Interactive Computer Graphics: Lecture 2

Transformations for animation

The most useful operations: Previously defined transformation matrices

- Translation

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{pmatrix}$$

- Scaling

$$\begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x\, x \\ s_y\, y \\ s_z\, z \\ 1 \end{pmatrix}$$

Rotations about x, y and z axes.

$$\mathcal{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{R}_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{R}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotations about x, y and z axes.

$$
\mathcal{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\mathcal{R}_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

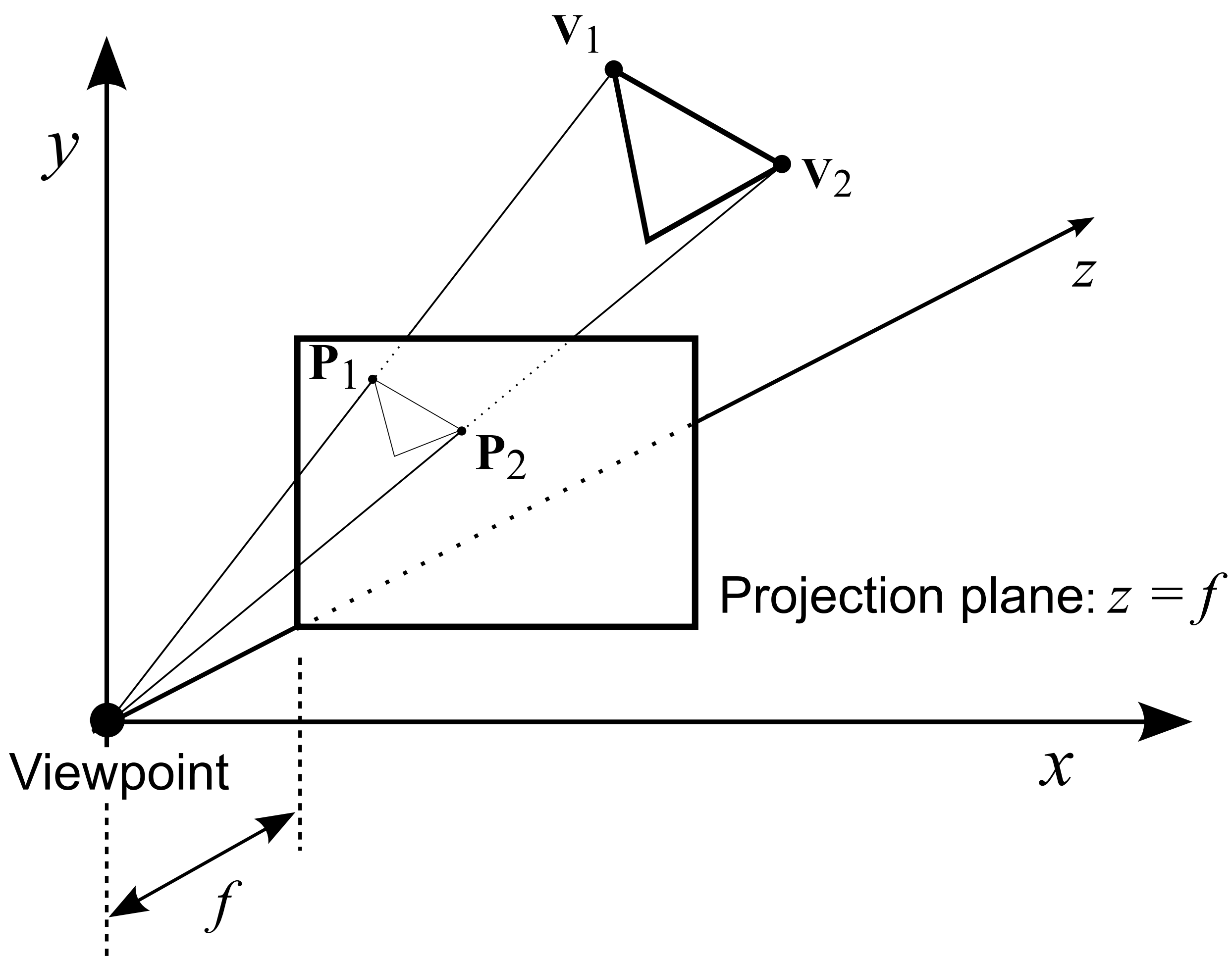
$$
\mathcal{R}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

We now consider more complex transformations which are combinations of translations, scalings and rotations

Flying sequences

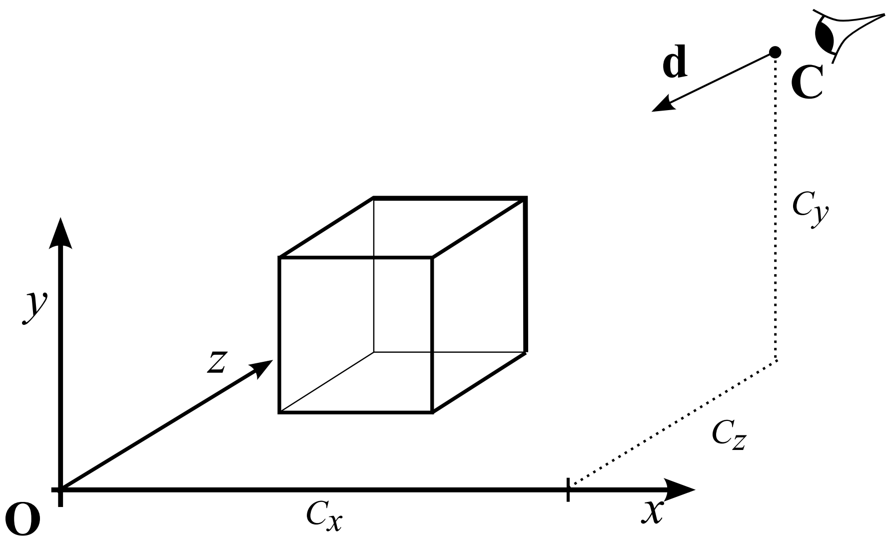
- In generating animated flying sequences we require the viewpoint to move around the scene.
- This implies a change of origin
- Let
  - the required viewpoint be $\mathbf{C} = (C_x, C_y, C_z)$
  - the required view direction be $\mathbf{d} = \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix}$

