Supplementary Material for "Wilson–Fisher Renormalization of Discrete Surface-Wave Turbulence"

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This Supplementary Material provides the scaling arguments and renormalization-group (RG) considerations underlying the main text. In particular, we justify the definition of the running coupling, the Wilson–Fisher beta function, the topology-dependent nonlinear transfer rate, and we provide additional steps supporting the Reynolds-number scaling of the integrated inertial spectral weight reported in the main text.

RUNNING COUPLING AND INERTIAL SPECTRA

We analyze the surface-velocity signal $v(t)$ measured by laser Doppler vibrometry through its power spectral density $S(\omega)$, defined by $\langle v^2 \rangle = \int_0^\infty S(\omega)\, d\omega$. In the inertial interval, weak wave turbulence predicts Kolmogorov–Zakharov (KZ) power laws $S(\omega) \sim \omega^{-p}$, with $p = 17/6$ for capillary waves and $p = 5/2$ for gravity waves [1, 2].

At scaling level, the velocity variance carried by a logarithmic band around ω is $v_\omega^2 \sim S(\omega)\,\Delta\omega$ with $\Delta\omega \sim \omega$, hence

$$v_\omega^2 \sim \omega\, S(\omega). \tag{S1}$$

Using $v_\omega \sim \omega a_\omega$ to relate velocity to surface-displacement amplitude a_ω then gives

$$a_\omega^2 \sim \frac{S(\omega)}{\omega}. \tag{S2}$$

DIMENSIONLESS NONLINEARITY AND RG COUPLING

For weakly nonlinear surface waves, the standard control of nonlinearity is the wave steepness $\epsilon(\omega) \sim k(\omega)a_\omega$, which measures the relative slope of the interface and sets the size of nonlinear frequency shifts and resonance broadening; accordingly, it is the small parameter underlying weak-turbulence closures [1, 2, 4]. Combining with Eq. (S2) yields

$$\epsilon(\omega)^2 \sim k(\omega)^2\, \frac{S(\omega)}{\omega}. \tag{S3}$$

This motivates the RG running coupling used throughout the main text,

$$g(\omega) \equiv \epsilon(\omega)^2, \tag{S4}$$

which is dimensionless, scale dependent, and small in the DSWT regime.

WILSON–FISHER BETA FUNCTION

Introducing the RG "time" $\ell = \ln(\omega/\omega_0)$, the inertial KZ–scaling and the dispersion relation $k(\omega) \sim \omega^\alpha$ imply

$$g(\omega) \sim \omega^{\,y_g}, \qquad y_g = 2\alpha - (p+1), \tag{S5}$$

where we used $g(\omega) = \epsilon(\omega)^2 \sim k(\omega)^2 S(\omega)/\omega$ [Eq. (S3)] and $S(\omega) \sim \omega^{-p}$. At scaling level, the RG flow of the coupling may be written in Wilson–Fisher form,

$$\beta(g) \equiv \frac{dg}{d\ell} = y_g\, g - B_N\, g^{\,N-1}, \tag{S6}$$

where N is the order of the resonant interaction ($N = 3$ for capillary-wave triads and $N = 4$ for gravity-wave tetrads), and $B_N > 0$ encodes nonlinear saturation. Besides the unstable Gaussian fixed point $g^\star = 0$, Eq. (S6) admits a nontrivial stable fixed point corresponding to the KZ inertial state discussed in the main text.

TOPOLOGY-DEPENDENT NONLINEAR TRANSFER RATE

In wave turbulence theory, spectral transfer is mediated by resonant N-wave interactions. The interaction vertex is of order $(N-2)$ in the wave amplitude, and the collision integral governing spectral evolution is quadratic in this vertex. As a result, the nonlinear transfer rate scales as

$$\tau_{\rm nl}^{-1}(\omega) \sim \omega\, \epsilon(\omega)^{2(N-2)}. \tag{S7}$$

Substituting $\epsilon(\omega)^2 \sim k(\omega)^2 S(\omega)/\omega$ [Eq. (S3)] into Eq. (S7) yields

$$\tau_{\rm nl}^{-1}(\omega) \sim \omega \left[\frac{k(\omega)^2}{\omega}\, S(\omega)\right]^{N-2}, \tag{S8}$$

which reproduces the bridge relation used in the main text in velocity-PSD form and makes explicit that the dependence on interaction topology enters solely through the exponent $N-2$.

