

Adjoint Computational Electromagnetics

Mohamed Kamal Abd Elrahman

6 October, Giza, Egypt

October 21, 2019

The adjoint method is used to inverse design linear and nonlinear photonic devices. The purpose of this handout is give the reader an introduction to the topic.

Motivation

In normal electromagnetic simulation, we are given a material distribution represented by $(\epsilon(x, y, z), \mu(x, y, z))$ and a current density source $\mathbf{J}(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Maxwell's equations are then solved using one of the computational techniques (FD, FEM, MoM) for the excited electromagnetic field (\mathbf{H}, \mathbf{E}) . For linear, isotropic, non magnetic and stationary media ¹ $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon \mathbf{E}$. Maxwell's equations take the form

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad (1a)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (1b)$$

We are asked to maximize the scalar quantity $G(\mathbf{E}, \mathbf{H})$ by varying the dielectric structure $\epsilon(x, y, z)$. G is often a measure of device performance: for example, the power flowing in certain direction or mode conversion efficiency. This is an inverse design problem, we start by defining an objective G and ask for the best material distribution to achieve this objective.

The gradient quantity $\frac{dG}{d\epsilon}$ is useful to compute. It could be used as a measure of robustness and sensitivity to any structure deformations like fabrication imperfections. In inverse design, it is the major quantity needed as it points in the direction of steepest ascent, from which we have an informed decision of how to update the dielectric structure.

$$\epsilon \leftarrow \epsilon + \alpha \frac{dG}{d\epsilon} \quad (2)$$

Where α is the step size in the gradient direction².

Assuming we have n number of design points, to calculate the sensitivity $\frac{dG}{d\epsilon}$ there are different approaches which vary in complexity and accuracy. One way is by using a finite difference approximation $\frac{dG}{d\epsilon} \approx \frac{\Delta G}{\Delta \epsilon}$. In this approach we need to carry n number of forward simulations. For a typical $100 \times 100 \times 100$ 3D finite difference grid, this means 1000000 simulations!. This approach also suffers from numerical discretization errors and the need to determine how small or

¹ In general ϵ and μ may be nonlinear anisotropic tensors. We restricted ourselves to linear isotropic media for simplicity and most material are nearly the case.

² Some algorithms attempt to optimize the step size so that the step maximally increases G

large is $\Delta\epsilon$. Automatic differentiation can be used to get the derivative exactly. However, there are two modes in automatic differentiation. The automatic forward mode, like finite difference, needs n number of forward simulations. In the automatic adjoint or reverse mode, two simulations are only needed regardless of the number of the design points. So, the adjoint method is by far much better in accuracy than finite difference and computational cost than automatic forward differentiation.

Problem Formulation

Before driving the adjoint equation, lets put Maxwell's in a matrix form:

$$\begin{bmatrix} \mu_0 & 0 \\ 0 & -\epsilon \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{H}}{\partial t} \\ \frac{\partial \mathbf{E}}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & \nabla \times \\ \nabla \times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{J} \end{bmatrix} \quad (3)$$

Which could be written as

$$[C] \frac{d\mathbf{X}}{dt} + [D]\mathbf{X} = \mathbf{S} \quad (4)$$

if the following definitions have been made $\mathbf{X} = \begin{bmatrix} \mathbf{H} \\ \mathbf{E} \end{bmatrix}$, $[C] =$

$$\begin{bmatrix} \mu_0 & 0 \\ 0 & -\epsilon \end{bmatrix}, [D] = \begin{bmatrix} 0 & \nabla \times \\ \nabla \times & 0 \end{bmatrix} \text{ and } \mathbf{S} = \begin{bmatrix} 0 \\ \mathbf{J} \end{bmatrix}$$

The initial condition for equation(4) is

$$I(\mathbf{X}(0), \epsilon) = \mathbf{0} \quad (5)$$

Our optimization problem can be then formulated as

$$\begin{aligned} \min_{\epsilon} \quad & G(\epsilon) = \int_0^T g(\mathbf{X}, \epsilon, t) dt \\ \text{s.t.} \quad & [C] \frac{d\mathbf{X}}{dt} + [D]\mathbf{X} - \mathbf{S} = 0 \\ & I(\mathbf{X}(0), \epsilon) = \mathbf{0} \end{aligned} \quad (6)$$

Where we have assumed our objective function can be written in this integral form $\int_0^T g(\mathbf{X}, \epsilon, t) dt$.

Equation (4) is commonly solved by discretization in space first and then solving the resulting system of Ordinary Differential Equations (ODEs). We can either start by replacing time derivatives with finite differences to arrive at a recursive set of spatial problems to be discretized. The discretization in space could be in done in real or spectral domains. Among the numerical methods, the finite difference method, the finite element method, and the method of moments form the core of computational electromagnetics. One of our goals is to show a simple one dimensional example with code for each

method along with its adjoint formulation. Interested readers can directly add an adjoint solver on top their solvers very easily and start doing inverse design.

Adjoint Method