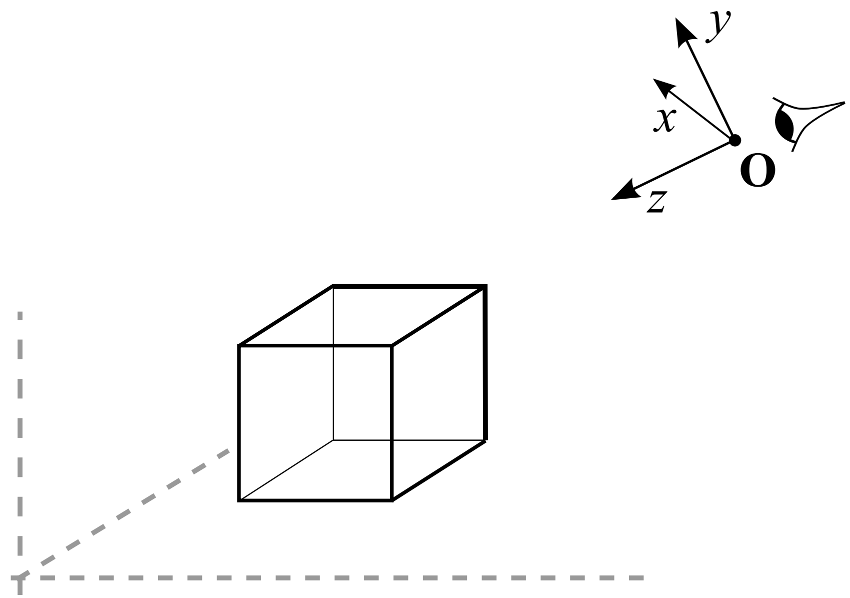
Recall the canonical form for perspective projection



We look along the z-axis and the the y-axis is 'up'

Transformation of viewpoint





Coordinate system for definition

Coordinate system for viewing

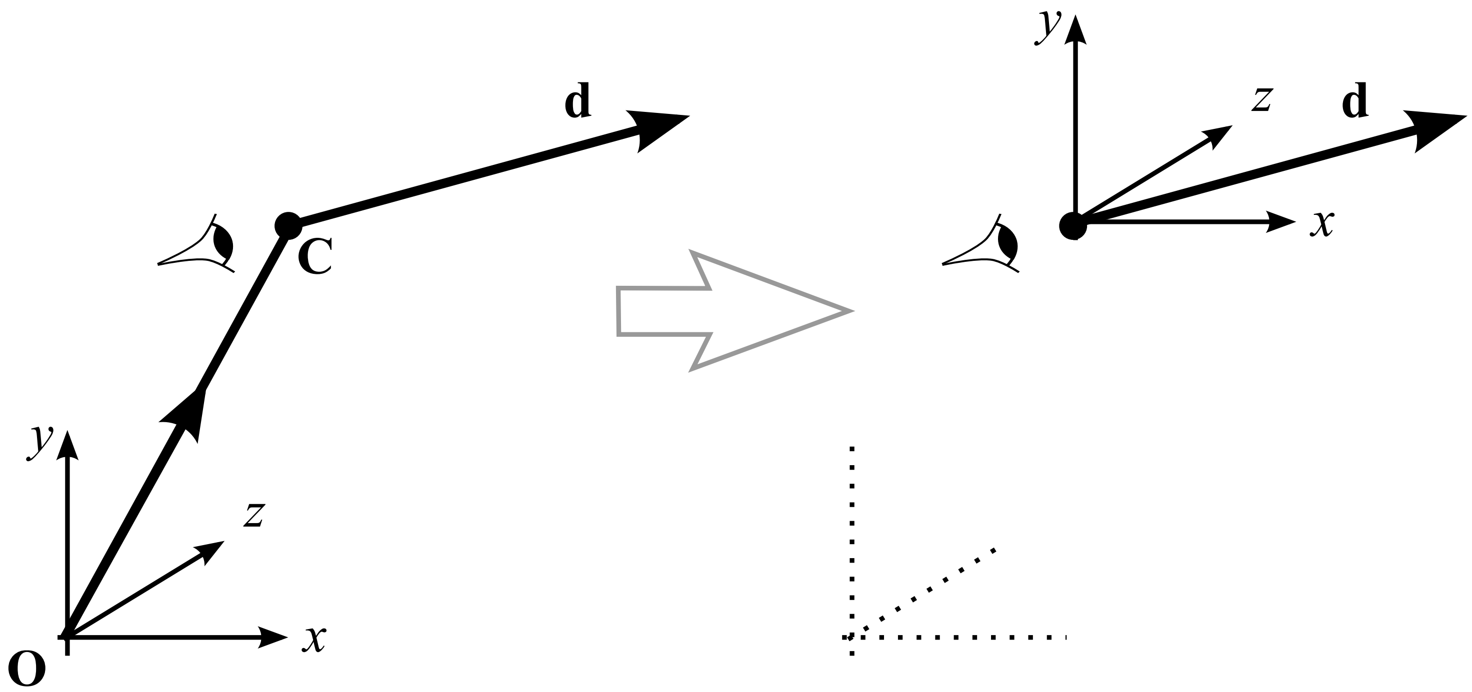
Flying Sequences

- The required transformation is in three parts:

