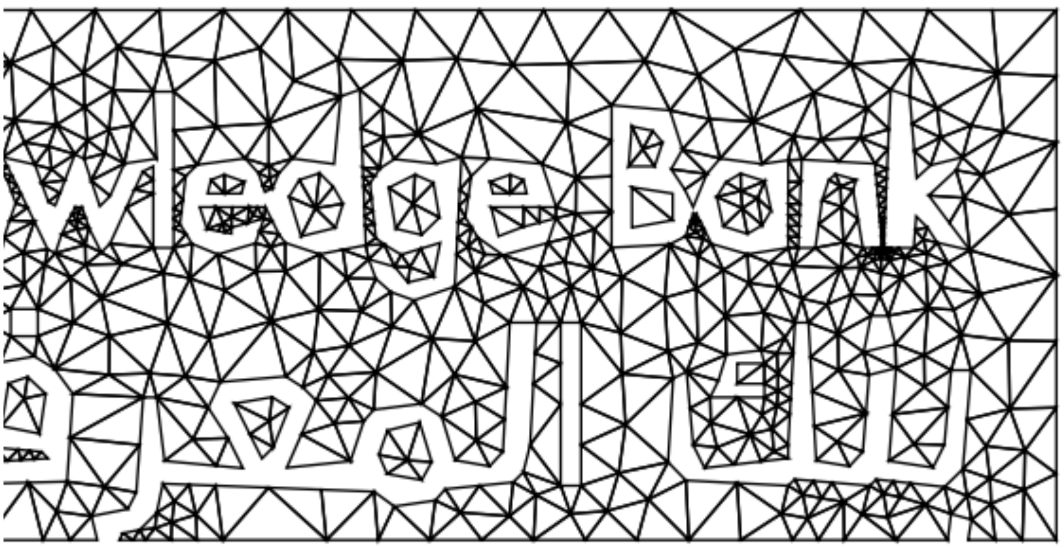
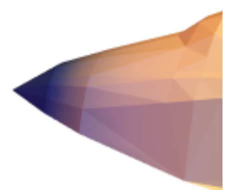
# Finite Element Programming with the Wolfram Language



M. K. AbdElrahman

Wolfram Research

# Outline

Introduction

Finite Element Data within NDSolve

Passing Finite Element Options to NDSolve

A Workflow Overview

The Partial Differential Equation Problem Setup

Stationary PDEs

# Introduction

**NDSolve** provides a high-level, one-step interface for solving partial differential equations with the finite element method. However, you may want to control the steps of the solution process with more detail. The **NDSolve`Fem`** package provides a lower-level interface that gives extensive control for each part of the solution process.

- To use the finite element functions, the package needs to be loaded.

In[ ]:=
```
Needs["NDSolve`FEM` "]
```

- The low-level functions in the **NDSolve`Fem`** package may be used for a variety of purposes:
  - To better understand what NDSolve does internally and how it finds solutions
  - To better understand what options are available, what their usages are and when they are beneficial
  - To intercept the solution process at various stages and provide access to intermediate data
  - To enable development of specific, finite element–based solvers, not only to solve PDEs but also other areas of numerics
  - To use NDSolve as an equation preprocessor

## Finite Element Data within NDSolve

Set up the **NDSolve`StateData** object:

In[ ]:=
```
{state}=NDSolve`ProcessEquations [{Laplacian [u[x,y],{x,y}]==1,DirichletCondition [u[x,y]==0,True]},
```

Out[ ]=
```
{NDSolve`StateData [<SteadyState >]}
```

In[ ]:=
```
femdata =state["FiniteElementData "]
```

Out[ ]=
```
FiniteElementData [<1281>]
```

Compute the system solution:

In[ ]:=
```
NDSolve`Iterate [state]
```

The solution is then stored in the finite element data object:

