Introduction to the Adjoint Method

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Outline

Adjoint Sensitivity Analysis of Linear Systems

- Consider the Ordinary Differential Equation (ODE)

$$a(t,z)\frac{d^2\psi}{dt^2} + b(t,z)\frac{d\psi}{dt} + c(t,z)\psi = s(t) \tag{1.1}$$

- z is a design parameter that we want to change in order to minimize the objective function G

$$G = \int_0^T g(\psi, z)\, dt \tag{1.2}$$

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- To minimize G, we need the gradient with respect to the design parameter z.
- The design parameter z can then be updated with the steepest descent
$$z = z - \alpha \frac{dG}{dz} \tag{1.3}$$
- Local descent involves iteratively determining a descent direction ($\frac{dG}{dz}$) and then taking a step α in that direction and repeating that process until convergence or some termination condition is met

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- Lets modify the objective function G

$$G = \int_0^T g(\psi, z)\, dt + \int_0^T \lambda(t) R\, dt \tag{1.4}$$

- Where the residual $R = 0 = a(t,z)\frac{d^2\psi}{dt^2} + b(t,z)\frac{d\psi}{dt} + c(t,z)\psi - s(t)$
- The equation is still valid and for any $\lambda(t)$, since we are multiplying $\lambda(t)$ by zero and we just added a zero to the whole equation.
- $\lambda(t)$ is called the adjoint variable.

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- Using the chain rule to obtain the objective function gradient:

$$\frac{dG}{dz} = \int_0^T \frac{\partial g}{\partial z} + \frac{\partial g}{\partial \psi}\frac{\partial \psi}{\partial z} + \lambda \left[\frac{\partial a}{\partial z}\ddot{\psi} + a\frac{\partial \ddot{\psi}}{\partial z} + \frac{\partial b}{\partial z}\dot{\psi} + b\frac{\partial \dot{\psi}}{\partial z} + \frac{\partial c}{\partial z}\psi + c\frac{\partial \psi}{\partial z} \right] \tag{1.5}$$