    1. Translation of the origin
    2. Rotate about y-axis
    3. Rotate about x-axis

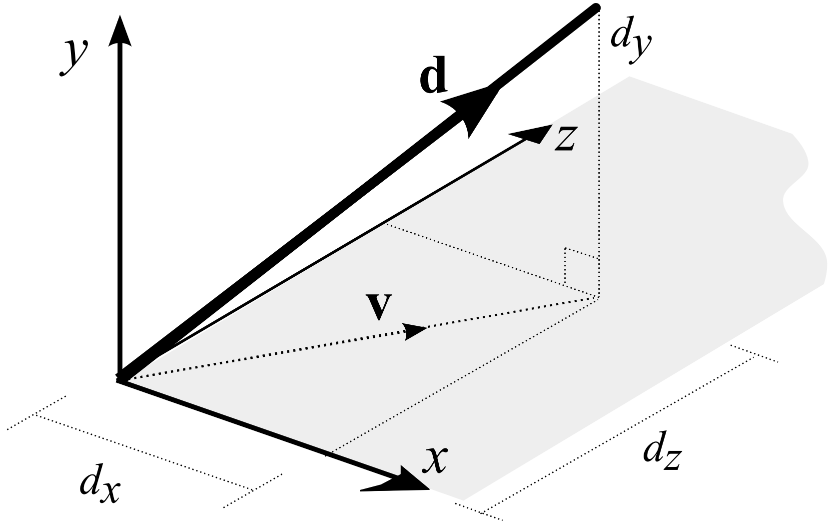
- The two rotations are to line up the z-axis with the view direction

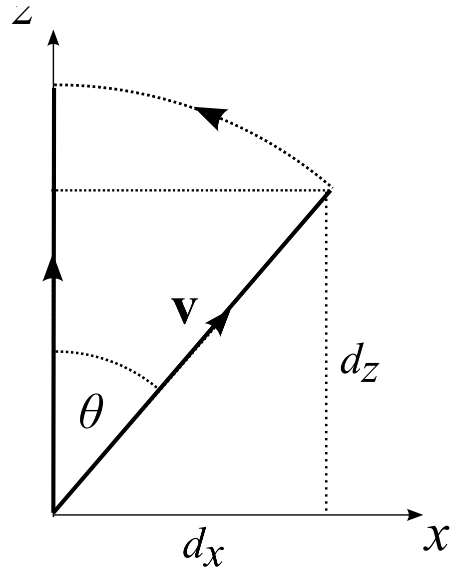
1. Translation of the Origin



$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & -C_x \\ 0 & 1 & 0 & -C_y \\ 0 & 0 & 1 & -C_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Rotate about y until $\mathbf{d}$ is in the y-z plane

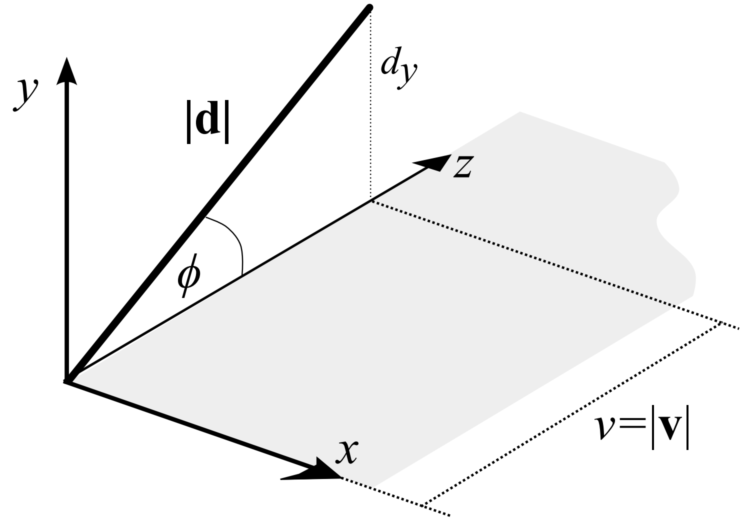


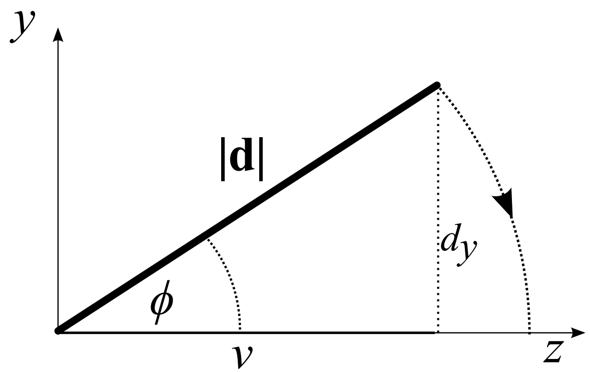


$$
\begin{aligned}
\|\mathbf{v}\| = v &= \sqrt{d_x^2 + d_z^2} \\
\cos\theta &= d_z / v \\
\sin\theta &= d_x / v
\end{aligned}
$$

$$
\mathcal{B} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_z/v & 0 & -d_x/v & 0 \\ 0 & 1 & 0 & 0 \\ d_x/v & 0 & d_z/v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

3. Rotate about x until $\mathbf{d}$ points along the z-axis





$$
\begin{aligned}
v &= \sqrt{d_x^2 + d_z^2} \\
\cos\phi &= v/|\mathbf{d}| \\
\sin\phi &= d_y/|\mathbf{d}|
\end{aligned}
$$

$$
\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v/|d| & -d_y/|d| & 0 \\ 0 & d_y/|d| & v/|d| & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Combining the matrices

- A single matrix that transforms the scene can be obtained from the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by multiplication

$$\mathcal{T} = \mathcal{C}\mathcal{B}\mathcal{A}$$

- And for every point $\mathbf{P}$ of the scene, we calculate

$$\mathbf{P}_t = \mathcal{T}\mathbf{P}$$

- The view is now in ‘canonical’ form and we can apply the standard perspective or orthographic projection.