MATCHING TO VISCOUS DISSIPATION: KOLMOGOROV CUTOFF SCALING (SUPPORT FOR FIGURE 2)

Viscous damping originates from the Navier–Stokes operator $\nu\nabla^2\boldsymbol{u}$ and yields, at scaling level, a universal dissipation rate

$$\tau_\nu^{-1}(k) \sim \nu k^2. \tag{S9}$$

The ultraviolet cutoff frequency is experimentally identified as the Kolmogorov cutoff ω_K (Fig. 1 of the main text) and its renormalized scaling with forcing is reported in Fig. 2. It is selected by the crossover condition

$$\tau_{\rm nl}(\omega_K) \sim \tau_\nu(\omega_K), \tag{S10}$$

which terminates the RG flow at finite scale.

To compare different fluids and dispersive regimes, we normalize the exit frequency by the viscous decay rate of the *injection* mode. Let $k_0 \equiv k(\omega_0)$ be the driven wavenumber selected by the dispersion relation at the fixed forcing frequency ω_0. We define the dimensionless ultraviolet coordinate

$$\bar{\Omega}_\nu \equiv \frac{\omega_K}{\nu k_0^2} \;=\; \frac{\text{exit rate}}{\text{viscous decay rate at injection}}, \tag{S11}$$

which measures how far the cascade extends before being arrested by viscous dissipation, in units of the injection-scale viscous rate $\tau_\nu^{-1}(k_0) \sim \nu k_0^2$. This normalization does not require k_0 to be invariant across fluids; it relies only on the microscopic Navier–Stokes operator evaluated at the forced mode.

The definition (S11) cleanly separates the RG-generated ultraviolet extent of the cascade, set by the balance (S10) at the exit scale, from the bare dissipative rate fixed at injection. Changing the precise numerical definition of k_0 by an $\mathcal{O}(1)$ factor rescales $\bar{\Omega}_\nu$ by a constant prefactor but leaves the measured power-law exponents unchanged. The observed Wegner-like crossover scaling in Fig. 2 therefore reflects a genuine Wilsonian property of the viscous cutoff, not an artifact of a particular injection-scale convention. Using the topology-dependent transfer rate in velocity-PSD form,

$$\tau_{\rm nl}^{-1}(\omega) \sim \omega \left[\frac{k(\omega)^2}{\omega}\, S(\omega)\right]^{N-2}, \tag{S12}$$

(see Eq. (S8)), the cutoff inherits its forcing dependence through the inertial KZ spectrum $S(\omega) \sim \Pi^{\chi}\omega^{-p}$, where Π is the conserved energy flux and χ is fixed by weak-turbulence theory [1, 2]. At fixed driving frequency ω_0, the injected kinetic-energy density $E = \frac{1}{2}\rho U_0^2$ sets the flux at scaling level, $\Pi \propto E$ (up to ω_0-dependent constants). Combining these scalings with $k(\omega) \sim \omega^{\alpha}$ in the matching condition (S10) yields the topology-dependent cutoff laws reported in the main text and validated by the collapse in Fig. 2,

$$\bar{\Omega}_\nu^{(C)} \sim \bar{E}^{2/3} \quad (N=3), \qquad \bar{\Omega}_\nu^{(G)} \sim \bar{E}^{1} \quad (N=4). \tag{S13}$$

For each forcing series performed at fixed ω_0, the ultraviolet reach may equivalently be parametrized by $\bar{\Omega}_\omega \equiv \omega_K/\omega_0$; however, $\bar{\Omega}_\nu$ is the more universal normalization across fluids, as it explicitly removes the fluid-dependent viscous rate at injection. In this sense, $\bar{\Omega}_\nu$ is a genuinely Wilsonian ultraviolet coordinate: it compares the RG-generated exit scale ω_K to the bare dissipative operator inherited from microscopic hydrodynamics. The reduced forcing is expressed as $\bar{E} \equiv E/E_0$, with $U_0 = A\omega_0$ and viscous reference energy

$$E_0 \equiv \rho \left(\frac{\nu}{\Lambda_0}\right)^2, \qquad \Lambda_0 \equiv \frac{2\pi}{k_0}. \tag{S14}$$

Here Λ_0 is the injection wavelength selected by the linear dispersion relation at ω_0. Physically, E_0 is the kinetic-energy density associated with viscous relaxation over one forcing wavelength, so that $\bar{E} \sim (U_0\Lambda_0/\nu)^2$ measures the distance from the viscous (Gaussian) regime using only injection-scale quantities, isolating the topology-controlled renormalization of the ultraviolet cutoff.

