$$

$$\left(\frac{1-m^2}{4m}\right)C = \frac{1-(0.66)^2}{4 \times 0.66} \times 0.424 \times 10^{-6} = 0.09\ \mu\text{F}$$

Fig. 15.38

The m-derived low-pass filter is shown in Fig. 15.38.

15.12 *m*-DERIVED HIGH-PASS FILTER

Figure 15.39 shows m-derived high-pass T and π filter.

Fig. 15.39 *m-derived high-pass filter*

Let

$$\mathbf{Z}_1 = \frac{1}{j\omega c}$$

$$\mathbf{Z}_2 = j\omega L$$

At resonant frequency,

$$\omega_0 \frac{L}{m} = \frac{1}{\omega_0 \left(\frac{4m}{1-m^2}\right) C}$$

$$\omega_0^2 = \frac{1}{\frac{L}{m}\left(\frac{4m}{1-m^2}\right)C} = \frac{1-m^2}{4LC}$$

$$\omega_0 = \frac{\sqrt{1-m^2}}{2\sqrt{LC}}$$

$$f_0 = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}} = f_\infty$$

For a high-pass filter,

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = f_c\sqrt{1-m^2}$$

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}$$

Fig. 15.40 *m-derived T-section high-pass filter*

The m-derived T-section high-pass filter is shown in Fig. 15.40.

Similarly, for an m-derived π section, the resonant circuit is formed by the series arm inductance and capacitance.

At resonant frequency,

$$\left(\frac{4m}{1-m^2}\right)\omega_0 L = \frac{1}{\frac{\omega_0}{m}C}$$

$$\omega_0^2 = \frac{1-m^2}{4LC}$$

$$\omega_0 = \frac{\sqrt{1-m^2}}{2\sqrt{LC}}$$

$$f_0 = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}} = f_\infty$$

For a high-pass filter,

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = f_c\sqrt{1-m^2}$$

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}$$

Fig. 15.41 *m-derived π-section high-pass filter*

The m-derived π section high-pass filter is shown in Fig. 15.41. The variation of attenuation phase constant and characteristic impedance with frequency are shown in Fig. 15.42 for an m-derived high-pass filter. The attenuation rises abruptly, becomes infinite at f_∞ and then decreases, thus giving a sharp cut-off.

Fig. 15.42 *Variation of (a) attenuation, (b) phase constant, (c) characteristic impedance of m-derived high-pass filter.*

The use of *m*-derived and constant-*k* high pass filters in series gives a sharp cut-off and high attenuation in the entire stop band. The half-section termination provides impedance matching.

Example 15.16 *Design an m-derived T-section high-pass filter with a cutoff frequency of 2 kHz, design impedance of 700 Ω and m = 0.6.*

Solution $f_C = 2$ kHz, $k = 700\ \Omega$, $m = 0.6$

For constant-*k* *T*-section high-pass filter,

$$L = \frac{k}{4\pi f_c} = \frac{700}{4\pi \times 2 \times 10^3} = 27.85 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 2000 \times 700} = 0.057\ \mu\text{F}$$

For an *m*-derived *T*-section high-pass filter,

$$\frac{L}{m} = \frac{27.85 \times 10^{-3}}{0.6} = 46.42 \text{ mH}$$

$$\frac{2C}{m} = \frac{2 \times 0.057 \times 10^{-6}}{0.6} = 0.19\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)C = \frac{4 \times 0.6}{1-(0.6)^2} \times 0.057 \times 10^{-6} = 0.21\ \mu\text{F}$$

The *m*-derived high-pass filter is shown in Fig. 15.43.

Fig. 15.43

Example 15.17 *Design the π-section of an m-derived high-pass filter having a design impedance of 600 Ω, cut-off frequency of 4 kHz and an infinite attenuation at 3.6 kHz.*

Solution $k = 600\ \Omega$, $f_c = 4$ kHz, $f_\infty = 3.6$ kHz

For a constant-*k* *π*-section high-pass filter,

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 4 \times 10^3} = 11.94 \text{ mH}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 4 \times 10^3 \times 600} = 0.033\ \mu\text{F}$$

For an *m*-derived *π*-section high-pass filter,

$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} = \sqrt{1-\left(\frac{3.6 \times 10^3}{4 \times 10^3}\right)^2} = 0.436$$

$$\frac{2L}{m} = \frac{2 \times 11.94 \times 10^{-3}}{0.436} = 54.77 \text{ mH}$$

$$\frac{C}{m} = \frac{0.033 \times 10^{-6}}{0.436} = 0.076\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)L = \frac{4\times 0.436}{1-(0.436)^2}\times 11.94\times 10^{-3} = 25.71 \text{ mH}$$

The m-derived π-section high-pass filter is shown in Fig. 15.44.

Fig. 15.44

15.13 TERMINATING HALF SECTIONS

A filter is composed of a number of sections. As the characteristic impedance of an equivalent T or π section does not match with each other, a half section is used for impedance matching between T and π sections.

The input impedance of the half section is same as the characteristic impedance of a T section, while the output impedance of the half section is same as the characteristic impedance of the π section. Hence, matching can be obtained if a half section is connected in between a T and π section.

15.13.1 Constant-*k* Half Sections

Figure 15.45(a) shows a constant-k T-section filter. If this section is bisected longitudinally, a half section is obtained as shown in Fig. 15.45 (b).

Fig. 15.45 *(a) Constant-k T-section (b) constant-k half T-section*

Similarly, a constant-k π section can be bisected into two half sections as shown in Fig. 15.46.

Fig. 15.46 *(a) Constant-k π-section (b) constant-k half π-section*

Figure 15.47 shows a constant-k half section. The image impedance of the section as seen from terminals 1-2 and terminals 3-4 can be found from the open circuit and short-circuit impedances.

$$\mathbf{Z}_{12} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}} = \sqrt{\frac{(2\mathbf{Z}_2)\left(2\mathbf{Z}_2\,\dfrac{\mathbf{Z}_1}{2}\right)}{2\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1}{2}}} = \sqrt{\frac{4\mathbf{Z}_1\,\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}} = \mathbf{Z}_{0\pi}$$

$$\mathbf{Z}_{34} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}} = \sqrt{\left(\frac{\mathbf{Z}_1}{2} + 2\mathbf{Z}_2\right)\left(\frac{\mathbf{Z}_1}{2}\right)}$$

$$= \sqrt{\mathbf{Z}_1\,\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)} = \mathbf{Z}_{0T}$$

Fig. 15.47 *Constant-k half section*

Thus, the half section has the impedance characteristics of a π section between terminals 1-2 and that of a T section between terminals 3-4. Hence, this half section can be used to match a π section to a T section. It can also be used to match a filter section to a terminating impedance which differs from the characteristic impedance of a π section.

15.13.2 *m*-derived Half Sections

Figure 15.48 shows and m-derived half section. The image impedance of the section as seen from terminals 1-2 and terminals 3-4 can be found from the open circuit and short-circuit impedances.

Fig. 15.48 *m-derived half section*

$$\mathbf{Z}_{12} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}}$$

$$= \sqrt{\frac{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)\left(m\dfrac{\mathbf{Z}_1}{2}\right)}{\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m} + \dfrac{m\mathbf{Z}_1}{2}}}$$

$$= \sqrt{\frac{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)^2\left(\dfrac{m\mathbf{Z}_1}{2}\right)}{\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m} + \dfrac{m\mathbf{Z}_1}{2}}}$$

$$= \sqrt{\frac{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m}\right)^2\left(\dfrac{m\mathbf{Z}_1}{2}\right)^2}{\left(\dfrac{1-m^2}{2m}\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2}{m} + \dfrac{m\mathbf{Z}_1}{2}\right)\left(\dfrac{m\mathbf{Z}_1}{2}\right)}}$$

$$= \sqrt{\frac{\left[\left(\dfrac{1-m^2}{4}\right)\mathbf{Z}_1^2 + \mathbf{Z}_1\,\mathbf{Z}_2\right]^2}{\dfrac{\mathbf{Z}_1^2}{4} - \dfrac{m^2\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\,\mathbf{Z}_2 + \dfrac{m^2\mathbf{Z}_1^2}{4}}}$$

$$= \frac{\mathbf{Z}_1\,\mathbf{Z}_2 + \dfrac{(1-m^2)}{4}\mathbf{Z}_1^2}{\sqrt{\mathbf{Z}_1\,\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1^2}{4}}}$$

$$= \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}_{0T}}\left[1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}(1-m)^2\right]$$

$$= \mathbf{Z}_{0\pi}\left[1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}(1-m)^2\right]$$

$$\mathbf{Z}_{34} = \sqrt{\mathbf{Z}_{oc}\,\mathbf{Z}_{sc}}$$

$$= \sqrt{\left[\frac{m\mathbf{Z}_1}{2}+\left(\frac{1-m^2}{2m}\right)\mathbf{Z}_1+\frac{2\mathbf{Z}_2}{m}\right]\left(\frac{m\mathbf{Z}_1}{2}\right)}$$

$$= \sqrt{\frac{m^2\mathbf{Z}_1^2}{4}+\frac{\mathbf{Z}_1^2}{4}-\frac{m^2\mathbf{Z}_1^2}{4}+\mathbf{Z}_1\,\mathbf{Z}_2}$$

$$= \sqrt{\mathbf{Z}_1\,\mathbf{Z}_2\left(1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}$$

$$= \mathbf{Z}_{0T}$$

Hence, the terminals 3-4 of the m-derived half section can be used to match the impedance of a constant-k T-section filter.

Example 15.18 *Find the values of shunt and series element of each half section of a constant-k T-section high-pass filter. The termination is 600 Ω and the filter is cut off below 20000 Hz.*

Solution $k = 600\ \Omega, \quad f_c = 20000$ Hz

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 20000} = 2.39 \text{ mH}$$

$$C = \frac{1}{4\pi f_c\, k} = \frac{1}{4\pi \times 20000 \times 600} = 0.0066\ \mu\text{F}$$

The half section consists of a capacitor of $2C$, i.e. 0.0133 μF in the series branch and an inductor of $2L$, i.e. 4.78 mH in the shunt branch as shown in Fig. 15.49.

Fig. 15.49

Example 15.19 *A prototype high-pass filter with $L = 4.77$ mH and $C = 0.0133\ \mu F$ has a cut-off frequency of 10 kHz and design impedance of 600 Ω. Design a suitable terminating half section with $m = 0.6$ such as to operate with same cut-off frequency.*

Solution $L = 4.77$ mH, $C = 0.0133$ μF, $f_c = 10$ kHz, $k = 600\ \Omega$

For m-derived T-section high-pass filter

$$\frac{L}{m} = \frac{4.77\times10^{-3}}{0.6} = 7.95 \text{ mH}$$

$$\frac{2C}{m} = \frac{2\times0.0133\times10^{-6}}{0.6} = 0.044\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)C = \frac{4\times0.6}{1-(0.6)^2}\times0.0133\times10^{-6} = 0.05\ \mu\text{F}$$

Fig. 15.50

The m-derived T-half section is shown in Fig. 15.50.
For m-derived π-section high-pass filter,

$$\frac{2L}{m} = \frac{2\times 4.77\times 10^{-3}}{0.6} = 15.9 \text{ mH}$$

$$\frac{C}{m} = \frac{0.0133\times 10^{-6}}{0.6} = 0.022\ \mu\text{F}$$

$$\left(\frac{4m}{1-m^2}\right)L = \frac{4\times 0.6}{1-(0.6)^2}\times 4.77\times 10^{-3} = 17.89 \text{ mH}$$

Fig. 15.51

The m-derived π-half section is shown in Fig. 15.51.

15.14 COMPOSITE FILTER

A constant-k filter does not give a sharp cut-off. An m-derived filter section has a sharp cut off, but its attenuation decreases for frequencies beyond f_∞. Moreover, two terminating half sections, one at each end, are needed for impedance matching. A filter composed of all the above sections is known as a *composite filter*. Figure 15.52 shows a composite low-pass filter.

Fig. 15.52 *Composite low-pass filter*

A composite filter consists of the following components:

(i) One or more constant-k sections to produce cut-off frequency, i.e. transition from pass band to the stop band at a specified frequency f_c.
(ii) One or more m-derived sections to give infinite attenuation at a frequency f_∞ near to f_c.
(iii) Two terminating m-derived half sections with $m = 0.6$ to provide matching with the source and the load.

Figure 15.53 shows the attenuation versus frequency characteristics of a constant-k low-pass filter, m-derived low-pass filter and a composite filter.

Fig. 15.53 *Attenuation characteristics*

For a composite filter, the attenuation rises very rapidly with frequency in the range f_c to f_∞ and falls only marginally with frequency.

Example 15.20 *Design a composite low-pass filter to have a cut-off frequency of 1000 Hz and a characteristic impedance of 600 Ω. Use one constant-k T section, one m-derived T section and two terminating half sections with m = 0.6. The frequency of infinite attenuation is 1050 Hz.*

Solution $f_c = 1000$ Hz, $k = 600\ \Omega$, $f_\infty = 1050$ Hz

For constant-*k T* section,

$$L = \frac{k}{\pi f_c} = \frac{600}{\pi \times 1000} = 0.19 \text{ H}$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi \times 1000 \times 600} = 0.53\ \mu\text{F}$$

Hence, the constant-*k T* section filter consists of an inductor of $\frac{L_1}{2}$, i.e. 0.095 H in series arm and a capacitor C_1, i.e. 0.53 μF in the shunt arm.
For the *m*-derived full *T* section,

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1 - \left(\frac{1000}{1050}\right)^2} = 0.305$$

$$\frac{mL}{2} = \frac{0.305 \times 0.19}{2} = 0.029 \text{ H}$$

$$mC = 0.305 \times 0.53 \times 10^{-6} = 0.162\ \mu\text{F}$$

$$\left(\frac{1 - m^2}{4m}\right) L = \frac{1 - (0.305)^2}{4 \times 0.305} \times 0.19 = 0.14 \text{ H}$$

For each terminating half section with $m = 0.6$,

$$\frac{mL}{2} = \frac{0.6 \times 0.19}{2} = 0.057 \text{ H}$$

$$\frac{mC}{2} = \frac{0.6 \times 0.53 \times 10^{-6}}{2} = 0.159\ \mu\text{F}$$

$$\left(\frac{1 - m^2}{2m}\right) L = \frac{1 - (0.6)^2}{2 \times 0.6} \times 0.19 = 0.101 \text{ H}$$

The composite low-pass filter is shown in Fig. 15.54.

Fig. 15.54

Example 15.21 *Design a composite high-pass filter having a characteristic impedance of 600 Ω and a cut-off frequency of 1000 Hz. Use one a constant-k T section, one m-derived T section with m = 0.2 and two m-derived terminating half sections with m = 0.6.*

Solution $k = 600\ \Omega, \quad f_c = 1000\ \text{Hz}$

For a constant-k T section,

$$L = \frac{k}{4\pi f_c} = \frac{600}{4\pi \times 1000} = 0.0477\ \text{H}$$

$$C = \frac{1}{4\pi f_c k} = \frac{1}{4\pi \times 1000 \times 600} = 0.133\ \mu\text{F}$$

Hence, the constant-k T section filter consists of an inductor of 0.0477 H in the shunt branch and a capacitor of $2C$, i.e. 0.266 μF in each series branch.

For an m-derived T section with $m = 0.2$,

$$\frac{2C}{m} = \frac{2 \times 0.133 \times 10^{-6}}{0.2} = 1.33\ \mu\text{F}$$

$$\frac{L}{m} = \frac{0.0477}{0.2} = 0.239\ \text{H}$$

$$\left(\frac{4m}{1-m^2}\right)C = \frac{4 \times 0.2}{1-(0.2)^2} \times 0.133 \times 10^{-6} = 0.11\ \mu\text{F}$$

For each terminating half section with $m = 0.6$,

$$\frac{2C}{m} = \frac{2 \times 0.133 \times 10^{-6}}{0.6} = 0.44\ \mu\text{F}$$

$$\frac{2L}{m} = \frac{2 \times 0.0477}{0.6} = 0.159\ \text{H}$$

$$\left(\frac{2m}{1-m^2}\right)C = \frac{2 \times 0.6}{1-(0.6)^2} \times 0.133 \times 10^{-6} = 0.25\ \mu\text{F}$$

The composite high-pass filter is shown in Fig. 15.55.

Fig. 15.55

15.15 ATTENUATOR

An attenuator is a two-port resistive network. It is used to reduce the signal level when used between a generator and load. Attenuators may be symmetrical or asymmetrical. They may provide fixed or variable attenuation. A fixed attenuator is also called a pad. The attenuation is expressed in decibels (dB).

$$\text{Attenuation in } dB = 10 \log_{10} \frac{P_1}{P_2}$$

where P_1 is the input power and P_2 is the output power.
For a properly matched network,

$$P_1 = I_1^2 R_0 = \frac{V_1^2}{R_0}$$

$$P_2 = I_2^2 R_0 = \frac{V_2^2}{R_0}$$

where R_0 is the characteristic resistance of the network.

Hence, $$\text{attenuation in } dB = 20 \log_{10} \frac{V_1}{V_2} = 20 \log_{10} \frac{I_1}{I_2}$$

Attenuation is also expressed in nepers as the natural logarithm of the voltage or current ratio.

$$\text{Attenuation in nepers } = \ln \frac{V_1}{V_2} = \ln \frac{I_1}{I_2} = \frac{1}{2} \ln \frac{P_1}{P_2}$$

If $$\frac{V_1}{V_2} = \frac{I_1}{I_2} = N \text{ then } \frac{P_1}{P_2} = N^2$$

$$\text{Attenuation in } dB = 20 \log_{10} N$$

$$N = \text{antilog}\left(\frac{dB}{20}\right)$$

where N is the attenuation in nepers.

Relation between Decibel and Neper

$$\text{Attenuation in nepers} = \ln\frac{V_1}{v_2}$$

$$\text{Attenuation in } dB = 20\log_{10}\frac{V_1}{V_2}$$

Changing the base of the algorithm,

$$\text{attenuation in } dB = 20\frac{\ln\frac{V_1}{V_2}}{\ln 10} = \frac{20}{2.303}\ln\frac{V_1}{V_2} = 8.686 \text{ (attenuation in neper)}$$

$$\text{Attenuation in nepers} = \frac{1}{8.686}\text{ (attenuation in } dB) = 0.115 \text{ (attenuation in } dB)$$

15.16 LATTICE ATTENUATOR

A lattice is a general type of symmetrical and balanced network. Any symmetrical balanced or unbalanced network can be transformed into an equivalent lattice network. Figure 15.56 shows a lattice network terminated through a characteristic resistance R_0.

The elements of a lattice attenuator can be specified in terms of characteristic impedance and propagation constant.

Fig. 15.56 *Lattice attenuator*

We know that $\quad Z_0 = \sqrt{Z_{oc}\, Z_{sc}}$

$$Z_{oc} = \frac{R_A + R_B}{2}$$

$$Z_{sc} = \frac{2R_A\, R_B}{R_A + R_B}$$

$$Z_0 = R_0 = \sqrt{Z_{oc}\, Z_{sc}} = \sqrt{\left(\frac{R_A + R_B}{2}\right)\left(\frac{2R_A\, R_B}{R_A + R_B}\right)} = \sqrt{R_A\, R_B}$$

Redrawing the network as shown in Fig. 15.57,

Applying KVL to the loop,

$$V_1 = R_0\, I_1 = R_A(I_1 - I) + R_0\, I_2 + R_A(I + I_2)$$

$$R_0\, I_1 = R_A(I_1 + I_2) + R_0\, I_2$$

$$(R_0 - R_A)I_1 = (R_A + R_0)I_2$$

$$\frac{I_1}{I_2} = \frac{R_A + R_0}{R_0 - R_A} = \frac{1 + \frac{R_A}{R_0}}{1 - \frac{R_A}{R_0}}$$

$$N = e^{\alpha} = \frac{I_1}{I_2} = \frac{1 + \frac{R_A}{R_0}}{1 - \frac{R_A}{R_0}} = \frac{1 + \sqrt{\frac{R_A}{R_0}}}{1 - \sqrt{\frac{R_A}{R_0}}}$$

Fig. 15.57 *Modified network*

$$\alpha = \log\left(\frac{1+\sqrt{\frac{R_A}{R_0}}}{1-\sqrt{\frac{R_A}{R_0}}}\right)$$

We can express R_A and R_B in terms of attenuation.

$$N\left(1-\frac{R_A}{R_0}\right) = 1+\frac{R_A}{R_0}$$

$$R_A = R_0\left(\frac{N-1}{N+1}\right)$$

Similarly, $R_B = R_0\left(\frac{N+1}{N-1}\right)$

These equations can be used to find the equivalent lattice network for T and π networks using the bisection theorem.