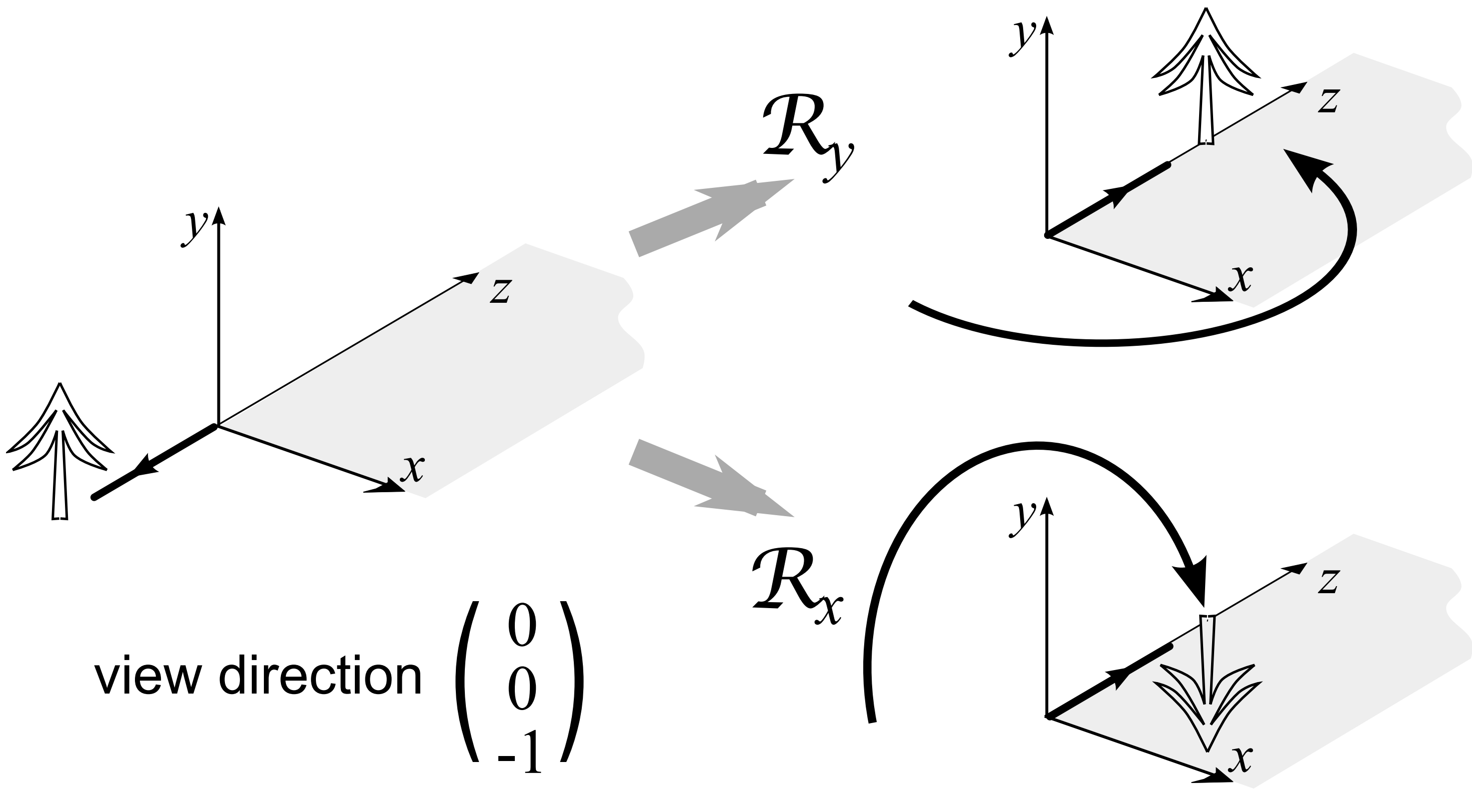
Verticals

- So far we have not looked at verticals

- Usually, the y direction is treated as vertical, and by doing the R_y transformation first, things work out correctly

- However it is possible to invert the vertical

Transformations and verticals



view direction $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Rotation about a general line

- Special effects, such as rotating a scene about a general line can be achieved by multiple transformations

- The transformation is formed by:
  - Making the line of rotation one of the Cartesian axes
  - Doing the rotation (about the chosen axis)
  - Restoring the line to its original place

Rotation about a general line

- The first part is achieved using the same matrices that we derived for the flying sequences

$$\mathcal{C}\mathcal{B}\mathcal{A}$$

- This rotates the general line so it is aligned with the z-axis.

- We then carry out the rotation about the z-axis then follow this by the inversion of the initial matrices.

- So the full matrix T of the combined transformation is

$$\mathcal{T} = \mathcal{A}^{-1}\mathcal{B}^{-1}\mathcal{C}^{-1}\mathcal{R}_z\mathcal{C}\mathcal{B}\mathcal{A}$$

Other effects

- Similar effects can be created using this approach

- e.g. to make an object shrink (and stay in place)

  1. Move the object to the origin
  2. Apply a scaling matrix
  3. Move the object back to where it was

Projection by matrix multiplication

- Usually projection and drawing of a scene comes after the transformation(s)

- It is therefore convenient to combine the projection with the other parts of the transformation

- So it is useful to have matrices for the projection operation

Orthographic projection matrix

- For (canonical) orthographic projection, we simply drop the z-coordinate:

$$\mathcal{M}_o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{M}_o \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix}$$

Perspective projection matrix

- Perspective projection of homogenous coordinates can also be done by matrix multiplication:

$$\mathcal{M}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix}$$

$$\mathcal{M}_p \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/f \end{pmatrix}$$

Perspective projection matrix: Normalisation

- Remember we can normalise homogeneous coordinates, so

$$\mathcal{M}_p \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/f \end{pmatrix} \text{ which is the same as } \begin{pmatrix} xf/z \\ yf/z \\ f \\ 1 \end{pmatrix}$$

- as required.

Projection matrices are singular

- Notice that both projection matrices are singular (i.e. ‘non-invertible’, zero-determinant, …)

$$\mathcal{M}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \qquad \mathcal{M}_o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- This is because a projection transformation cannot be inverted.
- Given a 2D image, we cannot in general reconstruct the original 3D scene.