In[ ]:=
```
state["SolutionData "]
```

Out[ ]=
```
{{None , NumericalRegion [ImplicitRegion [0 ≤ x ≤ 1 && 0 ≤ y ≤ 1, {x, y}], {{0, 1}, {0, 1}}],
  {-1., -1., -1., -1., -1., -0.25, -0.25, -0.25, -0.25, -0.25, -0.90625, -0.75,
   -0.53125, -0.90625, -0.75, -0.53125, -0.835938, -0.648438, -0.62625,
   -0.90625, -0.701172, -0.648438, -0.591797, -1., -0.960938, -0.960938,
   -0.794922, -0.873047, -0.835938, -0.935547, -0.398438, -0.466797,
   -0.873047, -0.960938, -0.935547, -0.398438, -0.58, -0.453438, -0.960938,
   -1., -0.960938, -0.960938, -1., -0.466797, -0.25, -1., -0.25, -0.453438,
   -0.398438, -0.25, -0.637422, -0.835938, -0.78625, -0.690938, -0.740547,
   -0.835938, -0.90625, -0.25, -0.398438, -0.75, -0.648438}, {}, {}, {}, {}, {}}}
```

Extract the solution as an **InterpolatingFunction** and plot the result:

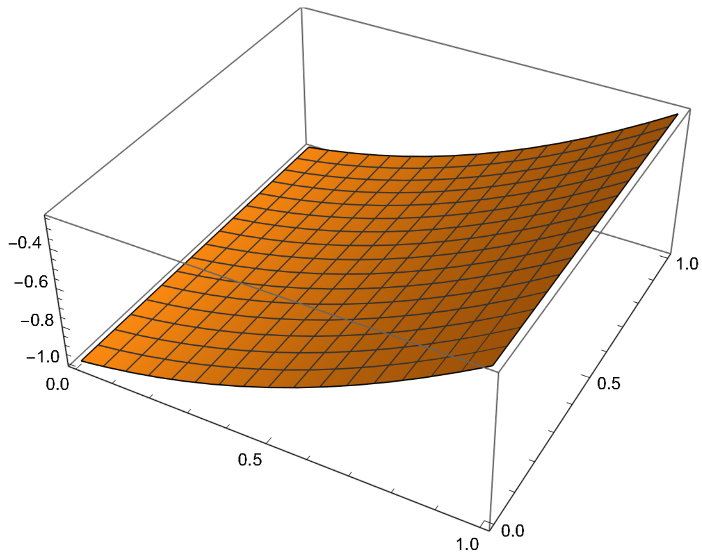
In[ ]:=
```
u/.NDSolve`ProcessSolutions [state]
```

Out[ ]=
```
InterpolatingFunction [ Domain: {{0., 1.}, {0., 1.}}
                        Output: scalar ]
```

In[ ∘ ]:=

```
Plot3D[%[x,y],{x,0,1},{y,0,1}]
```

Out[ ∘ ]=



# Passing Finite Element Options to NDSolve

Call NDSolve with finite element options specified:

```
NDSolve [{Laplacian [u[x,y],{x,y}]==1,DirichletCondition [u[x,y]==0,True]}, u, {x,0,1},{y,0,1},
Method →{"PDEDiscretization "→{"FiniteElement ","MeshOptions "→{"MaxCellMeasure "→0.1},
"IntegrationOrder "→5}}]
```

Out[ ]= {{u → InterpolatingFunction [ Domain : {{0., 1.}, {0., 1.}} Output : scalar ]}}

Solve a transient PDE with NDSolve and finite element options specified:

In[ ]:=
```
NDSolve [{D[u[t,x],t]-Laplacian [u[t,x],{x}]==t,DirichletCondition [u[t,x]==0,True],u[0,x]==0}, u, {t,0
Method →{
 "PDEDiscretization "→{"MethodOfLines ","SpatialDiscretization "→{"FiniteElement ",
"MeshOptions "→{"MaxCellMeasure "→0.1},"IntegrationOrder "→5}}}]
```

