Tutorial 1: 3D space, transformations and animations.

This tutorial uses the following notation:

- Position vectors are denoted by boldface capital letters: **P**, **Q**, **V** etc. Position vectors are the same as Cartesian coordinates, and represent position relative to the origin.

- Direction vectors are indicated by boldface lowercase letters **d**, **n** etc. Direction vectors are independent of any origin.

- Scalars are represented by italics: *a*, *b*, etc.

A plane is an object that is only defined in Cartesian space, however, each plane has a normal vector, whose size is non zero, and whose direction is at right angles to that plane. We can find a normal vector by taking the cross product of any two direction vectors which are parallel to the plane.

Analysis of 3D scenes

1. Given three points:

   $\mathbf{P}_1 = (10, 20, 5)$
   $\mathbf{P}_2 = (15, 10, 10)$
   $\mathbf{P}_3 = (25, 20, 10)$

   find two direction vectors which are parallel to the plane defined by $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$. Hence find a normal vector to the plane.

2. A plane is defined in vector terms by the equation:

   $$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_1) = 0$$

   where $\mathbf{P} = (x, y, z)$ is the locus of a point on the plane, and $\mathbf{P}_1$ is any point known to be in the plane.

   For the points given in part 1, expand the vector plane equation to find the Cartesian form of the plane equation, (i.e. $ax + by + cz + d = 0$).

   Verify that you get the same result using either $\mathbf{P}_1$ or $\mathbf{P}_2$.

3. Write a procedure, in any programming language you like, which takes as input three points and returns the coefficients of the Cartesian plane equation (*a*, *b*, *c* and *d*).

4. Starting from any point on a face of a polyhedron, an inner surface normal is a normal vector to the plane of the face whose direction points into the polyhedron.

A tetrahedron is defined by the three points of part 1, and a fourth point $\mathbf{P}_4 = (30, 20, 10)$. Determine whether the normal vector that you calculated in part 1 is an inner surface normal, and if not find the inner surface normal.

5. Two lines intersect at a point $\mathbf{P}_1$, and are in the directions defined by $\mathbf{d}_1$ and $\mathbf{d}_2$. Provided that $\mathbf{d}_1$ and $\mathbf{d}_2$ represent different directions, the two lines define a plane.

   Any point on the plane can be reached by travelling from $\mathbf{P}_1$ in direction $\mathbf{d}_1$ by some distance μ and then in direction $\mathbf{d}_2$ by a distance ν. Using this fact construct the parametric equation of any point on the plane of part 1 in terms of μ, ν, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$. By taking the dot product with a normal vector to the plane $\mathbf{n}$, show that the parametric plane equation is equivalent to the vector plane equation of part 2.

Animations

6. In a computer graphics animation scene an object is defined as a planar polyhedron. The object centre is located at position P = (0, 0, 10), and the scene is drawn, as normal, in perspective projection with the viewpoint at the origin and the view direction along the z-axis. Calculate the transformation matrix that will shrink the object in size by a factor of 0.8 towards its centre point.

7. Use your matrix of part 1 to check what happens to the points (0, 0, 10) and (0,0,5). Is your result what you expect?

8. In a different animation, the object, defined above is required to rotate clockwise, looking from the origin, while shrinking. In each successive frame it is to rotate by 15° while shrinking to 0.8 of its original size. The rotation axis is to be the z axis, and the shrinkage is, as before, towards the object's centre. Given that Cos(15°) = .97 and Sin(15°) = .26, what is the transformation matrix that will achieve this animation?

9. The scene above is to be drawn in perspective projection with the plane of projection being z = 2. Find the combined transformation that will do animation of part 1 followed by the perspective projection. Is your matrix singular?

10. Use your matrix to find the transformation and perspective projection of the points (0, 0, 10) and (0, 0, 5) in homogenous coordinates and then in Cartesian coordinates.

11. The scene is to be viewed from a moving viewpoint specified by its position **C** and a left-handed viewing coordinate system $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. At one point in the animation the view direction is $\boldsymbol{w} = (-1, 0, 0)^T$, and the viewpoint is given by $\mathbf{C} = (50, 10, -10)$. Given that the view is in the horizontal plane ($\boldsymbol{v} = (0, 1, 0)^T$) find the value of $\boldsymbol{u}$.