INTEGRATED INERTIAL SPECTRAL WEIGHT: REYNOLDS SCALING (SUPPORT FOR FIGURE 3)

We connect the experimentally accessible integrated inertial spectral weight

$$\Sigma_{\rm PSD} \equiv \int_{\omega_0}^{\omega_K} S(\omega)\, d\omega, \tag{S15}$$

to the Reynolds-number scaling reported in Fig. 3 of the main text. Here $S(\omega)$ is the surface-velocity PSD measured by LDV, so $\Sigma_{\rm PSD}$ has units of velocity squared and represents the total velocity variance accumulated in the direct cascade between the pump and the ultraviolet exit scale. The upper limit ω_K is retained because it marks the experimentally resolved exit from the inertial basin; however, for the steep inertial spectra observed here ($p > 1$), extending the integral to ∞ does not affect scaling exponents (the integral is pump-dominated).

1. Pump-dominated integral and flux prefactor

For a stationary direct energy-flux cascade, the inertial KZ spectrum takes the scaling form

$$S(\omega) \sim C_N\, \Pi^{\xi}\, \omega^{-p}, \tag{S16}$$

where Π is the constant energy flux through the inertial interval and ξ is fixed by weak-turbulence scaling for the relevant N-wave kinetic equation [1, 2]. Substituting Eq. (S16) into Eq. (S15) gives, for $p \neq 1$,

$$\Sigma_{\rm PSD} \sim \Pi^{\xi}\, \frac{\omega_K^{1-p} - \omega_0^{1-p}}{1-p}. \tag{S17}$$

Because $p > 1$ in both classes, the integral is dominated by its lower limit, so at scaling level

$$\Sigma_{\rm PSD} \sim \Pi^{\xi}\, \omega_0^{1-p} \qquad (p > 1), \tag{S18}$$

with ω_K contributing only a subleading correction $\propto \Pi^{\xi}\omega_K^{1-p}$.

2. Reduced response coordinate and Reynolds control

We define as reduced (dimensionless) cascade response the forcing-normalized integrated variance

$$\bar{\Sigma}_\omega \equiv \frac{\Sigma_{\rm PSD}}{(\Lambda_0\omega_0)^2} = \frac{\text{inertial velocity variance above the pump}}{\text{forcing-scale velocity squared}}, \qquad \Lambda_0 \equiv \frac{2\pi}{k_0}, \tag{S19}$$

where k_0 is the driven wavenumber selected by the dispersion relation at the fixed forcing frequency ω_0. This is the appropriate response coordinate for Fig. 3 because it removes only the trivial forcing-scale units and leaves viscosity to enter solely through renormalization. A viscous normalization, $\Sigma_{\rm PSD}/(\nu/\Lambda_0)^2$, would divide by the relevant perturbation itself and hard-wire an explicit ν^{-2} factor, masking the crossover physics.

In the pump-dominated regime $p > 1$, Eqs. (S18)–(S19) imply, at fixed ω_0 and Λ_0,

$$\bar{\Sigma}_\omega \sim \Pi^{\,\xi}, \tag{S20}$$

so the Reynolds dependence of $\bar{\Sigma}_\omega$ reflects how viscosity renormalizes the inertial flux (equivalently the injection-scale coupling).

The experimentally accessible control parameter is the injection Reynolds number

$$\mathrm{Re} = \frac{U_0 \Lambda_0}{\nu}, \qquad U_0 = A\omega_0. \tag{S21}$$

3. WF/Wegner scaling and quadratic response to the WF field

The Wilson–Fisher (WF) framework promotes the squared steepness $g(\omega) = \epsilon(\omega)^2$ to a running coupling whose flow is controlled by

$$\beta(g) = \frac{dg}{d\ell} = y_g g - B_N g^{\,N-1}, \qquad \ell \equiv \ln(\omega/\omega_0), \tag{S22}$$

with interaction topology fixing the first nonlinear saturation channel $g^{\,N-1}$ (triads: $N = 3$ for CW; tetrads: $N = 4$ for GW). The reduced response follows a Wegner crossover form,

$$\bar{\Sigma}_\omega(\mathrm{Re}) \;=\; \mathrm{Re}^{\kappa_N}\, \mathcal{F}_N\!\left(\frac{\mathrm{Re}}{\mathrm{Re}_\times}\right), \tag{S23}$$

where $\mathcal{F}_N$ is topology dependent and $\mathrm{Re}_\times$ is nonuniversal. The key closure supported by Fig. 3 is that the forcing-normalized integrated variance is quadratic in the coarse-grained WF field at injection,

$$\bar{\Sigma}_\omega \;\propto\; g_0^2, \qquad g_0 \equiv g(\omega_0). \tag{S24}$$

This dependence is dynamical: in weak WT the transfer amplitude is proportional to the interaction vertex and hence to the steepness ϵ, so the leading variance production above the pump scales as the square of that amplitude, $\sim \epsilon^2$. Since $g = \epsilon^2$, the leading cascade-induced contribution scales as $\delta\bar{\Sigma}_\omega \sim g_0^2$, consistent with the lowest non-vanishing self-renormalizations summarized in End Matter, Fig. 4.

