AE 502: Homework Project 3

Due Date: April 16, 2023, 11:59pm CT

Problem

Hill's approximation to the circular restricted three body problem (CR3BP) assumes the third body is infinitely far away compared to a satellite with negligible mass that orbits the primary body [Hill, 1878]. Similar to the CR3BP, the two massive bodies are still orbiting each other with a constant angular speed. The Hamiltonian of this problem $\mathcal{H}$ reads

$$\mathcal{H} = \frac{\mathbf{p}^2}{2} - \frac{1}{||\mathbf{q}||} - \mathbf{q}^T \Omega\, \mathbf{p}, \tag{1}$$

where $\mathbf{q} = Exp(t\Omega)\mathbf{r}$ and $\mathbf{p} = Exp(t\Omega)\mathbf{v}$ are canonical, time dependent transformations from non-rotating positions and velocities $\mathbf{r}$, $\mathbf{v}$ to generalized coordinates $\mathbf{q}$ and specific momenta $\mathbf{p}$ in the rotating system. Furthermore,

$$\Omega = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix}. \tag{2}$$

The system rotates around the vector $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ with angular speed $\omega = ||\boldsymbol{\omega}||$. For satellites close enough to the primary body, the dominant perturbation to the Keplerian motion around the primary turns out to be the rotation of the frame of reference caused by the other far-away massive body. This perturbation is represented by the term $-\mathbf{q}^T \Omega\, \mathbf{p}$. Note that the gravitational parameter has been set to unity.

1. (50 points) Assume the rotation rate of the frame is very low compared to the velocities in your system, e.g. ω=0.01 per system time unit (TU). Use Hori-Lie-Deprit perturbation theory to push the perturbation beyond the first order in ω. What do the new Kamiltonian and the corresponding equations of motion look like?

2. (50 points) A more convenient representation of the same system can be achieved through the use of Delaunay variables. In fact, the above Hamiltonian expressed in Delaunay variables simply reads

$$\mathcal{H}_D = -\frac{1}{2L^2} + \omega H \tag{3}$$

where

$$
\begin{aligned}
L &= na^2 && (4)\\
G &= L\,(1-e^2)^{1/2} && (5)\\
H &= G\cos i && (6)
\end{aligned}
$$

with a, e, i, n being the semimajor axis, eccentricity, inclination with respect to the orbital plane of the massive bodies, and the mean motion of the satellite around the primary, respectively. Determine the equations of motion and plot the orbit of the satellite in Cartesian, non-rotating space, i.e. $\mathbf{r}$(t), for a=1, e=0.5, i=45° for 100 time units.

3. (BONUS QUESTION - no partial credit - 50 points) Compare analytic solutions of question 1 to those of question 2 for at least 20 different initial conditions and integration times of 100 time units (TU) in two equinoctial element plots, namely h vs k and p vs q, where

$$
\begin{aligned}
h &= e\cdot\sin(\omega+\Omega) && (8)\\
k &= e\cdot\cos(\omega+\Omega) && (9)\\
p &= \tan(i/2)\sin\Omega && (10)\\
q &= \tan(i/2)\cos\Omega && (11)
\end{aligned}
$$

Reproduce the same plots for frame rotation frequencies $\omega = \{0.02, 0.1, 0.5\}\mathrm{TU}^{-1}$. What can you say about the behavior of the analytic and perturbation solutions?

Good luck!

Deliverables

Please write a short report and upload your code to your own GitHub repository! No extensions! Late submissions will not be graded!

References

George William Hill. Researches in the lunar theory. *American journal of Mathematics*, 1(1):5–26, 1878.

Solutions

1. Since the Hamiltonian of Problem 2 is equivalent to the Hamiltonian of Problem 1 we will base our approach off of the former.

$$\mathcal{H}_D = -\frac{1}{2L} + \omega H \tag{12}$$

Here, $\mathcal{H}_0 = -\frac{1}{2L}$ and $\mathcal{H}_1 = \omega H$. With the generating function W the homologic equation reads

$$\mathcal{K}_1 = \{\mathcal{H}_0, W\} + \mathcal{H}_1 = 0 \tag{13}$$

Since W