12. Hence, or otherwise, find the viewing transformation matrix.

Tutorial 1: Analysis of three dimensional (3D) space.

Solutions – Analysis of 3D scenes

Q1. The three points $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ can be used to generate two direction vectors in a number of different ways. For example, we could take

$$\mathbf{P}_2 - \mathbf{P}_1 = (5, -10, 5) \qquad\qquad \mathbf{P}_3 - \mathbf{P}_1 = (15, 0, 5)$$

to give the required direction vectors. Scaling a vector by a constant does not affect its direction so we can divide these direction vectors by 2 and 3 respectively to obtain the equivalent direction vectors:

$$(1, -2, 1) \qquad\qquad (3, 0, 1)$$

We can find the cross product to obtain the normal vector:

$$(1,\ -2,\ 1)\ \times\ (3,\ 0,\ 1)\ =\ (-2,\ 2,\ 6)$$

Scaling by a half gives the simpler normal vector (-1, 1, 3).

N.B. The rule for obtaining the cross product of two vectors can be represented in a number of different ways. For example

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

The same rule expressed using $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ notation is:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_2 b_3 - a_3 b_2\mathbf{i} + a_3 b_1 - a_1 b_3\mathbf{j} + a_1 b_2 - a_2 b_1\mathbf{k}$$

If you are familiar with matrices, the cross product can be represented as the determinant of a matrix:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Q2. We have a general point in the plane, $\mathbf{P}$, and a point $\mathbf{P}_1$ known to be in the plane from part 1:

$$\mathbf{P} = (x, y, z) \qquad\qquad \mathbf{P}_1 = (10, 20, 5)$$

The difference between these vectors gives a vector parallel to the plane:

$$\mathbf{P} - \mathbf{P}_1 = (x-10,\ y-20,\ z-5)$$

This vector is therefore perpendicular to the plane normal $\mathbf{n}$ = (-1, 1, 3), so we have:

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_1) = 0$$

Which can be used to find the Cartesian plane equation as follows:

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_1) = 0$$

$$\Rightarrow \quad (-1,\ 1,\ 3) \cdot (x-10,\ y-20,\ z-5) = 0$$

$$\Rightarrow \quad -x + 10 + y - 20 + 3(z-5) = 0$$

$$\Rightarrow \quad -x + y + 3z - 25 = 0$$

The same equation can be obtained using $\mathbf{P}_2$ instead:

$$\mathbf{P} - \mathbf{P}_2 = (x-15,\ y-10,\ z-10)$$

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_2) = 0$$

$$\Rightarrow \quad (-1,\ 1,\ 3) \cdot (x-15,\ y-10,\ z-10) = 0$$

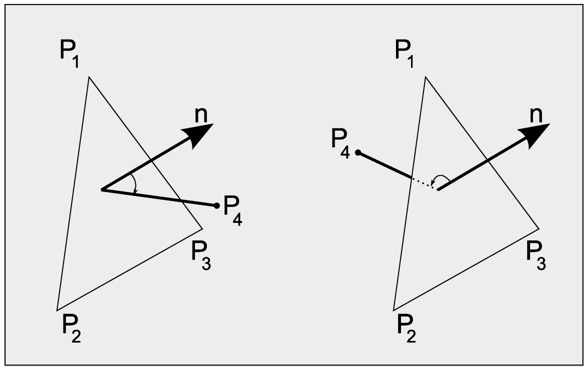
$$\Rightarrow \quad -x + 15 + y - 10 + 3(z-10) = 0$$