Out[ ]= {{u → InterpolatingFunction [ Domain : {{0., 1.}, {0., 1.}} Output : scalar ]}}

# A Workflow Overview

- The solution is found in three stages:
  - Initialization
  - Discretization
  - Solving

## Initialization

During the initialization stage, the PDE and boundary conditions are analyzed and classified into different components, and these results are stored in the **PDECoefficientData** and **BoundaryConditionData** objects, respectively. The finite element method data is set up and stored in the **FEMMethodData** object.

| | |
|---|---|
| InitializePDECoefficients | initialize partial differential equation coefficients |
| InitializeBoundaryConditions | initialize boundary conditions |
| InitializePDEMethodData | initialize partial differential equation method data |

Initialization functions.

## Discretization

In the second stage, the PDE and boundary conditions are discretized and stored in **DiscretizedPDE-Data** and **DiscretizedBoundaryConditionData**. "Discretized" essentially means that the continuous PDE and boundary conditions are approximated by discrete versions represented by so-called system matrices.

| | |
|---|---|
| DiscretizePDE | discretize initialized partial differential equations |
| DiscretizeBoundaryConditions | discretize initialized boundary conditions |

Discretization functions.

## Solving

The last step is then to merge the discretized PDE with the discretized boundary conditions (**DeployBoundaryCoditions**) and to use **LinearSolve** to solve the equations.

| | |
|---|---|
| DeployBoundaryConditions | deploy discretized boundary conditions into discretized partial differential equations |
| LinearSolve | solve linear systems of equations |

Solution stage functions.

Once the solution is found, InterpolatingFunction objects can be constructed.

| | |
|---|---|
| ProcessPDESolutions | generates InterpolatingFunction objects |

Post-processing functions.

# The Partial Differential Equation Problem Setup

To make a PDE susceptible to being solved by a numerical method such as the finite element method, three components are needed:

- A partial differential equation (PDE)
- A region
- Boundary conditions

Specify a partial differential equation operator:

In[ ]:=
```
op=Laplacian [u[x,y],{x,y}]-1
```

Out[ ]= $-1+u^{(0,2)}[x,\ y]+u^{(2,0)}[x,\ y]$

Create a region:

In[ ]:=
```
Ω=ImplicitRegion [0≤x≤1&&0≤y≤1,{x,y}]
```

Out[ ]= ImplicitRegion [0 ≤ x ≤ 1 && 0 ≤ y ≤ 1, {x, y}]

Set up boundary conditions at the left- and right-hand sides of the domain:

In[ ]:=
```
ΓD=DirichletCondition [u[x,y]==-1,x==0]
```

Out[ ]= DirichletCondition [u[x, y] == -1, x == 0]

In[ ]:=
```
ΓN=NeumannValue [-1+u[x,y],x==1]
```

Out[ ]= NeumannValue [-1 + u[x, y], x == 1]

## Use NDSolve to Solve the PDE

In[ ]:=
```
NDSolve [{op==ΓN,ΓD},u,{x,y}∈Ω]
```

Out[ ]= {{u → InterpolatingFunction [ Domain : {{0., 1.}, {0., 1.}} Output : scalar ]}}

## Use NDSolve`FEM` Functions

**1.** Extract the **NDSolve`StateData** object:

```
{state}=NDSolve`ProcessEquations  [{op==ΓN,ΓD},u,{x,y}x==1∈Ω,Method→{"PDEDiscretization  "->"Finite
```

Out[ ]= {NDSolve`StateData [<SteadyState >]}

**2.** Get the finite element data:

In[ ]:=
```
femdata = state["FiniteElementData "]
```

Out[ ]=
```
FiniteElementData [<1265>]
```

**3.** Inspect the properties:

In[ ]:=
```
femdata ["Properties "]
```

Out[ ]=
```
{BoundaryConditionData , FEMMethodData , PDECoefficientData , Properties }
```

The PDECoefficientData has been created by a call to **InitializePDECoefficients**. The BoundaryConditionData has been created by a call to **InitializeBoundaryConditions**, and the FEMMethodData has been created by a call to **InitializePDEMethodData**. These data objects hold data for subsequent computations. The creation of these data objects is discussed in the next section.