Bisection Theorem

A network is said to have been bisected when the open-circuited and short-circuited input impedances of the two bisected networks are equivalent. Also, the square root of the product of these impedances is the characteristic impedance of the original whole network.

The bisection theorem states that '*any symmetrical balanced or unbalanced network can be transformed into an equivalent lattice network. The series arm of the lattice is equal to the short-circuited impedance of the bisected network and the diagonal arm is equal to the open-circuited impedance of the bisected network'.*

Example 15.22 *Design a lattice attenuator if the characteristic resistance is 200 Ω and the attenuation is 20 dB.*

Solution $N(dB) = 20\,dB, \quad R_0 = 200\ \Omega$

$$N(dB) = 20\log_{10} N$$

$$20 = 20\log_{10} N$$

$$N = 10$$

$$R_A = R_0\left(\frac{N-1}{N+1}\right) = 200\left(\frac{10-1}{10+1}\right) = 163.6\ \Omega$$

$$R_B = R_0\left(\frac{N+1}{N-1}\right) = 200\left(\frac{10+1}{10-1}\right) = 244.5\ \Omega$$

Fig. 15.58

The lattice attenuator is shown in Fig. 15.58.

15.17 T-TYPE ATTENUATOR

A T-type attenuator is a symmetrical but unbalanced network and is shown in Fig. 15.59 (a). Its bisected network is shown in Fig. 15.59 (b).

Fig. 15.59 *(a) T-type attenuator (b) bisection of T-type attenuator*

From the bisection theorem, the series arm of the lattice network is equal to the short-circuited input resistance of the bisected network and the diagonal arm is equal to the open-circuited input resistance of the bisected network.

$$R_A = \frac{R_1}{2}$$

$$R_B = \frac{R_1}{2} + 2R_2$$

From lattice network,

$$R_A = R_0\left(\frac{N-1}{N+1}\right)$$

and

$$R_B = R_0\left(\frac{N+1}{N-1}\right)$$

Solving these equations,

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right)$$

$$2R_2 = R_0\left(\frac{N+1}{N-1} - \frac{N-1}{N+1}\right)$$

$$R_2 = \frac{R_0}{2}\left(\frac{4N}{N^2-1}\right) = \frac{2N\,R_0}{N^2-1}$$

Example 15.23 *If the ratio of R_1 and R_2 of a symmetrical T network is $\frac{1}{4}$, find the ratio of the input current to the output current. Also, calculate the attenuation in dB.*

Solution

$$\frac{R_1}{R_2} = \frac{1}{4}$$

For a *T*-type attenuator,

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right) \quad \text{(i)}$$

$$R_2 = R_0\left(\frac{2N}{N^2-1}\right) \quad \text{(ii)}$$

Dividing Eq. (ii) by Eq. (i),

$$\frac{\dfrac{2N}{N^2-1}}{2\left(\dfrac{N-1}{N+1}\right)} = \frac{1}{4}$$

$$\frac{2N}{N^2-1} = \frac{1}{2}\left(\frac{N-1}{N+1}\right)$$

$$\frac{2N}{(N+1)(N-1)} = \frac{1}{2}\left(\frac{N-1}{N+1}\right)$$

$$4N = (N-1)^2$$

$$N^2 - 6N + 1 = 0$$

$$N = 5.83 \quad \text{or} \quad N = 0.172$$

$$N \text{ in } dB = 20\log_{10} 5.83 = 15.31\ dB$$

or

$$N \text{ in } dB = 20\log_{10} 0.172 = -15.29\ dB$$

Since attenuation cannot be negative,

$$N = 5.83 \text{ nepers} = 15.31\ dB$$

$$\frac{I_1}{I_2} = N = 5.83$$

Example 15.24 *An attenuator is composed of symmetrical T-section having a series arm of 175 Ω and a shunt arm of 350 Ω. Find the characteristic impedance and attenuation in dB.*

Solution

$$\frac{R_1}{2} = 175\ \Omega, \quad R_1 = 350\ \Omega, \quad R_2 = 350\ \Omega$$

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right) = 350 \qquad \ldots\text{(i)}$$

$$R_2 = R_0\left(\frac{2N}{N^2-1}\right) = 350 \qquad \ldots\text{(ii)}$$

Dividing Eq. (ii) by Eq. (i),

$$\frac{\dfrac{2N}{N^2-1}}{2\left(\dfrac{N-1}{N+1}\right)} = 1$$

$$\frac{2N}{N^2-1} = 2\left(\frac{N-1}{N+1}\right)$$

$$\frac{N}{(N+1)(N-1)} = \frac{N-1}{N+1}$$

$$N = (N-1)^2$$

$$N^2 - 3N + 1 = 0$$

$$N = 2.618 \quad \text{or} \quad N = 0.3819$$

$$N \text{ in } dB = 20\log_{10} 2.618 = 8.36\ dB$$

$$\text{or } N \text{ in } dB = 20\log_{10} 0.3819 = -8.36\ dB$$

Since attenuation cannot be negative,

$$N = 2.618 \text{ nepers} = 8.36\ dB$$

Putting $N = 2.618$ in Eq. (i),

$$350 = 2R_0\left(\frac{2.618-1}{2.618+1}\right)$$

$$R_0 = 391.32\ \Omega$$

Example 15.25 *Design a T-type attenuator to give an attenuation of 60 dB and characteristic resistance of 500 Ω.*

Solution

$$N(dB) = 60\,dB, \quad R_0 = 500\ \Omega$$

$$N(dB) = 20\log_{10} N$$

$$60 = 20\log_{10} N$$

$$N = 1000$$

$$R_1 = 2R_0\left(\frac{N-1}{N+1}\right) = 2\times 500\left(\frac{1000-1}{1000+1}\right) = 998\ \Omega$$

$$R_2 = R_0\left(\frac{2N}{N^2-1}\right) = 500\left[\frac{2\times 1000}{(1000)^2-1}\right] = 1\ \Omega$$

Fig. 15.60

The *T*-type attenuator is shown in Fig. 15.60.

15.18 π-TYPE ATTENUATOR

A symmetrical π-type attenuator is shown in Fig. 15.61 (a). Its bisected network is shown in Fig. 15.61 (b).

Fig. 15.61 *(a) π-attenuator (b) Bisection of π-type attenuator*

From the bisection theorem, the series arm of the lattice network is equal to the short-circuited input resistance of the bisected network and the diagonal arm is equal to the open-circuit input resistance of the bisected network.

$$R_A = \frac{R_1\ R_2}{\frac{R_1}{2} + 2R_2}$$

$$R_B = 2R_2$$

From lattice network,

$$R_A = R_0\left(\frac{N-1}{N+1}\right)$$

$$R_B = R_0\left(\frac{N+1}{N-1}\right)$$

Solving these equations,

$$R_1 = \frac{2R_A R_B}{R_B - R_A} = R_0\left(\frac{N^2-1}{2N}\right)$$

$$R_2 = \frac{R_B}{2} = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right)$$

Example 15.26 *An attenuator is composed of a symmetrical π-section having series arm of 275 Ω and each shunt arm of 450 Ω. Find the characteristic impedance and attenuation in dB.*

Solution

$$R_1 = 275\ \Omega, \quad 2R_2 = 450\ \Omega, \quad R_2 = 225\ \Omega$$

$$R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 275 \qquad \ldots\text{(i)}$$

$$R_2 = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right) = 225 \qquad \ldots\text{(ii)}$$

Dividing Eq. (ii) by Eq. (i),

$$\frac{\dfrac{N^2-1}{2N}}{\dfrac{1}{2}\left(\dfrac{N+1}{N-1}\right)} = \frac{275}{225}$$

$$\frac{(N+1)(N-1)}{2N} = 0.611\left(\frac{N+1}{N-1}\right)$$

$$(N-1)^2 = 2N(0.611)$$

$$N^2 - 3.222N + 1 = 0$$

$$N = 2.872 \quad \text{or} \quad N = 0.348$$

$$N \text{ in } dB = 20\log_{10} 2.872 = 9.16\ dB$$

or

$$N \text{ in } dB = 20\log_{10} 0.348 = -9.16\ dB$$

Since attenuation cannot be negative,

$$N = 2.872 \text{ neper} = 9.16\ dB$$

Putting $N = 2.872$ in Eq. (i),

$$275 = R_0\left[\frac{(2.872)^2-1}{2(2.872)}\right]$$

$$R_0 = 217.92\ \Omega$$

Example 15.27 *Design a π-type attenuator to give attenuation of 20 dB and characteristic resistance of 500 Ω.*

Solution

$$N(dB) = 20\,dB, \quad R_0 = 500\ \Omega$$

$$N(dB) = 20\log_{10} N$$

$$20 = 20\log_{10} N$$

$$N = 10$$

$$R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 500\left[\frac{(10)^2-1}{2\times 10}\right] = 2475\ \Omega$$

$$R_2 = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right) = \frac{500}{2}\left(\frac{10+1}{10-1}\right) = 305.5\ \Omega$$

Fig. 15.62

The π-type attenuator is shown in Fig. 15.62.

15.19 LADDER-TYPE ATTENUATOR

When a number of symmetrical T-or is π-type attenuators are connected in cascade to provide the attenuation in steps, the resultant network is termed **ladder-type attenuator.** Figure 15.63 shows a ladder-type attenuator with three identical and symmetrical π-sections connected in cascade.

Fig. 15.63 *Ladder-type attenuator*

The output terminals are terminated with resistance R_0 which is the characteristic resistance of the network. The source end also has resistance R_0 so that it is a matched network. The net attenuation is the sum of the attenuation of individual sections between the input and output terminals.

Example 15.28 *Design a ladder-type attenuator for a load resistance of 500 Ω and an attenuation of 3 dB per step.*

Solution

$$N(dB) = 3\,dB, \quad R_0 = 500\ \Omega$$

$$N(dB) = 20\log_{10} N$$

$$3 = 20\log_{10} N$$

$$N = 1.413$$

$$R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 500\left[\frac{(1.413)^2-1}{2\times 1.413}\right] = 176.32\ \Omega$$

$$R_2 = \frac{R_0}{2}\left(\frac{N+1}{N-1}\right) = \frac{500}{2}\left(\frac{1.413+1}{1.413-1}\right) = 1460.65\ \Omega$$

Fig. 15.64

EXERCISES

15.1 A T-section low-pass filter has a series inductance of 80 mH and a shunt capacitance of 0.022 μF. Find the cut-off frequency and nominal impedance. Also design the equivalent π-section.

$$\left[\begin{array}{r} f_c = 7.587 \text{ kHz}, \quad k = 1.907 \text{ k}\Omega, L = 80 \text{ mH}, \\ \frac{C}{2} = 0.011 \ \mu\text{F} \end{array}\right]$$

15.2 Calculate the frequency of a constant-k, T-section low-pass filter having a cut-off frequency of 1000 Hz at which it has an attenuation of 10 dB.

$$[f = 1170 \text{ Hz}]$$

15.3 Design a low-pass constant-k (a) T-section, and (b) π-section filter with $f_c = 6$ kHz and $R_O = 500\ \Omega$. Calculate α and β for the filters for $f = 10$ kHz. Also determine the frequency at which the attenuation is 10 dB.

$$\left[\begin{array}{r} \alpha = 2.2 \text{ neper}, \quad \beta = 1.46 \text{ radians}, \\ \beta = \pi \text{ at } 10 \text{ kHz} T \rightarrow \frac{L}{2} = 13.25 \text{ mH}, \\ C = 0.106 \ \mu\text{F} \pi \rightarrow L = 26.5 \text{ mH}, \\ \frac{C}{2} = 0.053 \ \mu\text{F} \\ f = 7 \text{ kHz} \end{array}\right]$$

15.4 Design a π-section constant-k high-pass filter having a cut-off frequency $f_c = 8$ kHz and nominal characteristic impedance $k = 600\ \Omega$. Find (a) its characteristic impedance at a frequency of 12 kHz, (b) phase constant at frequency of 12 kHz, (c) attenuation at frequency of 800 Hz.

$$\left[\begin{array}{r} L = 5.971 \text{ mH}, \quad C = 0.016 \ \mu\text{F}, \\ Z_{O\pi} = 805\ \Omega, \quad \beta = 83.6°, \quad \alpha = 5 \text{ nepers} \end{array}\right]$$

15.5 Design a band-pass constant-k filter with $f_1 = 2$ kHz and $f_2 = 3$ kHz and $k = 500\ \Omega$.

$$\left[\begin{array}{r} L_1 = 159 \text{ mH}, \quad C_1 = 0.026 \ \mu\text{F}, \\ L_2 = 20.8 \text{ mH}, \quad C_2 = 0.637 \ \mu\text{F} \end{array}\right]$$

15.6 Design a band-stop constant-k filter with cut-off frequencies of 8 kHz and 12 kHz and nominal characteristic impedance of 500 Ω.

$$\left[\begin{array}{r} L_1 = 6.6 \text{ mH}, \quad C_1 = 0.0398 \ \mu\text{F}, \\ L_2 = 9.9 \text{ mH}, \quad C_2 = 0.0265 \ \mu\text{F} \end{array}\right]$$

15.7 Design an m-derived T-section low-pass filter with cut-off frequency $f_c = 1$ kHz, design impedance of 600 Ω and frequency of infinite attenuation of 1050 Hz.

$$[L = 191.08 \text{ mH}, \quad C = 0.53 \ \mu\text{F}, \quad m = 0.3]$$

15.8 Design an m-derived high-pass T-section filter having a design impedance of 600 Ω, cut-off frequency of 5 kHz and $m = 0.35$. Also determine the frequency of infinite attenuation.

$$\left[\begin{array}{r} \frac{2C}{m} = 0.148 \ \mu\text{F}, \quad \frac{L}{m} = 27.28 \text{ mH}, \\ \frac{4m}{1-m^2} C = 0.0414 \ \mu\text{F}, \quad f_\infty = 4.684 \text{ kHz} \end{array}\right]$$

15.9 In a constant-k band-pass filter, the ratio of capacitance in the shunt and series arms is 50:1 and the resonant frequency of both arms is 1000 Hz. Find the bandwidth of the filter.

[282.84 Hz]

15.10 Design a symmetrical lattice attenuator to have characteristic impedance of 800 Ω and attenuation of 20 dB.

[654.55 Ω, 977.78 Ω]

15.11 Design a T-type attenuator to have a characteristic resistance of 200 Ω and attenuation of 20 dB.

[163.6 Ω, 40.4 Ω]

15.12 Design a π-type attenuator to give 20 dB attenuation and characteristic impedance of 100 Ω.

[495 Ω, 122.22 Ω]

Objective–Type Questions

15.1 A 500 Ω is resistance attenuator is to have an attenuation of 20 db. The resistances of the series and shunt arms of the lattice network, respectively are

(a) 400 Ω, 500 Ω approxi
(b) 410 Ω, 510 Ω approx
(c) 410 Ω, 610 Ω approx
(d) none of the above

15.2 2. For the design of low-pass prototype filter Fig. 15.65 of with load resistance $R_L = 1\ \Omega$ and angular frequency $\omega = 1$ rad/s the values of L and C would be

Fig. 15.65

(a) 1 H and I F
(b) 1 H and 2 F
(c) 2 H and I F
(d) 2 H and 2 F

15.3 The passband of a typical filter network with Z_1 and Z_2 as the series and shunt-arm impedances is characerised by

(a) $-1 < \frac{Z_1}{4Z_2} < 0$ (b) $-1 < \frac{Z_1}{4Z_2} < 1$

(c) $0 < \frac{Z_1}{4Z_2} < 1$ (d) none of the above

15.4 In an m-derived low-pass filter, the value of m is

(a) $\sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}$ (b) $\sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2}$

(c) $\sqrt{1+\left(\frac{f_\infty}{f_c}\right)^2}$ (d) $\sqrt{1+\left(\frac{f_c}{f_\infty}\right)^2}$

15.5 For the lattice-type attenuater shown in Fig. 15.66, the characteristic impedance R_0 is

Fig. 15.66

(a) $\frac{R_1 + R_2}{2}$ (b) $\frac{2R_1R_2}{R_1 + R_2}$

(c) $\sqrt{R_1R_2}$ (d) $R_1 + \frac{R_2}{0}$

15.6 A T-type attenuator is designed for an attenuation of 40 dB and terminating resistance of 75 ohms. Which of the following values represent full series arm R_1 and shunt arm R_2?

(a) $R_1 \simeq 147\ \Omega$ (b) $R_1 \simeq 153\ \Omega$

(c) $R \simeq 1.5\ \Omega$ (d) $R_2 \simeq 3750\ \Omega$

Select the correct answer from the codes given below:

(a) 1 and 3 (b) 1 and 4
(c) 2 and 3 (d) 2 and 4

15.7 The application of the bisection theorem finds out an equivalent lattice network if the original network is

(a) symmetrical and balanced only
(b) unsymmetrical and balanced only
(c) symmetrical, balanced and unbalanced only
(d) symmetrical and unbalanced only

15.8 The attenuation of an attenuator is 20 dB. Its attenuation constant N is

(a) 1 (b) 2
(c) 10 (d) 20

15.9 Z_1 and Z_2 are the total series and shunt impedances of a T or π-filter. Consider the following zones of operation of filters and the conditions on the impedances

(a) Pass band 1. $\frac{Z_1}{4Z_2} < -1$

(b) Stop band 2. $\frac{Z_1}{4Z_2} = -1$

(c) Transition band 3. $-1 < \frac{Z_1}{4Z_2} < 0$

Tick the correct combination:

	A	B	C
(a)	1	2	3
(b)	3	1	2
(c)	3	2	1
(d)	2	3	1

15.10 Tick out the correct statement in case of a filter.

(a) Characteristic impedance is resistive in stop band

(b) Characteristic impedance is reactive in pass band

(c) Characteristic impedance is resistive in pass band

(d) None of the above

15.11 If L is the total series inductance and C the total shunt capacitacne of a T or π-type low-page filter, the pass band frequency range of the filter is given as

(a) 0 to $\frac{1}{2\pi\sqrt{LC}}$ Hz

(b) 0 to $\frac{1}{\pi\sqrt{LC}}$ Hz

(c) 0 to $\frac{2}{\pi\sqrt{LC}}$ Hz

(d) 0 to $\frac{1}{4\pi\sqrt{LC}}$ Hz

15.12 If C is the total series capatance, L is the total shunt inductance of a T or π-type high-pass filter, the frequency range for the stop band of the filter is

(a) 0 to $\frac{1}{2\pi\sqrt{LC}}$ Hz

(b) 0 to $\frac{1}{\pi\sqrt{LC}}$ Hz

(c) 0 to $\frac{2}{\pi\sqrt{LC}}$ Hz

(d) 0 to $\frac{2}{4\pi\sqrt{LC}}$ Hz

15.13 If f_1 and f_2 are the lower and uppper cut-off frequencies of the band pass filter, the series impedance Z_1 is

(a) capacitive at f_1

(b) inductive at f_1

(c) resistive at f_2

(d) none of the above

15.14 The phase constant β of a filter during stop band is

(a) Zero radian (b) 2

(c) π (d) 2π

Answers to Objective-Type Questions

15.1 (c)	15.2 (d)	15.3 (a)	15.4 (b)	15.5 (c)	15.6 (a)	15.7 (c)
15.8 (c)	15.9 (b)	15.10 (c)	15.11 (b)	15.12 (d)	15.13 (a)	15.14 (c)

16

Network Synthesis

16.1 INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

16.2 HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

   are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.
2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