$$\Rightarrow \quad -x + y + 3z - 25 = 0$$

Q3. The following pseudocode gives an example of how the coefficients of a Cartesian plane equation might be obtained using three given points in a plane:

```
TYPE Vector = Array [0..2] of REAL;

PROCEDURE PlaneEquation(P1,P2,P3: Vector;VAR a,b,c,d: REAL);

   VAR d1,d2: Vector;

   (* Find two vectors parallel to the plane *)
   FOR j:0 .. 2
      d1[j] = P2[j]-P1[j];
      d2[j] = P3[j]-P1[j];
   END FOR

   (* Find the normal vector to the plane n = [a,b,c] = d1 x d2 *)
   a := d1[1]*d2[2] - d1[2]*d2[1]
   b := d1[2]*d2[0] - d1[0]*d2[2]
   c := d1[0]*d2[1] - d1[1]*d2[0]
   (* take the dot product with P-P1 *)
   d := -(a*P1[0]+b*P1[1]+c*P1[2])

END PlaneEquation;
```

Q4. Let **n** be a surface normal to the tetrahedron. If it is an inner normal, it points in the same direction as a vector from the face defined by $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ to the fourth point $\mathbf{P}_4$. This is illustrated on the left in the following diagram:

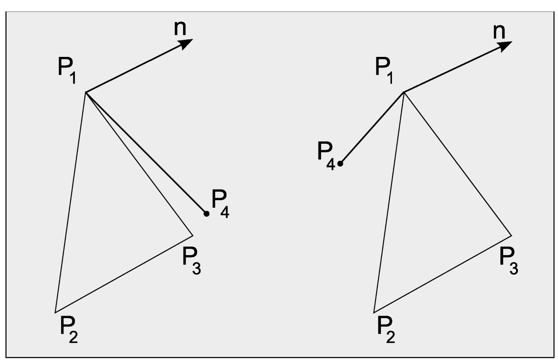


The right hand side illustrates the case where **n** is *not* an inner surface normal, i.e. an outward surface normal. From these diagrams, we can see that, wherever **n** is placed on the face, the angle it makes with the vector to $\mathbf{P}_4$ will be acute when **n** is an inner surface normal and obtuse otherwise. Using the fact that the dot product of two vectors **a**, **b** can be expressed in terms of the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

if $\theta < 90$, we have $\cos\theta > 0$ and hence $\mathbf{a} \cdot \mathbf{b} > 0$, otherwise we have $\mathbf{a} \cdot \mathbf{b} < 0$ when θ is obtuse.

It does not matter where **n** is placed in the plane, so we can place it at $\mathbf{P}_1$, the two cases can therefore be illustrated as follows:



Take the dot product of the normal vector and the vector from $\mathbf{P}_1$ to $\mathbf{P}_4$, i.e.:

$$\mathbf{n} = (-1,1,3)$$
$$\mathbf{P}_4 - \mathbf{P}_1 = (20,0,5)$$
$$\mathbf{n} \cdot (\mathbf{P}_4 - \mathbf{P}_1) = -20 + 15 = -5$$

Since the result is negative the angle between these two vectors is bigger than 90, and so the normal vector must be the outward surface normal. An inner surface normal can be obtained by negating the outer normal, i.e. the inner surface normal can be given by (1, -1, -3).

Q5. In the parametric plane equation, we have a starting point, which we can choose as $\mathbf{P}_1$, two parameters μ, ν and two direction vectors $\mathbf{d}_1$, $\mathbf{d}_2$ that are parallel to the plane. The direction vectors might be chosen as follows for example:

$$\mathbf{d}_1 = \mathbf{P}_2 - \mathbf{P}_1$$
$$\mathbf{d}_2 = \mathbf{P}_3 - \mathbf{P}_1$$

Putting it all together, a general point $\mathbf{P}$ can be expressed by the parametric plane equation:

$$\mathbf{P} = \mathbf{P}_1 + \mu\,\mathbf{d}_1 + \nu\,\mathbf{d}_2$$

We can take the dot product of both sides with the normal vector $\mathbf{n}$:

$$\mathbf{P} \cdot \mathbf{n} = \mathbf{P}_1 \cdot \mathbf{n} + \mu\,\mathbf{d}_1 \cdot \mathbf{n} + \nu\,\mathbf{d}_2 \cdot \mathbf{n}$$