# Stationary PDEs

## Manual Initialization

### Variable and solution data

Create the variable data with dependent and independent variable names:

In[ ]:=
```
vd=NDSolve`VariableData[{"DependentVariables"→{u},"Space"→{x,y}}]
```

Out[ ]= {None, {x, y}, {u}, {}, {}, {}, {}, {}}

Specify a **NumericalRegion**:

In[ ]:=
```
nr=ToNumericalRegion[Ω]
```

Out[ ]= NumericalRegion[ImplicitRegion[0 ≤ x ≤ 1 && 0 ≤ y ≤ 1, {x, y}], {{0, 1}, {0, 1}}]

Create the solution data with the **"Space"** component set:

In[ ]:=
```
sd=NDSolve`SolutionData[{"Space"→nr}]
```

Out[ ]= {None, NumericalRegion[ImplicitRegion[0 ≤ x ≤ 1 && 0 ≤ y ≤ 1, {x, y}], {{0, 1}, {0, 1}}], {}, {}, {}, {}, {}, {}}

# Stationary PDEs

## Manual Initialization

### InitializePDECoefficients

The coefficients of the model PDE:

In[ ]:=
```
op
```

Out[ ]= $-1+u^{(0,2)}[x,y]+u^{(2,0)}[x,y]$

In[ ]:=
```
coefficients ={"DiffusionCoefficients "→{{IdentityMatrix [2]}},"LoadCoefficients "→{{1}}}
```

Out[ ]=
```
{DiffusionCoefficients → {{{{1, 0}, {0, 1}}}}, LoadCoefficients → {{1}}}
```

Initialize the partial differential equation coefficients:

In[ ]:=
```
initCoeffs =InitializePDECoefficients [vd,sd,coefficients ]
```

Out[ ]=
```
PDECoefficientData [<1,2>]
```

Extract the system size and the spatial dimension of the initialized coefficients:

In[ ]:=
```
{initCoeffs ["SystemSize "],initCoeffs ["SpatialDimension "]}
```

Out[ ]=
```
{1, 2}
```

Extract the raw coefficients:

In[ ]:=
```
initCoeffs ["All"]
```

Out[ ]=
```
{{{{1}}, {{{{0}, {0}}}}, {{{{-1, 0}, {0, -1}}}}, {{{{0}, {0}}}}, {{{{0, 0}}}}, {{{0}}}, {{{0}}}, {{{0}}}}
```

During the initialization, the coefficients are classified into different categories of coefficients, such as stationary coefficients and transient coefficients:

In[ ]:=
```
initCoeffs ["Properties "]
```

Out[ ]=
```
{All, ConservativeConvectionCoefficients , Constraints , ConvectionCoefficients ,
 DampingCoefficients , DiffusionCoefficients , Discrete , DomainType ,
 IndexedDiscrete , LoadCoefficients , LoadDerivativeCoefficients ,
 MassCoefficients , MessageHead , Nonlinear , Parametric , Properties ,
 ReactionCoefficients , SpatialDimension , Stationary , SystemSize , Transient}
```

# Stationary PDEs

## Manual Initialization

### InitializeBoundaryConditions

In[ ∘ ]:=
```
initBCs = InitializeBoundaryConditions [vd, sd, {{ΓD, ΓN}}]
```

Out[ ∘ ]= BoundaryConditionData [<1,2>]

Boundary condition coefficients are classified into categories similar to the PDE coefficients:

In[ ∘ ]:=
```
initBCs ["Properties "]
```

Out[ ∘ ]=
```
{All, BoundaryTolerance , Constraints , Discrete , DomainType , IndexedDiscrete ,
 Nonlinear , Parametric , Properties , ScaleFactor , Stationary , Transient}
```