Homogenous coordinates as vectors

- We now take a second look at homogeneous coordinates, and their relation to vectors.
- In the previous lecture we described the fourth ordinate as a scale factor.

Homogeneous		Cartesian
$\begin{pmatrix} x \\ y \\ z \\ s \end{pmatrix}$	$\longrightarrow$	$\begin{pmatrix} x/s \\ y/s \\ z/s \end{pmatrix}$

Homogenous coordinates and vectors

- Homogenous coordinates fall into two types:

1. Position vectors
   - Those with non-zero final ordinate (s > 0).
   - Can be normalised into Cartesian form.

$$\begin{pmatrix} x \\ y \\ z \\ s \end{pmatrix}$$

2. Direction vectors
   - Those with zero in the final ordinate.
   - Have direction and magnitude.

$$\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

Adding direction vectors

- If we add two direction vectors we obtain a direction vector

$$\begin{pmatrix} x_i \\ y_i \\ z_i \\ 0 \end{pmatrix} + \begin{pmatrix} x_j \\ y_j \\ z_j \\ 0 \end{pmatrix} = \begin{pmatrix} x_i + x_j \\ y_i + y_j \\ z_i + z_j \\ 0 \end{pmatrix}$$

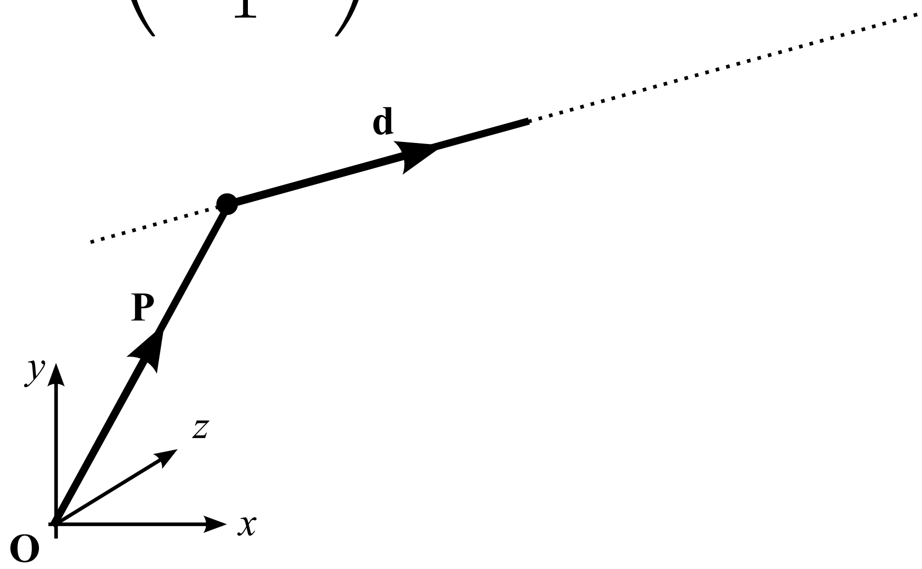
- This is the normal vector addition rule.

Adding position and direction vectors

- If we add a direction vector to a position vector we obtain a position vector:

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} X + x \\ Y + y \\ Z + z \\ 1 \end{pmatrix}$$

Nice result.
Ties in with definition of straight line in Cartesian space which uses a point and a direction



Adding two position vectors

- If we add two position vectors we obtain their mid-point

$$\begin{pmatrix} X_a \\ Y_a \\ Z_a \\ 1 \end{pmatrix} + \begin{pmatrix} X_b \\ Y_b \\ Y_b \\ 1 \end{pmatrix} = \begin{pmatrix} X_a + X_b \\ Y_a + Y_b \\ Z_a + Z_b \\ 2 \end{pmatrix} = \begin{pmatrix} (X_a + X_b)\,/\,2 \\ (Y_a + Y_b)\,/\,2 \\ (Z_a + Z_b)\,/\,2 \\ 1 \end{pmatrix}$$

- This is reasonable since adding two position vectors has no real meaning in vector geometry

The structure of a transformation matrix

- The bottom row is always 0 0 0 1
- The columns of a transformation matrix comprise three direction vectors and one position vector

Matrix	Direction vectors	Position vectors
$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} q_x \\ q_y \\ q_z \\ 0 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \\ 0 \end{pmatrix} \begin{pmatrix} s_x \\ s_y \\ s_z \\ 0 \end{pmatrix}$	$\begin{pmatrix} C_x \\ C_y \\ C_z \\ 1 \end{pmatrix}$

Characteristics of transformation matrices

- Direction vector: Zero, in the last ordinate $\Rightarrow$ not affected by the translation.

$$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix}$$

- Position vector: 1 in the last ordinate $\Rightarrow$ all vectors will have the same displacement.

$$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ 1 \end{pmatrix} = \begin{pmatrix} * + C_x \\ * + C_y \\ * + C_z \\ 1 \end{pmatrix}$$

- If we do not shear the object the three vectors $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ will remain orthogonal, ie:

$$\mathbf{q} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{s} = \mathbf{q} \cdot \mathbf{s} = \mathbf{0}$$

What do the individual columns mean?

- To see this, consider the effect of the transformation in simple cases.
- For example take the unit direction vectors along the Cartesian axes
  - e.g. along the x-axis, $\mathbf{i} = (1, 0, 0, 0)^T$

$$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} q_x \\ q_y \\ q_z \\ 0 \end{pmatrix}$$

What do the individual columns mean?

- The other axis transformations:

  Similarly, we find the following transformations of unit vectors $\mathbf{j}$ and $\mathbf{k}$

$$\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} r_x \\ r_y \\ r_z \\ 0 \end{pmatrix} \qquad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} s_x \\ s_y \\ s_z \\ 0 \end{pmatrix}$$

What do the individual columns mean?