Because the vectors $\mathbf{d}_1$, $\mathbf{d}_2$ are parallel to the plane, they are perpendicular to the normal vector by definition. Their dot product with $\mathbf{n}$ is therefore zero which means that the last two terms on the right hand side vanish. So we obtain:

$$\begin{aligned} & \mathbf{P} \cdot \mathbf{n} = \mathbf{P}_1 \cdot \mathbf{n} \\ \Rightarrow\quad & \mathbf{P} \cdot \mathbf{n} - \mathbf{P}_1 \cdot \mathbf{n} = 0 \\ \Rightarrow\quad & (\mathbf{P} - \mathbf{P}_1) \cdot \mathbf{n} = 0 \end{aligned}$$

This demonstrates that the parametric plane equation is equivalent to the vector plane equation given in part 2.

Solutions – Animations

Q6.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the required transformation with a single matrix is:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Q7.

The point (0, 0, 10) ‘moves’ to (0, 0, 10) i.e. it stays in the same place. This is expected because it is the centre of the object.

The point (0, 0, 5) goes to (0, 0, 6). This is a move towards the centre as expected.

Q8.

We want the rotation to be clockwise when viewed from the origin, i.e. when viewed from the negative side of the axis (because the object centre is at $z = 10$). So we need a value of $\theta = -15°$ in the rotation matrix R_z

$$\begin{matrix} \cos(15^\circ) \approx 0.97 \\ \sin(15^\circ) \approx 0.26 \end{matrix} \quad \text{so} \quad \begin{matrix} \cos(-15^\circ) \approx 0.97 \\ \sin(-15^\circ) \approx -0.26 \end{matrix}$$

Q8. (Contd.)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-15^\circ) & -\sin(-15^\circ) & 0 & 0 \\ \sin(-15^\circ) & \cos(-15^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.77 & 0.21 & 0 & 0 \\ -0.21 & 0.77 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q9.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0.4 & 1 \end{pmatrix}$$

It is singular. The last two rows are multiples of each other.

Q10.

$$\begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0.4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 5 \end{pmatrix}$$

in homogeneous coordinates which normalises into Cartesian coordinate $(0, 0, 2)$.

$$\begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0.4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix}$$

in homogeneous coordinates which normalises into Cartesian coordinate $(0, 0, 2)$.

So both points project to the origin in the plane of projection.

Q11.

<u>Method 1:</u> Brute force equation solving

We have the following identity for the left-handed system $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$

We know that

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

and would like to solve for $\mathbf{u}$. Writing $\mathbf{u} = (u_1, u_2, u_3)^{\mathrm{T}}$, we can obtain the equation

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -u_3 \\ 0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

which gives $u_3 = 1$ and $u_1 = 0$. We must also have $u_2 = 0$, because $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular), i.e. $\mathbf{u} \cdot \mathbf{v} = 0$.

Putting this all together gives $\mathbf{u} = (0, 0, 1)^{\mathrm{T}}$.

<u>Method 2:</u> Using the cross product identities (see revision notes on vector algebra).

For $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ to form a left-handed system, the following cross product formula holds

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}$$

Cyclic permutations of this formula also hold, i.e.

$$\mathbf{v} = \mathbf{w} \times \mathbf{u} \qquad \text{and} \qquad \mathbf{u} = \mathbf{v} \times \mathbf{w}$$

We can use the last of these formulas to find $\mathbf{u}$:

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which agrees with method 1.

Q12.

The transformation matrix can be written in terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\mathbf{C}$

$$\begin{pmatrix} u_x & u_y & u_z & -\mathbf{C} \cdot u \\ v_x & v_y & v_z & -\mathbf{C} \cdot v \\ w_x & w_y & w_z & -\mathbf{C} \cdot w \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And $\mathbf{C} = (50, 10, -10)$ so

$$\mathbf{C} \cdot u = -10$$
$$\mathbf{C} \cdot v = 10$$
$$\mathbf{C} \cdot w = -50$$

hence we write down the transformation matrix as:

$$\begin{pmatrix} 0 & 0 & 1 & 10 \\ 0 & 1 & 0 & -10 \\ -1 & 0 & 0 & 50 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$