Example 16.1 *State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.*

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Example 16.2 *Test whether the polynomial* $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^3 + 3s)s^4 + 5s^2 + 4\ \ (s \\
\qquad\quad \underline{s^4 + 3s^2} \\
\qquad\qquad 2s^2 + 4\Big)\, s^3 + 3s\left(\frac{1}{2}s\right. \\
\qquad\qquad\qquad\quad \underline{s^3 + 2s} \\
\qquad\qquad\qquad\qquad s)\ 2s^2 + 4\ (2s \\
\qquad\qquad\qquad\qquad\quad \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad 4\Big)\, s\left(\frac{1}{4}s\right. \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 16.3 *Test whether the polynomial* $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)\ n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2 + 2\,\Big)\,s^3 + 5s\,\Big(\,\frac{1}{4}s$$

$$\underline{s^3 + \frac{2}{4}s}$$

$$\frac{9}{2}s\,\Big)\,4s^2 + 2\,\Big(\,\frac{8}{9}s$$

$$\underline{4s^2}$$

$$2\,\Big)\,\frac{9}{2}s\,\Big(\,\frac{9}{4}s$$

$$\underline{\frac{9}{2}s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 16.4 *Test whether the polynomial $P(s) = s^4 + s^3 + 3s^2 + 2s + 12$ is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 3s^2 + 12$

Odd part of $P(s) = n(s) = s^3 + 2s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 2s\,\Big)\,s^4 + 3s^2 + 12\,\Big(\,s$$

$$\underline{s^4 + 2s^2}$$

$$s^2 + 12\,\Big)\,s^3 + \;2s\,\Big(\,s$$

$$\underline{s^3 + 12s}$$

$$-10s\,\Big)\,s^2 + 12\,\Big(\,-\frac{1}{10}s$$

$$\underline{s^2}$$

$$12\,\Big)\,-10s\,\Big(\,-\frac{10}{12}s$$

$$\underline{-10s}$$

$$0$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 16.5 *Prove that polynomial* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is not Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 2s^2 + 2$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$\begin{array}{l}
s^3 + 3s)s^4 + 2s^2 + 2 \ \ (s \\
\qquad\quad \underline{s^4 + 3s^2} \\
\qquad\qquad -s^2 + 2)s^3 + 3s(-s \\
\qquad\qquad\qquad \underline{s^3 - 2s} \\
\qquad\qquad\qquad\qquad 5s\Big) -s^2 + 2\left(-\frac{1}{5}s\right. \\
\qquad\qquad\qquad\qquad\qquad \underline{-s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2\Big) 5s\left(\frac{5}{2}s\right. \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{5s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 16.6 *Prove that polynomial* $P(s) = 2s^4 + 5s^3 + 6s^2 + 3s + 1$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^4 + 6s^2 + 1$

Odd part of $P(s) = n(s) = 5s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$\begin{array}{l}
5s^3 + 3s\Big) 2s^4 + 6s^2 + 1 \ \left(\frac{2}{5}s\right. \\
\qquad\quad \underline{2s^4 + \frac{6}{5}s^2} \\
\qquad\qquad \frac{24}{5}s^2 + 1\Big) 5s^3 + 3s\left(\frac{25}{24}s\right. \\
\qquad\qquad\qquad\qquad \underline{5s^3 + \frac{25}{24}s}
\end{array}$$

$$\frac{47}{24}s\,\Big)\,\frac{24}{5}s^2+1\,\Big(\,\frac{576}{235}s$$
$$\underline{\frac{24}{5}s^2}$$
$$1\,\Big)\,\frac{47}{24}s\,\Big(\,\frac{24}{47}s$$
$$\underline{\frac{47}{24}s}$$
$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.7 *Test whether the polynomial* $P(s)=s^4+7s^3+6s^2+21s+8$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+6s^2+8$

Odd part of $P(s)=n(s)=7s^3+21s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$7s^3+21s\,\Big)\,s^4+6s^2+8\,\Big(\,\frac{1}{7}s$$
$$\underline{s^4+3s^2}$$
$$3s^2+8\,\Big)\,7s^3+21s\,\Big(\,\frac{7}{3}s$$
$$\underline{7s^3+\frac{56}{3}s}$$
$$\frac{7}{3}s\,\Big)\,3s^2+8\,\Big(\,\frac{9}{7}s$$
$$\underline{3s^2}$$
$$8\,\Big)\,\frac{7}{3}s\,\Big(\,\frac{7}{24}s$$
$$\underline{\frac{7}{3}s}$$
$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.8 *Check whether* $P(s)=s^4+5s^3+5s^2+4s+10$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+5s^2+10$

Odd part of $P(s)=n(s)=5s^3+4s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
5s^3 + 4s \,\big)\, s^4 + 5s^2 + 10 \,\Big(\, \frac{1}{5}s \\
\qquad\qquad \underline{s^4 + \frac{4}{5}s^2} \\
\qquad\qquad\quad \frac{21}{5}s^2 + 10 \,\big)\, 5s^3 + 4s \,\Big(\, \frac{25}{21}s \\
\qquad\qquad\qquad\qquad \underline{5s^3 + \frac{250}{21}s} \\
\qquad\qquad\qquad\qquad\quad -\frac{166}{21}s \,\big)\, \frac{21}{5}s^2 + 10 \,\Big(\, -\frac{441}{830}s \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\frac{21}{5}s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 10 \,\big)\, -\frac{166}{21}s \,\Big(\, -\frac{166}{210}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-\frac{166}{21}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since the last two quotient terms are negative, the polynomial $P(s)$ is not Hurwitz.

Example 16.9 *Test whether the polynomial $s^5 + 3s^3 + 2s$ is Hurwitz.*

Solution Since the given polynomial contains odd functions only, it is not possible to perform a continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 9s^2 + 2$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
5s^4 + 9s^2 + 2 \,\big)\, s^5 + 3s^3 + 2s \,\Big(\, \frac{1}{5}s \\
\qquad\qquad \underline{s^5 + \frac{9}{5}s^3 + \frac{2}{5}s} \\
\qquad\qquad\quad \frac{6}{5}s^3 + \frac{8}{5}s \,\big)\, 5s^4 + 9s^2 + 2 \,\Big(\, \frac{25}{6}s
\end{array}
$$

$$
\begin{array}{l}
\underline{5s^4 + \frac{20}{3}s^2} \\
\quad \frac{7}{3}s^2 + 2\,\big)\,\frac{6}{5}s^3 + \frac{8}{5}s\,\big(\,\frac{18}{35}s \\
\qquad \underline{\frac{6}{5}s^3 + \frac{36}{35}s} \\
\qquad\quad \frac{20}{35}s\,\big)\,\frac{7}{3}s^2 + 2\,\big(\,\frac{49}{12}s \\
\qquad\qquad \underline{\frac{7}{3}s^2} \\
\qquad\qquad\quad 2\,\big)\,\frac{20}{35}s\,\big(\,\frac{10}{35}s \\
\qquad\qquad\qquad \underline{\frac{20}{35}s} \\
\qquad\qquad\qquad\quad 0
\end{array}
$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.10 *Test whether the polynomial P(s) is Hurwitz.*

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
5s^4 + 3s^2 + 1\,\big)\,s^5 + \ s^3 + s\,\big(\,\frac{1}{5}s \\
\qquad \underline{s^5 + \frac{3}{5}s^3 + \frac{1}{5}s} \\
\qquad\quad \frac{2}{5}s^3 + \frac{4}{5}s\,\big)\,5s^4 + \ 3s^2 + 1\,\big(\,\frac{25}{2}s \\
\qquad\qquad \underline{5s^4 + 10s^2} \\
\qquad\qquad\quad -7s^2 + 1\,\big)\,\frac{2}{5}s^3 + \ \frac{4}{5}s\,\big(\,-\frac{2}{35}s \\
\qquad\qquad\qquad \underline{\frac{2}{5}s^3 - \frac{2}{35}s}
\end{array}
$$

$$\frac{26}{35}s\Big)-7s^2+1\Big(-\frac{245}{26}s$$

$$\underline{-7s^2}$$

$$1\Big)\frac{26}{35}s\Big(\frac{26}{35}s$$

$$\underline{\frac{26}{35}s}$$

$$0$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

Example 16.11 *Test the polynomial P(s) of Hurwitz property.*

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4$$

Solution Even part of $P(s) = m(s) = s^6 + 8s^4 + 17s^2 + 4$

Odd part of $P(s) = n(s) = 3s^5 + 15s^3 + 12s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$3s^5+15s^3+12s\Big)s^6+8s^4+17s^2+4\Big(\frac{1}{3}s$$

$$\underline{s^6+5s^4+4s^2}$$

$$3s^4+13s^2+4\Big)3s^5+15s^3+12s\Big(s$$

$$\underline{3s^5+13s^3+4s}$$

$$2s^3+8s\Big)3s^4+13s^2+4\Big(\frac{3}{2}s$$

$$\underline{3s^4+12s^2}$$

$$s^2+4\Big)2s^3+8s\Big(2s$$

$$\underline{2s^3+8s}$$

$$0$$

The division has terminated abruptly (i.e., the number of partial quotients (that is four) is not equal to the order of polynomial (that is six) with common factor (s^2+4).

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4 = (s^2+4)(s^4 + 3s^3 + 4s^2 + 3s + 1)$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let
$$P_1(s) = s^2 + 4$$

Since it contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 2s$$
$$\frac{P_1(s)}{P_1'(s)} = \frac{s^2+4}{2s} = \frac{s^2}{2s} + \frac{4}{2s} = \frac{s}{2} + \frac{1}{\frac{s}{2}}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let
$$P_2(s) = s^4 + 3s^3 + 4s^2 + 3s + 1$$
$$m_2(s) = s^4 + 4s^2 + 1$$
$$n_2(s) = 3s^3 + 3s$$

By continued fraction expansion,

$$\begin{array}{l}
3s^3+3s\,\big)\,s^4+4s^2+1\,\big(\,\frac{1}{3}s \\
\quad\underline{s^4+\ s^2} \\
\qquad 3s^2+1)\ 3s^3+3s\ (s \\
\qquad\quad \underline{3s^3+s} \\
\qquad\qquad 2s\,\big)\,3s^2+1\,\big(\,\frac{3}{2}s \\
\qquad\qquad\quad \underline{3s^2} \\
\qquad\qquad\qquad 1)\ 2s\ (2s \\
\qquad\qquad\qquad\quad \underline{2s} \\
\qquad\qquad\qquad\qquad 0
\end{array}$$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.

Hence, $P(s) = (s^2+4)(s^4+3s^3+4s^2+3s+1)$ is Hurwitz.

Example 16.12 *Test whether the polynomial* $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + s^4 + 8s^2 + 4$

Odd part of $P(s) = n(s) = s^7 + 2s^5 + 4s^3 + 8s$

$$Q(s) = \frac{n(s)}{m(s)}$$

By continued fraction expansion,

$$2s^6 + s^4 + 8s^2 + 4 \,\big)\, s^7 + 2s^5 + 4s^3 + 8s \,\Big(\, \frac{1}{2}s$$

$$\underline{s^7 + \frac{1}{2}s^5 + 4s^3 + 2s}$$

$$\frac{3}{2}s^5 \qquad + 6s \,\big)\, 2s^6 + s^4 + 8s^2 + 4 \,\Big(\, \frac{4}{3}s$$

$$\underline{2s^6 \qquad + 8s^2}$$

$$s^4 \qquad + 4 \,\big)\, \frac{3}{2}s^5 + 6s \,\Big(\, \frac{3}{2}s$$

$$\underline{\frac{3}{2}s^5 + 6s}$$

$$0$$

Since the division has terminated abruptly it indicates a common factor $s^4 + 4$. The polynomial can be written as

$$P(s) = (s^4 + 4)(s^3 + 2s^2 + 2s + 1)$$

If both the factor are Hurwitz, $P(s)$ will be Hurwitz.
In the polynomial $(s^4 + 4)$, the terms s^3, s^2 and s are missing. Hence, it is not Hurwitz.
Therefore, $P(s)$ is not Hurwitz.

Example 16.13 *Test whether the polynomial $2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + 13s^4 + 56s^2 + 25$

Odd part of $P(s) = n(s) = s^5 + 6s^3 + 25s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^5 + 6s^3 + 25s \,)\, 2s^6 + 13s^4 + 56s^2 + 25 \,(\, 2s$$

$$\underline{2s^6 + 12s^4 + 50s^2}$$

$$s^4 + 6s^2 \quad + 25 \,)\, s^5 + 6s^3 + 25s \,(\, s$$

$$\underline{s^5 + 6s^3 + 25s}$$

$$0$$

The division has terminated abruptly.

$$P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25 = (s^4 + 6s^2 + 25)\,(2s^2 + s + 1)$$

Let $\qquad P_1(s) = s^4 + 6s^2 + 25$

Since $P_1(s)$ contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 4s^3 + 12s$$

By continued fraction expansion,

$$
\begin{array}{l}
4s^3 + 12s \,\big)\, s^4 + 6s^2 + 25 \,\Big(\, \dfrac{1}{4}s \\
\qquad\qquad \underline{s^4 + 3s^2} \\
\qquad\qquad 3s^2 + 25 \,\big)\, 4s^3 + \;\; 12s \,\Big(\, \dfrac{4}{3}s \\
\qquad\qquad\qquad\qquad \underline{4s^3 + \dfrac{100}{3}s} \\
\qquad\qquad\qquad\qquad -\dfrac{64}{3}s \,\big)\, 3s^2 + 25 \,\Big(\, -\dfrac{9}{64}s \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{3s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad 25 \,\big)\, -\dfrac{64}{3}s \,\Big(\, -\dfrac{64}{75}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-\dfrac{64}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since two of the quotient terms are negative, $P_1(s)$ is not Hurwitz.
We need not test the other factor $(2s^2 + s + 1)$ for being Hurwitz.
Hence, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $\quad P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

The Routh–Hurwitz array is given by,

$$
\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_n & b_{n-1} & b_{n-2} & \ldots \\
s^{n-3} & c_n & c_{n-1} & & \ldots \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
s^1 & . & & & \\
s^0 & . & & &
\end{array}
$$

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where
$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1}b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1}b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Example 16.14 *Test whether* $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 8 \\ s^3 & 7 & 21 & \\ s^2 & 3 & 8 & \\ s^1 & \frac{7}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.15 *Determine whether* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 2 \\ s^3 & 1 & 3 & \\ s^2 & -1 & 2 & \\ s^1 & 5 & 0 & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 16.16 *Test whether* $P(s) = s^5 + 2s^4 + 4s^3 + 6s^2 + 2s + 5$ *is Hurwitz.*

Solution The Routh array is given by,

s^5	1	4	2
s^4	2	6	5
s^3	1	−0.5	
s^2	7	5	
s^1	−1.21		
s^0	5		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 16.17 *Test whether the polynomial $P(s) = s^5 + s^3 + s$ is Hurwitz.*

Solution The given polynomial contains odd functions only.

$$P'(s) = 5s^4 + 3s^2 + 1$$

The Routh array is given by,

s^5	1	1	1
s^4	5	3	1
s^3	0.4	0.8	
s^2	−7	1	
s^1	0.86		
s^0	1		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 16.18 *Test whether the polynomial $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ is Hurwitz.*

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

s^8	1	5	2	3	1
s^7	8	30	8	6	0
s^6	1.25	1	2.25	1	
s^5	23.6	−6.4	−0.4	0	
s^4	1.33	2.27	1		
s^3	−46.6	−18.14	0		
s^2	1.75	1			
s^1	8.49				
s^0	1				

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 16.19 *Test whether $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.*

Solution The Routh array is given by,

s^5	1	45	44
s^4	12	60	48
s^3	40	40	
s^2	48	48	
s^1	0	0	
s^0			

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.

(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.

(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

s^5	1	45	44
s^4	12	60	48
s^3	40	40	
s^2	48	48	
s^1	96	0	
s^0	48		

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Example 16.20 *Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.*

Solution The Routh array is given by,

s^6	2	13	56	25
s^5	1	6	25	
s^4	1	6	25	
s^3	0	0	0	
s^2				
s^1				
s^0				

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

$$\begin{array}{c|cccc} s^6 & 2 & 13 & 56 & 25 \\ s^5 & 1 & 6 & 25 & \\ s^4 & 1 & 6 & 25 & \\ s^3 & 4 & 12 & & \\ s^2 & 3 & 25 & & \\ s^1 & -21.3 & & & \\ s^0 & 25 & & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 16.21 *Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & a & 3 \\ s^3 & 1 & 2 & \\ s^2 & a-2 & 3 & \\ s^1 & \dfrac{2a-7}{a-2} & & \\ s^0 & 3 & & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$
$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$
$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \dfrac{7}{2}$.

Example 16.22 *Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & 0 \\ s^0 & k & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e.,
$$\frac{6-k}{3} > 0$$
$$6-k > 0$$

i.e., $k < 6$ and $k > 0$
Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

16.3 POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.
(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,
$\text{Re}\, F(s) \geq 0 \qquad \text{for } \text{Re}(s) \geq 0$

16.3.1 Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\dfrac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

16.3.2 Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. $\text{Re}\, F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s-j\omega_0}+\frac{K_1^*}{s+j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s)=\frac{m_1(s)+n_1(s)}{m_2(s)+n_2(s)}=\frac{m_1+n_1}{m_2+n_2}$$

Multiplying $N(s)$ and $D(s)$ by $m_2 - n_2$,

$$F(s)=\frac{m_1+n_1}{m_2+n_2}\frac{m_2-n_2}{m_2-n_2}=\frac{m_1m_2-n_1n_2}{m_2^2-n_2^2}+\frac{m_2n_1-m_1n_2}{m_2^2-n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s)=\frac{m_1m_2-n_1n_2}{m_2^2-n_2^2}$$

$$\text{Od } F(s)=\frac{m_2n_1-m_1n_2}{m_2^2-n_2^2}$$

Now, substituting $s = j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s = j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\big|_{s=j\omega}=\text{Re } F(j\omega)$$

$$\text{Od } F(s)\big|_{s=j\omega}=j\,\text{Im } F(j\omega)$$

We have to test Re $F(j\omega) \geq 0$ for all ω.

The denominator of Re $F(j\omega)$ is always a positive quantity because

$$m_2^2-n_2^2\big|_{s=j\omega}\geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1m_2-n_1n_2\big|_{s=j\omega}=A(\omega^2)$$

should be positive and real for all $\omega \geq 0$.

Example 16.23 *Test whether* $F(s)=\dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s = -1$ and zero at $s = -3$ as shown in Fig. 16.1.

Thus, pole and zero are in the left half of the s-plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

Fig. 16.1

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (3)(1) - (s)(s)\,|_{s=j\omega} = 3 - s^2\,|_{s=j\omega} = 3 + \omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.24 *Test whether* $F(s) = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14} = \dfrac{(s+5)(s+1)}{(s+7)(s+2)}$

The function $F(s)$ has poles at $s = -7$ and $s = -2$ and zeros at $s = -5$ an $s = -1$ as shown in Fig. 16.2.

Thus, all the poles and zeros are in the left half of the s plane.

Fig. 16.2

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2 + 5)\,(s^2 + 14) - (6s)(9s)\,|_{s=j\omega} = s^4 - 35s^2 + 70 \;|_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.25 *Test whether* $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)} = \dfrac{s^3 + 8s^2 + 15s}{s^2 + 5s + 4}$

The function $F(s)$ has poles at $s = -1$ and $s = -4$ and zeros at $s = 0$, $s = -3$ and $s = -5$ as shown in Fig. 16.3.

Thus, all the poles and zeros are in the left half of the s plane.

Fig. 16.3

(b) There is no pole on the $j\omega$ axis, hence the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 8s^2$

Odd part of $N(s) = n_1 = s^3 + 15s$

Even part of $D(s) = m_2 = s^2 + 4$

Odd part of $D(s) = n_2 = 5s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (8s^2)(s^2+4) - (s^3+15s)(5s)\,|_{s=j\omega} = 3s^4 - 43s^2\,|_{s=j\omega} = 3\omega^4 + 43\omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.26 *Test whether* $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$

The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 16.4.

Thus, all the poles and zeros are on the $j\omega$ axis.

