Adjoint Computational Electromagnetics Eigenvalue Problems

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The sensitivities of eigenvalues and eigenvectors due to perturbations in the deign variables are derived. The focus here is on linear, isotropic, nonmagnetic and transparent materials [1].

[1] In transparent materials, both ϵ and μ are real and positive.

Problem Formulation

Maxwell's equation for reads

$$\nabla \times \mathbf{E} = -j\mu_0\omega\mathbf{H} \tag{1a}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon_0\epsilon(r)\mathbf{E} \tag{1b}$$

which can be combined into a single PDE [2]

[2] We will refer to this equation as the master equation.

$$\nabla \times \frac{1}{\epsilon}\nabla \times \mathbf{H} = \left(\frac{\omega}{c}\right)^2 \mathbf{H} \tag{2}$$

Equation. 2 is a standard eigenvalue problem of the form,

$$Ax = \lambda x \tag{3}$$

where we have identified the operator $A = \nabla \times \nabla \times$ and the eigenvector $x = \mathbf{E}$ with associated eigenvalue $\lambda = \omega^2$. The operator A can be shown to be hermitian and positive-semidefinite. The eigenmodes can be normalized

$$x^T x = 1 \tag{4}$$

Sensitivity Derivatives (Nondegenerate Case)

Our optimization problem can be formulated as

$$\begin{aligned} \min_{p} \quad & g(p, x, \lambda) \\ \text{s.t.} \quad & Ax = \lambda x \\ & x^T x = 1 \end{aligned} \tag{5}$$

where p is a vector of design parameters and g is the objective function we seek to minimize. To develop the eign-adjoint variable method, we first define the Lagrangian as

$$L = g(p, x, \lambda) + \kappa^T(A - \lambda)x + \sigma(1 - x^T x) \tag{6}$$

where σ and κ are the Lagrange multipliers, for eigenvalues and eigenvectors, respectively [3]. Direct differentiation of equation.6 with respect to the design parameters

[3] We will choose the Lagrange multipliers in a clever way to cancel the annoying derivative terms later.

$$\frac{dL}{dp} = \frac{\partial g}{\partial p} + \frac{\partial g}{\partial x}\frac{dx}{dp} + \frac{\partial g}{\partial \lambda}\frac{d\lambda}{dp} + \kappa^T\left(\frac{dA}{dp} - \frac{d\lambda}{dp}\right)x + \kappa^T(A-\lambda)\frac{dx}{dp} - 2\sigma\left(x^T\frac{dx}{dp}\right) \tag{7}$$

Collecting terms in $\frac{d\lambda}{dp}$ and $\frac{dx}{dp}$

$$\frac{dL}{dp} = \frac{\partial g}{\partial p} + \left(\frac{\partial g}{\partial x} + \kappa^T(A-\lambda) - 2\sigma x^T\right)\frac{dx}{dp} + \left(\frac{\partial g}{\partial \lambda} - \kappa^T x\right)\frac{d\lambda}{dp} + \kappa^T\frac{dA}{dp}x \tag{8}$$

Requiring κ and σ to satisfy the following equations,

$$\kappa^T x = \frac{\partial g}{\partial \lambda} \tag{9}$$

$$(A-\lambda)\kappa - 2\sigma x = -\left(\frac{\partial g}{\partial x}\right)^T \tag{10}$$

causes the implicit design variation of the Lagrangian to vanish. The second equation is problematic: $A-\lambda$ is singular, with a null space of x. So, we also require σ to make the solution orthogonal to the the null space: $x^T\left(\left(\frac{\partial g}{\partial x}\right)^T - 2\sigma x\right) = 0$. Then, $\sigma = \frac{x^T\left(\frac{\partial g}{\partial x}\right)^T}{2}$

Equation. 10 then becomes

$$(A-\lambda)\kappa - x^T\left(\frac{\partial g}{\partial x}\right)^T x = -\left(\frac{\partial g}{\partial x}\right)^T \tag{11}$$

This equation still has infinitely many solutions: $\kappa + \theta x$ is a solution for any θ. We can fix θ using equation. 9 and then $\theta = \frac{\partial g}{\partial \lambda}$. The sensitivity of g is finally,

$$\frac{dg}{dp} = \left(\kappa + \frac{\partial g}{\partial \lambda}x\right)^T\frac{dA}{dp}x \tag{12}$$

Thus the sensitivity of g is obtained in following steps:

1. Solving equation 3 for x
2. Solving equation 11 for κ
3. Substituting with x and κ in equation 12