# Stationary PDEs

## Manual Initialization

### InitializePDEMethodData

Initialize the finite element data with the variable and solution data:

```
In[ ]:= methodData =InitializePDEMethodData [vd ,sd]
```

```
Out[ ]= FEMMethodData [<1265 ,{2},4>]
```

Query degrees of freedom, interpolation order and integration order:

```
In[ ]:= {methodData ["DegreesOfFreedom "], methodData ["InterpolationOrder "], methodData ["IntegrationOrd
```

```
Out[ ]= {1265 , {2}, 4}
```

Initialize the finite element data with the variable and solution data, with options for the mesh generation:

```
In[ ]:= methodData =InitializePDEMethodData [vd ,sd ,Method →{"FiniteElement ","MeshOptions "→{"MaxCellMea
```

```
Out[ ]= FEMMethodData [<61 ,{2},4>]
```

Query degrees of freedom, interpolation order and integration order:

```
In[ ]:= {methodData ["DegreesOfFreedom "],
 methodData ["InterpolationOrder "], methodData ["IntegrationOrder "]}
```

```
Out[ ]= {61 , {2}, 4}
```

During the method initialization, an **ElementMesh** object will be generated and stored in the Numeri - calRegion object:

```
In[ ]:= mesh =nr["ElementMesh "]
```

```
Out[ ]= ElementMesh [{{0. , 1.}, {0. , 1.}}, {TriangleElement [<22>]}]
```

For one dependent variable, the degrees of freedom correspond to the number of coordinates in the mesh:

```
In[ ]:= Length [mesh ["Coordinates "]]=== methodData ["DegreesOfFreedom "]
```

```
Out[ ]= True
```

Increase the integration order:

```
In[ ]:= InitializePDEMethodData [vd, sd, Method → {"FiniteElement ", "IntegrationOrder "→5}]["IntegrationOrd
```

```
Out[ ]= 5
```

Inspect that the state object created through **NDSolve`ProcessEquations** generates the same FEM-MethodData object:

```
In[ ]:= {s1}=NDSolve`ProcessEquations [{op==Γ_N, Γ_D}, u, {x, y}∈Ω, Method ->{"FiniteElement ", "MeshOptions "→{"M
```

```
In[ ]:= s1["FiniteElementData "]["FEMMethodData "]=== methodData
```

```
Out[ ]= True
```

# Stationary PDEs

## Extracting the Initialized Finite Element Data from NDSolve`StateData

As an alternative to the manual initialization of the finite element method data, PDE coefficients and boundary conditions, **NDSolve`ProcessEquations** can be utilized for this purpose:

In[ ]:=
```
{state}=NDSolve`ProcessEquations[{op==ΓN,ΓD},u,{x,y}∈nr,Method→{"FiniteElement","MeshOptions"→
femdata=state["FiniteElementData"]
```

Out[ ]=
```
FiniteElementData[<61>]
```

Inspect that the objects created are the same:

In[ ]:=
```
femdata["FEMMethodData"]===methodData
```

Out[ ]=
```
True
```

# Stationary PDEs

## Discretization

### DiscretizePDE

In[ ]:=
```
discretePDE =DiscretizePDE [initCoeffs ,methodData ,sd]
```

Out[ ]= DiscretizedPDEData [<61>]

The display form of the DiscretizedPDEData shows the first dimension of the system matrices:

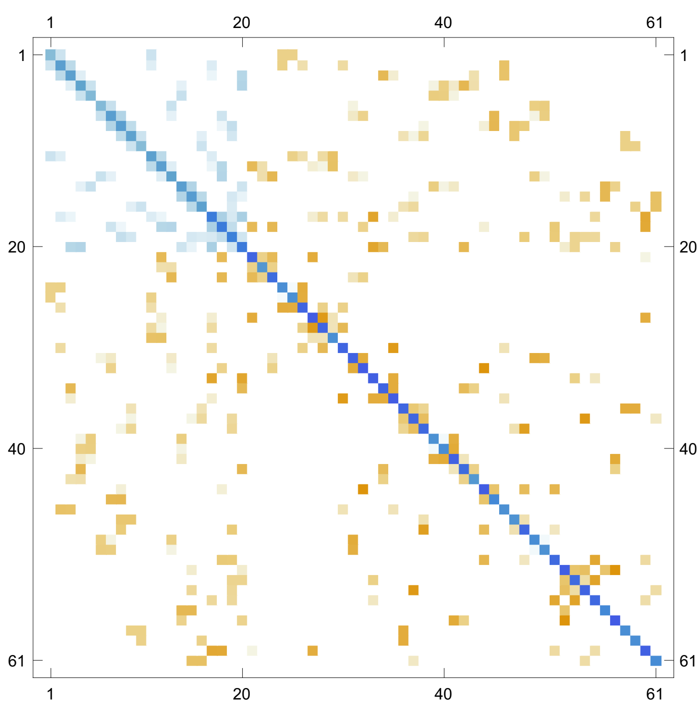
In[ ]:=
```
{load ,stiffness ,damping ,mass}=discretePDE ["SystemMatrices "]
```

Out[ ]= {SparseArray[ Specified elements : 61 Dimensions : {61, 1} ], SparseArray[ Specified elements : 571 Dimensions : {61, 61} ], SparseArray[ Specified elements : 0 Dimensions : {61, 61} ], SparseArray[ Specified elements : 0 Dimensions : {61, 61} ]}

Visualize the stiffness matrix:

In[ ]:=
```
MatrixPlot [stiffness ]
```

Out[ ]=



Extract the system matrices separately:

In[ ]:=
```
{discretePDE ["LoadVector "],discretePDE ["StiffnessMatrix "],discretePDE ["DampingMatrix "],discre
```

Out[ ]=
```
{SparseArray[ Specified elements : 61 Dimensions : {61, 1} ], SparseArray[ Specified elements : 571 Dimensions : {61, 61} ],
 SparseArray[ Specified elements : 0 Dimensions : {61, 61} ], SparseArray[ Specified elements : 0 Dimensions : {61, 61} ]}
```

# Stationary PDEs

## Discretization

### DiscretizeBoundaryCondition

**DiscretizeBoundaryConditions** computes a discretized version of the boundary conditions given. In a later step, the discretized boundary conditions are then deployed into the actual system matrices:

In[ ]:=
```
discreteBCs =DiscretizeBoundaryConditions [initBCs ,methodData ,sd]
```

Out[ ]= DiscretizedBoundaryConditionData [<61>]

Components from the generalized Neumann boundary value:

In[ ]:=
```
{discreteBCs ["LoadVector "], discreteBCs ["StiffnessMatrix "]}
```

Out[ ]= {SparseArray[ Specified elements : 9 Dimensions : {61, 1} ], SparseArray[ Specified elements : 33 Dimensions : {61, 61} ]}

Components from the **DirichletCondition** boundary condition:

In[ ]:=
```
{discreteBCs ["DirichletMatrix "], discreteBCs ["DirichletValues "]}
```

Out[ ]= {SparseArray[ Specified elements : 9 Dimensions : {9, 61} ], SparseArray[ Specified elements : 9 Dimensions : {9, 1} ]}

# Stationary PDEs

## Solution

### DeployBoundaryConditions

To have the discretized boundary conditions take effect, they need to be deployed into the system matrices. To save memory, this is done in place; no new matrices are generated:

In[ ]:=
```
DeployBoundaryConditions [{load ,stiffness },discreteBCs ]
```

Out[ ]=
```
DeployedBoundaryConditionData [<Insert>]
```

During the deployment of the boundary conditions, the contributions from **NeumannValue** are added to the stiffness and load matrix. Then the system matrices are modified such that the DirichletCondi - tion is satisfied.