Fig. 16.4

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{s^2+1}{s^3+4s} = \frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2* are called residues.

$$K_1 = s\,F(s)\,|_{s=0} = \left.\frac{s^2+1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$K_2 = (s+j2)F(s)\,|_{s=-j2} = \left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2} = \frac{-4+1}{(-j2)(-j2-j2)} = \frac{3}{8}$$

$$K_2^* = K_2 = \frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+1)(0) - (0)(s^3+4s)\,|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.27 *Test whether* $F(s) = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1} = \dfrac{2s^3 + 2s^2 + 3s + 2}{(s + j1)(s - j1)}$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 2s^2 + 2$

Odd part of $N(s) = n(s) = 2s^3 + 3s$

By continued fraction expansion,

$$\begin{array}{l}
2s^2 + 2 \,\big)\, 2s^3 + 3s \,\big(\, s \\
\qquad\qquad \underline{2s^3 + 2s} \\
\qquad\qquad\qquad s \,\big)\, 2s^2 + 2s \,\big(\, 2s \\
\qquad\qquad\qquad\qquad \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad 2 \,\big)\, s \,\big(\, \frac{1}{2}s \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s = -j1$ and $s = j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$
\begin{array}{l}
s^2+1\,\big)\,2s^3+2s^2+3s+2\,\big(\,2s+2 \\
\quad\underline{2s^3 \qquad\quad +2s} \\
\qquad 2s^2 + \;\; s+2 \\
\qquad \underline{2s^2 \qquad\quad +2} \\
\qquad\quad s
\end{array}
$$

$$F(s) = 2s+2+\frac{s}{s^2+1}$$

By partial-fraction expansion,

$$F(s) = 2s+2+\frac{K_1}{s+j1}+\frac{K_1^*}{s-j1}$$

$$K_1 = (s+j1)F(s)\,|_{s=-j1} = \frac{-j1}{-j1-j1} = \frac{1}{2}$$

$$K_1^* = K_1 = \frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = 2s^2+2$

Odd part of $N(s) = n_1 = 2s^3+3s$

Even part of $D(s) = m_2 = s^2+1$

Odd part of $D(s) = n_2 = 0$

$$A(\omega^2) = m_1m_2 - n_1n_2\,|_{s=j\omega} = (2s^2+2)(s^2+1)-(2s^3+3s)(0)\,|_{s=j\omega} = 2s^4+4s^2+2\;|_{s=j\omega} = 2(\omega^4-2\omega^2+1)$$
$$= 2(\omega^2-1)^2$$

$A(\omega^2) \geq 0$ for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.28 *Test whether* $F(s) = \dfrac{s^3+6s^2+7s+3}{s^2+2s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+6s^2+7s+3}{s^2+2s+1} = \dfrac{s^3+6s^2+7s+3}{(s+1)(s+1)}$

Since a numerator polynomial cannot be easily factorized, we will test whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 6s^2+3$

Odd part of $N(s) = n(s) = s^3+7s$

By continued fraction expansion,

$$
\begin{array}{l}
6s^2+3\,\big)\,s^3+7s\,\big(\,\frac{1}{6}s \\
\qquad\quad \underline{s^3+0.5s} \\
\qquad\qquad 6.5s\,\big)\,6s^2+3\,\big(\,0.92s \\
\qquad\qquad\quad \underline{6s^2} \\
\qquad\qquad\qquad 3\,\big)\,6.5s\,\big(\,2.17s \\
\qquad\qquad\qquad\quad \underline{6.5s} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that the zeros are in the left half of the s plane.

The function $F(s)$ has a double pole at $s = -1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 6s^2 + 3$

Odd part of $N(s) = n_1 = s^3 + 7s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 2s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (6s^2+3)(s^2+1) - (s^3+7s)(2s)\,|_{s=j\omega} = 4s^4 - 5s^2 + 3\,|_{s=j\omega} = 4\omega^4 + 5\omega^2 + 3$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.29 *Test whether* $F(s) = \dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $$F(s) = \frac{N(s)}{D(s)} = \frac{s^2+s+6}{s^2+s+1} = \frac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

The function $F(s)$ has zeros at $s = -\frac{1}{2} \pm j\frac{\sqrt{23}}{2}$ and poles at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 6$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s)\,|_{s=j\omega} = s^4 + 6s^2 + 6\,|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$
This condition is not satisfied.
Hence, the function $F(s)$ is not positve real.

Example 16.30 *Test whether* $F(s) = \dfrac{s^2+4}{s^3+3s^2+3s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+4}{s^3+3s^2+3s+1} = \dfrac{(s+j2)(s-j2)}{(s+1)^3}$

The function $F(s)$ has two zeros at $s = \pm j2$ and three poles at $s = -1$.
Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 4$
Odd part of $N(s) = n_1 = 0$
Even part of $D(s) = m_2 = 3s^2 + 1$
Odd part of $D(s) = n_2 = s^3 + 3s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+4)(3s^2+1) - (0)(s^3+3s)\,|_{s=j\omega} = 3s^4 + 13s^2 + 4\,|_{s=j\omega} = 3\omega^4 - 13\omega^2 + 4$$

For $\omega = 1$, $A(\omega)^2 = 3 - 13 + 4 = -6$
This condition is not satisfied.
Hence, the function $F(s)$ is not positive real.

Example 16.31 *Test whether* $F(s) = \dfrac{s^3+5s}{s^4+2s^2+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+5s}{s^4+2s^2+1} = \dfrac{s(s^2+5)}{(s^2+1)^2} = \dfrac{s(s+j\sqrt{5})(s-j\sqrt{5})}{(s\pm j1)(s\pm j1)}$

The function $F(s)$ has zeros at $s = 0$, $s = \pm j\sqrt{5}$ and two poles at $s = j1$ and two poles at $s = -j1$.

Thus, poles on the $j\omega$ axis are not simple.
Hence, the function $F(s)$ not positive real.

Example 16.32 *Test whether* $F(s) = \dfrac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$ *is positive real function.*

Solution

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$$

Here, it is easier to prove that $N(s)$ and $D(s)$ are Hurwitz.

By Routh array,

$$\begin{array}{c|ccc} s^4 & 1 & 1 & 2 \\ s^3 & 3 & 1 & \\ s^2 & \frac{2}{3} & 2 & \\ s^1 & -8 & & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, $N(s)$ is not Hurwitz. Thus, all the zeros are not in the left half of the s plane. The remaining two tests need not be carried out.
Hence, the function $F(s)$ is not positive real.

16.4 ELEMENTARY SYNTHESIS CONCEPTS

We know that impedances and admittances of passive networks are positive real functions. Hence, addition of impedances of the two passive networks gives a function which is also a positive real function. Thus, $Z(s) = Z_1(s) + Z_2(s)$ is a positive real function, if $Z_1(s)$ and $Z_2(s)$ are positive real functions. Similarly, $Y(s) = Y_1(s) + Y_2(s)$ is a positive real function, if $Y_1(s)$ and $Y_2(s)$ are positive real functions. There is a special terminology for synthesis procedure. We have,

$$Z(s) = Z_1(s) + Z_2(s)$$
$$Z_2(s) = Z(s) - Z_1(s)$$

Here, $Z_1(s)$ is said to have been removed from $Z(s)$ in forming the new function $Z_2(s)$ as shown in Fig. 16.5. If the removed network is associated with the pole or zero of the original network impedance then that pole or zero is also said to have been removed.
There are four important removal operations.

Fig. 16.5 *Network interpretation of the removal of impedance and admittance*

16.4.1 Removal of a Pole at Infinity

Consider an impedance function $Z(s)$ having a pole at infinity which means that the numerator polynomial is one degree greater than the degree of the denominator polynomial.

$$Z(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Hs + \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

where $\quad H = \dfrac{a_{n+1}}{b_n}$

Let $\quad Z_1(s) = Hs$

and $\quad Z_2(s) = \dfrac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Z(s) - Hs$

$Z_1(s) = Hs$ represents impedance of an inductor of value H. Hence, the removal of a pole at infinity corresponds to the removal of an inductor from the network of Fig. 16.6(a).

If the given function is an admittance function $Y(s)$, then $Y_1(s) = Hs$ represents the admittance of a capacitor $Y_C(s) = Cs$. The network for $Y_1(s)$ is a capacitor of value $C = H$ as shown in Fig. 16.6(b).

Fig. 16.6 *Network interpretation of the removal of a pole at infinity*

16.4.2 Removal of a Pole at Origin

If $Z(s)$ has a pole at the origin then it may be written as

$$Z(s) = \frac{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots + b_m s^m} = \frac{K_0}{s} + \frac{d_1 + d_2 s + \ldots + d_n s^{n-1}}{b_1 + b_2 s + \ldots + b_m s^{m-1}} = Z_1(s) + Z_2(s)$$

where $K_0 = \dfrac{a_0}{b_1}$

$Z_1(s) = \dfrac{K_0}{s}$ represents the impedance of a capacitor of value $\dfrac{1}{K_0}$.

If the given function is an admittance function $Y(s)$ then removal of $Y_1(s) = \dfrac{K_0}{s}$ corresponds to an inductor of value $\dfrac{1}{K_0}$.

Thus, removal of a pole from the impedance function $Z(s)$ at the origin corresponds to the removal of a capacitor, and from admittance function $Y(s)$ corresponds to removal of an inductor as shown in Fig. 16.7.

Fig. 16.7 *Network interpretation of the removal of a pole at origin*

16.4.3 Removal of Conjugate Imaginary Poles

If $Z(s)$ contains poles on the imaginary axis, i.e., at $s = \pm j\omega_1$ then $Z(s)$ will have factors $(s + j\omega_1)(s - j\omega_1) = s^2 + \omega_1^2$ in the denominator polynomial

$$Z(s) = \frac{p(s)}{(s^2 + \omega_1^2)\, q_1(s)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s + j\omega_1} + \frac{K_1^*}{s - j\omega_1} + Z_2(s)$$

For a positive real function, $j\omega$ axis poles must themselves be conjugate and must have equal, positive and real residues.

$$K_1 = K_1^*$$

Hence,
$$Z(s) = \frac{2K_1 s}{s^2 + \omega_1^2} + Z_2(s)$$

Thus,
$$Z_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}} = \frac{1}{Y_a + Y_b}$$

where $Y_a = \dfrac{s}{2K_1}$ is the admittance of a capacitor of value $C = \dfrac{1}{2K_1}$

and $Y_b = \dfrac{\omega_1^2}{2K_1 s}$ is the admittance of an inductor of value $L = \dfrac{2K_1}{\omega_1^2}$

If the given function is an admittance function $Y(s)$ then

$$Y_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{Z_a + Z_b} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}}$$

where $Z_a = \dfrac{s}{2K_1}$ is the impedance of an inductor of value $L = \dfrac{1}{2K_1}$ and $Z_b = \dfrac{\omega_1^2}{2K_1 s}$ is the impedance of a capacitor of value $C = \dfrac{2K_1}{\omega_1^2}$.

Thus, removal of conjugate imaginary poles from impedance function $Z(s)$ corresponds to the removal of the parallel combination of $L - C$ and from admittance function $Y(s)$ corresponds to removal of series combination of $L - C$ as shown in Fig. 16.8.

Fig. 16.8 *Network interpretation of the removal of conjugate imaginary poles*

16.4.4 Removal of a Constant

If a real number R_1 is subtracted from $Z(s)$ such that

$$Z_2(s) = Z(s) - R_1$$

$$Z(s) = R_1 + Z_2(s)$$

then R_1 represents a resistor.

If the given function is an admittance function $Y(s)$, then removal of $Y_1(s) = R_1$ represents a conductance of value R_1.

Thus, removal of a constant from impedance function $Z(s)$ corresponds to the removal of a resistance, and from admittance function $Y(s)$ corresponds to removal of a conductance.

Example 16.33 *Synthesize the impedance function* $Z(s) = \dfrac{s^3 + 4s}{s^2 + 2}$.

Solution

By long division of $Z(s)$,

$$s^2 + 2 \,\big)\, s^3 + 4s \,\big(\, s$$

$$\frac{s^3 + 2s}{2s}$$

$$Z(s) = s + \frac{2s}{s^2 + 2} = Z_1(s) + Z_2(s)$$

$Z_1(s) = s$ represents impedance of an inductor of value 1 H.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s^2 + 2}{2s} = \frac{s^2}{2s} + \frac{2}{2s} = \frac{1}{2}s + \frac{1}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = \dfrac{1}{2}s$ represents the admittance of a capacitor of value $\dfrac{1}{2}$ F.

$Y_4(s) = \dfrac{1}{s}$ represents the admittance of an inductor of value 1 H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.9.

Fig. 16.9

Example 16.34 *Realise the network having impedance function*

$$Z(s) = \frac{s^2 + 2s + 10}{s(s+5)}$$

Solution

By long division of $Z(s)$,

$$\begin{array}{l} s^2 + 5s \,\big)\, s^2 + 2s + 10 \,\big(\, \dfrac{2}{s} \\ \qquad\qquad\quad \underline{\quad 2s + 10 \quad} \\ \qquad\qquad\quad s^2 \end{array}$$

$$Z(s) = \frac{2}{s} + \frac{s^2}{s^2 + 5s} = \frac{2}{s} + \frac{s}{s+5} = Z_1(s) + Z_2(s)$$

$Z_1(s) = \dfrac{2}{s}$ represents the impedance of capacitor of value $\dfrac{1}{2}$F.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s+5}{s} = 1 + \frac{5}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = 1$ represents the admittance of a resistor of value 1 Ω.

$Y_4(s) = \dfrac{5}{s}$ represents the admittance of an inductor of value $\dfrac{1}{5}$H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.10.

Fig. 16.10

Example 16.35 *Realise the network having impedance function* $Z(s) = \dfrac{6s^3 + 5s^2 + 6s + 4}{2s^3 + 2s}$.

Solution By long division of $Z(s)$,

$$\begin{array}{l} 2s^3 + 2s \,\big)\, 6s^3 + 5s^2 + 6s + 4 \,\big(\, 3 \\ \qquad\qquad\quad \underline{6s^3 \qquad\quad + 6s} \\ \qquad\qquad\qquad\quad 5s^2 \qquad + 4 \end{array}$$

$$Z(s) = 3 + \frac{5s^2 + 4}{2s^3 + 2s} = Z_1(s) + Z_2(s)$$

$Z_1(s) = 3$ represents the impedance of a resistor of value 3 Ω.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{2s^3 + 2s}{5s^2 + 4}$$

By long division of $Y_2(s)$,

$$5s^2+4\,\Big)\,2s^3+2s\,\Big(\,\frac{2}{5}s$$

$$\underline{2s^3+\frac{8}{5}s}$$

$$\frac{2}{5}s$$

$$Y_2(s)=\frac{2}{5}s+\frac{\frac{2}{5}s}{5s^2+4}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\dfrac{2}{5}s$ represents the admittance of a capacitor of value $\dfrac{2}{5}$ F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{5s^2+4}{\frac{2}{5}s}=\frac{25s^2+20}{2s}=\frac{25}{2}s+\frac{10}{s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\dfrac{25}{2}s$ represents the impedance of an inductor of value $\dfrac{25}{2}$ H.

$Z_6(s)=\dfrac{10}{s}$ represents the impedance of a capacitor of value $\dfrac{1}{10}$ F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.11.

Fig. 16.11

Example 16.36

Realise the network having impedance function

$$Z(s)=\frac{s^4+10s^2+7}{s^3+2s}$$

Solution By long division of $Z(s)$,

$$s^3+2s\,\Big)\,s^4+10s^2+7\,\Big(\,s$$

$$\underline{s^4+2s^2}$$

$$8s^2+7$$

$$Z(s)=s+\frac{8s^2+7}{s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)=s$ represents the impedance of an inductor of value 1 H.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s^3+2s}{8s^2+7}$$

By long division of $Y_2(s)$,

$$8s^2+7\,\Big)\,s^3+2s\,\Big(\,\frac{1}{8}s$$

$$\underline{s^3+\frac{7}{8}s}$$

$$\frac{9}{8}s$$

$$Y_2(s)=\frac{1}{8}s+\frac{\frac{9}{8}s}{8s^2+7}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{1}{8}s$ represents the admittance of a capacitor of value $\frac{1}{8}$F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{8s^2+7}{\frac{9}{8}s}=\frac{64}{9}s+\frac{56}{9s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{64}{9}s$ represents the impedance of an inductor of value $\frac{64}{9}$H.

$Z_6(s)=\frac{56}{9s}$ represents the impedance of a capacitor of value $\frac{9}{56}$F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.12.

Fig. 16.12

Example 16.37 *Realise the network having admittance function* $Y(s)=\frac{4s^2+6s}{s+1}$.

Solution By long division of $Y(s)$,

$$s+1\,\Big)\,4s^2+6s\,\Big(\,4s$$

$$\underline{4s^2+4s}$$

$$2s$$

$$Y(s)=4s+\frac{2s}{s+1}=Y_1(s)+Y_2(s)$$

$Y_1(s)=4s$ represents the admittance of a capacitor of value 4 F.

$$Z_2(s)=\frac{1}{Y_2(s)}=\frac{s+1}{2s}=\frac{1}{2}+\frac{1}{2s}=Z_3(s)+Z_4(s)$$

$Z_3(s) = \frac{1}{2}$ represents the impedance of a resistor of value $\frac{1}{2}\,\Omega$.

$Z_4(s) = \frac{1}{2s}$ represents the impedance of a capacitor of value 2 F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.13.

Fig. 16.13

Example 16.38 *Realise the admittance function* $Y(s) = \frac{3+5s}{4+2s}$.

Solution By long division,

$$\begin{array}{r|l} 4+2s & 3+5s \left(\frac{3}{4} \right. \\ & \underline{3+\frac{3}{2}s} \\ & \frac{7}{2}s \end{array}$$

$$Y(s) = \frac{3}{4} + \frac{\frac{7}{2}s}{4+2s} = Y_1(s) + Y_2(s)$$

$Y_1(s) = \frac{3}{4}$ represents the admittance of a resistor of value $\frac{4}{3}\,\Omega$.

$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{4+2s}{\frac{7}{2}s} = \frac{8+4s}{7s} = \frac{8}{7s} + \frac{4}{7} = Z_3(s) + Z_4(s)$$

$Z_3(s) = \frac{8}{7s}$ represents the impedance of a capacitor of value $\frac{7}{8}$F.

$Z_4(s) = \frac{4}{7}$ represents the impedance of a resistor of value $\frac{4}{7}\,\Omega$.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.14.

Fig. 16.14

16.5 REALISATION OF *LC* FUNCTIONS

LC driving point immitance functions have the following properties.

1. It is the ratio of odd to even or even to odd polynomials.
2. The poles and zeros are simple and lie on the $j\omega$-axis.
3. The poles and zeros interlace on the $j\omega$-axis.
4. There must be either a zero or a pole at the origin and infinity.
5. The difference between any two successive powers of numerator and denominator polynomials is at most two. There cannot be any missing terms.

6. The highest powers of numerator and denominator polynomials must differ by unity; the lowest powers also differ by unity.

There are a number of methods of realising an *LC* function. But we will study only four basic forms—Foster I, Foster II, Cauer I and Cauer II forms. The Foster forms are obtained by partial-fraction expansion of $F(s)$, and the Cauer forms are obtained by continued fraction expansion of $F(s)$.

16.5.1 Foster Realisation

Consider a general *LC* function $F(s)$ given by

$$F(s) = \frac{H(s^2+\omega_1^2)(s^2+\omega_3^2)\ldots}{s(s^2+\omega_2^2)(s^2+\omega_4^2)\ldots}$$

where $0 \le \omega_1^2 < \omega_2^2 < \omega_3^2 \ldots$ and H is positive.

By partial-fraction expansion of $F(s)$,

$$F(s) = \frac{K_0}{s} + \frac{K_2}{s+j\omega_2} + \frac{K_2}{s-j\omega_2} + \ldots + K_\infty s$$

Combining terms with conjugate poles,

$$F(s) = \frac{K_0}{s} + \frac{2K_2 s}{s+\omega_2^2} \ldots + K_\infty s$$

where K_0, K_i and K_∞ are the residues of $F(s)$ at the origin, at $j\omega_i$ and at infinity respectively.
These residues are given by

$$K_0 = sF(s)\big|_{s=0}$$

$$K_i = \left.\frac{\left(s^2+\omega_i^2\right)F(s)}{2s}\right|_{s^2=-\omega_i^2}$$

$$K_\infty = \left.\frac{F(s)}{s}\right|_{s\to\infty}$$

Foster I Form If $F(s)$ represents an impedance function, it gives a series connection of impedances.

$$F(s) = Z(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2+\omega_2^2} \ldots + K_\infty s = Z_1(s) + Z_2(s) + \ldots + Z_n(s)$$

The first term $\frac{K_0}{s}$ represents the impedance of a capacitor of $\frac{1}{K_0}$ farad.

The last term $K_\infty s$ represents the impedance of an inductor of K_∞ henry.