- Transforming the origin:
  - If we transform the origin, $(0, 0, 0, 1)^{\mathrm{T}}$, we end up with the last column of the transformation matrix

$$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_x \\ C_y \\ C_z \\ 1 \end{pmatrix}$$

The meaning of a transformation matrix

Putting everything together …

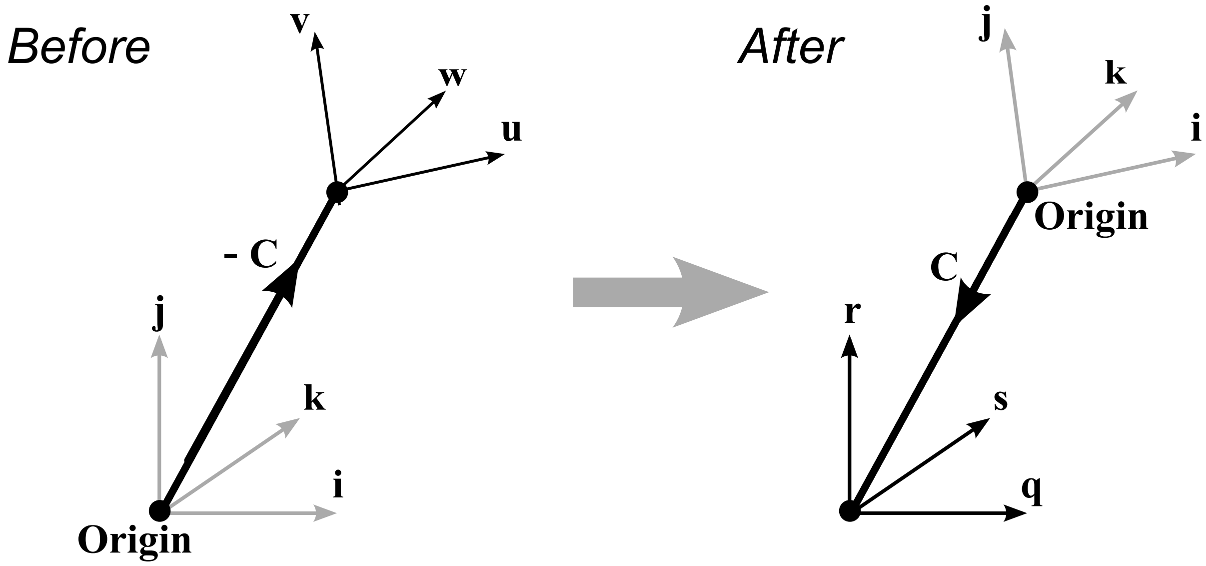
The columns are the original axis system after transforming to the new coordinate system

$$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{q} & \mathbf{r} & \mathbf{s} & \mathbf{C} \end{matrix}$$

q	transformed x-axis
r	transformed y-axis
s	transformed z-axis
C	transformed origin

Effect of a transformation matrix

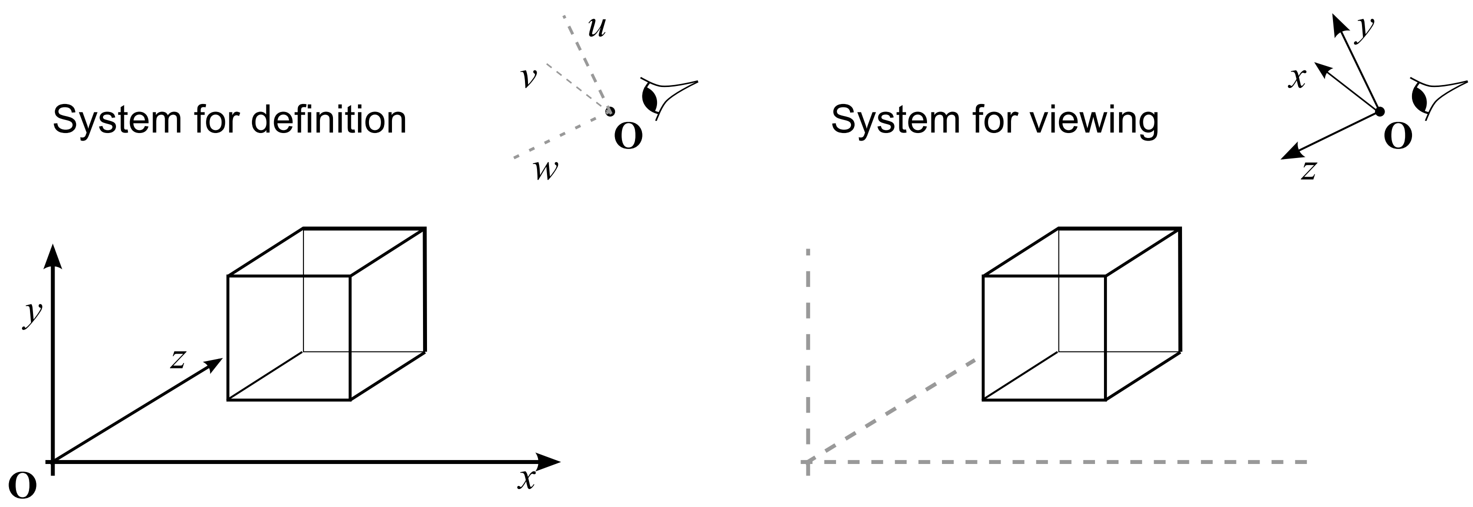


Tells us the old axes and origin in the new coordinate system.

$$\begin{pmatrix} q_x & r_x & s_x & C_x \\ q_y & r_y & s_y & C_y \\ q_z & r_z & s_z & C_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = [\mathbf{q} \quad \mathbf{r} \quad \mathbf{s} \quad \mathbf{C}]$$

What we want is the other way round

- Normally,
  - We are not given the transformation matrix that moves the scene to that coordinate system, we need to find it
  - We are given a view direction $\mathbf{d}$ and location $\mathbf{C}$