4. Topology-controlled Reynolds scaling of the cascade response

In the asymptotic WF regime ($\mathrm{Re} \gg \mathrm{Re}_\times$), the crossover function in Eq. (S23) saturates, so $\bar{\Sigma}_\omega \sim \mathrm{Re}^{\kappa_N}$. To determine κ_N, we write the injection-scale effective coupling as the single-parameter crossover ansatz $g_0 \sim \mathrm{Re}^{\gamma_N}$. The exponent γ_N is fixed by topology: the first nonlinear feedback that arrests the WF growth is the $g^{\,N-1}$ term in Eq. (S22), so the minimal injection-level equation of state is $\mathrm{Re} \propto g_0^{N-1}$, i.e.

$$g_0 \sim \mathrm{Re}^{\,1/(N-1)}. \tag{S25}$$

Combining Eq. (S25) with the quadratic response law Eq. (S24) gives

$$\bar{\Sigma}_\omega \sim \mathrm{Re}^{\,2/(N-1)}, \tag{S26}$$

hence $\kappa_N = 2/(N-1)$. Therefore,

$$\bar{\Sigma}_\omega^{(CW)} \sim \mathrm{Re}^1 \quad (N = 3), \qquad \bar{\Sigma}_\omega^{(GW)} \sim \mathrm{Re}^{2/3} \quad (N = 4), \tag{S27}$$

in agreement with the collapse in Fig. 3 of the main text.

RAW INTEGRATED INERTIAL SPECTRAL WEIGHT (SUPPLEMENTARY FIG. S1)

We report the non-normalized integrated inertial spectral weight

$$\Sigma_{\rm PSD} \equiv \int_{\omega_0}^{\omega_K} S(\omega)\, d\omega, \tag{S28}$$

where $S(\omega)$ is the LDV surface-velocity PSD and ω_K is the experimentally resolved viscous cutoff. Since $\Sigma_{\rm PSD}$ has units of velocity squared, it measures the total velocity variance accumulated in the direct cascade between the pump and the ultraviolet exit scale.

Supplementary Fig. S1 plots $\Sigma_{\rm PSD}$ versus the forcing velocity $U_0 = A\omega_0$ for gravity-wave (GW) and capillary-wave (CW) cascades across all fluids. In both classes, $\Sigma_{\rm PSD}$ increases monotonically with forcing, showing direct activation of the inertial response above the pump. The raw curves are not expected to collapse because $\Sigma_{\rm PSD}$ retains explicit dependence on the forcing scale and material parameters; the corresponding reduced response $\bar{\Sigma}_\omega$ and its Reynolds renormalization are reported in Fig. 3 of the main text.