The remaining terms, i.e., $\frac{2K_i s}{s^2+\omega_i^2}$ represent the impedance of a parallel combination of capacitor C_i and inductor L_i. For parallel combination of L_i and C_i,

$$Z(s) = \frac{1}{C_i s + \frac{1}{L_i s}} = \frac{\left(\frac{1}{C_i}\right)s}{s^2 + \frac{1}{L_i C_i}} = \frac{2K_i s}{s^2+\omega_i^2}$$

$$C_i = \frac{1}{2K_i} \text{ and } L_i = \frac{2K_i}{\omega_i^2}$$

Table 16.1 *Realisation of Foster-I form of LC network*

Impedance function	Element
$\frac{K_0}{s} = \frac{1}{C_0 s}$	$C_0 = \frac{1}{K_0}$
$\frac{2K_i s}{s^2+\omega_i^2} = \frac{\left(\frac{1}{C_i}\right)s}{s^2+\frac{1}{L_i C_i}}$	$L_i = \frac{2K_i}{\omega_i^2}$ $C_i = \frac{1}{2K_i}$
$K_\infty s = Ls$	$L_\infty = K_\infty$

The network corresponding to Foster I form is shown in Fig. 16.15.

Fig. 16.15 *Foster-I form of LC network*

If $Z(s)$ has no pole at the origin then capacitor C_0 is not present in the network. Similarly, if there is no pole at ∞, inductor L_∞ is not present in the network.

Foster II Form If $F(s)$ represents an admittance function, it gives the parallel combination of admittances.

$$F(s) = Y(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2+\omega_2^2} + \ldots + K_\infty s = Y_1(s) + Y_2(s) + \ldots Y_n(s)$$

The first term $\frac{K_0}{s}$ represents the admittance of an inductor of $\frac{1}{K_0}$ henry.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farad.

The remaining terms, i.e., $\frac{2K_i s}{s^2+\omega_i^2}$ represent the admittance of a series combination of an inductor L_i and a capacitor C_i.

For series combination of L_i and C_i,

$$Y(s) = \frac{1}{L_i s + \dfrac{1}{C_i s}} = \frac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}} = \frac{2K_i s}{s^2 + \omega_i^2}$$

$$L_i = \frac{1}{2K_i} \quad \text{and} \quad C_i = \frac{2K_i}{\omega_i^2}$$

Table 16.2 *Realisation of Foster-II form of LC network*

Admittance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{L_0 s}$	$L_0 = \dfrac{1}{K_0}$
$\dfrac{2K_i s}{s^2 + \omega_i^2} = \dfrac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}}$	$L_i = \dfrac{1}{2K_i}$ $C_i = \dfrac{2K_i}{\omega_i^2}$
$K_\infty s = Cs$	$C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 16.16.

If $Y(s)$ has no pole at the origin then inductor L_0 is not present. Similarly, if there is no pole at infinity, capacitor C_∞ is not present.

Fig. 16.16 *Foster-II form of LC network*

16.5.2 Cauer Realisation or Ladder Realisation

Cauer I Form Since the numerator and denominator polynomials of an *LC* function always differ in degrees by unity, there is always a zero or a pole at $s = \infty$. The Cauer I Form is obtained by successive removal of a pole or a zero at infinity from the function.

Consider an impedance function $Z(s)$ having a pole at infinity.

By removing the pole at infinity, we get

$$Z_2(s) = Z(s) - L_1 s$$

Now, $Z_2(s)$ has a zero at $s = \infty$. If we invert $Z_2(s)$, $Y_2(s)$ will have a pole at $s = \infty$.

By removing this pole,

$$Y_3(s) = Y_2(s) - C_2 s$$

Now $Y_3(s)$ has a zero at $s = \infty$, which we can invert and remove. This process continues until the remainder is zero. Each time we remove a pole, we remove an inductor or a capacitor depending on whether the function is an impedance or an admittance. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = L_1 s + \cfrac{1}{C_2 s + \cfrac{1}{L_3 s + \cfrac{1}{C_4 s + \ldots}}}$$

Thus, the final structure is a ladder network whose series arms are inductors and shunt arms are capacitors. The Cauer I network is shown in Fig. 16.17.

Fig. 16.17 *Cauer I form of LC network*

If the impedance function has zero at infinity, i.e., if degree of numerator is less than that of its denominator by unity, the function is first inverted and continued fraction expansion proceeds as usual. In this case, the first element is a capacitor as shown in Fig. 16.18.

Fig. 16.18 *Cauer-I form of LC network*

Cauer II Form Since the lowest degrees of numerator and denominator polynomials of LC function must differ by unity, there is always a zero or a pole at $s = 0$. The Cauer II form is obtained by successive removal of a pole or a zero at $s = 0$ from the function.

In this method, continued fraction expansion of $Z(s)$ is carried out in terms of poles at the origin by removal of the pole at the origin, inverting the resultant function to create a pole at the origin which is removed and this process is continued until the remainder is zero. To do this, we arrange both numerator and denominator polynomials in ascending order and divide the lowest power of the denominator into the lowest power of the numerator. Then we invert the remainder and divide again. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{L_2 s} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{L_4 s} + \ldots}}}$$

Thus, the final structure is a ladder network whose first element is a series capacitor and second element is a shunt inductor as shown in Fig. 16.19.

Fig. 16.19 *Cauer II form of LC network*

If the impedance function has a zero at the origin then the first element is a shunt inductor and the second element is a series capacitor as shown in Fig. 16.20.

Thus, the LC function $F(s)$ can be realised in four different forms. All these forms have the same number of elements and the number is equal to the number of poles and zeros of $F(s)$ includng any at infinity.

Fig. 16.20 *Cauer–II form of LC network*

Example 16.39 *State whether the following functions are driving point immittance of LC networks are not:*

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$ (b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

Solution

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$

The function $Z(s)$ has poles at $s = \pm j1$ and $s = \pm j\sqrt{3}$ and zeros at $s = 0$ and $s = \pm j2$ as shown in Fig. 16.21. Since the poles and zeros do not interlace on the $j\omega$ axis, the function $Z(s)$ is not an LC impedance function.

Fig. 16.21

(b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

The function $Z(s)$ has poles at $s = 0$ and $s = \pm j\sqrt{2}$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 16.22. The poles and zeros are simple and lie on the $j\omega$-axis. The poles and zeros interlace on the $j\omega$ axis. Hence, the function $Z(s)$ is an LC impedance function.

Fig. 16.22

Example 16.40 *Realise the Foster and Cauer forms of the following impedance function*

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = \pm j2$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 16.23.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the $j\omega$ axis. Poles and zeros are interlaced. Hence, the given function is an LC function.

Fig. 16.23

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. But degree of numerator is greater than degree of denominator. Hence, division is first carried out.

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{4s^4+40s^2+36}{s^3+4s}$$

$$s^3+4s)\ 4s^4+40s^2+36\ (4s$$

$$\frac{4s^4+16s^2}{24s^2+36}$$

$$Z(s) = 4s + \frac{24s^2+36}{s^3+4s} = 4s + \frac{24s^2+36}{s(s^2+4)}$$

By partial-fraction expansion,

$$Z(s) = 4s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = 4s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+4}$$

where
$$K_0 = sZ(s)\big|_{s=0} = \frac{4(1)(9)}{4} = 9$$

$$K_1 = \left.\frac{(s^2+4)Z(s)}{2s}\right|_{s^2=-4} = \frac{4(-4+1)(-4+9)}{2(-4)} = \frac{15}{2}$$

$$Z(s) = 4s + \frac{9}{s} + \frac{15s}{s^2+4}$$

The first term represents the impedance of an inductor of 4 H. The second term represents the impedance of a capacitor of $\frac{1}{9}$ F. The third term represents the impedance of a parallel LC network.

For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = \frac{1}{15}\text{F}$$

$$L = \frac{15}{4}\text{H}$$

The network is shown in Fig. 16.24.

Fig. 16.24

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+4)}{4(s^2+1)(s^2+9)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j3} + \frac{K_2^*}{s-j3} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+9}$$

where

$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+4)}{8(-1+9)} = \frac{3}{64}$$

$$K_2 = \frac{(s^2+9)}{2s}Y(s)\bigg|_{s^2=-9} = \frac{(-9+4)}{8(-9+1)} = \frac{5}{64}$$

$$Y(s) = \frac{\left(\frac{3}{32}\right)s}{s^2+1} + \frac{\left(\frac{5}{32}\right)s}{s^2+9}$$

These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{32}{3}\text{H} \qquad C_1 = \frac{3}{32}\text{F}$$

$$L_2 = \frac{32}{5}\text{H} \qquad C_2 = \frac{5}{288}\text{F}$$

The network is shown in Fig. 16.25.

Fig. 16.25

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{4s^4 + 40s^2 + 36}{s^3 + 4s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$s^3+4s\,\Big)\,4s^4+40s^2+36\,\Big(\,4s \leftarrow Z$$

$$\underline{4s^4+16s^2}$$

$$24s^2+36\,\Big)\,s^3+4s\,\Big(\,\frac{1}{24}s \leftarrow Y$$

$$\underline{s^3+\frac{3}{2}s}$$

$$\frac{5}{2}s\,\Big)\,24s^2+36\,\Big(\,\frac{48}{5}s \leftarrow Z$$

$$\underline{24s^2}$$

$$36\,\Big)\,\frac{5}{2}s\,\Big(\,\frac{5}{72}s \leftarrow Y$$

$$\underline{\frac{5}{2}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.26.

Fig. 16.26

Cauer II Form The Cauer II form is obtained from partial-fraction expansion about pole at origin.

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}=\frac{4s^4+40s^2+36}{s^3+4s}$$

The function $Z(s)$ has a pole at origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s)=\frac{36+40s^2+4s^4}{4s+s^3}$$

By continued fraction expansion,

$$4s+s^3\,\Big)\,36+40s^2+4s^4\,\Big(\,\frac{9}{s} \leftarrow \mathrm{Z}$$

$$\underline{36+\ \ 9s^2}$$

$$31s^2+4s^4\,\Big)\,4s+s^3\,\Big(\,\frac{4}{31s} \leftarrow Y$$

$$\underline{4s+\frac{16}{31}s^3}$$

$$\frac{15}{31}s^3\,\Big)\,31s^2+4s^4\,\Big(\,\frac{961}{15s} \leftarrow Z$$

$$\underline{31s^2}$$

$$4s^4 \Big) \frac{15}{31}s^3 \Big(\frac{15}{124s} \leftarrow Y$$

$$\frac{\frac{15}{31}s^3}{0}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.27.

Fig. 16.27

Example 16.41

Realise Foster forms of the LC impedance function

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} = \frac{s^4+4s^2+3}{s^3+2s}$$

$$s^3+2s \Big) s^4+4s^2+3 \Big(s$$

$$\frac{s^4+2s^2}{2s^2+3}$$

$$Z(s) = s + \frac{2s^2+3}{s^3+2s} = s + \frac{2s^2+3}{s(s^2+2)}$$

By partial-fraction expansion,

$$Z(s) = s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+2}$$

where

$$K_0 = sZ(s)|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_1 = \frac{(s^2+2)}{2s}Z(s)\Big|_{s^2=-2} = \frac{(-2+1)(-2+3)}{2(-2)} = \frac{1}{4}$$

$$Z(s) = s + \frac{\left(\frac{3}{2}\right)}{s} + \frac{\left(\frac{1}{2}\right)s}{s^2+2}$$

The first term represents the impedance of an inductor of 1 H. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of a parallel *LC* network.

For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = 2\text{F}$$

$$L = \frac{1}{4}\text{H}$$

The network is shown in Fig. 16.28.

Fig. 16.28

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j\sqrt{3}} + \frac{K_2^*}{s-j\sqrt{3}} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+3}$$

where

$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = \frac{(s^2+3)}{2s}Y(s)\bigg|_{s^2=-3} = \frac{-3+2}{2(-3+1)} = \frac{1}{4}$$

$$Y(s) = \frac{\left(\frac{1}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = 2\text{H}, \quad C_1 = \frac{1}{2}\text{F}$$

$$L_2 = 2\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$

The network is shown in Fig. 16.29.

Fig. 16.29

Example 16.42 *Realise Foster forms of the following LC impedance function:*

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s)=\frac{K_0}{s}+\frac{K_1}{s+j\sqrt{2}}+\frac{K_1^*}{s-j\sqrt{2}}+\frac{K_2}{s+j2}+\frac{K_2^*}{s-j2}=\frac{K_0}{s}+\frac{2K_1 s}{s^2+2}+\frac{2K_2 s}{s^2+4}$$

where

$$K_0=sZ(s)\big|_{s=0}=\frac{(1)(3)}{(2)(4)}=\frac{3}{8}$$

$$K_1=\frac{(s^2+2)}{2s}Z(s)\bigg|_{s^2=-2}=\frac{(-2+1)(-2+3)}{2(-2)(-2+4)}=\frac{1}{8}$$

$$K_2=\frac{(s^2+4)}{2s}Z(s)\bigg|_{s^2=-4}=\frac{(-4+1)(-4+3)}{2(-4)(-4+2)}=\frac{3}{16}$$

$$Z(s)=\frac{\frac{3}{8}}{s}+\frac{\left(\frac{1}{4}\right)s}{s^2+2}+\frac{\left(\frac{3}{8}\right)s}{s^2+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The other two terms represent the impedance of a parallel *LC* network.

For a parallel *LC* network,

$$Z_{LC}(s)=\frac{\left(\frac{1}{C}\right)s}{s^2+\frac{1}{LC}}$$

By direct comparison,

$$C_1=4\text{F},\quad L_1=\frac{1}{8}\text{H}$$

$$C_2=\frac{8}{3}\text{F},\quad L_2=\frac{3}{32}\text{H}$$

The network is shown in Fig. 16.30.

Fig. 16.30

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s)=\frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)}=\frac{s^5+6s^3+8s}{s^4+4s^2+3}$$

Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$s^4+4s^2+3)s^5+6s^3+8s(s$$

$$\underline{s^5+4s^3+3s}$$

$$2s^3+5s$$

$$Y(s)=s+\frac{2s^3+5s}{s^4+4s^2+3}=s+\frac{2s^3+5s}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s)=s+\frac{K_1}{s+j1}+\frac{K_1^*}{s-j1}+\frac{K_2}{s+j\sqrt{3}}+\frac{K_2^*}{s-j\sqrt{3}}=s+\frac{2K_1 s}{s^2+1}+\frac{2K_2 s}{s^2+3}$$

where
$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+2)(-1+4)}{2(-1+3)} = \frac{3}{4}$$
$$K_2 = \frac{(s^2+3)}{2s}Y(s)\bigg|_{s^2=-3} = \frac{(-3+2)(-3+4)}{2(-3+1)} = \frac{1}{4}$$
$$Y(s) = s + \frac{\left(\frac{3}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

The first term represents the admittance of capacitor of 1 F. The other two terms represent admittance of a series *LC* network. For a series *LC* network,
$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,
$$L_1 = \frac{2}{3}\text{H}, \quad C_1 = \frac{3}{2}\text{F}$$
$$L_2 = 2\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$

The network is shown in Fig. 16.31.

Fig. 16.31

Example 16.43 *Realise Cauer forms of the following LC impedance function:*
$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

Solution

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.
$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,
$$2s^3 + 2s)\,10s^4 + 12s^2 + 1\,(5s \leftarrow Z$$
$$\underline{10s^4 + 10s^2}$$
$$2s^2 + 1)\,2s^3 + 2s\,(s \leftarrow Y$$
$$\underline{2s^3 + s}$$
$$s)\,2s^2 + 1\,(2s \leftarrow Z$$
$$\underline{2s^2}$$
$$1)\,s\,(s \leftarrow Y$$
$$\frac{s}{0}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.32.

Fig. 16.32

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin.

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

The function $Z(s)$ has a pole at the origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s) = \frac{1 + 12s^2 + 10s^4}{2s + 2s^3}$$

By continued fraction expansion of $Z(s)$,

$$2s + 2s^3 \overline{\big)\, 1 + 12s^2 + 10s^4 \,\big(} \; \frac{1}{2s} \leftarrow \mathrm{Z}$$

$$\underline{1 + \quad s^2}$$

$$11s^2 + 10s^4 \overline{\big)\, 2s + 2s^3 \,\big(} \; \frac{2}{11s} \leftarrow Y$$

$$\underline{2s + \frac{20}{11}s^3}$$

$$\frac{2}{11}s^3 \overline{\big)\, 11s^2 + 10s^4 \,\big(} \; \frac{121}{2s} \leftarrow Z$$

$$\underline{11s^2}$$

$$10s^4 \overline{\big)\, \frac{2}{11}s^3 \,\big(} \; \frac{2}{110s} \leftarrow Y$$

$$\underline{\frac{2}{11}s^3}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in Cauer or ladder realisation. The network is shown in Fig. 16.33.

Fig. 16.33

Example 16.44 *Realise the following network function in Cauer I form:*

$$Z(s) = \frac{6s^4 + 42s^2 + 48}{s^5 + 18s^3 + 48s}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates the presence of a zero at infinity. The admittance function $Y(s)$ has a pole at infinity. Hence, the continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^5 + 18s^3 + 48s}{6s^4 + 42s^2 + 48}$$

By continued fraction expansion

$$6s^4 + 42s^2 + s^2 \Big) s^5 + 18s^3 + 48s \Big(\frac{1}{6}s \leftarrow Y$$

$$\underline{s^5 + \ 7s^3 + \ 8s}$$

$$11s^3 + 40s \Big) 6s^4 + 42s^2 + 48 \Big(\frac{6}{11}s \leftarrow Z$$

$$\underline{6s^4 + \frac{240}{11}s^2}$$

$$\frac{222}{11}s^2 + 30 \Big) 11s^3 + 40s \Big(\frac{121}{222}s \leftarrow Y$$

$$\underline{11s^3 + \frac{5808}{222}s}$$

$$\frac{3072}{222}s \Big) \frac{222}{11}s^2 + 48 \Big(\frac{49284}{33792}s \leftarrow Z$$

$$\underline{\frac{222}{11}s^2}$$

$$48 \Big) \frac{3072}{222}s \Big(\frac{128}{444}s \leftarrow Y$$

$$\underline{\frac{3072}{222}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.34.

Fig. 16.34

Example 16.45 *Realise Cauer II form of the function:*

$$Z_{LC}(s) = \frac{s(s^4 + 3s^2 + 1)}{3s^4 + 4s^2 + 1}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin. The given function has a zero at the origin. The admittance function $Y(s)$ has a pole at origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y_{LC}(s) = \frac{3s^4 + 4s^2 + 1}{s^5 + 3s^3 + s} = \frac{1 + 4s^2 + 3s^4}{s + 3s^3 + s^5}$$

By continued fraction expansion of $Y(s)$, we have

$$s + 3s^3 + s^5 \,\big)\, 1 + 4s^2 + 3s^4 \left(\frac{1}{s} \leftarrow Y\right.$$

$$\underline{1 + 3s^2 + s^4}$$

$$s^2 + 2s^4 \,\big)\, s + 3s^3 + s^5 \left(\frac{1}{s} \leftarrow Z\right.$$

$$\underline{s + 2s^3}$$

$$s^3 + s^5 \,\big)\, s^2 + 2s^4 \left(\frac{1}{s} \leftarrow Y\right.$$

$$\underline{s^2 + s^4}$$

$$s^4 \,\big)\, s^3 + s^5 \left(\frac{1}{s} \leftarrow Z\right.$$

$$\underline{s^3}$$

$$s^5 \,\big)\, s^4 \left(\frac{1}{s} \leftarrow Y\right.$$

$$\underline{s^4}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.35.

Fig. 16.35

Example 16.46 *Obtain the Cauer I form of realisation for the function*

$$Z_{LC}(s) = \frac{s^5 + 7s^3 + 10s}{s^4 + 5s^2 + 4}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z_{LC}(s)$ about pole at infinity.

$$
\begin{array}{l}
s^4+5s^2+4\,\big)\,s^5+7s^3+10s\,\big(\; s \leftarrow Z \\
\qquad \underline{s^5+5s^3+\;\;4s} \\
\qquad\quad 2s^3+6s\,\big)\,s^4+5s^2+4\,\big(\; \frac{1}{2}s \leftarrow Y \\
\qquad\qquad\quad \underline{s^4+3s^2} \\
\qquad\qquad\qquad 2s^2+4\,\big)\,2s^3+6s\,\big(\; s \leftarrow Z \\
\qquad\qquad\qquad\quad \underline{2s^3+4s} \\
\qquad\qquad\qquad\qquad 2s\,\big)\,2s^2+4\,\big(\; s \leftarrow Y \\
\qquad\qquad\qquad\qquad\quad \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad 4\,\big)\,2s\,\big(\; \frac{1}{2}s \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{2s} \\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.36.