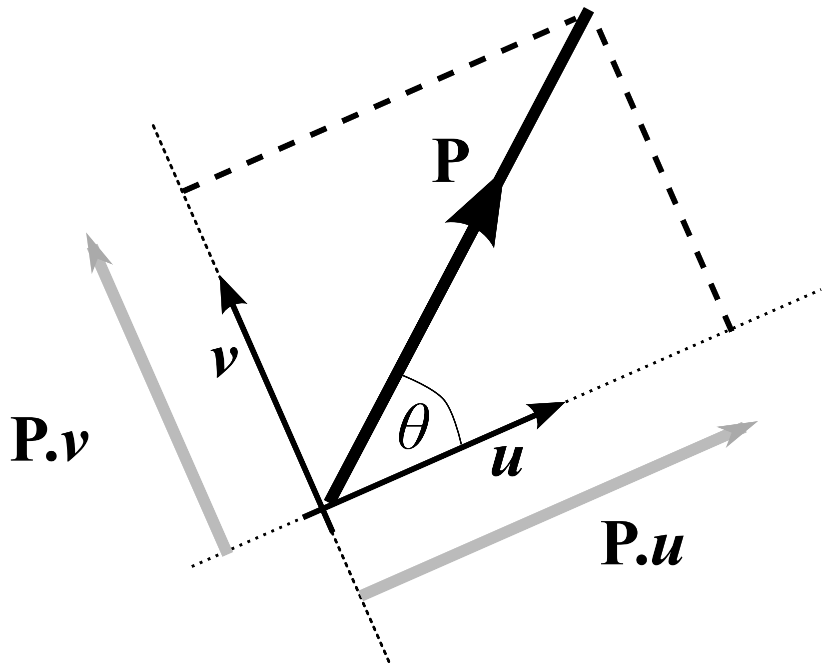
To see how to get the matrix, we introduce the idea of the dot product as a *projection*

The dot product as a projection

- The dot product is defined as

$$\mathbf{P} \cdot \boldsymbol{u} = |\mathbf{P}||\boldsymbol{u}| \cos \theta$$

- If $\boldsymbol{u}$ is
  - a unit vector then $\mathbf{P} \cdot \boldsymbol{u} = |\mathbf{P}| \cos \theta$
  - along a co-ordinate axis then $\mathbf{P} \cdot \boldsymbol{u}$ is the ordinate of $\mathbf{P}$ in the direction of $\boldsymbol{u}$

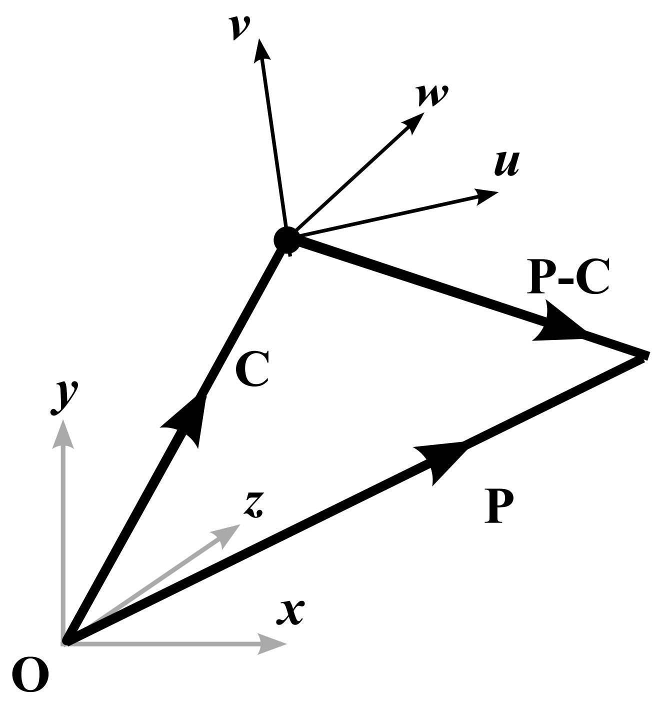


Changing axes by projection

- Extending the idea to three dimensions we can see that a change of axes can be expressed as projections using the dot product

For example, call the first coordinate of $\mathbf{P}$ in the new system $\mathbf{P}_x^{\ t}$

$$\mathrm{P}_x^{\ t} = (\mathbf{P} - \mathbf{C}) \cdot \boldsymbol{u}$$
$$= \mathbf{P} \cdot \mathbf{u} - \mathbf{C} \cdot \mathbf{u}$$



Transforming point $\mathbf{P}$

- Given point $\mathbf{P}$ in the (x, y, z) axis system, we can calculate the corresponding point in the (u, v, w) system as:

$$
\begin{aligned}
P_x^t &= (\mathbf{P} - \mathbf{C}) \cdot \boldsymbol{u} = \mathbf{P} \cdot \boldsymbol{u} - \mathbf{C} \cdot \boldsymbol{u} \\
P_y^t &= (\mathbf{P} - \mathbf{C}) \cdot \boldsymbol{v} = \mathbf{P} \cdot \boldsymbol{v} - \mathbf{C} \cdot \boldsymbol{v} \\
P_z^t &= (\mathbf{P} - \mathbf{C}) \cdot \boldsymbol{w} = \mathbf{P} \cdot \boldsymbol{w} - \mathbf{C} \cdot \boldsymbol{w}
\end{aligned}
$$