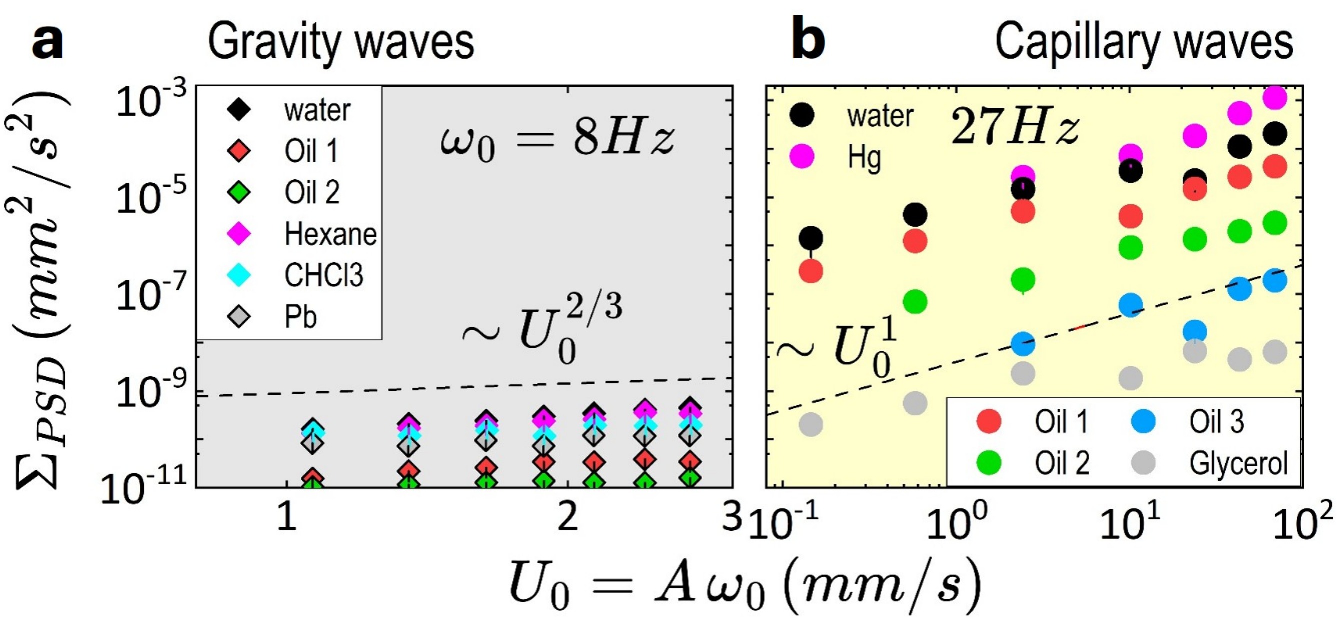

FIG. 1. **Supplementary Fig. S1: Raw integrated inertial spectral weight.** Integrated inertial spectral weight $\Sigma_{\rm PSD} = \int_{\omega_0}^{\omega_K} S(\omega)\, d\omega$ (LDV surface-velocity PSD) versus forcing velocity $U_0 = A\omega_0$ for gravity waves (left; $\omega_0 = 8\,\mathrm{Hz}$) and capillary waves (right; $\omega_0 = 27\,\mathrm{Hz}$), across all fluids (legend). The GW branch lies ~ 3 decades below the CW branch over the full forcing range. The forcing-normalized response $\bar{\Sigma}_\omega \equiv \Sigma_{\rm PSD}/(\Lambda_0\omega_0)^2$ and its Reynolds renormalization are shown in Fig. 3 of the main text.

ORTHOGONALITY OF RG COORDINATES: EXIT SCALE VERSUS RESPONSE

The two reduced observables used in the main text, the exit coordinate $\bar{\Omega}_\nu$ (Fig. 2) and the cascade response $\bar{\Sigma}_\omega$ (Fig. 3), probe complementary pieces of the WF structure. The exit coordinate $\bar{\Omega}_\nu = \omega_K/(\nu k_0^2)$ is normalized by the viscous decay rate of the injection mode, so it measures the ultraviolet reach of the inertial basin relative to the bare Navier–Stokes dissipative operator. By contrast, $\bar{\Sigma}_\omega = \Sigma_{\rm PSD}/(\Lambda_0\omega_0)^2$ is normalized only by forcing-scale quantities and measures the strength of the cascade, i.e. the velocity variance accumulated above the pump; using a viscous normalization here would mix the response with the relevant perturbation that drives the crossover. This asymmetric normalization is deliberate: $\bar{\Omega}_\nu$ quantifies where the flow stops (viscous arrest), whereas $\bar{\Sigma}_\omega$ quantifies how strongly it proceeds (nonlinear saturation). Together, $(\bar{\Omega}_\nu, \bar{\Sigma}_\omega)$ form an orthogonal pair of RG coordinates that separately resolve reach and intensity in DSWT.

SCOPE OF VALIDITY

The arguments presented here are valid at the scaling level within the weakly nonlinear DSWT regime, where (i) the inertial interval exhibits KZ-type power-law behavior, (ii) transfer is mediated predominantly by the resonant interaction topology ($N = 3$ for CW and effective $N = 4$ for GW), and (iii) viscous damping can be represented at scaling level by $\tau_\nu^{-1}(k) \sim \nu k^2$. The crossover matching $\tau_{\rm nl}(\omega_K) \sim \tau_\nu(\omega_K)$ is intended as a scaling criterion for the exit from the inertial basin.

Weak-nonlinearity diagnostic

We monitored the forcing-scale steepness $\epsilon_0 \equiv k_0 A$ as a dimensionless proxy for nonlinear frequency shifts and resonance broadening. Across all datasets reported here, ϵ_0 remains in the range

$$10^{-3} \lesssim \epsilon_0 \lesssim 10^{-1},$$

well below unity and firmly within the weakly nonlinear regime. In this range, the pump remains spectrally sharp, higher harmonics are subdominant, and a clear KZ-type inertial interval is resolved above ω_0.

As ϵ_0 approaches unity, nonlinear frequency shifts become comparable to linear mode spacing, leading to strong resonance broadening, cluster overlap, and eventual breakdown of topology-controlled transfer. Such conditions are expected to invalidate the present Wilson–Fisher crossover description and drive a transition toward strong (Philips-like) turbulence, a regime not explored in the present experiments.

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