Fig. 16.36

Example 16.47 *Synthesize the following LC impedance function in Cauer II form:*

$$Z(s)=\frac{s^3+2s}{s^4+4s^2+3}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin.

$$Z(s)=\frac{s(s^2+2)}{(s^2+3)(s^2+1)}$$

But $Z(s)$ has a zero at the origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y(s)=\frac{3+4s^2+s^4}{2s+s^3}$$

By continued fraction expansion of $Y(s)$,

$$
\begin{array}{l}
2s+s^3\,\big)\,3+4s^2+s^4\,\Big(\dfrac{3}{2s} \leftarrow Y \\
\qquad \underline{3+\dfrac{3}{2}s^2} \\
\qquad\quad \dfrac{5}{2}s^2+s^4\,\big)\,2s+s^3\,\Big(\dfrac{4}{5s} \leftarrow Z \\
\qquad\qquad\quad \underline{2s+\dfrac{4}{5}s^3} \\
\qquad\qquad\qquad \dfrac{s^3}{5}\,\big)\,\dfrac{5}{2}s^2+s^4\,\Big(\dfrac{25}{2s} \leftarrow Y \\
\qquad\qquad\qquad\quad \underline{\dfrac{5}{2}s^2} \\
\qquad\qquad\qquad\qquad s^4\,\big)\,\dfrac{1}{5}s^3\,\Big(\dfrac{1}{5s} \leftarrow Z \\
\qquad\qquad\qquad\qquad\quad \underline{\dfrac{1}{5}s^3} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.37.

Fig. 16.37

16.6 REALISATION OF *RC* FUNCTIONS

RC driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency nearest to the origin is a pole.
4. The highest critical frequency farthest to the origin is a zero.
5. Residues evaluated at the poles of $Z_{RC}(s)$ are real and positive.
6. The slope $\dfrac{d}{d\sigma}Z_{RC}$ is negative.
7. $Z_{RC}(\infty) < Z_{RC}(0)$.

RC functions can also be realised in four different ways. The impedance function of RC networks is given by,

$$Z(s) = \frac{H(s+\sigma_1)(s+\sigma_3)\ldots}{s(s+\sigma_2)\ldots}$$

16.6.1 Foster Realisation

Foster I Form The Foster I form is obtained by partial-fraction expansion of $Z(s)$.

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+\sigma_1} + \frac{K_2}{s+\sigma_2} + \ldots + K_\infty$$

where K_0, K_1, K_2, … K_∞ are residues of $Z(s)$.

$$K_o = sZ(s)\big|_{s=0}$$

$$K_i = (s+\sigma_i)Z(s)\big|s = -\sigma_i$$

$$K_\infty = \frac{Z(s)}{s}\bigg|_{s\to\infty}$$

The first term $\dfrac{K_0}{s}$ represents the impedance of a capacitor of $\dfrac{1}{K_o}$ farads.

The last term K_∞ represents the impedance of a resistor of K_∞ ohms.

The remaining terms, i.e., $\dfrac{K_i}{s+\sigma_i}$ represent the impedance of the parallel combination of resistor R_i and capacitor C_i. For parallel combinaton of R_i and C_i,

$$Z(s) = \frac{R_i\left(\dfrac{1}{C_i s}\right)}{R_i + \dfrac{1}{C_i s}} = \frac{K_i}{s+\sigma_i}$$

$$R_i = \frac{K_i}{\sigma_i} \text{ and } C_i = \frac{1}{K_i}$$

Table 16.3 *Realisation of Foster-I form of RC network*

Impedance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{C_0 s}$	$C_0 = \dfrac{1}{K_0}$
$\dfrac{K_i}{s+\sigma_i} = \dfrac{(R_i)\left(\dfrac{1}{C_i s}\right)}{R_i + \dfrac{1}{C_i s}}$	$Ri = \dfrac{K_i}{\sigma_i}$ $C_i = \dfrac{1}{K_i}$
$K_\infty = R_\infty$	$R_\infty = K_\infty$

The network corresponding to the Foster-I form is shown in Fig. 16.38.

Fig. 16.38 *Foster-I form of RC network*

Foster II Form The Foster-II form is obtained by partial fraction expansion of $Y(s)$. Since $Y(s) = \dfrac{1}{Z(s)}$ has negative residue at its pole, Foster II form is obtained by expanding $\dfrac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{K_o}{s} + \sum_{i=1}^{n} \frac{K_i}{(s+\sigma_i)} + K_\infty$$

Multiplying this equation by s,

$$Y(s) = K_o + \sum_{i=1}^{n} \frac{K_i s}{(s+\sigma_i)} + K_\infty s$$

The first term K_o represents the conductance of a resistor of $\frac{1}{K_o}$ ohms.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farads.

The remaining terms, i.e., $\frac{K_i s}{s+\sigma_i}$ represent the admittance of series combination of resistor R_i and capacitor C_i with $R_i = \frac{1}{K_i}$ ohms and $C_i = \frac{K_i}{\sigma_i}$ farads.

Table 16.4 *Realisation of Foster II form of RC network*

Admittance function	Element
$K_0 = \frac{1}{R_0}$	$R_o = \frac{1}{K_o}$
$\frac{K_i}{s+\omega_i} = \frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_i C_i}}$	$R_i = \frac{1}{K_i}$ $C_i = \frac{K_i}{\sigma_i}$
$K_\infty s = C_\infty s$	$C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 16.39.

Fig. 16.39 *Foster II form of RC network*

16.6.2 Cauer Realisation

Cauer I Form The Cauer I form is obtained by removal of the pole from the impedance function $Z(s)$ at $s = \infty$. This is the same as a continued fraction expansion of the impedance function about infinity. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = R_1 + \cfrac{1}{C_2 s + \cfrac{1}{R_3 + \cfrac{1}{C_4 s + \ldots}}}$$

The network is shown in Fig. 16.40.

In the network shown in Fig. 16.40, if $Z(s)$ has a zero at $s = \infty$, the first element is the capacitor C_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a pole at $s = 0$, the last element is C_n. If $Z(s)$ is a constant at $s = 0$, the last element is R_n.

Fig. 16.40 *Cauer-I form of RC network*

Cauer II Form The Cauer II form is obtained by removal of the pole from the impedance function at the origin. This is the same as a continued fraction expansion of an impedance function about the origin.

If the given impedance function has a pole at the origin, it is removed as a capacitor C_1. The reciprocal of the remainder function has a minimum value at $s = 0$ which is removed as a constant of resistor R_2. If the original impedance has no pole at the origin, then the first capacitor is absent and the process is repeated with the removal of the constant corresponding to the resistor R_2.

The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{R_2} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{R_4} + \ldots}}}$$

The network is shown in Fig. 16.41.

Fig. 16.41 *Caver-II form of RC network*

In the network shown in Fig. 16.41, if $Z(s)$ has a pole at $s = 0$, the first element is C_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_2. If $Z(s)$ has a zero at $s = \infty$, the last element is C_n. If $Z(s)$ is constant at $s = \infty$, the last element is R_n.

Example 16.48 *Determine whether following functions are RC impedance function or not.*

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$ *(b)* $\dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

Solution

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

The function Z(s) has poles at $s = 0$ and $s = -3$ and zeros at $s = -2$ and $s = -4$ as shown in Fig. 16.42. The poles and zeros are simple and located on the negative real axis of the s plane. The poles and zeros are interlaced. The lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC impedance function.

Fig. 16.42

(b) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

The function $Z(s)$ has poles at $s = -2$ and $s = -6$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 16.43. The poles and zeros are simple and located on the negative real axis of the s-plane. The poles and zeros are interlaced. But the lowest critical frequency nearest to the origin is not a pole, but zero. Hence, the function $Z(s)$ is not an RC impedance function.

Fig. 16.43

Example 16.49 *Realise the Foster and Cauer forms of the impedance function*

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 16.44.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the negative real axis. The poles and zeros are interlaced and the lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC function.

Fig. 16.44

Foster I Form The Foster I form is obtained by partial fraction expansion of impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{s^2+4s+3}{s^2+2s}$$

$$\begin{array}{r|l} s^2+2s & s^2+4s+3 \quad (1 \\ & \underline{s^2+2s} \\ & \quad 2s+3 \end{array}$$

$$Z(s) = 1 + \frac{2s+3}{s^2+2s} = 1 + \frac{2s+3}{s(s+2)}$$

By partial-fraction expansion,

$$Z(s) = 1 + \frac{K_1}{s} + \frac{K_2}{s+2}$$

where

$$K_1 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_2 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+1)(-2+3)}{-2} = \frac{1}{2}$$

$$Z(s) = 1 + \frac{\frac{3}{2}}{s} + \frac{\frac{1}{2}}{s+2}$$

The first term represents the impedance of a resistor of 1 Ω. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of parallel RC circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R = \frac{1}{4}\ \Omega$$

$$C = 2\ \text{F}$$

The network is shown in Fig. 16.45.

Fig. 16.45

Foster II Form The Foster II form is obtained by the partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s+2)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{s+2}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)}{(-3+1)} = \frac{1}{2}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}$$

$$Y(s) = \frac{\frac{1}{2}s}{s+1} + \frac{\frac{1}{2}s}{s+3}$$

These two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 2\ \Omega, \quad C_1 = \frac{1}{2}\ \text{F}$$

$$R_2 = 2\ \Omega, \quad C_2 = \frac{1}{6}\ \text{F}$$

The network is shown in Fig. 16.46.

Fig. 16.46

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{s^2+4s+3}{s^2+2s}$$

By continued fraction expansion,

$$s^2+2s\,\Big)\,s^2+4s+3\,\Big(\,1 \leftarrow Z$$

$$\underline{s^2+2s}$$

$$2s+3\,\Big)\,s^2+2s\,\Big(\,\frac{1}{2}s \leftarrow Y$$

$$
\begin{array}{l}
\underline{s^2 + \frac{3}{2}s} \\
\quad \frac{1}{2}s \,\big)\, 2s + 3 \,\big(\, 4 \leftarrow Z \\
\qquad \underline{2s} \\
\qquad\quad 3 \,\big)\, \frac{1}{2}s \,\big(\, \frac{1}{6}s \leftarrow Y \\
\qquad\qquad \underline{\frac{1}{2}s} \\
\qquad\qquad\quad 0
\end{array}
$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.47.

Fig. 16.47

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s) = \frac{3 + 4s + s^2}{2s + s^2}$$

By continued fraction expansion,

$$
\begin{array}{l}
2s + s^2 \,\big)\, 3 + 4s + s^2 \,\big(\, \frac{3}{2s} \leftarrow Z \\
\quad \underline{3 + \frac{3}{2}s} \\
\qquad \frac{5}{2}s + s^2 \,\big)\, 2s + s^2 \,\big(\, \frac{4}{5} \leftarrow Y \\
\qquad\quad \underline{2s + \frac{4}{5}s^2} \\
\qquad\qquad \frac{1}{5}s^2 \,\big)\, \frac{5}{2}s + s^2 \,\big(\, \frac{25}{2s} \leftarrow Z \\
\qquad\qquad\quad \underline{\frac{5}{2}s} \\
\qquad\qquad\qquad s^2 \,\big)\, \frac{1}{5}s^2 \,\big(\, \frac{1}{5} \leftarrow Y \\
\qquad\qquad\qquad\quad \underline{\frac{1}{5}s^2} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.48.

Fig. 16.48

Example 16.50 *Determine the Foster form of realisation of the RC impedance function.*

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where

$$K_0 = sZ(s)\big|_{s=0} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+1)(-2+3)}{(-2)(-2+4)} = \frac{(-1)(1)}{(-2)(2)} = \frac{1}{4}$$

$$K_2 = (s+4)Z(s)\big|_{s=-4} = \frac{(-4+1)(-4+3)}{(-4)(-4+2)} = \frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}$$

$$Z(s) = \frac{\frac{3}{8}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{3}{8}}{s+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The remaining terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = \frac{1}{8}\,\Omega, \qquad C_1 = 4 \text{ F}$$

$$R_2 = \frac{3}{32}\,\Omega, \quad C_2 = \frac{8}{3} \text{ F}$$

Fig. 16.49

The network is shown in Fig. 16.49.

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{s^2+6s+8}{s^2+4s+3}$$

Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$s^2+4s+3\big)\ s^2+6s+8\big(1$$

$$\underline{s^2+4s+3}$$

$$2s+5$$

$$\frac{Y(s)}{s}=1+\frac{2s+5}{s^2+4s+3}=1+\frac{2s+5}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s}=1+\frac{K_1}{s+1}+\frac{K_2}{s+3}$$

where

$$K_1=(s+1)\frac{Y(s)}{s}\bigg|_{s=-1}=\frac{(-1+2)(-1+4)}{(-1+3)}=\frac{(1)(3)}{2}=\frac{3}{2}$$

$$K_2=(s+3)\frac{Y(s)}{s}\bigg|_{s=-3}=\frac{(-3+2)(-3+4)}{(-3+1)}=\frac{(-1)(1)}{(-2)}=\frac{1}{2}$$

$$\frac{Y(s)}{s}=1+\frac{\frac{3}{2}}{s+1}+\frac{\frac{1}{2}}{s+3}$$

$$Y(s)=s+\frac{\frac{3}{2}s}{s+1}+\frac{\frac{1}{2}s}{s+3}$$

The first term represents the admittance of a capacitor of 1 F. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s)=\frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1=\frac{2}{3}\,\Omega,\quad C_1=\frac{3}{2}\,\text{F}$$

$$R_2=2\,\Omega,\quad C_2=\frac{1}{6}\,\text{F}$$

The network is shown in Fig. 16.50.

Fig. 16.50

Example 16.51 *Realise Foster forms of the following RC impedance function*

$$Z(s)=\frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$Z(s) = \frac{2s^2 + 12s + 16}{s^2 + 4s + 3}$$

$$s^2 + 4s + 3 \overline{\big)\, 2s^2 + 12s + 16 \,\big(}\; 2$$

$$\underline{2s^2 + 8s + 6}$$

$$4s + 10$$

$$Z(s) = 2 + \frac{4s + 10}{s^2 + 4s + 3} = 2 + \frac{4s + 10}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$Z(s) = 2 + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)Z(s)\big|_{s=-1} = \frac{2(-1+2)(-1+4)}{(-1+3)} = 3$$

$$K_2 = (s+3)Z(s)\big|_{s=-3} = \frac{2(-3+2)(-3+4)}{(-3+1)} = 1$$

$$Z(s) = 2 + \frac{3}{s+1} + \frac{1}{s+3}$$

The first term represents the impedance of a resistor of 2 Ω. The remaining terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\dfrac{1}{C_i}}{s + \dfrac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 3\ \Omega, \quad C_1 = \frac{1}{3}\ \text{F}$$

$$R_2 = \frac{1}{3}\ \Omega, \quad C_2 = 1\ \text{F}$$

The network is shown in Fig. 16.51.

Fig. 16.51

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\dfrac{Y(s)}{s}$.

$$Y(s) = \frac{(s+1)(s+3)}{2(s+2)(s+4)}$$

$$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{2s(s+2)(s+4)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where

$$K_0 = s\frac{Y(s)}{s}\bigg|_{s=0} = \frac{(1)(3)}{(2)(2)(4)} = \frac{3}{16}$$

$$K_1 = (s+2)\frac{Y(s)}{s}\bigg|_{s=-2} = \frac{(-2+1)(-2+3)}{2(-2)(-2+4)} = \frac{(-1)(1)}{2(-2)(2)} = \frac{1}{8}$$

$$K_2 = (s+4)\frac{Y(s)}{s}\bigg|_{s=-4} = \frac{(-4+1)(-4+3)}{2(-4)(-4+2)} = \frac{(-3)(-1)}{2(-4)(-2)} = \frac{3}{16}$$

$$\frac{Y(s)}{s} = \frac{\frac{3}{16}}{s} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{16}}{s+4}$$

$$Y(s) = \frac{3}{16} + \frac{\frac{1}{8}s}{s+2} + \frac{\frac{3}{16}s}{s+4}$$

The first term represents the admittance of a resistor of $\frac{16}{3}\,\Omega$. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1 = 8\,\Omega, \qquad C_1 = \frac{1}{16}\text{ F}$$

$$R_2 = \frac{16}{3}\,\Omega, \quad C_2 = \frac{3}{64}\text{ F}$$

The network is shown in Fig. 16.52.

Fig. 16.52

Example 16.52 *Obtain the Cauer forms of the RC impedance function*

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\,\big)\,s^2+8s+12\,\Big(\frac{1}{2} \leftarrow Z$$

$$\underline{s^2+4s+3}$$

$$4s+9\,\big)\,2s^2+8s+6\,\Big(\frac{1}{2}s \leftarrow Y$$

$$\underline{2s^2+\frac{9}{2}s}$$

$$\frac{7}{2}s+6\,\big)\,4s+9\,\Big(\frac{8}{7} \leftarrow Z$$

$$\underline{4s+\frac{48}{7}}$$

$$\frac{15}{7}\Big)\frac{7}{2}s+6\Big(\frac{49}{30}s \leftarrow Y$$

$$\underline{\frac{7}{2}s}$$

$$6\Big)\frac{15}{7}\Big(\frac{5}{14} \leftarrow Z$$

$$\underline{\frac{15}{7}}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.53.

Fig. 16.53

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s)=\frac{12+8s+s^2}{6+8s+2s^2}$$

By continued fraction expansion,

$$6+8s+2s^2\Big)12+8s+s^2\Big(2$$

$$\underline{12+16s+4s^2}$$

$$-8s-3s^2$$

Since negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\Big)6+8s+2s^2\Big(\frac{1}{2} \leftarrow Y$$

$$\underline{6+4s+\frac{1}{2}s^2}$$

$$4s+\frac{3}{2}s^2\Big)12+8s+s^2\Big(\frac{3}{s} \leftarrow Z$$

$$\underline{12+\frac{9}{2}s}$$

$$\frac{7}{2}s+s^2\Big)4s+\frac{3}{2}+s^2\Big(\frac{8}{7} \leftarrow Y$$

$$\underline{4s+\frac{8}{7}s^2}$$

$$\frac{5}{14}s^2\Big)\frac{7}{2}s+s^2\Big(\frac{49}{5s} \leftarrow Z$$

$$\frac{7}{2}s$$

$$s^2\Big)\frac{5}{14}s^2\Big(\frac{5}{14}\leftarrow Y$$

$$\frac{5}{14}s$$

$$0$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.54.

Fig. 16.54

Example 16.53 *Realise the RC impedance in Cauer I and Foster I forms*

$$Z(s)=\frac{s+4}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates presence of a zero at infinity. Hence, the admittance function $Y(s)$ has a pole at infinity.

$$Y(s)=\frac{s^2+8s+12}{s+4}$$

By continued fraction expansion,

$$s+4\Big)s^2+8s+12\Big(s\leftarrow Y$$

$$s^2+4s$$

$$4s+12\Big)s+4\Big(\frac{1}{4}\leftarrow Z$$

$$s+3$$

$$1\Big)4s+12\Big(4s\leftarrow Y$$

$$4s$$

$$12\Big)1\Big(\frac{1}{12}\leftarrow Z$$

$$\frac{1}{0}$$

The impedances are connected in series branches, whereas the admittances are connected in parallel branches. The network is shown in Fig. 16.55.

Fig. 16.55

Foster I Form The Foster I form is obtained by partial fraction expansion of $Z(s)$.

$$Z(s)=\frac{s+4}{(s+2)(s+6)}$$

By partial-fraction expansion,

$$Z(s)=\frac{K_1}{s+2}+\frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+4)}{(-2+6)} = \frac{1}{2}$$
$$K_2 = (s+6)Z(s)\big|_{s=-6} = \frac{(-6+4)}{(-6+2)} = \frac{1}{2}$$
$$Z(s) = \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s+6}$$

These two terms represent the impedance of a parallel *RC* circuit for which
$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,
$$R_1 = \frac{1}{4}\,\Omega, \quad C_1 = 2\text{ F}$$
$$R_2 = \frac{1}{12}\,\Omega, \quad C_2 = 2\text{ F}$$

The network is shown in Fig. 16.56.

Fig. 16.56

Example 16.54 *The RC driving-point impedance function is given as* $Z(s) = H\dfrac{(s+1)(s+4)}{s(s+3)}$ *Realise the impedance function in the ladder form, given* $Z(-2) = 1$.