After the boundary condition deployment, the system matrices have changed; the system matrices typically contain fewer elements:

In[ ]:=
```
{load ,stiffness }
```

Out[ ]=
```
{SparseArray[ Specified elements : 61  Dimensions : {61, 1} ], SparseArray[ Specified elements : 508  Dimensions : {61, 61} ]}
```

Components from the DirichletCondition boundary condition:

In[ ]:=
```
{discreteBCs ["DirichletMatrix "], discreteBCs ["DirichletValues "]}
```

Out[ ]=
```
{SparseArray[ Specified elements : 9  Dimensions : {9, 61} ], SparseArray[ Specified elements : 9  Dimensions : {9, 1} ]}
```

# Stationary PDEs

## Equation Solving

The system of equations can then be solved with LinearSolve:

In[ ]:=
```
Short[solution = LinearSolve[stiffness, load]]
```

Out[ ]//Short=
```
{{-1.}, {-1.}, {-1.}, {-1.}, {-1.}, {-0.25}, ≪50≫,
 {-0.90625}, {-0.25}, {-0.398438}, {-0.75}, {-0.648438}}
```

# Stationary PDEs

## Post-processing

As a post-processing step, an interpolation function can be created from the solution obtained from LinearSolve. For this, the solution data is updated with the solution found and **ProcessPDESolutions** then constructs the InterpolatingFunction:

In[ ]:=
```
NDSolve`SetSolutionDataComponent[sd,"DependentVariables",Flatten[solution]];
```

In[ ]:=
```
{ifun}=ProcessPDESolutions[methodData, sd]
```

Out[ ]=
```
{InterpolatingFunction[ Domain: {{0., 1.}, {0., 1.}} Output: scalar ]}
```

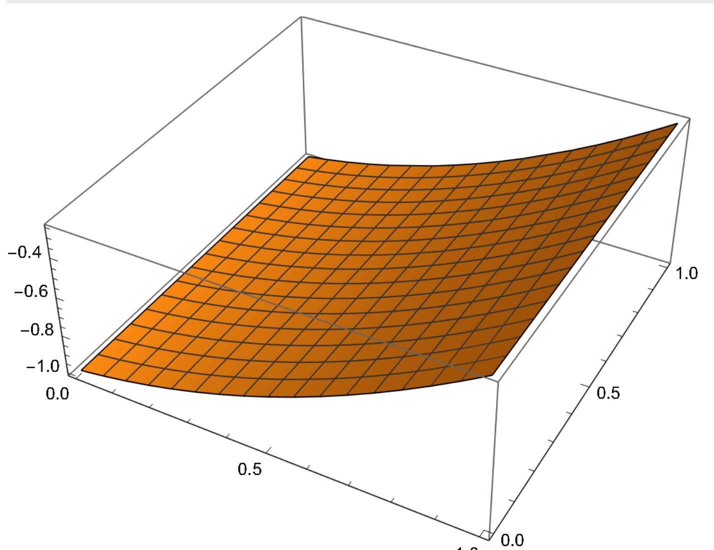
ProcessPDESolutions has the advantage that it also works well for multiple dependent variables. In the specific case at hand, an alternative to ProcessPDESolutions is to create the interpolation function directly with **ElementMeshInterpolation**. Here the first argument is the mesh and the second argument is the values at the mesh nodes:

In[ ]:=
```
ifun=ElementMeshInterpolation[mesh, solution]
```

Out[ ]=
```
InterpolatingFunction[ Domain: {{0., 1.}, {0., 1.}} Output: scalar ]
```

In[ ]:=
```
Plot3D[ifun[x, y],{x,y}∈mesh]
```

Out[ ]=



It is also possible to extract the ElementMesh from the interpolating function:

In[ ]:=
```
ifun["ElementMesh"]
```

Out[ ]=
```
ElementMesh[{{0., 1.}, {0., 1.}}, {TriangleElement[<22>]}]
```