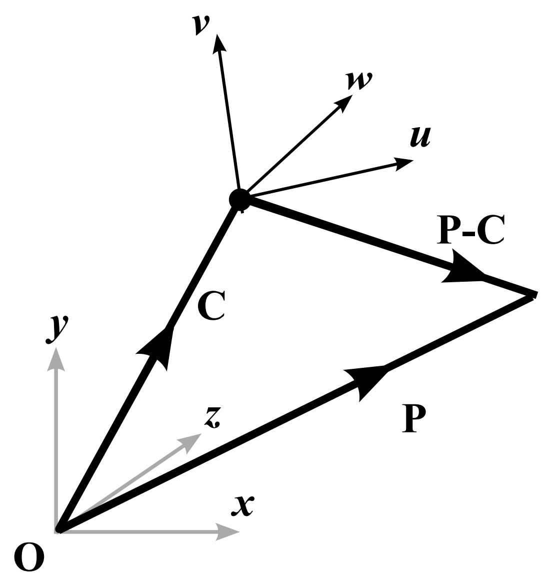
- Or, in matrix notation:

$$
\begin{pmatrix} P_x^t \\ P_y^t \\ P_z^t \\ 1 \end{pmatrix} = \begin{pmatrix} u_x & u_y & u_z & -\mathbf{C} \cdot \boldsymbol{u} \\ v_x & v_y & v_z & -\mathbf{C} \cdot \boldsymbol{v} \\ w_x & w_y & w_z & -\mathbf{C} \cdot \boldsymbol{w} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}
$$

Verticals revisited …

Unlike the previous analysis we now can control the vertical

i.e. we can assume the $\boldsymbol{v}$-direction is the vertical and constrain it in the software to be upwards

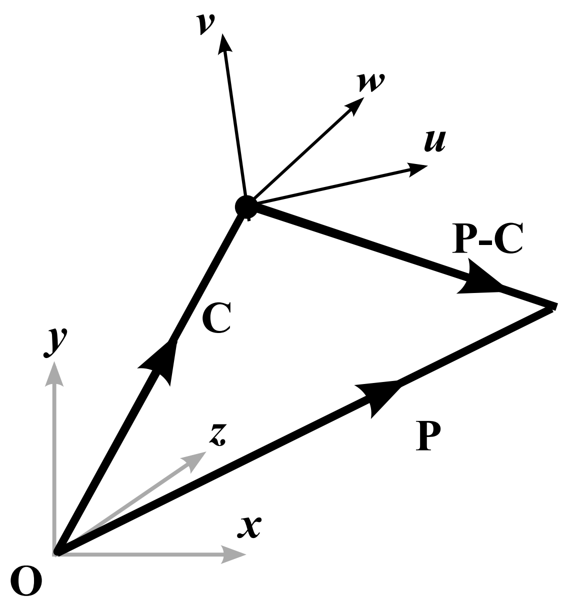


Back to flying sequences

- We now return to the original problem
  - Given a viewpoint point $\mathbf{C}$ and a view direction $\mathbf{d}$, we need to find the transformation matrix that gives us the canonical view.
  - We do this by first finding the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

We know that $\mathbf{d}$ is the direction of the new axis, so we can write immediately

$$\boldsymbol{w} =$$



Now the horizontal direction

- We can write $\boldsymbol{u}$ in terms of some vector $\mathbf{p}$ in the horizontal direction

$$\boldsymbol{u} = \frac{\mathbf{p}}{|\mathbf{p}|}$$

- To ensure that $\mathbf{p}$ is horizontal we set

$$p_y = 0$$

- so that $\mathbf{p}$ has no vertical component

And the vertical direction

- Let $\mathbf{q}$ be some vector in the vertical direction, we can then write $\mathbf{v}$ as

$$\Rightarrow \mathbf{v} = \frac{\mathbf{q}}{|\mathbf{q}|}$$

- $\mathbf{q}$ must have a positive y component, so we can say that

$$q_y = 1$$

So we have four unknowns

$$\begin{aligned} \mathbf{p} &= [p_x, 0, p_z] \quad \text{new horizontal} \\ \mathbf{q} &= [q_x, 1, q_z] \quad \text{new vertical} \end{aligned}$$

To solve for these we use the cross product and dot product.

We can write the view direction $\mathbf{d}$, which is along the new z axis, as

$$\mathbf{d} = \mathbf{p} \times \mathbf{q}$$

(We can do this because the magnitude of $\mathbf{p}$ is not yet set)

Evaluating the cross-product

$$\begin{aligned}
\mathbf{d} &= \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \mathbf{p} \times \mathbf{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ p_x & 0 & p_z \\ q_x & 1 & q_z \end{vmatrix} \\
&= -p_z\boldsymbol{i} + (p_z q_x - p_x q_z)\boldsymbol{j} + p_x\boldsymbol{k} = \begin{pmatrix} -p_z \\ p_z q_x - p_x q_z \\ p_x \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
d_x &= -p_z \\
d_y &= p_z q_x - p_x q_z \\
d_z &= p_x
\end{aligned}$$

So we can write vector $\mathbf{p}$ completely in terms of $\mathbf{d}$

$$\mathbf{p} = \begin{pmatrix} d_z \\ 0 \\ -d_x \end{pmatrix}$$

Using the dot product

- Lastly we can use the fact that the vectors $\mathbf{p}$ and $\mathbf{q}$ are orthogonal

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= 0 \\ \Rightarrow p_x q_x + p_z q_z &= 0 \end{aligned}$$

- And from the cross product (previous slide)

$$d_y = p_z q_x - p_x q_z$$

- So we have two simple linear equations to solve for $\mathbf{q}$ and write it in terms of the components of $\mathbf{d}$

The final matrix

- Once we have expressions for $\mathbf{p}$ and $\mathbf{q}$ in terms of the given vector $\mathbf{d}$, we have

$$\mathbf{u} = \frac{\mathbf{p}}{|\mathbf{p}|} \qquad \mathbf{v} = \frac{\mathbf{q}}{|\mathbf{q}|} \qquad \mathbf{w} = \frac{\mathbf{d}}{|\mathbf{d}|}$$

- We already know $\mathbf{C}$ as that is also given. So we can write down the matrix

$$\begin{pmatrix} u_x & u_y & u_z & -\mathbf{C}\cdot\boldsymbol{u} \\ v_x & v_y & v_z & -\mathbf{C}\cdot\boldsymbol{v} \\ w_x & w_y & w_z & -\mathbf{C}\cdot\boldsymbol{w} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$