Solution Putting $s = -2$,
$$Z(-2) = H\frac{(-2+1)(-2+4)}{(-2)(-2+3)} = H$$
$$H = 1$$
$$Z(s) = \frac{(s+1)(s+4)}{s(s+3)}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

By continued fraction expansion of $Z(s)$,
$$s^2+3s\,\big)\,s^2+5s+4\,\big(\,1 \leftarrow Z$$
$$\underline{s^2+3s}$$
$$2s+4\,\big)\,s^2+3s\,\big(\,\frac{1}{2}s \leftarrow Y$$
$$\underline{s^2+2s}$$
$$s\,\big)\,2s+4\,\big(\,2 \leftarrow Z$$
$$\underline{2s}$$
$$4\,\big)\,s\,\big(\,\frac{1}{4}s \leftarrow Y$$
$$\underline{s}$$
$$0$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.57.

Fig. 16.57

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s) = \frac{4 + 5s + s^2}{3s + s^2}$$

By continued fraction expansion,

$$3s + s^2 \,\Big)\, 4 + 5s + s^2 \,\Big(\, \frac{4}{3s} \leftarrow Z$$

$$\underline{4 + \frac{4}{3}s}$$

$$\frac{11}{3}s + s^2 \,\Big)\, 3s + s^2 \,\Big(\, \frac{9}{11} \leftarrow Y$$

$$\underline{3s + \frac{9}{11}s^2}$$

$$\frac{2}{11}s^2 \,\Big)\, \frac{11}{3}s + s^2 \,\Big(\, \frac{121}{6s} \leftarrow Z$$

$$\underline{\frac{11}{3}s}$$

$$s^2 \,\Big)\, \frac{2}{11}s^2 \,\Big(\, \frac{2}{11} \leftarrow Y$$

$$\underline{\frac{2}{11}s^2}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.58.

Fig. 16.58

Example 16.55 *An impedance function has the pole-zero diagram as shown in Fig. 16.59. Find the impedance function such that $Z(-4) = \frac{3}{4}$ and realise in Cauer I and Foster II forms.*

Fig. 16.59

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$.

$$Z(s) = H\frac{(s+1)(s+3)}{s(s+2)}$$

Putting $s = -4$,

$$Z(-4) = H\frac{(-4+1)(-4+3)}{(-4)(-4+2)} = H\frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}\text{H}$$

$$\frac{3}{4} = \frac{3}{8}\text{H}$$

$$H = 2$$

$$Z(s) = \frac{2(s+1)(s+3)}{s(s+2)} = \frac{2s^2+8s+6}{s^2+2s}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

By continued fraction expansion of $Z(s)$,

$$s^2+2s\,\big)\,2s^2+8s+6\,\big(\,2 \leftarrow Z$$

$$\underline{2s^2+4s}$$

$$4s+6\,\big)\,s^2+2s\,\big(\,\frac{1}{4}s \leftarrow Y$$

$$\underline{s^2+\frac{3}{2}s}$$

$$\frac{1}{2}s\,\big)\,4s+6\,\big(\,8 \leftarrow Z$$

$$\underline{4s}$$

$$6\,\big)\,\frac{1}{2}s\,\big(\,\frac{1}{12}s \leftarrow Y$$

$$\underline{\frac{1}{2}s}$$

$$0$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.60.

Fig. 16.60

Foster II Form The Foster II form is obtained by partial fraction expansion of $\frac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{s+2}{2(s+1)(s+3)}$$

By partial fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)}{2(-3+1)} = \frac{-1}{2(-2)} = \frac{1}{4}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{4}}{s+1} + \frac{\frac{1}{4}}{s+3}$$

$$Y(s) = \frac{\frac{1}{4}s}{s+1} + \frac{\frac{1}{4}s}{s+3}$$

Two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 4\ \Omega, \quad C_1 = \frac{1}{4}\ \text{F}$$

$$R_2 = 4\ \Omega, \quad C_2 = \frac{1}{12}\ \text{F}$$

Fig. 16.61

The network is shown in Fig. 16.61.

16.7 REALISATION OF *RL* FUNCTIONS

RL driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency is a zero which may be at $s = 0$.
4. The highest critical frequency is a pole which may be at infinity.
5. Residues evaluated at the poles of $Z_{RL}(s)$ are real and negative while that of $\frac{Z_{RL}(s)}{s}$ are real and positive.
6. The slope $\frac{d}{d\sigma} Z_{RL}$ is positive.
7. $Z_{RL}(0) < Z_{RL}(\infty)$.

The admittance of an inductor is similar to the impedance of a capacitor. Hence, properties of an *RL* admittance are identical to those of an *RC* impedance and vice-versa, i.e.,

$$Z_{RC}(s) = Y_{RL}(s)$$
$$Z_{RL}(s) = Y_{RC}(s)$$

An *RL* admittance can be considered as the dual of an *RC* impedance and vice-versa.

Example 16.56 *Indicate which of the following functions are either RL, RC or LC impedance functions.*

(a) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$ *(b)* $Z(s) = \dfrac{s(s+4)(s+8)}{(s+1)(s+6)}$

(c) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$ *(d)* $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+16)}$

Solution (a) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(b) $Z(s) = \dfrac{s(s+4)(s+8)}{(s+1)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

(c) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(d) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

Example 16.57 *Realise following RL impedance function in Foster-I and Foster-II form.*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{2(-2+1)(-2+3)}{(-2+6)} = -\frac{1}{2}$$
$$K_2 = (s+6)Z(s)\big|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6+2)} = -\frac{15}{2}$$

Since residues of $Z(s)$ are negative, partial fraction expansion of $\frac{Z(s)}{s}$ is carried out.

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

By partial fraction expansion,

$$\frac{Z(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_0 = s\frac{Z(s)}{s}\bigg|_{s=0} = \frac{2(1)(3)}{(2)(6)} = \frac{1}{2}$$
$$K_1 = (s+2)\frac{Z(s)}{s}\bigg|_{s=-2} = \frac{2(-2+1)(-2+3)}{(2)(-2+6)} = \frac{1}{4}$$
$$K_2 = (s+6)\frac{Z(s)}{s}\bigg|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6)(-6+2)} = \frac{5}{4}$$

$$\frac{Z(s)}{s} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{5}{4}}{s+6}$$
$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4}s}{s+2} + \frac{\frac{5}{4}s}{s+6}$$

The first term represents the impedance of the resistor of $\frac{1}{2}\,\Omega$. The other two terms represent the impedance of the parallel RL circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \quad L_1 = \frac{1}{8}\,\text{H}$$
$$R_2 = \frac{5}{4}\,\Omega, \quad L_2 = \frac{5}{24}\,\text{H}$$

Fig. 16.62

The network is shown in Fig. 16.62.

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

$$2s^2+8s+6\,\Big)\,s^2+8s+12\,\Big(\,\frac{1}{2}$$

$$\frac{s^2+4s+3}{4s+9}$$

$$Y(s)=\frac{1}{2}+\frac{4s+9}{2s^2+8s+6}=\frac{1}{2}+\frac{4s+9}{2(s+1)(s+3)}$$

By partial-fraction expansion,

$$Y_1(s)=\frac{4s+9}{2(s+1)(s+3)}=\frac{K_0}{s+1}+\frac{K_1}{s+3}$$

where

$$K_0=(s+1)Y_1(s)\big|_{s=-1}=\frac{(-4+9)}{2(-1+3)}=\frac{5}{4}$$

$$K_1=(s+3)Y_1(s)\big|_{s=-3}=\frac{(-12+9)}{2(-3+1)}=\frac{3}{4}$$

$$Y(s)=\frac{1}{2}+\frac{\frac{5}{4}}{s+1}+\frac{\frac{3}{4}}{s+3}$$

The first term represents the admittance of a resistor of 2 Ω. The other two terms represent the admittance of a series *RL* circuit. For a series *RL* circuit,

$$Y_{RL}(s)=\frac{\frac{1}{L_i}}{s+\frac{R_i}{L_i}}$$

By direct comparison,

$$R_1=\frac{4}{5}\,\Omega,\quad L_1=\frac{4}{5}\,\text{H}$$

$$R_2=4\,\Omega,\quad L_2=\frac{4}{3}\,\text{H}$$

The network is shown in Fig. 16.63.

Fig. 16.63

Example 16.58 *Find the Foster forms of the RL impedance function:*

$$Z(s)=\frac{(s+1)(s+4)}{(s+5)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of impedance function $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s}=\frac{(s+1)(s+4)}{s(s+5)(s+3)}$$

By partial-faction expansion,

$$\frac{Z(s)}{s}=\frac{K_0}{s}+\frac{K_1}{s+3}+\frac{K_2}{s+5}$$

where

$$K_0=s\frac{Z(s)}{s}\bigg|_{s=0}=\frac{(1)(4)}{(5)(3)}=\frac{4}{15}$$

$$K_1 = (s+3)\frac{Z(s)}{s}\bigg|_{s=-3} = \frac{(-3+1)(-3+4)}{(-3)(-3+5)} = \frac{(-2)(1)}{(-3)(2)} = \frac{1}{3}$$

$$K_2 = (s+5)\frac{Z(s)}{s}\bigg|_{s=-5} = \frac{(-5+1)(-5+4)}{(-5)(-5+3)} = \frac{(-4)(-1)}{(-5)(-2)} = \frac{2}{5}$$

$$\frac{Z(s)}{s} = \frac{\frac{4}{15}}{s} + \frac{\frac{1}{3}}{s+3} + \frac{\frac{2}{5}}{s+5}$$

$$Z(s) = \frac{4}{15} + \frac{\frac{1}{3}s}{s+3} + \frac{\frac{2}{5}s}{s+5}$$

The first term represents the impedance of the resistor of $\frac{4}{15}\,\Omega$. The other two terms represent the impedance of a parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{3}\,\Omega, \quad L_1 = \frac{1}{9}\,\text{H}$$

$$R_2 = \frac{2}{5}\,\Omega, \quad L_2 = \frac{2}{25}\,\text{H}$$

The network is shown in Fig. 16.64.

Fig. 16.64

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+5)(s+3)}{(s+1)(s+4)} = \frac{s^2+8s+15}{s^2+5s+4}$$

$$s^2+5s+4)s^2+8s+15(1$$

$$\underline{s^2+5s+4}$$

$$3s+11$$

$$Y(s) = 1 + \frac{3s+11}{(s+1)(s+4)}$$

By partial-fraction expansion,

$$Y_1(s) = \frac{K_0}{s+1} + \frac{K_1}{s+4}$$

where

$$K_0 = (s+1)Y_1(s)\big|_{s=-1} = \frac{(-3+11)}{(-1+4)} = \frac{8}{3}$$

$$K_1 = (s+4)Y_1(s)\big|_{s=-4} = \frac{(-12+11)}{(-4+1)} = \frac{1}{3}$$

$$Y_1(s) = \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

$$Y(s) = 1 + \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

The first term represents the admittance of a resistor of 1 Ω. The other two terms represent the admittance of a series *RL* circuit.

For a series *RL* circuit,

$$Y_{RL}(s) = \frac{\dfrac{1}{L_i}}{s + \dfrac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{3}{8}\ \Omega, \quad L_1 = \frac{3}{8}\ \text{H}$$

$$R_2 = 12\ \Omega, \quad L_2 = 3\ \text{H}$$

The network is shown in Fig. 16.65.

Fig. 16.65

Example 16.59 *Find the Cauer forms of the RL impedance function:*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by a continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)} = \frac{2s^2+8s+6}{s^2+8s+12}$$

By continued fraction expansion,

$$s^2+8s+12)2s^2+8s+6(2 \leftarrow Z$$

$$\underline{2s^2+16s+24}$$

$$-8s-18$$

Since a negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\Big)s^2+8s+12\Big(\frac{1}{2} \leftarrow Y$$

$$\underline{s^2+4s+3}$$

$$4s+9\Big)2s^2+8s+6\Big(\frac{1}{2}s \leftarrow Z$$

$$\underline{2s^2+\frac{9}{2}s}$$

$$\frac{7}{2}s+6\Big)4s+9\Big(\frac{8}{7} \leftarrow Y$$

$$\underline{4s+\frac{48}{7}}$$

$$\frac{15}{7}\Big)\frac{7}{2}s+6\Big(\frac{49}{30}s \leftarrow Z$$

$$\underline{\frac{7}{2}s}$$

$$6\Big)\frac{15}{7}\Big(\frac{15}{42} \leftarrow Y$$

$$\underline{\frac{15}{7}}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.66.

Fig. 16.66

Cauer II Form The Cauer II form is obtained from a continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\Big)6+8s+2s^2\Big(\frac{1}{2} \leftarrow Z$$

$$\underline{6+4s+\frac{1}{2}s^2}$$

$$4s+\frac{3}{2}s^2\Big)12+8s+s^2\Big(\frac{3}{s} \leftarrow Y$$

$$\underline{12+\frac{9}{2}s}$$

$$\frac{7}{2}s+s^2\Big)4s+\frac{3}{2}s^2\Big(\frac{8}{7} \leftarrow Z$$

$$\underline{4s+\frac{8}{7}s^2}$$

$$\frac{5}{14}s^2\Big)\frac{7}{2}s+s^2\Big(\frac{98}{10s} \leftarrow Y$$

$$\underline{\frac{7}{2}s}$$

$$s^2\Big)\frac{5}{14}s^2\Big(\frac{5}{14} \leftarrow Z$$

$$\underline{\frac{5}{14}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.67.

Fig. 16.67

Example 16.60 *Obtain the Foster I and Cauer I forms of the RL impedance function.*

$$Z(s) = \frac{s(s+4)(s+8)}{(s+1)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial fraction expansion of $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s} = \frac{(s+4)(s+8)}{(s+1)(s+6)}$$

Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$s^2 + 7s + 6)s^2 + 12s + 32(1$$

$$\frac{s^2 + 7s + 6}{5s + 26}$$

$$\frac{Z(s)}{s} = 1 + \frac{5s+26}{s^2+7s+6} = 1 + \frac{5s+26}{(s+1)(s+6)}$$

By partial-fraction expansion,

$$\frac{Z(s)}{s} = 1 + \frac{K_0}{s+1} + \frac{K_1}{s+6}$$

where

$$K_0 = \left.\frac{5s+26}{s+6}\right|_{s=-1} = \frac{-5+26}{-1+6} = \frac{21}{5}$$

$$K_1 = \left.\frac{5s+26}{s+1}\right|_{s=-6} = \frac{-30+26}{-6+1} = \frac{4}{5}$$

$$\frac{Z(s)}{s} = 1 + \frac{\frac{21}{5}}{s+1} + \frac{\frac{24}{5}}{s+6}$$

$$Z(s) = s + \frac{\frac{21}{5}s}{s+1} + \frac{\frac{4}{5}s}{s+6}$$

The first term represents the impedance of the inductor of 1 H. The other two terms represent the impedance of a parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \dfrac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{21}{5}\,\Omega, \quad L_1 = \frac{21}{5}\,\text{H}$$

$$R_2 = \frac{4}{5}\,\Omega, \quad L_2 = \frac{4}{30}\,\text{H}$$

Fig. 16.68

The network is shown in Fig. 16.68.

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{s^3 + 12s^2 + 32s}{s^2 + 7s + 6}$$

By continued fraction expansion,

$$s^2 + 7s + 6 \,\big)\, s^3 + 12s^2 + 32s \,\big(\, s \leftarrow Z$$

$$\underline{s^3 + 7s^2 + 6s}$$

$$5s^2 + 26s \,\big)\, s^2 + 7s + 6 \,\big(\, \frac{1}{5} \leftarrow Y$$

$$\underline{s^2 + \frac{26}{5}s}$$

$$\frac{9}{5}s + 6 \,\big)\, 5s^2 + 26s \,\big(\, \frac{25}{9}s \leftarrow Z$$

$$\underline{5s^2 + \frac{50}{3}s}$$

$$\frac{28}{3}s \,\big)\, \frac{9}{5}s + 6 \,\big(\, \frac{27}{140} \leftarrow Y$$

$$\underline{\frac{9}{5}s}$$

$$6 \,\big)\, \frac{28}{3}s \,\big(\, \frac{28}{18}s \leftarrow Z$$

$$\underline{\frac{28}{3}s}$$

$$0$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.69.

Fig. 16.69

Exercises

16.1 Test the following polynomials for Hurwitz property:

(i) $s^3 + s^2 + 2s + 2$
(ii) $s^4 + s^2 + s + 1$
(iii) $s^3 + 4s^2 + 5s + 2$
(iv) $s^4 + 7s^3 + 6s^2 + 21s + 8$
(v) $s^4 + s^3 + s + 1$
(vi) $s^7 + 3s^6 + 8s^5 + 15s^4 + 17s^3 + 12s^2 + 4s$
(vii) $s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$
(viii) $s^7 + 3s^5 + 2s^3 + s$
(ix) $s^5 + 2s^3 + s$
(x) $s^3 + 2s^2 + 4s + 2$
(xi) $s^4 + s^3 + 4s^2 + 2s + 3$
(xii) $s^5 + 8s^4 + 24s^3 + 28s^2 + 23s + 6$
(xiii) $s^7 - 2s^6 + 2s^5 + 9s^2 + 8s + 4$
(xiv) $s^7 + 3s^5 + 2s^3 + 3$
(xv) $s^5 + s^3 + s$
(xvi) $s^6 + 7s^4 + 5s^3 + s^2 + s$
(xvii) $s^4 + s^3 + 2s^2 + 3s + 2$

16.2 Determine whether the following functions are positive real:

(i) $\dfrac{s^3 + 5s}{s^4 + 2s^2 + 1}$

(ii) $\dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$

(iii) $\dfrac{2s^2 + 2s + 1}{s^3 + 2s^2 + s + 2}$

(iv) $\dfrac{s^4 + 3s^3 + s^2 + s + 2}{s^3 + s^2 + s + 1}$

(v) $\dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$

(vi) $\dfrac{s^2 + 2s + 1}{s^2 + 4s + 4}$

(vii) $\dfrac{s^3 + 2s^2 + 3s + 1}{s^3 + 2s^2 + s + 2}$

(viii) $\dfrac{s^3 + 2s^2 + s + 1}{s^2 + s + 1}$

(ix) $\dfrac{s + 4}{s^2 + 2s + 1}$

(x) $\dfrac{s^2 + 4s + 3}{s^2 + 6s + 8}$

(xi) $\dfrac{s^2 + 1}{s^3 + 4s}$

(xii) $\dfrac{s^4 + 2s^3 + 3s^2 + 1}{s^4 + s^3 + 3s^2 + 2s + 1}$

(xiii) $\dfrac{s^2 + 2s + 4}{(s+1)(s+3)}$

(xiv) $\dfrac{2s + 4}{s + 5}$

(xv) $\dfrac{s^2 + 2s}{s^2 + 1}$

(xvi) $\dfrac{s^2 + 4}{s^3 + 3s^2 + 3s + 1}$

16.3 Determine whether the following functions are *LC, RC* or *RL* function:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(iii) $Z(s) = \dfrac{(s+1)(s+8)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{Ks(s^2+4)}{(s^2+1)(s^2+3)}$

(v) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

(vi) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

(vii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(viii) $F(s) = \dfrac{(s+1)(s+2)}{s(s+3)}$

(ix) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(x) $Z(s) = \dfrac{(s+2)(s+4)}{(s+1)}$

(xi) $Y(s) = \dfrac{4(s+3)}{(s+1)(s+5)}$

(xii) $Y(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(xiii) $Z(s) = \dfrac{s(s^2+4)(s^2+16)}{(s^2+9)(s^2+25)}$

(xiv) $Z(s) = \dfrac{(s^2+1)(s^2+8)}{s(s^2+4)}$

16.4 Realise the following functions in Foster I form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $F(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(v) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

(vi) $Y(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

(vii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

16.5 Realise the following functions in Foster II form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $Z(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Y(s) = \dfrac{4(s^2+4)(s^2+25)}{s(s^2+16)}$

(v) $Z(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

16.6 Realise the following functions in Cauer I form:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(iii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(v) $F(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(vi) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{6(s+2)(s+4)}{s(s+3)}$

(viii) $Z(s) = \dfrac{s^3+2s}{s^4+4s^2+3}$

(ix) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(x) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

16.7 Realise the following function in Cauer II form:

(i) $F(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(ii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iv) $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(v) $F(s) = \dfrac{s^3+12s^2+32s}{s^2+7s+6}$

(vi) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(viii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

16.8 An impedance function has the pole-zero diagram as shown in Fig. 16.70 below. If $Z(-2) = 3$, synthesise the impedance function in Foster and Cauer forms.

