PHYS 3142 HW 8

Due date: 11:59 PM 10^{th} Apr. 2022

- Submit a report that includes your results and your python scripts
- Make sure your code can run
- Write comments in your code
- If you submit the assignment after the deadline or the report is missing, you can only get at most 80% of the full marks.
- If there is any kind of plagiarism, all students involved will get zero marks.

1 Chain of resistors (80 points)

Consider a long chain of resistors wired up like this:

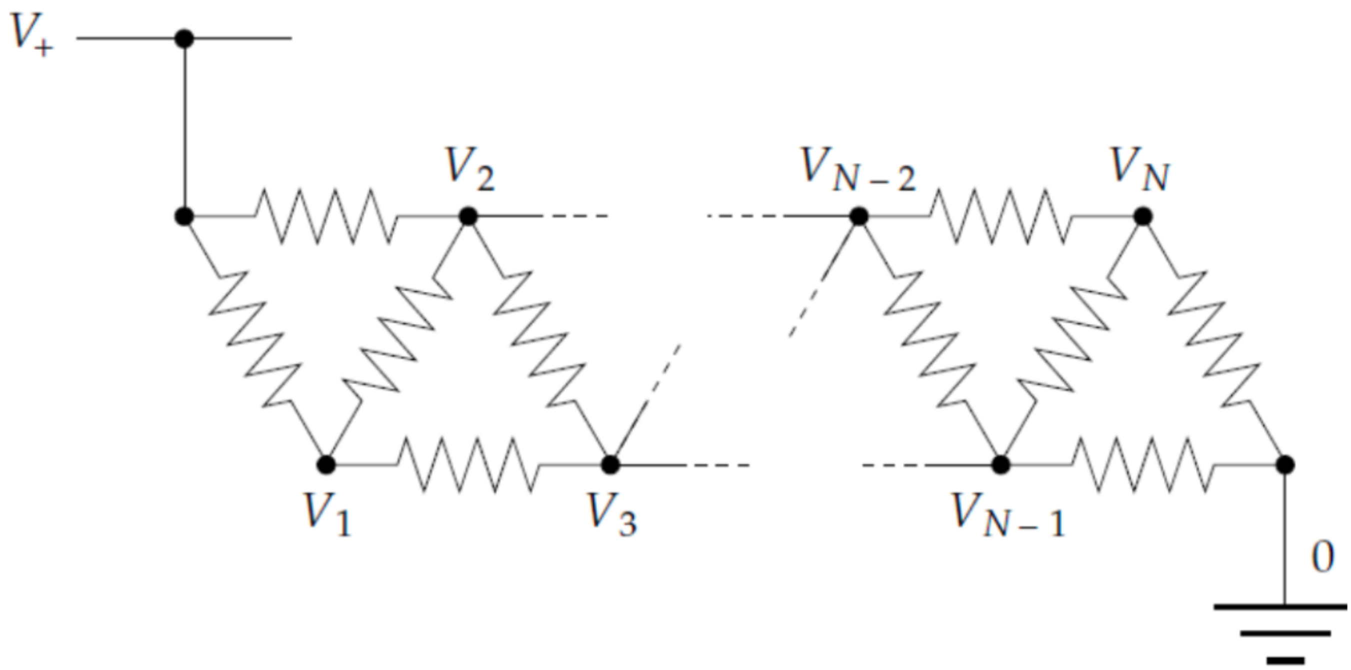

Figure 1: chain of resistors

All the resistors have the same resistance R. The power rail at the top is at voltage $V_+ = 5V$. The problem is to find the voltages $V_1 \dots V_N$ at the internal points in the circuit.

(a) Using Ohm's law and the Kirchhoff current law, which says that for any junction in an electrical circuit, the sum of currents flowing into that junction is equal to the sum of currents flowing out of that junction. Please show that when N=3, the voltages V_1, V_2, V_3 satisfy the equations

$$\begin{aligned} 3V_1 - V_2 - V_3 &= V_+ \\ 4V_2 - V_1 - V_3 &= V_+ \\ 3V_3 - V_1 - V_2 &= 0 \end{aligned} \tag{1}$$

Express these equations in vector form $Av = w$ and find the values of matrix A and the vector w. Fractorize the matrix A analytically using the LU decomposition. Then solve the simulaneous equations by either using the Gaussian elimination or LU decomposition. Also write a simple program and use numpy.linalg to check your result. [you may upload your hand writing by .jpg or .png files]

(b) If there are N internal junctions, show that the voltages $V_1 \dots V_N$ satisfy the equations:

$$\begin{gathered} 3V_1 - V_2 - V_3 = V_+ \\ -V_1 + 4V_2 - V_3 - V_4 = V_+ \\ -V_1 - V_2 + 4V_3 - V_4 - V_5 = 0 \\ \dots \\ -V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} = 0 \\ \dots \\ -V_{N-3} - V_{N-2} + 4V_{N-1} - V_N = 0 \\ -V_{N-2} - V_{N-1} + 3V_N = 0 \end{gathered} \tag{2}$$

Express these equations in vector form $Av = w$ and define a function to return the matrix A and the vector w for given N automatically. Show A and w when N=5 and 6.

(c) Write a program to use **solve** in **numpy.linalg** to solve the values of the V_i when there are $\mathrm{N} = 5, 6$ internal junctions with unknown voltages.

(d) Now solve V_i for the case where there are $N = 10000$ internal junctions. Use the **solve** function in **numpy.linalg** and also the **banded** function. Compare these two method by the calculation time. (find the banded.py file in the Lec. 14 folder)

2 exchange the rows (20 points)

For an arbitrary 4×4 matrix M, please find the transform matrix T manually that transform it into M', where the three rows of M are exchanged.

$$M = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{3}$$

$$M' = TM = \begin{bmatrix} m_{10} & m_{11} & m_{12} & m_{13} \\ m_{30} & m_{31} & m_{32} & m_{33} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{00} & m_{01} & m_{02} & m_{03} \end{bmatrix} \tag{4}$$

Optional

3 The rank of a binary matrix (10 points)

The rank of matrices has a lot of applications in physics. In information theory, to analyze the Renyi entropy[1] in Clifford circuits[2], we need to consider the rank of binary matrices. Please use the Gaussian elimination method to write a function which is capable of calculating the rank of an arbitrary binary matrix. Please find out the rank of a random binary matrix, printing out the original matrix and final matrix after the Gaussian elimination.

An example of Gaussian elimination for a binary matrix A:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{5}$$

After one step:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{6}$$

finally:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

So the rank of A should be 2 instead of 3, if this matrix is considered binary!

References

[1] Renyi entropy. https://en.wikipedia.org/wiki/R%C3%A9nyi_entropy.

[2] Clifford gates. https://en.wikipedia.org/wiki/Clifford_gates.