Fig. 16.70

16.9 An impedance function has the pole-zero diagram as shown in Fig. 16.71. Find the impedance function such that $Z(-4) = \dfrac{8}{3}$ and realise in Cauer forms.

Fig. 16.71

16.10 For the realisation of a given function $F(s)$.

$$F(s) = \frac{K_0}{s} + \sum_{i=1}^{n} \frac{sK_i}{(s^2+\omega_i^2)} + sK_\infty$$

where K_0, K_i (i = 1, 2, 3, … n) and K_∞ are constants.

(i) Mention the type of function (*RC*, *RL* or *LC*)

(ii) Given that $K_0 = 6$, $K_1 = 8$, $\omega_1 = 4$, $K_2 = 10$, $\omega_2 = 8$, $K_\infty = 5$, find the component values of realised network for $F(s) = Z(s)$ and $F(s) = Y(s)$. Draw neat diagrams.

Objective-Type Questions

16.1 The necessary and sufficient condition for a rational function $F(s)$ to be the driving-point impedance of an *RC* network is that all poles and zeros should be

(a) simple and lie on the negative real axis in the s-plane
(b) complex and lie in the left half of s-plane
(c) complex and lie in the right-half of s-plane
(d) simple and lie on the positive real axis of the s-plane

16.2 The number of roots of $s^3 + 5s^2 + 7s + 3 = 0$ in the left half of s-plane is

(a) zero (b) one
(c) two (d) three

16.3 The first and the last critical frequencies of a driving-point impedance function of a passive network having two kinds of elements, are a pole and a zero respectively. The above property will be satisfied by

(a) *RL* network only
(b) *RC* network only
(c) *LC* network only
(d) *RC* as well as *RL* network

16.4 The pole-zero pattern of a particular network is shown in Fig. 16.72. It is that of an

Fig. 16.72

(a) *LC* network (b) *RC* network
(c) *RL* network (d) none of these

16.5 The first critical frequency nearest to the origin of the complex frequency plane for an *RL* driving-point impedance function will be

(a) a zero in the left-half plane
(b) a zero in the right-half plane
(c) a pole in the left-half plane
(d) a pole in the right-half plane

16.6 Consider the following polynomials:

$P_1 = s^8 + 2s^6 + 4s^4$
$P_2 = s^6 - 3s^2 + 2s^2 + 1$
$P_3 = s^4 + 3s^3 + 3s^2 + 2s + 1$
$P_4 = s^7 + 2s^6 + 2s^4 + 4s^3 + 8s^2 + 8s + 4$

which one of these polymials is not Hurwitz?

(a) P_1 (b) P_2
(c) P_3 (d) P_4

16.7 For very high frequencies, the driving-point admittance function, $Y(s) = \dfrac{4(s+1)(s+3)}{s(s+2)(s+4)}$ behaves as

(a) a resistance of $\dfrac{3}{2}\ \Omega$
(b) a capacitance of 4 F
(c) an inductance of $\dfrac{1}{4}$ H
(d) an inductance of 4 H

16.8 The driving-point impedance $Z(s) = \dfrac{s+3}{s+4}$ behaves as

(a) a resistance of 0.75 Ω at low frequencies
(b) a resistance of 1 Ω at high frequencies
(c) both (a) and (b) above
(d) none of the above

16.9 An *RC* driving-point impedance function has zeros at $s = -2$ and $s = -5$. The admissible poles for the functions would be

(a) $s = 0, s = -6$ (b) $s = -1, s = -3$
(c) $s = 0, s = -1$ (d) $s = -3, s = -4$

16.10 Consider the following from the point of view of possible realisation as driving-point impedances using passive elements:

1. $\dfrac{1}{s(s+5)}$ 2. $\dfrac{s+3}{s^2(s+5)}$

3. $\dfrac{s^2+3}{s^2(s^2+5)}$ 4. $\dfrac{s+5}{s(s+5)}$

Of these, the realisable are

(a) 1, 2 and 4 (b) 1, 2 and 3
(c) 3 and 4 (d) none of these

16.11 The poles and zeros of a driving-point function of a network are simple and interlace on the negative real axis with a pole closest to the origin. It can be realised

(a) by an *LC* network
(b) as an *RC* driving point impedance
(c) as an *RC* driving point admittance
(d) only by an *RLC* network

16.12 If $F_1(s)$ and $F_2(s)$ are two positive real functions then the function which is always positive real, is

(a) $F_1(s)\,F_2(s)$ (b) $\dfrac{F_1(s)}{F_2(s)}$

(c) $\dfrac{F_1(s)F_2(s)}{F_1(s)+F_2(s)}$ (d) $F_1(s) - F_2(s)$

16.13 The circuit shown in Fig. 16.73 is

Fig. 16.73

(a) Cauer I form (b) Foster I form
(c) Cauer II form (d) Foster II form

16.14 For an *RC* driving-point impedance function, the poles and zeros

(a) should alternate on the real axis
(b) should alternate only on the negative real axis
(c) should alternate on the imaginary axis
(d) can lie anywhere on the left half-plane

Answers to Objective-Type Questions

16.1 (a)	16.2 (a)	16.3 (b)	16.4 (a)	16.5 (a)	16.6 (b)	16.7 (c)
16.8 (c)	16.9 (b)	16.10 (a)	16.11 (b)	16.12 (c)	16.13 (b)	16.14 (b)

Appendix: Active Filters

A-1 INTRODUCTION

Filters are frequency-selective networks that pass a specified band of frequencies and block or attenuate signals of frequencies outside this band. There are various advantages of active filters.

(i) Gain and frequency adjustment flexibility
(ii) No loading problem
(iii) More economical than passive filters

There are five types of filters.

(i) Low-Pass Filter (LPF)
(ii) High-Pass Filter (HPF)
(iii) Band-Pass Filter (BPF)
(iv) Band-Stop Filter (BSF)
(v) All Pass Filter

A-2 FIRST-ORDER LOW-PASS BUTTERWORTH FILTERS

Figure A.1 shows a first-order low-pass Butterworth filter.

Fig. A.1

Analysis

Fig. A.2

$$V_1 = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}$$

$$= \frac{1}{RCs + 1} V_{in}$$

For low-pass filter

$$V_0 = A_f V_1$$

$$= \frac{A_f}{RCs + 1} V_{in}$$

$$\frac{V_0}{V_{in}} = \frac{A_f}{RCs + 1}$$

Putting s $= j\omega$,

$$\frac{V_0}{V_{in}} = \frac{A_f}{1 + j\omega RC} = \frac{A_f}{1 + j2\pi fRC}$$

Let $f_H = \dfrac{1}{2\pi RC}$

$$\frac{V_0}{V_{in}} = \frac{A_f}{1 + j\left(\frac{f}{f_H}\right)}$$

where $\dfrac{V_0}{V_{in}}$ is the transfer gain of the filter as a function of frequency

A_f is the pass-band gain of the filter

f is the frequency of the input signal

$f_H = \dfrac{1}{2\pi RC}$ is the higher cut-off frequency of the filter.

$$\left|\frac{V_0}{V_{in}}\right| = \frac{A_f}{\sqrt{1 + \left(\frac{f}{f_H}\right)^2}}$$

The phase angle $\phi = -\tan^{-1}\left(\dfrac{f}{f_H}\right)$

At low frequency, $f < f_H$, $\left|\dfrac{V_0}{V_{in}}\right| \approx A_f$

At $f = f_H$, $\left|\dfrac{V_0}{V_{in}}\right| = \dfrac{A_f}{\sqrt{2}} = 0.707\ A_f$

For $f > f_H$, $\left|\dfrac{V_0}{V_{in}}\right| < A_f$

The low-pass filter has a constant gain A_f from 0 Hz to f_H. At f_H, the gain is 0.707 A_f, thereafter the gain decreases at a constant rate with an increase in frequency. The gain rolls off by 20 dB/decade or 6 dB/octave. The frequency f_H is called the higher cut-off frequency, –3 dB frequency, break frequency, or corner frequency.

Example A.1 *Design an LPF at a cutoff frequency of 1 kHz with a pass-band gain* of *2.*

Solution:

$$f_H = 1 \text{ kHz}$$
$$A_f = 2$$
$$f_H = \frac{1}{2\pi RC}$$

Let $C = 0.01\ \mu\text{F}$

$R = 15.9\ \text{k}\Omega$

$$A_f = 1 + \frac{R_f}{R_1}$$

Let $R_1 = 10\ \text{k}\Omega$

$R_f = 10\ \text{k}\Omega$

A.3 SECOND ORDER LOW-PASS BUTTERWORTH FILTERS

A stop-band response having 40 dB/decade roll-off is obtained with a second-order low-pass filter. A first-order low-pass filter can be converted into a second-order type by adding an additional *RC* network as shown in Fig. A.3.

Fig. A.3

The gain of second-order filter is set by R_1 and R_f while the higher cut-off frequency is determined by R_2, C_2, R_3 and C_3.

Analysis

Fig. A.4

$$\frac{V_A - V_{in}}{R_2} + \frac{V_A - V_0}{\frac{1}{C_2 s}} + \frac{V_A - V_1}{R_3} = 0$$

$$\left(\frac{1}{R_2} + \frac{1}{R_3} + C_2 s\right) V_A - \frac{1}{R_3} V_1 = \frac{V_{in}}{R_2} + C_2 s\, V_0 \qquad \ldots(1)$$

Also,

$$V_1 = \frac{\frac{1}{C_3 s}}{\frac{1}{C_3 s} + R_3} V_A$$

$$= \frac{1}{R_3 C_3 s + 1} V_A$$

$$V_A = (R_3 C_3 s + 1)\, V_1$$

Substituting V_A in Equation (1),

$$\left(\frac{1}{R_2} + \frac{1}{R_3} + C_2 s\right)(R_3 C_3 s + 1)\, V_1 - \frac{1}{R_3} V_1 = \frac{V_{in}}{R_2} + C_2 s\ V_0$$

$$\frac{\left[(R_2 + R_3 R_2 R_3 C_2 s)(R_3 C_3 s H) - R2\right]\ V_1}{R_2 R_3} = \frac{R_3 V_{in} + R_2 R_3 C_2 s V_0}{R_2 R_3}$$

$$V_1 = \frac{R_3 V_{in} + R_3 R_2 C_2 s V_0}{(R_3 C_3 s + 1)(R_2 + R_3 + R_3 R_2 C_2 s) - R_2}$$

$$V_0 = A_f V_1$$

$$V_0 = \frac{A_f [R_3 V_{in} + R_3 R_2 C_2 S V_0]}{(R_3 C_3 S + 1)(R_2 + R_3 + R_3 R_2 C_2 S) - R_2}$$

Simplifying the above equation,

$$\frac{V_0}{V_{in}} = \frac{A_f}{S^2 + \frac{(R_3 C_3 + R_2 C_3 + R_2 C_2 - A_f R_2 C_2)}{R_2 R_3 C_2 C_3} S + \frac{1}{R_2 R_3 C_2 C_3}}$$

For any second-order system,

$$\frac{V_0}{V_{in}} = \frac{A}{s^2 + 2 j \omega_n s + \omega_n^2}$$

For a second-order filter,

$$\omega_n^2 = \omega_H^2 = \frac{1}{R_2 R_3 C_2 C_3}$$

$$\omega_H = \frac{1}{\sqrt{R_2 R_3 C_2 C_3}}$$

$$f_H = \frac{1}{2\pi\sqrt{R_2 R_3 C_2 C_3}}$$

Except for having twice the roll of rate in the stop band, the frequency response of the second-order low-pass filter is identical to that of first-order type and the voltage gain magnitude equation is

$$\left|\frac{V_0}{V_{\text{in}}}\right| = \frac{A_f}{\sqrt{1+\left(\frac{f}{f_H}\right)^4}}$$

A.4 FIRST-ORDER HIGH-PASS BUTTERWORTH FILTERS

High-pass filters are obtained from low-pass filters by interchanging the frequency-determining resistors and capacitors in low-pass filters. The first-order high-pass filter is obtained from a first-order low-pass filter by interchanging R and C as shown in Fig. A.5.

Fig. A.5

Analysis

Fig. A.6

$$V_1 = \frac{R}{R + \frac{1}{Cs}} V_{in}$$

$$= \frac{RCs}{RCs + 1} V_{in}$$

For a high-pass filter,

$$V_0 = A_f V_1$$

$$= \frac{A_f RCs}{RCs + 1} V_{in}$$

$$\frac{V_0}{V_{in}} = \frac{A_f RCs}{RCs + 1}$$

Putting $s = j\omega$,

$$\frac{V_0}{V_{in}} = \frac{A_f j\omega Rc}{j\omega RC + 1}$$

$$= \frac{A_f j2\pi fRC}{j2\pi fRC + 1} = A_f \left[\frac{j\left(\frac{f}{f_L}\right)}{1 + j\left(\frac{f}{f_L}\right)} \right]$$

where $\frac{V_0}{V_{in}}$ is the transfer gain of the filter as a function of frequency.

A_f is the pass-band gain of the filter
f is the frequency of the input signal

$f_L = \frac{1}{2\pi RC}$ is the lower cut-off frequency of the filter

$$\left|\frac{V_0}{V_{in}}\right| = \frac{A_f \left(\frac{f}{f_L}\right)}{\sqrt{1 + \left(\frac{f}{f_L}\right)^2}}$$

The phase angle $\phi = 90° - \tan^{-1}\left(\frac{f}{f_L}\right)$

At low frequency, $f < f_L, \left|\frac{V_0}{V_{in}}\right| < A_f$

At $f = f_L, \left|\frac{V_0}{V_{in}}\right| = \frac{A_f}{\sqrt{2}} = 0.707\ A_f$

For $f > f_L, \left|\frac{V_0}{V_{in}}\right| \approx A_f$

The high-pass filter has a constant gain A_f from f_L to ∞. At f_L, the gain is 0.707 A_f, there after the gain decreases at a constant rate with an decrease in frequency. The gain rolls of by 20 dB/decade or 6 dB/octave. The frequency f_L is called the cut-off frequency, –3 dB frequency, break frequency or corner frequency.

Example A.2 *Design an HPF at a cut-off frequency of 1 kHz with a pass-band gain of 2.*

Solution

$$f_L = 1 \text{ kHz}$$

$$f_L = \frac{1}{2\pi RC}$$

Let

$$C = 0.01\ \mu\text{F}$$

$$R = 15.9\ \text{k}\Omega$$

$$A_f = 1 + \frac{R_f}{R_1}$$

Let $R_1 = 10\ \text{k}\Omega$

$R_f = 10\ \text{k}\Omega$

A.5 SECOND-ORDER HIGH-PASS BUTTERWORTH FILTERS

High-pass filters are obtained from low-pass filters by interchanging the frequency determining resistors and capacitors in low-pass filters. The second-order high-pass filter is obtained from second-order low-pass filter by interchanging *R* and *C* as shown in Fig. A.7.

Fig. A.7

The voltage gain magnitude equation of the second-order high-pass filter is given by,

$$\frac{V_0}{V_{in}} = \frac{A_f}{\sqrt{1 + \left(\frac{f_L}{f}\right)^4}}$$

A.6 BAND-PASS FILTERS

A band-pass filter has a pass band between two cut-off frequencies f_H and f_L such that $f_H > f_L$. Any frequency outside this pass-band is rejected by the filter. There are two types of bandpass filters :

1. Wide band-pass filter
2. Narrow band-pass filter

This classification is based on the value of Q, i.e. Fig. A.8 of merit or quality factor. If $Q < 10$ then it is a wide band-pass filter. If $Q > 10$, it is called narrow band-pass filter. Since Q is a measure of selectivity, larger the value of Q, more selective the filter is and the bandwidth decreases. The relationship between Q, the 3 dB bandwidth and the centre frequency f_c is given by

$$Q = \frac{f_c}{BW} = \frac{f_c}{f_H - f_L}$$

For a wide band-pass filter, the centre frequency f_c can be defined as $f_c = \sqrt{f_H \ f_L}$

In a narrow band-pass filter, the output voltage peaks at the centre frequency.

Wide Band-Pass Filters

A wide band-pass filter can be formed by cascading high-pass and low-pass sections as shown. To obtain a ±20 dB/decade band-pass, first order high-pass and first-order low-pass sections are cascaded. To obtain a ±40 dB/decade band-pass, second-order high-pass and second order low-pass section are cascaded. The circuit diagram of first-order wide band-pass filter is shown in Fig. A.8.

(a) First-order high-pass filter (b) First-order low-pass filter

Fig. 6.8

Fig. A.9

The voltage-gain magnitude of band-pass filter is equal to the product of the voltage-gain magnitudes of the high-pass and the low-pass filters.

$$\left|\frac{V_0}{V_{in}}\right| = A_{FT}\frac{\dfrac{f}{f_L}}{\sqrt{\left[1+\left(\dfrac{f}{f_L}\right)^2\right]\left[1+\left(\dfrac{f}{f_H}\right)^2\right]}}$$

where A_{FT} is the total pass-band gain
f is the frequency of the input signal
f_H is higher cut off frequency
f_L is lower cut off frequency

Example A.3 *Design a wide band-pass filter with f_L = 200 Hz, f_H = 1 kHz and a pass-band gain = 4.*

Solution For a low-pass filter,

$$f_H = 1 \text{ kHz}$$

$$f_H = \frac{1}{2\pi RC}$$

Let $C' = 0.01\ \mu\text{F}$

$R' = 15.9\ \text{k}\Omega$

For a high-pass filter,

$$f_L = 200 \text{ kHz}$$

Let $C = 0.05\ \mu\text{F}$

$$R = \frac{1}{2\pi f_L C} = 15.9\ \text{k}\Omega$$

Since the pass-band gain is 4, the gain of the high-pass as well as the low-pass filter can be set at 2.

$$A_f = 2$$

$$A_f = 1 + \frac{R_f}{R_1}$$

Let $R_1 = R_1' = 10\ \text{k}\Omega$

$R_f = R_f' = 10\ \text{k}\Omega$

A.7 BAND-STOP FILTERS

In a band-stop filter, frequencies are attenuated in the stop band. Fig. A.10 shows a wideband reject filter. A band-stop filter can be constructed by connecting a high-pass filter and low-pass filter in parallel and then connecting this output to the adder circuit.

Fig. A.10

To realise the band-stop response, the lower cut-off frequency f_L of high-pass filter must be larger than the high cut-off frequency f_H of the low-pass filter. Also, the pass-band gain of both the filters must be equal. The frequency response of a band-stop filter is shown in Fig. A.10.

A.8 ALL-PASS FILTERS

An all-pass filter passes all signal frequencies without attenuation but with varying phase shifts. Figure. A.11 shows an all-pass filter.

Fig. A.11

With Terminal 1 grounded and voltage V_{in} applied to Terminal 2,

$$V'_0 = -\frac{R_f}{R_1} V_{in}$$

$$= -V_{in} \; (R_f = R_1)$$

With Terminal 2 grounded and voltage V_{in} applied to Terminal 1,

$$V''_0 = \left(1 + \frac{R_f}{R_1}\right) V_1$$

$$= 2V_1 \; (R_f = R_1)$$

$$V_1 = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}$$

$$= \frac{1}{RCs + 1} V_{in}$$

$$V''_0 = \frac{2}{RCs + 1} V_{in}$$

By superposition principle,

$$V_0 = V'_0 + V''_0$$

$$= -V_{in} + \frac{2}{RCs + 1} V_{in}$$

$$V_0 = V_{in}\left(-1 + \frac{2}{RCs + 1}\right)$$

$$\frac{V_0}{V_{in}} = \frac{1 - RCs}{1 + RCs}$$

Putting $s = j\omega$

$$\frac{V_0}{V_{in}} = \frac{1 - j\omega RC}{1 + j\omega RC}$$

$$\left|\frac{V_0}{V_{in}}\right| = 1$$

The phase angle $\phi = -2 \tan^{-1} \omega RC$

Figure A.12 shows a phase shift of 30° between the input V_{in} and output V_o. For fixed values of R and C, the phase angle ϕ changes from 0 to −180° as the frequency f is varied from 0 to ∞.

Fig. A.12

When R and C are interchanged, the phase shift between input and output becomes positive. Thus, the output V_o leads the input V_{in}.

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